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BASIC PROPERTIES OF REAL ANALYTIC AND SEMIANALYTIC GERMS.

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We give an expository account of the basic features of analytic and semianalytic germs. The main results treated here are: Risler's Nullstellensatz, the Curve Selection Lemma and the Finiteness Theorem for semianalytic germs. The method to prove them consists of a real Local Parametrization Theorem and an analytic version of Thom's General Lemma. Finally, all these properties suggest that real spectra behave for analytic algebras as properly as for finitely generated algebras over $\mathbb{R}$.

1. Let us recall first some definitions. Consider an open set $\Omega \subset \mathbb{R}^n$.

An analytic function $f: \Omega \to \mathbb{R}$ is a mapping with a convergent power series expansion at each point of Ω. Then f is differentiable (C^∞) in Ω.

An analytic set $X \subset \Omega$ is one such that for each $x \in \Omega$ there are analytic functions $f_i: U \to \mathbb{R}$ (defined in a neighborhood U of x) with

$$(1.1) \qquad X \cap U = \{f_1=0,\dots,f_r=0\}$$

(and, of course, one single function suffices: $f = \Sigma f_i^2$). Then X is closed in Ω.

A <u>semianalytic set</u> $Z \subset \Omega$ is one such that for each $x \in \Omega$ there are analytic functions $g_{ij}, f_i : U \to \mathbb{R}$ with

$$(1.2) \qquad X \cap U = \bigcup_{i=1}^{r} \{g_{i1} > 0,\dots,g_{ir} > 0, f_i=0\}$$

Let M be a subset of Ω. A <u>regular point of dimension d</u> of M is one $x \in M$ such that there are analytic functions $f_i : U \to \mathbb{R}$ as in 1.1 verifying also

$$(1.3) \qquad \operatorname{rank} \left[\frac{\partial f_i}{\partial x_j}(x)\right] = r = n-d.$$

This number d is well defined and only depends on the couple M,x. It is denoted by $\dim_x M$. The set of all regular points of M is open in M, and denoted by M^{o}. The function $x \mapsto \dim_x M$ is constant on each connected component of M^{o}.

An <u>analytic manifold</u> is a subset $M \subset \Omega$ such that $M = M^{o}$. The basic results of the calculus on differential manifolds can be translated for analytic manifolds, with two essential differences:

- Partitions of unity are not available in the analytic category. This affects to globalization problems and we shall not enter in such questions here.

- The <u>identity principle</u>, which is at the basis of the precedent lack. It can be formulated as follows

(1.4) Let X be an analytic subset of Ω and M a connected analytic manifold. If $X \cap M$ has non-empty interior in M, then $X \supset M$.

2. Analytic germs.- In order to study functions and sets at a given point, say the origin $0 \in \mathbb{R}^n$, we introduce the notion of germ.

(2.1) Two analytic functions $f: U \to \mathbb{R}$, $g: V \to \mathbb{R}$ defined in some neighborhoods of 0 are identified if they coincide on another neighborhood $W \subset U \cap V$. This way we define the germ of f at 0, which is denoted by f_o. These analytic function germs form an R-algebra, $\mathcal{O}_n$, canonically isomorphic to the ring of convergent power series, $\mathbb{R}\{x\}$, in the variables $x = (x_1,\dots,x_n)$.

(2.2) Two sets, $X,Y \subset \mathbb{R}^n$ are equivalent if $X \cap U = Y \cap U$ for some neighborhood U of 0. The class of X for this relation is called germ at 0 of X and denoted by X_o. Finite unions, intersections, are well defined for set-germs via representatives. Also the complements and the inclusion relation, and the standard properties are checked immediately. Just the same for topological notions like closure, interior, etc.

Now we introduce two operators:

(2.3) If $E \subset \mathcal{O}_n$, then $E.\mathcal{O}_n = \{f_{1,o},\dots,f_{r,o}\}\mathcal{O}_n$ for some analytic functions $f_1,\dots,f_r: U \to \mathbb{R}$ and we put $V(E)$ = germ at 0 of $\{f_1=0,\ldots,f_r=0\}$. (This does not depend on the choice of generators).

(2.4) If X_o is a set-germ, we denote by $I(X_o)$ the ideal of all germs of analytic functions $f\colon U \to \mathbb{R}$ with $f|X \cap U \equiv 0$.

Thus we turn our attention to analytic germs, i.e. the germs of analytic sets of open neighborhoods of 0. They are the germs $X_o = V(E)$ for $E \subset \mathcal{O}_n$, and verify $X_o = VI(X_o)$. An analytic germ is irreducible if it is not a union of proper analytic subgerms. It is easy to see that X_o is irreducible if and only if $I(X_o)$ is prime. Besides, by means of the associated prime ideals of $I(X_o)$, we decompose in a unique way every analytic X_o into a finite union of irreducible germs, none of which is contained in the others.

To an analytic germ $X_o \subset \mathbb{R}^n_o$ there is associated an analytic algebra $\mathcal{O}[X_o] = \mathbb{R}\{x\}/I(X_o)$. The correspondence $X_o \mapsto \mathcal{O}[X_o]$ gives in fact an equivalence of categories. For instance: X_o is regular (i.e. 0 is a regular point of X) if and only if $\mathcal{O}[X_o]$ is a regular ring. In that case $\dim \mathcal{O}[X_o] = \dim_0 X$. This suggest the following

Definition (2.5).- $\dim Z_o = \dim \mathcal{O}[X_o]$, $X_o = VI(Z_o)$, Z semianalytic.

(Later on we shall find a geometric characterization of the dimension).

3. Risler's Nullstellensatz ([R]).- Like in the complex case the interesting question about V and I is to compute IV(I) for $I \subset \mathcal{O}_n$. In the real setting this computation is more complicated, and was done by Risler. He proved:

Theorem (3.1).- IV(I) = real-radical of I.

To deduce this result one must obtain first a good parametric description of an analytic germ: the Local Parametrization Theorem. Then a common topic in Real Geometry comes to the stage: Orderings on function rings and Specialization Criteria, which bring naturally to our study the semianalytic germs, i.e. the germs of semianalytic sets of open neighborhoods of 0.

4. Local Parametrization ([M], [N]).- It is a refined geometric version of a purely algebraic result concerning analytic algebras.

Let p be a prime ideal of $\mathcal{O}_n = \mathbb{R}\{x\}$, set $r = \mathrm{ht}\,p$, $d = n-r$, $x' = (x_1,\dots,x_d)$.

Noether's normalization lemma (4.1).- After a (generic) linear change of coordinates the following conditions hold true:

(1) The canonical homomorphism $A = \mathbb{R}\{x'\} \to \mathbb{R}\{x\}/p = B$ is finite and injective.

(2) $\chi = x_{d+1} \bmod p$ is a primitive element of $L = \mathrm{qf}B$ over $K = \mathrm{qf}A$ and the polynomial $P = \mathrm{Irr}(\chi, K) \in K[x_{d+1}]$ is

actually a Weierstrass polynomial in $A[x_{d+1}]$, i.e.
$P = x_{d+1}^{p}+a_1(x')x_{d+1}^{p-1}+\ldots+a_p(x')$, $a_i(x') \in A$, $a_i(0) = 0$,
p = degree (P).

(3) The discriminant $\delta \in A \setminus \{0\}$ of P is a universal denominator:

$$\delta . B \subset A+A.\chi+\ldots+A.\chi^{p-1}$$

(A proof can be found in [T]).

To translate this into geometric terms, set $X_o = V(p)$ and choose representatives X, δ, P of all involved data, defined in some open polycylinder $U \subset \mathbb{R}^n$. The inclusion $\mathbb{R}\{x'\} \to \mathbb{R}\{x\}$ is, of course, represented by the linear projection $\pi: \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d} \to \mathbb{R}^d$ and we put $W = \pi(U)$. Then

<u>Local Parametrization Theorem</u> (4.2).- The polycylinder U can be chosen arbitrarily small verifying:

(1) The restriction $\pi: X \to W$ is proper and finite-to-one (although not necessarily onto).

(2) $X \setminus \{\delta=0\}$ has finitely many connected components and each one of them is an analytic manifold of dimension d, adherent to 0.

(3) The restriction of π to each connected component D of $X \setminus \{\delta=0\}$ is an analytic diffeomorphism onto a connected component of $W \setminus \{\delta=0\}$.

<u>Sketch of proof</u>.- The main part of the argument, which we

hint here, refers to item (2). We start by finding some suitable equations for $X\setminus\{\delta=0\}$. This is done as follows. Since δ is an universal denominator, 4.1.3, there are polynomials $Q_j \in A[x_{d+1}]$, $j=d+2,\dots,n$, such that

$$\delta(x')x_j = Q_j(x',x_{d+1}) \bmod p;$$

but also $\quad P(x',x_{d+1}) = 0 \quad \bmod p,$

and from this system we deduce, after some technical reductions of U and W a description for $X\setminus\{\delta=0\}$:

$$(4.3)\quad X\setminus\{\delta=0\} = \{x \in W\times\mathbb{R}^{n-d} : P(x',x_{d+1})=0, \delta(x')x_j=Q_j(x',x_{d+1}), \delta(x')\neq 0\}$$

Then take a point $a' = \pi(a)$, $a \in X\setminus\{\delta=0\}$ and let C stand for the connected component of a' in $W\setminus\{\delta=0\}$. To find the component D lying over C via π we only need to solve $P(x',x_{d+1}) = 0$ for $x' \in C$, since the coordinates $x_{d+2},\dots,x_n$ are easily obtained then. But we can solve $P(x',x_{d+1}) = 0$ locally by the implicit function theorem, because $\frac{\partial P}{\partial x_{d+1}}(x',x_{d+1}) \neq 0$ if $\delta(x') \neq 0$ and $P(x',x_{d+1}) = 0$. To glue together these local solutions we use the fact that for all $x' \in C$, the polynomials in x_{d+1}, $P(x',x_{d+1}) \in \mathbb{R}[x_{d+1}]$ have the same number of real roots, and so we can label them. This way we have analytic functions $h_\ell : C \to \mathbb{R}$ such that $P(x',h_\ell(x')) = 0$, $x' \in C$. Clearly

$$D_\ell = \{x : x' \in C,\ x_{d+1}=h_\ell(x'), x_j=Q_j(x',h_\ell(x'))/\delta(x')\}$$

are the connected components of $X\setminus\{\delta=0\}$ lying over C and $D = D_\ell$ for some ℓ.

Finally we must prove that there are finitely many C's. This is an assertion on $W \setminus \{\delta=0\}$ and is deduced by induction on d with an argument similar to the precedent one with the D_ℓ's (some standard use of Weierstrass' theorem is needed previously, see [M] or [N]).

Let us remark here that the method above can be considered the initial step to stratify the set of zeroes of a Weierstrass polynomial: we drop all multiple roots and work with the implicit function theorem. This analysis can be carried on to get more precise descriptions of semianalytic germs (cf. n°8).

Before continuing we get an important consequence of 4.2:

Semicontinuity of the dimension (4.4).- Let X be an analytic set, $0 \in X$. Then:

$$\dim_0 X = \text{upper} \lim_{x \to 0, x \in X^\circ} \dim_x X$$

Indeed, one reduces to $I(X_o) = \mathfrak{p}$ prime, using the decomposition into irreducible components. Then $\delta \notin I(X_o)$ and $X_o \setminus V(\delta) \neq \emptyset$. So 0 is adherent to $X \setminus \{\delta=0\}$, and 4.4 follows immediately.

Let us only remark that the same holds true for a semianalytic set Z and the proof is the same, after some manipulations with the local equations of Z.

5. Proof of Risler's Nullstellensatz.- It is clear that the statement in 4.2 has no meaning if $X \setminus \{\delta=0\} = \emptyset$. As a matter of fact this is a key aspect of the proof of Risler's theorem. Let us first remark a useful reality criterion (compare with the proof of this lemma given in [R]).

Lemma (5.1).- If $\dim X_o = d$, then $p = I(X_o)$.

In other words, if the germ $V(p)$ has the right dimension, then $p = IV(p)$. For, in any case: $n-d = \mathrm{ht}p \leqq \mathrm{ht}I(X_o) =$ $= n - \dim X_o$, so that under that hypothesis on the dimension we conclude $\mathrm{ht}p = \mathrm{ht}I(X_o)$; p being prime, both ideals coincide.

Now we notice that $\dim X_o = d$ is just the same that $X_o \not\subset \{\delta=0\}$. Otherwise, we would have: $\dim X_o =$ $= \dim V(p+\delta.\mathbb{R}\{x\}) \leqq n - \mathrm{ht}(p+\delta.\mathbb{R}\{x\}) < n - \mathrm{ht}p = d$, since $\delta \notin p$. Thus $\dim X_o = d$ is equivalent to $X \setminus \{\delta=0\} \neq \emptyset$ (for arbitrarily small U, as usual).

Finally, in view of 4.3, the latter condition can be written:

(5.2) There are $x' \in W \setminus \{\delta=0\}$ arbitrarily close to $0' \in \mathbb{R}^d$ such that the polynomial in x_{d+1}, $P(x',x_{d+1}) \in \mathbb{R}[x_{d+1}]$, has some real root.

So, assume p is real. To prove Risler's Nullstellensatz we only need to check 5.2. This turns out to be a specialization problem. Indeed, $P(x',x_{d+1})$, as a polynomial in

$\mathbb{R}\{x'\}[x_{d+1}] = A[x_{d+1}]$, has the root $\chi = x_{d+1} \bmod p \in B =$ $= \mathbb{R}\{x\}/p$ and B is formally real. By means of Sturm's theorem, the existence of real roots is tested by looking at the signs of some series in $A = \mathbb{R}\{x'\}$. Thus 5.2 follows from a standard specialization criterion:

(5.3) Let $f_1,\dots,f_m \in \mathbb{R}\{x'\}$ be positive in some total ordering in $\mathbb{R}\{x'\}$. Then there are $x' \in \mathbb{R}^d$ arbitrarily close to $0' \in \mathbb{R}^d$ such that $f_1(x')>0,\dots,f_m(x')>0$.

The proof of this lemma is most similar to the one in the algebraic case. As usual in dealing with analytic germs, Weierstrass' theorem makes possible an induction on d (see [R]).

6. <u>General Specialization Criterion</u>.- The ideas in Risler's proof can be extended a bit to formulate a specialization result that includes the Nullstellensatz and 5.3.

Consider a prime ideal $p \subset \mathbb{R}\{x\}$ and $X_o = V(p)$. Let X_o^* stand for the maximum dimension locus of X_o:

$$X_o^* = \text{germ at } 0 \text{ of } \{x \in X: \dim_x X = \dim_o X\}$$

(Later on we shall see that X_o^* is semianalytic. By now we remark that $\overline{X \setminus \{\delta=0\}}$ is a representative of X_o^*. The result we are to state could be equivalently formulated with $X_o \setminus V(\delta)$ instead of X_o^*).

<u>Specialization lemma</u> (6.1).- Let $f_1,\dots,f_m \in \mathbb{R}\{x\}$. Their classes mod p are positive in some total ordering of $\mathbb{R}\{x\}/p$ if and only if the germ $\{f_1 > 0,\dots,f_m > 0\} \cap X_o^*$ is not empty.

The proof consists of applying the Nullstellensatz to the germ

$$Y_o = (X_o \times \mathbb{R}_o^m) \cap \{t_i^2 = f_i,\ i=1,\dots,m\} \subset \mathbb{R}_o^n \times \mathbb{R}_o^m ,$$

where $t_1,\dots,t_m$ are the coordinates in $\mathbb{R}_o^m$. This gives an implication, and the converse follows from Serre's criterion.

As a consequence also Risler's solution to Hilbert's 17th Problem for $X_o = \mathbb{R}_o^n$ can be generalized to any singular X_o:

<u>Hilbert's 17th Problem</u>.- Let $f \in \mathcal{O}[X_o]$ be an analytic function germ on an irreducible X_o. Then f is a sum of squares of meromorphic function germs on X_o if and only if f is ≥ 0 on X_o^*.

This can have further generalizations (non-negativity criterions, McEnerney [McE], Ruiz [Rz]), which translate to germs the algebraic results by Stengle ([St]).

7. <u>The Curve Selection Lemma</u>.- A very interesting aspect of the Specialization lemmas and Nullstellensatz is their connection with that essential keystone in analytic geometry. In the end the contents of n$^{\underline{os}}$ 5 and 6 are a step before the curve selection. We shall present here a proof that shows clearly this connection.

First of all, according to Merrien [Mn] and Lassalle, [Ls], we may very easily repeat all proofs in the previous sections to get.

<u>Lemma</u> (7.1).- Let $p \subset \mathbb{R}\{x\}$ be a prime ideal and $f_1,\dots,f_m \in \mathbb{R}\{x\}$. The classes mod p of $f_1,\dots,f_m$ are positive in some total ordering in $\mathbb{R}\{x\}/p$ if and only if there is an analytic homomorphism $\psi: \mathbb{R}\{x\} \to \mathbb{R}\{t\}$ such that

(1) $\ker\psi \supset p$; (2) $\psi(f_1)>0,\dots,\psi(f_m)>0$

($\mathbb{R}\{t\}$ is endowed with the ordering $t > 0$).

The idea supporting this is quite natural. In a germ we do not have any points, except the 0, which is useless because it is <u>always there</u>. Consequently, dimension one appears as the smallest dimension of a subgerm. One realizes, thus, that in our context "points" are curve germs; or more accurately, homomorphisms $\mathbb{R}\{x\} \to \mathbb{R}\{t\}$, like ψ in 7.1. Since the real closure of $\mathbb{R}\{t\}$ is well-known (Puiseux' theorem), a good theory can be developed for these "points".

Now we deduce the classical Curve Selection Lemma from 7.1:

<u>Curve Selection</u> (7.2).- Let Z be a semianalytic subset of an open set $\Omega \subset \mathbb{R}^n$, and $0 \in \Omega$ a point adherent to Z. Then there is an analytic curve $[0,\varepsilon) \to \mathbb{R}^n$ such that:

$$c(0) = 0 \ ; \quad c(t) \in Z, \text{ all } t \neq 0$$

For the proof we may suppose $Z = \{g_1>0,\dots,g_m>0,f=0\}$ for some analytic functions g_i, f. Set $X = \{f=0\}$. Without loss of generality we can also assume

$$p = I(Z_o) = I(X_o) \text{ and } X_o \text{ irreducible.}$$

Then, apply Local Parametrization, and

$$Z_o \setminus \{\delta=0\} = \{g_1>0,\dots,g_m>0\} \cap X_o \setminus \{\delta=0\} \subset \{g_1>0,\dots,g_m>0\} \cap X_o^*$$

$$Z_o \setminus \{\delta=0\} \neq \emptyset$$

For, if 0 were not adherent to $Z \setminus \{\delta=0\}$, δ would belong to $I(Z_o) = p$, contradiction. Hence, by 6.1, there is a total ordering in $\mathbb{R}\{x\}/p$ making positive the classes of the g_i's. It follows from 7.1 that there is

$$\psi: \mathbb{R}\{x\} \to \mathbb{R}\{t\}, \quad x = (x_1,\dots,x_n) \mapsto x(t) = (\psi(x_1),\dots,\psi(x_n))$$

such that

$$g_i(x(t)) = \psi(g_i) > 0 \quad \& \quad f(x(t)) = \psi(f) = 0.$$

Clearly, $c: [0,\varepsilon) \to \mathbb{R}^n: t \mapsto x(t)$ is the curve we sought.

8. Separating families.- We shall describe now the analytic version of this useful notion, well known in the algebraic context (see [H]). We begin with some terminology, which must be very carefully chosen in our setting.

(8.1) Let $f_1,\dots,f_m$ stand for some convergent power series in $\mathbb{R}\{x\}$ as well as for the associated analytic functions $f_1,\dots,f_m: U \to \mathbb{R}$ defined in some neighborhood U of $0 \in \mathbb{R}^n$.

For any choice of signs $\theta = (\theta_1, \ldots, \theta_m)$:

$$\theta_i = \{t>0\} \quad \text{or} \quad \{t=0\} \quad \text{or} \quad \{t<0\},$$

we put

$$\Sigma_U(\theta) = \{x \in U: f_1(x) \in \theta_1, \ldots, f_m(x) \in \theta_m\}$$

We shall write: $\theta^= = (\theta_1 \cup \{0\}, \ldots, \theta_m \cup \{0\})$, i.e. $\theta^=$ is obtained from θ by relaxing inequalities.

<u>Definition</u> (8.2).- We say that $f_1, \ldots, f_m$ is a <u>separating family</u> if there are arbitrarily small neighborhoods U of $0 \in \mathbb{R}^n$ such that for any choice of θ with $\Sigma_U(\theta) \neq \emptyset$ it holds:

(1) $\Sigma_U(\theta)$ is connected

(2) The closure of $\Sigma_U(\theta)$ in U is obtained by relaxing inequalities, i.e. it coincides with $\Sigma_U(\theta^=)$.

Then, the main fact is

<u>Thom's general lemma</u> (8.3).- Any finite collection $f_1, \ldots, f_s$ can be extended to a separating family.

We shall sketch a proof of 8.3 in Section 9, and give here some interesting consequences.

Let Z be a semianalytic set of Ω = open in $\mathbb{R}^n$. Fix a point $0 \in \Omega$. Then there are analytic functions $g_{ij}, h_i: U \to \mathbb{R}$ in a neighborhood U of 0 with:

$$Z \cap U = \bigcup_{i=1}^{s} \{g_{i1} > 0, \ldots, g_{ir} > 0, h_i = 0\}$$

Now we can extend the collection of all g_{ij}, h_i to a separating family $f_1, \ldots, f_m$. This means that <u>after a reduction of U</u> the sets

$$\Gamma = \Sigma_U(\theta) \neq \emptyset$$

verify the conditions in 8.2 (of course, we only consider the non-empty Γ's). As the functions defining $Z \cap U$ are among the $f_1, \ldots, f_m$, we have:

$$Z \cap U = \bigcup_{\Gamma \subset Z} \Gamma$$

Furthermore, as there are finitely many Γ's, there is a neighborhood $W \subset U$ of 0 with:

$$Z \cap W \subset \bigcup_{0 \in \overline{\Gamma}} \Gamma \subset Z \cap U .$$

It follows that when $0 \in Z$, $\bigcup_{0 \in \overline{\Gamma}} \Gamma$ is a connected (not necessarily open!) neighborhood of 0 in $Z \cap U$. In other words:

(8.4) Z is locally connected.

On the other hand, if Z^* is a connected component of Z we have

$$Z^* \cap U = \bigcup_{\Gamma \subset Z} (\Gamma \cap Z^*) = \bigcup_{\Gamma \subset Z^*} \Gamma$$

because each Γ is connected. This shows

(8.4) Z^* is a semianalytic set of Ω.

Finally, there must be fewer Z^*'s than Γ's and so:

(8.5) The family of all connected components of Z is locally finite.

Actually we get more, namely

(8.6) For U small enough the number of connected components of $Z \cap U$ adherent to 0 remains constant, and so we have a well defined notion of connected components of the semianalytic germ Z_o. This connected components are, of course, semianalytic germs.

Now we look at the closure of Z (in Ω). Clearly:

$$\overline{Z} \cap U = \bigcup_{\Gamma \subset Z} (\overline{\Gamma} \cap U)$$

But by 8.2.2 we see that $\overline{Z} \cap U$ is semianalytic and defined by non-strict inequalities. Thus:

<u>Finiteness Theorem</u> (8.7).- The closure (resp. interior) of Z in Ω is semianalytic, and given locally by non-strict (resp. strict) inequalities.

(For the "open part", just take complements). For instance, as was quoted in n°6, the maximum dimension locus of an analytic germ is semianalytic. Actually we can get more:

(8.8) The set Z^d of d-dimensional points in Z is semianalytic.

First notice that it suffices to prove 8.8 for the maximum dimension d, up to an induction argument. On the other hand some manipulations with the local equations of Z at a fixed point $0 \in \Omega$ show that the essential case is the following:

$$Z_o = \{f_1>0,\dots,f_r>0\} \cap X_o \qquad I(X_o) = I(Z_o)$$

where X_o is an irreducible analytic germ. Then one applies Local Parametrization and there is some δ with

$$\emptyset \neq Z_o \setminus \{\delta=0\} = \{f_1>0,\dots,f_r>0\} \cap X_o \setminus \{\delta=0\}$$

$$\dim X_o \cap \{\delta=0\} < \dim X_o = d$$

But in this situation, $Z_o^d = \overline{Z_o \setminus \{\delta=0\}}$, because if $x \in Z \setminus \overline{Z \setminus \{\delta=0\}}$, there is an open neighborhood W of x with

$$Z \cap W \subset X \cap \{\delta=0\}$$

and for x close to 0 it follows $\dim_x Z = \dim_x Z \cap W \leqq$ $\leqq \dim_x X \cap \{\delta=0\} < d$. Consequently, by 8.7, Z_o^d is semianalytic.

9. Proof of Thom's general lemma.- It is essentially the same as in the algebraic case over the reals, but here we need to control the neighborhoods we consider at each step. The argument that follows is due to Fernández-Recio ([FR]).

We work by induction on the number n of variables. For n=1 the assertion is immediate, because $\{f_1,\dots,f_m,t\}$ is separating. Indeed, take any open interval U with $0 \in U \subset \mathbb{R}$, such that 0 is the only possible zero of the f_i's (if any) in U. The conditions in 8.2 are evident.

Now assume the result for fewer than n variables. First a linear change of coordinates and Weierstrass' Preparation Theorem allow us to suppose

$$f_i(x',x_n) = x_n^{p_i}+a_{i1}(x')x_n^{p_i-1}+\ldots+a_{ip_i}(x') \in \mathbb{R}\{x'\}[x_n]$$

where $a_{i\ell}(0) = 0$, $x' = (x_1,\ldots,x_{n-1})$. Of course the $a_{i\ell}$'s define analytic functions in some open neighborhood W of $0 \in \mathbb{R}^{n-1}$. We also consider the derivatives:

$$f_{ij} = \frac{\partial^j f}{\partial x_n^j} \in \mathbb{R}\{x'\}[x_n],$$

which are Weierstrass polynomials too, defined in $W \times \mathbb{R}$.

The next step is to produce a suitable partition of W. By a nice argument of Lojasiewicz, we find analytic functions $g_1,\ldots,g_r$: $W \to \mathbb{R}$ such that for any choice of signs $\theta' = (\theta'_1,\ldots,\theta'_r)$, over the set

$$\Sigma_W(\theta') = \{x' \in W: g_1(x') \in \theta'_1,\ldots,g_r(x') \in \theta'_r\}$$

any product $\Psi = \Pi f_{ij}$ of f_{ij}'s has a constant number of real roots, say $N = N(\Psi,\theta')$, i.e.:

(9.1) For each $x' \in \Sigma_W(\theta')$ the polynomial in x_n, $\Psi(x',x_n) \in \mathbb{R}[x_n]$ has exactly $N = N(\Psi,\theta')$ distinct real roots (eventually $N = +\infty$).

We remark that the $g_1,\ldots,g_r$ are obtained from the coefficients of the Ψ's by a suitable linear algebra trick. ([Ł] p.106).

Finally, we apply the induction hypothesis and extend $g_1,\ldots,g_r$ to a separating family $g_1,\ldots,g_s \in \mathbb{R}\{x'\}$. We claim that $\{g_k,f_{ij}\}$ is separating.

For, we take an open neighborhood of $0 \in \mathbb{R}^{n-1}$, $U' \subset W \subset \mathbb{R}^{n-1}$, such that the conditions in 8.2 hold for U' and the g_k's. Consider $\varepsilon > 0$. We may assume that for each $x' \in U'$ the polynomials in x_n, $f_{ij}(x',x_n) \in \mathbb{R}[x_n]$, have all their roots in $|x_n| < \varepsilon$ (continuity of the roots and the fact $f_{ij}(0,x_n) = x_n^{p_{ij}}$). Then set $U = U'x(-\varepsilon,\varepsilon)$. Obviously U can be arbitrarily small, so we only have to study a non-empty set

$$\emptyset \neq \Gamma = \Sigma_U(\theta) = \{x=(x',x_n)\in U = U'x(-\varepsilon,\varepsilon):g_k(x')\in\theta_k,f_{ij}(x)\in\theta_{ij}\}$$

where $\theta = (\theta_k,\theta_{ij})$ is a choice of signs.

If we put $\theta' = (\theta_k)$, then

$$\Gamma' = \Sigma_{U'}(\theta') = \{x' \in U': g_k(x') \in \theta_k\} \supset \Pi(\Gamma) \neq \emptyset$$

and so it is a connected subset of U' whose closure in U' is obtained by relaxing inequalities. We are to built up Γ from Γ' using the roots of the polynomial

$$\Psi = \Pi\{f_{ij}: f_{ij} \not\equiv 0 \text{ on } \Gamma'\}$$

Since $U' \subset W$, and $\Psi \not\equiv 0$ on Γ', it has by 9.1 $N < +\infty$ real roots all through Γ'. This Γ' being connected, there are continuous functions

$$\xi_1,\ldots,\xi_N: \Gamma' \to \mathbb{R}$$

such that

(9.2) For each $x' \in \Gamma'$, $\xi_1(x') <\ldots< \xi_N(x')$ are the real roots of $\Psi(x',x_n) \in \mathbb{R}[x_n]$. (see [Ł] p.109).

Notice that if $N = 0$, then $\Gamma = \Gamma' x(-\varepsilon,\varepsilon)$ and the result is immediate. On the other hand, we know that $-\varepsilon < \xi_1$, $\xi_N < \varepsilon$ because of the condition on the roots of the f_{ij}. Put $\xi_o \equiv -\varepsilon$, $\xi_{N+1} \equiv \varepsilon$ and consider the sets

$$B = \{x \in \Gamma' x(-\varepsilon,\varepsilon): \xi_\ell(x') < x_n < \xi_{\ell+1}(x')\} \qquad 0 \leq \ell \leq N$$

$$B = \{x \in \Gamma' x(-\varepsilon,\varepsilon): \xi_\ell(x') = x_n\} \qquad 1 \leq \ell \leq N.$$

It is easy to check that the roots of every f_{ij} along Γ' are among the ξ_ℓ's and so it has constant sign on B. Thus some B is contained in Γ. But from Thom's lemma for polynomials in one single variable it follows (see [Ł] p.109) that in fact $B = \Gamma$, and so Γ is connected. We also deduce $\pi(\Gamma) = \Gamma'$.

Finally let us show that the closure of Γ in U is exactly the set Γ^* obtained by relaxing inequalities. The inclusion to be seen is, of course, $\Gamma^* \subset \overline{\Gamma}$.

Consider $a = (a',a_n) \in \Gamma^* \cap U$. By induction hypothesis $a' \in \overline{\Gamma}'$. We claim $(\{a'\} \times [-\varepsilon,\varepsilon]) \cap \overline{\Gamma} \neq \emptyset$. Indeed, otherwise by compactness there is an open neighborhood G of a' such that $G \times (-\varepsilon,\varepsilon)$ does not meet Γ. It follows $G \cap \Gamma' = \emptyset$, since $\pi(\Gamma) = \Gamma'$. Contradiction.

Now, by Thom's lemma for polynomials in one variable, the set

$$I = \{x_n \in \mathbb{R}: f_{ij}(a',x_n) \in \theta_{ij}\}$$

is either a point or an open interval with closure

$$\overline{I} = \{x_n \in R: f_{ij}(a',x_n) \in \theta_{ij} \cup \{0\}\}$$

As the polynomials $f_{ij}(a',x_n) \in \mathbb{R}[x_n]$ have all their roots in $(-\varepsilon,\varepsilon)$ it is easy to check that $I \subset \overline{I} \subset (-\varepsilon,\varepsilon)$. To end the proof we distinguish two cases:

(9.3) $I = \overline{I} = \{a_n\}$.

Take a sequence $x'_p \in \Gamma'$ convergent to x'. As $\Gamma' = \pi(\Gamma)$, there is a <u>bounded</u> sequence $x_{np} \in \mathbb{R}$ with $(x'_p,x_{np}) \in \Gamma$. By compactness, we may assume x_{np} converges to some $c_n \in \mathbb{R}$. Since $f_{ij}(x'_p,x_{np}) \in \theta_{ij}$ we have $f_{ij}(a',c_n) \in \theta_{ij} \cup \{0\}$, and so $c_n \in \overline{I}$, i.e. $c_n = a_n$ and (a',a_n) is a limit point of Γ.

(9.4) $I \neq \{a_n\}$.

Then, as $a_n \in \overline{I}$, I is an open interval, and no θ_{ij} is the equality. On the other hand a' is the limit of a sequence $x'_q \in \Gamma'$ and a_n the one of a sequence $x_{np} \in I$. Since all the θ_{ij} are open conditions, $f_{ij}(a',x_{np}) \in \theta_{ij}$ and $a' = \lim_q x'_q$, there is a subsequence x'_p of x'_q with $f_{ij}(x'_p,x_{np}) \in \theta_{ij}$. This means $(x'_p,x_{np}) \in \Gamma$ and so a is a limit point of Γ.

10. <u>Final remark on the real spectrum</u>.- Let $X_o \subset \mathbb{R}^n_o$ be an analytic germ and $\mathcal{O}[X_o]$ its associated analytic algebra. Then we have a correspondence

$$\left\{\begin{matrix}\text{semianalytic}\\ \text{subgerms}\\ \text{of } X_o\end{matrix}\right\} \overset{\sigma}{\rightarrow} \left\{\begin{matrix}\text{constructible}\\ \text{sets}\\ \text{of } \mathrm{Spec}_R \mathcal{O}[X_o]\end{matrix}\right\}$$

defined in the obvious way.

The results quoted in this survey can be used to show that σ behaves like in the algebraic case (see [CR]): it is a bijection that preserves unions, intersections, complements..., as well as topological operations: closures, interiors, connected components... In particular, it may be interesting to point out the following consequence of 8.7:

(10.1) A constructible set of $\mathrm{Spec}_R \mathcal{O}[X_o]$ has finitely many connected components.

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