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Publications des séminaires de mathématiques et informatique de Rennes, 1980, fascicule S3
« Colloque d'algèbre », , p. 63-81
<http://www.numdam.org/item?id=PSMIR_1980___S3_63_0>

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Robert M. FOSSUM

0. INTRODUCTION

This lecture reports on work done jointly with H.-B. FOXBY and B. IVERSEN. Details are found in [FFI]. The interest in this work began when we tried to formalize the following problem : let A be a noetherian local ring. Consider bounded complexes of projective A-modules of finite type which have homology of finite length and maps from this category to abelian groups that are additive over exact sequences of complexes and one equal on complexes that are homotopically equivalent. Find a universal group for these maps and generators (and relations) for this group. While studying this problem we rediscovered some relative K-groups of Bass [B], found a generalization of Mennicke symbols and used some of our techniques to calculate anew some groups of divisors. Since the details are available in preprint form and may he available soon in published form, this report will attempt only to give the highlight of our work. This report is divided into three parts that are described below.

§1. Ⓔ-Theory. This section contains definitions and alludes to some basic properties of family of abelian groups that FOXBY and I denote by Ⓔ (a Danish vowel that is close, in most languages, to "a" or "e" or somewhere in between, that has the sound of e in the english "bed", and that is phonetically written "ε"). One of the groups in the family is the group mentioned in the paragraph above.

§2. Characteristic classes and Mennicke symbols. This section contains some mention of the relations between groups in the family above and relative K-groups. There is also an indication of the construction of relative Mennicke symbols.

§3. Examples. In this section examples are mentioned of calculations of some of the groups involved.

Again I want to emphasize that this is work done together with FOXBY and IVERSEN. I thank my wife Barbara for her inspiration. This research was partially supported by the United States National Science Foundation.

§ 1. (Æ)-THEORY

Let A be a commutative ring. Let $\underline{\mathcal{C}}.(A)$ denote the category of complexes of projective A-modules of finite type.

Recall that a morphism $f. : P. \longrightarrow Q.$ in $\underline{\mathcal{C}}.(A)$ is a family of homomorphisms $f_i : P_i \longrightarrow Q_i$ such that

$$\begin{array}{ccc} P_i & \xrightarrow{d_i^P} & P_{i-1} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i-1}} \\ Q_i & \xrightarrow{d_i^Q} & Q_{i-1} \end{array}$$

commutes for each i. Such a map induces a homomorphism $H.(f.) : H.(P.) \to H.(Q.)$ on homology. A morphism f. is a <u>quasi-isomorphism</u> if $H.(f.)$ is an isomorphism.

A sequence

$$0 \longrightarrow P. \longrightarrow Q. \xrightarrow{g.} R. \longrightarrow 0$$

of complexes is <u>exact</u> if the sequences

$$0 \longrightarrow P_i \xrightarrow{f_i} Q_i \xrightarrow{g_i} R_i \longrightarrow 0$$

are <u>split</u> exact for each i.

Let $\underline{\mathcal{C}}^b.(A)$ denote the category of bounded complexes in $\underline{\mathcal{C}}.(A)$. Of $P. \in \underline{\mathcal{C}}^b.(A)$, the <u>length</u> of P. is defined by

$$\text{lenght}(P.) = \sup\{i : P_i \neq 0\} - \inf\{i \mid P_i \neq 0\} .$$

Let $\underline{\mathcal{S}}$ be a Serre subcategory of the category of A-modules. We denote by

$\mathcal{C}.(A,\mathcal{S})$ and by $\mathcal{C}^b_.(A,\mathcal{S})$ the subcategories of $\mathcal{C}.(A)$ and $\mathcal{C}^b_.(A)$ consisting of those complexes whose homology modules are objects in $\mathcal{S}$. (For example if $\mathcal{S}$ is the category of A-modules, then $\mathcal{C}.(A,\mathcal{S}) = \mathcal{C}.(A)$. If $\mathcal{S} = \{0\}$, then $\mathcal{C}.(A,\mathcal{S})$ is the category of exact complexes, that is $H.(P.) \equiv 0$).

Note that $\mathcal{C}.(A,\mathcal{S})$ and $\mathcal{C}^b_.(A,\mathcal{S})$ are Serre subcategories in that if

$$0 \longrightarrow P. \longrightarrow Q. \longrightarrow R. \longrightarrow 0$$

is an exact sequence of complexes and any two of the complexes are in $\mathcal{C}.(A,\mathcal{S})$, then the third is also. Note also that by using quasi-isomorphisms or the homology category, we can get triangulated categories in the sense of Verdier [V].

Either using triangulated categories or <u>ab initio</u> we define an abelian group $\textcircled{Æ}(A,\mathcal{S})$ to be a certain Grothendieck group.

<u>THEOREM</u> 1.1. *There is an abelian group* $\textcircled{Æ}(A,\mathcal{S})$ *and an assignment* $\textcircled{æ} : \mathcal{C}^b_.(A,\mathcal{S}) \longrightarrow \textcircled{Æ}(A,\mathcal{S})$ *such that :*

1.2. *if* $0 \longrightarrow P. \longrightarrow Q. \longrightarrow R. \longrightarrow 0$ *is exact in* $\mathcal{C}^b_.(A,\mathcal{S})$ $\textcircled{æ}(Q.) = \textcircled{æ}(P.) + \textcircled{æ}(R.)$.

1.3. *If* $P.$ *is in* $\mathcal{C}^b_.(A,\mathcal{S})$ *and* $H.(P.) \equiv 0$, *then* $\textcircled{æ}(P.) = 0$.

1.4. *The pair* $(\textcircled{Æ}(A,\mathcal{S}), \textcircled{æ})$ *is universal for such assignments.*

(This means, of course, that if $\langle\ \rangle : \mathcal{C}^b_.(A,\mathcal{S}) \longrightarrow L$ is such an assignment to an abelian groupe L, then there is a unique $\textcircled{æ'} : \textcircled{Æ}(A,\mathcal{S}) \longrightarrow L$ such that

$$\begin{array}{ccc} & \xrightarrow{\textcircled{æ}} & \textcircled{Æ}(A,\mathcal{S}) \\ \mathcal{C}^b_.(A,\mathcal{S}) & & \downarrow \textcircled{æ'} \\ & \xrightarrow[\langle\ \rangle]{} & L \end{array}$$

commutes).

Recall that a complex can be shifted. if $P. \in \mathcal{C}.(A)$, then for $n \in Z$ we define a new complex $P.[n]$ by

$$(P.[n])_i := P_{n+i}$$

with differential $d_i^{P[n]} = (-1)^n \, d_{n+i}^P$.

Using mapping cones and an inductive definition of the total complex of a double complex, a very useful result concerning the classes of a total complex is obtained. While no direct use is made in this report, the result is, in some sense, critical in making calculations.

<u>THEOREM</u> 1.5. *Let* P.. *be a bounded double complex such that* $P_i. \in \underline{\mathcal{P}}^b_{\cdot}(A,\underline{\mathcal{S}})$ *for each* i. *Then the total complex*

$$\mathrm{Tot.}(P..) \in \underline{\mathcal{P}}^b_{\cdot}(A,\underline{\mathcal{S}})$$

and

$$\textcircled{\ae}(\mathrm{Tot.}(P..)) = \sum_i (-1)^i \; \textcircled{\ae}(P_i.)$$

in

$$\textcircled{\AE}(A,\underline{\mathcal{S}}) \; .$$

<u>COROLLARY</u> 1.6. *Of* $P. \in \underline{\mathcal{P}}^b_{\cdot}(A,\underline{\mathcal{S}})$, *then*

$$\textcircled{\ae}(P.[n]) = (-1)^n \; \textcircled{\ae}(P.) \; .$$

<u>COROLLARY</u> 1.7. *Of* P.. *is a bounded complex and* $P_i \in \underline{\mathcal{P}}^b_{\cdot}(A,\underline{\mathcal{S}})$ *and* $P._j \in \underline{\mathcal{P}}^b_{\cdot}(A,\underline{\mathcal{S}})$ *for all* i *and* j, *then*

$$\sum_i (-1)^i \, \textcircled{\ae}(P_i.) = \sum_j (-1)^j \, \textcircled{\ae}(P._j) \quad \textit{in } \textcircled{\AE}(A,\underline{\mathcal{S}})$$

This last result is particularly useful when considering a square of A-modules,

$$\begin{array}{ccc} P_1 & \xrightarrow{d_1} & P_o \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_o} \\ Q_1 & \xrightarrow{e_1} & Q_o \end{array} \; .$$

Suppose the four complexes are in $\underline{\mathcal{P}}.(A,\underline{\mathcal{S}})$. Then, denoteng the class of the complex by $\textcircled{\ae}(f)$, where f is the map, we obtain

$$-\textcircled{\ae}(f_1) + \textcircled{\ae}(f_o) = -\textcircled{\ae}(d_1) + \textcircled{\ae}(e_1) \; ,$$

or

$$\textcircled{\ae}(f_o) + \textcircled{\ae}(d_1) = \textcircled{\ae}(e_1) + \textcircled{\ae}(f_1) \; ,$$

or

$$\textcircled{\ae}(f_o \circ d_1) = \textcircled{\ae}(e_1 \circ f_1)$$

where $e_1 \circ f_1,\ f_o \circ d_1 : P_1 \longrightarrow Q_o$.

We have in mind two categories $\mathcal{S}$. Let $\underline{\underline{fl}}$ denote the category of A-modules of finite length. Then $P. \in \mathcal{C}^b_{\cdot}(A,\underline{\underline{fl}})$ if and only if $H_i(P.)$ has finite lenght for each i. There is an obvious map $\underline{\chi} : \mathcal{C}^b_{\cdot}(A,\underline{\underline{fl}}) \longrightarrow \mathbb{Z}$ given by

$$\underline{\chi}(P.) := \Sigma(-1)^i \text{ length } (H_i(P.))$$

and this clearly factors through (Æ)$(A,\underline{\underline{fl}})$. In some special cases we can determine the kernel of this homomorphism (cf. § 3).

Let S be a multiplicative subset of A. We can consider those modules M for which $S^{-1}M = (0)$. Then $\mathcal{C}^b_{\cdot}(A,\mathcal{S})$ consists of those complexes P. such that $S^{-1}P.$ is exact. The associated group is denoted by (Æ)$(A,S^{-1}A)$. In § 3 we will indicate how useful this group is for classifying singularities.

We introduce another family of abelian groups. Let $F : \underline{\mathcal{Q}} \longrightarrow \underline{\mathcal{B}}$ be an additive functor form the additive category $\underline{\mathcal{Q}}$ to the additive category $\underline{\mathcal{B}}$. Define two categories Co(F) and Hot($\underline{\mathcal{Q}}$,F) as follows :

The category Co(F) consists of triples

$$\pi = (P_1, p, P_o)$$

as objects, where $P_i \in \underline{\mathcal{Q}}$ and $p : F(P_1) \longrightarrow F(P_o)$ is an isomorphism in $\underline{\mathcal{B}}$. Morphisms in Co(F) are pairs

$$f. = (f_1, f_o) : \underline{\pi} \longrightarrow \underline{\rho} := (R_1, r, P_o)$$

where $f_i : P_i \longrightarrow R_i$ and the diagram

$$\begin{array}{ccc} FP_1 & \xrightarrow{\ p\ } & FP_o \\ {\scriptstyle Ff_1}\downarrow & & \downarrow{\scriptstyle Ff_o} \\ FR_1 & \xrightarrow{\ r\ } & FR_o \end{array}$$

is commutative.

The category Hot($\underline{\mathcal{Q}}$,F) consists of bounded complexes P. of objects in $\underline{\mathcal{Q}}$ such that the transforms under F are contractible in $\underline{\mathcal{B}}$; morphisms are maps of complexes. (Recall that a bounded complex Q. in $\underline{\mathcal{B}}$ is <u>contractible</u> if there are morphisms $S_i : Q_i \longrightarrow Q_{i+1}$ such that

$$S_{i-1} \circ d_i^Q + d_{i+1}^Q \circ S_i = Id_{Q_i}$$

for each i. Call S. the contraction)..

Suppose $P. \in Hot(\underline{\mathcal{Q}},F)$ with contraction S. We construct an object

(1.6) $\underline{\pi}(P.,S.)$ in $Co(F)$

by $\pi(P.,S.) = (P_{odd},P,P_{even})$

where

$$P_{odd} := \coprod_i P_{2i+1} = \begin{matrix} \vdots \\ \oplus \\ P_{2i-1} \\ \oplus \\ P_{2i+1} \\ \oplus \\ \vdots \end{matrix} \quad \text{and} \quad P_{even} := \coprod_i P_{2i} = \begin{matrix} \vdots \\ \oplus \\ P_{2i} \\ \oplus \\ P_{2i+2} \\ \oplus \\ \vdots \end{matrix}$$

and $p : F(P_{odd}) \longrightarrow F(P_{even})$ is given by

$$\begin{pmatrix} \vdots & & \\ F(d_{2i-1}) & 0 & 0 \\ S_{2i-1} & F(d_{2i+1}) & 0 \\ 0 & S_{2i+1} & F(d_{2i+3}) \\ \cdots & \vdots & \vdots \end{pmatrix} .$$

$$\text{Since} \quad \begin{pmatrix} \cdots & S_{2n-2} & F(d_{2n}) & 0 \\ & 0 & S_{2n} & F(d_{2n+2}) \\ & \cdots & & \end{pmatrix} \circ p = \begin{pmatrix} Id & & 0 \\ & Id & \\ * & & \end{pmatrix} ,$$

it follows that p is an isomorphism and that we have determined an element in Co(F).

Now define two groups denoted respectively by $K_o(\underline{\mathcal{Q}},F)$ and $K_o(Hot(\underline{\mathcal{Q}},F))$ whose existences are guaranteed by the following theorems.

THEOREM 1.7. *Let* $F : \underline{\mathcal{A}} \longrightarrow \underline{\mathcal{B}}$ *as above. There is an abelian group* $K_o(\underline{\mathcal{A}},F)$ *and an assigment* $[\ \ ] : Co(F) \longrightarrow K_o(\underline{\mathcal{A}},F)$ *satisfying.*

1.8. If $\underline{\pi}$ and $\underline{\rho}$ are composible in the sense that $P_o = R_1$, and denoting by $\underline{\rho\pi}$ the composition then

$$[\underline{\rho\pi}] = [\underline{\rho}] + [\underline{\pi}] \quad .$$

1.9. If $0 \longrightarrow \underline{\pi} \xrightarrow{f.} \underline{\rho} \xrightarrow{g.} \underline{\sigma} \longrightarrow 0$ is exact (which means that

$$P_i \xrightarrow{f_i} R_i \xrightarrow{g_i} S_i$$

are split exact in $\underline{\mathcal{A}}$), then

$$[\underline{\rho}] = [\underline{\pi}] + [\underline{\sigma}] \ .$$

1.10. The pair $(K_o(\underline{\mathcal{A}},F), [\ ])$ is universal for assignments satisfying 1.8 and 1.9.

THEOREM 1.11. *There is an abelian group* $K_o(Hot(\underline{\mathcal{A}},F))$ *and an assignment* $< >: Hot(\underline{\mathcal{A}},F) \longrightarrow K_o(Hot(\underline{\mathcal{A}},F))$ *satisfying.*

1.12. If P. is contractible in $\underline{\mathcal{A}}$, then $<P.> = 0$.

1.13. If $0 \longrightarrow P. \longrightarrow Q. \longrightarrow R. \longrightarrow 0$ is exact $Hot(\underline{\mathcal{A}},F)$, then

$$<Q.> = <P.> + <R.> \ .$$

1.14. The pair $(K_o(Hot(\underline{\mathcal{A}},F),< >)$ is universal for maps from $Hot(\underline{\mathcal{A}},F)$ to abelian groups satisfying 1.12 and 1.13.

We note that the triple $\underline{\pi}(P.,S.)$ for P. in $Hot(\underline{\mathcal{A}},F)$ gives an element in Co(F) and hence in $K_o(\underline{\mathcal{A}},F)$. The main theorem in this part of theory is that this element is independant of the contraction S.

THEOREM 1.15. *Suppose* P. *is in* $Hot(\underline{\mathcal{A}},F)$. *The element* $\underline{\pi}(P.,S.)$ *is independant of the contraction* S. *That is to say, if* t. *is another contraction , then*

$$[\underline{\pi}(P.,S.)] = [\underline{\pi}(P.,t.)] \ .$$

DEFINITION 1.16. *The map induced from* Hot($\underline{\mathcal{Q}}$,F) *to* $K_o(\underline{\mathcal{Q}},F)$ *is denoted by wh and is called the whitehead maps, in order to indicate the influence of whitehead on this subject. The whitehead map satisfies* 1.12 *and* 1.13 *and therefore there is a unique map*

$$wh : K_o(\mathrm{Hot}(\underline{\mathcal{Q}},F)) \longrightarrow K_o(\underline{\mathcal{Q}},F)$$

satisfying $wh\langle P.\rangle = [\pi(P.,S.)]$.

In case F is localizing in the sense of Gabriel, then it can be shown that wh is an isomorphism. In particular, when this is the case, then $K_o(\mathrm{Hot}(\underline{\mathcal{Q}},F))$ is generated by complexes of length 1. We return later to this remark.

It is not the intention here to list all the functorial properties of there groups Ⓐ(A,$\underline{\mathcal{S}}$), $K_o(\mathrm{Hot}(\underline{\mathcal{Q}},F))$ and $K_o(\underline{\mathcal{Q}},F)$. Some of them are obvious. One question that comes immediately to mind is the relation that might exist between Ⓐ(A,$\underline{\mathcal{S}}$) and some $K_o(\mathrm{Hot}(\underline{\mathcal{Q}},F))$. For example, let $\underline{\mathcal{Q}}$ be the additive category generated by the projective A-modules of finite type. Is there a fonctor $F : \underline{\mathcal{Q}} \longrightarrow \underline{\mathcal{B}}$ such that $\mathrm{Hot}(\underline{\mathcal{Q}},F) = \underline{\mathcal{C}}^b.(A,\underline{\mathcal{S}})$? Is there an F that is localizing ?

§ 2. CHARACTERISTIC CLASSES AND GENERALIZED MENNICKE SYMBOLS.

In this section A denotes a ring with 1 (not necessarily commutative). We work with right A-modules.

DEFINITION 2.1. *A pair* (F,$\underline{f}$) *is a based* A*-module if* F *is a free module with ordered basis* $f = (f_1,\ldots,f_n)$. *If* (F,$\underline{f}$),(G,$\underline{g}$) *is a pair of based* A*-modules, then the direct sum* $\left(\begin{smallmatrix}F\\ \oplus\\ G\end{smallmatrix}, \begin{smallmatrix}f\\ g\end{smallmatrix}\right)$ *is a based* A*-module with basis*

$$\binom{\underline{f}}{\underline{g}} = \binom{f_1}{0},\ldots,\binom{f_n}{0},\binom{0}{g_1},\ldots,\binom{0}{g_m} .$$

A homomorphism $f : (F,\underline{f}) \longrightarrow (G,\underline{g})$ of based modules is a homomorphism of the underlying A-modules.

A bounded complex of based A-modules F. is <u>exact</u> if the complex of underlying A-modules is exact and

$$\sum_i (-1)^i \operatorname{rk}(F_i) = 0.$$

Suppose F. is exact. Construct based A-modules

$$F_{odd} = \begin{matrix} \vdots \\ F_{2i-1} \\ \oplus \\ F_{2i+1} \\ \vdots \end{matrix} \quad \text{and} \quad F_{even} = \begin{matrix} \vdots \\ \oplus \\ F_{2i-2} \\ \oplus \\ F_{2i} \\ \oplus \\ \vdots \end{matrix}$$

and a matrix $W(S.) : F_{odd} \longrightarrow F_{even}$ defined by this complex as follows : let S. be a contraction of F. (which exists because the complex is exact and consists of free modules). Then each S_i has a matrix with respect to the bases of F_i and F_{i+1}. Then set

$$W(S.) = \begin{pmatrix} \vdots & & & \\ d_{2n-1} & & & \\ S_{2n-1} & d_{2n+1} & 0 & \\ 0 & S_{2n+1} & & 0 \\ & 0 & & \end{pmatrix} .$$

As before, it follows that $W(S.) \in GL_N(A)$. Using a proof similar to that used to show the independence of the Whitehead map in the previous section, we can show that the class of W(S.) in $K_1(A)$ is independent of the contraction.

<u>THEOREM</u> 2.2. *The class of* W(S.) *in* $K_1(A)$ *is independent of* S. .

<u>DEFINITION</u> 2.3. *Call the element obtained in* $K_1(A)$ *the whitehead torsion of the complex* F. *and denote it by* wt(F.).

Using techniques developed in examing the properties for <u>wh</u>, it possible to write formulas for wt(F.[n]) and wt(Tot.(F..) wich are similar to those found in 1.6. and 1.7.

This construction is used to define generalized Mennicke symbols. For the remainder of this section we suppose A is a commutative ring.

If $(E,\underline{e})$ is a based module, then the exterior powers $(\Lambda^p E, \Lambda\underline{e})$ become based modules by ordering the tensors $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_p}$ lexiographically. Suppose $(a_1,\dots,a_n) \in A^n$ and that this row in unimodular. Then the Koszul complex $K.(a_1,\dots,a_n)$ becomes a based complex which is exact.

DEFINITION 2.4. *If* $(a_1,\dots,a_n)$ *is a unimodular row in* A^n*, the the generalized* n^{th} *order Mennicke symbol, denoted by* $[a_1,\dots,a_n]$*, is defined by*

$$[a_1,\dots,a_n] := wt(K.(a_1,\dots,a_n)).$$

THEOREM 2.5. *The* n^{th} *order Mennicke symbol is*

i) *symmetric,*

ii) *multilinear, and*

iii) *unimodular.*

It is clear what it means to be symmetric. To be multilinear means

$$[ab,a_2,\dots,a_n] = [a,a_2,\dots,a_n]\ [b,a_2,\dots,a_n] .$$

While to be unimodular means

$$[a_1+X_1a_n,\dots,a_{n-1}+X_{n-1}a_n,a_n] = [a_1,\dots,a_n]$$

for any $(X_1,\dots,X_{n-1}) \in A^{n-1}$.

Examples. For n = 2, we obtain

$$[a_1,a_2] = \left[\begin{pmatrix} a_1 & a_2 \\ X_1 & X_2 \end{pmatrix}\right]$$

where $a_1X_2 - a_2X_1 = 1$.

For n = 3, suppose $a_1X_3 - a_2X_2 + a_3X_1 = 1$. Then

$$[a_1,a_2,a_3] = \left[\begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ X_2 & X_3 & 0 & a_3 \\ -X_1 & 0 & X_3 & -a_2 \\ 0 & -X_1 & -X_2 & a_1 \end{pmatrix}\right].$$

When n=4, then for $a_1X_4-a_2X_3+a_3X_2-a_4X_1 = 1$ we obtain

$$[a_1,a_2,a_3,a_4] = \left[\left(\begin{array}{cccc|cccc} (a_1 & a_2 & a_3 & a_4) & 0 & 0 & 0 & 0 \\ \hline X_3 & X_4 & 0 & 0 & a_3 & a_4 & 0 & 0 \\ -X_2 & 0 & -X_4 & 0 & -a_2 & 0 & -a_4 & 0 \\ X_1 & 0 & 0 & X_4 & 0 & -a_2 & -a_3 & 0 \\ 0 & -X_2 & -X_3 & 0 & a_1 & 0 & 0 & a_4 \\ 0 & X_1 & 0 & -X_3 & 0 & a_1 & 0 & -a_3 \\ 0 & 0 & X_1 & X_2 & 0 & 0 & a_1 & a_2 \\ \hline 0 & 0 & 0 & 0 & (X_1 & X_2 & X_3 & X_4) \end{array}\right)\right].$$

<u>Example</u>. Let $\mathbb{S}^k$ denote the k-sphere $\mathbb{R}^{k+1}$ given as the locus of $X_1^2+X_2^2+\ldots+X_{k-1}^2 = 1$. Let A_k denote the ring of complex valued continuous functions on S^k. From [M] and the periodicity theorem it follows that

$$SK_1(A_k) = \begin{cases} \mathbb{Z} & k = 3,5,7,\ldots \\ 0 & \text{otherwise.} \end{cases}$$

Suppose k is odd, say $k = 2n-1$. Then $\mathbb{R}^{k+1} = \mathbb{C}^n$ with coordinate functions $Z_1,\ldots Z_n$. Furthermore $\mathbb{S}^{2n-1}$ is the locus of $Z_1\overline{Z}_1+\ldots+Z_n\overline{Z}_n = 1$. Hence $(Z_1,\ldots,Z_n)$ is a unimodular row in A_{2n-1} (where we restrict the Z_i).

<u>THEOREM</u> 2.6. *For* $n \geqslant 2$, *the Mennicke symbol* $[Z_1,\ldots,Z_n]$ *generates* $SK_1(A_{2n-1})$.

Let $B_k = C[X_1,X_2,\ldots,X_{k+1}]/(X_1^2+X_2^2+\ldots+X_{k+1}^2 = 1)$. When $k+1 = 2n$, set $Z_m = X_{2m-1} + iX_{2m}$ and $\overline{Z}_m = X_{2m-1}-iX_{2m}$ for $m = 1,2,\ldots,n$.

Then $B_{2n-1} = \mathbb{C}[Z_1,\overline{Z}_1,\ldots,Z_n,\overline{Z}_n]/(Z_1\overline{Z}_1+\ldots+Z_n\overline{Z}_n-1)$.
There is an injection $B_{2n-1} \hookrightarrow A_{2n-1}$ and hence a homomorphism

$$SK_1(B_{2n-1}) \longrightarrow SK_1(A_{2n-1}).$$

The generalized Mennicke symbol $[Z_1,\ldots,Z_n]$ lives in $SK_1(B_{2n-1})$ and hence we conclude that

$$SK_1(B_{2n-1}) \neq 0$$

for $n \geq 2$.

The interseted reader is referred to [FFI] for further computations and the demonstrations of there results.

§ 3. EXAMPLES.

In this last section we indicate how sensitive the group Ⓐ(A,$\underline{\underline{fl}}$) is to the properties of A. We assume that A is noetherian. Suppose $\underline{\mathcal{S}}$ is a Serre subcategory of the category of A-modules of finite type. Let $G_o(\underline{\mathcal{S}})$ denote the Grothendieck group on $\underline{\mathcal{S}}$.

DEFINITION 3.1. *The Euler characteristic of a complex* P. *in* $\underline{\mathcal{C}}^b_\cdot(A,\underline{\mathcal{S}})$ *is defined to be*

$$\underline{\chi}(P.) := \sum_i (-1)^i \, [H_i(P.)] ,$$

where $[H_i(P.)]$ *denotes the class of* $H_i(P.)$ *is* $G_o(\underline{\mathcal{S}})$.

There is induced a unique homomorphism

$$\underline{\chi} : Ⓐ(A,\underline{\mathcal{S}}) \longrightarrow G_o(\underline{\mathcal{S}})$$

such that

$$\underline{\chi}(ⓐ(P.)) = \underline{\chi}(P.).$$

THEOREM 3.2. *Let* A *be a regular local ring. Then*

$$\underline{\chi} : Ⓐ(A,\underline{\underline{fl}}) \longrightarrow \mathbb{Z} = G_o(\underline{fl})$$

is an isomorphism.

Problem 3.3.

a) Compute Ⓐ(A,$\underline{\underline{fl}}$) for A a regular, but not necessarily local, ring.

b) Compute $\textcircled{Æ}(A,\underline{fl})$ for all local rings.

We mention at this point a problem suggested by L. SZPIRO. If P. is a bounded complex with finite length homology, where we assume A is a local ring, then the same is true of its dual $\mathrm{Hom}^{\cdot}_{A}(P.,A) =: P^{\vee}_{\cdot}$.
If P. is exact then so is P. . Also if

$$0 \longrightarrow P. \longrightarrow Q. \longrightarrow R. \longrightarrow 0$$

is exact, then

$$0 \longrightarrow R^{\vee}_{\cdot} \longrightarrow Q^{\vee}_{\cdot} \longrightarrow P^{\vee}_{\cdot} \longrightarrow 0$$

is exact. We get induced an involution $^{\vee} : \textcircled{Æ}(A,\underline{fl}) \longrightarrow \textcircled{Æ}(A,\underline{fl})$. satisfying $\textcircled{æ}(P^{\vee}_{\cdot}) = \textcircled{æ}(P.)^{\vee}$.

<u>Problem</u> 3.4. (Szpiro). Let χ be as in definition 3.1. What is the relation between $\chi(P.)$ and $\chi(P^{\vee}_{\cdot})$? More generally what properties does $^{\vee}$ have as a homomorphism of $\textcircled{Æ}(A,\underline{fl})$? (Cf [PS]) where a formula is discussed in the graded case).

Consider now the case where S is a multiplicatively closed subset of A. We want to consider $\textcircled{Æ}(A,S^{-1}A)$. The theory developed in § 1 can be used. Let P(A) = be the category of projective A-modules of finite type and let $\underline{\mathcal{B}}$ be $S^{-1}A$-modules, the functor $S : P(A) \longrightarrow \underline{\mathcal{B}}$ is just $P \longrightarrow S^{-1}P$. It is clearly localizing. Hence $\textcircled{Æ}(A,S^{-1}A) = K_o(\mathrm{Hot}(P(A),S)) = K_o(A,S^{-1}A)$. There is a resulting exact sequence [B] of K-groups

$$\begin{array}{ccccccccc}
K_1(A) & \longrightarrow & K_1(S^{-1}A) & \longrightarrow & \textcircled{Æ}(A,S^{-1}A) & \longrightarrow & K_o(A) & \longrightarrow & K_o(S^{-1}A) \\
\downarrow \det & & \downarrow \det & & \downarrow \det & & \downarrow \det & & \downarrow \det \\
\underline{\mu}(A) & \longrightarrow & \underline{\mu}(S^{-1}A) & \longrightarrow & \mathrm{Pic}(A,S^{-1}A) & \longrightarrow & \mathrm{Pic}(A) & \longrightarrow & \mathrm{Pic}(S^{-1}A)
\end{array}$$

which fits into the diagram above. (Let $\underline{\mu}(A)$ be the group of units of A, let $\underline{\alpha}(A)$ be the additive group of A).

Apply this to the case

- A is a reduced local ring of dimension 1
- S is the set of non-zero divisors in A

- B is the normal closure of A in $S^{-1}A$, which we assume to be of finite type as an A-module. Then the exact sequences of K-theory become, respectively,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \underline{\mu}(A) & \longrightarrow & \underline{\mu}(S^{-1}A) & \longrightarrow & Ⓐ(A,S^{-1}A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow & & \\ 1 & \longrightarrow & \underline{\mu}(B) & \longrightarrow & \underline{\mu}(S^{-1}A) & \longrightarrow & Ⓐ(B,S^{-1}A) & \longrightarrow & 0 \end{array}$$

which yields an exact sequence

$$0 \longrightarrow \underline{\mu}(B)/\underline{\mu}(A) \longrightarrow Ⓐ(A,S^{-1}A) \longrightarrow Ⓐ(B,S^{-1}B) \longrightarrow 0 .$$

Since B is a semi local regular ring, the local Euler maps induce an isomorphism

$$Ⓐ(B,S^{-1}B) \simeq \mathbb{Z}^b$$

where b is the number of branches of A, or the number of maximal ideals of B.

<u>THEOREM</u> 3.5. *Of the residue class field* k *of* A *is algebraically closed, then*

$$K_o(A,S^{-1}A) = Ⓐ(A,S^{-1}A) = Q(k) \times \mathbb{Z}^b$$

where Q(k) *is an abelian group that is* g *divisible for all* g *with* (g, char k) = 1 *and* b *is the number of branches of* A.

Let $\underline{f}$ be the conductor from B to A. Then the diagram

$$\begin{array}{ccc} A & \longrightarrow & A/\underline{f} \\ \downarrow & & \downarrow \\ B & \longrightarrow & B/\underline{f} \end{array}$$

is cartesian and so there results an isomorphism

$$\underline{\mu}(B)/\underline{\mu}(A) \simeq \underline{\mu}(B/\underline{f})/\underline{\mu}(A/\underline{f}) .$$

Suppose $\underline{\Lambda}$ is a subsemi group of $\mathbb{N}_o = \mathbb{N} \cup \{0\}$. Let $\underline{\Delta} = \mathbb{N}_o - \underline{\Lambda}$. Suppose further that $\underline{\Lambda}\mathbb{Z} = \mathbb{Z}$ (i.e. the greatest common divisor of elements in $\underline{\Lambda}$ is 1). This is the same as to say $\underline{\Delta}$ is finite. Let $f+1 = \min n : n \notin \Delta\}$. Suppose k is a field. Let $A_{\underline{\Delta}}$ = be the subring of k[[t]] consisting of those power series $a_o + a_1t + a_2t^2 + \ldots =: p(t)$ such that $a_i = 0$ if i $\underline{\Delta}$. Set B = k[[t]]. Then B is the normal closure of A and the conductor $\underline{f}$ of $A_{\underline{\Delta}}$ in B is the principal idal $t^{f+1}B = \{\rho(t) : a_o = a_1 = .- = a_f = 0\}$. Then $B/\underline{f} = k[t]/t^{f+1}[t]$ and

$A_{\underline{\Delta}}/\underline{\mathcal{F}}$ is the subring consisting of those truncated polynomials whose coefficients of t^i are zero unless $i \in \Lambda$.

Let R be a commutative ring. For an integer $n \in \mathbb{N}$, let $W_n(R)$ denote the set of $n \times n$ unipotent upper triangular matrice of the form

$$\begin{pmatrix} 1 & a_1 & a_2 & a_3 & a_n \\ 0 & 1 & a_1 & a_2 & a_{n-1} \\ \vdots & & & & \\ 0 & 0 & 0 \dots 1 & & a_1 \\ 0 & 0 & 0 \dots 0 & & 1 \end{pmatrix} =: (1,a_1,a_2,\dots,a_n) .$$

If $(1,a_1,\dots,a_n)$ dans $(1,b_1,\dots,b_n)$ are two such matrices, then the product is $(1,c_1,\dots,c_n)$ where

$$c_1 = a_1 + b_1$$
$$c_2 = a_2 + a_1 b_1 + b_2$$
$$\vdots$$
$$c_n = a_n + a_{n-1}b_1 + a_{n-2}b_2 + \dots + a_1 b_{n-1} + b_n .$$

Now note that $f(t) \in R[t]/(t^{n+1})$ is a unit if and only if $f(0)$ in R is a unit. Write such an element as

$$a_o g(t) = a_o(1+c_1 t+\dots+c_n t^n) .$$

Define a map

$$\underline{\mu}(R[T]/(t^{n+1})) \longrightarrow \underline{\mu}(R) \times W_n(R)$$

by

$$a_o g(t) \longrightarrow (a_o,(1,c_1,\dots,c_n)) .$$

(This looks like the matrix

$$\begin{pmatrix} a_o & a_1 & \dots & a_n \\ 0 & a_o & a_1 & a_{n-1} \\ \vdots & & & \\ 0 & 0 & a_o & a_1 \\ 0 & 0 & 0 & a_o \end{pmatrix}$$

where $a_i = a_o c_i$). This map is clearly an isomorphism. If $\underline{\Delta}$ in as above and $(R[t]/(t^{f+1}))_{\underline{\Delta}}$ denotes those polynomials whose coefficients of t^d are zero for d in $\underline{\Delta}$, then $W_{f,\underline{\Delta}}(R)$, the subgroup of $W_f(R)$ for which $a_i = 0$ if $i \in \underline{\Delta}$, is in the image of $\mu(R[t]/(t^{f+1}))_{\underline{\Delta}})$. Hence

$$\underline{\mu}(R[t]/(t^{f+1})/\underline{\mu}((R[t]/(t^{f+1}))_{\underline{\Delta}}) \cong W_f(R)/W_{f,\underline{\Delta}}(R) .$$

Note that multiplication by an element of the form $(1,0,0,\ldots,0,a_r,a_{r+1},\ldots,a_f)$ leaves the first r coordinates of $(1,b_1,\ldots,b_f)$ unchanged. So given $(1,b_1,b_2,\ldots,b_f)$ in $W_f(R)$, there is an element u in $W_{f,\underline{\Delta}}(R)$ such that $u(1,b_1,\ldots,b_f) = (1,c_1,c_2,\ldots,c_f)$ where $c_i = 0$ if $i \notin \underline{\Delta}$. Let $\alpha_{\underline{\Delta}}(R) := W_f(R)/W_{f,\underline{\Delta}}(R)$. Then $\alpha_{\underline{\Delta}}$ is an algebraic group of dimension $\mathrm{Card}(\underline{\Delta})$ and $\alpha_{\underline{\Delta}} = \mathrm{Spec}(Z[X_1,\ldots,X_d])$ where $\mathrm{Card}\ \underline{\Delta} = d$.

<u>Example</u> 1. Take $\underline{\Delta} = \{1,2,3,\ldots,f\}$. So the semi-group is $S = \{0,f+1,f+2,\ldots\}$. Then

$$\alpha_{\underline{\Delta}} = W_f .$$

Thus for $A_{\underline{\Delta}} = \{a_o + a_{f+1}t^{f+1} + a_{f+2}t^{f+2} + \ldots\}$ we get

$$Ⓐ(A_{\underline{\Delta}},L) = W_f(k) \times Z .$$

<u>Example</u> 2. The cusp. Take $\underline{\Delta} = \{1\}$. Then

$$A_{\underline{\Delta}} = k[[t^2,t^3]] = k[[X,Y]]/(y^2-X^3).$$

Then

$$Ⓐ(A_{\underline{\Delta}},L) = \underline{\alpha}(k) \times \mathbb{Z}.$$

<u>Example</u> 3. Take $\underline{\Delta} = \{1,3\}$. Then

$$A_{\underline{\Delta}} = k[[t^2,t^5]] = k[[X,y]]/(Y^2-X^5).$$

Then

$$Ⓐ(A_{\underline{\Delta}},L) = \underline{\alpha}_{\{1,3\}}(k) \times \mathbb{Z} .$$

Now $\underline{\alpha}_{\{1,3\}}(k) = k \times k$. We consider an element $(a_1,a_3) = (1,a_1,0,a_3)$ mod $W_{4,\underline{\Delta}}(k)$. So $(a_1,a_3) \oplus (b_1,b_3) \equiv (1,a_1,0,a_3)(1,b_1,0,b_3)$ mod $W_{4,\underline{\Delta}}(k)$. Then

$$(1,a_1,0,a_3)(1,b_1,0,b_3) = (1,a_1+b_1,a_1b_1,a_3+b_3) .$$

Mutiply this by $(1,0,-a_1b_1,0)$ to get

$$\equiv (1, a_1+b_1, 0, a_3+b_3-(a_1b_1)(a_1+b_1)) = (a_1+b_1, a_3+b_3-a_1b_1(a_1+b_1)).$$

So in $\alpha_{\{1,3\}}(h)$ we have

$$(a_1, a_3) \oplus (b_1, b_3) = (a_1+b_1, a_3+b_3-a_1b_1(a_1+b_1))$$

(which is addition of 3-Witt vectors of length 2).

Example 4. Take $\underline{\Delta} = \{1,3,5\}$. Then

$$A_{\underline{\Delta}} = k[[t^2, t^7]] = k[[X,Y]]/(Y^2-X^7)$$

and

$$Ⓐ(A_{\underline{\Delta}}, L) = \underline{\alpha}_{\{1,3,5\}}(k) \times \mathbb{Z} .$$

The multiplication in $\underline{\alpha}_{\{1,3,5\}}(k)$ is

$$(a_1, a_3, a_5).(b_1, b_3, b_5) = (a_1+b_1, a_3+b_3-a_1b_1(a_1+b_1), a_5+b_5-a_1b_1(a_3+b_3) + [(a_1b_1)^2-(a_1b_3+a_3b_1)](a_1+b_1)) .$$

The computations for the rational cases are the same. Then $B' = k[t]_{(t)}$ with field of fractions $k(t)$. Then $B' = k[[t]] \cap k(t)$. Let $A'_{\underline{\Delta}} = A_{\underline{\Delta}} \cap B'$. Then

$$Ⓐ(A'_{\underline{\Delta}}, k(t)) = \alpha_{\underline{\Delta}}(k) \times \mathbb{Z} .$$

Example 5. The node. This plane curve is the locus of $y^2 = X^2(X-1)$. We suppose $(-1)^{1/2} \in k$ and that $\frac{1}{2} \in k$. In $k[[X]]$ let $t = (X-1)^{1/2} = (-1)^{1/2}(1-x)^{1/2}$. Then $y^2-X^2(X-1) = y^2-X^2t^2 = (y-Xt)(y+Xt)$. Note that t is a unit. Let $u = y-Xt$, $v = y+Xt$. Then $k[[X,y]] = k[[u,v]]$. The local ring at the origin of $y^2 = X^2(X-1)$ completed is then $A = k[[u,v]]/(uv)$. The normal closure in the total ring of quotients is $B = k[[r]] \times k[[s]]$ with

$$f(u,v) \longmapsto (f(r,0), f(0,s))$$

for $f(u,v) \in A$. The conductor $\underline{\mathcal{F}} = (r,s)B = (u,v)A$. Hence $B/\underline{\mathcal{F}} = k \times k$ and $A/\underline{\mathcal{F}} = k$, the map $A/\underline{\mathcal{F}} \longrightarrow B/\underline{\mathcal{F}}$ being the diagonal. Hence

$$\mu(B/\underline{\mathcal{F}})/\mu(A/\underline{\mathcal{F}}) \cong \underline{\mu}(k) ,$$

the map being

$$(a,b) \longmapsto ab^{-1}$$

for $(a,b) \in \underline{\mu}(k) \times \underline{\mu}(k) = \mu(B/\mathcal{F})$. There fore

$$\text{Ⓐ}(A,L) = \underline{\mu}(k) \times \mathbb{Z}^2 .$$

Example 6. Consider the homomorphism

$$k[[X,y]] \longrightarrow k[[S_1]] \times k[[S_2]] \times k[[S_3]] = B$$

given by $X \longmapsto (S_1, 0, S_2)$

$y \longmapsto (0, S_2, S_3)$.

It is clear that $Xy(X-y)$ is in the kernel. Since $k[[X,y]]$ is factorial and the image ring is reduced of dimension 1, the kernel is the intersection of prime principal ideals. Hence the ideal generated by $Xy(X-y)$ is the kernel. Also let $A = k[[X,y]]/(Xy(X-y))$. Then B is the normal closure of A in its total quotient ring. The conductor is $(S_1^2, S_2^2, S_3^2)B = (x,y)^2 A$. In this case

$$\mu(B/\mathcal{F}) = (\mu(k) \times \alpha(k)) \times (\mu(k) \times \alpha(k)) \times (\mu(k) \times \alpha(k))$$

and

$$\mu(A/\mathcal{F}) = \mu(k) \times \alpha(k) \times \alpha(k).$$

The map is given by

$$(u,a,b) \longmapsto ((u,a),(u,d),(u,a+b))$$

for $u \in \underline{\mu}(k)$, $(a,b) \in \underline{\alpha}(k) \times \underline{\alpha}(k)$. (The élément (u,a,b) corresponds to the unit

$$u(1+ax+by) \quad \text{in} \quad A/\mathcal{F} .$$

The element

$$((u_1,a_1),(u_2,a_2),u_3,a_3))$$

corresponds to the unit

$$(u_1(1+a_1s_1), u_2(1+a_2s_2), u_3(1+a_3s_3))$$

in $B/\mathcal{F}$). Hence

$$\underline{\mu}(B/\mathcal{F})/\underline{\mu}(A/\mathcal{F}) \simeq \underline{\mu}(k) \times \underline{\mu}(k) \times \underline{\alpha}(k)$$

the map being given by

$$((u_1,a_1),(u_2,a_2),(u_3,a_3)) \longmapsto (u,u_3^{-1},u_2u_3^{-1},a_1+a_2-a_3) .$$

Hence

$$\text{Ⓐ}(A,L) \simeq \underline{\mu}(k) \times \underline{\mu}(k) \times \underline{\alpha}(k) \times \mathbb{Z}^3 .$$

(This example was told to we by F. Orecchia).

We constrast these local examples with the case of global example, again the cusp.

Example 7. Consider the cusp $k[X,y]/(y^2-X^3) = A$ with normal closure $B = k[t]$. We get exact sequences

$$\begin{array}{ccccccccccc} K_1(A) & \longrightarrow & K_1(k(t)) & \longrightarrow & \textcircled{Æ}(A,k(t)) & \longrightarrow & K_0A & \longrightarrow & K_0(k(t)) & \longrightarrow & 0 \\ \downarrow & & \downarrow = & & \downarrow & & \downarrow & & \downarrow = & & \\ K_1(B) & \longrightarrow & K_1(k(t)) & \longrightarrow & \textcircled{Æ}(B,k(t)) & \longrightarrow & K_0(B) & \longrightarrow & K_0(k(t)) & \longrightarrow & 0 \end{array}$$

as before. We obtain the isomorphism

$$\textcircled{Æ}(A,L) \cong \text{Div } B \times \widetilde{K_0(A)},$$

where $\widetilde{K_0(A)} = \text{Ker}(K_0(A) \xrightarrow{rk} \mathbb{Z})$. It follows that $\widetilde{K_0(A)} = \text{Pic}A = \underline{\alpha}(k)$. Thus

$$\textcircled{Æ}(A,L) \cong \text{Div } B \times \underline{\alpha}(k),$$

the $\alpha(k)$ corresponding to the one singular point.

Concluding remarks : These calculations shoned give some indication of the role played by the groups $\textcircled{Æ}(A,\underline{\underline{fl}})$, and the importance of considering complexes rather than modules.

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