

J. A. NITSCHÉ

**Convergence of Finite Element Galerkin Approximations
on Galerkin Problems**

Publications des séminaires de mathématiques et informatique de Rennes, 1978, fascicule S4

« Journées éléments finis », , p. 1-9

http://www.numdam.org/item?id=PSMIR_1978__S4_A8_0

© Département de mathématiques et informatique, université de Rennes, 1978, tous droits réservés.

L'accès aux archives de la série « Publications mathématiques et informatiques de Rennes » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

0. Introduction

Let the model problem

$$\begin{aligned} \dot{u} - \Delta u &= f && \text{in } \Omega \times (0, T] \quad , \\ u &= 0 && \text{on } \partial\Omega \times (0, T] \quad , \\ u_{t=0} &= u_0 && \text{in } \Omega \end{aligned}$$

be given. Here $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with $\partial\Omega$ sufficiently smooth. With the help of a finite-element-approximation-space $S_h \subseteq \overset{\circ}{H}_1$ the standard Galerkin approximation $u_h = u_h(t) \in S_h$ is defined by

$$\begin{aligned} (\dot{u}_h, \chi) + D(u_h, \chi) &= (f, \chi) && \text{for } \chi \in S_h \quad , \\ u_h(0) &= Q_h u_0 \quad . \end{aligned}$$

Here (\cdot, \cdot) is the $L_2(\Omega)$ -scalar product, $D(\cdot, \cdot)$ the Dirichlet integral and Q_h an appropriate projection.

The aim of this paper is to derive L_∞ -estimates for the error $e = e_h = u - u_h$. For the corresponding elliptic case this problem was solved by SCOTT, NITSCHÉ about three years ago. There is a certain feeling that the proofs for the parabolic case would (resp. should) be more or less a direct consequence.

Besides the case of one space-dimension seemingly only BRAMBLE-SCHATZ-THOMÉE-WAHLBIN have attacked this problem. Their approach is to rewrite the Galerkin-equations in the form

$$e + T_h \dot{e} = (I - R_h) u \quad .$$

Here to any f the element $U_h = T_h f = R_h(-\Delta^{-1}f)$ is the Ritz-approximation on $-\Delta^{-1}f$ defined by $U_h \in S_h$ and

$$D(U_h, \chi) = (f, \chi) \quad \text{for } \chi \in S_h .$$

We may also write

$$e = -\Delta^{-1}\dot{e} + (I-R_h)\Delta^{-1}\dot{e} + (I-R_h)u .$$

In this way L_∞ -estimates for e are reduced to L_∞ -estimates of the Ritz-method applied to u on the one hand and to $\Delta^{-1}\dot{e}$ on the other and L_∞ -estimates of $\Delta^{-1}\dot{e}$. The last term is bounded in L_∞ if \dot{e} belongs to L_p with $p > n/2$. In this way

$$\|e\|_{L_\infty} \leq c\|\dot{e}\|_{L_p} + \dots$$

is shown. Repeating this argument an estimate of the type

$$\|e\|_{L_\infty} \leq c\|\partial_t^v e\|_{L_2} + \dots$$

can be derived. This finally leads to an optimal order of convergence - with respect to the subspaces - , but depending on the dimension the norms of the solution u entering the right hand side are stringent.

Independent of the space-dimension the validity of the estimates can be shown:

$$\|e\|_{L_\infty(L_\infty)} \leq c h^m \left\{ \|u\|_{L_\infty(W_\infty^m)} + \|\dot{u}\|_{L_\infty(W_\infty^m)} + \|\ddot{u}\|_{L_2(W_\infty^m)} \right\} .$$

The proofs are given in an article to appear in R.A.I.R.O., anal. numer. Here we discuss the case of $N = 3$ space dimensions in some detail. The general case is only sketched.

1. Notations, L_2 -Projection

In the following we will use the standard notations of the theory of partial differential equations. In addition we will work with weighted norms resp. semi-norms introduced three years ago here in Rennes.

For $x_0 \in \Omega$ and $\rho > 0$ let

$$\mu = |x - x_0|^2 + \rho^2 .$$

The weighted semi-norms are defined by

$$\|\nabla^k u\|_\alpha^2 = \sum_{|\xi|=k} \iint_{\Omega} \mu^{-\alpha} |D^\xi u|^2 dx .$$

The first result of the mentioned paper was related to the L_2 -projection:

Theorem 1: The L_2 -projection onto a finite-element-space is bounded in the α -norm for any $\alpha \in \mathbb{R}$:

$$\|P_h u\|_\alpha \leq c_\alpha \|u\|_\alpha .$$

Generalizing this result in the same way it can be shown.

Theorem 2: Let S_h be a finite-element-space and P_h be the L_2 -projection onto S_h . Then

$$\|u - P_h u\|_\alpha + h \|\nabla(u - P_h u)\|_\alpha \leq \inf_{\chi \in S_h} \{ \|u - \chi\|_\alpha + h \|\nabla(u - \chi)\|_\alpha \} .$$

This theorem guarantees the simultaneous approximation property of the L_2 -projection with respect to the L_2 - and H_1 -norm.

2. A priori Estimates in Weighted Norms

We will use the identity

$$D(u, \mu^{-\alpha} u) = \|\nabla u\|_\alpha^2 + \iint u \nabla u \nabla \mu^{-\alpha} = \|\nabla u\|_\alpha^2 - \frac{1}{2} \iint u^2 \Delta \mu^{-\alpha} .$$

Now by direct differentiations we have

$$\Delta \mu^{-\alpha} = -2\alpha \mu^{-\alpha-2} (N\rho^2 + (N-2\alpha-2)r^2) .$$

For $0 < \alpha < N/2 - 1$ therefore $\Delta \mu^{-\alpha}$ is negative and

$$\mu^{\alpha+1} |\Delta \mu^{-\alpha}|$$

is bounded and bounded away from zero:

Lemma 1: Let $0 < \alpha < N/2 - 1$. Then for any $u \in \overset{\circ}{H}_1$

$$\|u\|_{\alpha+1}^2 + \|\nabla u\|_\alpha^2 \leq c D(u, \mu^{-\alpha} u) .$$

The case $\alpha = N/2 - 1$ is of special interest. Then

$$\Delta \mu^{-\alpha} = -N(N-2) \rho^2 \mu^{-\alpha-2} .$$

Lemma 2: Let $\alpha = N/2 - 1$. Then for any $u \in \overset{\circ}{H}_1$

$$c \rho^2 \|u\|_{\alpha+2}^2 + \|\nabla u\|_\alpha^2 \leq D(u, \mu^{-\alpha} u) .$$

For finite elements the L_∞ -norm is bounded by the weighted norms if x_0 and ρ are chosen properly. Especially we have for $\rho = \gamma h$ with γ fixed: Let $\varphi \in S_h$ and $\alpha = N/2 - 1$. Then

$$\|\varphi\|_{L_\infty} \leq \sup_{x_0 \in \Omega} \rho \|\varphi\|_{\alpha+2}.$$

3. Error Estimates for the Galerkin Method in Case of $N = 3$ Space Dimensions.

The defining relation of the error $e = u - u_h$ is

$$(\dot{e}, \chi) + D(e, \chi) = 0 \quad \text{for } \chi \in S_h.$$

Now we introduce the Ritz-approximation $U_h = R_h u$ and use the splitting

$$\begin{aligned} e &= (u - U_h) - (u_h - U_h) \\ &= \varepsilon - \phi \end{aligned}$$

with $\phi \in S_h$. Further we assume the initial condition

$$u_h(0) = R_h u_0.$$

Therefore we have

$$\phi(0) = 0.$$

The defining relation for ϕ is

$$(\dot{\phi}, \chi) + D(\phi, \chi) = (\dot{\varepsilon}, \chi) \quad \text{for } \chi \in S_h.$$

Using the estimates of Section 2 we get for $\alpha = N/2 - 1 = 1/2$

$$c^{-1} \left\{ \rho^2 \|\Phi\|_{\alpha+2}^2 + \|\nabla \Phi\|_{\alpha}^2 \right\} \leq D(\Phi, \mu^{-\alpha} \Phi)$$

and

$$D(\Phi, \mu^{-\alpha} \Phi) = D(\Phi, \mu^{-\alpha} \Phi - \chi) - (\dot{\epsilon} - \dot{\Phi}, \mu^{-\alpha} \Phi - \chi) + (\dot{\epsilon} - \dot{\Phi}, \mu^{-\alpha} \Phi) .$$

If we choose $\chi = P_h(\mu^{-\alpha} \Phi)$ the L_2 -projection then the middle term vanishes mostly. By pure approximation arguments we get

Lemma

$$\inf_{\chi \in S_h} D(\Phi, \mu^{-\alpha} \Phi - \chi) \leq c \left(\frac{h}{\rho} \right)^2 \left\{ \rho^2 \|\Phi\|_{\alpha+2}^2 + \|\nabla \Phi\|_{\alpha}^2 \right\} .$$

For $\rho = \gamma h$ with a proper γ this gives

$$\rho^2 \|\Phi\|_{\alpha+2}^2 + \|\nabla \Phi\|_{\alpha}^2 \leq c (\dot{\epsilon} - \dot{\Phi}, \mu^{-\alpha} \Phi) .$$

The left hand side is a bound of $\|\Phi\|_{L_{\infty}}^2$ if x_0 is chosen properly. The right hand side can be estimated by

$$(\dot{\epsilon} - \dot{\Phi}, \mu^{-\alpha} \Phi) \leq \|\Phi\|_{L_{\infty}} \iint \mu^{-\alpha} |\dot{\epsilon} - \dot{\Phi}| \leq \|\Phi\|_{L_{\infty}} \|\dot{\epsilon} - \dot{\Phi}\| \left\{ \iint \mu^{-2\alpha} \right\}^{1/2}$$

In the case of $N = 3$ we have $2\alpha = 1$ and the last integral is bounded. This gives because of $e = \epsilon - \Phi$

Theorem 3: Let $N = 3$. Then for any fixed time t

$$\|e\|_{L_{\infty}} \leq c \left\{ \|\epsilon\|_{L_{\infty}} + \|\dot{\epsilon}\|_{L_2} + \|\ddot{\epsilon}\|_{L_2} \right\} .$$

The L_2 -error estimates

$$\|e\|_{L_\infty(L_2)} \leq \|\epsilon\|_{L_\infty(L_2)} + c\|\dot{\epsilon}\|_{L_2(L_2)}$$

are well-known. We apply this with e replaced by \dot{e} and ϵ replaced by $\dot{\epsilon}$. In this way we come to

Theorem 4: Let $N = 3$. Then

$$\|e\|_{L_\infty(L_\infty)} \leq c\left\{\|\epsilon\|_{L_\infty(L_\infty)} + \|\dot{\epsilon}\|_{L_\infty(L_2)} + \|\dot{\dot{\epsilon}}\|_{L_2(L_2)}\right\}.$$

For u sufficiently regular this leads to optimal error estimates of the Galerkin approximation.

4. Two Types of Error Estimates in Arbitrary Dimensions.

Using the splitting $e = \epsilon - \phi$ of above we get for arbitrary N of space dimensions with $\alpha = N/2 - 1$ similar to above with $\chi = P_h(\mu^{-\alpha}\phi)$

$$\begin{aligned} (\dot{\phi}, \phi)_\alpha + \|\nabla\phi\|_\alpha^2 + k\rho^2\|\phi\|_{\alpha+2}^2 &= \\ &= D(\phi, \mu^{-\alpha}\phi - \chi) - (\dot{\epsilon}, \mu^{-\alpha}\phi - \chi) + (\dot{\epsilon}, \phi)_\alpha. \end{aligned}$$

By approximation arguments we come to

$$(\dot{\phi}, \phi)_\alpha \leq c\left\{\|\phi\|_\alpha^2 + \|\dot{\epsilon}\|_\alpha^2\right\}$$

if $\rho \geq \gamma h$ with γ properly chosen. This leads to the interesting a priori inequality

$$\|\phi(t)\|_\alpha^2 \leq \|\phi(0)\|_\alpha^2 + c \int_0^t \|\dot{\epsilon}(\tau)\|_\alpha^2 d\tau.$$

A first - but non-optimal - L_∞ -result is the consequence.

If x_0 is chosen properly then $\|\dot{\phi}\|_\alpha$ is a bound of $h\|\dot{\phi}\|_{L_\infty}$. Therefore we get

$$\|\dot{\phi}\|_{L_\infty(L_\infty)}^2 \leq c h^{-2} \|\dot{\varepsilon}\|_{L_2(\alpha)}^2 = c h^{-2} \int_0^T \|\dot{\varepsilon}\|_\alpha^2 d\tau .$$

The α -norm is bounded by the L_p -norm for $p > N$. Therefore we have

Theorem 5: Let $N > 3$ be arbitrary and $p > N$. Then

$$\|e\|_{L_\infty(L_\infty)} \leq \|\varepsilon\|_{L_\infty(L_\infty)} + c h^{-1} \|\dot{\varepsilon}\|_{L_2(L_p)}$$

Similarly we can get

$$\|\dot{\phi}(t)\|_\alpha^2 \leq \|\dot{\phi}(0)\|_\alpha^2 + c \int_0^t \|\ddot{\varepsilon}(\tau)\|_\alpha^2 d\tau .$$

For $\dot{\phi}(0) = 0$ we find

$$\|\dot{\phi}(0)\|_\alpha \leq c \|\dot{\varepsilon}(0)\|_\alpha$$

and in this way

$$\sup \|\dot{\phi}(t)\|_\alpha \leq \sup \|\dot{\varepsilon}(t)\|_\alpha + c \left\{ \int_0^T \|\ddot{\varepsilon}\|_\alpha^2 d\tau \right\}^{1/2} .$$

The link to L_∞ -estimates of $\dot{\phi}$ and therefore of e is

Theorem 6: Let $N > 3$ and $\alpha = N/2 - 1$. For $\rho \geq \gamma h$ with γ chosen properly

$$\|\phi\|_{\alpha+2} + \|\nabla\phi\|_{\alpha+1} \leq c \rho^{-1} \|\varepsilon - \dot{\phi}\|_{\alpha} .$$

The proof of this theorem is quite lengthy, the lines of it still being the same as in the paper three years ago.

Since now $\alpha + 2 = N/2 + 1$ for $x_0 \in \Omega$ appropriate and $\rho = \gamma h$ we get

$$\|\phi\|_{L_{c_0}} \leq c \rho \|\phi\|_{\alpha+2} .$$

This finally gives the error estimate stated at the end of the introduction.

In the mentioned paper to appear in R.A.I.R.O. a detailed bibliography is given. We suppress this here.