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ELEMENTS FINIS EQUILIBRE POUR LES PLAQUES PLASTIQUES

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INTRODUCTION

Nous appliquons la méthode d'éléments finis mixtes de Hellan-Herrmann [1-5] à des problèmes de flexion de plaques. Cette méthode, équivalente à celle, non-conforme, de Morley [8] dans le cas de charges concentrées, est une des plus simples pour ce type de problèmes.

Nous considérons successivement les cas élastiques, élasto-plastiques et visco-plastiques; ce dernier permettant de déterminer avec précision la charge limite du cas élasto-plastique [6-7] .

Pour résoudre les problèmes élasto-plastiques et visco-plastiques (non-linéaires) nous utilisons des algorithmes décrits dans [7] , et qui consistent, à chaque itération, à résoudre un problème élastique et à projeter sur le convexe de plasticité approché K_h.

Les éléments de ce convexe K_h sont des moments de flexion, notés M_{ij}.Selon que l'on impose au moment normal $M_{ij}n_in_j$ de se raccorder ou non aux interéléments, la projection sur K_h est un problème d'optimisation avec contraintes globales ou locales. C'est à dire résoluble globalement (très cher) ou élément par élément (coût négligeable).

Notons que la première approche est naturelle quand on utilise la formulation mixte-équilibre (Hermann), alors que la deuxième semble plus naturelle avec la formulation non conforme (Morley).

Nous rappelons les résultats de convergence obtenus dans les différents cas, et décrivons les algorithmes de résolution.

Des résultats numériques sont donnés dans [15] .

1 - PROBLEMS TO BE SOLVED.

From Love-Kirchoff's theory, the elastic flexure of a thin plate, the section of which is a domain $\Omega \subset \mathbb{R}^2$, and submitted to a transverse load $s f$, where $f \in L^2(\Omega)$ and $s \in \mathbb{R}$, can be written as follows,

(1) $$M_{ij} = \lambda \Delta u \ \delta_{ij} + 2\mu \frac{\partial^2 u}{\partial x_i \partial x_j} \quad ,$$

(2) $$\frac{\partial^2 M_{ij}}{\partial x_i \partial x_j} = s f \quad ;$$

with boundary conditions,

(3) $$u = 0 \ , \ \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \ ,$$

if the plate is clamped, or

(4) $$u = 0 \ , \ M_{ij} n_i n_j = 0 \quad \text{on} \quad \partial\Omega \ ,$$

if the plate is simply supported. We use the summation convention for repeated indices :

- M_{ij} denotes the flexure moments ($i, j = 1, 2$ and $M_{ij} = M_{ji}$) ;
- u denotes the transverse displacement ;
- n_i denotes the components of the unit normal vector to $\partial\Omega$, boundary of Ω ;
- δ_{ij} denotes the Kronecker tensor ;
- λ, μ denote some elasticity coefficients of the plate (one has $\lambda = E\, e^3\, \nu/12(1-\nu)$ and $\mu = E\, e^3 /24(1+\nu)$, where E is Young's modulus, ν Poisson's ratio, e the thickness of the plate).

VARIATIONAL FORMULATION OF THE ELASTIC PROBLEM.

We call H the space of flexure moments,

$$H = \left\{ N \in (L^2(\Omega))^4, \text{ symmetrical : } N_{ij} = N_{ji} \right\} ,$$

endowed with the scalar product

$$(M, N) = \int_\Omega (\lambda \, tr(M) \, tr(N) + 2\mu \, M.N) \, dX ,$$

where $tr(N)$ denotes the trace of the tensor N , and $M.N \equiv M_{ij} N_{ij}$. The norm associated to this scalar product is the *energy norm*.

We call H' the dual of H , and

$$<Q,M> = \int_\Omega Q.M \, dX ,$$

the duality pairing between H and H'. Here, $Q.M$ denotes the standard Euclidean scalar product of $\mathbb{R}^4$.

Let us denote by $H^k(\Omega)$ the Sobolev space of functions, the derivatives of which, up to the order k , are in $L^2(\Omega)$. We call V the space of admissible displacements

$$V = \left\{ v \in H^2(\Omega) \ : \ v = 0 \ \text{ on } \Gamma \text{ and } (*) \ \frac{\partial v}{\partial n} = 0 \ \text{ on } \Gamma \right\} ;$$

$D : V \to H'$ is the linear operator defined by $D_{ij}(v) = \dfrac{\partial^2}{\partial x_i \partial x_j}$;

$$E(s) = \left\{ N \in H : < Dv, N > = s \int_\Omega f.v \, dX , \ \forall v \in V \right\}$$

is then the set of flexure moments in *equilibrium* under the external load $s f$.

A variational formulation of the elastic problem (1), (2), (3) or (4), is the following

(5)
$$\begin{cases} \textit{Find } \{M, u\} \in E(s) \times V \textit{ such that} \\ (M, N) - < Du, N > = 0 , \ \forall N \in H . \end{cases}$$

This problem has a unique solution [3, 4].

WEAK VARIATIONAL FORMULATION.

In view of the elasto-plastic problem, we shall also consider a weaker variational formulation of the elastic problem :

(*) if the plate is clamped.

(6) $$\begin{cases} \text{Find } \{M, u\} \in E(s) \times H_0^1(\Omega) \text{ such that} \\ (M, N) - [u, D^* N] = 0, \ \forall N \in \widetilde{H} , \end{cases}$$

where

$H_0^1(\Omega) = \left\{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma \right\}$;

$[\ ,\ ]$ denotes the duality pairing between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$;

D^* is the transpose of D ;

$\widetilde{H} = \left\{ N \in H : D^* N \in H^{-1}(\Omega) \text{ and } (*) \ M_{ij} n_i n_j = 0 \text{ on } \Gamma \right\}$.

This variational formulation is called weak since, after an integration by part, we may require less regularity on one of the variables (the displacement u), but of course more for the other one (the bending moment M). It is convenient to use it for the elasto-plastic problem (as we shall do) since, in this case, the displacement is only in $H^1(\Omega)$, due to the possible appearance of plastic hinges.

ELASTO-PLASTIC PROBLEM.

We introduce the set of plastically admissible flexure moments,

$$K = \left\{ N \in H : F(N(x)) \leqslant 0 \quad \text{a.e.}, \ x \in \Omega \right\} ,$$

where $F : \mathbb{R}^4 \to \mathbb{R}$ is a convex function called plasticity criterion. For metal plates, the Von Misès criterion is often used :

(7) $$F(N) \equiv N_{11}^2 + N_{22}^2 - N_{11} N_{22} + 3 N_{12}^2 - \frac{3}{4} M_P^2 ,$$

where M_P is a constant coefficient depending on the material.

For elasto-plastic plates, the linear constituent law (1) should be replaced by the following (Hencky's law) :

(8) $$(M, N-M) - \langle Du, N-M \rangle \geqslant 0 , \ \forall N \in K .$$

(*) if the plate is simply supported.

To find $\{M, u\} \in E(s) \times V$ satisfying (8), it is *necessary* that the scale factor s be not too big (i.e. the external load be smaller than the limit load). We assume that

(9) *there exists* $\theta > 1$ *and* $\chi \in E(s)$ *such that* $\theta\chi \in K$.

However, this hypothesis is not *sufficient*, since the displacement may not be in $H^1(\Omega)$ (see counter-example in [5]).

The existence of a solution to the following weak problem is proved in [5] under the additionnal assumption that $\chi \in \tilde{H}$:

(10)
$$\begin{cases} \textit{Find } \{M, u\} \in (K \cap E(s)) \times H_0^1(\Omega) \textit{ such that} \\ (M, N-M) - [u, D^* N - D^* M] \geq 0 \ , \ \forall N \in K \cap \tilde{H} \ . \end{cases}$$

THE LIMIT LOAD PROBLEM.

We can determine the ultimate coefficient α such that the hypothesis (9) holds. More precisely, we define

$$\alpha = \sup \left\{ s \in \mathbb{R} \ : K \cap E(s) \ \text{ is not empty} \right\} ,$$

then $s < \alpha$ is equivalent to (9).

To compute α , we use the fact that α is also the upper bound of the coefficients s such that $u_s = 0$, where u_s is the solution of the visco-plastic problem

(11)
$$\begin{cases} \textit{Find } \{M_s, u_s\} \in E(s) \times V \\ (J'(M_s), N) - [u_s, D^* N] = 0 \ , \ \forall N \in \tilde{H} \ , \end{cases}$$

where $J'(N) \equiv N - \Pi N$, and $\Pi : H \to K$ is the projection operator onto K .

Our method to compute α consists of a binary search. For given s , solve problem (11). If $u_s \neq 0$, $s > \alpha$, otherwise $s \leq \alpha$.

Note that (11) is a well set problem for *any* s .

2 - APPROXIMATION VIA THE MIXED HELLAN-HERRMANN METHOD.

The idea of mixed finite element methods consists of approximating directly the weak (mixed) variational formulations (6), (10) or (11).

Let $\mathcal{T}_h$ be a triangulation of Ω , h being a parameter such as the diameter of the greatest triangle. We call

$$V_h = \left\{ v_h \in C_0(\overline{\Omega}) \ : \ \forall\, T \in \mathcal{T}_h \text{ , the restriction of } v_h \text{ to } T \text{ is linear affine} \right\} ,$$

where $C_0(\overline{\Omega})$ denotes the set of continuous functions, defined on $\overline{\Omega}$ and which vanish on Γ .

Let γ denotes an edge of the triangulation $\mathcal{T}_h$, and n the unit normal vector to γ , we let

$$R_n(N) \equiv N_{ij}\, n_i\, n_j \ .$$

We introduce the space H_h of the approximate bending moments (approximating $\tilde{H}$) :

$$H_h = \Big\{ N_h \in H \ : \ \forall\, T \in \mathcal{T}_h \text{ , the restriction of } N_h \text{ to } T \text{ is constant ;}$$
$$R_n(N_h) \text{ is continuous between 2 adjacent elements, and } (*)\ R_n(N_h) = 0 \text{ on } \Gamma \Big\}.$$

We check that V_h and H_h are finite-dimensional. The dimension of V_h is equal to the number of internal vertices, that of H_h to the number of (internal (*)) edges of the triangulation. In fact, if the value of $R_n(N_h)$ is prescribed on any edge γ of $\mathcal{T}_h$, the value of N_h is fixed everywhere.

Let $M(\Omega)$ be the dual of $C_0(\overline{\Omega})$ (space of bounded measures on Ω). We denote again by [,] the duality pairing between both spaces. We check [3] that

a) $N \in H_h \Rightarrow D^* N \in M(\Omega)$;

b) $[v, D^* N] = - \sum_{T \in \mathcal{T}_h} \int_{\partial T} R_n(N) \frac{\partial v}{\partial n}\, d\gamma \ , \ \forall\, T \in \mathcal{T}_h \ , \forall\, v \in V_h$.

(*) if the plate is simply supported.

Let

$$E_h(s) = \left\{ N_h \in H_h : [v_h , D^* N_h] = s \int_\Omega f \, v_h \, dX \; , \; \forall v_h \in V_h \right\} .$$

The approximate problems associated to (6), (10) and (11) are respectively

(12)
$$\begin{cases} \text{Find } \{M_h , u_h\} \in E_h(s) \times V_h \text{ such that} \\ (M_h , N) - [u_h , D^* N] = 0 \; , \; \forall N \in H_h \; , \end{cases}$$

the approximate *elastic* problem ;

(13)
$$\begin{cases} \text{Find } \{M_h , u_h\} \in (K \cap E_h(s)) \times V_h \text{ such that} \\ (M_h , N - M_h) - [u_h , D^* N - D^* M_h] \geq 0 \; , \; \forall N \in K \cap H_h \; , \end{cases}$$

the approximate *elasto-plastic* problem ;

(14)
$$\begin{cases} \text{Find } \{M_h , u_h\} \in E_h(s) \times V_h \text{ satisfying} \\ (J'_h(M_h) , N) - [u_h , D^* N] = 0 \; , \; \forall N \in H_h \; , \end{cases}$$

the approximate *visco-plastic* problem, where we choose $J'_h = J'$. In such a case, we compute, via a binary search,

$$\alpha_h = \sup \left\{ s \in \mathbb{R} : u_h(\text{sol of } (14)) = 0 \right\} .$$

CONVERGENCE RESULTS.

For the elastic problem, we have the error estimates (see [3])

$$\| M - M_h \| \leq C h \; ,$$
$$\| u - u_h \|_1 \leq C h \; ,$$

(where $\| \; \|_s$ denotes the norm of $H^s(\Omega)$), which hold both in the clamped and simply supported case, for Ω convex, polyhedral or not, as long as the boundary is sufficiently regular (*).

(*) In the non convex case, it is necessary to choose the triangulation in a suitable way. Some difficulties may arise, due to the lack of regularity of the solution.

For the elasto-plastic problem, there is only a convergence result for M_h to M [5]. Finally, for the visco-plastic problem, we have the convergence of u_h to u, and then $\limsup_{h \to 0} \alpha_h \leq \alpha$. However, the convergence of α_h to α seems to be an open problem.

REMARK 2.1 : We have also used $J'_h(N) = N - \Pi_h N$ in (14), where $\Pi_h : H_h \to K_h$ denotes the projection operator onto $K_h = K \cap H_h$.

In this case, if f is a sum of concentrated loads, we have $\alpha_h \leq \alpha$ for any h.

As we have mentioned it in the introduction, this choice leads to long computing times, since the projection onto K_h is not an easy problem.

Even with $J'_h(N) = N - \Pi N$, we can get the same property $\alpha_h \leq \alpha$ if we define α_h in the following way :

$$\alpha_h = \sup \left\{ s \in \mathbb{R} : M_h \text{ (sol of (14))} \in K \right\} . \quad \blacksquare$$

3 - ALGORITHMIC ASPECTS.

3.1 - FOR THE ELASTIC PROBLEM.

We denote by A the matrix of the scalar product $(\ ,\ )$ on H_h, and by B the matrix of the bilinear form $[v_h, D^* N_h]$ on $V_h \times H_h$. We shall denote again by M_h and u_h the column-vectors constituted with the components of M_h and u_h on the natural bases of H_h and V_h. The approximate elastic problem then consists of solving the linear system of equations

(15)
$$\begin{cases} -A\,M_h + B^T u_h = 0\ ; \\ B\,M_h = b\ , \end{cases}$$

where b is the column-vector representing the linear form (f, v_h) on V_h.

The matrix of this linear system

$$\mathcal{A} = \begin{pmatrix} -A & B^T \\ B & 0 \end{pmatrix}$$

is invertible [4] and symmetric, but not positive definite. We shall therefore solve (15) via a Gaussian elimination with partial pivoting.

The matrix $\mathcal{A}$ is sparse, and can even have a band structure, if the unknowns are ordered suitably (like any other stiffness matrix), which we have performed (see Fig. 1). Actually, pivoting and double precision have not been necessary, and the frontal method could be used.

Note that, since this mixed method is equivalent to Morley's method [8], the linear system (15) can be transformed, up to a simple change of the unknowns, into a linear system of equations with a symmetric positive definite matrix.

3.2 - ALGORITHM FOR SOLVING THE ELASTO-PLASTIC PROBLEM.

The application of the gradient algorithm to the nonlinear displacement problem corresponding to (13) gives the following algorithm :
$u^0 \in V_h$ and $\rho > 0$ being given, and by induction $u^n \in V_h$, compute $\{M^{n+1}, u^{n+1}\}$ solution of

5)

$$\begin{cases} A\, M^{n+1} - \frac{1}{\rho}\, B^T u^{n+1} = A\, M^{n+\frac{1}{2}} - \frac{1}{\rho}\, B^T u^n \\ B\, M^{n+1} = b \quad , \end{cases}$$

where $M^{n+\frac{1}{2}}$ *is the projection of* $B^T u^n$ *either on* K , *or on* K_h .

Recall that the convergence is proved for $0 < \rho < 2$ (see [7]).

The linear system of equations (16) has almost the same matrix as the elastic problem (15). Considerable computing time is saved by performing and storing the $L\,U$ factorization of this matrix.

There is no difficulty to compute the projection of $B^T u^n$ on K , since this can be done elementwise . In the Von Misès case, via a duality method, it is sufficient to solve a one variable non linear equation on each element, for which we have used Newton's method.

The projection on K_h is another matter, since K_h is a global convex set. We have applied Uzawa's algorithm to the Lagrangian

$$\mathcal{L}(N, \lambda) = \frac{1}{2} \|N\|^2 - [u^n, D^* N] + \int_\Omega \lambda\, F(N)\, dX \ ,$$

where λ is piecewise constant and positive.

This gives the following algorithm : λ^0 *and* $\theta > 0$ *being given and by induction* λ^k , *compute* N^k *minimizing* $\mathcal{L}(N, \lambda^k)$ *and* $\lambda^{k+1} = (\lambda^k + \theta\, F(N^k))^+$
where the superscrit $+$ indicates the positive part of a function.

The computation of N^k requires the solution of a linear system of equations, the matrix of which depends on λ^k . This explains why the projection on K_h is costly. To avoid this, we eventually used the penalty method, and a nonlinear conjugate gradient algorithm [14] to solve the penalized

problem, however without noticeable improvement of the computing time.

3.3 - ALGORITHM FOR SOLVING THE VISCO-PLASTIC PROBLEM.

We use again the gradient algorithm with auxiliary operator :

$M^0 \in H_h$ *and* $\rho > 0$ *being given ; by induction* $M^n \in H_h$ *being given, compute* $\{M^{n+1}, u^{n+1}\}$ *solution of*

(17) $$\begin{cases} A\,M^{n+1} - \rho\,B^T u^{n+1} = A\,M^{n+\frac{1}{2}} \\ B\,M^{n+1} = b \end{cases}$$

where $M^{n+\frac{1}{2}} = M^n - \rho\,J'_h(M^n)$.

Recall that the convergence is proved for $0 < \rho < 2$ (see [7]).

As we have mentionned it before, there are again 2 possibilities for the choice of J'_h , one of them involves a projection on K , and the other, on K_h .

The first possibility has revealed to be by far cheaper in computing time.

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