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ON A CLASS OF WEIGHTED SOBOLEV'S SPACES.

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ON A CLASS OF WEIGHTED SOBOLEV SPACES

I - CASE OF THE HALF-LINE \mathbb{R}_+ .

For an integer $m \in \mathbb{N}$, two real numbers α and $\beta > 0$ and an interval I of \mathbb{R}_+ , we consider the space :

$$V_{\alpha, \beta}^m(I) = \{u \in \mathcal{D}'(I) ; t^\alpha u \in L^2(I) , t^\beta D_t^m u \in L^2(I)\}$$

equipped by the canonical norm.

Proposition I.1 :

If $u \in V_{\alpha, \beta}^m(0, T)$, where T is a real number strictly positive, we have :

(i) $t^{\beta-j} D_t^{m-j} u \in L^2(0, T)$ for $0 \leq j \leq \text{Min}(j_0, m)$ with $j_0 = [\beta + \frac{1}{2}]_-$;

(ii) $t^{\beta-j_0} D_t^{m-j} u \in L^2(0, T)$ for $j_0 + 1 \leq j \leq m$ if $j_0 + 1 \leq m$;

(iii) $u \in H^{m-\beta}(0, T)$ if $\beta - m \neq \text{integer} + \frac{1}{2}$.

The notation $[A]_-$ means the greatest integer $< A$.

Proof : Let ψ be an indefinitely differentiable function such that $\psi(t) = 1$ if $t \leq \frac{T}{2}$ and $\psi(t) = 0$ if $t > \frac{3T}{4}$. Put $v = \psi u$; then $v \in V_{\alpha, \beta}^m(\mathbb{R}_+)$ with bounded support.

Using the Hardy's inequality, we obtain (i).

Again for (ii) : we have $t^{\beta-j_0} D_t^{m-j_0} v \in L^2(\mathbb{R}_+)$, also $t^{\beta-j_0+1} D_t^{m-j_0} v \in L^2(\mathbb{R}_+)$ and by the Hardy's inequality, we get $t^{\beta-j_0} D_t^{m-j_0-1} v \in L^2(\mathbb{R}_+)$;

repeating the same argument, we obtain (ii).

If $\beta > m$, it results from (i) that $t^{\beta-m} u \in L^2(\mathbb{R}_+)$ and consequently

if $\beta - m \neq \text{integer} + \frac{1}{2}$, we have $([4]) u \in H^{m-\beta}(\mathbb{R}_+)$.

If $\beta < m$, then $j_0 \leq m$ and $-\frac{1}{2} < \beta - j_0 \leq \frac{1}{2}$. Then, two cases must be distinguished according to $-\frac{1}{2} < \beta - j_0 \leq 0$ and $0 < \beta - j_0 \leq \frac{1}{2}$.

First case :

$0 < \beta - j_0 \leq \frac{1}{2}$. We have : $t^{\beta-j_0} D_t^{m-j_0-1} v$ and $t^{\beta-j_0} D_t^{m-j_0} v \in L^2(\mathbb{R}_+)$ (see that $0 < \beta - j_0$ and $\beta \leq m$ implies $j_0+1 \leq m$). Then, we have $t^{1/2} D_t^{m-j_0-1} v$ and $t^{1/2} D_t^{m-j_0} v \in L^2(\mathbb{R}_+)$, and now we prove that these two conditions imply $D_t^{m-j_0-1} v \in L^2(\mathbb{R}_+)$.

Lemma I-1 :

$([1])$. If $u \in V_{1/2, 1/2}^1(\mathbb{R}_+)$, then $u \in L^2(\mathbb{R}_+)$.

Proof :

If $u \in \mathcal{D}(\mathbb{R}_+)$, we can write :

$$|u(t)|^2 = 2 \operatorname{Re} \int_t^{+\infty} u(\sigma) \overline{u'(\sigma)} d\sigma$$

and using the Fubini's theorem, it comes :

$$\int_0^{+\infty} |u|^2 dt \leq -2 \operatorname{Re} \int_0^{+\infty} \sigma u(\sigma) \overline{u'(\sigma)} d\sigma \leq \int_0^{+\infty} t |u(t)|^2 dt + \int_0^{+\infty} t |u'(t)|^2 dt.$$

At last, by the density of $\mathcal{D}(\mathbb{R}_+)$ in the space $V_{1/2, 1/2}^1(\mathbb{R}_+)$, we get the lemma I.1.

Now, we prove that $D_t^{m-j_0-1} v \in H^\varepsilon(\mathbb{R}_+)$ with $\varepsilon = 1 - (\beta - j_0)$. For that, put $D_t^{m-j_0-1} v = f$ and $D_t^{m-j_0} v = F$ and compute :

$$\int_0^{+\infty} \int_0^{+\infty} \frac{|f(x) - f(y)|^2}{|x-y|^{2\varepsilon+1}} dx dy = \int_0^{+\infty} \int_0^{+\infty} \frac{|f(x+t) - f(x)|^2}{t^{2\varepsilon+1}} dx dt .$$

But,

$$f(x+t) - f(x) = \int_0^t F(x+\sigma) d\sigma .$$

Then,

$$\int_0^{+\infty} \frac{|f(x+t) - f(x)|^2}{t^{2\epsilon+1}} dt = \int_0^{+\infty} \frac{1}{t^{2\epsilon+1}} \left| \int_0^{+\infty} F(x+\sigma) d\sigma \right|^2 dt ,$$

and using the Hardy's inequality,

$$\leq C \int_0^{+\infty} \frac{1}{t^{2\epsilon-1}} |F(x+t)|^2 dt .$$

(C is a constant).

But,

$$\begin{aligned} \int_0^{+\infty} \frac{1}{t^{2\epsilon-1}} |F(x+t)|^2 dt &= \int_x^{+\infty} \frac{1}{|y-x|^{2\epsilon-1}} |F(y)|^2 dy \\ &= x^{-2(\epsilon-1)} \int_1^{+\infty} \frac{1}{|\sigma-1|^{2\epsilon-1}} |F(\sigma x)|^2 d\sigma \end{aligned}$$

and using the Fubini's theorem, it comes :

$$\int_0^{+\infty} x^{-2(\epsilon-1)} \left(\int_1^{+\infty} \frac{1}{|\sigma-1|^{2\epsilon-1}} |F(\sigma x)|^2 d\sigma \right) dx = \int_1^{+\infty} \frac{\sigma^{2(\epsilon-1)-1}}{|\sigma-1|^{2\epsilon-1}} d\sigma \cdot \int_0^{+\infty} y^{-2(\epsilon-1)} |F(y)|^2 dy$$

then, $D_t^{m-j_0-1} v \in H^\epsilon(\mathbb{R}_+)$ and $v \in H^{m-\beta}(\mathbb{R}_+)$.

Second case :

$-\frac{1}{2} < \beta - j_0 \leq 0$. The case $\beta - j_0 = 0$ being trivial, we can assume that

$-\frac{1}{2} < \beta - j_0 < 0$. Then, $\frac{1}{2} < \beta - j_0 + 1 < 1$ and we have $D_t^{m-j_0} v \in L^2(\mathbb{R}_+)$

and $t^{\beta-j_0+1} D_t^{m-j_0+1} v \in L^2(\mathbb{R}_+)$. By the same calculus as before we get that

$D_t^{m-j_0} v \in H^\epsilon(\mathbb{R}_+)$ with $\epsilon = -(\beta - j_0)$ and finally $v \in H^{m-\beta}(\mathbb{R}_+)$.

The proposition I.1 is proved.

Remark I.1 :

We can improve the result of the proposition I.1 when $\beta - \alpha > m$, in fact we have : if $\beta - \alpha > m$ and if $u \in V_{\alpha, \beta}^m(0, T)$, then $t^{\alpha + \frac{j}{m}(\beta - \alpha)} D_t^j u \in L^2(0, T)$ for $j = 0, \dots, m$. The proof is analogous to that of the following proposition I.2.

Proposition I.2 :

If $\beta - \alpha < m$ and if $u \in V_{\alpha, \beta}^m(T, +\infty)$ where T is a real number > 0 , then :

$$t^{\alpha + \frac{j}{m}(\beta - \alpha)} D_t^j u \in L^2(T, +\infty) \quad \text{for } j = 0, \dots, m.$$

Proof :

It will be made in two steps.

First step :

Reduction to the case $\alpha = 0$.

Lemma I.2 :

If $u \in V_{\alpha, \beta}^m(T, +\infty)$, then : $t^{\beta - m + j} D_t^j u \in L^2(T, +\infty)$.

Proof :

If $\beta < \frac{1}{2}$, obviously we have $u \in H^m(T, +\infty)$ and then $t^{\beta - j} D_t^{m-j} u \in L^2(T, +\infty)$ for $j = 0, \dots, m$.

If $\beta > \frac{1}{2}$, then, as in the proposition I.1, we get that $t^{\beta - j} D_t^{m-j} u \in L^2(T, +\infty)$ for $0 \leq j \leq \text{Min}(j_0, m)$ with $j_0 = [\beta + \frac{1}{2}]_-$. At last, since $D_t^{m-j} u \in L^2(T, +\infty)$ for $j = 0, \dots, m$, we get that $t^{\beta - j} D_t^{m-j} u \in L^2(T, +\infty)$ for $j = j_0 + 1, \dots, m$ if $j_0 + 1 \leq m$ ($\beta - j$ is negative).

Lemma I.3 :

The map $u \longrightarrow t^\alpha u$ is an isomorphism from $V_{\alpha, \beta}^m(T, +\infty)$ onto $V_{0, \beta-\alpha}^m(T, +\infty)$.

Proof :

Let u be an element of $V_{\alpha, \beta}^m(T, +\infty)$, we put $v = t^\alpha u$; then $t^{\beta-\alpha} D_t^m v(t) = \sum_{j=0}^m a_j \cdot t^{\beta-j} D_t^{m-j} u(t)$ and by the lemma I.2, it results that $v \in V_{0, \beta-\alpha}^m(T, +\infty)$.

Conversely, let v be an element of $V_{0, \beta-\alpha}^m(T, +\infty)$, we put $u = t^{-\alpha} v$; then $t^\beta D_t^m u(t) = \sum_{j=0}^m a_j \cdot t^{\beta-\alpha-j} D_t^{m-j} v(t)$ and by the lemma I.2, it results that $u \in V_{\alpha, \beta}^m(T, +\infty)$.

Second step :

We assume $\alpha = 0$.

We use the change of variable $y = t^{\frac{m-\beta}{m}}$ and of function $w(y) = y^{\beta/2(m-\beta)} u(t)$.

By induction on p , we show that, for $0 \leq p \leq m$, we have :

$$D_y^p w(y) = y^{\beta/2(m-\beta)} \sum_{j=0}^p a_{jp} \cdot t^{j-p+\frac{\beta}{m}} D_t^j u(t) .$$

where $a_{pp} \neq 0$. By the lemma I.2, we get $D_y^m w \in L^2(Y, +\infty)$ where $Y = T^{\frac{m-\beta}{m}}$ and consequently $w \in H^m(Y, +\infty)$ since $w \in L^2(Y, +\infty)$. Then, $D_y^p w \in L^2(Y, +\infty)$ for $p = 0, \dots, m$ and using the precedent formula, we get, by induction on p and since $j-p+\frac{\beta}{m} < \frac{j}{m}$ for $j < p$, that $t^{\frac{j}{m}} D_t^j u \in L^2(T, +\infty)$ for $j = 0, \dots, m$.

The proposition I.2 is proved.

We now apply these results to a sub-class of Sobolev spaces with weights which we will be useful for the following : let be $m \in \mathbb{N}$, $-\sigma$ and δ two real numbers > 0 such that $\sigma+m \geq 0$ and $\sigma+\delta m \geq 0$, we consider the space:

$$W_{\sigma, \delta}^m(\mathbb{R}_+) = \{u \in H^{-\sigma}(\mathbb{R}_+) ; t^{\sigma+\delta k+j} D_t^j u \in L^2(\mathbb{R}_+) \text{ for } \sigma+\delta k+j \geq 0 \text{ and } k+j \leq m\}$$

equipped by the canonical norm.

By the propositions I.1 and I.2, this space coincide with the space $V_{\sigma+\delta m, \sigma+m}^m(\mathbb{R}_+)$.

We now give the Sobolev's theorem for the spaces $W_{\sigma, \delta}^m(\mathbb{R}_+)$.

Proposition I.3 : we have :

i) If $u \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, u is continuous on \mathbb{R}_+ and there exists a constant $C > 0$ such that for every $u \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, for every $t > 0$, we have :

$$(1.1) \quad |u(t)| \leq C \cdot t^{-\frac{\sigma+m}{2m}} \|u\|_{W_{\sigma, \delta}^m}^{1/2m} \|u\|_{L^2}^{1-1/2m};$$

(ii) We assume $-\sigma > \frac{1}{2}$, then : if $u \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, u is continuous and bounded on \mathbb{R}_+ and there exists a constant $C > 0$ such that for every $u \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, for every $t > 0$, we have :

$$(1.2) \quad |u(t)| \leq C \cdot \|u\|_{W_{\sigma, \delta}^m}^{-1/2\sigma} \|u\|_{L^2}^{1+\frac{1}{2\sigma}};$$

$$(1.3) \quad |u(t)| \leq C \cdot t^{-(\sigma+\delta m) + \frac{1}{2}(\delta-1)} \|u\|_{W_{\sigma, \delta}^m}.$$

Proof :

(i) At first, we apply the usual Sobolev's theorem : if $v \in H^m(\mathbb{R}_+)$ with $m \geq 1$, then v is continuous on $\overline{\mathbb{R}_+}$ and there exists a constant $C > 0$ such that for every $v \in H^m(\mathbb{R}_+)$, for every $t \geq 0$, we have :

$$|v(t)|^2 \leq C \left\{ \int_0^{+\infty} |D_t^m v(\tau)|^2 d\tau + \int_0^{+\infty} |v(\tau)|^2 d\tau \right\}.$$

If $w \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, the function v defined by $v(\tau) = w(\tau+t)$ belongs to $H^m(\mathbb{R}_+)$ for every $t > 0$. Since $-\sigma > 0$ and $\sigma+m > 0$, then $m > 1$ and for every $w \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, for every $t > 0$, we have :

$$|w(t)|^2 \leq C. \left\{ \int_t^{+\infty} |D_t^m w(\tau)|^2 d\tau + \int_t^{+\infty} |w(\tau)|^2 d\tau \right\}.$$

Now, let u be an element of $W_{\sigma, \delta}^m(\mathbb{R}_+)$ and we apply the precedent inequality to the function w defined by $w(\tau) = u(\lambda\tau)$ where λ is a positive constant. Then, there exists a constant $C > 0$ such that, for every $u \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, for every $t > 0$, for every $\lambda > 0$, we have :

$$(1.4) \quad |u(t)|^2 \leq C/\lambda \left\{ \int_t^{+\infty} |\lambda^m D_t^m u(\tau)|^2 d\tau + \int_t^{+\infty} |u(\tau)|^2 d\tau \right\},$$

and since $t \leq \tau$, we get :

$$|u(t)|^2 \leq C/\lambda \left\{ \int_t^{+\infty} \lambda^{2m} t^{-2(\sigma+m)} |\tau^{\sigma+m} D_t^m u(\tau)|^2 d\tau + \int_t^{+\infty} |u(\tau)|^2 d\tau \right\}.$$

Choosing $\lambda = t^{\frac{\sigma+m}{m}}$, a fortiori we obtain :

$$|u(t)|^2 \leq C. t^{-\frac{\sigma+m}{m}} \left\{ \int_0^{+\infty} |\tau^{\sigma+m} D_t^m u(\tau)|^2 d\tau + \int_0^{+\infty} |u(\tau)|^2 d\tau \right\}.$$

Now, we apply this inequality to the function v defined by $v(\tau) = u(\lambda\tau)$

where λ is a constant > 0 :

$$|u(\lambda t)|^2 \leq C. t^{-\frac{\sigma+m}{m}} \frac{1}{\lambda} \left\{ \int_0^{+\infty} \lambda^{-2\sigma} |\tau^{\sigma+m} D_t^m u(\tau)|^2 d\tau + \int_0^{+\infty} |u(\tau)|^2 d\tau \right\}.$$

Putting $\lambda = r^{1/2\sigma}$, we get for every $u \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, for every $t > 0$, for every $r > 0$, we have :

$$|u(t r^{1/2\sigma})|^2 \leq C. (t r^{1/2\sigma})^{-\frac{\sigma+m}{m} \frac{1}{r^{2m}} - 1} \left\{ \int_0^{+\infty} |\tau^{\sigma+m} D_t^m u(\tau)|^2 d\tau + r \int_0^{+\infty} |u(\tau)|^2 d\tau \right\}.$$

Finally, there exists a constant $C > 0$ such that, for every $t > 0$, for every $r > 0$, for every $u \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, we have :

$$|u(t)|^2 \leq C. t^{-\frac{\sigma+m}{m}} r^{\frac{1}{2m}-1} \left\{ \|u\|_{W_{\sigma, \delta}^m}^2 + r \|u\|_{L^2}^2 \right\}.$$

Taking $r = \frac{\|u\|_{L^2}^2}{\|u\|_{W_{\sigma, \delta}^m}^2}$, we obtain the inequality (1.1).

(ii) If $-\sigma > \frac{1}{2}$, the Sobolev's theorem imply that if $v \in H^{-\sigma}(\mathbb{R}_+)$, then v is continuous and bounded on $\overline{\mathbb{R}_+}$ and there exists a constant $C > 0$ such that for every $v \in H^{-\sigma}(\mathbb{R}_+)$, for every $t \geq 0$, we have :

$$|v(t)|^2 \leq C. \|v\|_{H^{-\sigma}(\mathbb{R}_+)}^2$$

But, from the proposition I.1, the space $V_{0, \sigma+m}^m(\mathbb{R}_+)$ is continuously imbedded in $H^{-\sigma}(\mathbb{R}_+)$, then, for every $t \geq 0$, for every $v \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, we have :

$$|v(t)|^2 \leq C. \left\{ \int_0^{+\infty} |\tau^{\sigma+m} D_t^m u|^2 d\tau + \int_0^{+\infty} |u(\tau)|^2 d\tau \right\}.$$

Using the same change of functions as before, we get that for every $u \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, for every $t > 0$, for every $r > 0$, we have :

$$|u(t)|^2 \leq C. r^{-1 - \frac{1}{2\sigma}} \left\{ \|u\|_{W_{\sigma, \delta}^m}^2 + r \|u\|_{L^2}^2 \right\}.$$

We obtain the inequality (1.2) in taking $r = \frac{\|u\|_{L^2}^2}{\|u\|_{W_{\sigma, \delta}^m}^2}$.

To have the inequality (1.3), we start from the inequality (1.4) in which

we choose $\lambda = \left(\int_t^{+\infty} |u(\tau)|^2 d\tau \right)^{1/2m} \left(\int_t^{+\infty} |D_t^m u(\tau)|^2 d\tau \right)^{-1/2m}$, that gives :

$$|u(t)|^2 \leq C. \left(\int_t^{+\infty} |D_t^m u|^2 d\tau \right)^{1/2m} \left(\int_t^{+\infty} |u(\tau)|^2 d\tau \right)^{1-1/2m};$$

after that, we remark that, since $t \leq \tau$, we have :

$$\int_t^{+\infty} |D_t^m u|^2 d\tau \leq t^{-2(\sigma+m)} \int_t^{+\infty} \tau^{2(\sigma+m)} |D_t^m u|^2 d\tau \leq t^{-2(\sigma+m)} \|u\|_{W_{\sigma,\delta}^m}^2$$

and

$$\int_t^{+\infty} |u(\tau)|^2 d\tau \leq t^{-2(\sigma+\delta m)} \int_t^{+\infty} \tau^{2(\sigma+\delta m)} |u(\tau)|^2 d\tau \leq t^{-2(\sigma+\delta m)} \|u\|_{W_{\sigma,\delta}^m}^2$$

hence the inequality (1.3).

II - CASE OF THE HALF SPACE \mathbb{R}_+^n , $n > 1$.

Let m be an integer, $-\sigma$ and δ two real numbers > 0 such that $\sigma+m \geq 0$ and $\sigma+\delta m \geq 0$, we consider the space :

$$W_{\sigma,\delta}^m(\mathbb{R}_+^n) = \{u \in L^2(\mathbb{R}_+^n); t^{\sigma+\delta|\alpha|+j} D_t^j D_x^\alpha u \in L^2(\mathbb{R}_+^n) \text{ for } \sigma+\delta|\alpha|+j \geq 0 \text{ and } |\alpha|+j \leq m\}$$

equipped by the canonical norm.

The space $\mathcal{D}(\overline{\mathbb{R}_+^n})$ is dense in the space $W_{\sigma,\delta}^m(\mathbb{R}_+^n)$ (cf [2] for example) and also we have :

$$W_{\sigma,\delta}^m(\mathbb{R}_+^n) = \{u \in \mathcal{D}'(\mathbb{R}_+^n); t^{\text{Max}(0, \sigma+\delta|\alpha|+j)} D_t^j D_x^\alpha u \in L^2(\mathbb{R}_+^n) \text{ for } |\alpha|+j \leq m\}.$$

Proposition II.1. we have :

i) if $m > n/2$ and if $u \in W_{\sigma,\delta}^m(\mathbb{R}_+^n)$, then u is continuous on \mathbb{R}_+^n and there exists a constant $C > 0$ such that, for every $u \in W_{\sigma,\delta}^m(\mathbb{R}_+^n)$, for every $(t,x) \in \mathbb{R}_+^n$, we have :

$$(2.1) \quad |u(t,x)| \leq C. t^{-\frac{\tau+m}{2m} - \frac{n-1}{2m}(\sigma+\delta m)} \|u\|_{W_{\sigma,\delta}^m}^{n/2m} \|u\|_{L^2}^{1-n/2m};$$

(ii) If $\text{Min}(-\sigma, -\sigma/\delta) > n/2$ and if $u \in W_{\sigma, \delta}^m(\mathbb{R}_+^n)$, then u is continuous and bounded on \mathbb{R}_+^n and there exists a constant $C > 0$ such that for every $u \in W_{\sigma, \delta}^m(\mathbb{R}_+^n)$, for every $(t, x) \in \mathbb{R}_+^n$, we have :

$$(2.2) \quad |u(t, x)| \leq C \cdot \|u\|_{W_{\sigma, \delta}^m}^{-\frac{1+\delta(n-1)}{2\sigma}} \|u\|_{L^2}^{1+\frac{1+\delta(n-1)}{2\sigma}}.$$

Proof :

The proof is analogous to those made in the chapter I. (i) ; at first, we apply the usual Sobolev's theorem : if $v \in H^m(\mathbb{R}_+^n)$ with $m > n/2$ then v is continuous on $\overline{\mathbb{R}_+^n}$ and there exists a constant $C > 0$ such that for every $v \in H^m(\mathbb{R}_+^n)$, for every $(t, x) \in \mathbb{R}_+^n$, we have :

$$|u(t, x)|^2 \leq C \cdot \left\{ \sum_{j+|\alpha|=m} \int_{\mathbb{R}_+^n} |D_t^j D_x^\alpha v(\tau, y)|^2 d\tau dy + \int_{\mathbb{R}_+^n} |v(\tau, y)|^2 d\tau dy \right\}.$$

If $w \in W_{\sigma, \delta}^m(\mathbb{R}_+^n)$, the function v defined by : $v(\tau, y) = w(\tau+t, y)$ belongs to $H^m(\mathbb{R}_+^n)$ for every $t > 0$. Hence, for every $w \in W_{\sigma, \delta}^m(\mathbb{R}_+^n)$, for every $(t, x) \in \mathbb{R}_+^n$, we have :

$$|w(t, x)|^2 \leq C \cdot \left\{ \sum_{|\alpha|+j=m} \int_t^{+\infty} \int_{\mathbb{R}^{n-1}} |D_t^j D_x^\alpha w(\tau, y)|^2 d\tau dy + \int_t^{+\infty} \int_{\mathbb{R}^{n-1}} |w(\tau, y)|^2 d\tau dy \right\}.$$

Let now u an element of $W_{\sigma, \delta}^m(\mathbb{R}_+^n)$ and apply the precedent inequality to the function w defined by : $w(\tau, y) = u(\lambda\tau, \mu y)$ where λ and μ are two constants. Hence, there exists a constant $C > 0$ such that, for every $u \in W_{\sigma, \delta}^m(\mathbb{R}_+^n)$, for every $(t, x) \in \mathbb{R}_+^n$, for every λ and $\mu > 0$, we have :

$$|u(t, x)|^2 \leq C \lambda^{-1} \mu^{n-1} \cdot \left\{ \sum_{|\alpha|+j=m} \int_t^{+\infty} \int_{\mathbb{R}^{n-1}} \lambda^{2j} \mu^{2(m-j)} |D_t^j D_x^\alpha u(\tau, y)|^2 d\tau dy + \int_t^{+\infty} \int_{\mathbb{R}^{n-1}} |u(\tau, y)|^2 d\tau dy \right\},$$

and since $t \leq \tau$, that gives:

$$|u(t,x)|^2 \leq C \frac{t}{\lambda \cdot \mu^{n-1}} \times$$

$$\times \left\{ \sum_{|\alpha|+j=m} \int_t^{+\infty} \int_{\mathbb{R}^{n-1}} \lambda^{2j} \mu^{2(m-j)} t^{-2(\sigma+\delta(m-j)+j)} |\tau^{\sigma+\delta} |\alpha|+j D_t^j D_x^\alpha u|^2 d\tau dy \right.$$

$$\left. + \int_t^{+\infty} \int_{\mathbb{R}^{n+1}} |u(\tau,y)|^2 d\tau dy \right\}.$$

choosing $\lambda = t^{\frac{\sigma+m}{m}}$ and $\mu = t^{\frac{\sigma+\delta m}{m}}$, a fortiori we get :

$$|u(t,x)|^2 \leq C \cdot t^{-\frac{\sigma+m}{m} - \frac{n-1}{m}(\sigma+\delta m)} \left\{ \sum_{|\alpha|+j=m} \int_{\mathbb{R}_+^n} |\tau^{\sigma+\delta} |\alpha|+j D_t^j D_x^\alpha u|^2 d\tau dy \right.$$

$$\left. + \int_{\mathbb{R}_+^n} |u(\tau,y)|^2 d\tau dy \right\}.$$

We now apply this inequality to the function v defined by : $v(\tau,y) = u(\lambda\tau, \mu x)$ where λ and μ are some constants :

$$|u(\lambda t, \mu x)|^2 \leq C \times$$

$$\times \frac{t}{\lambda \cdot \mu^{n-1}} \left\{ \sum_{|\alpha|+j=m} \int_{\mathbb{R}_+^n} \lambda^{-2(\sigma+\delta(m-j))} \mu^{2(m-j)} |\tau^{\sigma+\delta} |\alpha|+j D_t^j D_x^\alpha u|^2 d\tau dy \right.$$

$$\left. + \int_{\mathbb{R}_+^n} |u|^2 d\tau dy \right\}.$$

Putting $\lambda = r^{1/2\sigma}$ and $\mu = \lambda^\delta$, we deduce that for every $u \in W_{\sigma, \delta}^m(\mathbb{R}_+^n)$, for every $(t,x) \in \mathbb{R}_+^n$, for every $r > 0$, we have :

$$|u(tr^{1/2\sigma}, xr^{\delta/2\sigma})|^2 \leq C x$$

$$x_{(tr^{1/2\sigma})}^{-\frac{\sigma+m}{m} - \frac{n-1}{m}(\sigma+\delta m)} r^{n/2m-1} \left\{ \sum_{|\alpha|+j=m} \int_{\mathbb{R}_+^n} |\tau^{\sigma+\delta} |\alpha|+j D_t^j D_x^\alpha u|^2 d\tau dy \right. \\ \left. + r \int_{\mathbb{R}_+^n} |u|^2 d\tau dy \right\}.$$

Finally, there exists a constant $C > 0$ such that, for every $(t, x) \in \mathbb{R}_+^n$, for every $r > 0$, for every $u \in W_{\sigma, \delta}^m(\mathbb{R}_+^n)$, we have :

$$|u(t, x)|^2 \leq C. t^{-\frac{\sigma+m}{m} - \frac{n-1}{m}(\sigma+\delta m)} r^{n/2m-1} \left\{ \|u\|_{W_{\sigma, \delta}^m}^2 + r \|u\|_{L^2}^2 \right\}.$$

The inequality (2.1) results from this in choosing $r = \|u\|_{W_{\sigma, \delta}^m}^2 / \|u\|_{L^2}^2$.

(ii), we begin to show the

Lemma II-1 :

We have the algebraic and topologic imbedding :

$$W_{\sigma, \delta}^m(\mathbb{R}_+^n) \subset H^{\text{Min}(-\sigma, -\sigma/\delta)}(\mathbb{R}_+^n).$$

Proof :

By the chapter I, we know $V_{\sigma+\delta m, \sigma+m}^m(\mathbb{R}_+) \subset H^{-\sigma}(\mathbb{R}_+)$, hence, there exists a constant $C > 0$ such that, for every $v \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, we have :

$$\int_{-\infty}^{+\infty} (1+\tau^2)^{-\sigma} |F(Pv)|^2 d\tau \leq C. \left\{ \int_0^{+\infty} |t^{\sigma+m} D_t^m v|^2 dt + \int_0^{+\infty} |t^{\sigma+\delta m} v|^2 dt \right\},$$

where F means the Fourier transform in the variable t and P a linear and continuous extension operator from $H^{-\sigma}(\mathbb{R})$ (for example, P can be taken as the Babitch extension).

If $v \in W_{\sigma, \delta}^m(\mathbb{R}_+^n)$, the function $u(t) = v(t\Lambda^{-1/\delta})$, where Λ is positive constant, belongs to $W_{\sigma, \delta}^m(\mathbb{R}_+^n)$; for every $\Lambda > 0$, we have :

$$\int_{-\infty}^{+\infty} (\Lambda^{2/\delta} + \tau^2)^{-\sigma} |F(Pv)|^2 d\tau \leq C. \left\{ \int_0^{+\infty} |t^{\sigma+m} D_t^m v|^2 dt + \Lambda^{2m} \int_0^{+\infty} |t^{\sigma+\delta m} v|^2 dt \right\} .$$

Let now u be an element of $\mathcal{D}(\overline{\mathbb{R}_+^n})$ and for every $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$, we consider the function $v(t) = \hat{u}(t, \xi)$, where Λ means the Fourier transform in the variable $x \in \mathbb{R}^{n-1}$; then $F(Pv)(\tau) = \mathcal{F}Pu(\tau, \xi)$, where \mathcal{F} means the Fourier transform in the variable (t, x) in \mathbb{R}^n and from the precedent inequality, we deduce, taking $\Lambda = |\xi|$ and after integrate in ξ over \mathbb{R}^{n-1} , that there exists a constant $C > 0$ such that for all $u \in \mathcal{D}(\overline{\mathbb{R}_+^n})$, we have : putting $\sigma^* = \text{Min}(-\sigma, -\sigma/\delta)$,

$$\|Pu\|_{H^{-\sigma^*}(\mathbb{R}_+^n)} \leq C. \|u\|_{W_{\sigma, \delta}^m(\mathbb{R}_+^n)}$$

and then :

$$\|u\|_{H^{-\sigma^*}(\mathbb{R}_+^n)} \leq C. \|u\|_{W_{\sigma, \delta}^m(\mathbb{R}_+^n)} .$$

The space $\mathcal{D}(\overline{\mathbb{R}_+^n})$ being dense in the space $W_{\sigma, \delta}^m(\mathbb{R}_+^n)$, we have proved the lemma II-1.

Now, if $\text{Min}(-\sigma, -\sigma/\delta) > n/2$ and if $u \in W_{\sigma, \delta}^m(\mathbb{R}_+^n)$, then u is continuous and bounded on $\overline{\mathbb{R}_+^n}$ and there exists a constant $C > 0$ such that for every $u \in W_{\sigma, \delta}^m(\mathbb{R}_+^n)$, for every $(t, x) \in \mathbb{R}_+^n$, we have :

$$|u(t, x)|^2 \leq C. \left\{ \sum_{|\alpha|+j=m} \int_{\mathbb{R}_+^n} \tau^{2(\sigma+\delta(m-j)+j)} |D_t^j D_x^\alpha u(\tau, y)|^2 d\tau dy + \int_{\mathbb{R}_+^n} |u(\tau, y)|^2 d\tau dy \right\} .$$

Then, we do the change of variable of (i), that gives :

$$|u(t,x)|^2 \leq C \lambda^{\mu n-1} \times$$

$$\times \left\{ \sum_{|\alpha|+j=m} \int_{\mathbb{R}_+^n} \lambda^{-2(\sigma+\delta(m-j))} \mu^{2(m-j)} |\tau^{-2(\sigma+\delta(m-j)+j)} D_t^j D_x^\alpha u(\tau,y)|^2 d\tau dy \right.$$

$$\left. + \int_{\mathbb{R}_+^n} |u(\tau,y)|^2 d\tau dy \right\};$$

we choose $\lambda = r^{1/2\sigma}$ and $\mu = \lambda^\delta$, that gives :

$$|u(t,x)|^2 \leq C. r^{-\frac{2\sigma+1+\delta(n-1)}{2\sigma}} \left\{ \|u\|_{W_{\sigma,\delta}^m(\mathbb{R}_+^n)}^2 + r \|u\|_{L^2(\mathbb{R}_+^n)}^2 \right\},$$

and taking $r = \frac{\|u\|_{W_{\sigma,\delta}^m}^2}{\|u\|_{L^2}^2}$, we get the inequality (2.2).

Proposition II.2 :

Let ℓ be an integer, $0 \leq \ell < -\sigma - \frac{1}{2}$; then the map $u \longrightarrow \gamma_\ell u = D_t^\ell u(t=0) : \mathcal{D}'(\mathbb{R}_+^n) \longrightarrow (\mathbb{R}^{n-1})'$ can be extended in a linear and continuous map from $W_{\sigma,\delta}^m(\mathbb{R}_+^n)$ into $H^{-\frac{2(\sigma+\ell)+1}{2\delta}}(\mathbb{R}^{n-1})$.

Proof :

It comes, by the chapter I, that there exists a constant $C > 0$ such that, for every $v \in W_{\sigma,\delta}^m(\mathbb{R}_+)$, we have :

$$|D_t^\ell v(0)|^2 \leq C. \left\{ \int_0^{+\infty} |t^{\sigma+m} D_t^m v|^2 dt + \int_0^{+\infty} |t^{\sigma+\delta m} v|^2 dt \right\}.$$

If $v \in W_{\sigma,\delta}^m(\mathbb{R}_+)$, the function $u(t) = v(t \Lambda^{-\frac{1}{\sigma}})$, where Λ is a positive constant, belongs to $W_{\sigma,\delta}^m(\mathbb{R}_+)$; hence here exists a constant $C > 0$ such that for every

$v \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, for every $\Lambda > 0$, we have :

$$\Lambda^{-\frac{2(\sigma+\ell)+1}{2\delta}} |D_t^\ell v(0)|^2 \leq C \cdot \left\{ \int_0^{+\infty} |t^{\sigma+m} D_t^m v|^2 dt + \Lambda^{2m} \int_0^{+\infty} |t^{\sigma+\delta m} v|^2 dt \right\}.$$

Let now u be an element of $\mathcal{D}(\overline{\mathbb{R}_+^n})$, and for every $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$, we consider the function $v(t) = \hat{u}(t, \xi)$, where Λ is the Fourier transform in the variable $x \in \mathbb{R}^{n-1}$; as in lemma II-1, we deduce that :

$$\| \gamma_\ell u \|_H - \frac{2(\sigma+\ell)+1}{2\delta} \leq C \cdot \| u \|_{W_{\sigma, \delta}^m}.$$

It will be very useful for the following to have an inequality of type "compactness" for the spaces $W_{\sigma, \delta}^m$:

Proposition II.3.

Let m be an integer ≥ 1 and put $\delta_1 = \text{Min}(1, \delta)$. There exists a constant $C > 0$ such that, for every $\epsilon > 0$, for every $u \in W_{\sigma, \delta}^m(\mathbb{R}_+^n)$, with $\text{supp } u \subset \{|t| \leq 1\}$, we have :

$$(2.3) \quad \| u \|_{W_{\sigma+\delta_1, \delta}^{m-1}} \leq C \cdot \left\{ \epsilon \cdot \| u \|_{W_{\sigma, \delta}^m} + \epsilon^{-(m-1)} \| u \|_{L^2} \right\}.$$

Proof :

We begin to establish a lemma :

Lemma II-2 :

The map $u \longrightarrow \left\{ \| t^{\sigma+m} D_t^m u \|_{L^2}^2 + \sum_{|\alpha|=m} \| t^{\sigma+\delta m} D_x^\alpha u \|_{L^2}^2 + \| u \|_{L^2}^2 \right\}^{1/2}$ is an equivalent norm for the space $W_{\sigma, \delta}^m(\mathbb{R}_+^n)$.

Proof :

Let k and j be some integers such that $\sigma + \delta k + j > 0$ and $k + j \leq m$. From the chapter I, it results that if $v(t) \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, then $t^{\sigma + \delta k + j} D_t^j v \in L^2(\mathbb{R}_+)$ and :

$$\int_0^{+\infty} |t^{\sigma + \delta k + j} D_t^j v|^2 dt \leq C. \left\{ \int_0^{+\infty} |t^{\sigma + m} D_t^m v|^2 dt + \int_0^{+\infty} |t^{\sigma + \delta m} v|^2 dt \right\}$$

where C is a constant > 0 which does not depend on v .

If $v \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, the function $u(t) = v(t \Lambda^{-1/\sigma})$, where Λ is a positive constant belongs to $W_{\sigma, \delta}^m(\mathbb{R}_+)$; hence, there exists a constant $C > 0$ such that for every $v \in W_{\sigma, \delta}^m(\mathbb{R}_+)$, for every Λ , we have :

$$(2.4) \quad \Lambda^{2k} \int_0^{+\infty} |t^{\sigma + \delta k + j} D_t^j v|^2 dt \leq C. \left\{ \int_0^{+\infty} |t^{\sigma + m} D_t^m v|^2 dt + \Lambda^{2m} \int_0^{+\infty} |t^{\sigma + \delta m} v|^2 dt \right\}.$$

Let now u be an element of $\mathcal{D}(\overline{\mathbb{R}_+^n})$ and for every $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$, we consider the function $v(t) = \hat{u}(t, \xi)$, where Λ means the Fourier transform in the variable $x \in \mathbb{R}^{n-1}$, and from the precedent inequality, we deduce, taking $\Lambda = |\xi|$ and after integration in ξ over \mathbb{R}^{n-1} , that there exists a constant $C > 0$ such that for every $u \in \mathcal{D}(\overline{\mathbb{R}_+^n})$, we have :

$$\|u\|_{W_{\sigma, \delta}^m}^2 \leq C. \left\{ \| |t^{\sigma + m} D_t^m u| \|_{L^2}^2 + \sum_{|\alpha| = m} \| |t^{\sigma + \delta m} D_x^\alpha u| \|_{L^2}^2 + \|u\|_{L^2}^2 \right\}.$$

The space $\mathcal{D}(\overline{\mathbb{R}_+^n})$ being dense in the space $W_{\sigma, \delta}^m(\overline{\mathbb{R}_+^n})$, the lemma II-2 is a consequence of this inequality and the Banach's theorem.

Proof of the proposition II-3 :

From the inequality (2.4) in which we take $j = m-1$, $k = 1$ and $\Lambda^{-1} = \varepsilon > 0$, we deduce that :

$$\int_0^{+\infty} |t^{\sigma+\delta+m-1} D_t^{m-1} v|^2 dt \leq C \times$$

$$\times \left\{ \varepsilon^2 \int_0^{+\infty} |t^{\sigma+m} D_t^m v|^2 dt + \varepsilon^{-2(m-1)} \int_0^{+\infty} |t^{\sigma+\delta m} v|^2 dt \right\}.$$

We apply this inequality to the function $v(t) = \hat{u}(t, \xi)$ for $u \in \mathcal{D}(\overline{\mathbb{R}_+^n})$ and $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$, we integrate in ξ over \mathbb{R}^{n-1} , that gives :

$$(2.5) \quad \left\| |t^{\sigma+\delta+m-1} D_t^{m-1} u|^2 \right\|_{L^2(\mathbb{R}_+^n)}^2 \leq C \cdot \left\{ \varepsilon^2 \left\| |t^{\sigma+m} D_t^m u|^2 \right\|_{L^2(\mathbb{R}_+^n)}^2 + \varepsilon^{-2(m-1)} \left\| |u|^2 \right\|_{L^2(\mathbb{R}_+^n)}^2 \right\}$$

if $\text{supp } u \subset \{|t| \leq 1\}$.

Besides, we know that there exists a constant $C > 0$ such that for every $\varepsilon > 0$, for every $v(x) \in H^m(\mathbb{R}^{n-1})$, we have :

$$(2.6) \quad \sum_{|\alpha|=m-1} \int_{\mathbb{R}^{n-1}} |D_x^\alpha v|^2 dx \leq C \cdot \left\{ \varepsilon^2 \sum_{|\alpha|=m} \int_{\mathbb{R}^{n-1}} |D_x^\alpha v|^2 dx + \varepsilon^{-2(m-1)} \int_{\mathbb{R}^{n-1}} |v|^2 dx \right\}.$$

Then, we use this inequality to the function $v(x) = u(t, x)$, $t > 0$, where $u \in \mathcal{D}(\overline{\mathbb{R}_+^n})$; we multiply by $t^{\sigma+\delta m}$, and we integrate in $t > 0$ over \mathbb{R}_+ , that gives :

$$(2.7) \quad \sum_{|\alpha|=m-1} \left\| |t^{\sigma+\delta m} D_x^\alpha u|^2 \right\|_{L^2(\mathbb{R}_+^n)}^2 \leq C \cdot \left\{ \varepsilon^2 \sum_{|\alpha|=m} \left\| |t^{\sigma+\delta m} D_x^\alpha u|^2 \right\|_{L^2(\mathbb{R}_+^n)}^2 \right.$$

$$\left. + \varepsilon^{-2(m-1)} \left\| |u|^2 \right\|_{L^2(\mathbb{R}_+^n)}^2 \right\}.$$

if $\text{supp } u \subset \{|t| \leq 1\}$.

The inequality (2.3), for $\delta \leq 1$, is a consequence of (2.5) and (2.7).

For $\delta \geq 1$, we replace the inequality (2.5) by the inequality :

$$(2.8) \quad \left\| |t^{\sigma+m} D_t^{m-1} u|^2 \right\|_{L^2(\mathbb{R}_+^n)}^2 \leq C \cdot \left\{ \varepsilon^2 \left\| |t^{\sigma+m} D_t^m u|^2 \right\|_{L^2(\mathbb{R}_+^n)}^2 + \varepsilon^{-2(m-1)} \left\| |u|^2 \right\|_{L^2(\mathbb{R}_+^n)}^2 \right\}.$$

if $\text{supp } u \subset \{|t| \leq 1\}$. This inequality is easy to prove like for (2.5).

After, in (2.7), we multiply by $t^{2(\sigma+1+\delta(m-1))}$ and we choose $\varepsilon = \eta t^{\delta-1}$, $\eta > 0$, and we achieve as before.

III - CASE OF A BOUNDED OPEN SET Ω OF \mathbb{R}^n , $n > 1$.

Let Ω be a bounded open set of \mathbb{R}^n , with boundary Γ . We assume that Ω is a compact C^∞ manifold. We give $\psi: \mathbb{R}^n \rightarrow \mathbb{R}$ a C^∞ function such that :

$$(3.1) \quad \begin{cases} \Omega = \{x \in \mathbb{R}^n ; \psi(x) > 0\} , \\ \Gamma = \{x \in \mathbb{R}^n ; \psi(x) = 0\} , \\ \text{grad } \psi(x) \neq 0 \text{ for } x \in \Gamma , \end{cases}$$

where $\text{grad } \psi(x) = \left(\frac{\partial \psi}{\partial x_1}(x), \dots, \frac{\partial \psi}{\partial x_n}(x) \right)$ is the gradient vector associated to ψ .

Let $(X_i)_{0 \leq i \leq q}$ be some vector fields with C^∞ coefficients on \mathbb{R}^n such that :

$$(3.2) \quad X_0 \text{ is transversal to } \Gamma \text{ on } \Gamma, \text{ ie : } (X_0 \psi)(x) \neq 0 \text{ for } x \in \Gamma ;$$

$$(3.3) \quad X_i \text{ is tangent to } \Gamma \text{ on } \Gamma \text{ for } i = 1, \dots, q, \text{ ie : } (X_i \psi)(x) = 0 \text{ for } x \in \Gamma ;$$

$$(3.4) \quad \text{for every } x \in \bar{\Omega}, \text{ the rank of the system } (X_i(x))_{0 \leq i \leq q} \text{ is equal to } n.$$

Let m be an integer, $-\sigma$ and δ two real numbers > 0 such that $\sigma+m \geq 0$ and $\sigma+\delta m \geq 0$, we consider the space:

$$W_{\sigma, \delta}^m(\Omega) = \{u \in L^2(\Omega) ; \psi^{\text{Max}(0, \sigma+\delta m, \alpha)} X^\alpha u \in L^2(\Omega) \text{ for } |\alpha| \leq m \}$$

equipped by the canonical norm. We have used the notation $X^\alpha = X_0^{\alpha_0} \dots X_q^{\alpha_q}$ for $\alpha = (\alpha_0, \dots, \alpha_q) \in \mathbb{N}^{q+1}$ and $\langle \delta, \alpha \rangle = \delta \sum_{i=1}^q \alpha_i + \alpha_0$.

Proposition III-1.

With the precedent assumptions, we have :

- (i) $W_{\sigma, \delta}^m(\Omega) \subset H_{loc}^m(\Omega)$;
- (ii) for every $\phi \in C^\infty(\bar{\Omega})$ and for every $u \in W_{\sigma, \delta}^m(\Omega)$, we have : $\phi u \in W_{\sigma, \delta}^m(\Omega)$.

Proof :

(i) With the assumption (3.4), for every $x_0 \in \Omega$, there exists a neighbourhood $V(x_0)$ of x_0 in Ω in which we can write :

$$\frac{\partial}{\partial x_k} = \sum_{i=0}^q \beta_i^k(x) X_i$$

for $k = 1, \dots, n$ with some convenient functions β_i^k which are C^∞ in $V(x_0)$ and we can easily get (i).

(ii) Let ϕ be a C^∞ function on $\bar{\Omega}$ and $u \in W_{\sigma, \delta}^m(\Omega)$. Then $\phi u \in L^2(\Omega)$ and for $|\alpha| \leq m$, we have :

$$X^\alpha(\phi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (X^\beta \phi) (X^{\alpha-\beta} u)$$

it results that $\psi^{\text{Max}(0, \sigma + \langle \delta, \alpha \rangle)} X^\alpha(\phi u) \in L^2(\Omega)$, that is to say $\phi u \in W_{\sigma, \delta}^m(\Omega)$.

Remark III-1 :

It is easy to prove that the space $W_{\sigma, \delta}^m(\Omega)$ does not depend of the choice of the vector fields $(X_i)_{0 \leq i \leq q}$ satisfying the conditions (3.2), (3.3), (3.4).

Proposition III-2 :

We have :

(i) If $m > n/2$ and if $u \in W_{\sigma, \delta}^m(\Omega)$, then u is continuous on Ω and there exists a constant $C > 0$ such that, for every $u \in W_{\sigma, \delta}^m(\Omega)$, for every $x \in \Omega$, we have :

$$(3.5) \quad |u(x)| \leq C \cdot \psi(x)^{-\frac{\sigma+m}{2m} - \frac{n-1}{2m}(\sigma+\delta m)} \|u\|_{W_{\sigma, \delta}^m}^{n/2m} \|u\|_{L^2}^{1-n/2m};$$

(ii) if $\text{Min}(-\sigma, -\sigma/\delta) > n/2$ and if $u \in W_{\sigma, \delta}^m(\Omega)$, then u is continuous and bounded on Ω there exists a constant $C > 0$ such that for every $u \in W_{\sigma, \delta}^m(\Omega)$, for every $x \in \Omega$, we have :

$$(3.6) \quad |u(x)| \leq C \cdot \|u\|_{W_{\sigma, \delta}^m}^{-\frac{1+\delta(n-1)}{2\sigma}} \|u\|_{L^2}^{1 + \frac{1+\delta(n-1)}{2\sigma}}.$$

Proof :

(i) With the proposition III-1 and by a partition of unity the inequality (3.5) can be only obtained for functions $u \in W_{\sigma, \delta}^m(\Omega)$ with support in a neighbourhood of the boundary Γ of Ω .

Let x_0 be a point of Γ ; from the properties (3.1), we see that there exists a neighbourhood $V(x_0)$ of x_0 in \mathbb{R}^n and a diffeomorphism $\Theta = (\theta_1, \dots, \theta_n)$ with $\theta_n = \psi$ from $V(x_0)$ on to the unit ball of \mathbb{R}^n such that :

$$(3.7) \quad \left\{ \begin{array}{l} \Theta(V \cap \Omega) = B_+ = \{y \in \mathbb{R}^n; |y| \leq 1, y_n > 0\}; \\ \Theta(V \cap \Gamma) = B_0 = \{y \in \mathbb{R}^n; |y| \leq 1, y_n = 0\}; \\ X_0(\theta_k) = 0 \text{ in } V \text{ for } k = 1, \dots, n-1. \end{array} \right.$$

In these conditions, if $u \in W_{\sigma, \delta}^m(\Omega)$ with $\text{supp } u \subset V$ and if $v = u \circ \Theta^{-1}$, then $v \in W_{\sigma, \delta}^m(\mathbb{R}_+^n)$ with $\text{supp } v \subset \overline{B_+}$. In fact, it suffices for that to remark that by the diffeomorphism Θ , the vector fields $(X_i)_{0 \leq i \leq n}$ are become the vector fields $(I_i)_{0 \leq i \leq n}$ with :

$$(3.8) \quad I_0 = \alpha \frac{\partial}{\partial y_n}, \quad \alpha(y) \neq 0 \text{ for } y \in B = \{y \in \mathbb{R}^n; |y| \leq 1\};$$

$$(3.9) \quad I_i = I_i^t + [(X_i \varphi) \cdot \Phi^{-1}] \frac{\partial}{\partial y_n} \text{ for } i = 1, \dots, q,$$

where I_i^t means an homogeneous differential operator of order 1, with C^∞ coefficients in the variables y_1, \dots, y_{n-1} ;

$$(3.10) \quad \text{for every } y \in B = \{y \in \mathbb{R}^n; |y| \leq 1\}, \text{ the rank of the system } (I_i)_{0 \leq i \leq q} \text{ is equal to } n.$$

Hence, the inequality (3.5) comes from the inequality (2.1) and the proposition II-1.

(ii) In the same way, the inequality (3.6), at the boundary comes from the inequality (2.2) of the proposition II-1.

In the interior, it comes from the fact that if $u \in W_{\sigma, \delta}^m(\Omega)$, then $u \in H_{loc}^m(\Omega)$ and then too belongs to $H_{loc}^{m'}(\Omega)$ where $m' = -\frac{\sigma n}{1 + \delta(n-1)}$; in fact, since $\sigma + m \geq 0$ and $\sigma + \delta m \geq 0$, we have $m' \leq m$. Then, the inequality (3.6), in the interior, is a consequence of the classical inequality :

$$|u(x)| \leq C \cdot \|u\|_{H^{m'}}^{n/2m'} \cdot \|u\|_{L^2}^{1-n/2m'}.$$

Proposition III-3 :

Let ℓ be an integer, $0 \leq \ell < -\sigma - \frac{1}{2}$; then, the map $u \longrightarrow \gamma_\ell u = \frac{\partial^\ell u}{\partial n^\ell} \Big|_\Gamma : \mathcal{D}(\Omega) \longrightarrow$

$\mathcal{D}(\Gamma)$ can be extended in a linear and continuous map from $W_{\sigma, \delta}^m(\Omega)$ into

$$H^{-\frac{2(\sigma+\ell)+1}{2\delta}}(\Gamma).$$

$\frac{\partial}{\partial n}$ means the derivative along that unit normal vector to Γ , interior in Ω).

This proposition comes from the proposition II-2.

Proposition III-4 :

Let m be an integer ≥ 1 and $\delta_1 = \text{Min}(1, \delta)$. There exists a constant $C > 0$ such that, for every $\varepsilon > 0$, for every $u \in W_{\sigma, \delta}^m(\Omega)$, we have :

$$(3.11) \quad \|u\|_{W_{\sigma+\delta_1, \delta}^{m-1}} \leq C \cdot \{ \varepsilon \|u\|_{W_{\sigma, \delta}^m} + \varepsilon^{-(m-1)} \|u\|_{L^2} \}.$$

Proof :

As before, we see that the inequality (3.11) at the boundary comes from the inequality (2.3) and, in the interior, from the classical inequality for the usual Sobolev spaces :

$$\|u\|_{H^{m-1}} \leq C \cdot \{ \varepsilon \|u\|_{H^m} + \varepsilon^{-(m-1)} \|u\|_{L^2} \}.$$

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