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Piecewise Monotonic Transformations and Exactness

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§0) Introduction and notations

The goal of the following pages is to describe the structure of the tail-field of piecewise monotonic transformations on [0,1] (pw.m.t.) as they have been considered before by Lasota and Yorke [1], Bowen [2], and Kowalski [3-5], and to derive from this description sufficient conditions for exactness.

In §1) we will give some definitions and state the results, in §2) we will study some special properties of the invariant measures of pw.m.t.'s, and in §3) we will describe some basic properties of such transformations, that similarly can be found in the papers of most of the authors having dealt with them, and give a construction of a set of points with a certain "good" behaviour with regard to the singularities of the transformation. Together with a rather special lemma of Lebesgue-density-theorem-type, that is proved in §4), the results of §2) and §3) will allow us in §5) to "expand" the high density that a set of the tail-field possesses in a small interval to bigger intervals with a certain minimal length. This leads us to the desired result. §6) finally contains some results (without proof) for piecewise expanding transformations of higher dimensional spaces that can be proved by applying the same basic ideas.

In the sequel, $([0,1], \mathcal{B}, \lambda^1)$ always denotes the unit-interval equipped with Lebesgue-measure.

For a set $M \subseteq [0,1]$, $\text{dia}(M) := \sup\{|x-y| \mid x,y \in M\}$, and for $x \in [0,1]$, $r>0$, we will denote by $S_r(x) := \{y \in [0,1] \mid |x-y| < r\}$ the open unit-ball with radius r centered at x.

Let $\text{Int}(A) = \mathring{A}$ be the interior of a set $A \subseteq [0,1]$, and for $r>0$ let $\text{Int}_r(A) := \{x \in A \mid S_r(x) \subseteq A\}$.

$\dot{T}$ is the derivative of a function T on $[0,1]$.

For a system $\mathcal{M}$ of sets, $\bigcup\mathcal{M} := \{x \mid \exists M \in \mathcal{M} : x \in M\}$.

For $h:[0,1] \longrightarrow \mathbb{R}$ we denote by $h(x^+) := \lim\limits_{\substack{y \to x \\ y > x}} h(y)$ and by $h(x^-) := \lim\limits_{\substack{y \to x \\ y < x}} h(y)$.

§1) Definitions and results

<u>Definition:</u>

$T:[0,1]\longrightarrow[0,1]$ is called a piecewise monotonic transformation (pw.m.t.), if there exists a partition $\mathcal{P}=\{P_1,\ldots,P_N\}$ of $[0,1]$ into subintervals such that for each $P_i\in\mathcal{P}$:

i) T is C^1 on P_i ,

ii) $|\dot{T}_{|P_i}|\geq\alpha>1$ for a constant α ,

iii) $\dot{T}_{|P_i}$ is Lipschitz-continuous,

iv) (as a consequence of ii) and iii):)

$$\|\dot{T}\|:=\sup\left\{|\dot{T}(x)|\;\middle|\;x\in\bigcup_{i=1}^{N}\mathrm{Int}(P_i)\right\}<\infty .$$

A result of Lasota and Yorke [1] shows, that each pw.m.t. T on $[0,1]$ has an invariant measure $\mu=h\cdot\lambda^1$ with a density function h that can be chosen to be of bounded variation.

Denoting by $\mathcal{O}\!l_\infty(T):=\left\{A\in\mathcal{B}\;\middle|\;T^{-k}(T^k(A))=A\quad(k\in\mathbb{N})\right\}$ the tail-field of the transformation T, we can now state the main result:

<u>Theorem:</u>

Let $T:[0,1]\longrightarrow[0,1]$ be a piecewise monotonic transformation and $\mu=h\cdot\lambda^1$ the above mentioned T-invariant measure. Then

1) h can be chosen to be lower semi-continuous and can be bounded below on its support by a positive constant.

2) $\mathcal{O}\!l_\infty(T)$ is generated μ-mod 0 by a finite number of atoms, each of which is μ-mod 0 a finite union of open intervals.

3) The number of atoms of $\mathcal{O}\!l_\infty(T)$ is $\leq(N-1)\cdot\min\left\{\frac{1}{\log_2\alpha},\frac{1}{\alpha-1}\right\}$, where N is the number of elements of the partition .

<u>Corollary 1:</u>

There is a power T^p of T such that $T^p(A) = A$ μ-mod 0 for each $A \in \mathfrak{A}_\omega(T)$ and $T^p_{|A}$ is exact for each atom A of $\mathfrak{A}_\omega(T)$.

Some immediate consequences of the theorem have been proved by Kowalski before: Theorem 5 of [5] gives an analogous description of the ergodic atoms of T.

With some additional considerations we can derive the following corollary:

<u>Corollary 2:</u>

a) For N odd (i.e. $N \geq 3$) and $\alpha > \frac{N+1}{2}$, T is exact, while for $\alpha = \frac{N+1}{2}$ T need not even be ergodic (example 1).

b) For N even we have:

   $\alpha > \frac{N}{2} \Longrightarrow T$ ergodic, while for $\alpha = \frac{N}{2}$ T need not be ergodic (ex.2).

   $\alpha > \sqrt{\frac{N}{2}(\frac{N}{2}+1)} \Longrightarrow T$ exact, while for $\alpha = \sqrt{\frac{N}{2}(\frac{N}{2}+1)}$ T need not be exact (example 3).

c) For N=2, $p \in \mathbb{N}$, and $\alpha > \sqrt[p]{2}$, $\mathfrak{A}_\omega(T)$ has at most p-1 atoms, while for $\alpha = \sqrt[p]{2}$ $\mathfrak{A}_\omega(T)$ may have p atoms (example 4).

(For N=2 and p=2 cf. Bowen [2].)

The counterexamples mentioned in this corollary are given now:

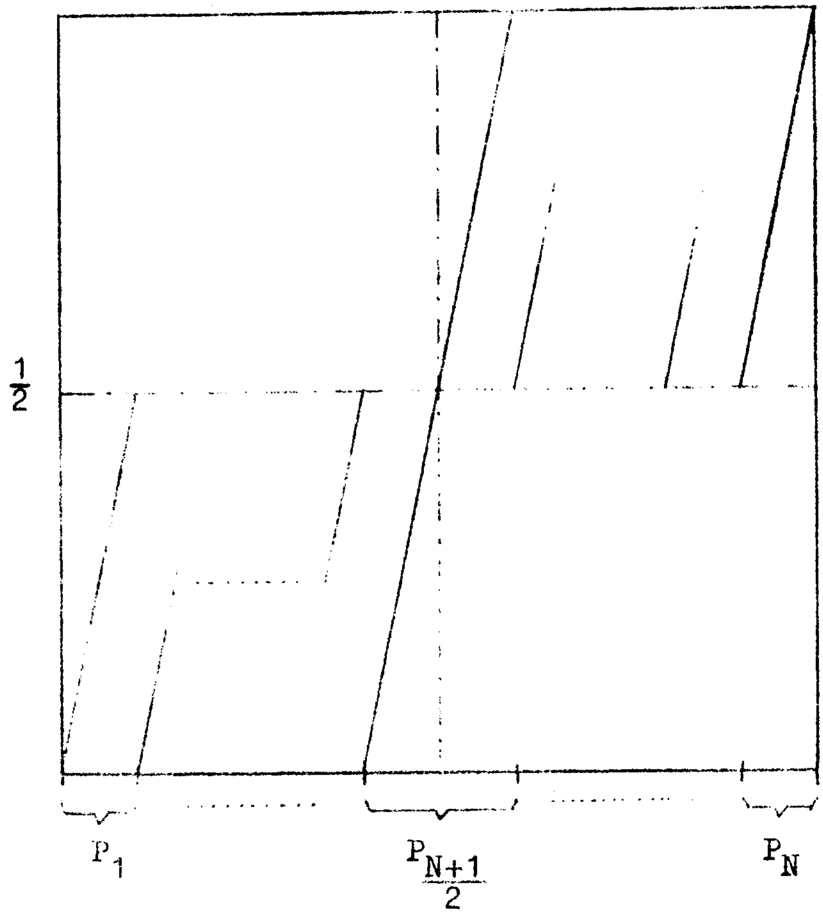
Example 1:

N odd, $N \geq 3$, $\alpha = \frac{N+1}{2}$

$\lambda^1(P_i) = \frac{1}{N+1}$ for $i \neq \frac{N+1}{2}$ and

$\lambda^1(P_i) = \frac{2}{N+1}$ for $i = \frac{N+1}{2}$

The Lebesgue-measure is the invariant measure.



Example 2:

For N even, $N \geq 4$, and $\alpha = \frac{N}{2}$, the same construction can serve as a counterexample for ergodicity by introducing an arbitrary additional (unnecessary!) singularity (e.g. at $\frac{1}{2}$).

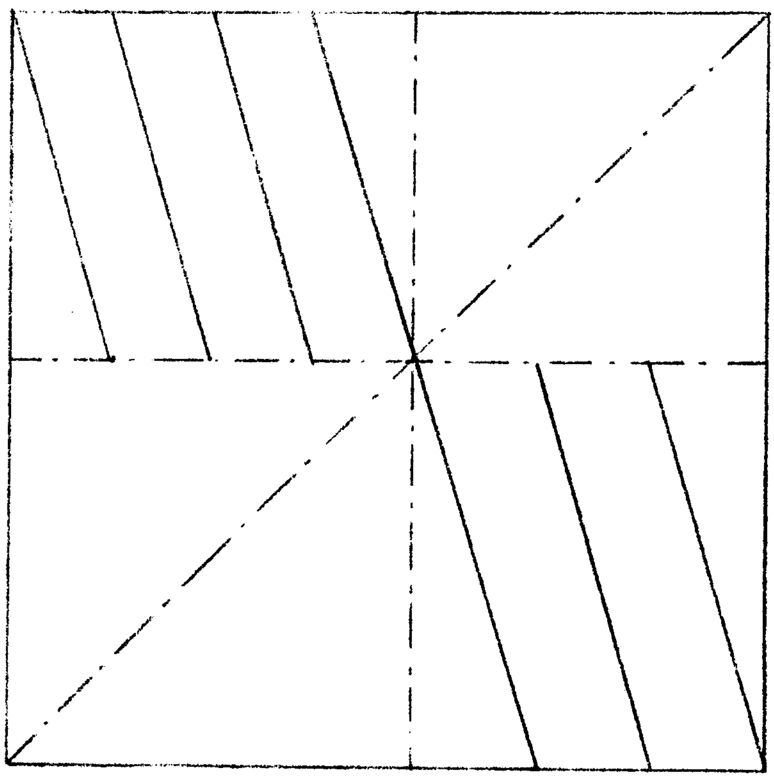
<u>Example 3:</u> N even, $N \geq 4$, $\alpha = \sqrt{\frac{N}{2}(\frac{N}{2}+1)}$, put $n := \frac{N}{2}$.

$$a_i = \begin{cases} i \cdot \frac{1}{n+1} \cdot \frac{\alpha}{n+\alpha} & (i=0,\dots,n) \\ 1 - (N-i)\,\frac{1}{n} \cdot \frac{n}{\alpha+n} & (i=n+1,\dots,N) \end{cases}$$

$$T(x) = \begin{cases} 1 - \alpha(x - a_i) & (a_i \leq x < a_{i+1} \text{ and } i \leq n) \\ \frac{\alpha}{\alpha+n} - \alpha(x - a_i) & (a_i \leq x < a_{i+1} \text{ and } i \geq n+1) \end{cases}$$

The invariant measure $\mu = h \cdot \lambda^1$ is given by

$$h(x) = \begin{cases} \frac{\alpha+n+1}{N+1} & (x \leq \frac{\alpha}{\alpha+n}) \\ \frac{\alpha+n}{N+1} & (x > \frac{\alpha}{\alpha+n}) \end{cases}$$



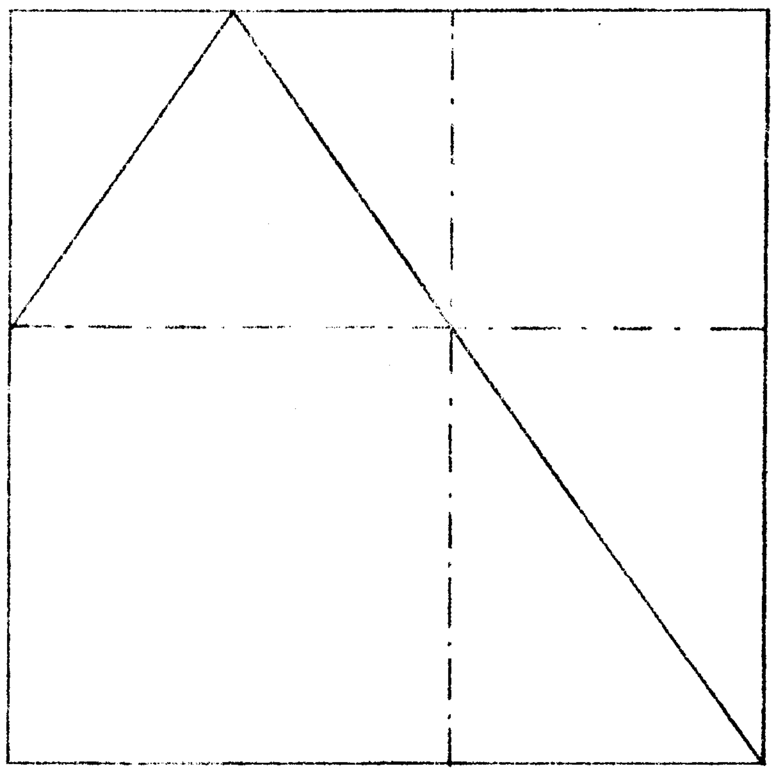
(Picture for N=6)

Example 4a: N=2, p=2, $\alpha = \sqrt{2}$.

$$T(x) = \begin{cases} 2 - \sqrt{2} + \sqrt{2}\cdot x & (0 \leq x \leq 1 - \frac{1}{\sqrt{2}}) \\ 1 - \sqrt{2}\cdot(x - 1 + \frac{1}{\sqrt{2}}) & (1 - \frac{1}{\sqrt{2}} \leq x \leq 1) \end{cases}$$

The invariant measure is $\mu = h\cdot\lambda^1$ with

$$h(x) = \begin{cases} \frac{1}{2(2-\sqrt{2})} & (0 \leq x \leq 2 - \sqrt{2}) \\ \frac{1}{2(\sqrt{2}-1)} & (2 - \sqrt{2} \leq x \leq 1) \end{cases}$$



<u>Example 4b:</u> N=2, p>2, $\alpha = 2^{1/p}$

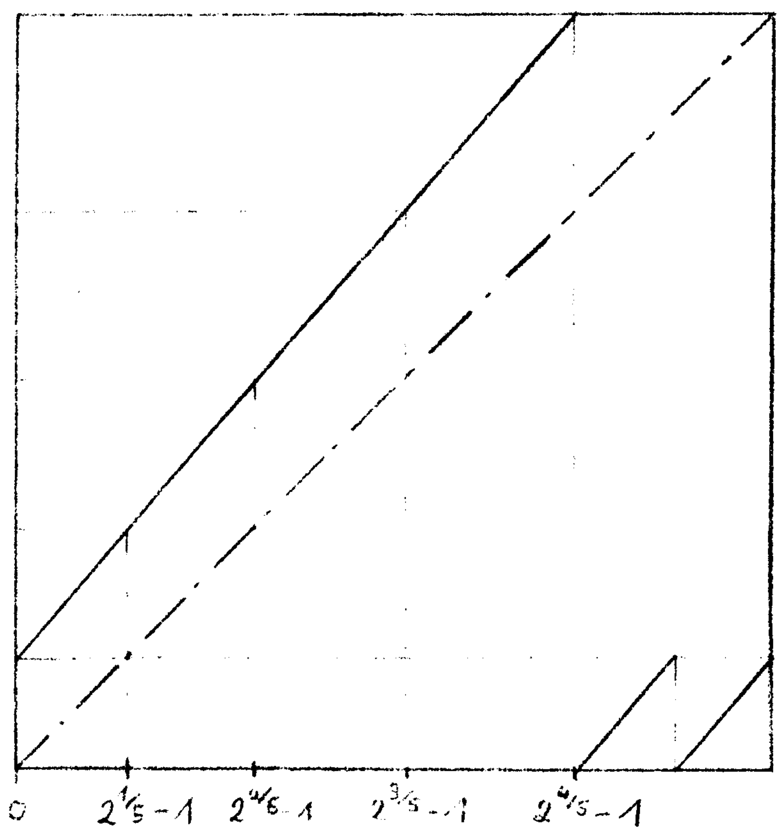
$$T(x) = \begin{cases} (2^{1/p} - 1 + x\cdot 2^{1/p}) \bmod 1 & (0 \leq x < \frac{1}{2^{1/p}}) \\ (2^{1/p}\cdot x) \bmod 1 & (\frac{1}{2^{1/p}} \leq x < 1) \end{cases}$$

Although this is not really an example for N=2, we can consider it as N=2 by identifying the unit-interval with the unit-circle, since

$T(0) = 2^{1/p} - 1 = T(1)$ and $\dot{T}(0) = 2^{1/p} = \dot{T}(1)$.

The invariant measure $\mu = h\cdot\lambda^1$ is given by

$$h(x) = \frac{1}{p(2^{i/p} - 2^{(i-1)/p})} \quad \text{for } 2^{(i-1)/p} - 1 \leq x < 2^{i/p} - 1$$



(The picture is for p=5)

§2) On the density function

From now on let T be a piecewise monotonic transformation. As already mentioned, Lasota and Yorke proved in [1] the existence of a T-invariant measure $\mu = h\cdot\lambda^1$ for which the density function h can be chosen to be of bounded variation. That means that $h(x^+)$ and $h(x^-)$ exist for each $x\in[0,1]$ and $h(x^+) = h(x^-) = h(x)$ for all but at most countably many $x\in[0,1]$. Therefore, by changing the value of h at the at most countably many discontinuities, we may assume that

i) h is lower semi-continuous with $h(x) = \min\{h(x^+), h(x^-)\}$ for all $x\in[0,1]$, and

ii) h is bounded, i.e. $\|h\|_\infty < \infty$.

(In fact, these two properties of h are the only ones we will refer to later on.)

<u>Remark:</u> Because of the T-invariance of μ, we have for each

$$A\in\mathcal{B}: \mu(A) \leq \mu(T^{-1}(T(A))) = \mu(T(A)).$$

Let us denote by $0 = a_0 < a_1 < \dots < a_N = 1$ the points generating the partition $\mathcal{P}$, and without loss of generality we will assume that $Int(P_i) = (a_{i-1}, a_i)$ $(i=1,\dots,N)$.

Put $X := supp(h) := \{x\in[0,1] \mid h(x) > 0\}$.

<u>Lemma 1:</u> i) $T(X\setminus\{a_0,\dots,a_N\}) \subseteq X$

ii) $\lambda^1(X\setminus T(X\setminus\{a_0,\dots,a_N\})) = 0$

Proof: Let $x\in X\setminus\{a_0,\dots,a_N\}$ and w.l.o.g. $x\in Int(P_i)$. Then there exists an $\varepsilon > 0$ such that $S_\varepsilon(x) \subseteq P_i$ and $h(y) > \frac{1}{2}\cdot h(x) > 0$ for all $y\in S_\varepsilon(x)$, since h is lower semi-continuous. W.l.o.g. we can assume that $h(Tx) = h((Tx)^+)$. Then for δ, $0<\delta<\varepsilon$, either

$T([x,x+\delta))$ or $T((x-\delta,x])$ is of the kind $[Tx,Tx+\delta')$, and we will call that one TV_δ. W.l.o.g. again, let us assume that $V_\delta = [x,x+\delta)$.

Then a) $Tx \in TV_\delta$, $\operatorname{dia}(TV_\delta) \longrightarrow 0 \quad (\delta \to 0)$ and

b) $\int_{TV_\delta} h \, d\lambda^1 = \mu(TV_\delta) \geq \mu(V_\delta) = \int_{V_\delta} h \, d\lambda^1 \geq \frac{1}{2} \cdot h(x) \cdot \lambda^1(V_\delta)$

$\geq \frac{1}{2} \cdot h(x) \cdot \|\dot{T}\|^{-1} \cdot \lambda^1(TV_\delta)$,

and since $TV_\delta = [Tx,Tx+\delta')$ and $h(Tx) = h((Tx)^+)$, we can conclude that $h(Tx) > 0$, i.e. $Tx \in X$, proving i).

From i) it follows that

$\mu(X) = \mu(X \setminus \{a_0,\dots,a_N\}) \leq \mu(T(X \setminus \{a_0,\dots,a_N\})) \leq \mu(X)$

which proves ii), since X is the support of μ.

<u>Lemma 2:</u> $X = \operatorname{supp}(h)$ is a finite union of open intervals.

(See Kowalski 4 .)

Proof: X is open since h is lower semi-continuous. Therefore X is an at most countable disjoint union of open intervals:

$X = \sum_{I \in \mathfrak{J}} I$. We must show that $\mathfrak{J}$ is finite.

Let $\mathfrak{J}_0 := \{ I \in \mathfrak{J} \mid I \cap \{a_0,\dots,a_N\} \neq \emptyset \}$. $\mathfrak{J}_0$ is finite and $\bigcup \mathfrak{J}_0 \setminus \{a_0,\dots,a_N\}$ is a finite union of open intervals:

$\bigcup \mathfrak{J}_0 \setminus \{a_0,\dots,a_N\} = \sum_{I \in \mathfrak{J}_1} I$, $\mathfrak{J}_1$ finite. With $\hat{\mathfrak{J}} := (\mathfrak{J} \setminus \mathfrak{J}_0) \cup \mathfrak{J}_1$,

T is continuously differentiable on each $I \in \hat{\mathfrak{J}}$, and TI again is an open interval with $\lambda^1(TI) \geq \alpha \cdot \lambda^1(I)$, $\alpha > 1$, for each $I \in \hat{\mathfrak{J}}$.

Since $I \subseteq X \setminus \{a_0,\dots,a_N\}$ for $I \in \hat{\mathfrak{J}}$, the open interval TI is contained in X (see lemma 1), such that there is an $I' \in \mathfrak{J}$ with $TI \subseteq I'$.

Now let $c := \min \{ \lambda^1(I) \mid I \in \mathfrak{J}_1 \} > 0$ and $\mathfrak{J}_c := \{ I \in \hat{\mathfrak{J}} \mid \lambda^1(I) \geq c \}$. Then

i) $\mathfrak{J}_1 \subseteq \mathfrak{J}_c \subseteq \hat{\mathfrak{J}}$, ii) $\mathfrak{J}_c$ is finite, and

iii) $T(\bigcup \mathfrak{J}_c) \subseteq \bigcup \mathfrak{J}_c$ since $I \in \mathfrak{J}_c \Longrightarrow \lambda^1(TI) \geq \alpha \cdot c > c \Longrightarrow TI \subseteq \bigcup \mathfrak{J}_c$.

Assume now that $\mathcal{J}_c \neq \hat{\mathcal{J}}$. Choose $\tilde{I} \in \hat{\mathcal{J}} \setminus \mathcal{J}_c$ in such a way that $\lambda^1(\tilde{I})$ is maximal in $\hat{\mathcal{J}} \setminus \mathcal{J}_c$. Since $\tilde{I} \in \hat{\mathcal{J}}$, $\lambda^1(T\tilde{I}) \geq \alpha \cdot \lambda^1(\tilde{I}) > \lambda^1(\tilde{I})$, and therefore the interval $I' \in \mathcal{J}$ containing $T\tilde{I}$ is not an element of $\hat{\mathcal{J}} \setminus \mathcal{J}_c$.

$\Longrightarrow I' \in (\mathcal{J}_0 \cup \mathcal{J}_c) \cap \mathcal{J}$.

$\Longrightarrow T\tilde{I} \subseteq I' \subseteq \bigcup \mathcal{J}_0 \cup \bigcup \mathcal{J}_c \subseteq \bigcup \mathcal{J}_1 \cup \{a_0, \dots, a_N\} \cup \bigcup \mathcal{J}_c$

$\subseteq \bigcup \mathcal{J}_c \cup \{a_0, \dots, a_N\}$ since $\mathcal{J}_1 \subseteq \mathcal{J}_c$

$\Longrightarrow \mu(\bigcup \mathcal{J}_c \setminus T\tilde{I}) = 0$

$\Longrightarrow \mu(\tilde{I} \cup \bigcup \mathcal{J}_c) \leq \mu(T(\tilde{I} \cup \bigcup \mathcal{J}_c)) \leq \mu(T\tilde{I} \cup \bigcup \mathcal{J}_c)$

$= \mu(\bigcup \mathcal{J}_c)$

$\Longrightarrow \mu(\tilde{I}) = 0$ since $\tilde{I} \in \hat{\mathcal{J}} \setminus \mathcal{J}_c$ such that $\tilde{I} \cap \bigcup \mathcal{J}_c = \emptyset$

$\Longrightarrow \lambda^1(\tilde{I}) = 0$ since $\tilde{I} \subseteq \text{supp}(h)$,

which is a contradiction to $\tilde{I}$ being an open interval.

So we have $\mathcal{J}_c = \hat{\mathcal{J}}$, and since $\mathcal{J}_c$ is finite, $\hat{\mathcal{J}}$ is finite and so is $\mathcal{J}$.

<u>Lemma 3:</u> There is a constant $C>0$ such that $h_{|X} > C$.

(This proves prt 1 of the theorem.)

Proof: Let $X = \sum_{I \in \mathcal{J}} I$ be a finite disjoint union of open intervals, $\hat{X} := X \setminus \{a_0, \dots, a_N\}$, and $\hat{X} = \sum_{J \in \mathfrak{J}} J$ be a finite union of open intervals, too.

T is continuously differentiable on each $J \in \mathfrak{J}$, and, by the same arguments as in the preceding proof, for each $J \in \mathfrak{J}$ there exists an $I \in \mathcal{J}$ with $TJ \subseteq I$.

Letting (c,d) be any interval in $\mathcal{J}$ or $\mathfrak{J}$ we will associate to its endpoints two classes of "standard intervals" (c.o.s.i.) $\mathcal{C}_c = \{(c, c+\varepsilon) \mid \varepsilon > 0\}$ and $\mathcal{C}_d = \{(d-\varepsilon, d) \mid \varepsilon > 0\}$ and call c and d the "endpoints" of $\mathcal{C}_c$ and $\mathcal{C}_d$ respectively.

Between these classes we establish a relation "$\rightsquigarrow$":

Let $\mathcal{C}, \mathcal{C}'$ be classes of standard intervals.

$\mathcal{C} \rightsquigarrow \mathcal{C}'$ iff $TU \in \mathcal{C}'$ for each sufficiently small $U \in \mathcal{C}$.

This relation has the following properties:

1) If $\mathcal{C}'$ is a c.o.s.i. associated to an endpoint of an interval $I \in \mathfrak{J}$, then there is at least one c.o.s.i. $\mathcal{C}$ such that $\mathcal{C} \rightsquigarrow \mathcal{C}'$.

Proof: Let $I \in \mathfrak{J}$ be arbitrary and $\mathcal{C}'$ be a c.o.s.i. associated to an endpoint of I. For each $J \in \mathfrak{J}$ either $TJ \subseteq I$ or $TJ \cap I = \emptyset$. Since $\lambda(I \setminus T\hat{X}) \leq \lambda(X \setminus T\hat{X}) = 0$ (see lemma 1), there must be a $J \in \mathfrak{J}$ with $TJ \in \mathcal{C}'$, and the assertion follows immediately.

2) If $I \in \mathfrak{J}$, c' an endpoint of I, $\lim\limits_{\substack{x \to c' \\ x \in I}} h(x) = 0$, $\mathcal{C}'$ the c.o.s.i. associated to c', $\mathcal{C} \rightsquigarrow \mathcal{C}'$, and c the endpoint of $\mathcal{C}$, then $\lim\limits_{\substack{x \to c \\ x \in U}} h(x) = 0$ for each $U \in \mathcal{C}$.

The proof works with the same arguments used in the proof of lemma 1.

3) In the situation of 2) the lower semicontinuity of h implies that $h(c) = 0$, i.e. $c \notin X$, and this in turn implies that c is an endpoint of an interval $I \in \mathfrak{J}$. Denoting by

$K_0 := \{ \mathcal{C} \mid \mathcal{C}$ c.o.s.i. associated to an endpoint c of an $I \in \mathfrak{J}$ with $\lim\limits_{\substack{x \to c \\ x \in I}} h(x) = 0 \}$

we can conclude that:

4) $\mathcal{C}' \in K_0$, $\mathcal{C} \rightsquigarrow \mathcal{C}' \Longrightarrow \mathcal{C} \in K_0$

Now let us assume that $K_0 \neq \emptyset$.

Combining 1) and 4) we see that for each $\mathcal{C}' \in K_0$ there is at least one $\mathcal{C} \in K_0$ with $\mathcal{C} \rightsquigarrow \mathcal{C}'$.

On the other hand it is trivial that for each $\mathcal{C} \in K_o$ there can be at most one $\mathcal{C}' \in K_o$ with $\mathcal{C} \rightsquigarrow \mathcal{C}'$, such that, since K_o is finite, "$\rightsquigarrow$" is a bijective relation on K_o.

Let $\mathcal{C}$ be any element of K_o. Then there are $\mathcal{C}_o, \ldots, \mathcal{C}_n \in K_o$ uniquely determined, such that $\mathcal{C} = \mathcal{C}_o \rightsquigarrow \mathcal{C}_1 \rightsquigarrow \ldots\ldots \rightsquigarrow \mathcal{C}_n = \mathcal{C}$. Choosing $U \in \mathcal{C} = \mathcal{C}_n$ small enough we can achieve that $T^{-i}U$ is an open interval with $T^{-i}U \in \mathcal{C}_{n-i}$ and $T^{-i}U \subseteq X$ $(i=0,\ldots,n)$. In particular we have $T^{-n}U \in \mathcal{C}_o = \mathcal{C}$ with $\lambda^1(T^{-n}U) < \alpha^{-n} \cdot \lambda^1(U)$, such that, by induction, we get a sequence of intervals $(T^{-k \cdot n}U)_{k \in \mathbb{N}}$ in $\mathcal{C}$ with $\lambda^1(T^{-k \cdot n}U) < \alpha^{-k \cdot n} \cdot \lambda^1(U)$ yielding the following chain of inequalities:

$$\mu(U) = \mu(T^{-k \cdot n}U) \le \|h\|_\infty \cdot \lambda^1(T^{-k \cdot n}U) < \|h\|_\infty \cdot \alpha^{-k n} \cdot \lambda^1(U) \qquad (k \in \mathbb{N})$$

$$\Longrightarrow \mu(U) = 0 \Longrightarrow \lambda^1(U) = 0 \qquad \text{since } U \subseteq X,$$

contradicting the fact that $U \in \mathcal{C}$ is an open interval.

Therefore the assumption $K_o \neq \emptyset$ must be false, and we can conclude that $\lim\limits_{\substack{x \to c \\ x \in I}} h(x) > 0$ for each of the finitely many end-points of intervals $I \in \mathcal{J}$. Because of this and since h is lower semi-continuous, a compactness-argument shows that there is a $C > 0$ with $h_{|X} \ge C$.

§3) On partitions generated by T and $\mathcal{P}$

Remember that $\mathcal{P} = \{P_1, \dots, P_N\}$ is the partition of $[0,1]$ into intervals of smoothness of T. By $\mathcal{P}_n$ we will denote the partition of $[0,1]$ into intervals on which T^n is C^1, i.e. $\mathcal{P}_n = \bigvee_{k=0}^{n-1} T^{-k}\mathcal{P}$ for $n \geq 1$. Elements of $\mathcal{P}_n$ can be written as

$$\Delta_n(j_1, \dots, j_n) := P_{j_1} \cap T^{-1}P_{j_2} \cap \dots \cap T^{-(n-1)}P_{j_n}, \quad j_\nu \in \{1, \dots, N\}.$$

(Remark: We will use the symbol $\Delta_n(j_1, \dots, j_n)$ to denote the above intersection even if it is void.)

Define $\Delta_0 := [0,1]$ and $\mathcal{P}_0 := \{[0,1]\} = \{\Delta_0\}$.

Sometimes, we will write elements of $\mathcal{P}_n$ simply as Δ, but only if this does not cause any confusion.

Since for each $n \in \mathbb{N}_0$: $\bigcup_{\Delta \in \mathcal{P}_n} \Delta = \bigcup \mathcal{P}_n = [0,1]$, the symbol $\Delta_n[x]$ can be defined for each $x \in [0,1]$ to denote that $\Delta \in \mathcal{P}_n$ for which $x \in \Delta$.

<u>Lemma 4:</u>

1) For $m, n \geq 0$, $x \in [0,1]$ holds:

$$\Delta_{m+n}[x] = \Delta_m[x] \cap T^{-m}\Delta_n[T^m x]$$

2) For $0 \leq l \leq n$ and $x \in [0,1]$ is $T^l(\Delta_n[x]) \subseteq \Delta_{n-l}[T^l x]$.

3) Let $M \subseteq [0,1]$, $\Delta \in \mathcal{P}_n$, and $x, y \in \Delta \cap T^{-n}M$.
   Then $|x - y| \leq \alpha^{-n} \cdot |T^n x - T^n y| \leq \alpha^{-n} \cdot \mathrm{dia}(M)$.

4) There is a constant $S > 0$ such that for all $A, B \in \mathcal{B}$ and all $\Delta \in \mathcal{P}_n$ with $\lambda^1(\Delta \cap B) > 0$:

$$\frac{\lambda^1(T^n(\Delta \cap A))}{\lambda^1(T^n(\Delta \cap B))} \leq S \cdot \frac{\lambda^1(\Delta \cap A)}{\lambda^1(\Delta \cap B)}$$

5) If $P \in \mathcal{P}$ and $A \subseteq [0,1]$ is a closed set with $A \subseteq TP$, then $T^{-1}(A) \cap P$ is closed.

Proof: 1), 2), and 3) are immediate.

4) is established by a straightforward computation using the exponential expansion of T^n on sets $\Delta \in \mathcal{P}_n$ and the Lipschitz-continuity of the restrictions $T_{|P_i}$. (Compare the proof of lemma 3/ii) in Bowen [2].)

To show 5) observe that P and TP are intervals and that $T_{|P}$ is a homeomorphism between them. As a closed subset of $[0,1]$, A is compact. Therefore $(T_{|P})^{-1}(A)$ also is compact, but $(T_{|P})^{-1}(A) = T^{-1}(A) \cap P$, such that this set is closed in $[0,1]$.

Later on it will be essential to find, given a fixed $k_o \in \mathbb{N}$, as many points $x \in [0,1]$ as possible satisfying for arbitrary $k \geq k_o$ not only the inclusion $T^{k-k_o}(\Delta_k[x]) \subseteq \Delta_{k_o}[T^{k-k_o}x]$ (see lemma 4/2) but also the inversion.

The following construction will provide us with such points x:

<u>Construction:</u> Let $M \in \mathcal{B}$ and $k_o \in \mathbb{N}$ be so that $T^{-k_o}M \cap \Delta$ is closed (it may be empty!) for each $\Delta \in \mathcal{P}_{k_o}$.

Then we define sets $M_{k_o}, M_{k_o+1}, \ldots\ldots$ inductively by

i) $M_{k_o} := T^{-k_o}M$ and

ii) $M_{l+1} := T^{-1}(M_l \cap I(\alpha^{-1}\cdot \mathrm{dia}(M)))$ $(l \geq k_o)$,

where $I(r) := \complement\left(\bigcup_{P \in \mathcal{P}} (TP \setminus \mathrm{Int}_r(TP))\right)$ $(r > 0)$.

Remark: In the inductive step of this construction we first cut off from M_l those points being too close to the endpoints of any TP they are contained in, and then take the preimage under T.

<u>Lemma 5:</u> For the sets obtained in the above construction the following is true:

1) $T^{l-k}(M_l) \subseteq M_k \subseteq T^{-k}(M)$ for $l \geq k \geq k_o$

2) Let $l \geq k_o$ and $x \in M_l$. Then for each $A \subseteq T^{-k_o}(M)$ holds

$$T^{l-k_o}(\Delta_l[x] \cap T^{-(l-k_o)}(A)) = \Delta_{k_o}[T^{l-k_o}x] \cap A$$

3) There is a constant $H > 0$ for which

$$\mu(M_l) \geq \mu(M) - H \cdot \mathrm{dia}(M) \cdot \alpha^{-k_o} \qquad (l \geq k_o).$$

4) $\Delta_l[x] \cap T^{-l}(M)$ is closed for $l \geq k_o$ and $x \in M_l$.

Remark: Assertion 3) gives a hint, why things change in dimensions > 1: In higher dimensions, Lebesgue-measure and diameter of an interval are not so closely related as in the one-dimensional case.

Proof of the lemma:

1) is obvious.

2) is proved by induction on l:

<u>$l=k_o$:</u> is trivial

<u>$l \Rightarrow l+1$:</u> $T^{l+1-k_o}(\Delta_{l+1}[x] \cap T^{-(l+1-k_o)}(A))$

$$= T^{l+1-k_o}(\Delta_1[x] \cap T^{-1}\Delta_l[Tx] \cap T^{-1}(T^{-(l-k_o)}(A)))$$

(see lemma 4/1)

$$= T^{l-k_o}(T\Delta_1[x] \cap \Delta_l[Tx] \cap T^{-(l-k_o)}(A))$$

$$(*) \quad \subseteq T^{l-k_o}(\Delta_l[Tx] \cap T^{-(l-k_o)}(A))$$

$$= \Delta_{k_o}[T^{l+1-k_o}x] \cap A$$

by inductive hypothesis, since $Tx \in T(M_{l+1}) \subseteq M_l$.

To show the revers inclusion of (*) let $z \in \Delta_l[Tx] \cap T^{-(l-k_o)}(A)$. From lemma 4/3 it follows that $|z - Tx| < \alpha^{-l} \cdot \mathrm{dia}(M)$, since $Tx \in M_l \subseteq T^{-l}(M)$ and $z \in T^{-(l-k_o)}(M)$ by assumption, and we obtain: $z \in S_{\alpha^{-1}\cdot \mathrm{dia}(M)}(Tx)$.

Additionally we have:

$\Longrightarrow$ $x \in M_{l+1}$

$\Longrightarrow$ $Tx \in I(\alpha^{-1}\cdot \mathrm{dia}(M))$

$\Longrightarrow$ $\forall P \in \mathcal{P}: Tx \notin TP \setminus \mathrm{Int}_{\alpha^{-1}\cdot \mathrm{dia}(M)}(TP)$

$\Longrightarrow$ $Tx \in \mathrm{Int}_{\alpha^{-1}\cdot \mathrm{dia}(M)}(T\Delta_1[x])$, since $\Delta_1[x] \in \mathcal{P}$,

and we can conclude that $z \in T\Delta_1[x]$, thus obtaining immediately the desired reversion of (✱).

3) For $l \geq k_0$ we have

$$\begin{aligned}\mu(M_{l+1}) &= \mu(M_l \cap I(\alpha^{-1}\cdot \mathrm{dia}(M)))\\ &\geq \mu(M_l) - \mu\Big(\bigcup_{P\in\mathcal{P}}(TP\setminus \mathrm{Int}_{\alpha^{-1}\cdot \mathrm{dia}(M)}(TP))\Big)\\ &\geq \mu(M_l) - N\cdot 2\cdot \|h\|_\infty \cdot \alpha^{-1}\cdot \mathrm{dia}(M)\ ,\end{aligned}$$

and by induction:

$$\begin{aligned}\mu(M_l) &\geq \mu(M_{k_0}) - N\cdot 2\cdot \|h\|_\infty\cdot \mathrm{dia}(M)\cdot\sum_{i=k_0}^{l-1}\alpha^{-i}\\ &\geq \mu(M) - H\cdot \mathrm{dia}(M)\cdot \alpha^{-k_0} \qquad \text{since } M_{k_0} = T^{-k_0}(M),\end{aligned}$$

where $H = N\cdot 2\cdot \|h\|_\infty \dfrac{\alpha}{\alpha-1}$.

4) is proved again by induction on l:

<u>$l=k_0$:</u> is valid by assumption.

<u>$l\Rightarrow l+1$:</u> For $x \in M_{l+1}$ is $Tx \in M_l$ and

$$\begin{aligned}&\Delta_{l+1}[x] \cap T^{-(l+1)}(M)\\ &= \Delta_1[x] \cap T^{-1}(\Delta_l[Tx] \cap T^{-l}(M))\ ,\end{aligned}$$

where $\Delta_l[Tx] \cap T^{-l}(M)$ is closed by inductive hypothesis. Because of lemma 4/5) it suffices to show that $\Delta_l[Tx] \cap T^{-l}(M) \subseteq T(\Delta_1[x])$. But this follows directly from the following two facts:

a) $\mathrm{dia}(\Delta_l[Tx] \cap T^{-l}(M)) < \alpha^{-1}\cdot \mathrm{dia}(M)$ (lemma 4/3)

b) $x \in M_{l+1} \Longrightarrow Tx \in \mathrm{Int}_{\alpha^{-1}\cdot \mathrm{dia}(M)}(T(\Delta_1[x]))$, what already has been proved under 2). End of proof of lemma 5.

Denoting by $\partial(T\mathcal{P}) := \bigcup_{P\in\mathcal{P}} \partial(TP)$ the finite set of endpoints of the intervals TP ($P\in\mathcal{P}$), $Z_{k_o} := \bigcup_{k=0}^{k_o-1} T^k(\partial(T\mathcal{P}))$ is a finite set for all $k_o\in\mathbb{N}$.

Since $X = \mathrm{supp}(h)$ is a finite union of open intervals, $X\setminus Z_{k_o}$ also can be written as a finite disjoint union of open intervals: $X\setminus Z_{k_o} = \bigcup_{i=1}^{r} R_i$, R_i open intervals.

<u>Lemma 6:</u>

Let $M\subseteq R_i$, $i\in\{1,\dots,r\}$, be a compact interval , and $0\leq l\leq k_o$. Then for each $\Delta\in\mathcal{P}_l$ with $\Delta\cap T^{-l}(M)\neq\emptyset$, $\Delta\cap T^{-l}(M)$ is a compact interval and $T^l(\Delta\cap T^{-l}(M)\cap T^{-l}(A)) = M\cap A$ for each $A\subseteq[0,1]$.

Proof by induction on l:

<u>l=0:</u> is trivial since $\Delta\in\mathcal{P}_0 \Longrightarrow \Delta = [0,1]$

<u>$l\Rightarrow l+1$:</u> ($l\leq k_o-1$)

Assume that $\Delta\in\mathcal{P}_{l+1}$, $\Delta\cap T^{-(l+1)}(M)\neq\emptyset$. There are $P\in\mathcal{P}$ and $\Delta'\in\mathcal{P}_l$ with $\Delta = P\cap T^{-1}\Delta'$. $\Longrightarrow$ $\Delta'\cap T^{-l}(M)\neq\emptyset$.

$\Longrightarrow$(by inductive hypothesis): $\Delta'\cap T^{-l}(M)$ is a compact interval with $T^l(\Delta'\cap T^{-l}(M)) = M$. Moreover we have

1) $\Delta'\cap T^{-l}(M)\cap \partial(T\mathcal{P}) = \emptyset$, since $M\cap Z_{k_o} = \emptyset$.

2) $TP\cap\Delta'\cap T^{-l}(M)\neq\emptyset$, since $P\cap T^{-1}(\Delta'\cap T^{-l}M) = \Delta\cap T^{-(l+1)}M\neq\emptyset$.

From 1) and 2) follows: $\Delta'\cap T^{-l}(M)\subseteq TP$.

Since $T_{|P}$ is a homeomorphism between P and TP,

$\Delta\cap T^{-(l+1)}(M) = {T_{|P}}^{-1}(\Delta'\cap T^{-l}(M))$ is a compact interval and

$T^{l+1}(\Delta\cap T^{-(l+1)}(M)) = T^l(\Delta'\cap T^{-l}(M)) = M$.

For $A\subseteq[0,1]$ we finally have:

$T^{l+1}(\Delta\cap T^{-(l+1)}M\cap T^{-(l+1)}A) = T^{l+1}(\Delta\cap T^{-(l+1)}M)\cap A = M\cap A$.

§4) Something similar to a Lebesgue-density theorem

As indicated in the introduction, the main idea to prove exactness-properties is the following:
Imagine that for a measurable set A, a small interval $U \subseteq \Delta \in \mathcal{P}_n$, and a small $\delta > 0$ holds: $(*)$ $\quad \lambda^1(U \setminus A) < \delta \cdot \lambda^1(U)$.
Then $\lambda^1(T^nU \setminus T^nA) < \varsigma \cdot \delta \cdot \lambda^1(T^nU)$, where T^nU is an interval much bigger than U. For sets A of the tail-field of T, such a property will be enough to prove that $\mu(A)$ is "sufficiently" large. What we have to show is that situations as described by $(*)$ really occur. This is done in the following lemma:

<u>Lemma 7:</u>
Let $J \subseteq \mathbb{N}$ be an infinite index set, $Q_l, \tilde{Q}_l \in \mathcal{B}$ with $Q_l \subseteq \tilde{Q}_l$ $(l \in J)$, and $d > 0$ a constant, such that for all $l \in J$ and $x \in Q_l$ the following holds:

i) $\Delta_l[x] \cap \tilde{Q}_l$ is closed,

ii) $\lambda^1(\Delta_l[x] \cap \tilde{Q}_l) \geq d \cdot \lambda^1(\Delta_l[x])$.

Let $Q^* := \bigcap_{k \in J} \bigcup_{\substack{l \geq k \\ l \in J}} Q_l$.

Then for each $A \in \mathcal{B}$ with $\lambda^1(Q^* \cap A) > 0$ and $\varepsilon > 0$ there is an $x \in Q^* \cap A$ such that

$$\forall l_0 \in \mathbb{N}_0 \; \exists l \in J, l \geq l_0 : \begin{cases} x \in Q_l & \text{and} \\ \lambda^1(\Delta_l[x] \cap \tilde{Q}_l \cap \complement A) \leq \varepsilon \cdot \lambda^1(\Delta_l[x] \cap \tilde{Q}_l) \end{cases}$$

Proof: Let us assume that the statement of the lemma is false. Then there is an $A \in \mathcal{B}$ with $\lambda^1(Q^* \cap A) > 0$ and an $\varepsilon > 0$ such that
$\forall x \in Q^* \cap A \; \exists l_0 \in \mathbb{N} \; \forall l \in J, l \geq l_0:$

$$x \in Q_l \Longrightarrow \lambda^1(\Delta_l[x] \cap \tilde{Q}_l \cap \complement A) > \varepsilon \cdot \lambda^1(\Delta_l[x] \cap \tilde{Q}_l) .$$

By v we denote

$$v := \bigcup_{l\in J} \left\{ \Delta\in\mathcal{P}_l \;\middle|\; \Delta\cap Q^*\cap A \neq \emptyset,\ \ \Delta\cap Q_l \neq \emptyset,\ \lambda^1(\Delta\cap\tilde{Q}_l\cap \complement A) > \varepsilon\cdot\lambda^1(\Delta\cap\tilde{Q}_l)\right\}$$

and by $\mathcal{P}_\infty := \bigcup_{n=0}^{\infty} \mathcal{P}_n$.

<u>Assertion 1:</u>

If $\mathcal{O}\subseteq[0,1]$ is open and $v_{\mathcal{O}} := \{\Delta\in\mathcal{P}_\infty \mid \Delta\subseteq\mathcal{O}\}$, then there is an at most countable set $f\subseteq v\cap v_{\mathcal{O}}$ of pairwise disjoint sets for which $Q^*\cap A\cap\mathcal{O} \subseteq \bigcup f \subseteq \mathcal{O}$.

Proof: Since $v\cap v_{\mathcal{O}} \subseteq \mathcal{P}_\infty$ is partially ordered by inclusion, it makes sense to define $f := \{\Delta\in v\cap v_{\mathcal{O}} \mid \Delta \text{ maximal in } v\cap v_{\mathcal{O}}\}$, and the following are valid:

1) $f\subseteq v\cap v_{\mathcal{O}} \subseteq \mathcal{P}_\infty$, consequently f is at most countable.

2) $\Delta_1, \Delta_2\in f,\ \Delta_1\cap\Delta_2\neq\emptyset \Longrightarrow \Delta_1\subseteq\Delta_2$ or $\Delta_2\subseteq\Delta_1$ (a property of ...)

$\Longrightarrow \Delta_1=\Delta_2$ since $\Delta_1, \Delta_2\in f$.

So the elements of f are pairwise disjoint.

3) $\bigcup f = \bigcup(v\cap v_{\mathcal{O}})$:

$\bigcup f \subseteq \bigcup(v\cap v_{\mathcal{O}})$ is trivial.

On the contrary let $\Delta\in v\cap v_{\mathcal{O}}$. Since $\Delta\subseteq\Delta'$ for at most finitely many $\Delta'\in\mathcal{P}_\infty$, there exists a maximal $\Delta'\in v\cap v_{\mathcal{O}}$ with $\Delta\subseteq\Delta'$, such that $\Delta\subseteq\Delta'\subseteq\bigcup f$. That means $\bigcup(v\cap v_{\mathcal{O}})\subseteq\bigcup f$.

4) From 3) it follows that $\bigcup f = \bigcup(v\cap v_{\mathcal{O}}) \subseteq \mathcal{O}$ by definition of $v_{\mathcal{O}}$.

5) Let $x\in Q^*\cap A\cap\mathcal{O}$. Then

a) $\exists l_0\in\mathbb{N}\ \forall l\in J, l\geq l_0 : x\in Q_l \Longrightarrow \lambda^1(\Delta_l[x]\cap\tilde{Q}_l\cap\complement A) > \varepsilon\cdot\lambda^1(\Delta_l[x]\cap\tilde{Q}_l)$

(by assumption)

b) $\forall l\in\mathbb{N}:\ x\in Q^*\cap A\cap\Delta_l[x]$

c) $\forall l\in\mathbb{N}\ \exists l'\in J, l'\geq l : x\in Q_{l'}$ since $x\in Q^*$

d) $x\in\mathcal{O}$, $\mathcal{O}$ open $\Longrightarrow \exists l_1\in\mathbb{N}\ \forall l\geq l_1 : \Delta_l[x]\subseteq\mathcal{O}$

That is why there is a $k\geq l_0, l_1$ with $k\in J$ and a') $\lambda^1(\Delta_k[x] \cap \tilde{Q}_k \cap \complement A) > \varepsilon \cdot \lambda^1(\Delta_k[x] \cap \tilde{Q}_k)$

b') $Q^* \cap A \cap \Delta_k[x] \neq \emptyset$

c') $x \in Q_k$, i.e. $\Delta_k[x] \cap Q_k \neq \emptyset$

d') $\Delta_k[x] \subseteq \mathcal{O}$,

such that $\Delta_k[x] \in v \cap v_{\mathcal{O}}$, what in turn implies that

$$x \in \Delta_k[x] \subseteq \bigcup(v \cap v_{\mathcal{O}}) = \bigcup f.$$

Since $x \in Q^* \cap A \cap \mathcal{O}$ has been arbitrarily chosen, the proof of assertion 1 is complete.

Now let $\hat{v} := \bigcup_{l \in J} \left\{ \Delta \cap \tilde{Q}_l \,\middle|\, \Delta \in v \cap \mathcal{P}_l \right\}$.

By assumption, all $\hat{\Delta} \in \hat{v}$ are closed, since $\Delta \in v \cap \mathcal{P}_l \implies \Delta \cap Q_l \neq \emptyset$.

<u>Assertion 2:</u>

For each $\delta > 0$ there is an at most countable set $\hat{g} \subseteq \hat{v}$ of pairwise disjoint sets with the following properties:

i) $\lambda^1((Q^* \cap A) \setminus \bigcup \hat{g}) = 0$,   ii) $\lambda^1(\bigcup \hat{g}) \leq (1+\delta) \cdot \lambda^1(Q^* \cap A)$.

Proof: We will construct the family $\hat{g}$ inductively:

By regularity of λ^1 we can find an open set $\mathcal{O} \supseteq Q^* \cap A$ in such a way that $\lambda^1(\mathcal{O}) \leq (1+\delta) \cdot \lambda^1(Q^* \cap A)$.

<u>n=0:</u> Let $\hat{g}_o := \emptyset$

<u>n⇒n+1:</u> We will assume that $\hat{g}_n \subseteq \hat{v}$ has been constructed with the following properties:

1) $\hat{g}_n$ is a finite (or void) collection of pairwise disjoint sets.

2) $\bigcup \hat{g}_n \subseteq \mathcal{O}$

3) $\bigcup \hat{g}_n$ is closed.

4) $0 < \lambda^1((Q^* \cap A) \setminus \bigcup \hat{g}_n) \leq (1 - \frac{d}{4})^n \cdot \lambda^1(Q^* \cap A)$

(These 4 conditions are trivially satisfied by $\hat{g}_o$.)

In case $\lambda^1((Q^* \cap A) \setminus \bigcup \hat{g}_n) = 0$ the construction can be finished here by setting $\hat{g} := \hat{g}_n$.

Otherwise we choose an open set $U \subseteq \mathcal{O}$ with $Q^* \cap A \subseteq U$ and

$\lambda^1(U \setminus (Q^* \cap A)) \leq \frac{d}{2} \cdot \lambda^1((Q^* \cap A) \setminus \bigcup \hat{g}_n)$.

Then for $\mathcal{O}_{n+1} := U \setminus \bigcup \hat{g}_n$ the following holds:

i) $\mathcal{O}_{n+1}$ is open, $\mathcal{O}_{n+1} \subseteq \mathcal{O}$

ii) $(Q^* \cap A) \setminus \bigcup \hat{g}_n \subseteq \mathcal{O}_{n+1}$

iii) $\lambda^1(\mathcal{O}_{n+1} \setminus ((Q^* \cap A) \setminus \bigcup \hat{g}_n)) \leq \frac{d}{2} \cdot \lambda^1((Q^* \cap A) \setminus \bigcup \hat{g}_n)$

Let $v_{\mathcal{O}_{n+1}} := \{\Delta \in \mathcal{P}_\infty \mid \Delta \subseteq \mathcal{O}_{n+1}\}$. By assertion 1, there is an at most countable collection $f_{n+1} \subseteq v \cap v_{\mathcal{O}_{n+1}}$ of pairwise disjoint sets, for which $Q^* \cap A \cap \mathcal{O}_{n+1} \subseteq \bigcup f_{n+1} \subseteq \mathcal{O}_{n+1}$.

Putting $\hat{f}_{n+1} := \bigcup_{l \in J} \{\Delta \cap \tilde{Q}_l \mid \Delta \in f_{n+1} \cap \mathcal{P}_l\}$, we have

a) $\hat{f}_{n+1} \subseteq \hat{v}$ consists of at most countably many pairwise disjoint, closed sets,

b) $\bigcup \hat{f}_{n+1} \subseteq \bigcup f_{n+1} \subseteq \mathcal{O}_{n+1}$

c) $\lambda^1(\bigcup \hat{f}_{n+1}) = \sum_{\Delta \cap \tilde{Q}_l \in \hat{f}_{n+1}} \lambda^1(\Delta \cap \tilde{Q}_l) \geq d \cdot \sum_{\Delta \in f_{n+1}} \lambda^1(\Delta)$

[by assumption ii) of the lemma, since $\Delta \in f_{n+1} \subseteq v \Longrightarrow \Delta \cap Q_l \neq \emptyset$.]

$= d \cdot \lambda^1(\bigcup f_{n+1}) \geq d \cdot \lambda^1(Q^* \cap A \cap \mathcal{O}_{n+1})$

$= d \cdot \lambda^1(Q^* \cap A \cap (U \setminus \bigcup \hat{g}_n))$

(∗) $= d \cdot \lambda^1((Q^* \cap A) \setminus \bigcup \hat{g}_n)$ since $Q^* \cap A \subseteq U$,

and we get the following estimation:

$\lambda^1(((Q^* \cap A) \setminus \bigcup \hat{g}_n) \setminus \bigcup \hat{f}_{n+1})$

$\leq \lambda^1(\mathcal{O}_{n+1} \setminus \bigcup \hat{f}_{n+1})$ by ii) above

$= \lambda^1(\mathcal{O}_{n+1}) - \lambda^1(\bigcup \hat{f}_{n+1})$ by b) above

$= \lambda^1(\mathcal{O}_{n+1} \setminus ((Q^* \cap A) \setminus \bigcup \hat{g}_n)) + \lambda^1(\mathcal{O}_{n+1} \cap ((Q^* \cap A) \setminus \bigcup \hat{g}_n)) - \lambda^1(\bigcup \hat{f}_{n+1})$

$\leq (\frac{d}{2} + 1 - d) \cdot \lambda^1((Q^* \cap A) \setminus \bigcup \hat{g}_n)$ by (∗) and iii) above

$= (1 - \frac{d}{2}) \cdot \lambda^1((Q^* \cap A) \setminus \bigcup \hat{g}_n)$

Therefore, there exists a finite subset $\hat{h}_{n+1} \subseteq \hat{f}_{n+1}$ with

a') $\hat{h}_{n+1} \subseteq \hat{v}$ consists of finitely many pairwise disjoint, closed sets,

b') $\bigcup \hat{h}_{n+1} \subseteq \mathcal{O}_{n+1} = U \setminus \bigcup \hat{g}_n \subseteq \mathcal{O}$,

c') $\lambda^1(((Q^* \cap A) \setminus \bigcup \hat{g}_n) \setminus \bigcup \hat{h}_{n+1}) \leqslant (1 - \frac{d}{4}) \cdot \lambda^1((Q^* \cap A) \setminus \bigcup \hat{g}_n)$,

d') as a finite union of closed sets, $\bigcup \hat{h}_{n+1}$ itself is closed.

Put $\hat{g}_{n+1} := \hat{g}_n \cup \hat{h}_{n+1}$. Then

1') $\hat{g}_{n+1} \subseteq \hat{v}$ is a finite collection of pairwise disjoint sets.

2') $\bigcup \hat{g}_{n+1} \subseteq \mathcal{O}$

3') $\bigcup \hat{g}_{n+1}$ is closed,

4') $$\begin{aligned} \lambda^1((Q^* \cap A) \setminus \bigcup \hat{g}_{n+1}) &= \lambda^1(((Q^* \cap A) \setminus \bigcup \hat{g}_n) \setminus \bigcup \hat{h}_{n+1}) \\ &\leqslant (1 - \tfrac{d}{4}) \cdot \lambda^1((Q^* \cap A) \setminus \bigcup \hat{g}_n) \\ &\leqslant (1 - \tfrac{d}{4})^{n+1} \cdot \lambda^1(Q^* \cap A) \end{aligned}$$

5') $\bigcup \hat{g}_n \subseteq \bigcup \hat{g}_{n+1}$

Putting $\hat{g} := \bigcup_{n \in \mathbb{N}} \hat{g}_n$ we get:

1") $\hat{g} \subseteq \hat{v}$ is an at most countable collection of pairwise disjoint sets.

2") $\bigcup \hat{g} \subseteq \mathcal{O}$ implying that $\lambda^1(\bigcup \hat{g}) \leqslant \lambda^1(\mathcal{O}) \leqslant (1+\delta) \cdot \lambda^1(Q^* \cap A)$

4") $\lambda^1((Q^* \cap A) \setminus \bigcup \hat{g}) = \lim_{n \to \infty} \lambda^1((Q^* \cap A) \setminus \bigcup \hat{g}_n) \leqslant \lim_{n \to \infty} (1 - \frac{d}{4})^n = 0$

thus accomplishing the proof of assertion 2.

Now, the assumption that the statement of the lemma is false can easily be led to a contradiction:

Applying assertion 2 with $\delta = \frac{\varepsilon}{2}$ guarantees the existence of a set $\hat{g} \subseteq \hat{v}$ with the properties listed there, and we can conclude:

$$
\begin{aligned}
(1+\tfrac{\varepsilon}{2})\cdot \lambda^1(Q^*\cap A) &\geq \lambda^1(\cup\hat{g}) = \lambda^1(\cup\hat{g}\cap A) + \lambda^1(\cup\hat{g}\cap \complement A)\\
&\geq \lambda^1(Q^*\cap A) + \sum_{\Delta\cap\tilde{Q}_1\in\hat{g}} \lambda^1(\Delta\cap\tilde{Q}_1\cap\complement A)\\
&\geq \lambda^1(Q^*\cap A) + \sum_{\Delta\cap\tilde{Q}_1\in\hat{g}} \varepsilon\cdot\lambda^1(\Delta\cap\tilde{Q}_1)\\
&\qquad\qquad \text{since } \Delta\cap\tilde{Q}_1\in\hat{g} \Longrightarrow \Delta\in V\,,\\
&= \lambda^1(Q^*\cap A) + \varepsilon\cdot\lambda^1(\cup\hat{g}) \geq \lambda^1(Q^*\cap A) + \varepsilon\cdot\lambda^1(Q^*\cap A)\\
&= (1+\varepsilon)\cdot\lambda^1(Q^*\cap A)\,,
\end{aligned}
$$

contradicting $\varepsilon > 0$ and $\lambda^1(Q^*\cap A) > 0$, and the proof of the lemma is complete.

§5) The tail-field of a piecewise monotonic transformation

Remember that $\mathfrak{A}_\infty(T) = \left\{ A \in \mathfrak{B} \,\middle|\, T^{-k}(T^k(A)) = A \quad (k \in \mathbb{N}) \right\}$ is the tail-field of T and the notation introduced at the end of §3: $Z_{k_o} = \bigcup_{k=0}^{k_o-1} T^k(\partial(T\mathcal{P}))$, $X \setminus Z_{k_o} = \bigcup_{i=1}^{r} R_i$, R_i open intervals.

<u>Lemma 8:</u>

$k_o \in \mathbb{N}$ can be chosen in such a way that for each component R_i of $X \setminus Z_{k_o}$ as above and each $A \in \mathfrak{A}_\infty(T)$ the following holds: For each $\varepsilon > 0$ and each infinite index set $J \subseteq \mathbb{N}$ there is an infinite subset $J(\varepsilon) \subseteq J$ such that for each $j \in J(\varepsilon)$

$$\lambda^1(R_i \cap T^j A) \leqslant \varepsilon \cdot \lambda^1(R_i) \qquad \text{or} \qquad \lambda^1(R_i \cap T^j \complement A) \leqslant \varepsilon \cdot \lambda^1(R_i) \ .$$

Proof: Choosing $k_o \in \mathbb{N}$ so big that $H \cdot \alpha^{-k_o} \leqslant \frac{1}{2} \cdot C$ (where C and H are the constants from lemmas 3) and 5) respectively,) and a compact, nonvoid interval $M \subseteq R_i$ with $\lambda^1(R_i \setminus M) \leqslant \frac{\varepsilon}{2} \cdot \lambda^1(R_i)$, lemma 6) tells us that $T^{-k_o}M$ satisfies the assumptions of the construction in §3, which gives us sets $M_{k_o}, M_{k_o+1}, M_{k_o+2}, \ldots$ with the properties listed in lemma 5).
Without loss of generality we will assume that $J \subseteq \{k_o, k_o+1, k_o+2, \ldots\}$.
In order to apply lemma 7) to this situation for the case $Q_l = M_l$ and $\tilde{Q}_l = T^{-l}M$ $(l \in J)$ we first must check that the conditions of lemma 7) are satisfied:

a) $M_l \subseteq T^{-l}M$ by lemma 5/1).

b) $\Delta_l[x] \cap T^{-l}M$ is closed for each $x \in M_l$ $(l \in J)$ by lemma 5/4).

c) Let $l \in J$, $x \in M_l$. If $\lambda^1(\Delta_l[x]) > 0$, we have

$$\frac{\lambda^1(\Delta_l[x] \cap T^{-l}M)}{\lambda^1(\Delta_l[x])} \geqslant \frac{1}{S} \cdot \frac{\lambda^1(T^l(\Delta_l[x] \cap T^{-l}M))}{\lambda^1(T^l(\Delta_l[x]))} \qquad \text{by lemma 4/4)}$$

$\geq \frac{1}{S} \cdot \lambda^1(T^{k_o}(\Delta_{k_o}[T^{1-k_o}x] \cap T^{-k_o}M))$ by lemma 5/2)

$= \frac{1}{S} \cdot \lambda^1(M)$ by lemma 6), since

$T^{1-k_o}x \in T^{1-k_o}M_1 \subseteq M_{k_o} \subseteq T^{-k_o}M$ such that

$\Delta_{k_o}[T^{1-k_o}x] \cap T^{-k_o}M \neq \emptyset$.

> 0 since M is a nonempty interval.

Therefore we can take for the constant d in lemma 7): $d = \frac{1}{S} \cdot \lambda^1(M)$.

d) By 3) of lemma 5) we have for each $l \in J$:

$\mu(M_l) \geq \mu(M) - H \cdot \mathrm{dia}(M) \cdot \alpha^{-k_o}$

$\geq C \cdot \lambda^1(M) - H \cdot \lambda^1(M) \cdot \alpha^{-k_o}$ since $M \subseteq X$ and $h_{|X}$ C,

$> \frac{C}{2} \cdot \lambda^1(M)$ by choice of k_o,

For $M^* := \bigcap_{k \in J} \bigcup_{\substack{l \geq k \\ l \in J}} M_l$, consequently $\mu(M^*) \geq \frac{C}{2} \cdot \lambda^1(M) > 0$.

So, for $A \in \mathfrak{A}_\infty(T)$ two cases, not mutually excluding, can arise:

Case I: $\lambda^1(M^* \cap A) > 0$ Case II: $\lambda^1(M^* \cap \complement A) > 0$

Both are treated in the same way, so, without loss of generality we will have a closer look at case I:

Lemma 7) tells us that for each $\varepsilon > 0$ there is an $x \in M^* \cap A$ such that $\forall l_0 \in \mathbb{N} \ \exists l \in J, l \geq l_0$:

$$(\S) \quad \begin{cases} x \in M_l \quad \text{and} \\ \lambda^1(\Delta_l[x] \cap T^{-l}M \cap \complement A) \leq \frac{\varepsilon}{2S} \cdot \lambda^1(\Delta_l[x] \cap T^{-l}M) . \end{cases}$$

Since for each $l \in J$

$$\frac{\lambda^1(M \cap T^l \complement A)}{\lambda^1(M)}$$

$$= \frac{\lambda^1(T^{k_o}(\Delta_{k_o}[T^{1-k_o}x] \cap T^{-k_o}M \cap T^{-k_o}(T^l \complement A)))}{\lambda^1(T^{k_o}(\Delta_{k_o}[T^{1-k_o}x] \cap T^{-k_o}M))} \quad \text{by lemma 6)}$$

$$= \frac{\lambda^1(T^l(\Delta_l[x] \cap T^{-l}M \cap \complement A))}{\lambda^1(T^l(\Delta_l[x] \cap T^{-l}M))} \qquad \text{by lemma 5/2)}$$

$$\leq S \cdot \frac{\lambda^1(\Delta_l[x] \cap T^{-l}M \cap \complement A)}{\lambda^1(\Delta_l[x] \cap T^{-l}M)} \qquad \text{ly lemma 4/4),}$$

it follows from (§) that for each $l_0 \in \mathbb{N}$ there is an $l \in$ [illegible], $l \geq l_0$ such that $\lambda^1(M \cap T^l A) \leq \frac{\varepsilon}{2} \cdot \lambda^1(M)$.

Since $M \subseteq R_i$ with $\lambda^1(R_i \setminus M) \leq \frac{\varepsilon}{2} \cdot \lambda^1(R_i)$, the proof of the lemma is complete.

Now we can turn to the

<u>proof of the theorem:</u>

Applying lemma 8) inductively to all R_i we can obtain:

Let $A \in \mathfrak{A}_\infty(T)$. Then

$\forall \varepsilon > 0 \ \forall J \subseteq \mathbb{N}, |J| = \infty \ \exists l = l(\varepsilon) \in J \ \forall i = 1, \dots, r$:

$$(*) \quad \begin{cases} \text{either} & \lambda^1(R_i \cap T^l A) < \varepsilon \cdot \lambda^1(R_i) \\ \text{or} & \lambda^1(R_i \cap T^l \complement A) < \varepsilon \cdot \lambda^1(R_i) \end{cases}$$

Since $R_i \subseteq X$ for all R_i and $C \cdot \lambda^1(B) \leq \mu(B) \leq \|h\|_\infty \cdot \lambda^1(B)$ for all measurable $B \subseteq X$, we also have:

(**) (*) is valid for μ instead of λ^1.

But for $\mu(A) > 0$ and $\varepsilon < \mu(A)$ it is impossible, because of the T-invariance of μ, that for all R_i, i=1,...,r, holds

$$\mu(R_i \cap T^l A) < \varepsilon \cdot \mu(R_i) \ .$$

Therefore, for each $\varepsilon > 0$ there is at least one R_i with

$\mu(R_i \cap T^{l(\varepsilon)} \complement A) <$ [illegible] ${}^1(R_i)$, such that

$$\mu(T^{l(\varepsilon)}A) > (1-\varepsilon)\cdot\lambda^1(R_i) \quad ,$$

and since $\mu(T^l A) = \mu(A)$ $(l \in \mathbb{N},\ A \in \mathfrak{A}_\infty(T))$, we have for each $A \in \mathfrak{A}_\infty(T)$ with $\mu(A) > 0$:

$$\mu(A) \geq \min\{\lambda^1(R_i) \mid i=1,\dots,r\} > 0 \quad .$$

So $\mathfrak{A}_\infty(T)$ is generated μ-mod 0 by a finite number of atoms.

Let $A \in \mathfrak{A}_\infty(T)$ be such an atom.
Then $T^l A \in \mathfrak{A}_\infty(T)$ are atoms too $(l \in \mathbb{N})$, and consequently there exists a $p = p(A) \in \mathbb{N}$ with $T^p A = A$ μ-mod 0.
Applying (**) with the special index set $J := p\cdot\mathbb{N}$ shows immediately that for each R_i $(i=1,\dots,r)$

$$\text{either} \quad \mu(R_i \cap A) = 0 \qquad \text{or} \qquad \mu(R_i \setminus A) = 0 \; ,$$

thus proving that A is μ-mod 0 a finite union of open intervals (part 2 of the theorem).

Denote by L_i the biggest open interval contained μ-mod 0 in $T^i A$ $(i=0,\dots,p(A)-1)$ and by n_i the number of singularities from $\{a_1,\dots,a_{N-1}\}$ contained in L_i. For each $i=0,\dots,p-1$, n_i must satisfy

$$\frac{\alpha\cdot\lambda^1(L_i)}{n_i+1} \leq \lambda^1(L_{i+1}) \quad ,$$

since n_i singularities divide L_i into n_i+1 open subintervals at least one of which has length $\geq \frac{\lambda^1(L_i)}{n_i+1}$, such that the image under T of such an interval is an interval with length $\geq \alpha\cdot\frac{\lambda^1(L_i)}{n_i+1}$ contained in $T^{i+1}A$ μ-mod 0.
From the relations

$$(\S) \qquad \rho_i\cdot\alpha \leq (n_i+1)\;, \quad \rho_i = \frac{\lambda^1(L_i)}{\lambda^1(L_{i+1})} \qquad (i=0,\dots,p-1)$$

with $\prod_{i=0}^{p-1}\rho_i = 1$ (since $L_p = L_0$), we can independently deduce two estimates for p:

1) $\alpha^p \leq \prod_{i=0}^{p-1} (n_i+1) \Longrightarrow p \cdot \log_2\alpha \leq \sum_{i=0}^{p-1} \log_2(n_i+1) \leq \sum_{i=0}^{p-1} n_i$

$$\Longrightarrow \quad p \leq \frac{1}{\log_2\alpha} \cdot \sum_{i=0}^{p-1} n_i$$

2) $p + \sum_{i=0}^{p-1} n_i = \sum_{i=0}^{p-1} (n_i+1) \geq \alpha \cdot \sum_{i=0}^{p-1} \rho_i \geq p \cdot \alpha$, since $\prod_{i=0}^{p-1} \rho_i = 1$

$$\Longrightarrow \quad p \leq \frac{1}{\alpha - 1} \cdot \sum_{i=0}^{p-1} n_i$$

Both estimates are valid for each cycle $A, TA, \ldots\ldots, T^{p(A)-1}A$ of atoms of $\mathfrak{A}_\omega(T)$. Thus, since there are only N-1 singularities of T, we have

$\left|\text{atoms}(\mathfrak{A}_\infty(T))\right| \leq \frac{N-1}{\log_2\alpha}$ and $\left|\text{atoms}(\mathfrak{A}_\infty(T))\right| \leq \frac{N-1}{\alpha - 1}$.

This is statement 3) of the theorem.

Corollary 1) follows immediately from 2) of the theorem.

For the proof of corollary 2) we need another estimate that can be obtained in an analogous way as the one above, only much simpler (see Kowalski [3]):

Let $\lceil\alpha\rceil := \min\{n \in \mathbb{N} \mid n \geq \alpha\}$. Then the number of ergodic atoms of T is $\leq (N-1) \cdot \frac{1}{\lceil\alpha\rceil - 1}$.

In order to show exactness of T we need $\left|\text{atoms}(\mathfrak{A}_\infty(T))\right| < 2$, for which - by 3) of the theorem - a sufficient condition is: $\alpha > \frac{N+1}{2}$. This proves a) of corollary 2).

Now let us assume $N \geq 4$, N even, and $\alpha > \sqrt{\frac{N}{2}(\frac{N}{2}+1)}$. Then $\lceil\alpha\rceil \geq \frac{N}{2}+1$ and by the estimation of the number of ergodic atoms above we see that T is ergodic. Similarly, 3) of the theorem tells us that the number of atoms of $\mathfrak{A}_\infty(T)$ is ≤ 2. Assuming that this number is $=2$ we could specialise (§) on p. 28 to the following relation:

$[\varrho \cdot \alpha] \leq n_0 + 1$, $\left[\frac{1}{\varrho} \cdot \alpha\right] \leq n_1 + 1$, $n_0 + n_1 = N - 1$ for a suitable ϱ.

$\Longrightarrow$ $[\varrho \cdot \alpha] + \left[\frac{1}{\varrho} \cdot \alpha\right] \leq N + 1$

but simple considerations show that this is impossible for $\alpha > \sqrt{\frac{N}{2}(\frac{N}{2} + 1)}$ and arbitrary $\varrho > 0$.

So T is exact, and b) of corollary 2) is proved for $N \neq 2$.

The case $N = 2$ is proved together with c):

3) of the theorem tells us that for $N = 2$ and $\alpha > \sqrt[p]{2}$ holds:

$|\text{atoms}(\mathcal{O}l_{\alpha}(T))| \leq \frac{1}{\log_2 \alpha} < p$ thus proving c), and for $p = 2$ we also have the remaining case from part b).

§6) Remark on higher dimensions

With the same basic idea we can prove results for higher dimensional spaces too, for example the following one for transformations on $E^2 := \{(x_1, x_2) \mid 0 \leq x_1, x_2 \leq 1\}$.

Let $\mathcal{P} = \{P_1, \ldots, P_N\}$ be a finite partition of E^2. We call $\mathcal{P}$ **smooth**, if the boundary of each of the P_i consists of finitely many C^1-curves. Then we can state .

Proposition:

Let $\mathcal{P}$ be a smooth partition of E^2, $\alpha > 1$, and $T: E^2 \longrightarrow E^2$ a transformation satisfying:

i) For each $P \in \mathcal{P}$, $T_{|P}$ is C^1, $\forall x \in \mathring{P}: \|(DT_{|P}(x))^{-1}\| \leq \alpha^{-1}$, and the Jacobian of $DT_{|P}(x)$ as a function of x is Lipschitz-continuous on $\mathring{P}$.

ii) T posseses an invariant measure $\mu = h \cdot \lambda^1$ with $\|h\|_\infty < \infty$.

Then:

I) If there is a set $M \subseteq E^2$ with the properties

a) $\mu(\mathring{M}) > 0$ and

b) $\forall x \in M \ \forall \delta > 0: \ \mu(\bigcup_{n \in \mathbb{N}} T^n(S_\delta(x))) = 1$,

then (T, μ) is ergodic.

II) If there is a set $M \subseteq E^2$ with the properties

a) $\mu(\mathring{M}) > 0$ and

b) $\forall x \in M \ \forall \delta > 0: \ \sup_{n \in \mathbb{N}} \mu(T^n(S_\delta(x))) = 1$,

then (T, μ) is exact.

Analogous results hold for transformations in n-dimensional spaces. (For dimension 1 cf. ...en [1].)

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