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Cylindrical Stochastic Integral

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* CYLINDRICAL STOCHASTIC INTEGRAL

by

M. METIVIER and J. PELLAUMAIL

Summary

In the first part of this study, we construct the stochastic integral $\int Y\, d\tilde{M}$ where Y is a "weakly" predictable process and $\tilde{M}$ is a "cylindrical" square integrable martingale. This last notion generalises the case where $\tilde{M}$ is a "white noise in time and space".

In the second part, this construction is extended, when M is not square integrable, by a regionalization procedure (Cf. [18]') ; this procedure is a generalization of the classical procedure of localization.

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A FIRST PART

CONSTRUCTION OF THE STOCHASTIC INTEGRAL WITH RESPECT TO PROCESSES OF LINEAR FUNCTIONNALS

INTRODUCTION

For the purpose of studying stochastic partial differential equations it is worth considering perturbations which are "white noise in time and in space". The mathematical expression of such and object is a cylindrical measure, or a linear random functional as studied for example in [1] or [8] . Considering the special case of "cylindrical brownian motion", several authors defined a stochastic integral with respect to such a stochastic process (cf. for example [7] and [11]). In [7] the operator valued processes, which are integrated with respect to the cylindrical brownian motion, are such that the integral process is a (Hilbert valued) Martingale.

The purpose of this part is to show that, in a very general context, it is possible to developp a theory of stochastic integration with respect to "cylindrical martingales", which extends in a natural way the classical L^2-stochastic integral with respect to square integrable martingales (real or Hilbert valued) as studied in [10], [14], [18] for example. This part generalizes and completes [14]'.

Hypotheses and notations are given in the first paragraph. In the second one, the notion of cylindrical martingale is defined and the particular case of "white noise in time and space" is specially studied. In the third paragraph, we define and study a process Q : the role of this process is analognous to the role of the "quadratic variation" for a real square integrable martingale. The stochastic integral is constructed in the fourth paragraph. The case where the processes considered are Hilbert-space valued is more specially studied in the fifth paragraph.

I - NOTATIONS

I-1 In all this
vall $[0,1]$ in $\mathbb{R}^+$, a basic pr
$(\mathcal{F}_t)_{t \in T \times \mathbb{R}^+}$ of sub-σ-algeb
assumption : $\mathcal{F}$ is P-comple
for every t.

$\mathcal{R}$ will mean the
where $s \leq t$, $s,t \in T$ and $F \in$

$\mathcal{X}$ will be the a

$\mathcal{P}$ is the σ-alge
predictable subsets of $T \times \Omega$.

I-2 $\mathcal{M}$ is the s
following properties :

(i) $(M_t)_{t \in T}$
process

(ii) $(M_t)_{t \in T}$

(iii) $(M_t)_{t \in T}$
is to say
pond to t

(iv) $M_o = 0$

If M and N are t
it is well-known that $\mathcal{M}$ is
(cf. [17]).

I-3 $\mathbb{H}$ and $\mathbb{G}$ wil
written : $||\cdot||_{\mathbb{H}}$, $||\cdot||_{\mathbb{G}}$,
denote the topological dual
if not otherwise specified,

If $\mathbb{H}$ and $\mathbb{G}$ are H
will be denoted by $<.,.>_{\mathbb{H}}$,
confusion.

We recall that the algebraic tensor product $\mathbb{H} \otimes \mathbb{G}$ can be endowed with several norms, giving rise to several completions of $\mathbb{H} \otimes \mathbb{G}$:

- $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ is the completion for a norm such that every continuous bilinear mapping $b : (\mathbb{H} \otimes \mathbb{G}) \to \mathbb{K}$ can be factorized in a unique way as $b = u_b \circ \Pi$ where Π is the canonical inbedding $\Pi(x,y) = x \otimes y$ and u_b is a continuous linear mapping from $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ into $\mathbb{K}$, with same norm as b. The norm $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ is often called the trace-norm and denoted $||.||_{Tr}$. Recall that if $\mathbb{G} = \mathbb{H}$ is an Hilbert space and $b(x,y) = \langle x,y\rangle_{\mathbb{H}}$ the corresponding linear form u_b on $\mathbb{H} \hat{\otimes}_1 \mathbb{H}$ is called the trace-form and denoted Tr.

- If $\mathbb{H}$ and $\mathbb{G}$ are Hilbert spaces, $\mathbb{H} \hat{\otimes}_2 \mathbb{G}$ is a Hilbert space with scalar product the extension of $\langle x \otimes y,\ x' \otimes y'\rangle = \langle x,x'\rangle_{\mathbb{H}} \cdot \langle y,y'\rangle_{\mathbb{G}}$.

- $\mathbb{H} \hat{\otimes}_\varepsilon \mathbb{G}$ is a Banach space, the norm of which will be more easily described later.

The three topologies induced by the three considered topological tensor product on $\mathbb{H} \otimes \mathbb{G}$ are comparable and we have the canonical continuous injection.

$$\mathbb{H} \hat{\otimes}_1 \mathbb{G} \hookrightarrow \mathbb{H} \hat{\otimes}_2 \mathbb{G} \hookrightarrow \mathbb{H} \hat{\otimes}_\varepsilon \mathbb{G} .$$

I-4 There is a unique injective linear mapping of $\mathbb{H} \otimes \mathbb{G}$ into the vector space of linear operators with finite range from $\mathbb{H}$ into $\mathbb{G}$, associating to $x \otimes y$ the operator $h \leadsto \langle x,h\rangle_{\mathbb{H}}\, y$. This linear mapping has extensions which are :

1°) isometry from $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ onto $\mathcal{L}_1(\mathbb{H}\ ;\ \mathbb{G})$, the Banach space of nuclear operators from $\mathbb{H}$ into $\mathbb{G}$ with the trace norm ;

2°) isometry from $\mathbb{H} \hat{\otimes}_2 \mathbb{G}$ onto $\mathcal{L}_2(\mathbb{H}\ ;\ \mathbb{G})$ the Hilbert space of Hilbert-Schmidt operators from $\mathbb{H}$ into $\mathbb{G}$ with the Hilbert-Schmidt scalar product ;

3°) isometry from $\mathbb{H} \hat{\otimes}_\varepsilon \mathbb{G}$ onto $\mathcal{L}_c(\mathbb{H}\ ;\ \mathbb{G})$, the Banach space of compact operators with the usual norm of bounded operators.

In as much $x \otimes y$ can be identified with a bilinear continuous form on $(\mathbb{H} \times \mathbb{G})$ or a continuous linear form on $\mathbb{H} \otimes \mathbb{G}$, through the formula

$$\langle x \otimes y,\ x' \otimes y'\rangle = \langle x,x'\rangle_{\mathbb{H}} \cdot \langle y,y'\rangle_{\mathbb{G}}$$

there is also a continuous
into an isometry from ($\mathbb{H} \hat{\otimes}$
bounded operators from $\mathbb{H}$ i
fact the one which associa
linear operator $\tilde{b}$ in $\mathcal{L}$($\mathbb{H}$

I-5 We shall
continuous operators from
topology $\sigma(\mathbb{G}, \mathbb{G}')$. If u
u is defined as a linear c
$\sigma(\mathbb{G}', \mathbb{G})$ into H' with its

I-6 Random v
measurable mapping from Ω
ty $E(||X||^2_{\mathbb{H}}) < \infty$, then ω
variable with values in $\mathbb{H}$
is an integrable mapping f
and is called the covarian

If to X is mo
$h \leadsto \langle X,h\rangle$ from $\mathbb{H}$ into $L^2_{\mathbb{R}}$(
And it can be shown that c
$L^2_{\mathbb{H}}(\Omega, \mathfrak{F}, P)$ there can be a
such that $\langle X,h\rangle = \tilde{X}(h)$ a.s
Schmidt norm $||\tilde{X}||_2$ of $\tilde{X}$

I-7 To abbre

$\mathcal{X}^p$ =

$\mathcal{X}^p_t$ =

I-8 The norm
in $\mathcal{L}_2(\mathbb{H}\ ;\ \mathbb{G})$: $||\cdot||_{H.S}$,

II - CYLINDRICAL MARTINGALE

II-1 Definition

If H is a Banach space, we shall say that $\tilde{M}$ is a 2-cylindrical martingale on H' if $\tilde{M}$ is an element of $\mathcal{L}(H', \mathcal{M})$, that is to say that $\tilde{M}$ is a linear continuous mapping from H' into $\mathcal{M}$ (for the strong topology of H' and the Hilbert space topology of $\mathcal{M}$).

II-2 Quadratic Doléans's measure

Let $\tilde{M}$ be a 2-cylindrical martingale on the Banach space H. We consider the function m defined on $[\mathcal{R} \times (H' \otimes H')]$ by :

$$\forall A = (F\times]s,t]) \in \mathcal{R} \ , \quad \forall (h,g) \in (H' \times H') \ ,$$

$$[m(A)](h \otimes g) = E \{1_F [\tilde{M}_t(h).\tilde{M}_t(g) - \tilde{M}_s(h).\tilde{M}_s(g)]\}$$

It is well-known that, for each element (h,g) of $(H' \times H')$, there is a unique real measure defined on $\mathcal{P}$ which is an extension of $m(.)(h \otimes g)$.

Then, this extension m defines a mapping from $\mathcal{P}$ into the algebraic dual of $(H' \otimes H')$.

We shall say that this extension m is the quadratic Doléans' measure of $\tilde{M}$. In fact, we are essentially interested in the case where the total variation of m is finite, m being considered as an application of $\mathcal{P}$ into the Banach space $(H' \hat{\otimes}_1 H')'$, dual of the tensor product $H' \otimes_1 H'$.

That is the case (Cf.II-5 below) in particular when $\tilde{M}$ is a "white noise in time and in space" : in our context, the mathematical definition of such a process is the following :

II-3 Cylindrical brownian process (definition)

Let H be a Hilbert space and $\tilde{w}$ a cylindrical martingale on H. We shall say that $\tilde{w}$ is a cylindrical brownian process if, for each finite family $(h_k)_{1\le k\le n}$ constituted of elements of H, $(\tilde{w}_t(h_k))_{1\le k\le n}$ is an n-dimensionnal brownian motion such that $E[w_t(h_i).\ w_t(h_j)] = t \langle h_i\ ,\ h_j\rangle_H$ for all pairs $(h_i\ ,\ h_j)$ of elements of H.

II-4 Propos

brownian motion)

Let H be a H

motion on H. Then, we have

1°) *$\tilde{w}$ is an isometry*

2°) *For all elements*

we have :

$E\ \{1_F\ [\tilde{w}_t($

3°) *Let m be the quad*

variation of m is

4°) *Let $Q = \frac{dm}{dv}$ be t*

v. Then, for all

of $(H \times H)$,

$Q\ [h \otimes g]$ (

Proof

1°) For each element

$\langle\tilde{w}(h),\ \tilde{w}(g)$

then $\tilde{w}$, considere

2°) We consider $h \in H$

$\tilde{w}_t(h) - \tilde{w}_s(h)$ be

$E\ \{1_F.\ [\tilde{w}_t$

$= P(F).(t-$

3°) Let u be an eleme

where $(a_i)_{i\in I}$ i

tic Doléans's mea

we have :

$$\langle m(A),u\rangle = \sum_{(i,j)\in I\times I} \lambda_{i,j} \cdot E\,\{1_F\cdot[\tilde{w}_t(a_i)\,\tilde{w}_t(a_j) - \tilde{w}_s(a_i)\,\tilde{w}(a_j)]\}$$

$$= \sum_{i\in I} \lambda_{i,i} \cdot (P\otimes\mu)(A) \leq (P\otimes\mu)(A).||u||_1$$

then, $P\otimes\mu = v$ is the total variation of m considered as a measure with values in $(H\,\hat{\otimes}_1\,H)'$.

4°) Moreover, for each pair (h,g) of elements of H,
$m(F\times]s,t])(h\otimes g) = (P\otimes\mu)(F\times]s,t]).\langle h,g\rangle_H$ and this proves the 4°).

III - THE PROCESS Q

III-1 Hypotheses

For all the following parts, we consider a Banach space H and a 2-cylindrical martingale $\tilde{M}$ on H. We note m the quadratic Doléans's measure of $\tilde{M}$. We suppose that the total variation v of m is finite.

We writte Q the weakly predictable process, $(H\,\hat{\otimes}_1\,H)'$-valued, Radon-Nikodym derivative of m with respect to v.

III-2 Proposition

Let v be a positive measure defined on the tribe of predictable sets. Let V be the increasing "natural" process (cf. [5]) associated to v. Let r be a real measure defined on the tribe of predictable sets : we suppose then $|r| \leq v$ if $|r|$ is the total variation of r. Let Q be the predictable Radon-Nikodym derivative of r with respect to v. For each element ω of Ω , the real function Q(.,ω) is a borelian function. Then, we can define the process $(R_t)_{t\in T}$ by $R_t(\omega) = \int_{]o,t]} Q(s,\omega).dV_s(\omega)$ (this integral being calculated by trajectories). Then, the process $(R_t)_{t\in T}$ is the "natural" process associated to Ω.

Proof

Let $\mathcal{P}$ be the cla
for each trajectory ω, the rea
to the Borel tribe ; $\mathcal{P}$ is a v
sing sequence of elements of
element of $\mathcal{P}$; moreover, all
Then, $\mathcal{P}$ is the set of all pre
defined. If $Q = 1_A$ with $A\in\mathcal{R}$
Then, the same property is sat
by linearity and dominated con
of [14]).

To prove that R i
sufficient to prove that the D
it is sufficient to prove that
have :

r(A) = E [

This property is
satisfied for each bounded pre
convergence.

III-3 Theorem (p

Let B be a separa
on the tribe of predictable se
that m is σ-additive for the t
is finite. Then, this total va
and weakly predictable process
to v. Let V be the increasing
B'-valued process defined by

$S_t(\omega) = \int_]$

this integral being a "weak in

Then, for each el
an indistinguability , the "
measure <m,x> .
We call S the nat

Proof

The σ-additivity of v and m is a well-known property for vector measures. Then Q is well-defined by a "weak" Radon-Nikodym theorem (cf. [12]). The end of the theorem is a corollary of the proposition III-2 above.

IV - CONSTRUCTION OF THE STOCHASTIC INTEGRAL

IV-1 Introduction

The purpose of this part is to define the stochastic integral $\int Y.d\tilde{M}$ where Y is a "weakly predictable" process with values in $\mathcal{L}(\mathbb{H}, \mathbb{G}_\sigma)$ (cf. I-5 above), and where $\tilde{M}$ is a cylindrical martingale on $\mathbb{H}$ such that the total variation of its quadratic Doléans's measure is finite.

For all this part, $\mathbb{H}$ and $\mathbb{G}$ are Banach spaces, $\mathbb{H}$ being reflexive, and $\tilde{M}$ is an element of $\mathcal{L}(\mathbb{H}', \mathcal{M})$. We suppose than $\mathbb{H}'$ is a separable space. We note m the quadratic Doléans's measure of $\tilde{M}$ and v the total variation of m where m is considered as a $(\mathbb{H}' \otimes_1 \mathbb{H}')'$-valued measure. We suppose than $v(\Omega \times T) < +\infty$. We note Q the predictable process, Radon-Nikodym derivative of m with respect to v.

We shall say that a $\mathcal{L}(\mathbb{H}, \mathbb{G}_\sigma)$-valued process Y is "weakly predictable" if, for each element (h,g) of $\mathbb{H} \times \mathbb{G}'$, $\langle Y(h), g\rangle$ is a real predictable process.

IV-2 $\mathcal{E}$-step process and stochastic integral associated

We note $\mathcal{E}$ the set of the processes Y such that $Y = \sum_{i\in I} u_i . 1_{A(i)}$ where $(u_i)_{i\in I}$ is a finite family of elements of $\mathcal{L}(\mathbb{H}, \mathbb{G}_\sigma)$ and $(A(i))_{i\in I}$ is an associated family of elements of $\mathcal{A}$.

We remark that, in this situation, we can suppose that the sets A(i) belong to $\mathcal{R}$ and are pairwise disjoint (cf. [18]).

Let Y be an element of $\mathcal{E}$ with $Y = \sum_{i\in I} u_i . 1_{A(i)}$ where, for each element i of I, $A(i) = (F(i) \times]s(i), t(i)])$. For each element g of $\mathbb{G}'$, let $\tilde{Z}(g)$ be the real martingale defined by :

$[\tilde{Z}(g)]_t = \sum_{i\in I} \{1_{F(i)}$

where u_i^* is the adjoint o

This defined a
$\int Y.d\tilde{M}$ *and call the stocha*

The problem is
processes Y as done, for exa
strict sense.

For this extens

IV-3 Remark

Let Y be an ele
sets $(A(i))_{i\in I}$ are pairwis
then, we have :

$||\tilde{Z}(g)||^2_{\mathcal{M}} = \sum_{i\in I} E\{$

the random variables $1_{F(i)}\cdot$
gonal in $L_2(\Omega, \mathcal{F}, P)$, we have

$||Z(g)||^2_{\mathcal{M}} = \int_{\Omega\times T} [Y^*$

where Y^* is the adjoint of Y,

finally, we obtain :

$||Z(g)||^2_{\mathcal{M}} = \int_{\Omega\times T} Q\ [Y$

The fundamental idea is to co
this formula.

IV-4 Lemma (top

We consider u ∈

be the adjoint of v. We suppose that H' is separable. Let $\{x_n\}_{n>o}$ *be a sequence of elements of H', dense in H'. Let* H'_n *be the vector space generated by* $\{x_k\}_{1\leq k\leq n}$. *Let* π'_n *a projector of H' onto* H'_n *which is a contraction. Then, we have :*

$$u\left[(v^* \circ g)\otimes(v^* \circ g)\right] = \lim_{n\to\infty} u\left[(\pi'_n \circ v^* \circ g)\otimes(\pi'_n \circ v^* \circ g)\right]$$

and $$\lim_{n\to\infty} u\,\{\left[(v^* \circ g) - (\pi'_n \circ v^* \circ g)\right]^{\otimes 2}\} = 0$$

<u>Proof</u>

If we consider v, g and $\varepsilon > o$, there exists $k > o$ and $w \in H'_k$ such that $||v^* \circ g - w||_{H'} \leq \varepsilon$; this implies , $\forall\, n \geq k$:

$$||\pi'_n \circ v^* \circ g - v^* \circ g||_{H'} \leq ||\pi'_n \circ v^* \circ g - w||_{H'} + ||v^* \circ g - w||_{H'}$$

$$\leq ||\pi'_n \circ (v^* \circ g - w)||_{H'} + \varepsilon$$

$$\leq 2\varepsilon$$

then $\pi'_n \circ v^* \circ g$ converges strongly to $v^* \circ g$ in H'. The lemma follows from the continuity of u for the strong topology of H' and from the continuity of the mapping $(x,y) \rightsquigarrow (x \otimes y)$ for the " trace norm " on $(H' \hat{\otimes}_1 H')$.

IV-5 <u>Preliminary proposition</u>

1°) We suppose that H' is a separable Banach space. Let Q be a $(H' \hat{\otimes}_1 H')'$*-valued and weakly predictable process. Let Y be a "weakly predictable"* $\mathcal{L}(H, G_\sigma)$*-valued process (cf. the end of IV-1 above). Let* Y^* *be the adjoint of Y (with values in* $\mathcal{L}(G'_\sigma, H'_\sigma)$*). Then, for each element g of G', the process* $Q\left[Y^*(g) \otimes Y^*(g)\right]$ *is a real (positive) predictable process.*

2°) We have a

the condition "for each el

in the place of the condit

<u>Proof</u>

We prove the

guous. We consider Q, Y an

projectors as in the lemma

$\lim_{n\to\infty} Q\left[(\pi'_n \circ Y^* \circ g)\right.$

But, for each n, $Q\left[(\pi'_n \circ\right.$

process. Then, the same is

IV-6 <u>Définit</u>

Let $\mathcal{Q}$ *be as in IV-*

denote by $\mathcal{Q}_g$ *the vector sp*

(i) $\forall (t,\omega) \in]0,1] \otimes \Omega$, *Y*

in H and range in G

(cf. below)

(ii) $Q\left[Y^*(g) \otimes Y^*(g)\right]$ *i*

infinite)

We give now t

is "well-defined".

1°/ Let (t,ω) be an elemen

in the following case

let $Q^{1/2}(t,\omega)$ be the s

if $\mathcal{D}[X(t,\omega)] \supset$ Range

is extendable into a b

$Q\,[Y^*(g) \otimes Y^*(g)]\ (t,\omega)$ is well-defined by :

$$Q\,[Y^*(g) \otimes Y^*(g)]\ (t,\omega) = [Y(t,\omega) \circ Q^{1/2}(t,\omega)]^*\ (g)$$

2°/ If Y is a process with values in $\mathcal{L}(\mathbb{H}, \mathbb{G}_\sigma)$ "weakly predictable" (cf. the end of IV-1 above) then the conditions (i) and (ii) above are satisfied (cf. the proposition IV-5 above).

IV-7 <u>Definitions of N_g and $\bar{\mathcal{E}}_g$</u>

For each element g of $\mathbb{G}'$ and for each process Y belonging to Q_g, we note : $N_g(Y) = \{ \int_{\Omega\times T} Q\ [Y^*(g) \otimes Y^*(g)]\,.\ dv \}^{1/2}$

This quantity, finite or infinite, is well-defined (cf. IV-6 above). If $\tilde{M}$ is a cylindrical brownian motion (cf. II-3 above), we remark that

$$N_g(Y) = \{ \int_{\Omega x T} ||Y^*(g)||^2 .\ dv\}^{1/2}$$

We note $\bar{\mathcal{E}}_g$ the adherence of $\mathcal{E}$, for the semi-norm N_g above, in the set of all "weakly predictable" processes Y.

The mapping $Y \rightsquigarrow \int Y.d\tilde{M}(g) = \tilde{Z}(g)$, defined for $Y \in \mathcal{E}$, admits an unique extension to $\bar{\mathcal{E}}_g$ which is a linear continuous mapping from $\bar{\mathcal{E}}_g$ (with the topology associated to N_g) into $\mathcal{M}$ (cf. the remark IV-3 above).

IV-8 <u>Definition of $\bar{\mathcal{E}}^b$</u>

We shall say that a process Y belong to $\bar{\mathcal{E}}^b$ if the two following properties are satisfied :

(i) for each element g of $\mathbb{G}'$, $Y \in \bar{\mathcal{E}}_g$ (cf. IV-7 above)

(ii) $\Big\{ \underset{g \in \mathbb{G}', ||g|| \leq}{Sup}$

The mapping Y $\rightsquigarrow$
mapping from $\bar{\mathcal{E}}^b$ (with the to
For each element Y of $\bar{\mathcal{E}}^b$, we sh
associated to Y and we shall ca
respect to $\tilde{M}$.

IV-9 <u>Theorem</u>

We consider the
each element g of $\mathbb{G}'$, $\bar{\mathcal{E}}_g$ (cf.
processes Y with values in $\mathcal{L}($
(cf. the end of IV-1) and such

Moreover, $\bar{\mathcal{E}}^b$ (c
processes with values in $\mathcal{L}(\mathbb{H},$
$N^b(Y) < +\infty$ (where $N^b(Y) =$ $_{g \in \mathbb{G}}$

<u>Proof</u>

The second part
the first part.

To prove the firs
process Y such that $N_g(Y) < +\infty$
a sequence of projectors as in

$A_n = \{(\omega,t) : Q\ [(\pi'_n \circ Y^*$

$Y_n = 1_{A(n)} .\ [Y \circ (\pi'_n)^*]$

(Y^* is well-defined because

The sequence of

converges to zero (cf. the lemma

$Q\ [(\pi'_n \circ Y^* \circ g - Y^* \circ g)^{\otimes}$

Then, we have $\lim_{n\to\infty} N_g(Y-Y_n) = 0$ by the Lebesgue dominated convergence theorem. Moreover it is easily seen than Y_n belongs to $\mathcal{E}_g$ since $<Y_n, g>$ is strongly predictable and $N_g(Y_n) < +\infty$. Then Y belongs also to $\bar{\mathcal{E}}_g$.

IV-10 <u>Theorem</u>

We consider the hypotheses given in IV-1. In this case, the mapping $Y \rightsquigarrow \int Y.d\tilde{M}$ *is a linear isometry from* $\bar{\mathcal{E}}^b$ *(with the semi-norm* $N_b(.)$*) into* $\mathcal{L}(G', \mathcal{M})$ *(cf. the end of I-4).*

Moreover if Y is an element of $\bar{\mathcal{E}}^b$, *the quadratic Doléans's measure z of* $\tilde{Z} = \int Y.d\tilde{M}$ *is the* $(G' \hat{\otimes}_1 G')'$*-valued measure defined by*

$$z(g_1 \otimes g_2) = \int \{ Q\, [Y^*(g_1) \otimes Y^*(g_2)] \}\, dv$$

for each element $(g_1 \otimes g_2)$ *of* $G' \otimes G'$

If $\mathbb{H}$ *and* $\mathbb{G}$ *are reflexive Banach spaces, the process* $Q\,[Y^*(.) \otimes Y^*(.)]$ *is a* $(G' \hat{\otimes}_1 G')'$*-valued process.*

In this case, the total variation r of z (considered as a $(G' \hat{\otimes}_1 G')'$*-valued measure) is such that* $dr = ||Q\,[Y^*(.) \otimes Y^*(.)]\,||_{(G' \hat{\otimes}_1 G')'}\, dv$

Then, this total variation is σ-finite.

<u>Proof</u>

1°) The norm of $\int Y.d\tilde{M}$ considered as an element of $\mathcal{L}(\mathbb{G}', \mathcal{M})$ is equal to

$$||\int Y.dM||_{\mathcal{L}(\mathbb{G}', \mathcal{M})} = \sup_{g \in \mathbb{G}', ||g|| \le 1} N_g(Y)$$

$$= ||Y||_b$$ then the mapping $Y \rightsquigarrow \int Y.d\tilde{M}$ is an isometry.

2°) Let Y be an element of $\mathcal{E}$ with $Y = \sum_{i \in I} u_i . 1_{A(i)}$ where the sets $(A(i))_{i \in I}$ are pairwise disjoint. Let i be an element of I and let B be an element of $\mathcal{R}$ contained in A(i) with $B = F \times]s,t]$. For each element (g_1, g_2) of $(\mathbb{G}' \times \mathbb{G}')$, we have :

$$(z(B))(g_1 \otimes g_2) = E\,\{1_F . [(\tilde{M}_t \circ u_i^* \circ g_1)(\tilde{M}_t \circ u_i^* \circ g_2) - (\tilde{M}_s \circ u_i^* \circ g_1)(\tilde{M}_s \circ u_i^* \circ g_2)]\}$$

$$= \int_B Q\,[Y^*(g_1) \otimes Y^*(g_2)]\, dv$$

Then the same equality is true ...
element Y of $\bar{\mathcal{E}}^b$ by linearity ...

3°) For each (t,ω) ...
mapping from $\mathbb{G}'$ with the topolo...
$\sigma(\mathbb{H}', \mathbb{H})$.

If $\mathbb{H}$ and $\mathbb{G}$ are ref...
$Y^*(t,\omega)$ is a continuous linear ...
into $\mathbb{H}'$ with its strong topolo...

Then, $Q\,[Y^*(.) \otimes$ ...
$(\mathbb{G}' \hat{\otimes}_1 \mathbb{G}')'$, weakly predictable ...

<u>V - MORE WHEN $\mathbb{H}$ AND $\mathbb{G}$ ARE HILBE...</u>

V-1 <u>Introduction</u>

If $\mathbb{H}$ is a separabl...
be a little simplified ; actua...
lemma IV-4, it is convenient to...
the orthogonal projector on the...

Moreover, we have ...

V-2 We consider t...
suppose that $\mathbb{H}$ is a separable ...
in the cône of positive element...
$Q^{1/2}$, with values in the set ...
such that, for each $(t,\omega,h) \in$ ...

$||Q_t^{1/2} \circ h||^2_{\mathbb{H}}(\omega) =$ ...

Then we have :

$N_g(Y) = \int ||Q^{1/2}$ ...

If $\tilde{M}$ is a cylindrical brownian ...
(t,ω,h) of $(T \times \Omega \times \mathbb{H})$, $[Q_t^{1/2}$ ...

V-3 Condition to obtain a genuine process

In the preceding parts, we have supposed that $\vec{M}$ is an element of $\mathcal{L}(H', \mathcal{M})$.

If H is a Hilbert space and if $\vec{M}$ is an element of $\mathcal{L}_2(H', \mathcal{M})$ (cf. I-4 above), there exists a genuine process M, with values in H, such that, for each element h of H, $\vec{M}(h) = \langle M,h \rangle$. In this case, the quadratic Doléans's measure of M takes its values in $(H \otimes_\varepsilon H)' = H \hat{\otimes} H$ which identifies itself as such, as a subspace of $(H \hat{\otimes} H)'$. Moreover, (cf. [16]), Q takes its values in $H \hat{\otimes} H$ and is strongly predictable.

If $\vec{M}$ is an element of $\mathcal{L}(H', \mathcal{M})$ and if H and G are Hilbert spaces, it is interesting to obtain a sufficient condition on Y such that $\vec{Z} = \int Y.d\vec{M}$ is an element of $\mathcal{L}_2(H', \mathcal{M})$: in this case, there exists a genuine process Z associated to $\vec{Z}$ as above.

The following theorem gives such a sufficient condition.

V-4 Theorem

We consider the hypotheses given in IV-1 and we suppose that H and G are separable Hilbert spaces.

Let Y be a $\mathcal{L}(H, G_\sigma)$-valued (cf. I-5) process weakly predictable (cf. the end of IV-1).

a) *The process $||Y \circ Q^{1/2}||_{H.S.}$ is a real predictable process.*

b) *Then, we can define :*

$$N_2(Y) = \{ \int_{T\times\Omega} ||Y \circ Q^{1/2}||^2_{H.S} \, . \, dv \}^{1/2} \leq +\infty$$

$$\bar{\mathcal{E}}^2 = \{ Y : Y \in \bar{\mathcal{E}}^b , N_2(Y) < +\infty \}$$

c) *The mapping $Y \rightsquigarrow N_2(Y)$ is an hilbertian semi-norm on $\bar{\mathcal{E}}^2$ associated with the positive bilinear form defined by, if $(g_k)_{k\geq 0}$ is an orthonormal basis of G :*

$$(Y_1 , Y_2) \rightsquigarrow \int_{\Omega\times T} \{ \sum_{k=0}^{\infty} Q [Y_1^*(g_k) \otimes Y_2^*(g_k)] \}.dv$$

d) *The mapping $Y \rightsquigarrow \int Y.$ [...] $\mathcal{L}_2(G, \mathcal{M})$; then, if [...] process Z, with value[...]*

$$\langle Z,g \rangle = \vec{Z}(g) \ldots$$

e) *Moreover, if Y is an [...] z of $\vec{Z} = \int Y.d\vec{M}$ is t[...]*

$$z(g_1 \otimes g_2) = \ldots$$

The $(G \hat{\otimes} G)$-valued "n[...] to the $(H \hat{\otimes} H)'$-value[...]

$$\langle z \rangle_t = \int_0^t d \ldots$$

Proof

a) Let $(g_k)_{k\geq 0}$ an orth[...]

$$(||Y \circ Q^{1/2}||\ldots$$

then $||Y \circ Q^{1/2}||_{H.S}$ [...] process (finite or in[...]

b) For each element g of [...] vector space of $\mathcal{L}(H,$ [...] end of IV-1) processe[...]

c) is evident. If $\tilde{Q}$ is c[...] considered positive l[...]

$$(Y_1, Y_2) \rightsquigarrow \ldots$$

d) Let Y be an element of $\mathcal{E}$ with $Y = \sum_{i \in I} u_i . 1_{A(i)}$ and, $\forall\, i \in I$, $A(i) = F(i) \times]s(i),t(i)]$ (the sets $(A(i))_{i \in I}$ being pairwise disjoint) ; let $(g_n)_{n \geq o}$ an orthonormal basis in $\mathbb{G}$. The square of the norm of $\int Y.dM$ in $\mathcal{L}_2(G, \mathcal{H})$ is equal to :

$$\left(\left\| \int Y.d\tilde{M} \right\|_{\mathcal{L}_2(\mathbb{G},\mathcal{H})} \right)^2 = \left(\left\| \int Y.d\tilde{M} \right\|_{H.S.} \right)^2$$

$$= \sum_{n=o}^{\infty} \left\| \left(\int Y.d\tilde{M} \right)(g_n) \right\|^2_{\mathcal{H}}$$

$$= \sum_{n=o}^{\infty} \left\{ \sum_{i \in I} E \left\{ 1_{F(i)} \left[(\tilde{M}ou_i^* og_n)_{t(i)} - (\tilde{M}ou_i^* og_n)_{s(i)} \right]^2 \right\} \right\}$$

$$= \sum_{i \in I} \int_{F(i)\times]s(i),t(i)]} \left\{ \sum_{n \geq o} \left\| Q^{1/2} \circ u_i^*(g_n) \right\|^2_{\mathbb{H}} \right\} dv$$

$$= \int_{T\times\Omega} \left\| Q^{1/2} \circ Y^* \right\|^2_{H.S} . dv$$

then the mapping $Y \rightsquigarrow \int Y.d\tilde{M}$ is an isometry .

e) The proof of e) is the same as the proof of IV-10 2°) (cf. also, V-3).

B S E

LOCALIZATI

I - DOOB-MEYER DECOMPOSITION TH

I-1 Definition

Let us writte $\mathcal{A}^p_t$

Banach space. Then a family $\tilde{X}$

every t, will be called a p-pro

If for every h $\in \mathcal{B}$

gale, the process $\tilde{X}$ *will be cal*

generalization of the definitio

I-2 Doleans' meas

We extend here the

in [5] for real sub-martingale

quasi-martingales (see, for ex.,

To every process $\tilde{X}$

the additive functions $\tilde{\alpha}_X$ with

table rectangles by

(I-2-1) $\tilde{\alpha}_X( ]s,t] \times F) = E$

Such a function on

algebra $\mathcal{A}$ generated by $\mathcal{R}$. We

I-3 Definition

If the additive fu

the norm of $\mathcal{B}'$*), the process* $\tilde{X}$

a *cylindrical quasi-martingale*

This clearly gener

We have then the

I-4 Proposition

For $\tilde{X}$ to be a generalized quasi-martingale, it is necessary and sufficient that the family of real additive measures (α_X^h) associated with the real processes $(\tilde{X}(h))_{||h||\leq 1}$ be of bounded variation, and that the set of those variations $|\alpha_X^h|$ has a supremum in the ordered set of bounded positive measures.

Proof

This comes from the fact that the total variation of $\vec{\alpha}_X$ can be approximated by sums of the type

$$\sum_i E(1_{F_i} |X_{t_i}(h_i) - X_{s_i}(h_i)|) \qquad h_i \in \mathbb{B},\ ||h_i|| \leq 1$$

while the supremum of the variations can be approximated by sums of the type

$$\sum_i |\alpha_X^{h_i}\,]s_i,t_i] \times F_i| = \sum_i |E\{1_{F_i}[X_{t_i}(h_i) - X_{s_i}(h_i)]\}|$$

It is easily seen that both supremum coincide.

I-5 Doob-Meyer decomposition theorem

Let $\tilde{X}$ be a cylindrical quasi-martingale on B (cf. I-3 above). Let $\alpha_{\tilde{X}}$ be the Doleans's measure of $\tilde{X}$ and let v be the total variation of $\alpha_{\tilde{X}}$. Let $(V_t)_{t\in T}$ be the real "natural" process associated to v. Let $(Z_t)_{t\in T}$ be the B'-valued and weak predictable process, Radon-Nikodym derivative of $\alpha_{\tilde{X}}$ with respect to v. Let $(Y_t)_{t\in T}$ the process defined by :

$$Y_t = \int_0^t Z_s(\omega)\ .\ d\,V_s(\omega)$$

This integral being a weak integral calculated "by trajectories". Let $\tilde{Y}$ be the cylindrical process associated to Y by $\tilde{Y}(h) = \langle Y,h\rangle$.

Then, $\tilde{X} - \tilde{Y}$ is a cylindrical martingale. Moreover, for each element h of B, $\langle Y,h\rangle$ is the "natural" process associated to $\tilde{X}(h)$, up to an indistinguashibility.

Then to define the stochastic
to define the stochastic inte
jectories") and with respect

Proof

This theorem is
This theorem generalizes theo

II - LOCALIZATION

II-1 Stopped cy

Let σ be a stopp
Banach space $\mathbb{B}$. Let $\tilde{Z}$ the cyl

$\forall\, h \in \mathbb{B}'$, $[\tilde{Z}(h)$

(where $[\tilde{X}(h)]_{t\wedge\sigma}$ is the rea

Then, we shall sa
at σ and we shall note $\tilde{Z}_t$ = (

II-2 Local cyli

We shall say tha
a local cylindrical quasi-mar
ce $(\sigma_n)_{n>0}$ of stopping times

(i) $\lim_{n\to\infty} P\,[\sigma$

(ii) $\forall\, n$, *the*
drical ma

It is easily see
stochastic integral, Doob-Mey
local cylindrical process as
gives a sufficient condition

II-3 Proposition

We consider a Banach space such that its dual $\mathbb{B}'$ *is separable and a cylindrical process* $\tilde{X}$ *on* $\mathbb{B}$ *such that :*

(i) *for each element h of* $\mathbb{B}'$, $\tilde{X}(h)$ *is a local square integrable real martingale,*

(ii) *for each stopping time* σ, *the mapping* $(\tilde{X}_{t \wedge \sigma})$ *from* $\mathbb{B}'$, *with its strong topology, into* $L^2(\Omega, \mathfrak{F}, P)$, *with its strong topology, is continuous.*

Then, $\tilde{X}$ *is a local 2-cylindrical martingale.*

Proof

Let a be the positive measure defined on $\Omega \times T$ by $a = P \otimes \mu$ where μ is the Lebesgue measure on T. Let $(h_n)_{n>o}$ be a sequence of elements of $\mathbb{B}'$, dense in $\mathbb{B}'$. For each integer n>o, let $(\sigma'(n,k))_{k>o}$ be an increasing sequence of stopping times such that, for each integer k>o, $a(]\sigma'(n,k), 1]) \leq 2^{-(n+k)}$ and the real process $\tilde{M}(h_n)$ stopped a $\sigma'(n,k)$ is a square integrable martingale. For each integer k, let be $\sigma(k)$ the stopping time defined by

$$\sigma(k) = \inf_{n>o} \sigma'(n,k) \text{ . We have :}$$

$$a( ]\sigma(k), 1]) \leq \sum_{n>o} a( ]\sigma'(n,k), 1]) \leq 2^{-k}$$

then the increasing sequence $(\sigma(k))_{k>o}$ satisfies the properties II-2-(i) and (ii).

III - CYLINDRICAL REGIONAL 2-DISTRIBUTION PROCESS

III-1 Definition

Let $\tilde{X}$ *be a cylindrical process on the Banach space* $\mathbb{B}$. *We shall say that* $\tilde{X}$ *is a cylindrical regional 2-distribution process if there exists a sequence* $(F(n))_{n>o}$ *of elements of* $\mathfrak{F}$ *and a sequence* $(a_n)_{n>o}$ *of* σ*-finite positive measures on* $\mathfrak{P}$ *such that*

(i) *for each element h of* $\mathbb{B}'$ *such that* $||h|| \leq 1$, *for each element A of* $\mathfrak{P}$ *and for each* real $\mathcal{E}$*-step process with* $\sup_{t,\omega} |Y_t(\omega)| \leq 1$

we have $\left(||1_{F(n)} \cdot \int_A Y.d\,\tilde{X}(h)||_{L^2(\Omega,\mathfrak{F},P)} \right)^2 \leq a_n(A)$

(ii) *for each i*

As in the real c

of a $\mathcal{L}(H, G_\sigma)$-valued process

this stochastic integral for e

Then, it is sufficient to defi

cylindrical 2-distribution pro

III-2 Definition

Let $\tilde{X}$ *be a cylin*

say that $\tilde{X}$ *is a cylindrical 2-*

real measure a on $\mathfrak{P}$ *such that*

for each element A of $\mathfrak{P}$ *and*

$\sup_{t,\omega} |Y_t(\omega)| \leq 1$ *we have* (

We remark that,

different from the definition

It is convenient for our purpo

III-3 The proces

Let $\mathbb{H}$ be a separ

normal basis in $\mathbb{H}$. Let $\tilde{X}$ be a

a σ-finite positive measure on

$(\mathbb{H} \times \mathfrak{R})$ and for each real $\mathcal{E}$-

$(|| \int_B Y.d\,\tilde{X}(h)$

For each integer n, we conside

m such that, for each real $\mathcal{E}$-

we have $(|| \int_B Y.d\,\tilde{X}(h_n) ||_{L^2}$

of J_n). Let a_n be the lower bo

derivative of a_n with respect

of the Radon-Nikodym derivati

For each element $(h \otimes h')$ of $\mathbb{H} \otimes \mathbb{H}$ with $h = \sum_{k>o} \alpha_k . h_k$ and $h' = \sum_{k>o} \alpha'_h . h_k$ and for each element ω of Ω we define :

$$Q(h \otimes h')(\omega) = \sum_{k>o} \alpha_k . \alpha'_k . S_k(\omega)$$

By hypothesis, for each integer k, $|S_k(\omega)| \leq 1$.
Then, for each element ω of Ω, $Q(.)(\omega)$ is a real linear mapping defined on $(\mathbb{H} \otimes \mathbb{H})$ and this mapping is continuous for the trace-norm on $\mathbb{H} \otimes \mathbb{H}$: then, this mapping is extendable in a real linear continuous mapping defined on $(\mathbb{H} \hat{\otimes}_1 \mathbb{H})$ that we shall note also $Q(.)(\omega)$. The process Q is a $(\mathbb{H} \hat{\otimes}_1 \mathbb{H})'$-valued process.

III.4 The stochastic integral

Then, it is easily seen that the construction of the stochastic integral given in the paragraph A-IV above can be extended in the present case. Notably, if $\mathbb{G}$ is a Hilbert space, if Y a $\mathcal{A}$-step process with values in $\mathcal{L}(\mathbb{H}, \mathbb{G}_\sigma)$, for each element A of $\mathcal{A}$, for each element g of $\mathbb{G}$ with $||g|| \leq 1$, we have

$$\left(\left|\left| \int_A Y^*(g) . d\tilde{X} \right|\right|_{L^2(\Omega, \mathcal{F}, P)} \right)^2 \leq \int_A Q\left[Y^*(g) \otimes Y^*(g)\right] . da$$

(result analoguous to that of the remark A-IV - 3).

Let Y be a $\mathcal{L}(\mathbb{H}, \mathbb{G}_\sigma)$-valued weakly predictable process such that there exists an increasing sequence $(A(n))_{n>o}$ of element of $\mathcal{P}$ with :

$$\sup_{g \in \mathbb{G}, ||g|| \leq 1} \left\{ \int_{\Omega \times T} Q\left[Y^*(g) \otimes Y^*(g)\right] . da < +\infty \right\}$$

Then the stochastic integral $\tilde{Z} = \int Y . d\tilde{X}$ can be defined as in A-IV and $\tilde{Z}$ is a cylindrical 2-distribution process.

IV - EXAMPLE

This example
a Hilbert space $\mathbb{H}$, the tota
m of $\tilde{M}$ is not necessarily fi
m considered as a measure w

Construction of the example

Let (t_n) be
space, (e_n) an orthonormal
tingale :

$$\tilde{M}_t(h) = \sum_n 1_{[t_{n+1},}$$

where (β_t) is a usual real
$(\tilde{M}_t(e_n))_{t \in \mathbb{R}^+}$ is a process
path-wise constant on $[t_n, \infty$
$L^2(\Omega, \mathcal{F}_t, P)$ with

$$E\,|\tilde{M}_t(h)|^2 \leq ||h|$$

defining a process with zer
the partition ($]t_{n+1}, t_n]$ ×
that, $\tilde{X}$ beeing the process

$$\sum_n E\,||1_\Omega(\tilde{X}_{t_n} - \tilde{X}_{t}$$

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