E. ZENHDER

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The stability problem of Hamiltonian systems, such as the celestial N-body system, has been attacked again and again over the last hundred years. Its analytical difficulties have to do with the presence of so-called small divisors. The surprising and famous break-through came within the last two decades through the work of Kolmogorov, Arnold and Moser, known under the apocryph KAM - theory. (All the references can be found in a recent survey on this subject [1].) The main goal of this lecture is to report on some of this well-known progress and on some more recent results.

In order to outline the main ideas underlying the KAM - theory I start from a very special situation, a so-called integrable system. It is given by an exact Hamiltonian vectorfield $v_{h_0} = (\partial_y h_0)\,\partial_x - (\partial_x h_0)\,\partial_y$ on the symplectic manifold $M^{2n} = T^n \times J$, $J \subset \mathbb{R}^n$ an open cube, where we assume the Hamiltonfunction h_0 to depend only on y : $h_0 = h_0(y)$. The Hamiltonian equations are then basically linear and the flow is explicitly given for all times $\phi_t(x,y) = (x + t\,\partial_y h_0(y), y)$. So, geometrically, the manifold M^{2n} is foliated into an n-parameter family of tori $T_y := T^n \times \{y\}$, $y \in J$, which are invariant under the flow ϕ and the restriction of the flow onto each single torus is a Kronecker flow ϕ_t / T_y : $x \mapsto x + t\omega$, with the frequencies $\omega := \partial_y h_0(y)$. We assume that these frequencies vary from torus to torus, requiring that

$$\mathrm{Det}\left(\partial_y^2 h_0(y)\right) \neq 0, \quad y \in J, \tag{1}$$

as it turns out, this postulated nonlinearity has a stabilizing effect under a perturbation. Integrable systems are, of course, very rare; as examples I just mention the two-body problem and the geodesic flow on ellipsoids with distinct axis. There are, however, many Hamiltonian systems, such as the restricted

3-body problem, or more general, the N-body problem of celestial mechanics, which are in a certain sense close to such an integrable situation. Moreover, in general, every Hamiltonian system in a very small neighbourhood of a linear stable periodic orbit is close to an integrable system. For a long time one has, therefore, been bothered by the following question: What happens to the invariant foliation under a small perturbation? Or - to put it differently - what does the flow of Hamiltonian vectorfield v_h look like for <u>all times</u>, if h is a Hamiltonfunction, periodic in $x = (x_1, \ldots, x_n)$ and close to h_0 ? Recalling the transformation theory of Hamiltonain vectorfields, one could try - following a first naive impulse - to find a symplectic mapping φ on M^{2n} which would transform our perturbed system into an integrable system:

$$h \circ \varphi(\xi, \eta) = h_1(\eta)$$

Remember that a mapping φ is called symplectic, if $\varphi^*(v_h) = v_{h \circ \varphi}$ for all functions h.
In general no such φ satisfying (2) exists. As a matter of fact, the orbit structure of such a system is very complicated. As it turns out, the difficulty lies in the fact that, due to the resonances and the Hamiltonian character, solutions of completely different behaviour over an <u>infinite</u> time interval coexist side by side clustering on each other. In order to separate conceptually and analytically solutions of different type, we introduce the concept of a subsystem of a given system. Here a pair (N, v), N a compact manifold and v a vectorfield on it, is called a subsystem of (M, V) if there is an embedding $f: N \to f(N) \subset M$ carrying v onto V restricted to $f(N)$:

$$V \circ f = Tf \circ v.$$

Geometrically this just means that $f(N)$ is an embedded submanifold which is invariant under the flow $\phi_t[Y]$, and on $f(N)$ this flow is described by the flow $\varphi_t[v]$ of the subsystem : $\phi_t \circ f = f \circ \varphi_t$, for all $t \in \mathbb{R}$. The basic idea of the KAM - theory is now to give up the hopeless task of describing all solutions of v_h by means of solving the initial value problem; instead one tries to fill up the perturbed systems with well known subsystems, which, as it happens, leads to a boundary value problem. Candidates for such systems are easily found in the unperturbed system which is foliated into a family of subsystems, all of which are Kronecker systems (T^n, ω). However, according to their frequencies, these sybsystems react completely differently to a perturbation. If the frequencies satisfy some resonance conditions, like

4) $$< \omega, \lambda > \; = 0 \; , \quad \lambda \in \mathbb{Z}^n \setminus \{0\} \; ,$$

then, the invarinat torus will break apart under a perturbation. The KAM - theory states on the other hand, that all those Kroneckersystems of the integrable system will survive under a perturbation which have frequencies ω which are not only rationally independent but also satisfy the following set of inequalities

$$| < \omega, k > | \quad \geq \quad \nu \; |k|^{-\tau}$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$, and for some $\nu > 0$ and $\tau \geq (n-1)$ where $|k| := \sum_{i=1}^{n} |k_i|$. Every solution on such a torus covers the torus densly, as is, of course, well known. In other words: tori of the integrable systems with frequencies (5) have, under a Hamiltonian perturbation, a continuation as parameterized invariant manifold on which the flow is conjugate to the unperturbed Kronecker flow. Now, since the set of real numbers for which no such ν and τ exist, is a set of measure zero, and since by assumption $y \longmapsto \partial_y h_0(y) = \omega$ is a local diffeomorphism, such frequencies occur in abbundance in our perturbed system.

In order to formulate this KAM-result more precisely and to outline the basic idea of the proof we pick one single Kronecker-system of the integrable system, say $(T^n \times \{\mathring{y}\}\,;\ \omega = \partial_y h_0(\mathring{y}))$ whose frequencies satisfy (5), we may assume $\mathring{y} = 0$. Hence, as unperturbed Hamiltonfunction h_0 in the neighbourhood of this invariant torus we consider the following function which we let now also depend on x :

6) $$h_0(x,y) = c_0 + \langle \omega, y\rangle + \frac{1}{2}\langle Q_0(x)y, y\rangle + O_3(|y|).$$

Generalizing (1), we require that $\mathrm{Det}\,([Q_0]) \neq 0$, where $[\ ]$ stands for the meanvalue of the symmetric matrixvalued function Q_0 on T^n. Let us denote for our fixed ω the set of functions which are in such a normalform (6) by N_ω ; we therefore write for the particular h_0, $h_0 \in N_\omega(c_0, Q_0)$. The KAM-statement then can be rephrased by saying that this set N_ω is in the set of all Hamiltonfunctions on M structurally stable under a subgroup of symplectic diffeomorphisms of M, that is to say, if h is any function close to $h_0 \in N_\omega(c_0, Q_0)$, then there is a symplectic diffeomorphism φ, such that

7) $$h \circ \varphi \in N_\omega(c, Q),$$

for a (c, Q) close to (c_0, Q_0) . So, in contrast to (2), not the whole system is structurally stable but only the Kronecker-system we singled out with fixed frequencies, chosen in advance. The second and higher order terms in y (6) depend also on x. Geometrically (7) says in particular that (T_0, ω) is a sub-system of v_h , 0 refers to $\eta = 0$. Indeed, using the fact that φ is symplectic we have for the flows for all $t \in \mathbb{R}$:

8) $$\phi_t[v_h] \circ \varphi \,|\, T_0 = \varphi \circ \phi_t[v_{h\circ\varphi}] \,|\, T_0 .$$

Reformulating (7) as a functional equation for functions on the torus this existence statement can presently by derived from a generalized implicit function theorem of Moser-Nash type, [4], abstracting the new methods and techniques introduced by the KAM-theory in order to deal with the small divisor problem. Such theorems are designed for nonlinear functional equations, where the linearized problem has an approximate solution losing moreover some derivatives, however, with quantitative estimates. This loss of derivatives in the problem at hand is due to the small divisors and depends on τ in (5). The basic idea of such theorems is a modifications of Newton's rapidly converging iteration algorithm in the framework of families of linear spaces. One has to replace the right inverse of the linearized which does not exist with an approximate right inverse, (the existence of an approximate right inverse is based on the algebraic structure of the problem (7)), combined with an analytic smoothing technique with very quantitative estimates; in order to catch up the loss of derivatives one has at each iteration step. So, the existence of the above subsystems is indeed very much related to excessive smoothness requirements on the Hamiltonian function h, roughly h has to have a little more than $2n$ derivatives, $2n$ the dimension of the symplectic manifold. In order to continue the Kronecker system (T,ω) we kept the frequencies fixed. It is sufficient to merely keep the frequency ratio fixed. One can make use of this extra freedom to ensure that the continued invariant torus lies on an energy surface ~~which~~ with fixed energy value, say $\varepsilon \in \mathbb{R}$. Here one singles out for fixed ω satisfying (5) and fixed ε, the following subset $N_{\omega,\varepsilon}$ of normalforms: $h \in N_{\omega,\varepsilon}(\mu, Q)$, iff:

$$h(x,y) = \varepsilon + \mu < \omega, y > + \tfrac{1}{2} < Q(x)y, y > + O_3(|y|),$$

where $\mu \neq 0$, and where the nondegeneracy assumption is different, namely one requires that $Det\,[\,Q_\omega\,] \neq 0$, where

0) $$Q_\omega(x) = \begin{pmatrix} 0 & \omega_1 & \cdots & \omega_n \\ \omega_1 & & & \\ \vdots & & Q(x) & \\ \omega_n & & & \end{pmatrix} .$$

The fact that this subset $N_{\omega,\varepsilon}$ is structurally stable means that the Kroneckersystem survives with the same frequency ratio and lies on a energy surface with fixed energy value ε.
From the normalforms (6) and (8) one easily gains some more information on the orbit structure ~~on~~ in a neighbourhood of the invariant torus T_o. Namely an easy application of an old fixed point theorem by Birkhoff yields that T_o is contained in the closure of the set of periodic orbits of v_h: $T_o \subset \overline{Per(v_h)}$. In this sense all the solutions on the invariant torus T_o, which are by definition quasi periodic, are limits of periodic solutions.

The KAM-theory of course provides not only one single invariant torus but gives a whole set of such Kroneckersystems according to different choices of ω satisfying (5). Since the complement of these frequencies is a set of measure zero, one might at first expect that the invariant tori so found ~~xxxx~~ leave out a set of measure zero only. However, this is <u>not</u> the case, since the smallness condition on the perturbation in (7) in order to continue the subsystem with a chosen frequency (5), depends heavily on ν and τ, $|h - h_o| \leq \varepsilon(\nu, \tau)$, τ measuring the loss of derivatives. Yet the following assertion can be proved, if we denote with S the set in the phase space, covered by the continued invariant tori, then for any $\varepsilon > 0$,

) $$m\,(M \setminus S) < \varepsilon\, m(M)$$

if, loosely speaking, $|h - h_o| < \delta(\varepsilon)$. Here m denotes the Lionville (Lebesque) measure on M. <u>By the same</u> token one gets the curious statement that $m\,(\overline{Per(v_h)}) > 0$.

The KAM-theory extends the above ideas to the equilibrium problem of any Hamiltonian vectorfield v_h. In what sense is a linear stable equilibrium point which I assume to be O,

(12) $$v_h(0) = 0$$

actually stable? Since the vectorfield is Hamiltonian, a necessary condition for the stability is that all the eigenvalues of the, in O, linearized system are purely imaginary; we assume them to be distinct and label them such that $\alpha_1, \ldots \alpha_n, -\alpha_1, \ldots -\alpha_n$, are all the eigenvalues. If we now exclude the following low order resonances:

13) $$\sum_{k=1}^{n} j_k \alpha_k \neq 0 \quad , 0 < |j| \leq 4,$$

then we can go close to an integrable situation. For there exists an analytic symplectic transformation such that the Hamiltonfunction h in 0 has the following Birkhoff normalform:

14) $$h(\xi,\eta) = \sum_{k=1}^{n} i\,\alpha_k p_k + \langle Qp, p\rangle + O_5$$

where $p_k = \frac{1}{2}(\xi_k^2 + \eta_k^2)$, O_5 is a power series in ξ, η containing only terms of order ≥ 5 in ξ and η.
The KAM-theory then says that if

15) $$\mathrm{Det}\,(Q) \neq 0,$$

and if h is sufficiently smooth $(h \in C^{\ell},\ \ell > 2n+2)$, then for any $\varepsilon > 0$ there is an open neighbourhood $U(\varepsilon)$ of O containing an invariant set S, consisting of Kronecker subsystems, such that $S \subset \mathrm{Per}(v_h)$ and

$$m\,\big(U(\varepsilon)\setminus S\big) < \varepsilon\, m\,\big(U(\varepsilon)\big).$$

The important point of this statement is that one has under only finitely many conditions, namely (13),(15), on the low order coefficients of the Hamiltonfunction a weakened type of stability, a kind of stability in measure; aside from a small proportion all solutions in the open small neighbourhood $U(\varepsilon)$ remain in it for all times. This is in particular not just a generic statement As one sees, if the system is not linear, that is to say, if $\mathrm{Det}\,(Q) \neq 0$, then only low order resonances are harmful. One concludes of course that a Hamiltonian flow, possessing one single such equilibrium point is certainly not ergodic. If the degree of freedom is only two, $n = 2$, then the invariant tori bound open regions in the 3-dimensional energy surface, hence the KAM-theory yields in this case a criterion for topological stability. Indeed, if (13) holds true, and if

$$Q_{11}\,\alpha_2^2 - 2Q_{12}\,\alpha_1\alpha_2 + Q_{22}\,\alpha_1^2 \neq 0,$$

then the equilibrium point is topologically stable. The condition (17) is precisely the condition we imposed in order to continue the Kroneckersystems on fixed energy surfaces (10). However, if the degree of freedom is greater than 2, $n > 2$, then the complement of the set of invariant tori on the energy surfaces is connected and examples show that the set of escaping solutions may not be empty, if the system has no positive definite integral, of course. This leads us to the still wide open problem for $n > 2$, namely, what is really going on in the complement of the Kroneckersystems, in the so-called zone of instability, where the resonances (4) play a crucial role and ~~xxdx~~ give, as is to be expected, rise to a very unstable and unpredictalbe orbit - structure which can be described by topological embeddings of Bernoulli flows, Markov flows and so on, with hyperbolic structures.

So far we have considered invariant tori of dimension precisely half of that of the phase space M^{2n} and the question arises whether one can extend such a perturbation theory to other dimensions and other subsystems. Indeed, a perturbation theory of tori of all dimensions $r < n$ has recently been established by S. Graff. By a different method this theory has been generalized [4] to the case of exact symplectic diffeomorphisms instead of Hamiltonian vectorfields. The theory, however, applies only to unstable tori, so called whiskered tori. This theory yields in particular unstable quasiperiodic solutions in every neighbourhood of a parti~~cular~~al hyperbolic equilibrium point. To outline the idea, we consider on the symplectic manifold $M^{2n} := T^r \times J^r \times \mathbb{R}^s \times \mathbb{R}^s$, $r + s = n$, the following set of normalforms $h_0 \in \widehat{N}_\omega (c_0, Q_0, \Omega_0)$:

)

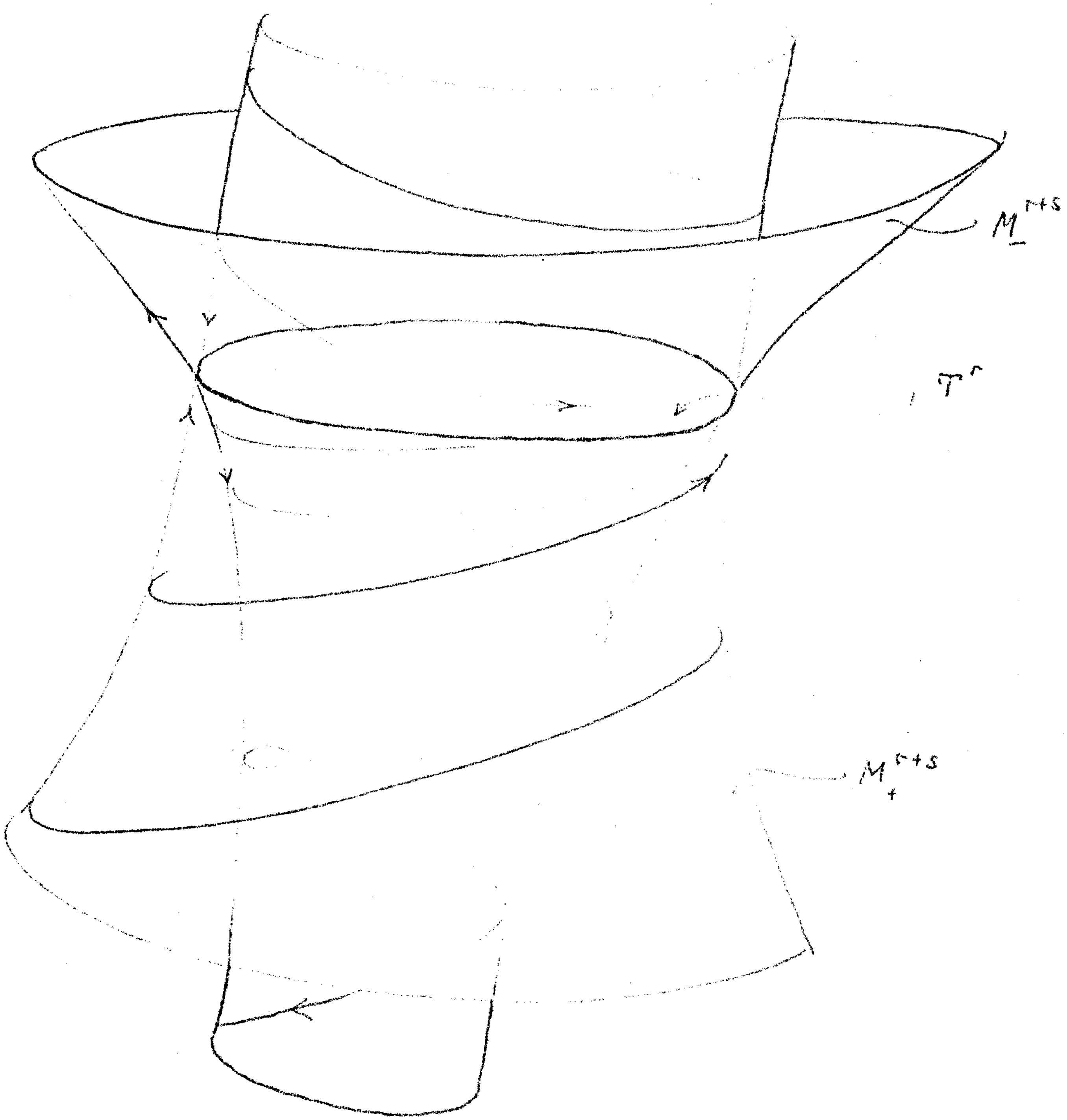
$$h_0(x,y,p,q) = c_0 + \langle \omega, y \rangle + \tfrac{1}{2} \langle Q_0(x) y, y \rangle + \langle \Omega_0(x) p, q \rangle + O_3\left(|y| + |p| + |q|\right),$$

where $\omega = (\omega_1, \dots \omega_r)$ <u>fixed</u> and satisfying conditions like (5), $\operatorname{Det} [Q_0] \neq 0$, and where Ω_0 is a $(s \times s)$-matrixvalued function on the torus T^r, such that

9)

$$\operatorname{Re} \left(\Omega_0(x) z, z \right) \geq k \, |z|^2 ,$$

for some $k > 0$ and all $z \in \mathbb{C}^s$. The condition (19) just says that the eigenvalues of the symmetric part of $\Omega_0(x)$ lie in the right half plane. Note that we do not impose any further conditions on the eigenvalues. These normalforms (18) describe symplectic hyperbolic Kronecker-subsystems illustrated by the following picture in which we do not indicate the unpleasant neutral directions. (To get the picture just drop for simplicity the O_3 terms in (18), and assume Q_0 and Ω_0 to be independent of x):


M_-^{r+s}
T^r
M_+^{r+s}

From the invariant torus T^r $(y = p = q = 0)$, on which we have a Kroneckerflow with r frequencies ω, there issue two invariant manifolds M_+^{r+s} and M_-^{r+s} consisting of solutions which approach the invariant torus T^r at an exponential rate as $t \to \pm\infty$, $T^r = M_+ \cap M_-$. In addition, an orbit on these whiskers which are Lagrangian submanifolds, not only tends to the invariant torus T^r but approximates a particular orbit on the invariant torus. If now h is real analytic (which is, from the mathematical point of view, the interesting case, requiring techniques that go beyond the ones used previously), and close to h_0, then there is a symplectic diffeomorphism φ on M^{2n} such that $h \circ \varphi \in \hat{N}_\omega(c, Q, \Omega)$. Hence the whole geometric, symplectic and analytic structure of the subsystem survives under a perturbation. The fact that the invariant whiskers are analytic is related to the fact that on the invariant torus the flow is linear. This statement is one of the main ingredients one needs to prove the following, now merely <u>generic</u> statement which sheds some light on what is going on in between the invariant tori of dimension n in M^{2n}.

Generically a linear, stable equilibrium point $(n \geq 2)$ is a cluster point of whiskered tori T^r for all dimesnions $1 \leq r < n$. Moreover, each such torus T^r lies in the <u>closure of the</u> set of unstable periodic orbits : $T^r \subset \overline{UPer(v_h)}$. [5].

Analogously, one has for the exact symplectic diffeomorphisms on M^{2n} : Generically a linear stable fixed point $(n \geq 2)$ is a cluster point of invariant whiskered tori T^r, $1 \leq r < n$, and each such hyperbolic torus T^r lies in the closure of the set of unstable periodic points of the mapping [5]. The proof is based on the study of high order resonance tori T^n on which one has $(n-r)$ independent resonance conditions (4). These statements can be considered as a first little step towards proving the conjecture that generically a linear stable equilibrium point, which, as one knows, is stable in measure, is actually topologically unstable. Here, of course, one excludes again the trivial case of definite Hamiltonian functions where one has topological stability.
A prove would follow an idea by Arnold, according to which one would construct a transition chain of such whiskered tori, intersecting

transversally and would so establish an escaping solution on the energy surface, sneaking out between all the invariant n-dimensional tori. Such a process would be very slow, as a recent estimate by Nekhoroshev [3] indicates. Roughly speaking, he proves that if

$$h(x,y) = h_0(y) + \varepsilon h_1(x,y)$$

is real analytic on $M = T^n \times J$, and if h_0 is in a precise sense generic, (the Taylorcoefficients of h_0 do not satisfy certain algebraic equations), then there are constants $C > 0$, $0 < a, b < 1$, depending on h_0 such that for <u>all</u> initial conditions $x(0)$, $y(0)$, one has the following strong estimate for the solutions

$$|y(t) - y(0)| < \varepsilon^b, \quad t \in [0, \exp(C\varepsilon^{-a})]$$

The generic conditions on h_0 exclude of course resonance strips on which the solution runs away.

CONCLUSION:

From our discussion now emerges the following concept of stability for systems close to integrable systems. Before the KAM - theory the conventional view was that in the phase space there are open regions in which the solutions remain stable and bounded, while outside of such regions the solutions behave unboundedly or escape. The KAM - theory replaced this rather crude picture with the following intricate modell: In the phase space one finds complicated cantor sets which can be compared to a sponge. The solid part of this sponge, of positive measure, consists of Kronecker subsystems on which the solutions are bounded for all times, while the solutions lying in the tiny holes of the sponge behave in an unpredictable and unstable way. Here, in this zone one would expect to find whiskered tori and Anosov subsystems which for would give rise to a very slow diffusion through the sponge.

Such a strange modell emerges, of course, only, if one studies the motion for <u>all times</u> and not merely for a reasonable but finite time interval.

Still open remains the problem of what happens if the perturbation gets large. Here numerical calculations indicate that the KAM - results hold for perturbations which are several magnitudes larger. This phenomenon is not understood. It may be that the KAM - theory, a highly sophisticated existence theory, is, in an way, too fine and too precise to give an insight into this question. One expects here that under larger perturbations the solid part of the sponge will shrink, while the region of instability blows up, showing up regions of positive measure on which the system behaves like a Bernoulli system. Also not yet clear is the precise and crucial role of the smoothness requirements one needs to establish the stable phenomena in this picture. Also the dependence of this modell on the number of degrees of freedom has not yet been investigated.

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