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A COMMON GENERALIZATION OF TOPOLOGICAL AND MEASURE-THEORETIC ENTROPY

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Nowadays ergodic theory is split into two branches: measure-theoretic and topological, according to the methods used.
In both branches there are similar results, proved using similar ideas. Therefore it is natural to look for a common generalization.
For theorems connecting spectral and mixing properties of dynamical systems Nagel [2], [3], [4] has found an appropriate generalization in terms of Banach lattices: an abstract dynamical system is a triple (E,u,T), where E is a Banach lattice with quasi-interior point $u \in E_+$ and $T: E \to E$ is a lattice homomorphis satisfying $Tu = u$ (this definition is slightly different from that given in [2]).
For theorems concerning entropy and related questions, other mathematical structures are used: If one looks into the entropy sections of Walters' book [8], for example, the measure-theoretical and topological proofs of many analogous theorems look very similar and these proofs are based on lattice methods.
Therefore I have defined entropy for a dynamical lattice (see definition 1.1.).
This definition has two advantages:

1) Given an abstract dynamical system (E,u,T), the lattice of all closed ideals in E yields a dynamical lattice (see 1.3.), whose entropy reduces to the usual entropy in both the measur theoretic and the topological case (see 1.4.).

2) In this definition of entropy it is necessary to define the entropy for not necessarily disjoint covers, even in the measure-theoretic case. But this fact allows an easy proof of Goodwyn's theorem [1] by means of a generalized version of the Kolmogoroff-Sinai theorem (see 3.4.).

In the following I want to give the basic definitions and theorems for the entropy of dynamical lattices and to sketch the proof of Goodwyn's theorem.

1. dynamical lattices

1.1.Definition: A dynamical lattice is a triple (V,m,f), where

V is a distributive lattice with 0 and 1,

$m:V \to \mathbb{R}_+$ satisfies $m(0)=0$ and:

$m(a)=0 \Rightarrow m(a \vee b)=m(b)$ for every $a,b \in V$,

$f:V \to V$ satisfies $f(0)=0$, $f(1)=1$ and:

$m(a)=0 \Rightarrow m(f(a))=0$ for every $a \in V$.

1.2.Definition: Two dynamical laticces (V,m,f) and (V',m',f') are called isomorphic, if there is a lattice isomorphism $\phi:V \to V'$ satisfying $\phi \circ f = f' \circ \phi$ and $m' \circ \phi = m$.

1.3.Definition: Let (E,u,T) be an abstract dynamical system.

Let V be the lattice of all closed (lattice-)ideals in E (see[6])

$$m:\begin{cases} V \to \mathbb{R}_+ \\ I \mapsto \sup\{\|x\|: x \in I \cap [0,u]\} \end{cases}, \quad f:\begin{cases} V \to V \\ I \mapsto \langle T(I) \rangle \end{cases};$$

where $\langle A \rangle$ denotes the closed ideal generated by A.

Then (V,m,f) is called the dynamical lattice of closed ideals associated to (E,u,T).

By the entropy of (E,u,T) we mean the entropy of the associated dynamical lattice of closed ideals.

<u>1.4.</u>: In the <u>toplogical case</u> we have a <u>topological dynamical system</u> (X,φ), i.e. a compact Hausdorff space X and a continu mapping $\varphi:X \to X$. Here we set $E:=C(X)$, $u=1$ and $T(f):=f\circ\varphi$. For this abstract dynamical system we get (using 1.3.) $V=\{\text{open sets in } X\}$, $m(a)=m_1(a):=\begin{smallmatrix}0 \text{ if } a=0\\ 1 \text{ if } a\neq 0\end{smallmatrix}$ and $f=\varphi^{-1}$.
In the <u>measure-theoretic case</u> we have a <u>dynamical system</u> (X,Σ,μ,φ), i.e. a probability space (X,Σ,μ) and a measurable measure-preserving mapping $\varphi:X\to X$. Here we set $E:=L^1(X,\Sigma,\mu)$ and again $u=1$, $T(f):=f\circ\varphi$. For this abstract dynamical syst we get V isomorphic to the measure-algebra $\Sigma/\mathcal{N}$ ($\mathcal{N}$ denoting t μ-nullsets), $m=\mu$ and $f=\varphi^{-1}$.

2. <u>entropy</u>

<u>2.1.Definition</u>: Let (V,m,f) be a dynamical lattice.

1) A finite subset α of V is called a <u>cover</u>, if $\sup\alpha=1$.
2) The set $\tilde{V}$ of all covers is ordered by:
   $\alpha\leq\beta$ (β is a <u>refinement</u> of α) if and only if for every $b\in$ there is an $a\in\alpha$ such that $b\leq a$.
3) $\alpha\vee\beta:=\{a\wedge b : a\in\alpha, b\in\beta\}$ and $\alpha^n:=\bigvee_{i=0}^{n-1} f^i(\alpha)$.
4) Let α be a cover and $k:=\sum_{a\in\alpha} m(a)$, then we set
   $h^*(\alpha):=-\sum_{a\in\alpha}\frac{m(a)}{k}\log\frac{m(a)}{k}$.
5) $\hat{h}(\alpha):=\sup\{h^*(\beta): \beta\geq\alpha, N(\beta)\leq N(\alpha)\}$, $N(\alpha)$ denoting the number of elements $a\in\alpha$ such that $m(a)\neq 0$.
6) $h(\alpha):=\inf\{\sum_{i=1}^{n}\hat{h}(\beta_i): \bigvee_{i=1}^{n}\beta_i\geq\alpha,\ n\in\mathbb{N}\}$.
7) $h(f,\alpha):=\underline{\lim}\ h(\alpha^n)/n$, $H(f,\alpha):=\overline{\lim}\ h(\alpha^n)/n$.
8) $h(V,m,f):=\sup\{h(f,\alpha): \alpha\in\tilde{V}\}$, $H(V,m,f):=\sup\{H(f,\alpha) : \alpha\in\tilde{V}$

$h(V,m,f)$ is called the <u>entropy</u> of (V,m,f).

2.2. Remarks: a) It can be proved, that in many cases $h(f,\alpha) = H(f,\alpha)$ holds for every cover α [5].

b) Step 5 of the definition should be explained:
In the measure-theoretic case we want to get the measure entropy, therefore it should be sufficient to consider disjoint covers. Now if V is a Boolean algebra and α any cover, there is a disjoint refinement β of α with $N(\beta) \leq N(\alpha)$, but if α is already disjoint, then α is the only such refinement. Therefore in step 6 we have
$h(\alpha) = \inf\{\sum_{i=1}^{n} \hat{h}(\beta_i) : \bigvee_{i=1}^{n} \beta_i \succ \alpha,\ \beta_i \text{ disjoint},\ n \in \mathbb{N}\}$ and
$\hat{h}(\beta) = h^*(\beta)$ for disjoint β.

c) In this general context the entropy still has many of the well-known properties of the usual entropies:

2.3. Theorem [5]: a) If (V,m,f) and (V',m',f') are isomorphic, they have the same entropy.

b) Let (V,m,f) be a dynamical lattice, where f is a lattice isomorphism such that $m \circ f = m$, then $h(V,m,f) = H(V,m,f)$ and $h(V,m,f^k) = |k| \cdot h(V,m,f)$ for $k \in \mathbb{Z}$

c) In the topological case (see 1.4.) $h(V,m,f)$ is equal to the topological entropy.

d) In the measure-theoretic case $h(V,m,f)$ is equal to the measure entropy.

3. generators

Let me define pseudometrics on V and $\tilde{V}$:

3.1. Definition: a) Given $a,b \in V$ let $\delta(a,b) := \inf\{m(d) : d \vee a = d \vee b\}$.

b) Given $\alpha, \beta \in \tilde{V}$ with $|\alpha| \leq |\beta|$ (say) let $d(\alpha,\beta) = d(\beta,\alpha) :=$
$= \inf\{\sum_{a \in \alpha} \delta(a,\pi(a)) + \sum_{b \notin \pi(\alpha)} m(b) : \pi : \alpha \to \beta \text{ injective}\}$.

3.2.Definition: Given two covers α,β I shall write $\alpha \underset{\varepsilon}{\lessapprox} \beta$, if there is a cover α' satisfying $d(\alpha,\alpha') < \varepsilon$ and $\alpha' \preceq \beta$.

3.3.Definition: A cover β is called a generator, if for every cover α and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $\alpha \underset{\varepsilon}{\lessapprox} \beta^n$.
A subset W of V is called generating, if for every cover α and every $\varepsilon > 0$ there is a cover $\beta \subseteq W$ such that $\alpha \underset{\varepsilon}{\lessapprox} \beta$.

With these notions we can prove a generalized version of the well-known Kolmogoroff-Sinai theorem (along the lines of [7], see especially Lemma 5.8) [5].

3.4.Theorem: Let (V,m,f) be a dynamical lattice, V a Boolean algebra, m monotone ($a \leq b \Rightarrow m(a) \leq m(b)$) and subadditive ($m(a \vee b) \leq m(a)+m(b)$) and $m \circ f = m$, then

a) $h(f,\beta) = h(V,m,f)$ for every generator β.

b) $h(V,m,f) = \sup\{h(f,\beta): \beta \in \tilde{V}, \beta \subseteq W\}$ for every generating $W \subseteq V$.

4. Goodwyns theorem

4.1.: Finally I will sketch a new proof of Goodwyn's theorem [1]: Given a topological dynamical system (X,φ) and a φ-invariant regular Borel-measure μ on X, the topological entropy h_t of φ is $\geq$ the measure entropy h_μ of φ with respect to μ.

According to 2.3. the topological entropy h_t is $h(V,m_1,f)$, where $V = \{\text{open sets in } X\}$ and $f = \varphi^{-1}$, and the measure entropy is $h(\Sigma,\mu,f)$ where Σ denotes the σ-algebra of Borel-sets. Since μ is regular, V is a generating subset of Σ. Therefore we have (3.4.b):

$$(*)\quad h(\Sigma,\mu,f) = \sup\{h(f,\alpha): \alpha \in \tilde{\Sigma}, \alpha \subseteq V\} = \sup\{h(f,\alpha): \alpha \text{ open cover of } X\}$$

If α is an open cover of X, clearly $h^*(\alpha)$ computed for (V,m_1,f) is $\log N(\alpha)$, which is $\geq h^*(\alpha)$ computed for (Σ,μ,f).
Therefore $h(f,\alpha)$ computed for (V,m_1,f) is $\geq h(f,\alpha)$ computed for (Σ,μ,f) (according to definition 2.1.).
So we can continue (*):

$$h(\Sigma,\mu,f) = \sup\{h(f,\alpha):\alpha\in\tilde{\Sigma},\ \alpha\leq V\} \leq \sup\{h(f,\alpha):\alpha\in\tilde{V}\} = h(V,m_1,f).$$

With the same ideas the following generalization of Goodwyn's theorem can be proved [5]:

<u>4.2.Theorem</u>: Let X be a compact Hausdorff space and (E,u,T) an abstract dynamical system satisfying:

a) C(X) is a dense T-invariant sublattice of E.

b)The norm of E is order-continuous.

c) u is the function $1\in C(X)$.

d) T is an isometry.

Then $T|_{C(X)}$ corresponds to a homeomorphism $\varphi: X\to X$ by means of $Tf = f\circ\varphi$, and the topological entropy of φ is $\geq$ the entropy of (E,u,T).

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