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SPECTRUM OF MEASURABLE FLOWS

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Let $\{T_t\}$ be a measurable flow (one parameter group of measure preserving transformations satisfying standard measurability condition) on the (Lebesgue) probability space (X,Σ,μ). Denote by H the generator of the group $U_t = e^{-itH}$ of induced unitaries and let $sp(H)$ be its spectrum.

Let $P(x)$ be the period of $x \in X$ (if x is aperiodic, $P(x) = \infty$). We have the

Proposition.

$$\text{If } ||P||_\infty = \infty \text{ , sp } (H) = \mathbb{R}$$

Corollary. If $\{T_t\}$ has an aperiodic component and in particular, if T^t is ergodic and $T^t = \mathbb{1}$ only if $t = 0$, then $sp\ (H) = \mathbb{R}$.

Proof: We first give an a argument which only establishes the latter half of the corollary. The argument, which is a generalizatic of the standard proof that the discrete spectrum of an ergodic transformation forms a group, may be applied to prove the appropriate results for automorphisms of C^*-algebras.

Let $sp\ \psi$ denote the support of $\psi \in L^2(X,\mu)$ in the spectral representation determined by H . Then

Lemma 1. If $\psi, \phi \in L^{\infty}(X,\mu)$ then $sp(\psi\phi) \subset sp\psi + sp\phi$.

Proof. For $\psi \in L^2(X,\mu)$ let

$$\hat{\psi}^{\tau}(E) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iEt}\, e^{-\frac{\tau}{2}t^2}(U_t\psi)\, dt \text{ (Bochner Integral)}$$

By the spectral theorem :

$$\hat{\psi}^{\tau}(E) = \delta_E^{\tau}(H)\psi$$

where

$$\delta_E^{\tau}(x) = \frac{1}{\sqrt{2\pi\tau}}\, e^{-(\chi-E)^2/2\tau} .$$

We claim that if $\lim_{t\to 0} ||\hat{\psi}^{\tau}(E)||_2 = 0$ for all E in an open set θ, then $sp(\psi)$ and θ are disjoint. [By the spectral theorem :

$$\int_{\theta'} dE\, ||\hat{\psi}^{\tau}(E)||_2^2 = \frac{1}{2\pi\tau}\int_{\theta'} dE \int d\mu_{\psi}(E')\, e^{-(E'-E)^2/\tau}$$
$$= \frac{1}{2\pi\tau}\int d\mu_{\psi}(E')\int_{\theta'} dE\, e^{-\frac{(E'-E)^2}{\tau}} \sim \frac{1}{\sqrt{4\pi\tau}}\mu_{\psi}(\theta') .$$

Therefore if $||\hat{\psi}^{\tau}(E)||_2 \to 0$ uniformly on θ' , $\mu_{\psi}(\theta') = 0$. Hence, by Egoroff's theorem, $\mu_{\psi}(\theta) = 0$ (here μ_{ψ} is the spectral measure belonging to ψ)] .

Now $\widehat{(\psi\phi)}^{\tau}(E_0) = \hat{\psi}^{\tau/2} * \hat{\phi}^{\tau/2}(E_0) = \int \hat{\psi}^{\tau/2}(E_0-E)\, \hat{\phi}^{\tau/2}(E)\, dE$,

and

$$||\widehat{(\psi\phi)}^{\tau}(E_0)||_2 \leq \int ||\hat{\psi}^{\tau/2}(E_0-E)\, \hat{\phi}^{\tau/2}(E)||_2\, dE \leq \frac{\text{const}}{\tau^2}\, e^{\delta^2/2\tau}\int e^{-\text{const}'E^2} dE \underset{t\to 0}{\longrightarrow} 0$$

if $E_0 \notin sp\psi + sp\phi$, where

$$\delta = \inf_E \text{dist}(E, sp\phi) \vee \text{dist}(E_0-E, sp\psi) = \frac{1}{2}\text{dist}(E_0, sp\psi + sp\phi).$$

In the above we have used these estimates :

$$||\hat{\psi}^{\tau}(E)||_{\infty} \leq \int\left(\frac{1}{2\pi}\, e^{-\tau\, \tau^2/2} dt\right) ||\psi||_{\infty} = \frac{1}{\sqrt{2\pi\tau}}||\psi||_{\infty}$$

$$||\hat{\psi}^{\tau}(E)||_2 \leq \frac{1}{\sqrt{2\pi\tau}}||\psi||_2\, e^{-[\text{dist}(E, sp\psi)]^2/2\tau}$$

(spectral theorem) and

$$||\psi\phi||_2 \leqslant ||\psi||_\infty \, ||\phi||_\infty \, ||\psi||_2 \, ||\phi||_2$$

Theorem.

Let T_t *be ergodic. Then* $sp(H)$ *forms a group.*

Proof. Let $C\psi(x) = \bar{\psi}(x)$. Then $C\,U_t = U_t\,C$, i.e. $CH = -HC$. Therefore if $\lambda \in sp(H)$, $-\lambda \in sp(H)$. Let

$$E_1 \in \sigma(H) \; , \; sp\,\psi \quad (E_1 - \tfrac{1}{n},\, E_1 + \tfrac{1}{n}) \, , ||\psi||_2 = 1$$

$$E_2 \in \sigma(H) \; , \; sp\,\phi \quad (E_2 - \tfrac{1}{n},\, E_2 + \tfrac{1}{n}) \, , ||\phi||_2 = 1 \quad .$$

By ergodicity,

$$\frac{1}{T}\int_0^T dt \int d\mu|\psi||\phi(T_t x)| \xrightarrow[T \to \infty]{} (\int d\mu|\psi|)(\, d|\phi|) > 0 \; ,$$

so we may assume that $\psi\phi \neq 0$. By lemma 1, $\psi\phi$ is therefore an approximate eigenvector of H of eigenvalue $E_1 + E_2$, so that $E_1 + E_2 \in sp(H)$.

Now if $sp(H)$ is discrete, it must be of the form

$$\{n\,E_0 \mid n = \ldots,-2,-1,0,1,2,\ldots\}$$

and $T_\tau = \mathbb{1}$ for $\tau = 2\pi/E_0$. If $sp(H)$ is not discrete, it must be $\mathbb{R}$, since it is closed.

The proposition itself should follow from the decomposition of T_t into its ergodic components. We instead prove it directly, using an entirely different approach.

Assume first that T_t has a nontrivial aperiodic component. This may be represented as a special flow [], i.e. as a flow built under a function $f \geqslant \delta > 0$ and aperiodic automorphism (X_0,μ_0,T_0). (X,μ) is identified with the "region under the function f on (X_0,μ_0)" with measure given by $\mu \times$ Lebesgue measure.

For $t > 0$, $T_t(x,s) = (x,s+t)$ for $s+t \leqslant f(x)$ and

$$T_t(x,s) = (T_0 x, s+t-f(x)) \text{ otherwise.}$$

Since T_0 is aperiodic, for any $n > 0$ there exists a set A with $\mu_0(A) > 0$ and $A, T_0(A), \ldots T_0^{n-1}(A)$ disjoint [ ] . Therefore there exist in X "rectangular tubes" of arbitrarily large length in which T_t induces a uniform translation. Consequently we may construct approximate eigenvectors (approximate plane waves e^{its}) corresponding to any $\lambda \in \mathbb{R}$. Finally, if T_t has no aperiodic component, we may construct approximate eigenfunctions is above by employing the representation of T_t , as the flow built under the function $\pi(x)$ and the identity automorphism. Here $\pi(x)$ is the period of $x \in X_0$. Since $||P(x)||_\infty = \infty$, we obviously have "tubes" of arbitrarily large length on which the motion is a uniform translation.