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Integration with Respect to Processes of Linear Functionals

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INTEGRATION WITH RESPECT TO PROCESSES
OF LINEAR FUNCTIONALS
by M. METIVIER

SUMMARY
The notion of "intégrale stochastique radonifiante" as introduced by B. GAVEAU in C.R.A.S. Paris t. 276 - 1973 for the cylindrical brownian motion is studied in the general context of generalized quasi-martingales. This gives rise to a theory in which the notions and formulas of the Kunita-Watanabe theory for real square integrable martingales have a natural extension.

INTRODUCTION
For the purpose of studying stochastic partial differential equations it is worth considering perturbations which are "white noise in time and in space". The mathematical expression of such an object is a cylindrical measure, or a linear random functional as studied for example in \[1\] or \[s\]. Considering the special case of "cylindrical brownian motion", several authors defined a stochastic integral with respect to such a stochastic process (cf. for example \[7\] and \[7\]). In \[7\] the operator valued processes, which are integrated with respect to the cylindrical brownian motion, are such that the integral process is a (Hilbert valued) Martingale in an ordinary sense; and in \[11\], a Girsanov-theorem is obtained in such a situation.

The purpose of studying stochastic partial differential equations is to develop a theory of stochastic integration with respect to square integrable martingales, which extends in a natural way the classical integral with respect to square integrable martingales as studied in \[10\], \[14\], \[18\].

After the necessary definitions we give in §3 a Doob-Meyer type theorem for generalized quasi-martingales.

In §4 the stochastic integral is defined, and an isometry formula as in \[16\] is given. The class of operator-valued processes which are integrated is wide. The values of the process (as in \[16\]) are necessarily continuous operators. But if, on the contrary, Hilbert-Schmidt valued the integral process is an ordinary valued martingale.

In §5 an Ito's formula is given for those generalized quasi-martingale. But in this first draft of the work, the formula is stated only in the case of continuous quasi-martingales.
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LINEAR STOCHASTIC FUNCTIONALS - NOTATIONS

In all this paper we will assume that $T$ is a closed or open interval in $\mathbb{R}^+$, a basic probability space $(\Omega, \mathcal{F}, P)$ and an increasing family $\mathcal{F}_t$ of sub-$\sigma$-algebras of $\mathcal{F}$ with the usual following completion assumption is $P$-complete and all the $P$-null sets in $\mathcal{F}_t$ for every $t$.

Let $\mathcal{S}$ be the set of "predictable rectangles" $[s,t] \times \mathcal{G}_t$, where $s \leq t$, $s,t \in T$ and $\mathcal{G}_t \subseteq \mathcal{F}_t$.

$\mathcal{S}$ will be the algebra of subsets of $T \times \Omega$ generated by $\mathcal{S}$.

The algebra generated by $\mathcal{S}$, i.e.: the $\sigma$-algebra of predictable subsets of $T \times \Omega$.

$B, C, K$ will denote real Hilbert spaces, all of them separable (in our context this is no restriction). The scalar product in those spaces will be denoted by $\langle , \rangle$, $\langle ., . \rangle$ ... or simply $\langle ., . \rangle$ if there is no possible confusion. The norm will be written: $\| \cdot \|_B$.

If $\mathcal{H}$ is a Banach space, then $\mathcal{H}'$ will denote the algebraic tensor product on $\mathcal{H} \otimes C$ is a Hilbert space with scalar product an extension of $\langle x \otimes y, x \otimes y \rangle = \langle x, x \rangle_B \langle y, y \rangle_C$.

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The three topologies induced by the three considered topological tensor product on $\mathcal{H} \otimes C$ are comparable and we have the canonical continuous injection.

Moreover an important theorem of Schatten (see for example [19] chap. 48) says that the function $(x \otimes y, x' \otimes y') \mapsto \langle x, x' \rangle \langle y, y' \rangle$ can be extended as a continuous bilinear form on $(\mathcal{H} \otimes C \times H \otimes C)$ and that for $\mathcal{H}' \otimes C$ this duality $\mathcal{H} \otimes C$ identifies itself with the Banach dual of $\mathcal{H} \otimes C$.

$I-3$ There is a unique injective linear mapping of $\mathcal{H} \otimes C$ into the vector space of linear operators with finite range from $\mathcal{H}$ into $C$, associating to $x \otimes y$ the operator $h \mapsto \langle h, y \rangle$. This linear mapping has the following extensions which are:

1°) isometry from $\mathcal{H} \otimes C$ onto $\mathcal{L}_1(\mathcal{H} \otimes C)$, the Banach space of nuclear operators from $\mathcal{H}$ into $C$ with the trace norm.

2°) isometry from $\mathcal{H} \otimes C$ onto $\mathcal{L}_2(\mathcal{H} \otimes C)$, the Banach space of Hilbert-Schmidt operators from $\mathcal{H}$ into $C$ with the Hilbert-Schmidt scalar product.

3°) isometry from $\mathcal{H} \otimes C$ onto $\mathcal{C}(\mathcal{H} \otimes C)$, the Banach space of compact operators with the usual norm.

In as much $x \otimes y$ can be identified with a bilinear form on $(\mathcal{H} \times C)$ or a continuous linear form on $L^2(\mathcal{H}; C)$, there is also a continuous linear extension of the preceding linear mapping into an isometry from $(\mathcal{H} \otimes C)'$ onto $\mathcal{L}^\infty(\mathcal{H}; C)$, the Banach space of bounded operators from $\mathcal{H}$ into $C$ with the usual norm. (This isometry is in fact the one which associates to a bounded bilinear $b$ on $(\mathcal{H} \times C)$ the bounded linear operator $b$ in $\mathcal{L}^\infty(\mathcal{H}; C)$ such that $\langle b(x), y \rangle = b(x,y)$.)

$I-4$ Random variables with values in $\mathcal{H}$ will be strongly measurable mappings from $\Omega$ into $\mathcal{H}$. If such a random variable $X$ has the property $\mathbb{E}(\|X\|_\mathcal{H}^2) < \infty$, then $\omega \mapsto X(\omega) \otimes X(\omega)$ is a strongly measurable random variable with values in $\mathcal{H} \otimes \mathcal{H}$, and as $\|x \otimes y\|_\mathcal{H} = \|x\|_\mathcal{H} \|y\|_C$.
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1.3.2

... is an integrable mapping from \( \mathcal{F} \) into \( B \otimes \mathcal{W} \). As a consequence, \( \mathcal{E}(X(g)X) \) is called the covariance of the variable \( X \).

If \( X \) is moreover associated the continuous mapping \( \tilde{X} : I \to H \), this mapping appears to be Hilbert-Schmidt. And it can be shown that conversely, to a mapping \( \tilde{X} \) linear from \( H \) into \( L^2(\Omega, \mathcal{F}, P) \), there can be associated a random variable \( X \) with values in \( X \), such that \( \langle X, h \rangle = X(h) \) a.s, if and only if \( \tilde{X} \) is Hilbert-Schmidt. The Hilbert-Schmidt norm \( \|X\|_H \) is then equal to \( \sqrt{\mathcal{E}(\|X\|_H^2)} \).

To abbreviate the writing we will write

\[
\mathcal{H}_p = L^p(\Omega, \mathcal{F}, P) \quad p \geq 0
\]

The norm in \( L^p(\mathcal{H}; \mathcal{E}) \) will be written \( \| \cdot \|_p \), the norm in \( L^p(\mathcal{H}; \mathcal{E}) \) : \( \| \cdot \|_b \).

II - PROCESSES OF LINEAR FUNCTIONALS

II-1 Definition 1

Let us write \( \mathcal{L}_p = L^p(\Omega, \mathcal{F}_t, P) \) where \( p \geq 0 \). Let \( \mathcal{B} \) be a Banach space Then a family \( X = (X_t)_{t \in T} \) in \( \mathcal{L}_p \) \( \mathcal{B} \) is called a \( p \)-process of stochastic linear functionals on \( \mathcal{B} \). The process \( X \) is called a \( p \)-cylindrical martingale.

For every \( h \in \mathcal{B} \), the real process \( \langle \tilde{X}_t(h) \rangle \) \( \mathcal{B} \) is a martingale, called a \( p \)-cylindrical martingale.

What has been said in the first paragraph, if \( \tilde{X}_t \in \mathcal{L}_p(\mathcal{F}_t; \mathcal{B}) \) for every \( t \) and \( h \), \( \tilde{X}_t(h) = \langle X_t, h \rangle \) where \( X_t \) \( \mathcal{B} \) are integrable stochastic processes with values in \( \mathcal{B} \), as defined in \([10]\) and \([14]\).

II-2 Doleans' measure of a process of linear functionals

We extend here the concept of Doleans' measure as first defined in \([5]\) for real sub-martingale and extended since then to vector-valued quasi-martingales (see for ex. \([15]\)).

To every process \( \tilde{X} \) of S.L.F. on the Banach space \( \mathcal{B} \), we associate the additive functions \( \tilde{a}_X \) with values in \( \mathcal{B} \) and defined on the set \( S \) of predictable rectangles by

\[
\tilde{a}_X([s,t] x F) = \mathcal{E}(X_t - X_s) \in \mathcal{B}
\]

Such a function on \( \mathcal{B} \) has clearly an additive extension to the algebra \( \mathcal{C} \) generated by \( \mathcal{S} \). We call \( \tilde{a}_X \) again.

Definition 3

If the additive function \( \tilde{a}_X \) on \( \mathcal{B} \) has bounded variation (for the norm of \( \mathcal{B} \)), the process \( \tilde{X} \) of stochastic linear functionals will be called a generalized quasi-martingale.

This clearly generalizes the classical definition. We have then the

Proposition 1

For \( \tilde{X} \) to be a generalized quasi-martingale, it is necessary and sufficient that the family of real processes \( \langle X_t \rangle \) associated with the real processes \( \langle \tilde{X}_t \rangle \) of bounded variation, and that the set of the...
ordered set of bounded positive measures.

from the fact that the total variation of $\zeta$ can be of the type

$$\int \left| X_t (h_i) - X_{s_i} (h_i) \right| \, \mathbb{1}_{h_i \in \mathcal{B}, \ |h_i| | \leq 1}$$

the variations can be approximated by sums of the type

$$\sum_{i} \int_{s_i,t_i} \left| X_{t_i} (h_i) - X_{s_i} (h_i) \right|$$

both supremum coincide.

are integrable cylindrical martingales

\[ \tilde{M} \] be a 2-cylindrical martingale on the Banach space the i-process of S.L.F on $\mathbb{B} \otimes \mathbb{B}$ defined by

$$\tilde{M} (h \otimes g) = \tilde{M}_c (h) \cdot \tilde{M}_c (g)$$

ar that if $\tilde{M}$ is the process of S.L.F associated with $\tilde{M}$ with values in $\mathbb{B}$, $\tilde{X}$ is the process of S.L.F. on the ordinary sense process $(\tilde{M} \otimes \tilde{M}^c) \in \mathbb{B} \otimes \mathbb{B}$

true (cf. for example [15]) that, if $\tilde{M}$ is right continuous sense quasi-martingale, with associated $\sigma$-additive (with values in $\mathbb{B} \otimes \mathbb{B}$)

$$\left( s, t \right] \rightarrow \mathbb{E} \left[ \mathbb{I}_F \left( n, \mathbb{1} \right) \right]$$

d quasi-martingale.

following example shows that for a 2-cylindrical martingale that $\tilde{M} \otimes \tilde{M}$ is a generalized quasi-martingale.

Example 1

Let $(\tau_n)$ be a decreasing sequence to zero. Let $(e_n)$ an orthonormal basis of $H$ and let us define the martingale:

$$\tilde{M}_c (h) = \sum_{n} \left[ \tau_{n+1}, \tau_n \right] \mathbb{1} \left( e_n, \mathbb{1} \right) \cdot \frac{1}{\sqrt{\tau_n - \tau_{n+1}}} (B)$$

where $(B)$ is a usual real standard Brownian motion. For every $e_n$ is a process with independent increments, zero on $[0, \tau_n]$, constant on $[\tau_n, \infty[$. For every $h \in H$, the above series in $L^2 (\Omega, \mathcal{F}_t, P)$ with

$$\mathbb{E} |\tilde{M}_c (h)|^2 \leq ||h||^2_H$$

defining a process with zero-mean independent increments. By the partition $(\left[ \tau_{n+1}, \tau_n \right] \times \mathbb{1} \left( e_n, \mathbb{1} \right))_{n \geq 0}$ of $[0, \tau_1] \times \mathbb{1}$, it is seen that, $\tilde{X}$ being the process of S.L.F. above defined:

$$\sum_{n} \mathbb{E} \left[ \mathbb{I}_F \left( \mathbb{1}_n, \mathbb{1} \right) \right]$$

We will then give the following

Definition 4

A 2-cylindrical martingale $\tilde{N}_c$ on a Banach called a square integrable cylindrical martingale (S.I.C. Martingale) if the process of S.L.F. (improperly) denoted by $\tilde{N} \otimes \tilde{N}$, defined

$$\tilde{N} (h \otimes g) = \tilde{N} (h) \cdot \tilde{N} (g)$$

is a quasi-martingale.

The additive measure $\tilde{N} \otimes \tilde{N}$ on $\mathbb{B}$ will be the quadratic measure of $\tilde{N}_c$ and its variation the control measure.

Example 2

The Brownian process, associated with the unitary matrix on a Hilbert space $H$ is a S.I.C. Martingale : let us...
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of S.L.F on $B$, such that, for every finite set of vectors in $tfl$, the process $(\hat{W}_t(h_1), \ldots, \hat{W}_t(h_n))$ is an $n$-dimensional Gaussian process with independent increments, and the bilinear form $(h, g)$ is the scalar product $\langle h, g \rangle$. In this case, the quadratic measure $\alpha$ of $\hat{W}$ associates to every $\bar{F}$ a one dimensional measure with values in $(E(\mathcal{F}_t \otimes \mathcal{F}_s))$, proportional to the product measure $t \otimes P$ where $t$ is the Lebesgue measure on $\mathbb{R}$.

Proposition 2 let us assume that $B$ is reflexive.

III - DECOMPOSITION THEOREM FOR A GENERALIZED QUASI-MARTINGALE

For the purpose of transformation formulas in stochastic integration, Doob-Meyer's decomposition theorem play an essential role. We intend to give an extension of such decomposition theorem in our setting.

All this rests on the following:

III-1 Theorem 1

Let $E'$ be the Banach dual of a separable Banach algebra $E$. If $\alpha$ be a $\sigma$-additive measure on the predictable sets, with bounded variation, such that for every evanescent set, $\alpha(\{\omega\}) = 0$.

Let $\hat{\alpha}(E', E)$ denote the Mackey topology on $E'$ for which $E$ is the dual of $E$.

Then:

1) There exists a stochastic process $V$, with values in $E'$, with right continuous paths for $\hat{\alpha}(E', E)$, uniquely up to indistinguishability such that

(i) $\hat{V}_t$ is weakly integrable in $E'$ for $t \in [0, T]$.

(ii) For every $\omega \in \Omega$, the intercall-function $[a, t] \rightarrow V_t(\omega) - V_a(\omega)$ can be extended to the Borel sets of $[a, t]$, with values in $E'$ with bounded variation (for the norm of $E'$) and $\sigma$-additive.

(iii) If $E(1_p | F^p)$ denotes a left continuous (then predictable) version of the real martingale $(E(1_p | F^p))_{t \in [0, T]}$, for every $\omega \in \Omega$ $V_t \in E(1_p | F^p)$.

2) The process $V$ just defined is predictable as a process in $(E', \hat{\alpha}(E', E))$, and $\|V_t\|_{E'}$ is predictable almost surely.
If for every \( A \subseteq E \) we define (integrating on each path of \( V \)):

\[
m(A) = \int_A(s,.) \, dV(\cdot)
\]

a stochastic measure (cf. [16] or [17]) with bounded variation \( |m| \). 

Proof

Let us first remark that, to the difference with the situation in [14] and [18], \( E' \) is not assumed to be separable. The proof, nevertheless, goes along the same line. We sketch it, insisting only upon the needed modifications.

As in [14], for every \( t \), the mapping on \( \mathcal{F}_t \) defined through:

\[
a_t(F) = \int_0^t \mathbb{E}(\chi_F(\omega)) \, \alpha(du,\omega)
\]

is a \( \mathcal{F}_t \)-additive measure with values in \( E' \), with

\[
|a_t| \leq \mathbb{E}(\chi_F(\omega)) \mathbb{E} \left( |\alpha| \right)
\]

The real measure on the right side of this inequality is positive, and then \( a_t \) is a process with bounded variation, and such that \( a_t(F) = 0 \) implies \( a_t(U) = 0 \). Then there exists (cf. [13]) a density \( U_t \) from \( \Omega \) to \( E' \) for the topology \( \sigma(E', E) \). Because of the separability of \( E \), \( \| U_t \| = \mathbb{E} \left( |\alpha| \right) \) is \( \mathcal{F}_t \)-measurable.

Using the separability of \( E \), it is proved, exactly as in [18], that, for each \( t \), \( U_t \) can be modified on a \( \mathcal{P} \)-null set into a process \( V_t \) such that for every \( y \in E \), the real process \( \langle y, V_t \rangle \) has right continuous paths with bounded variation. This implies the \( \sigma \)-additivity of the interval-function \( [s,t] \mapsto \langle y, V_t(\omega) - V_s(\omega) \rangle \), and (cf. chap. III) the extendability of this function into a measure \( \mu \) on the \( \sigma \)-algebra of \( [0,T] \), with values in a ball of \( E' \) (because of the bounded variation of the paths). This proves the part 1) of the theorem, except for bounded variation of the paths.

Part 2) is a mere consequence of the fact that (iii) implies the naturality of each real process \( \langle y, V_t \rangle \) and then its predictability. The predictability of \( |V_t| \) follows from the separability of \( E \).

As to part 3) the only thing to prove is that \( m \) is \( \mathcal{P} \)-measurable. From (III-1-3) it is clear that \( |m| \leq |a| \). But the converse inequality is a consequence of:

\[
|a_t(F)| = \sup \mathbb{E} \left( |\alpha| \right) \mathbb{E} \left( \mathbb{E}(\chi_F(\omega)) \right)
\]

As now, for every subdivision \( 0 = t_0 < t_1 < \cdots < t_n = T \) of the interval \([0,T] \),

\[
\mathbb{E} \left( \left| |V_{t_j} - V_{t_{j-1}}| \right| \right) \leq |m| \mathbb{E} \left( \mathbb{E}(\chi_F(\omega)) \right)
\]

The bounded variation of the paths is immediate and the proof is complete.

### III-2 Corollary (Decomposition theorem)

Let \( X \) be a generalized quasi-martingale on a separable Banach space \( E \). Then there exists a uniquely defined (up to indistinguishability) process \( V \), with values in \( E' \), with the following properties:

- \( V_s = 0 \) if \( s < 0 \)
- \( V_T = X \)
- \( \mathbb{E} \left( |V_T| \right) < \infty \)

\( V \) is the Pettis space of weakly integrable mappings in \( E' \),

\[
\mathbb{E} \left( |V_T| \right) = \sup \mathbb{E} \left( |<\cdot,V>\right) .
\]
(i) $V$ has paths with bounded variation (for the norm of $B'$) and right continuous for $T(B', B)$.

(ii) For every $y \in B'$, $<y, V>$ is a real predictable process.

Thus $\bar{V}$ the $\mathbb{F}$-process of S.L.F. on $\mathbb{B}$ associated with $V$, so $X - \bar{V}$ is a cylindrical martingale.

Proof

It is an immediate consequence of theorem 1, because saying that the difference $X - V$ of two processes of S.L.F. is a martingale is equivalent to saying that $X$ and $V$ have the same Doob's* measure. And $V$ is uniquely determined by theorem 1 as the process with properties (i), (ii) and (iii) corresponding to the measure $a_{\mathbb{B}}$.

Definition 5

Let $\alpha$ be a $\sigma$-additive measure as in theorem 1 (resp. $\bar{X}$ be a martingale as in corollary). The process $V$ of the theorem (resp. the natural process of the measure $\alpha$) will be called the natural process of the measure $\alpha$ (resp. of the martingale $\bar{X}$).

Proposition 3

Let $V$ be the natural process of the measure $\alpha$, as in theorem 1, and $m$ the measure as in part 3) of the theorem.

For every $h \in L^p(\Omega, \mathbb{F}_t, \mathbb{P})$ and every predictable bounded process $\psi$ with values in $L^p(\mathbb{B}', \mathbb{E})$ where $\mathbb{E}$ is a Banach space,

$$ \int \psi \, dm = \int \mathbb{E}(h, \mathbb{F}_t) \psi(u, w) \alpha(du, dw) $$

for every predictable process $\psi$, with values in $L^p(\mathbb{B}', \mathbb{E}')$ where $\mathbb{E}'$ is the dual of a Banach space $\mathbb{E}$, with the property

$$ \int ||\psi|| \, d\alpha < \infty $$

the natural process $W$ of the measure $\beta$ defined by

$$ \beta(A) = \int_A \mathbb{P} \, dx $$

is such that for $\mathbb{P}$-almost all $\omega$:

$$ W_t(\omega) = \int_0^t \psi(s, \omega) \, dV(s, \omega) $$

But this is $1^*$ with $E = E'$.

Definition 6

Let $M$ be a right continuous S.I.C. martingale. We will write $<M>$ the natural process of the quadratic measure $\alpha$ of $M$.

III-3 Local cylindrical quasi-martingale

Let $\bar{X}$ be a $\sigma$-process of S.L.F. For a stopping time $T$, for any $h \in \mathbb{B}$ and $t \in [0, T]$, $h \mathbb{E}[0, T]$ $(t, \omega) \bar{X}(\omega) \in L^0(\Omega, \mathbb{F}_t, \mathbb{P})$. The thus defined $\sigma$-process denoted by $\mathbb{E}[0, T] \bar{X}$, we will then have the natural extension of the classical definition.
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Definition 7
A process of S.L.F. X will be called a local cylindrical quasi-martingale (resp. a local S.I.C. martingale) if there exists an increasing sequence \((\tau(n))_n \in \mathcal{B}\) of stopping times, and a corresponding sequence of cylindrical quasi-martingales (resp. S.I.C. martingales) \((X^n)_{n = 1}^{\infty} = T \ a.s \) and

\[ \forall n \quad 1_{[0, \tau(n)]} X^n = 1_{[0, \tau(n)]} X_n \]

that the previous results can be extended to local cylindrical quasi-martingales or S.I.C. martingales as in the real case.

IV - STOCHASTIC INTEGRAL WITH RESPECT TO A GENERALIZED QUASI-MARTINGALE

A local generalized quasi-martingale is the sum of a process which define vector measures on \(\mathcal{T}\), and of a local generalized martingale. The problem of defining the stochastic integral with respect to a generalized quasi-martingale, reduces, as in the classical case, to defining the integral with respect to a local generalized martingale.

Passing over from the integral with respect to a square integrable martingale to a generalized martingale, which can be localized into square integrable martingales through a suitable increasing sequences \((\tau^n)\) of stopping times, goes exactly as in the real classical case. So we will restrict ourselves to integrating with respect to square integrable martingales, extending the isometry formula proved in

IV-1 Theorem 2
Let \(M\) be a square integrable cylindrical martingale, on a Hilbert space \(\mathcal{H}\) with quadratic measure \(\mathcal{Q}\) and control measure \(\mathcal{P}\). Then there exists a process \(Q\) with values in the unit ball of \((\mathcal{H} \otimes \mathcal{H})'\) with the following properties.

a) \(Q\) is weakly predictable and weakly integrable (for the duality \(\langle (\mathcal{H} \otimes \mathcal{H})', \mathcal{H} \otimes \mathcal{H} \rangle\), with values in the cone of symmetric positive elements of \((\mathcal{H} \otimes \mathcal{H})'\).

b) \(\forall \mathcal{A} \in \mathcal{A} \subset \omega \quad \mathcal{P}(\mathcal{A}) = \int_{\mathcal{A}} Q d\mathcal{P} \)

\(Q\) is unique up to a (weak) \(\lambda\)-equivalence.

Proof
This is a mere application of a weak Radon-Nikodym theorem for vector-measures (see [13] th. 7). In fact it is immediate that \(\mathcal{A} \subset \omega \), when \(\mathcal{P}\) is the unit ball of \((\mathcal{H} \otimes \mathcal{H})'\).

IV-2 Remark
If for every \(t\), \(\mathcal{M}_t \in (\mathcal{H} \otimes \mathcal{H})'\), is associated with an ordinary \(\mathcal{H}\)-valued martingale, \(\alpha\) takes values in \((\mathcal{H} \otimes \mathcal{H})'\), which identifies itself as such, as a subspace of \((\mathcal{H} \otimes \mathcal{H})'\). In this case (cf. [16] ), \(Q\) takes its values in even strongly predictable.

IV-3 Definition of spaces \(\mathcal{H}_2^\mathcal{A}(\mathcal{H}; \mathcal{A})\) and \(\mathcal{H}_2^\mathcal{A}(\mathcal{H}; \mathcal{A})\)
Let \(\mathcal{Q}\) be a process in \((\mathcal{H} \otimes \mathcal{H})'\), weakly predictable and integrable (for the duality \(\langle (\mathcal{H} \otimes \mathcal{H})', \mathcal{H} \otimes \mathcal{H} \rangle\), with values in the cone of positive elements of \((\mathcal{H} \otimes \mathcal{H})'\). We recall (cf. §1-3 above) that \(\mathcal{Q}(t, \omega)\) denotes the bounded linear operator associated with

\[ Q(t, \omega) : \mathcal{H}_1 \to \mathcal{H}_2, \quad h_1 \mapsto Q(t, \omega)h_1. \]

From the separability of \(\mathcal{H}\), it is clear that weak predictability is equivalent to the strong predictability of \(\mathcal{Q}(t, \omega)\) as \(h_1\) values in \(\mathcal{H}_1\), for every \(h_2 \in \mathcal{H}_2\).

Let \(\mathcal{E}\) be an Hilbert space. We will define the space of stochastic processes as the space of those processes \(X\) such that:

\[ \forall (t, \omega) \in \mathcal{E} \times \Omega, \quad X(t, \omega) \] is a linear operator.
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1) \( \text{Range of } Q^+ \) and range in \( E \), such that the linear \( Q^+(t,w) \) is extendable into a bounded linear operator \( \lambda(t,w) \rightarrow Q^+(t,w) \circ X(t,w) g \) is strongly predictable \( \lambda(t,w) \rightarrow Q^+(t,w) \) is the adjoint mapping of \( X \).

The adjoint also the space \( L^Q_\Theta(H;E) \) of stochastic processes \( X \) such that

\[
(\omega) \in [0,T] \times \Omega, \quad X(t,w) \text{ is a linear operator with domain } \mathcal{D}(Q^+), \text{ range in } E, \text{ such that the linear } Q^+(t,w) \text{ is extendable into a Hilbert Schmidt operator }
\]

\[ (t,w) \rightarrow X(t,w) \circ Q^+(t,w) \text{ is strongly predictable } \]

\[
\int \|X \circ Q^+\|_H^2 \, d\lambda < \infty
\]

Properties of spaces \( A^X_\Theta(H;E) \) and \( A^Q_\Theta(H;E) \)

\[ 1°) \text{Let } \lambda \text{ be a bounded positive measure on } ([0,T] \times \Omega, \mathcal{G}), \text{ the mapping } X \mapsto \left( \sup_{[0,T] \times \Omega} \int \|Q^+ \circ X^*(g)\|_H^2 \, d\lambda \right)^\frac{1}{2} \text{ is a complete semi-norm on } A^Q_\Theta(H;E), \text{ for which this space is complete.} \]

\[ 2°) \text{The mapping } X \mapsto \left( \int_{[0,T] \times \Omega} \|X \circ Q^+\|_H^2 \, d\lambda \right)^\frac{1}{2} \text{ is an Hilbertian semi-norm on } A^Q_\Theta(H;E), \text{ associated with the positive bilinear form } (X,Y) \mapsto \text{Tr}(X \circ Q^+ \circ Y^*) \, d\lambda. \]

Proof

The fact that the above mappings are semi-norms (the second one being a prehilbertian one) is immediate.

1°) Let us consider now a Cauchy sequence \( (X_n) \).

Because of the separability of \( E \), it is a subsequence \( (X_{n_k}) \) and a linear operator \( Y(t,w) \) such that

\[
\sup_k \int \|Q^+ \circ X_{n_k}(t,w) \circ X_{n_k}(t,w)g\|_H^2 \, d\lambda < \infty
\]

From the Banach-Steinhaus theorem \( Y(t,w) \) is bounded.

The existence of \( X(t,w) \) (possibly non continuous) defined on \( \text{Range } Q^+(t,w) \) is evident. And moreover, the inequality

\[
\sup_k \int \|Q^+ \circ X_{n_k}(t,w) \circ X_{n_k}(t,w)g\|_H^2 \, d\lambda < \sup_k \int \|Q^+ \circ X_{n_k}(t,w) \circ X_{n_k}(t,w)g\|_H^2 \, d\lambda
\]

and the Cauchy property of the sequence \( (X_n) \), it is close to \( X \) in \( A^Q_\Theta(H;E) \).

2°) If \( (X_n) \) is a Cauchy sequence in \( A^Q_\Theta(H;E) \), \( (X_n \circ Q^+) \) is a Cauchy sequence in the space \( L^2([0,T] \times \Omega, E) \), where \( E \) is the Hilbert space \( L^2(H;E) \).

Moreover the limit \( X \circ Q^+ \) from the first part of the proof is a Cauchy sequence in \( A^Q_\Theta(H;E) \) too. This proves the theorem.

Remark

In \([16]\) proposition 3, the part 2° of the theorem was proved when \( Q^+ \) is Hilbert-Schmidt.
In what follows we will call $T^\lambda(W;\mathcal{C})$ the set of processes of the following form:

$$X(t,\omega) = \sum_{i=1}^{n} \mathbb{1}_{[t_i, t_{i+1})} X_i(t,\omega),$$

where $n$ is any integer $n < s$, $\mathbb{1}_I$ is the indicator function of the interval $I$, and $X_i(t,\omega)$ is in $\mathcal{C}(H;\mathcal{C})$.

The vector spaces $\Lambda^\lambda_T$ and $\Lambda^\lambda_T^\omega$ are, moreover, always endowed with the above semi-norms, and, as usually done, we will consider without changing the name, the associated separated Banach spaces of equivalence classes of processes. So, when speaking of a process $X$ in $\Lambda^\lambda_T^\omega$ (resp. $\Lambda^\lambda_T$) we will mean a process in $\Lambda^\lambda_T$ defined up to an equivalence.

**Theorem 4**

The closure of $T^\lambda(W;\mathcal{C})$ in $L^\infty_T^\lambda(W;\mathcal{C})$ (resp. $L^\infty_T^\lambda(W;\mathcal{C})$) contains all the processes $X$ with values in $\mathcal{C}(H;\mathcal{C})$ strongly predictable for the uniform norm of $\mathcal{C}(H;\mathcal{C})$ and such that $\|X\|_b < K$.

**Proof**

We first remark that if $X$ is strongly predictable as a process with values in $\mathcal{C}(H;\mathcal{C})$ (with its operator norm), and such that $\|X\|_b < K$ for all $n$ with $n < s$, then:

$$\|X(t,\omega) - X_n(t,\omega)\|_b^2 \to 0$$

as $n \to \infty$.

Next, if $X$ satisfies the hypothesis of the theorem, it is easily seen to be approximated in $L^\infty_T^\lambda(W;\mathcal{C})$ by the sequence $X_n = X$, $n = 1, 2, \ldots$.

The last same approximation works for a process with values in $\mathcal{C}_l^\lambda(W;\mathcal{C})$, strongly predictable for the norm of $\mathcal{C}_l^\lambda(W;\mathcal{C})$ and such that $\|X\|_b < K$ by a square integrable function $\tilde{Q}$.

Suppose now that $X$ is any process in $\Lambda^\lambda_T^\omega$, such that $\|X\|_b < K$ and $(e_n)$ is an orthogonal basis of $H$.

$$X_n = X \circ \Pi_n$$

where $\Pi_n$ is the orthogonal projection in $H$ generated by $\{e_n\}$. The process $X_n$ is clearly strongly predictable as a process with values in $\mathcal{C}_l^\lambda(W;\mathcal{C})$ (with its uniform norm) and such that $\|X\|_b < K$ for all $n$ with $n < s$.

**Remark**

When $\tilde{Q}$ is nuclear the part of this theorem concerning $\mathcal{C}_l^\lambda(W;\mathcal{C})$ has been proved in [16] prop. 1.

**Definition 7**

If $\tilde{Q}$ is the process associated with a
Stochastic integral with respect to a square-integrable cylindrical martingale

Theorem 5
Let \( M \) be a square-integrable cylindrical martingale on the separable Hilbert space \( \mathcal{H} \) and let \( \mathcal{E} \) be another separable Hilbert space. For every process \( \mathbf{X} \) where \( \mathbf{X} \) is a predictable rectangle and \( \mathbf{X} \in \mathcal{H} \) we define \( \mathbf{X} d\mathbf{M} \) as the L.S.F on \( \mathcal{E} \):

\[
\mathbf{X} \mathbf{dM} = \left( \sum_{i=1}^{n} \mathbf{X}(s_i, t_i) \right) \mathbf{u}_i^*(\mathbf{g})
\]

where \( \mathbf{u}_i^*(\mathbf{g}) \) is an orthonormal basis of \( \mathcal{E} \). Using again the martingale property of \( \mathbf{M} \) we get

\[
||\mathbf{X} \mathbf{dM}||_2 = \sum_{i=1}^{n} ||\mathbf{X}(s_i, t_i) \mathbf{u}_i^*(\mathbf{g})||_2
\]

which proves the isometry.
This last integral is defined pathwise as the integral with values in \( L^1(0,T; L^1(\Omega, E)) \) respect to a vector valued measure \( d\tilde{M}(u) \) w.r.t. to the bilinear mapping \( f : E_1 \times E_2 \rightarrow \mathbb{F} \) and relativel\( y \) to the bilinear mapping \( f : B_g \times B_g \rightarrow B_g \).

3°) If \( \tilde{M} \) is continuous, then \( \tilde{Y} \) is continuous.

**Proof**

1°) Saying that \( \tilde{Y} = (\int_0^T X \, d\tilde{M}) \) is a right continuous square integrable cylindrical martingale comes to saying that, for every \( g \in \mathcal{E} \), \( (\tilde{Y}_t(g)) \) is a right continuous real martingale.

We will prove it for a process \( X = (X_t) \), \( 0 \leq t \leq T \) and \( u \in \mathcal{E} \), and, because of the linearity and continuity of the mapping \( (\int_0^T X \, d\tilde{M}) \), it will appear immediately to be true for any \( X \in \mathcal{H}_T \).

Let us prove first that \( \forall 0 \leq s < t \leq T \),

\[
E \left[ (\tilde{Y}_t(g) - \tilde{Y}_s(g)) \cdot 1_{[s,T]} \right] = 0.
\]

For a particular \( X \) of the above form,

\[
E \left[ (\tilde{Y}_t(g) - \tilde{Y}_s(g)) \cdot 1_{[s,T]} \right] = E \left[ 1_{[s,T]} \cdot \tilde{M} \right] = 0.
\]

The martingale property of \( \tilde{M} \) gives the martingale property of \( \tilde{Y} \). As, for

\[
\tilde{Y}_t(g) = 1_{[s,t]} \cdot \tilde{M}(u(g)) - \tilde{M}(u(g))
\]
Continuity of the mapping $t \mapsto Y_t(g)$ is clear.

By linearity and density we get immediately (IV-5-3) the martingale property for a general $X \in H_T^b(H; G)$. Assume that a sequence $Y^n$ is such that, for any $t$, $(Y^n_t)$ converges to $Y_t$ in $L^2(G; L^2(\Omega, F_t, P))$. Then, using the classical procedures that, for any $g$, we can deduce the right continuity in $L^2$ of the real sequence $(Y_t(g))_{t \in [0,T]}$ from the right continuity of $(g)_t \in [0,T]$. In the particular case when $Y \in H_T^b(H; G)$, it follows from theorem 6 that

$$\sup E \|X \circ M_t(g)\|_{L^2} = \sup E \|X \circ M_t(g)\|_{L^2}$$

we get formula (IV-5-1) for all $X \in H_T^b(H; G)$ and $\Lambda = \Lambda_{t \in [0,T]}$. The formula for all predictable $\Lambda$ follows from the additivity in $\Lambda$ of both members of (IV-5-1).

The Pettis integrability of $X \otimes X$ with respect to $\Lambda$ proves that $X \otimes X$ is Pettis integrable on $\Omega$-almost all paths $\omega$ with respect to $M$. The cylindrical process $\Phi_t = \int_0^t \langle X \circ M_t(g), g \rangle $ satisfies

$$\alpha_t(\Lambda_{s \in [0,T]} X_t, g) = E \langle X \circ M_t(g) \rangle$$

which proves formula (IV-5-2).

3°) It is sufficient to prove that, for all $g \in G$, $Y(g)$ is a martingale and continuous since $Y(g)$ is a martingale. If $M$ is continuous, the real process $X = \int_0^t X \circ M_t(g)$ is continuous, and $X = u \cdot \Delta$, where $\Delta \in \mathcal{F}$ and $u \in \mathcal{F}$. Then $Y(g)$ is continuous and $\Delta$ is true in the general case by linearity and

Remark

If $M$ is a local cylindrical square integrable martingale and if $X \in L_T^b(H; G)$ (resp. $X \in H_T^b(H; G)$), the process $X \circ M_t(g)$ is a local cylindrical martingale (resp. there is a local square integrable $G$-valued martingale $Y$ associated to the local cylindrical martingale $\tilde{M} = \int_0^t X \circ M_t$).
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CHANGE OF VARIABLE. ITO'S FORMULA

An Ito's formula has been given for the stochastic integral with respect to a cylindrical Brownian motion. We want only to note this immediate consequence of known general "Ito's formula example in [9]. We will restrict ourselves to the continuous case a general "Ito's formula" as stated for example in [9] and [14]. Let Y be a square integrable martingale with continuous paths of bounded variation. Let Y be a square integrable martingale with continuous paths. If f is a mapping of \( F \times G \) into a Hilbert space \( K \), once continuously differentiable in the first variable, with derivative denoted by \( D_x f(x,y) \in C(F;K) \) bounded on any bounded set in \( F \times G \), then we have the following formula, expressing equality up to indistinguishability between two processes:

\[
(V,Y) = (V_0,Y_0) + \int_0^t D_x f(V_s,Y_s) dV_s + \int_0^t D_y f(V_s,Y_s) dY_s
\]

and integrals being taken pathwise, while the second is a stochastic integral.

If instead of the bilinear form \( D_2 f(x,y) \) we consider the associated continuous linear mapping \( D_2 f(x,y) \in \mathcal{L}(G;K) \) the last integral can be written

\[
\frac{1}{2} \int_0^t \langle V_s, Y_s \rangle d \mu_{xy} + \int_0^t Y_s D_2 f(V_s,Y_s) d\mu_{xy}
\]

In the particular case when \( M^* \) is a cylindrical Brownian motion \( B^* \) as in [7], the process of S.L.F \( <M^*>_s \) reduces to an ordinary linear mapping \( \sigma \) (the covariance) of \( \mathcal{L}(K;\mathbb{R}) \). The above formula reduces, when moreover \( V = 0 \), to the formula of [7]:

\[
(\Phi(Y_t) = \Phi(Y_0) + \int_0^t \Phi(Y_t) dX_t + \frac{1}{2} \int_0^t \sigma(Y_t) ds
\]

Using the formulas of theorem 6 we get immediately the following result:

\[
\frac{1}{2} \int_0^t \langle V_s, Y_s \rangle d\mu_{xy} + \int_0^t Y_s D_2 f(V_s,Y_s) d\mu_{xy}
\]

Instead of the bilinear form \( D_2 f(x,y) \) we consider the linear mapping \( \sigma(x,y) \in \mathcal{L}(G;G) \), the
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