

CONSTANCE VAN EEDEN

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A one-sample analogue of a theorem of Jurečkova^v

by

Constance van Eeden^{*}

Université de Montréal

I. INTRODUCTION.

The purpose of this note is to prove that if, for each $v=1,2,\dots,X_{v,1},\dots,X_{v,n_v}$ are a random sample from a distribution symmetric around 0, then the signed-rank statistic

$$T_v(\theta) = \sum_{i=1}^{n_v} p_{v,i} \Psi \left(\frac{R_{|X_{v,i} - q_{v,i}\theta|}}{n_v + 1} \right) \operatorname{sgn} (X_{v,i} - q_{v,i}\theta),$$

where $R_{|X_{v,i} - q_{v,i}\theta|}$ is the rank of $|X_{v,i} - q_{v,i}\theta|$ among

$|X_{v,1} - q_{v,1}\theta|, \dots, |X_{v,n_v} - q_{v,n_v}\theta|$, is under certain conditions on the

common distribution of the $X_{v,i}$, on the constants $p_{v,i}$, $q_{v,i}$ and on the function Ψ , asymptotically approximately a linear function of θ in the sense that

$$\lim_{n_v \rightarrow \infty} P \left\{ \sup_{|\theta| \leq C} |T_v(\theta) - T_v(0) + \theta K \sum_{i=1}^{n_v} p_{v,i} q_{v,i}| \geq \epsilon \sigma(T_v(0)) \right\} = 0,$$

for every $C > 0$ and every $\epsilon > 0$, where K is a constant depending on the common distribution of the $X_{v,i}$ and on the function Ψ .

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An analogous result was proved by Jurečková^v [2] for the statistic

$$S_v(\theta) = \sum_{i=1}^{n_v} c_{v,i} \varphi \left(\frac{R_{X_{v,i} - d_{v,i}\theta}}{n_v + 1} \right),$$

where $R_{X_{v,i} - d_{v,i}\theta}$ is the rank of $X_{v,i} - d_{v,i}\theta$ among

$X_{v,1} - d_{v,1}\theta, \dots, X_{v,n_v} - d_{v,n_v}\theta$ and where, for each $v=1,2,\dots$, the $X_{v,i}$ are independently and identically distributed.

For the proof of our result some lemmas are needed which are given in section 2 ; one of these lemmas is a generalization of Theorem 5 of Lehmann [6] ; two of the lemmas are analogous to Corollary 1 and 2 of Lehmann [6] . The main results and their proofs are given in section 3.

II. SOME LEMMAS.

Let $i_1, \dots, i_n$ and $j_1, \dots, j_n$ each be a permutation of the numbers $1, \dots, n$ and let $\epsilon_1, \dots, \epsilon_n, \delta_1, \dots, \delta_n$ each be $+1$ or -1 such that $(i_k, \epsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies

$$\text{Condition } A_n : \begin{cases} 1. \delta_k = 1 \implies \epsilon_k = 1 \\ 2. \{ \ell < k, \delta_k = 1, j_\ell < j_k \} \implies i_\ell < i_k \\ 3. \{ \ell < k, \epsilon_k = -1, j_\ell > j_k \} \implies i_\ell > i_k \end{cases}$$

For fixed $M (1 \leq M \leq n)$ define

$$(2, 1) \quad a_{M,1} > a_{M,2} > \dots > a_{M,K_M}$$

as the ordered values of those i_k among $i_{n-M+1}, i_{n-M+2}, \dots, i_n$ for which $\epsilon_k = +1$ and

$$(2, 2) \quad b_{M,1} > b_{M,2} > \dots > b_{M,L_M}$$

as the ordered values of those j_k among $j_{n-M+1}, j_{n-M+2}, \dots, j_n$ for which $\delta_k = +1$. Obviously, by Condition $A_n.1$, $K_M \geq L_M$, further $K_M \leq M$.

Further define

$$(2, 3) \quad c_{M,1} > c_{M,2} > \dots > c_{M,M-K_M}$$

as the ordered values of those i_k among $i_{n-M+1}, i_{n-M+2}, \dots, i_n$ for which $\epsilon_k = -1$ and

$$(2, 4) \quad d_{M,1} > d_{M,2} > \dots > d_{M,M-L_M}$$

as the ordered values of those j_k among $j_{n-M+1}, j_{n-M+2}, \dots, j_n$ for which $\delta_k = -1$.

Lemma 2, 1. If $(i_k, \epsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies Condition A_n , then

$$(2,5) \quad \begin{cases} b_{M,\ell} \leq a_{M,\ell} & \ell=1, \dots, L_M \\ c_{M,\ell} \leq d_{M,\ell} & \ell=1, \dots, M-K_M \end{cases} \quad M=1, \dots, n$$

Proof : The proof will be given in four parts.

1) The lemma is true for $M = 1$ and any $n \geq 1$. To prove this, notice that by Condition $A_n.1$ it is sufficient to prove that

$$(2, 6) \quad \begin{cases} j_n \leq i_n & \text{if } \delta_n = 1 \\ j_n \geq i_n & \text{if } \epsilon_n = -1 \end{cases}$$

This can be seen as follows.

$$(2, 7) \quad \begin{cases} j_n = (\# \text{ of } j_k \leq j_n) = n - (\# \text{ of } j_k > j_n) \\ i_n = (\# \text{ of } i_k \leq i_n) = n - (\# \text{ of } i_k > i_n) \end{cases}$$

By Condition $A_n.2$

$$(2, 8) \quad (\# \text{ of } j_k \leq j_n) \leq (\# \text{ of } i_k \leq i_n) \text{ if } \delta_n = 1$$

and by Condition $A_n.3$

$$(2, 9) \quad (\# \text{ of } j_k > j_n) \leq (\# \text{ of } i_k > i_n) \text{ if } \epsilon_n = -1$$

2) If the lemma is true for some (n, M) then the lemma is true for $(n+1, M)$. To see this consider, for some $n \geq 1$,

$(i_k, \epsilon_k, j_k, \delta_k)_{k=1}^{n+1}$ satisfying Condition A_{n+1} . From $(i_k, \epsilon_k, j_k, \delta_k)_{k=1}^{n+1}$ derive $(i'_k, \epsilon_k, j'_k, \delta_k)_{k=2}^{n+1}$, satisfying Condition A_n , as follows. Let

$$(2, 10) \quad \begin{cases} r_k = \text{rank of } i_k \text{ among } (i_1, i_k) \\ s_k = \text{rank of } j_k \text{ among } (j_1, j_k) \end{cases} \quad k = 2, \dots, n+1$$

and let

$$(2, 11) \quad \begin{cases} i'_k = i_k - (r_k - 1) \\ j'_k = j_k - (s_k - 1) \end{cases} \quad k = 2, \dots, n+1$$

Then $i'_2, \dots, i'_{n+1}$ and $j'_2, \dots, j'_{n+1}$ are each permutations of the numbers $1, \dots, n$ and from

$$(2, 12) \quad \begin{cases} i_k < i_\ell \iff i'_k < i'_\ell \\ j_k < j_\ell \iff j'_k < j'_\ell \end{cases} \quad k, \ell = 2, \dots, n+1$$

it then follows that $\{i'_k, \epsilon_k, j'_k, \delta_k\}_{k=2}^{n+1}$ satisfies Condition A_n .

For fixed $M \leq n$ let $a'_{M,\ell}, b'_{M,\ell}, c'_{M,\ell}, d'_{M,\ell}, L'_M$ and K'_M be defined, as in (2 ; 2) - (2 ; 4), for $(i'_k, \epsilon_k, j'_k, \delta_k)_{k=n+2-M}^{n+1}$ and let $a_{M,\ell}, b_{M,\ell}, c_{M,\ell}, d_{M,\ell}, K_M$ and L_M be so defined for

$(i_k, \epsilon_k, j_k, \delta_k)_{k=n+2-M}^{n+1}$; then $L_M = L'_M$ and $K_M = K'_M$. Assuming the lemma to be true for (n, M) we have

$$(2 ; 13) \quad \begin{cases} b'_{M,\ell} \leq a'_{M,\ell} & \ell = 1, \dots, L_M \\ c'_{M,\ell} \leq d'_{M,\ell} & \ell = 1, \dots, M - K_M \end{cases}$$

Now let ℓ_0 be the number of $b_{M,\ell} > j_1$; then by (2 ; 11)

$$(2 ; 14) \quad b'_{M,\ell} = \begin{cases} b_{M,\ell} - 1 & \ell = 1, \dots, \ell_0 \\ b_{M,\ell} & \ell = \ell_0 + 1, \dots, L_M \end{cases}$$

Let k_0 be the number of $a_{M,\ell} > i_1$; then by (2 ; 11)

$$(2 ; 15) \quad a'_{M,\ell} = \begin{cases} a_{M,\ell} - 1 & \ell = 1, \dots, k_0 \\ a_{M,\ell} & \ell = k_0 + 1, \dots, L_M \end{cases}$$

Further, by Condition $A_{n+1} \cdot 2$, $\ell_0 \leq k_0$. From (2 ; 13) - (2 ; 15) it then follows that

$$(2 ; 16) \quad b_{M,\ell} \leq a_{M,\ell} \quad \ell = 1, \dots, L_M.$$

The proof that

$$(2 ; 17) \quad c_{M,\ell} \leq d_{M,\ell} \quad \ell = 1, \dots, M - K_M$$

is analogous, using Condition $A_{n+1} \cdot 3$.

3) If the lemma is true for some $n \geq 2$ with $M = n-1$, then the lemma is true for the same n with $M = n$. This can be seen as follows.

Assuming the lemma to be true for $M = n-1$ we have

$$(2 ; 18) \quad \begin{cases} b_{n-1,\ell} \leq a_{n-1,\ell} & \ell = 1, \dots, L_{n-1} \\ c_{n-1,\ell} \leq d_{n-1,\ell} & \ell = 1, \dots, n-1-K_{n-1} \end{cases}$$

and it will be proved that

$$(2 ; 19) \quad \begin{cases} 1. b_{n,\ell} \leq a_{n,\ell} & \ell = 1, \dots, L_n \\ 2. c_{n,\ell} \leq d_{n,\ell} & \ell = 1, \dots, n-K_n \end{cases}$$

The following three cases can be distinguished

a) $\delta_1 = \epsilon_1 = -1$. Then $L_n = L_{n-1}$, $K_n = K_{n-1}$, $b_{n,l} = b_{n-1,l}$ ($l=1, \dots, L_n$) and $a_{n,l} = a_{n-1,l}$ ($l=1, \dots, K_n$), so that (2 ; 19.1) is obvious. Further $(a_{n,l}, l=1, \dots, K_n, c_{n,l}, l=1, \dots, n-K_n)$ and $(b_{n,l}, l=1, \dots, L_n, d_{n,l}, l=1, \dots, n-L_n)$ are each permutations of the numbers $1, \dots, n$ so that (2 ; 19.2) follows from (2 ; 19.1)

b) $\delta_1 = -1$, $\epsilon_1 = 1$. Then $L_n = L_{n-1}$, $K_n = K_{n-1} + 1$, $b_{n,l} = b_{n-1,l}$ ($l=1, \dots, L_n$) and $c_{n,l} = c_{n-1,l}$ ($l=1, \dots, n-K_n$). To prove (2 ; 19.1) let k_0 be the number of $a_{n-1,l}$ ($l=1, \dots, K_{n-1}$) larger than i_1 ; then

$$(2 ; 20) \quad a_{n,l} = \begin{cases} a_{n-1,l} & l = 1, \dots, k_0 \\ i_1 & l = k_0 + 1 \\ a_{n-1,l-1} & l = k_0 + 2, \dots, K_n \end{cases}$$

If $L_n \leq k_0 \leq K_{n-1}$ then (2 ; 19.1) is immediate. If $0 \leq k_0 < L_n = L_{n-1}$, then (2 ; 19.1) is immediate for $l=1, \dots, k_0$. Further

$$(2 ; 21) \quad b_{n,k_0+1} = b_{n-1,k_0+1} \leq a_{n-1,k_0+1} < i_1 = a_{n,k_0+1}$$

and for $l = k_0 + 2, \dots, L_n$

$$(2 ; 22) \quad b_{n,l} = b_{n-1,l} \leq a_{n-1,l} = a_{n,l+1} \leq a_{n,l}$$

The proof of (2 ; 19.2) is analogous.

c) $\delta_1 = \epsilon_1 = 1$. Then $L_n = L_{n-1} + 1$, $K_n = K_{n-1} + 1$, $c_{n,l} = c_{n-1,l}$ ($l=1, \dots, n-K_n$) and $d_{n,l} = d_{n-1,l}$ ($l=1, \dots, n-L_n$) so that (2 ; 19.2) is obvious. Further (see a)) (2 ; 19.1) follows from (2 ; 19.2)

4) The lemma now follows by induction on M . According to part. 1 of the proof, the lemma is true for $M = 1$ and any $n \geq 1$. Let M_0 be an integer ≥ 1 and assume the lemma is true for $M = M_0$ and any $n \geq M_0$, then it will be proved that the lemma is true for $M = M_0 + 1$ and any $n \geq M_0 + 1$. This can be seen as follows. According to the induction hypothesis the lemma is true for $n = M_0 + 1$ and $M = M_0$; according to part 3 of the proof this implies the truth for $n = M_0 + 1$ and $M = M_0 + 1$; according to part

2 of the proof this implies the truth for $M = M_0 + 1$ and any $n \geq M_0 + 1$.

Q. E. D.

In Lemma 2 , 1 it was shown that Condition A_n is sufficient for (2 , 5) to hold for each $M = 1, \dots, n$. For (2 , 5) to hold for a particular value of M it is obviously sufficient that $(i_k, \epsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies

$$\text{Condition } A_{n,M} \left\{ \begin{array}{l} \text{For each } k \geq n-M+1 \\ 1. \delta_k = 1 \implies \epsilon_k = 1 \\ 2. \text{ for each } l \leq k-1 (\delta_k = 1, j_l < j_k) \implies i_l < i_k \\ 3. \text{ for each } l \leq k-1 (\epsilon_k = -1, j_l > j_k) \implies i_l > i_k \end{array} \right.$$

Further, if $(i_k, \epsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies Condition $A_{n,M}$ for $M = M_0$ then $(i_k, \epsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies Condition $A_{n,M}$ for all $M \leq M_0$, which proves the following lemma.

Lemma 2 , 2. If $(i_k, \epsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies Condition $A_{n,M}$ for $M = M_0$, then

$$(2, 23) \quad \left\{ \begin{array}{ll} a_{M,l} \leq b_{M,l} & l=1, \dots, L_M \\ c_{M,l} \leq d_{M,l} & l=1, \dots, M-K_M \end{array} \right. \quad 1 \leq M \leq M_0$$

Lemma 2 , 3. If h is nondecreasing and nonnegative and if $(i_k, \epsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies Condition $A_{n,M}$ for $M = M_0$, then

$$(2, 24) \quad \left\{ \begin{array}{ll} \sum_{l=n+1-M}^n h(i_l) \geq \sum_{l=n+1-M}^n h(j_l) & \epsilon_l > 0 \\ \sum_{l=n+1-M}^n h(i_l) \leq \sum_{l=n+1-M}^n h(j_l) & \delta_l > 0 \end{array} \right. \quad 1 \leq M \leq M_0$$

Proof : Because h is nondecreasing, it follows from Lemma 2 , 2 that, for $1 \leq M \leq M_0$,

$$(2, 25) \quad \begin{cases} 1. h(b_{M,l}) \leq h(a_{M,l}) & l=1, \dots, L_M \\ 2. h(c_{M,l}) \leq h(d_{M,l}) & l=1, \dots, M-K_M \end{cases}$$

From (2, 25.1) and the fact that h is non negative it follows that, for $1 \leq M \leq M_0$,

$$(2, 26) \quad \sum_{\substack{l=n+1-M \\ \delta_l > 0}}^n h(j_l) = \sum_{l=1}^{L_M} h(b_{M,l}) \leq \sum_{l=1}^{L_M} h(a_{M,l}) \leq \sum_{l=1}^{K_M} h(a_{M,l}) = \\ = \sum_{\substack{l=n+1-M \\ \epsilon_l > 0}}^n h(i_l)$$

From (2, 25.2) and the fact that h is non negative it follows that, for $1 \leq M \leq M_0$,

$$(2, 27) \quad \sum_{\substack{l=n+1-M \\ \epsilon_l < 0}}^n h(i_l) = \sum_{l=1}^{M-K_M} h(c_{M,l}) \leq \sum_{l=1}^{M-K_M} h(d_{M,l}) \leq \sum_{l=1}^{M-L_M} h(d_{M,l}) = \\ = \sum_{\substack{l=n+1-M \\ \delta_l < 0}}^n h(j_l). \quad \text{Q. E. D.}$$

Remark. In the two special cases, where $\delta_k = 1$ for all k or $\epsilon_k = -1$ for all k , Lemma 2, 1 reduces to Theorem 5 of Lehmann [6]. Further, in each of these special cases, Lemma 2, 3 is analogous to Corollary 1 of Lehmann [6].

Lemma 2, 4. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be n numbers satisfying

$$(2, 28) \quad 0 \leq \alpha_1 \leq \dots \leq \alpha_n,$$

let h be non decreasing and non negative and let $(i_k, \epsilon_k, j_k, \delta_k)_{k=1}^n$ satisfy

$$(2, 29) \quad \begin{cases} 1. (\delta_k = 1, \alpha_k > 0) \implies \epsilon_k = 1 \\ 2. (\delta_k = 1, \alpha_k > 0, l < k, j_l < j_k) \implies i_l < i_k \\ 3. (\epsilon_k = -1, \alpha_k > 0, l < k, j_l > j_k) \implies i_l > i_k \end{cases}$$

then

$$(2, 30) \quad \sum_{k=1}^n \alpha_k \epsilon_k h(i_k) \geq \sum_{k=1}^n \alpha_k \delta_k h(j_k).$$

Proof : The following proof is analogous to Lehmann's proof of his Corollary 2 in [6].

(2, 30) is obviously true if $\alpha_k = 0$ for all $k = 1, \dots, n$, so in the following it will be supposed that $\alpha_k > 0$ for at least one k . Further, since h is non negative,

$\sum_{l=1}^n h(l) \geq 0$ and $\sum_{l=1}^n h(l) = 0$ if and only if $h(l) = 0$ for all $l = 1, \dots, n$, in which case (2, 30) is obvious. In the following it will be supposed that $\sum_{l=1}^n h(l) > 0$.

Let $0 \leq \beta_1 < \beta_2 < \dots < \beta_T$ be the different values of $\alpha_1, \dots, \alpha_n$ and let $n_t (t=1, \dots, T)$ be the number of α_k equal β_t . Further let

$N_t = \sum_{s=1}^t n_s$ ($t=1, \dots, T$) and $N_0 = 0$. Consider the random variables X and Y each taking the values $(-\beta_T, -\beta_{T-1}, \dots, -\beta_1, \beta_1, \dots, \beta_{T-1}, \beta_T)$

with

$$(2, 31) \quad \left\{ \begin{array}{l} 1. P(X \leq -\beta_s) = \frac{\sum_{l=N_{s-1}+1}^{N_T} h(i_l)}{\sum_{l=1}^n h(l)} \quad \epsilon_l < 0 \\ 2. P(X \leq \beta_s) = 1 - \frac{\sum_{l=N_s+1}^{N_T} h(i_l)}{\sum_{l=1}^n h(l)} \quad \epsilon_l > 0 \end{array} \right. \quad s=1, \dots, T$$

and

$$(2, 32) \quad \left\{ \begin{array}{l} 1. P(Y \leq -\beta_s) = \frac{\sum_{\ell=N_s-1+1}^{N_T} h(j_\ell)}{\sum_{\ell=1}^n h(\ell)} \\ 2. P(Y \leq \beta_s) = 1 - \frac{\sum_{\ell=N_s+1}^{N_T} h(j_\ell)}{\sum_{\ell=1}^n h(\ell)} \end{array} \right. \quad s = 1, \dots, T$$

where, if $\beta_1 = 0$, $P(X \leq 0)$ and $P(Y \leq 0)$ are defined by (2, 31.2) and (2, 32.2) respectively.

If $\beta_1 > 0$, condition (2, 29) reduces to Condition A_n and from Lemma 2, 3 it then follows that

$$(2, 33) \quad P(X \leq x) \leq P(Y \leq x) \text{ for all } x.$$

If $\beta_1 = 0$, condition (2, 29) is Condition $A_{n,M}$ for $M = N_T - N_1 = n - n_1$, so that in this case (2, 24) holds for $M \leq n - n_1$, which proves (2, 33)

From (2, 33) it follows that

$$(2, 34) \quad E X \geq E Y,$$

which is equivalent to

$$(2, 35) \quad \sum_{s=1}^T \beta_s \sum_{\ell=N_{s-1}+1}^{N_s} \epsilon_\ell h(i_\ell) \geq \sum_{s=1}^T \beta_s \sum_{\ell=N_{s-1}+1}^{N_s} \delta_\ell h(j_\ell),$$

which is equivalent to

$$(2, 36) \quad \sum_{\ell=1}^n \alpha_\ell \epsilon_\ell h(i_\ell) \geq \sum_{\ell=1}^n \alpha_\ell \delta_\ell h(j_\ell) \quad \text{Q. E. D.}$$

III, MAIN RESULTS.

Let, for each $v = 1, 2, \dots, X_{v,1}, \dots, X_{v,n_v}$ be independently and identically distributed random variables with common distribution function $F(x)$ satisfying

$$(3, 1) \quad \left\{ \begin{array}{l} 1. F(x) \text{ has an absolutely continuous density } f(x) \\ 2. \int_0^1 \varphi_F^2(u) du < \infty, \text{ where } \varphi_F(u) = - \frac{f'(F^{-1}(u))}{f(F^{-1}(u))} (0 \leq u \leq 1) \\ \text{and where } f' \text{ is the derivative of } f \\ 3. f(x) = f(-x) \text{ for all } x. \end{array} \right.$$

Let $\Psi(u)$ ($0 \leq u \leq 1$) be a function satisfying

$$(3, 2) \quad \left\{ \begin{array}{l} 1. \Psi(u) \text{ can be written as the sum of two functions } \Psi_1(u) \\ \text{and } \Psi_2(u) \text{ where } \Psi_1(u) \text{ is non decreasing and non negative} \\ \text{and } \Psi_2(u) \text{ is non increasing and non positive} \\ 2. \int_0^1 \Psi_i^2(u) du < \infty (i = 1, 2) \text{ and } \int_0^1 \Psi^2(u) du > 0 \end{array} \right.$$

Let $p_{v,1}, \dots, p_{v,n_v}$ and $q_{v,1}, \dots, q_{v,n_v}$ be vectors of constants satisfying

$$(3, 3) \quad \left\{ \begin{array}{l} 1. \sum_{i=1}^{n_v} p_{v,i}^2 > 0 \\ \max_{1 \leq i \leq n_v} p_{v,i}^2 \\ 2. \lim_{v \rightarrow \infty} \frac{\max_{1 \leq i \leq n_v} p_{v,i}^2}{\sum_{i=1}^{n_v} p_{v,i}^2} = 0, \end{array} \right.$$

$$(3, 4) \quad \left\{ \begin{array}{l} 1. \sum_{i=1}^{n_v} q_{v,i}^2 \leq M \text{ for some positive number } M \\ \text{independent of } v \\ 2. \lim_{v \rightarrow \infty} \max_{1 \leq i \leq n_v} q_{v,i}^2 = 0 \end{array} \right.$$

and, for each $v = 1, 2, \dots$, either

$$(3, 5) \quad \begin{cases} 1. p_{v,i} q_{v,i} \geq 0 & \text{for all } i = 1, \dots, n_v \\ 2. (|p_{v,i}| - |p_{v,i'}|) (|q_{v,i}| - |q_{v,i'}|) \geq 0 & \text{for all } i, i' = 1, \dots, n_v \end{cases}$$

or,

$$(3, 6) \quad \begin{cases} 1. p_{v,i} q_{v,i} \leq 0 & \text{for all } i = 1, \dots, n_v \\ 2. (|p_{v,i}| - |p_{v,i'}|) (|q_{v,i}| - |q_{v,i'}|) \geq 0 & \text{for all } i, i' = 1, \dots, n_v \end{cases}$$

Let $R_{|X_{v,i} - q_{v,i}\theta|}$ be the rank of $|X_{v,i} - q_{v,i}\theta|$ among

$|X_{v,1} - q_{v,1}\theta|, \dots, |X_{v,n_v} - q_{v,n_v}\theta|$, let

$$(3, 7) \quad \text{sgn } u = \begin{cases} 1 & \text{if } u > 0 \\ -1 & \text{if } u < 0 \end{cases}$$

and let

$$(3, 8) \quad T_v(\theta) = \sum_{i=1}^{n_v} p_{v,i} \Psi \left(\frac{R_{|X_{v,i} - q_{v,i}\theta|}}{n_v + 1} \right) \text{sgn}(X_{v,i} - q_{v,i}\theta).$$

Theorem 3 ; 1. If $F(x)$ is continuous, if $\Psi(u)$ is non decreasing and non negative then, for each $v = 1, 2, \dots$, $T_v(\theta)$ is with probability one a non increasing step function of θ if (3 ; 5) holds and a non decreasing step function of θ if (3 ; 6) holds :

Proof : In the proof the index v will be omitted. The proof will be given for the case that (3 ; 5) holds. The result for the case that (3 ; 6) holds is then obvious.

If $F(x)$ continuous, $T(\theta)$ is, with probability one, not well defined only for those values of θ satisfying $\theta = -\frac{x_i}{q_i}$ for some i with $q_i \neq 0$ and for those values of θ satisfying $|x_i - q_i\theta| = |x_{i'} - q_{i'}\theta|$ for some pair (i, i') with $q_i \neq 0$ or $q_{i'} \neq 0$. These values of θ where $T(\theta)$ is not well defined, define a finite number of intervals for θ within each of which $T(\theta)$ is independent of θ .

Now consider two values θ_1 and θ_2 of θ for which $T(\theta)$ is well defined and let $\theta_1 < \theta_2$. Then it will be proved that $T(\theta_1) \geq T(\theta_2)$.

Without loss of generality the X_1 can be numbered in such a way that

$|p_1| \leq \dots \leq |p_n|$ Then, by (3, 5.2), $|q_1| \leq \dots \leq |q_n|$. Write $T(\theta)$ as

$$(3, 9) \quad T(\theta) = \sum_{k=1}^n |p_k| \left(\frac{R |X_k - q_k \theta|}{n+1} \right) \operatorname{sgn} p_k (X_k - q_k \theta),$$

where, for $p_k = 0$, $\operatorname{sgn} p_k (X_k - q_k \theta)$ is defined as 1.

Now apply Lemma 2, 4 with, for $k = 1, \dots, n$

$$(3, 10) \quad \begin{cases} \alpha_k = |p_k| \\ \epsilon_k = \operatorname{sgn} p_k (X_k - q_k \theta_1) & \delta_k = \operatorname{sgn} p_k (X_k - q_k \theta_2) \\ j_k = R |X_k - q_k \theta_1| & j_k = R |X_k - q_k \theta_2| \end{cases}$$

Then $T(\theta_1) \geq T(\theta_2)$ if (2, 29) is satisfied. That (2, 29) is satisfied

can be seen from the following steps a), b) and c)

a) (2, 29.1) is identical with

$$\left\{ p_k (X_k - q_k \theta_2) > 0, p_k \neq 0 \right\} \implies p_k (X_k - q_k \theta_1) > 0$$

which follows immediately from (3, 5.1) and

$$p_k (X_k - q_k \theta_1) = p_k (X_k - q_k \theta_2) + p_k q_k (\theta_2 - \theta_1)$$

b) (2, 29.2) is identical with

$$\left\{ p_k (X_k - q_k \theta_2) > 0, p_k \neq 0, \ell < k, |X_\ell - q_\ell \theta_2| < |X_k - q_k \theta_2| \right\} \implies$$

$$|X_\ell - q_\ell \theta_1| < |X_k - q_k \theta_1|$$

This can be seen as follows. We have

$$-\frac{p_k}{|p_k|} (X_k - q_k \theta_2) < X_\ell - q_\ell \theta_2 < \frac{p_k}{|p_k|} (X_k - q_k \theta_2)$$

so that, using (3, 5),

$$\begin{aligned}
 x_{\ell} - q_{\ell} \theta_1 &< \frac{p_k}{|p_k|} (x_k - q_k \theta_1) + (\theta_2 - \theta_1) \left(q_{\ell} - \frac{p_k}{|p_k|} q_k \right) \\
 &= \frac{p_k}{|p_k|} (x_k - q_k \theta_1) + (\theta_2 - \theta_1) (q_{\ell} - |q_k|) \leq \\
 &\leq \frac{p_k}{|p_k|} (x_k - q_k \theta_1)
 \end{aligned}$$

also

$$\begin{aligned}
 x_{\ell} - q_{\ell} \theta_1 &> - \frac{p_k}{|p_k|} (x_k - q_k \theta_1) + (\theta_2 - \theta_1) \left(q_{\ell} + \frac{p_k}{|p_k|} q_k \right) \\
 &= - \frac{p_k}{|p_k|} (x_k - q_k \theta_1) + (\theta_2 - \theta_1) (q_{\ell} + |q_k|) \geq \\
 &\geq - \frac{p_k}{|p_k|} (x_k - q_k \theta_1),
 \end{aligned}$$

so that $|x_{\ell} - q_{\ell} \theta_1| \leq |x_k - q_k \theta_1|$.

c) (2, 29.3) is identical with

$$\left\{ p_k (x_k - q_k \theta_2) < 0, p_k \neq 0, \ell < k, |x_{\ell} - q_{\ell} \theta_2| > |x_k - q_k \theta_2| \right\} \implies |x_{\ell} - q_{\ell} \theta_1| > |x_k - q_k \theta_1|.$$

The **proof** of this is analogous to that for (2, 29.2). Q. E. D.

A special case of Theorem 3, 1 with $\Psi(u) = u$ and

$p_{v,i} = q_{v,i}$ ($i = 1, \dots, n_v$) was proved by Koul ([5], Lemma 2, 2).

Theorem 3, 2. If (3, 1) - (3, 4) and (3, 5) or (3, 6) are satisfied then

$$(3, 11) \quad \lim_{v \rightarrow \infty} P \left\{ \sup_{|\theta| \leq C} |T_v(\theta) - T_v(0)| + \theta K \sum_{i=1}^{n_v} p_{v,i} q_{v,i} |> \varepsilon \sigma(T_v(0)) \right\} = 0,$$

where $K = \int_0^1 \Psi(u) \varphi_F\left(\frac{u+1}{2}\right) du$.

Proof : The index v will be omitted in the proof. It is sufficient to prove the theorem for the case where $\Psi_2(u) = 0$ for all u . Further the proof will be given for the case where (3, 5) holds ; the result for

the case where (3 , 6) holds is then obvious.

The proof is analogous to the proof of Jurečková of her Theorem 3 , 1 in [2] . As in her case it can be supposed without loss of generality that $\sum_{i=1}^n p_i^2 = 1$ and it can be seen, using the result of Hájek and Šidák ([1] , Theorem 1. 7) that it is sufficient to prove

$$\lim_{v \rightarrow \infty} P \left\{ \sup_{|\theta| \leq C} |T(\theta) - T(o) + \theta K \sum_{i=1}^n p_i q_i| > \epsilon \right\} = 0$$

As in Jurečková's proof and using the results of Hájek and Šidák ([1] , section VI. 2. 5) it can be proved that for any fixed set of points $\theta_1, \dots, \theta_r$

$$\lim_{v \rightarrow \infty} P \left\{ |T(\theta_i) - T(o) + \theta_i K \sum_{i=1}^n p_i q_i| \leq \epsilon \text{ for all } i = 1 \dots r \right\} = 1$$

Further, for a fixed $C > 0$, choosing $\theta_1, \dots, \theta_r$ with

$$-C = \theta_1 < \theta_2 < \dots < \theta_{r-1} < \theta_r = C$$

and

$$K | \theta_{i+1} - \theta_i | \leq \frac{1}{2} \epsilon \frac{1}{\sqrt{M}},$$

where M is the constant in (3 , 4), it can be seen, as in Jurečková's proof and using theorem 3,1 above, that

$$\left\{ |T(\theta_i) - T(o) + \theta_i K \sum_{i=1}^n p_i q_i| \leq \frac{\epsilon}{2} \text{ for all } i = 1, \dots, r \right\} \implies$$

$$\sup_{|\theta| \leq C} |T(\theta) - T(o) + \theta K \sum_{i=1}^n p_i q_i| \leq \epsilon$$

Q. E. D.

The conditions on the $p_{v,1}$ and $q_{v,1}$ in Theorem 3 , 2 can be weakened as follows. (see also Jurečková [2] , Remark, page 1897). For every sequence of pairs of vectors $(p_{v,1}, \dots, p_{v,n_v})$, $(q_{v,1}, \dots, q_{v,n_v})$ it is possible to find a sequence of quadruplets of vectors

$$(p_{v,1}^{(\ell)}, \dots, p_{v,n_v}^{(\ell)}) , \ell = 1, 2, 3, 4 \text{ such that for each } v = 1, 2, \dots$$

$$(3, 12) \quad \left\{ \begin{array}{l} 1. \quad p_{v,i} = \sum_{\ell=1}^4 p_{v,i}^{(\ell)} \quad i = 1, \dots, n_v \\ 2. \quad p_{v,i}^{(\ell)} q_{v,i} \geq 0 \text{ for } \ell = 1, 2, i = 1, \dots, n_v \\ \quad p_{v,i}^{(\ell)} q_{v,i} \leq 0 \text{ for } \ell = 3, 4, i = 1, \dots, n_v \\ 3. \quad (|p_{v,i}^{(\ell)}| - |p_{v,i'}^{(\ell)}|) (|q_{v,i}| - |q_{v,i'}|) \geq 0 \quad \ell = 1, 2, 3, 4 \\ \quad \text{and } i, i' = 1, \dots, n_v \end{array} \right.$$

That this is possible can be seen as follows. For every pair of vectors $(p_{v,1}, \dots, p_{v,n_v})$, $(q_{v,1}, \dots, q_{v,n_v})$ one can find $\alpha_{v,1}, \dots, \alpha_{v,n_v}$, $\beta_{v,1}, \dots, \beta_{v,n_v}$ such that $p_{v,i} = \alpha_{v,i} + \beta_{v,i}$ and

$$(\alpha_{v,i} - \alpha_{v,i'}) (|q_{v,i}| - |q_{v,i'}|) \geq 0 \quad \text{for all } i, i' = 1, \dots, n_v$$

$$(\beta_{v,i} - \beta_{v,i'}) (|q_{v,i}| - |q_{v,i'}|) \leq 0$$

Further one can find $\gamma_v \geq 0$ such that $\alpha_{v,i} + \gamma_v \geq 0$, $\beta_{v,i} - \gamma_v \leq 0$ for all $i = 1, \dots, n_v$. By taking $p'_{v,i} = \alpha_{v,i} + \gamma_v$, $p''_{v,i} = \beta_{v,i} - \gamma_v$

one has found $p'_{v,1}, \dots, p'_{v,n_v}$ and $p''_{v,1}, \dots, p''_{v,n_v}$ such that $p_{v,i} = p'_{v,i} + p''_{v,i}$, $p'_{v,i} \geq 0$, $p''_{v,i} \leq 0$ ($i = 1, \dots, n_v$) and

$$(|p'_{v,i}| - |p'_{v,i'}|) (|q_{v,i}| - |q_{v,i'}|) \geq 0$$

$$\text{all } i, i' = 1, \dots, n_v$$

$$(|p''_{v,i}| - |p''_{v,i'}|) (|q_{v,i}| - |q_{v,i'}|) \geq 0$$

Further, if $p_{v,1}, \dots, p_{v,n_v}$ and $q_{v,1}, \dots, q_{v,n_v}$ satisfy the condition that the $p_{v,i}$ all have the same sign and

$$(3, 13) \quad (|p_{v,i}| - |p_{v,i'}|) (|q_{v,i}| - |q_{v,i'}|) \geq 0 \quad \text{all } i, i' = 1, \dots, n_v$$

then one can find $p'_{v,1}, \dots, p'_{v,n_v}$, $p''_{v,1}, \dots, p''_{v,n_v}$ such that

$$(3 ; 14) \quad \left\{ \begin{array}{l} 1. p_{v,i} = p'_{v,i} + p''_{v,i} \\ 2. p'_{v,i} q_{v,i} \cdot p_{v,i} \geq 0, \quad p''_{v,i} q_{v,i} \cdot p_{v,i} \leq 0 \quad i=1, \dots, n_v \\ 3. (|p'_{v,i}| - |p'_{v,i+1}|) (|q_{v,i}| - |q_{v,i+1}|) \geq 0 \\ \quad (|p''_{v,i}| - |p''_{v,i+1}|) (|q_{v,i}| - |q_{v,i+1}|) \geq 0. \end{array} \right.$$

This can be done as follows. Suppose, without loss of generality,

$|q_{v,i}| \leq |q_{v,i+1}| \quad i = 1, \dots, n_v - 1$ and take

$$p'_{v,i} = 2i p_{v,i} \frac{q_{v,i}}{|q_{v,i}|} \quad p''_{v,i} = \left[1 - 2i \frac{q_{v,i}}{|q_{v,i}|} \right] p_{v,i},$$

where $\frac{q_{v,i}}{|q_{v,i}|}$ is taken as 1 if $q_{v,i} = 0$. Then

$$p'_{v,i} q_{v,i} p_{v,i} = 2i p_{v,i}^2 |q_{v,i}| \geq 0$$

$$p''_{v,i} q_{v,i} p_{v,i} = \left[q_{v,i} - 2i |q_{v,i}| \right] p_{v,i}^2 \leq 0$$

Further, using (3 ; 13) ,

$$|p'_{v,i+1}| - |p'_{v,i}| = (2i+1) |p_{v,i+1}| - 2i |p_{v,i}| \geq |p_{v,i}| \geq 0$$

and, again using (3 ; 13) ,

$$|p''_{v,i+1}| - |p''_{v,i}| \geq |p_{v,i}| \left\{ \left| 1 - (2i+2) \frac{q_{v,i+1}}{|q_{v,i+1}|} \right| - \left| 1 - 2i \frac{q_{v,i}}{|q_{v,i}|} \right| \right\} \geq 0,$$

because $\left| 1 - 2i \frac{q_{v,i}}{|q_{v,i}|} \right|$ is non decreasing in i .

Further it is clear that, if $p_{v,1}, \dots, p_{v,n_v}$ satisfies

$\sum_{i=1}^{n_v} p_{v,i}^2 > 0$ for each v (condition 3 ; 3.1), then, for each v , there

exists an ℓ ($\ell = 1, 2, 3, 4$) such that $\sum_{i=1}^{n_v} \left\{ p_{v,i}^{(\ell)} \right\}^2 > 0$. Also, if

$p_{v,i}$ is written as $\sum_{\ell=1}^4 p_{v,i}^{(\ell)}$, $T_v(0)$ can be written as the sum of four

statistics and (3 ; 11) remains true, if it is true for each of these

four statistics and

$$(3;15) \quad \sum_{k=1}^4 \sum_{i=1}^{n_v} \{p_{v,i}^{(k)}\}^2 \leq M_1 \sum_{i=1}^{n_v} p_{v,i}^2$$

for some M_1 independent of v . Further (3;11) is true for each of these four statistics if (3;1), (3;2) and (3;4) are satisfied and the $p_{v,i}^{(k)}$ satisfy (3;12) and

$$(3;16) \quad \left\{ \begin{array}{l} 1. \text{ for at least one } k \\ \sum_{i=1}^{n_v} \{p_{v,i}^{(k)}\}^2 > 0 \quad \text{for each } v \\ 2. \text{ For each } k \text{ for which 1. is not satisfied} \\ \sum_{i=1}^{n_v} \{p_{v,i}^{(k)}\}^2 = 0 \quad \text{for each } v \\ 3. \text{ For each } k \text{ for which 1. is satisfied} \\ \lim_{v \rightarrow \infty} \frac{\max_{1 \leq i \leq n_v} \{p_{v,i}^{(k)}\}^2}{\sum_{i=1}^{n_v} \{p_{v,i}^{(k)}\}^2} = 0 \end{array} \right.$$

This proves the following theorem.

Theorem 3, 3 : If (3, 1), (3, 2) and (3, 4) are satisfied, if there exist $p_{v,1}^{(1)}, \dots, p_{v,n_v}^{(k)}$ ($k=1, 2, 3, 4$) such that (3;12), (3;15) and (3;16) are satisfied then (3, 11) holds.

A special case of Theorem 3, 2 with $p_{v,i} = q_{v,i} = \frac{1}{\sqrt{n_v}}$ was

used by Kraft and van Eeden [3], [4] to find the asymptotic properties of linearized estimates based on signed ranks for the one sample location problem.

Koul [5] proves a theorem analogous to Theorem 3, 2 for the p variate case where $R |X_{v,i} - q_{vi} \theta|$ is replaced by

$$R |X_{v,i} - \sum_{j=1}^p q_{v,ij} \theta_j| \quad . \text{ He considers the case where } p_{v,i} = q_{v,i}$$

for some j and all $i = 1, \dots, n_j$; further in his case $\Psi(u) = u$
and his conditions on F are stronger than (3, 1)

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