

AUTOMORPHISM GROUPS OF POLYCYCLIC-BY-FINITE GROUPS AND ARITHMETIC GROUPS

by OLIVER BAUES *and* FRITZ GRUNEWALD

ABSTRACT

We show that the outer automorphism group of a polycyclic-by-finite group is an arithmetic group. This result follows from a detailed structural analysis of the automorphism groups of such groups. We use an extended version of the theory of the algebraic hull functor initiated by Mostow. We thus make applicable refined methods from the theory of algebraic and arithmetic groups. We also construct examples of polycyclic-by-finite groups which have an automorphism group which does not contain an arithmetic group of finite index. Finally we discuss applications of our results to the groups of homotopy self-equivalences of $K(\Gamma, 1)$ -spaces and obtain an extension of arithmeticity results of Sullivan in rational homotopy theory.

CONTENTS

1. Introduction	214
1.1 The main results	214
1.2 Outline of the proofs and more results	216
1.3 Cohomological representations	219
1.4 Applications to the groups of homotopy self-equivalences of spaces	220
2. Prerequisites on linear algebraic groups and arithmetic groups	223
2.1 The general theory	223
2.2 Algebraic groups of automorphisms	226
3. The group of automorphisms of a solvable-by-finite linear algebraic group	228
3.1 The algebraic structure of $\text{Aut}_q(\mathbf{H})$	229
3.2 Arithmetic subgroups of $\text{Aut}_q(\mathbf{H})$	231
3.3 Some closed subgroups of $\text{Aut}_q(\mathbf{H})$	232
4. The algebraic hull of a polycyclic-by-finite wfn-group	233
4.1 A faithful rational representation of $\text{Aut}(\Gamma)$	235
4.2 Thickenings of Γ in \mathbf{H}_Γ	235
4.3 The automorphism group of the thickening	236
5. Thickenings of Γ admit a supplement	236
6. Lemmas from group theory	238
7. Unipotent shadows of Γ	239
8. Arithmetic subgroups of $\text{Aut}(\Gamma)$	242
9. The automorphism group of Γ as a subgroup of $\text{Aut}_q(\mathbf{H}_\Gamma)$	245
10. Extensions and quotients of arithmetic groups	247
11. The arithmeticity of $\text{Out}(\Gamma)$	253
11.1 The case of polycyclic-by-finite wfn-groups	253
11.2 The case of a general polycyclic-by-finite group	255
12. Polycyclic groups with non-arithmetic automorphism groups	258
12.1 Automorphism groups of semi-direct products	258
12.2 Examples	259
13. Cohomology representations of $\text{Out}(\Gamma)$	261
13.1 Automorphisms of Lie algebra cohomology	262
13.2 Computation of $H^*(\Gamma, \mathbf{C})$ via geometry and $\text{Aut}(\Gamma)$ -actions	263
13.3 Rational action of $\text{Out}_q(\mathbf{H}_\Gamma)$ on $H^*(\Gamma, \mathbf{C})$	265

1. Introduction

1.1. The main results. — We write $\text{Aut}(\Gamma)$ for the group of automorphisms of a group Γ . The subgroup consisting of automorphisms induced by conjugations with elements of Γ is denoted by Inn_Γ . It is normal in $\text{Aut}(\Gamma)$ and the quotient group

$$\text{Out}(\Gamma) := \text{Aut}(\Gamma)/\text{Inn}_\Gamma$$

is called the outer automorphism group of Γ . This paper is devoted to the detailed study of the groups $\text{Aut}(\Gamma)$ and $\text{Out}(\Gamma)$ in case Γ is polycyclic-by-finite. Here we say that a group Γ is \mathcal{E} -by-finite whenever \mathcal{E} is a property of groups and when Γ has a subgroup of finite index having property \mathcal{E} . As one of our main results we prove:

Theorem 1.1. — *For any polycyclic-by-finite group Γ , $\text{Out}(\Gamma)$ is an arithmetic group.*

To explain the concept of an arithmetic group we recall that a \mathbf{Q} -defined linear algebraic group G is a subgroup $G \leq \text{GL}(n, \mathbf{C})$ ($n \in \mathbf{N}$) which is also an affine algebraic set defined by polynomials with rational coefficients in the natural coordinates of $\text{GL}(n, \mathbf{C})$. If \mathbf{R} is a subring of \mathbf{C} we put $G(\mathbf{R}) := G \cap \text{GL}(n, \mathbf{R})$. We have $G = G(\mathbf{C})$. Let G be a \mathbf{Q} -defined linear algebraic group. A subgroup $\Gamma \leq G(\mathbf{Q})$ is called an *arithmetic subgroup* of G if Γ is commensurable with $G(\mathbf{Z})$. An abstract group Δ is called *arithmetic* if it is isomorphic to an arithmetic subgroup of a \mathbf{Q} -defined linear algebraic group. Two subgroups Γ_1, Γ_2 of $\text{GL}(n, \mathbf{C})$ are called *commensurable* if their intersection $\Gamma_1 \cap \Gamma_2$ has finite index both in Γ_1 and Γ_2 . These definitions are taken from [35].

In case Γ is a finitely generated nilpotent group, Segal [41] observed that $\text{Out}(\Gamma)$ contains an arithmetic subgroup of finite index. Such a group need itself not be arithmetic, see [21] for examples. Our Theorem 1.1 is a strengthening of Segal's result even in this restricted case. The results of Sullivan [44] imply for finitely generated nilpotent groups a still weaker result on their outer automorphism groups. We shall come back to this in Section 1.4. A related result, for any polycyclic-by-finite group, is proved by Wehrfritz in [46]. He shows that $\text{Out}(\Gamma)$ admits a faithful representation into $\text{GL}(n, \mathbf{Z})$, for some n , in this general case.

A role model for these results is the case $\Gamma = \mathbf{Z}^n$ ($n \in \mathbf{N}$). Here we have

$$\text{Aut}(\mathbf{Z}^n) = \text{Out}(\mathbf{Z}^n) = \text{GL}(n, \mathbf{Z})$$

and both $\text{Aut}(\mathbf{Z}^n)$ and $\text{Out}(\mathbf{Z}^n)$ are arithmetic groups. More generally, $\text{Aut}(\Gamma)$ is arithmetic for every finitely generated nilpotent group Γ , see [40, Corollary 9, Chapter 6]. This result was obtained for torsion-free finitely generated nilpotent groups by Auslander and Baumslag, see [4–6]. We generalize it to the case of finitely generated nilpotent-by-finite groups in Corollary 1.11. But this is not the general picture. We now describe a polycyclic group so that $\text{Aut}(\Gamma)$ does not contain an arithmetic sub-

group of finite index. To do this we choose $d \in \mathbf{N}$ not a square, and let $\mathbf{K} = \mathbf{Q}(\sqrt{d})$ be the corresponding real quadratic number field and write $x \mapsto \bar{x}$ for the non-trivial element of the Galois group of \mathbf{K} over \mathbf{Q} . Consider the subring $\mathcal{O} = \mathbf{Z} + \mathbf{Z}\sqrt{d} \leq \mathbf{K}$. By Dirichlet's unit theorem we may choose a unit $\epsilon \in \mathcal{O}^*$ which is of infinite order and satisfies $\epsilon\bar{\epsilon} = 1$. Define

$$(1) \quad D_\infty := \langle A, \tau \mid \tau^2 = (A\tau)^2 = 1 \rangle$$

to be the infinite dihedral group. It is easy to see that $F = \mathcal{O} \times \mathbf{Z}$ obtains the structure of a D_∞ -module by defining

$$(2) \quad A \cdot (m, n) = (\epsilon m, n), \quad \tau \cdot (m, n) = (\bar{m}, -n) \quad (m \in \mathcal{O}, n \in \mathbf{Z}).$$

We denote the corresponding split extension by

$$\Gamma(\epsilon) := F \rtimes D_\infty.$$

The group $\Gamma(\epsilon)$ is polycyclic and even an arithmetic group, but we have:

Theorem 1.2. — *The automorphism group of $\Gamma(\epsilon)$ does not contain an arithmetic group of finite index.*

The statement of Theorem 1.2 is stronger than just saying that the automorphism group of $\Gamma(\epsilon)$ is not arithmetic, since there are groups (see [21]) which are not arithmetic but which contain a subgroup of finite index which has this property. Our examples show that the structure of the automorphism group varies dramatically when Γ is replaced by one of its subgroups of finite index. It is for example easy to see (also from results in Subsection 1.2) that the automorphism group of the subgroup of $\Gamma(\epsilon)$ which is generated by F and A is an arithmetic group. More generally, as we will explain below, every polycyclic-by-finite group has a finite index subgroup with an arithmetic automorphism group.

Theorem 1.2 is complemented by a result of Merzljakov [30] who has proved, for any polycyclic group Γ , that $\text{Aut}(\Gamma)$ has a faithful representation into $\text{GL}(n, \mathbf{Z})$ for some $n \in \mathbf{N}$. This generalizes also to the case of polycyclic-by-finite groups, see [47].

Let $F = \text{Fitt}(\Gamma)$ be the Fitting subgroup of Γ . This is the maximal nilpotent normal subgroup of Γ . Since F is characteristic in Γ (that is normalized by every automorphism of Γ), we may define

$$(3) \quad A_{\Gamma/F} := \{\phi \in \text{Aut}(\Gamma) \mid \phi|_{\Gamma/F} = \text{id}_{\Gamma/F}\}.$$

As our main structural result on the automorphism group of a general polycyclic-by-finite group we show:

Theorem 1.3. — *Let Γ be a polycyclic-by-finite group. Then $A_{\Gamma|\mathbb{F}}$ is a normal subgroup of $\text{Aut}(\Gamma)$ and $A_{\Gamma|\mathbb{F}}$ is an arithmetic group. There exists a finitely generated nilpotent group $B \leq \text{Aut}(\Gamma)$ which consists of inner automorphisms such that*

$$A_{\Gamma|\mathbb{F}} \cdot B \leq \text{Aut}(\Gamma)$$

is a subgroup of finite index in the automorphism group of Γ .

This theorem is a more precise version of [40, Theorem 2, Chapter 8] and of certain similar theorems contained in [1]. It has many finiteness properties of $\text{Aut}(\Gamma)$ as a consequence, see [40, Chapter 8]. The fact that $\text{Aut}(\Gamma)$ is finitely presented was first proved by Auslander in [1].

1.2. Outline of the proofs and more results. — We shall describe now the general strategy in our proofs and more results on the structure of the automorphism group $\text{Aut}(\Gamma)$ of a general polycyclic-by-finite group Γ . We take the basic theory of polycyclic-by-finite groups for granted. As a reference for our notation and results we use the book [40].

A polycyclic-by-finite group Γ has a maximal finite normal subgroup τ_Γ . We say that Γ is a *wfn-group* if $\tau_\Gamma = \{1\}$. We first consider the case of polycyclic-by-finite groups which are wfn-groups and later reduce to this case.

Let now Γ be a polycyclic-by-finite group which is a wfn-group. We use the construction from [8] of a \mathbf{Q} -defined solvable-by-finite linear algebraic group \mathbf{H}_Γ which contains Γ in its group of \mathbf{Q} -rational points. The study of the functorial construction $\Gamma \mapsto \mathbf{H}_\Gamma$ traces back to work of Mostow [31, 32]. The algebraic group \mathbf{H}_Γ has special features which allow us to identify the group of algebraic automorphisms of \mathbf{H}_Γ , which we call $\text{Aut}_a(\mathbf{H}_\Gamma)$, with a \mathbf{Q} -defined linear algebraic group \mathcal{A}_Γ . In general, the group of algebraic automorphisms of a \mathbf{Q} -defined algebraic group is an extension of a linear algebraic by an arithmetic group, see [12]. The functoriality of \mathbf{H}_Γ leads to a natural embedding

$$(4) \quad \text{Aut}(\Gamma) \leq \text{Aut}_a(\mathbf{H}_\Gamma) = \mathcal{A}_\Gamma.$$

These steps are carried out in detail in Sections 3 and 4.

After preparations in Sections 5 and 7 we prove the following in Section 9.

Theorem 1.4. — *Let Γ be a polycyclic-by-finite group which is a wfn-group. Then $\text{Aut}(\Gamma)$ is contained in $\mathcal{A}_\Gamma(\mathbf{Q})$. A subgroup of finite index in $\text{Aut}(\Gamma)$ is contained in $\mathcal{A}_\Gamma(\mathbf{Z})$. The group $A_{\Gamma|\mathbb{F}}$ is an arithmetic subgroup in its Zariski-closure in \mathcal{A}_Γ .*

Theorem 1.4 is the main step in the proof of Theorem 1.3 which is given in Section 11.2.

In Section 10 we develop, starting from results in [21], a theory of proving arithmeticity for linear groups which are related to arithmetic groups in certain ways. This shows that the situation built up in Theorems 1.4 and 1.3 implies that $\text{Out}(\Gamma)$ is an arithmetic group for polycyclic-by-finite groups which are wfn-groups: To construct a \mathbf{Q} -defined algebraic group \mathcal{O}_Γ which contains $\text{Out}(\Gamma)$ as an arithmetic subgroup we start off by considering the Zariski-closure \mathcal{B} of $A_{\Gamma|\mathbb{F}}$ in \mathcal{A}_Γ . We then take the quotient group \mathcal{B}/\mathcal{C} where \mathcal{C} is the Zariski-closure of $\text{Inn}_\mathbb{F}$ in \mathcal{A}_Γ . The image of $A_{\Gamma|\mathbb{F}}$ in \mathcal{B}/\mathcal{C} is an arithmetic subgroup. We show that $\text{Out}(\Gamma)$ is a finite extension group of the quotient $A_{\Gamma|\mathbb{F}}/\text{Inn}_\mathbb{F}$ latexdocs its central subgroup $D = (A_{\Gamma|\mathbb{F}} \cap \text{Inn}_\Gamma)/\text{Inn}_\mathbb{F}$. We modify the algebraic group \mathcal{B}/\mathcal{C} to obtain a \mathbf{Q} -defined linear algebraic group \mathcal{E} such that the group D is unipotent-by-finite, and \mathcal{E} still contains an isomorphic copy of $A_{\Gamma|\mathbb{F}}/\text{Inn}_\mathbb{F}$ as an arithmetic subgroup. The construction ensures that $(A_{\Gamma|\mathbb{F}} \cap \text{Inn}_\Gamma)/\text{Inn}_\mathbb{F}$ is arithmetic in its Zariski-closure \mathcal{D} in \mathcal{E} and we can prove that $\text{Out}(\Gamma)$ is arithmetic in a finite extension group \mathcal{O}_Γ of the quotient \mathcal{E}/\mathcal{D} . The final definition of \mathcal{O}_Γ is contained in Section 11.1.

To prepare for later applications we introduce some more notation. The group of inner automorphisms $\text{Inn}_{\mathbf{H}_\Gamma}$ is by our constructions a \mathbf{Q} -closed subgroup of the group of algebraic automorphisms $\mathcal{A}_\Gamma = \text{Aut}_a(\mathbf{H}_\Gamma)$. We call the quotient

$$(5) \quad \text{Out}_a(\mathbf{H}_\Gamma) := \mathcal{A}_\Gamma / \text{Inn}_{\mathbf{H}_\Gamma}$$

the algebraic outer automorphism group of \mathbf{H}_Γ . It is again a \mathbf{Q} -defined linear algebraic group, and we obtain a group homomorphism

$$(6) \quad \pi_\Gamma : \text{Out}(\Gamma) \rightarrow \text{Out}_a(\mathbf{H}_\Gamma)$$

by restricting the quotient homomorphism $\mathcal{A}_\Gamma \rightarrow \text{Out}_a(\mathbf{H}_\Gamma)$ to the subgroup $\text{Aut}(\Gamma) \leq \mathcal{A}_\Gamma$. We then prove in Section 11.1:

Theorem 1.5. — *Let Γ be a polycyclic-by-finite group which is a wfn-group. Then there is a \mathbf{Q} -defined linear algebraic group \mathcal{O}_Γ which contains $\text{Out}(\Gamma)$ as an arithmetic subgroup and a \mathbf{Q} -defined homomorphism $\pi_{\mathcal{O}_\Gamma} : \mathcal{O}_\Gamma \rightarrow \text{Out}_a(\mathbf{H}_\Gamma)$ such that the diagram*

$$(7) \quad \begin{array}{ccc} \text{Out}(\Gamma) & \longrightarrow & \mathcal{O}_\Gamma \\ \pi_\Gamma \downarrow & \searrow \pi_{\mathcal{O}_\Gamma} & \\ \text{Out}_a(\mathbf{H}_\Gamma) & & \end{array}$$

is commutative. The kernel of π_Γ is finitely generated, abelian-by-finite and is centralized by a finite index subgroup of $\text{Out}(\Gamma)$. If Γ is nilpotent-by-finite then the kernel of π_Γ is finite.

We shall start the discussion of the general situation now. Let Γ be a polycyclic-by-finite group, possibly with τ_Γ non-trivial. Building on an idea of [40, Chapter 6, exercise 10] for nilpotent groups, we show in Section 11.2:

Proposition 1.6. — *Let Γ be polycyclic-by-finite group and let τ_Γ denote its maximal finite normal subgroup. Then the groups $\text{Aut}(\Gamma)$ and $\text{Aut}(\Gamma/\tau_\Gamma)$ are commensurable. If $\text{Aut}(\Gamma/\tau_\Gamma)$ is an arithmetic group then $\text{Aut}(\Gamma)$ is arithmetic too. If $\text{Aut}(\Gamma)$ is an arithmetic group then $\text{Aut}(\Gamma/\tau_\Gamma)$ contains a subgroup of finite index which is an arithmetic group.*

Recall that two abstract groups G and G' are called *commensurable* if they contain finite index subgroups $G_0 \leq G$ and $G'_0 \leq G'$ which are isomorphic. The examples in [21] show that a group can contain an arithmetic subgroup of finite index without being an arithmetic group. We do not know whether this phenomenon can arise in the situation of Proposition 1.6. As proved in [21], all arithmetic subgroups in algebraic groups which do not have a quotient isomorphic to $\text{PSL}(2)$ have only arithmetic groups as finite extensions.

The groups $\text{Out}(\Gamma)$ and $\text{Out}(\Gamma/\tau_\Gamma)$ satisfy a weaker equivalence relation (they are called *S-commensurable*, see Section 1.4).

Proposition 1.7. — *The natural homomorphism $\text{Out}(\Gamma) \rightarrow \text{Out}(\Gamma/\tau_\Gamma)$ has finite kernel and maps $\text{Out}(\Gamma)$ onto a finite index subgroup of $\text{Out}(\Gamma/\tau_\Gamma)$.*

Clearly, Γ/τ_Γ is a wfn-group. From Theorem 1.5 we already know that $\text{Out}(\Gamma/\tau_\Gamma)$ is an arithmetic group. We further know from [46] that $\text{Out}(\Gamma)$ is isomorphic to a subgroup of $\text{GL}(n, \mathbf{Z})$, for some $n \in \mathbf{N}$, and hence is residually finite. Recall that a group G is called *residually finite* if for every $g \in G$ with $g \neq 1$ there is a subgroup of finite index in G not containing g . We complete the proof of Theorem 1.1 in the general case by the following easy result.

Proposition 1.8. — *Let A be a residually finite group and E a finite normal subgroup in A . If A/E is arithmetic then A is an arithmetic group.*

Indeed, A embeds as a subgroup of finite index in the arithmetic group $A/E \times A/N$ where N is a normal subgroup of A with $N \cap E = \{1\}$ (cf. [40], Chapter 6, Lemma 7).

In [16] Deligne constructs lattices (in non-linear) Lie groups which map onto arithmetic groups with finite kernel but which are not residually finite. The above gives a strong converse to this result.

To contrast the examples from Theorem 1.2 we formulate now a simple condition on a polycyclic-by-finite group which implies that $\text{Aut}(\Gamma)$ is an arithmetic group. We write $\lambda : \Gamma \rightarrow \tilde{\Gamma} = \Gamma/\tau_\Gamma$ for the projection homomorphism. We define

$$(8) \quad \tilde{F}(\Gamma) := \lambda^{-1}(\text{Fitt}(\tilde{\Gamma})).$$

From general theory we infer that $\tilde{F}(\Gamma)$ is normal in Γ and that $\Gamma/\tilde{F}(\Gamma)$ is an abelian-by-finite group. Suppose that $\Sigma \leq \Gamma$ is a normal subgroup of finite index with $\tilde{F}(\Gamma) \leq \Sigma$ and the additional property that $\Sigma/\tilde{F}(\Gamma)$ is abelian. Then the finite group $\mu(\Gamma, \Sigma) = \Gamma/\Sigma$ acts through conjugation on the abelian group $\Sigma/\tilde{F}(\Gamma)$. We prove:

Theorem 1.9. — *Let Γ be a polycyclic-by-finite group. Assume that there is a normal subgroup $\Sigma \leq \Gamma$ which is of finite index, and such that $\Sigma/\tilde{F}(\Gamma)$ is abelian and $\mu(\Gamma, \Sigma) = \Gamma/\Sigma$ acts trivially on $\Sigma/\tilde{F}(\Gamma)$. Then $\text{Aut}(\Gamma)$ is an arithmetic group.*

Theorem 1.9 has the following immediate corollaries.

Corollary 1.10. — *Let Γ be a polycyclic-by-finite group. If $\Gamma/\text{Fitt}(\Gamma)$ is nilpotent then $\text{Aut}(\Gamma)$ is an arithmetic group.*

Corollary 1.11. — *The automorphism group of a finitely generated and nilpotent-by-finite group is an arithmetic group.*

Corollary 1.12. — *Every polycyclic-by-finite group has a finite index subgroup whose automorphism group is arithmetic.*

Analogue of Theorem 1.9 and Corollary 1.10 have been proved in [19] for arithmetic polycyclic-by-finite groups. The book [40] contains in Chapter 8 various results which also show the arithmeticity of $\text{Aut}(\Gamma)$, but the hypotheses on Γ are a lot stronger than ours.

1.3. Cohomological representations. — Let Γ be a torsion-free polycyclic-by-finite group, and \mathbf{R} a commutative ring. Let $H^*(\Gamma, \mathbf{R})$ denote the cohomology of Γ with (trivial) \mathbf{R} -coefficients. Since inner automorphisms act trivially on cohomology, the outer automorphism group $\text{Out}(\Gamma)$ is naturally represented on the cohomology space $H^*(\Gamma, \mathbf{R})$. The finite dimensional complex vector space $H^*(\Gamma, \mathbf{C})$ comes with an \mathbf{Z} -structure (and a resulting \mathbf{Q} -structure) given by the image of the base change homomorphism $H^*(\Gamma, \mathbf{Z}) \rightarrow H^*(\Gamma, \mathbf{C})$. This allows us to identify $\text{GL}(H^*(\Gamma, \mathbf{C}))$ with a \mathbf{Q} -defined linear algebraic group. The representation

$$(9) \quad \rho : \text{Out}(\Gamma) \rightarrow \text{GL}(H^*(\Gamma, \mathbf{C}))$$

is integral with respect to the above \mathbf{Z} -structure. Moreover, we define in Section 13.3, building on geometric methods developed in [8], a \mathbf{Q} -defined homomorphism

$$(10) \quad \eta : \text{Out}_a(\mathbf{H}_\Gamma) \rightarrow \text{GL}(H^*(\Gamma, \mathbf{C}))$$

which extends the homomorphism ρ via our homomorphism π_{θ_Γ} from Proposition 1.5. By composition, we obtain a \mathbf{Q} -defined homomorphism

$$(11) \quad \rho_{\theta_\Gamma} := \eta \circ \pi_{\theta_\Gamma} : \theta_\Gamma \rightarrow \text{GL}(H^*(\Gamma, \mathbf{C})).$$

We collect all of this together in the following theorem:

Theorem 1.13. — *Let Γ be a torsion-free polycyclic-by-finite group. Then there is a \mathbf{Q} -defined homomorphism $\rho_{\theta_\Gamma} : \theta_\Gamma \rightarrow \mathrm{GL}(\mathrm{H}^*(\Gamma, \mathbf{C}))$ such that the diagram*

$$(12) \quad \begin{array}{ccc} \mathrm{Out}(\Gamma) & \longrightarrow & \theta_\Gamma \\ \rho \downarrow & \swarrow \rho_{\theta_\Gamma} & \\ \mathrm{GL}(\mathrm{H}^*(\Gamma, \mathbf{C})) & & \end{array}$$

is commutative.

Thus, the Zariski-closure of the image of $\mathrm{Out}(\Gamma)$ in $\mathrm{GL}(\mathrm{H}^*(\Gamma, \mathbf{C}))$ is a \mathbf{Q} -closed subgroup and we have:

Theorem 1.14. — *Let Γ be a torsion-free polycyclic-by-finite group. Then the cohomology image $\rho(\mathrm{Out}(\Gamma))$ is an arithmetic subgroup of its Zariski-closure in $\mathrm{GL}(\mathrm{H}^*(\Gamma, \mathbf{C}))$. The kernel of ρ is a finitely generated subgroup of $\mathrm{Out}(\Gamma)$. If Γ is in addition nilpotent then the kernel of ρ is nilpotent by-finite.*

1.4. Applications to the groups of homotopy self-equivalences of spaces. — Let Γ be a group. The motivation to study the structure of $\mathrm{Out}(\Gamma)$ comes partially from topology since $\mathrm{Out}(\Gamma)$, for example, is naturally isomorphic to the group of homotopy classes of homotopy self-equivalences of any $\mathbf{K}(\Gamma, 1)$ -space (see [39]). A substantial theory using this tie between algebra and topology has been developed starting from the work of Sullivan [44]. We shall explain here the connections of our work with Sullivan's theory as described in [44] and we also mention the additions we can make to Sullivan's theory.

Sullivan considers spaces \mathbf{X} which have a nilpotent homotopy system of finite type, that is, the homotopy groups

$$(\pi_1(\mathbf{X}); \pi_2(\mathbf{X}), \pi_3(\mathbf{X}), \dots)$$

are all finitely generated, $\pi_1(\mathbf{X})$ is torsion-free nilpotent and acts nilpotently on all $\pi_i(\mathbf{X})$ ($i \geq 2$). Sullivan associates to such a space \mathbf{X} the minimal model of the \mathbf{Q} -polynomial forms of some complex representing \mathbf{X} . This is a finitely generated nilpotent differential algebra defined over \mathbf{Q} , it is called \mathcal{X} . Under the further assumption that \mathbf{X} is either a finite complex or that the homotopy groups $\pi_i(\mathbf{X})$ are trivial, for almost all $i \in \mathbf{N}$, he uses this construction to prove important results on the group $\mathrm{Aut}(\mathbf{X})$ of classes of homotopy self-equivalences of the space \mathbf{X} , cf. [44, Theorem 10.3].

Let us specialize, for a moment, to the case where \mathbf{X} is a $\mathbf{K}(\Gamma, 1)$ -space, Γ a torsion-free finitely generated nilpotent group. The results of Sullivan can then be related to what we prove. The following table contains a dictionary between the objects defined by Sullivan and those appearing in our theory.

Sullivan	algebraic theory
$\text{Aut}(\mathbf{X})$ \mathcal{X} $\text{Aut}(\mathcal{X})$	$\text{Out}(\Gamma)$ \mathfrak{g} $\text{Aut}(\mathfrak{g})$
$\text{Aut}_{\mathbf{Q}}(\mathbf{X}) = \text{Aut}(\mathcal{X})/\text{inner automorphisms}$	$\mathcal{O}_{\Gamma} \xrightarrow{\pi_{\mathcal{O}_{\Gamma}}} \text{Aut}(\mathfrak{g})/\text{Inn}_{\mathfrak{g}}$

Here \mathfrak{g} is the Lie algebra of the Malcev-completion of $\pi_1(\mathbf{X}) = \Gamma$. The associated DGA \mathcal{X} coincides with the Koszul-complex of \mathfrak{g} . The \mathbf{Q} -defined linear algebraic group \mathcal{O}_{Γ} is defined in the previous subsection. It admits a \mathbf{Q} -homomorphism $\pi_{\mathcal{O}_{\Gamma}}$ with finite kernel onto the group $\text{Aut}(\mathfrak{g})/\text{Inn}_{\mathfrak{g}}$ of Lie algebra automorphisms of \mathfrak{g} modulo inner automorphisms.

We mention the concept of *S-commensurability* of groups appearing in [44]. This is the equivalence relation amongst groups which is generated by the operations of taking quotients with finite kernel and finite index subgroups.

We shall now go through the four statements about $\text{Aut}(\mathbf{X})$ which are contained in [44, Theorem 10.3], specialized to a $\mathbf{K}(\Gamma, 1)$ space \mathbf{X} , and rephrase them in our context:

In Theorem 10.3 (i), it is stated that $\text{Aut}(\mathbf{X})$ is *S-commensurable with a full arithmetic subgroup of $\text{Aut}_{\mathbf{Q}}(\mathbf{X})$* . We prove (see Theorem 1.5), quite equivalently, that there is a homomorphism $\pi_{\Gamma} : \text{Out}(\Gamma) \rightarrow \text{Aut}(\mathfrak{g})/\text{Inn}_{\mathfrak{g}}$ which has finite kernel and an arithmetic subgroup as image. We additionally show that $\text{Out}(\Gamma)$ is isomorphic to an arithmetic subgroup in a \mathbf{Q} -defined linear algebraic group \mathcal{O}_{Γ} which is a finite extension group of $\text{Aut}(\mathfrak{g})/\text{Inn}_{\mathfrak{g}}$. (Examples ([16], [21]) show that the class of arithmetic groups is not closed under any of the two operations used to define S-commensurability. Hence the information on $\text{Aut}(\mathbf{X}) = \text{Out}(\Gamma)$ contained in [44, Theorem 10.3 i]), for our restricted case, is weaker than what follows from our results. Most dramatically there are groups which can be mapped onto an arithmetic group with a finite kernel which are not even residually finite. As Serre [42] remarks, localization results in [45] imply that the group $\text{Aut}(\mathbf{X})$ is residually finite. We wonder whether the general theory can be extended similarly to our results.)

In Theorem 10.3 (ii) it is proved that *the natural action of $\text{Aut}(\mathbf{X})$ on the integral homology is compatible with an algebraic matrix representation of $\text{Aut}_{\mathbf{Q}}(\mathbf{X})$ on the vector spaces of rational homology*. We consider here cohomology instead of homology and we obtain a similar result. We prove that the natural action of $\text{Out}(\Gamma) = \text{Aut}(\mathbf{X})$ on the image of integral cohomology is compatible with an algebraic matrix representation of \mathcal{O}_{Γ} on the vector spaces of rational cohomology (see Theorem 1.5 and Section 1.3). This has the important consequence that the stabilizers of cohomology classes are arithmetic groups.

By Theorem 10.3 (iii), the reductive part of $\text{Aut}_{\mathbf{Q}}(\mathbf{X})$ is faithfully represented on the natural subspace of homology generated by maps of spheres into \mathbf{X} . We recall (see Proposition 13.2) that the reductive part of $\text{Aut}(\mathfrak{g})/\text{Inn}_{\mathfrak{g}}$ is faithfully represented on the cohomology vector spaces. Of course, Sullivan's result implies that the representation is already faithful on the first homology space. We obtain a similar result for the action of \mathcal{O}_{Γ} on the first cohomology space.

In Theorem 10.3 (iv) it is proved that as we vary \mathbf{X} through finite complexes, $\text{Aut}(\mathbf{X})$ runs through every commensurability class of groups containing arithmetic groups. Here our results give something different and new since Sullivan uses simply connected spaces ($\pi_1(\mathbf{X}) = \{1\}$) which have non-vanishing higher homotopy groups to show his existence result:

Let \mathcal{N} -commensurability be the equivalence relation for groups which is generated by the operations of taking quotients with finitely generated nilpotent-by-finite groups as kernel and by taking finite index subgroups. We can prove:

Proposition 1.15. — *As we vary \mathbf{X} through all finite $\mathbf{K}(\Gamma, 1)$ complexes, Γ a finitely generated nilpotent group, $\text{Aut}(\mathbf{X})$ runs through every \mathcal{N} -commensurability class of groups containing an arithmetic group.*

The result just mentioned follows from our theory of cohomological representations and from the existence theorems in [15]. We do not provide details here.

Having discussed the relation of our results to Sullivan's we can mention the following generalization for spaces \mathbf{X} which are $\mathbf{K}(\Gamma, 1)$ spaces for torsion-free polycyclic-by-finite groups. In fact, we can collect the results described in Sections 1.2 and 1.3 to show:

Theorem 1.16. — *Let Γ be a torsion-free polycyclic-by-finite group and let \mathbf{X} a $\mathbf{K}(\Gamma, 1)$ space. Let $\text{Aut}(\mathbf{X})$ be the group of classes of homotopy self-equivalences of \mathbf{X} . Then the following hold:*

- i) $\text{Aut}(\mathbf{X})$ is an arithmetic group.
- ii) The action of $\text{Aut}(\mathbf{X})$ on integral cohomology is compatible with an algebraic matrix representation of the \mathbf{Q} -defined linear algebraic \mathcal{O}_{Γ} on the vector spaces of cohomology.
- iii) The stabilizer of an integral (rational) cohomology class is arithmetic.

Again, we wonder whether our Theorem 1.16 has an extension which does not assume the vanishing of the higher homotopy groups.

As a corollary, using well known finiteness results for arithmetic groups we get:

Corollary 1.17. — *Let Γ be a torsion-free polycyclic-by-finite group and let \mathbf{X} be a $\mathbf{K}(\Gamma, 1)$ -space. Let $\text{Aut}(\mathbf{X})$ be the group of classes of homotopy self-equivalences of \mathbf{X} . Then the following hold:*

- i) *The group $\text{Aut}(\mathbf{X})$ is residually finite.*
- ii) *The group $\text{Aut}(\mathbf{X})$ is finitely presented.*
- iii) *The group $\text{Aut}(\mathbf{X})$ contains only finitely many conjugacy classes of finite subgroups.*
- iv) *Let μ be a finite group acting by group automorphisms on $\text{Aut}(\mathbf{X})$, then the cohomology set $\mathbf{H}^1(\mu, \text{Aut}(\mathbf{X}))$ is finite.*
- v) *The group $\text{Aut}(\mathbf{X})$ is of type WF.*

For the definition and discussion of cohomology sets, see Section 6. The properties i), ..., v) all follow from the fact that $\text{Aut}(\mathbf{X})$ is an arithmetic group. Property i) is well known to hold for finitely generated linear groups, for ii), iii) see [9], iv) is proved in [21] (for a weaker form see [12]). The last item, property WF, means that every torsion-free subgroup Δ of finite index in $\text{Aut}(\Gamma)$ appears as the fundamental group of a finite $\mathbf{K}(\Delta, 1)$. See [42] for discussion of this property of arithmetic groups and further references. It is not known whether groups which are finite extensions of a WF-group inherit the property WF, see [43, 14]. Note also that properties ii), iii) are compatible with S-commensurability whereas i), iv) are not.

Let \mathbf{X} be a $\mathbf{K}(\Gamma, 1)$ -space for a group Γ . The automorphism group of Γ can be naturally identified with the group of classes of pointed homotopy self-equivalences $\text{Aut}^*(\mathbf{X})$ (see [39]). In case Γ is finitely generated torsion-free nilpotent it was well known that $\text{Aut}^*(\mathbf{X})$ is an arithmetic group. Our results from Theorem 1.9 extend this to a much larger class of groups. Our examples constructed in Section 12.2 and Theorem 1.2 give rise to $\mathbf{K}(\Gamma, 1)$ -spaces \mathbf{X} , where $\pi_1(\mathbf{X})$ is nice and arithmetic whereas $\text{Aut}^*(\mathbf{X})$ is far from being an arithmetic group.

Acknowledgement

The authors wish to thank Hans-Joachim Baues, Vladimir Platonov, Dan Segal and Wilhem Singhof for helpful conversations concerning the contents of this article.

2. Prerequisites on linear algebraic groups and arithmetic groups

In the sequel we shall use a certain amount of the theory of linear algebraic groups and also of the theory of arithmetic groups. We briefly review here what we need and also add certain consequences of the general theory. Our basic references are [11, 13, 35].

2.1. The general theory. — We use the usual terminology of Zariski-topology. Thus a linear algebraic group \mathcal{A} is a Zariski-closed subgroup of $\text{GL}(n, \mathbf{C})$, for some $n \in \mathbf{N}$. It is \mathbf{Q} -defined if it is \mathbf{Q} -closed. We use the shorter term \mathbf{Q} -closed for what should be called closed in the Zariski-topology with closed subsets being those affine

algebraic sets defined by polynomials with coefficients in \mathbf{Q} . If \mathbf{R} is a subring of \mathbf{C} , we put $\mathcal{A}(\mathbf{R}) = \mathcal{A} \cap \mathrm{GL}(n, \mathbf{R})$. We denote the connected component of the identity in a linear algebraic group \mathcal{A} by \mathcal{A}° . The connected component \mathcal{A}° always has finite index in \mathcal{A} and \mathcal{A} is called *connected* if $\mathcal{A} = \mathcal{A}^\circ$. If \mathcal{A} is defined over \mathbf{Q} , its group of \mathbf{Q} -points $\mathcal{A}(\mathbf{Q})$ is Zariski-dense in \mathcal{A} .

A *homomorphism of algebraic groups* is a morphism of the underlying affine algebraic varieties which is also a group homomorphism. A homomorphism is *defined over \mathbf{Q}* (or *\mathbf{Q} -defined*) if the corresponding morphism of algebraic varieties is defined over \mathbf{Q} . It is called a *\mathbf{Q} -defined isomorphism* if its inverse exists and is also a homomorphism defined over \mathbf{Q} . An *automorphism* is an isomorphism of a linear algebraic group to itself. We also use the more abstract concept of a \mathbf{Q} -defined linear algebraic group. It is well known that a \mathbf{Q} -defined affine variety equipped with a group structure given by \mathbf{Q} -defined morphisms is \mathbf{Q} -isomorphic with a \mathbf{Q} -closed subgroup of $\mathrm{GL}(n, \mathbf{C})$, for some $n \in \mathbf{N}$. As usual we write $\mathbf{G}_a = \mathbf{C}$ for the additive and $\mathbf{G}_m = \mathbf{C}^*$ for the multiplicative group. Quotients exist in the category of \mathbf{Q} -defined linear algebraic groups. That is given a \mathbf{Q} -closed normal subgroup \mathcal{N} of a \mathbf{Q} -defined linear algebraic group \mathcal{A} the quotient group \mathcal{A}/\mathcal{N} is a \mathbf{Q} -defined linear algebraic group and the natural map $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{N}$ is a \mathbf{Q} -defined homomorphism. We say that \mathcal{A} is the *almost direct product* of two Zariski-closed subgroups \mathcal{B}, \mathcal{C} if \mathcal{B} and \mathcal{C} centralize each other, their intersection $\mathcal{B} \cap \mathcal{C}$ is finite and if $\mathcal{A} = \mathcal{B} \cdot \mathcal{C}$ holds.

Let now \mathcal{A} be a \mathbf{Q} -defined linear algebraic group, we write $\mathbf{U}_{\mathcal{A}}$ for its *unipotent radical*. This is the largest Zariski-closed unipotent normal subgroup in \mathcal{A} , it is \mathbf{Q} -closed. The algebraic group \mathcal{A} is called *reductive* if $\mathbf{U}_{\mathcal{A}} = \{1\}$. In particular, the quotient $\mathcal{A}^{\mathrm{red}} = \mathcal{A}/\mathbf{U}_{\mathcal{A}}$ is reductive, for all linear algebraic groups \mathcal{A} . We write $\mathcal{A}^{\mathrm{sol}}$ for the solvable radical of \mathcal{A} . This is the largest connected Zariski-closed solvable normal subgroup in \mathcal{A} , it is \mathbf{Q} -closed. A connected linear algebraic group \mathcal{A} is called *semisimple* if $\mathcal{A}^{\mathrm{sol}} = \{1\}$.

A linear algebraic group is called a *d-group* if it consists of semisimple elements only. A Zariski-closed subgroup which is a d-group is called a *d-subgroup*. A d-group is reductive and abelian-by-finite. A *torus* is a linear algebraic group isomorphic to \mathbf{G}_m^n , for some $n \in \mathbf{N}$. If \mathcal{S} is a d-group then \mathcal{S}° is a torus.

Let \mathcal{A} be a \mathbf{Q} -defined linear algebraic group. Then the following hold (see [11–13, 35]).

- AG1:** There exists a \mathbf{Q} -closed reductive complement $\mathcal{R} \leq \mathcal{A}$, that is, $\mathcal{A} = \mathbf{U}_{\mathcal{A}} \cdot \mathcal{R}$ is a semi-direct product of subgroups, and $\mathcal{R} \cong \mathcal{A}^{\mathrm{red}}$. In particular, we also have $\mathcal{A} \cong \mathbf{U}_{\mathcal{A}} \rtimes \mathcal{A}^{\mathrm{red}}$. Here the symbol \cong indicates \mathbf{Q} -isomorphism of linear algebraic groups, and \rtimes indicates a semi-direct product.
- AG2:** All \mathbf{Q} -closed reductive complements are conjugate by elements of $\mathbf{U}_{\mathcal{A}}(\mathbf{Q})$.
- AG3:** Let \mathcal{A} be a \mathbf{Q} -defined connected and reductive linear algebraic group. Then \mathcal{A} is the almost direct product of its center, which is a \mathbf{Q} -closed

d-subgroup, and the commutator subgroup $[\mathcal{A}, \mathcal{A}]$, which is \mathbf{Q} -closed, connected and semisimple.

- AG4:** Let \mathcal{A} be a \mathbf{Q} -defined connected and semisimple linear algebraic group. Then its group $\text{Aut}_a(\mathcal{A})$ of algebraic automorphisms is a \mathbf{Q} -defined linear algebraic group and $\text{Inn}_{\mathcal{A}}$ is its connected component.
- AG5:** Let \mathcal{A} be a \mathbf{Q} -defined connected and reductive linear algebraic group. Let \mathcal{B} be \mathbf{Q} -closed normal subgroup of \mathcal{A} . Then there is a \mathbf{Q} -closed subgroup \mathcal{C} of \mathcal{A} which centralizes \mathcal{B} , satisfies $\mathcal{A} = \mathcal{B} \cdot \mathcal{C}$ such that $\mathcal{B} \cap \mathcal{C}$ is finite. That is, \mathcal{A} is the almost direct product of \mathcal{B} and \mathcal{C} .
- AG6:** Let \mathcal{S} be a commutative d-group and G a finite group of \mathbf{Q} -defined automorphisms of \mathcal{S} . Let $\mathcal{S}_1 \leq \mathcal{S}$ be a G -invariant \mathbf{Q} -closed subgroup. Then there is a G -invariant \mathbf{Q} -closed subgroup $\mathcal{S}_2 \leq \mathcal{S}$ such that \mathcal{S} is the almost direct product of \mathcal{S}_1 and \mathcal{S}_2 .
- AG7:** Let $\mathcal{T} \leq \mathcal{A}$ be a torus. Then the centralizer $Z_{\mathcal{A}}(\mathcal{T})$ of \mathcal{T} in \mathcal{A} has finite index in the normalizer $N_{\mathcal{A}}(\mathcal{T})$.

Statement AG6 is proved by using the category equivalence between the category of \mathbf{Q} -defined commutative d-groups and the category of continuous, \mathbf{Z} -finitely generated modules for the absolute Galois group of \mathbf{Q} (see [11, §8]) together with Maschke's theorem. The last fact AG7 is called the *rigidity of tori*, see [11, Corollary 2 of III.8] for a proof.

We shall add some remarks concerning the structure of solvable-by-finite groups. A linear algebraic group \mathbf{H} is solvable-by-finite iff its identity component \mathbf{H}° is solvable. A Cartan subgroup \mathbf{D} of \mathbf{H} is by definition the normalizer of a maximal torus \mathbf{T} in \mathbf{H} .

For a \mathbf{Q} -defined, solvable-by-finite linear algebraic group \mathbf{H} the following hold:

- SG1:** There are maximal tori $\mathbf{T} \leq \mathbf{H}$ which are \mathbf{Q} -closed.
- SG2:** Let \mathbf{T} be a maximal torus of \mathbf{H} and $\mathbf{D} = N_{\mathbf{H}}(\mathbf{T})$. Then $\mathbf{D}^\circ = N_{\mathbf{H}^\circ}(\mathbf{T})$ equals the centralizer of \mathbf{T} in \mathbf{H}° and is a connected nilpotent group. Moreover, \mathbf{D} contains a maximal d-subgroup \mathbf{S} with $\mathbf{S}^\circ = \mathbf{T}$.
- SG3:** Let $\mathbf{T} \leq \mathbf{H}$ be a maximal torus which is \mathbf{Q} -closed. Then $\mathbf{D} := N_{\mathbf{H}}(\mathbf{T})$ is \mathbf{Q} -closed and contains a maximal d-subgroup \mathbf{S} of \mathbf{H} which is \mathbf{Q} -closed and which satisfies $\mathbf{S}^\circ = \mathbf{T}$.
- SG4:** Let \mathbf{S} be a maximal d-subgroup of \mathbf{H} . Then $\mathbf{H} = \mathbf{U}_{\mathbf{H}} \cdot \mathbf{S} = \mathbf{U}_{\mathbf{H}} \rtimes \mathbf{S}$.
- SG5:** All maximal tori and also all maximal d-subgroups in \mathbf{H} are conjugate by elements of $[\mathbf{H}^\circ, \mathbf{H}^\circ] \leq \mathbf{U}_{\mathbf{H}}$.
- SG6:** All \mathbf{Q} -closed maximal tori and all \mathbf{Q} -closed maximal d-subgroups are conjugate by elements of $[\mathbf{H}^\circ, \mathbf{H}^\circ](\mathbf{Q})$.
- SG7:** Let \mathbf{D} be a Cartan subgroup of \mathbf{H} and $\mathbf{F} \leq \mathbf{U}_{\mathbf{H}}$ a normal subgroup of \mathbf{H} which contains $[\mathbf{H}^\circ, \mathbf{H}^\circ]$; then we have $\mathbf{H} = \mathbf{F} \cdot \mathbf{D}$.

For almost all of this see [11, Chapter III]. Statement SG6 is contained in [12], for a detailed proof see [19, Section 2].

We have defined the notion of an arithmetic subgroup of a \mathbf{Q} -defined linear algebraic group and that of an arithmetic group in the beginning of the introduction. We shall often have to use the behaviour of arithmetic subgroups under \mathbf{Q} -defined homomorphisms. To describe this, let $\rho : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ be a \mathbf{Q} -defined homomorphism between \mathbf{Q} -defined linear algebraic groups $\mathcal{A}_1, \mathcal{A}_2$ and $A \leq \mathcal{A}_1$ a subgroup. Furthermore we suppose that ρ is surjective. We have:

- AR1:** If A is an arithmetic subgroup of \mathcal{A}_1 then $\rho(A)$ is an arithmetic subgroup of \mathcal{A}_2 .
- AR2:** Suppose that ρ is injective, then A is an arithmetic subgroup of \mathcal{A}_1 if and only if $\rho(A)$ is commensurable with $\mathcal{A}_2(\mathbf{Z})$.
- AR3:** Every abelian subgroup of an arithmetic group is finitely generated.
- AR4:** Let \mathcal{A} be a \mathbf{Q} -defined group and $\mathcal{A}_1, \mathcal{A}_2$ \mathbf{Q} -defined subgroups such that $\mathcal{A} = \mathcal{A}_1 \rtimes \mathcal{A}_2$ is a semi-direct product. Let $A \leq \mathcal{A}(\mathbf{Q})$ be a subgroup. Assume there exist subgroups A_1, A_2 of A such that $A = A_1 \rtimes A_2$ is a semi-direct product. If A_1 and A_2 are arithmetic subgroups in \mathcal{A}_1 and \mathcal{A}_2 , respectively, then A is an arithmetic subgroup of \mathcal{A} .

Statement AR1 is proved in [10] and AR2 is a consequence of it. Statement AR2 shows that the notion of an arithmetic subgroup does not depend on the particular embedding of the ambient algebraic group into $\mathrm{GL}(n, \mathbf{C})$. Statement AR3 follows from Mal'cev's theorem on solvable subgroups of $\mathrm{GL}(n, \mathbf{Z})$, see [40], Chapter 2. Statement AR4 is a consequence of AR1.

2.2. Algebraic groups of automorphisms. — Let G be a group acting by automorphisms on a group A and let $B \leq A$ be a subgroup of A . We write

$$\begin{aligned} N_G(B) &:= \{g \in G \mid g(B) = B\}, \\ Z_G(B) &:= \{g \in G \mid g(b) = b \text{ for all } b \in B\} \end{aligned}$$

for the normalizer and centralizer of B in G .

Definition 2.1. — Let \mathcal{A} be a \mathbf{Q} -defined linear algebraic group and let G be a group which acts by \mathbf{Q} -defined automorphisms on \mathcal{A} . Then G normalizes the unipotent radical $\mathbf{U}_{\mathcal{A}}$ and we say that G acts as an algebraic group of automorphisms on \mathcal{A} if $Z_G(\mathcal{T})$ has finite index in $N_G(\mathcal{T})$ for every \mathbf{Q} -defined torus \mathcal{T} of $\mathcal{A}^{\mathrm{red}} = \mathcal{A}/\mathbf{U}_{\mathcal{A}}$.

Note that $G = \mathrm{GL}(n, \mathbf{Z})$ acts by \mathbf{Q} -defined automorphisms on the torus $\mathcal{A} = \mathbf{G}_m^n$. But for $n \geq 2$ this is not an algebraic group of automorphisms on \mathcal{A} . Positive examples are given in the next lemma.

Lemma 2.2. — Let \mathcal{B} be a \mathbf{Q} -defined linear algebraic group and $\mathcal{A} \leq \mathcal{B}$ a \mathbf{Q} -closed subgroup. Let further $\mathbf{G} \leq \mathcal{B}(\mathbf{Q})$ be a subgroup which normalizes \mathcal{A} . Then \mathbf{G} acts by conjugation on \mathcal{A} as an algebraic group of automorphisms.

Proof. — Replacing \mathcal{B} by the Zariski-closure of $\mathcal{A} \cdot \mathbf{G}$, we may assume that \mathcal{A} is normal in \mathcal{B} . Then $\mathbf{U}_{\mathcal{A}}$ is also normal in \mathcal{B} and the image of \mathbf{G} in $\mathcal{B}/\mathbf{U}_{\mathcal{A}}$ acts by conjugation on $\mathcal{A}/\mathbf{U}_{\mathcal{A}}$. Thus the lemma follows from the rigidity of tori (AG7).

Lemma 2.3. — Let \mathcal{A} be a \mathbf{Q} -defined linear algebraic group. Let \mathbf{G} be an algebraic group of automorphisms on \mathcal{A} . Then $Z_{\mathbf{G}}(\mathcal{S})$ has finite index in $N_{\mathbf{G}}(\mathcal{S})$ for every \mathbf{Q} -closed commutative d -subgroup \mathcal{S} of $\mathcal{A}/\mathbf{U}_{\mathcal{A}}$.

Proof. — To prove the lemma, we may replace \mathcal{A} by $\mathcal{A}/\mathbf{U}_{\mathcal{A}}$. We consider a \mathbf{Q} -closed commutative d -subgroup \mathcal{S} of \mathcal{A} . Its connected component \mathcal{S}° is a \mathbf{Q} -defined torus in \mathcal{A}° . We have $N_{\mathbf{G}}(\mathcal{S}) \leq N_{\mathbf{G}}(\mathcal{S}^{\circ})$. Our hypothesis implies that $Z_1 = N_{\mathbf{G}}(\mathcal{S}) \cap Z_{\mathbf{G}}(\mathcal{S}^{\circ})$ has finite index in $N_{\mathbf{G}}(\mathcal{S})$.

There is a finite Zariski-closed subgroup $\mathcal{E} \leq \mathcal{S}$ such that $\mathcal{S} = \mathcal{S}^{\circ} \cdot \mathcal{E}$. Let \mathcal{E}_0 be the kernel of the homomorphism from \mathcal{S} to \mathcal{S} which sends $x \in \mathcal{S}$ to $x^{|\mathcal{E}|}$. It is finite, contains \mathcal{E} and is normalized by $N_{\mathbf{G}}(\mathcal{S})$. Trivially $Z_2 = N_{\mathbf{G}}(\mathcal{S}) \cap Z_{\mathbf{G}}(\mathcal{E}_0)$ has finite index in $N_{\mathbf{G}}(\mathcal{S})$. Since $Z_1 \cap Z_2$ is contained in $Z_{\mathbf{G}}(\mathcal{S})$, this proves that $Z_{\mathbf{G}}(\mathcal{S})$ is of finite index in $N_{\mathbf{G}}(\mathcal{S})$.

The following lemmata describe natural operations which can be performed with algebraic groups of automorphisms.

Lemma 2.4. — Let \mathcal{A} be a \mathbf{Q} -defined linear algebraic group and \mathcal{B} a \mathbf{Q} -closed normal subgroup in \mathcal{A} . Let \mathbf{G} be an algebraic group of automorphisms of \mathcal{A} which normalizes \mathcal{B} . Then \mathbf{G} acts as a group of algebraic automorphisms on \mathcal{B} and also on \mathcal{A}/\mathcal{B} .

Proof. — This is clear for \mathcal{B} , since \mathcal{B} is normal in \mathcal{A} . To prove that \mathbf{G} acts as an algebraic group of automorphisms on \mathcal{A}/\mathcal{B} , we use a \mathbf{G} -invariant decomposition AG5. We leave the details to the reader.

Remark that, in general, the converse of the statements of Lemma 2.4 is not true. Nevertheless, we have:

Lemma 2.5. — Let \mathcal{A} and \mathcal{B} be \mathbf{Q} -defined linear algebraic groups, and let \mathbf{G} act as an algebraic group of automorphisms on \mathcal{A} and on \mathcal{B} . Then \mathbf{G} acts as an algebraic group of automorphisms on the product $\mathcal{C} = \mathcal{A} \times \mathcal{B}$.

Proof. — Let \mathbf{G} act on \mathcal{C} by the product action. Let \mathcal{T} be a torus in $\mathcal{C}^{\text{red}} = \mathcal{A}^{\text{red}} \times \mathcal{B}^{\text{red}}$ and put $\mathbf{G}' = N_{\mathbf{G}}(\mathcal{T})$. The torus \mathcal{T} is contained in the direct product of factors $\mathcal{T}_{\mathcal{A}} \leq \mathcal{A}^{\text{red}}$, $\mathcal{T}_{\mathcal{B}} \leq \mathcal{B}^{\text{red}}$ which are stabilized by \mathbf{G}' . Hence, both are central-

ized by a finite index subgroup of G' . Therefore, \mathcal{T} is centralized by this finite index subgroup of G' .

Lemma 2.6. — *Let \mathcal{A} be a \mathbf{Q} -defined linear algebraic group and G a group of \mathbf{Q} -defined automorphisms of \mathcal{A} . Suppose $L, H \leq G$ are subgroups such that L is normal in G and $G = L \cdot H$ holds. If both L and H are algebraic groups of automorphisms of \mathcal{A} then also G has this property.*

Proof. — We consider G as a group of \mathbf{Q} -defined automorphisms of $\mathcal{B} = (\mathcal{A}/\mathbf{U}_{\mathcal{A}})^{\circ}$. We have the almost direct decomposition $\mathcal{B} = \mathcal{Z} \cdot [\mathcal{B}, \mathcal{B}]$, where \mathcal{Z} is the center of \mathcal{B} . The group G normalizes both \mathcal{Z} and $[\mathcal{B}, \mathcal{B}]$. Let \mathcal{T} be a \mathbf{Q} -defined torus of \mathcal{B} . If $\mathcal{T} \leq [\mathcal{B}, \mathcal{B}]$ then $Z_G(\mathcal{T})$ has finite index in $N_G(\mathcal{T})$ since G contains a subgroup of finite index which acts by inner automorphisms on $[\mathcal{B}, \mathcal{B}]$ (AG4). Now assume $\mathcal{T} \leq \mathcal{Z}$. By our assumption on L and H it follows that $Z_G(\mathcal{Z}^0)$ has finite index in G (since G normalizes \mathcal{Z}^0), and, a fortiori, $Z_G(\mathcal{T})$ has finite index in $N_G(\mathcal{T})$. The case of a general \mathcal{T} can be treated by projecting \mathcal{T} on the factors of the almost direct product decomposition AG3.

In Section 10 the following fact plays a crucial role.

Proposition 2.7. — *Let \mathcal{A} be a connected \mathbf{Q} -defined linear algebraic group, equipped with an algebraic group of automorphisms G . Then there are:*

- i) a \mathbf{Q} -closed, G -invariant central d -subgroup \mathcal{Z}_1 in \mathcal{A} and
- ii) a \mathbf{Q} -closed, G -invariant subgroup \mathcal{A}_1 of \mathcal{A} with unipotent-by-finite center

such that \mathcal{A} is decomposed as the almost direct product of \mathcal{Z}_1 and \mathcal{A}_1 .

Proof. — Let \mathcal{A}^{red} be a maximal reductive subgroup of \mathcal{A} which is defined over \mathbf{Q} . Let \mathcal{Z} be the center of \mathcal{A}^{red} and let \mathcal{Z}_1 be the maximal torus in the center of \mathcal{A} . Thus \mathcal{Z}_1 is \mathbf{Q} -closed in \mathcal{A} , it is contained in \mathcal{Z} , and it is normalized by G .

By Lemma 2.3, the group G_1 of automorphisms of \mathcal{Z} which is induced by the quotient action of G on the isomorphic image of \mathcal{Z} in $\mathcal{A}/\mathbf{U}_{\mathcal{A}}$ is finite. Hence, by AG6, we may choose a \mathbf{Q} -closed and G_1 -invariant subgroup $\mathcal{Z}_2 \leq \mathcal{Z}$ such that \mathcal{Z} is the almost direct product of \mathcal{Z}_1 and \mathcal{Z}_2 . This shows that \mathcal{Z}_1 and $\mathcal{A}_1 = \mathbf{U}_{\mathcal{A}} \cdot \mathcal{Z}_2 \cdot [\mathcal{A}^{\text{red}}, \mathcal{A}^{\text{red}}]$ satisfy the requirements of the lemma.

3. The group of automorphisms of a solvable-by-finite linear algebraic group

Let \mathcal{A} be a linear algebraic group defined over \mathbf{Q} . We let

$$(13) \quad \text{Aut}_a(\mathcal{A}), \text{Aut}_{a, \mathbf{Q}}(\mathcal{A})$$

denote the group of all automorphisms of \mathcal{A} , respectively the group of all \mathbf{Q} -defined automorphisms of \mathcal{A} . We also need the following concept.

Definition 3.1. — We say that a linear algebraic group \mathcal{A} has a strong unipotent radical if the centralizer $Z_{\mathcal{A}}(\mathbf{U}_{\mathcal{A}})$ of its unipotent radical $\mathbf{U}_{\mathcal{A}}$ is contained in $\mathbf{U}_{\mathcal{A}}$.

Given a \mathbf{Q} -defined solvable-by-finite linear algebraic group \mathbf{H} with a strong unipotent radical we will identify $\text{Aut}_a(\mathbf{H})$ in a natural way with a \mathbf{Q} -defined linear algebraic group. We obtain special features of this identification which we will need later.

For a general linear algebraic group \mathcal{A} , the group $\text{Aut}_a(\mathcal{A})$ is a group of type ALA, that is, $\text{Aut}_a(\mathcal{A})$ is an extension of an affine algebraic group by an arithmetic group, see [12]. Our approach is a variation of corresponding results in [12]. A construction similar to ours, but in a more restricted situation, is contained in [19].

3.1. *The algebraic structure of $\text{Aut}_a(\mathbf{H})$.* — We will assume here that \mathbf{H} is a \mathbf{Q} -defined solvable-by-finite linear algebraic group which has a strong unipotent radical $\mathbf{U} := \mathbf{U}_{\mathbf{H}}$. Let \mathfrak{u} denote the Lie algebra of \mathbf{U} . The Lie algebra \mathfrak{u} is defined over \mathbf{Q} and $\text{Aut}(\mathfrak{u}) \leq \text{GL}(\mathfrak{u})$ is a \mathbf{Q} -defined linear algebraic group. The exponential map

$$(14) \quad \exp : \mathfrak{u} \rightarrow \mathbf{U}$$

is a \mathbf{Q} -defined isomorphism of varieties. Thus, via the bijective homomorphism

$$(15) \quad \text{Aut}_a(\mathbf{U}) \ni \Phi \mapsto \exp^{-1} \circ \Phi \circ \exp \in \text{Aut}(\mathfrak{u})$$

the group $\text{Aut}_a(\mathbf{U})$ attains a natural structure of a linear algebraic group which is defined over \mathbf{Q} .

We let $\mathbf{S} \leq \mathbf{H}$ be a maximal \mathbf{Q} -closed d-subgroup and obtain the decomposition $\mathbf{H} = \mathbf{U} \cdot \mathbf{S}$ described in SG4 of Section 2. We define

$$(16) \quad \text{Aut}_a(\mathbf{H})_{\mathbf{S}} := \{\Phi \in \text{Aut}_a(\mathbf{H}) \mid \Phi(\mathbf{S}) = \mathbf{S}\} \leq \text{Aut}_a(\mathbf{H}).$$

Lemma 3.2. — The restriction map $\Phi \mapsto \Phi|_{\mathbf{U}}$ identifies $\text{Aut}_a(\mathbf{H})_{\mathbf{S}}$ with a \mathbf{Q} -defined closed subgroup of $\text{Aut}_a(\mathbf{U})$.

Proof. — Since \mathbf{H} has a strong unipotent radical the restriction map is injective on $\text{Aut}_a(\mathbf{H})_{\mathbf{S}}$. Let $\text{Ad}(\mathbf{S})$ denote the image of \mathbf{S} under the restriction of the adjoint representation to \mathfrak{u} . Then $\text{Ad}(\mathbf{S})$ is a \mathbf{Q} -defined subgroup of $\text{Aut}(\mathfrak{u})$. Note that the obvious homomorphism $\mathbf{S} \rightarrow \text{Ad}(\mathbf{S})$ is a \mathbf{Q} -defined isomorphism since \mathbf{H} has a strong unipotent radical (see also [19], Lemma 2.3). It is clear that, via the exponential map, the restriction of $\text{Aut}_a(\mathbf{H})_{\mathbf{S}}$ to \mathbf{U} is isomorphic to the normalizer of $\text{Ad}(\mathbf{S})$ in $\text{Aut}(\mathfrak{u})$. Since $\text{Ad}(\mathbf{S})$ is a \mathbf{Q} -defined group, its normalizer and centralizer are Zariski-closed subgroups of $\text{Aut}(\mathfrak{u})$. By the Galois criterion for rationality (see [11, AG 14]) the normalizer and centralizer of $\text{Ad}(\mathbf{S})$ are defined over \mathbf{Q} . In particular, $\text{Aut}_a(\mathbf{H})_{\mathbf{S}}$ restricts to a \mathbf{Q} -defined subgroup of $\text{Aut}_a(\mathbf{U})$.

Thus the lemma furnishes a natural structure of \mathbf{Q} -defined linear algebraic group on $\mathrm{Aut}_a(\mathbf{H})_{\mathbf{S}}$. Since $\mathrm{Aut}_a(\mathbf{H})_{\mathbf{S}}$ acts on \mathbf{U} by \mathbf{Q} -defined morphisms, the semi-direct product $\mathbf{U} \rtimes \mathrm{Aut}_a(\mathbf{H})_{\mathbf{S}}$ is an affine algebraic group over \mathbf{Q} , and thus is also equipped with the natural structure of a \mathbf{Q} -defined linear algebraic group, see [11]. Let $\mathrm{Inn}_u \in \mathrm{Aut}_a(\mathbf{H})$ denote the inner automorphism corresponding to $u \in \mathbf{U}$, that is, $\mathrm{Inn}_u(h) = uhu^{-1}$, for all $h \in \mathbf{H}$. We consider the homomorphism

$$(17) \quad \Theta : \mathbf{U} \rtimes \mathrm{Aut}_a(\mathbf{H})_{\mathbf{S}} \longrightarrow \mathrm{Aut}_a(\mathbf{H}), \quad (u, \Phi) \mapsto \mathrm{Inn}_u \circ \Phi.$$

Lemma 3.3. — *The homomorphism Θ is surjective.*

Proof. — The group \mathbf{H} decomposes as a semi-direct product of \mathbf{Q} -defined algebraic groups $\mathbf{H} = \mathbf{U} \cdot \mathbf{S} \cong \mathbf{U} \rtimes \mathbf{S}$. All maximal d-subgroups of \mathbf{H} are conjugate by elements of \mathbf{U} , and \mathbf{Q} -defined d-subgroups are conjugate by elements of $\mathbf{U}(\mathbf{Q})$ (see Section 2). Therefore, the homomorphism Θ is onto $\mathrm{Aut}_a(\mathbf{H})$.

Lemma 3.4. — *The kernel of the homomorphism Θ is a \mathbf{Q} -defined unipotent subgroup of $\mathbf{U} \rtimes \mathrm{Aut}_a(\mathbf{H})_{\mathbf{S}}$.*

Proof. — In fact, $\ker \Theta = \{(u, \mathrm{Inn}_u^{-1}) \mid \mathrm{Inn}_u \in \mathrm{Aut}_a(\mathbf{H})_{\mathbf{S}}\}$ is a unipotent subgroup of $\mathbf{U} \rtimes \mathrm{Aut}_a(\mathbf{H})_{\mathbf{S}}$. Since the homomorphism Θ corresponds to a \mathbf{Q} -defined action of $\mathbf{U} \rtimes \mathrm{Aut}_a(\mathbf{H})_{\mathbf{S}}$ on the variety \mathbf{U} , its kernel is a \mathbf{Q} -defined subgroup.

Let

$$(18) \quad \mathcal{A}_{\mathbf{H}} := \mathbf{U} \rtimes \mathrm{Aut}_a(\mathbf{H})_{\mathbf{S}} / \ker \Theta$$

denote the quotient group of $\mathbf{U} \rtimes \mathrm{Aut}_a(\mathbf{H})_{\mathbf{S}}$ by the kernel of Θ . The group $\mathcal{A}_{\mathbf{H}}$ has a natural structure of a \mathbf{Q} -defined linear algebraic group, such that

$$(19) \quad \mathcal{A}_{\mathbf{H}}(\mathbf{Q}) = \mathbf{U} \rtimes \mathrm{Aut}_a(\mathbf{H})_{\mathbf{S}}(\mathbf{Q}) / (\ker \Theta)(\mathbf{Q}).$$

Therefore, we have:

Theorem 3.5. — *The homomorphism $\mathcal{A}_{\mathbf{H}} \rightarrow \mathrm{Aut}_a(\mathbf{H})$ which is induced by (17) naturally identifies $\mathrm{Aut}_a(\mathbf{H})$ with the complex points of the \mathbf{Q} -defined linear algebraic group $\mathcal{A}_{\mathbf{H}}$.*

By arguments using the above setup, the rational points of the corresponding affine algebraic group are naturally interpreted in the following way:

Proposition 3.6. — *Under the homomorphism $\mathcal{A}_{\mathbf{H}} \rightarrow \mathrm{Aut}_a(\mathbf{H})$ which is induced by (17) the group of rational points $\mathcal{A}_{\mathbf{H}}(\mathbf{Q})$ of $\mathcal{A}_{\mathbf{H}}$ corresponds to the group $\mathrm{Aut}_{a, \mathbf{Q}}(\mathbf{H})$ of \mathbf{Q} -defined automorphisms of \mathbf{H} .*

Proof. — Let $\Phi \in \mathcal{A}_{\mathbf{H}}(\mathbf{Q})$. It is clear from our construction that Φ preserves the group of rational points $\mathbf{H}(\mathbf{Q}) \leq \mathbf{H}$ which is Zariski-dense in \mathbf{H} (cf. [11, §18.2]). By the Galois-criterion for rationality (compare [11, AG 14]), Φ is defined over \mathbf{Q} .

Conversely, assume $\Phi \in \text{Aut}_a(\mathbf{H})_{\mathbf{Q}}$ is a \mathbf{Q} -defined automorphism of \mathbf{H} . Then, by AG2, there exists $v \in \mathbf{U}(\mathbf{Q})$ such that $\Psi = \text{Inn}_v \circ \Phi \in \text{Aut}_a(\mathbf{H})_{\mathbf{S}}$, and Ψ is defined over \mathbf{Q} as well. The exponential correspondence shows that $\Psi \in \text{Aut}_a(\mathbf{H})_{\mathbf{S}}(\mathbf{Q})$. It follows that $\Phi \in \text{Aut}_a(\mathbf{H})(\mathbf{Q})$.

Henceforth, we will identify $\text{Aut}_a(\mathbf{H})$ with (the complex points of) the linear algebraic group $\mathcal{A}_{\mathbf{H}}$, and $\Theta : \mathbf{U} \rtimes \text{Aut}_a(\mathbf{H})_{\mathbf{S}} \rightarrow \mathcal{A}_{\mathbf{H}} = \text{Aut}_a(\mathbf{H})$ becomes a morphism of algebraic groups which is defined over \mathbf{Q} .

3.2. Arithmetic subgroups of $\text{Aut}_a(\mathbf{H})$. — We keep the notation of the previous subsection. In particular, \mathbf{H} denotes a \mathbf{Q} -defined solvable-by-finite linear algebraic group with a strong unipotent radical $\mathbf{U} = \mathbf{U}_{\mathbf{H}}$. Let further \mathbf{S} be a maximal \mathbf{Q} -defined d-subgroup of \mathbf{H} .

Let $\theta \leq \mathbf{U}(\mathbf{Q})$ be a finitely generated subgroup which is Zariski-dense in \mathbf{U} . Then θ is an arithmetic subgroup of the \mathbf{Q} -defined group \mathbf{U} . We explain now how the choice of θ gives rise to an arithmetic subgroup Λ_{θ} of $\text{Aut}_a(\mathbf{H}) = \mathcal{A}_{\mathbf{H}}$.

Definition 3.7. — We define the following subgroups of $\text{Aut}_a(\mathbf{H})$:

$$\begin{aligned} (20) \quad \mathcal{A}_{\theta} &:= \{\Phi \in \text{Aut}_a(\mathbf{H}) \mid \Phi(\theta) = \theta\}, & \Lambda_{\theta} &:= \text{Aut}_a(\mathbf{H})_{\mathbf{S}} \cap \mathcal{A}_{\theta}, \\ (21) \quad \Lambda_{\theta} &:= \Theta(\theta \rtimes \Lambda_{\theta}) \leq \mathcal{A}_{\mathbf{H}}(\mathbf{Q}). \end{aligned}$$

Lemma 3.8. — Let $\theta \leq \mathbf{U}(\mathbf{Q})$ be a finitely generated subgroup which is Zariski-dense in \mathbf{U} . Then the group Λ_{θ} is an arithmetic subgroup of $\text{Aut}_a(\mathbf{H})_{\mathbf{S}}$.

Proof. — As explained in Subsection 3.1, the group $\text{Aut}_a(\mathbf{H})_{\mathbf{S}}$ is \mathbf{Q} -isomorphic to the normalizer $N_{\text{Aut}(\mathfrak{u})}(\text{Ad}(\mathbf{S}))$ via the isomorphism (15). Embedding $\text{Aut}(\theta)$ as a subgroup of $\text{Aut}_a(\mathbf{U})$, it corresponds to

$$N_{\text{Aut}(\mathfrak{u})}(\log \theta) = \{\Psi \in \text{Aut}(\mathfrak{u}) \mid \Psi(\log \theta) \subseteq \log \theta\} \leq \text{Aut}(\mathfrak{u}).$$

The group $N_{\text{Aut}(\mathfrak{u})}(\log \theta)$ stabilizes the lattice L in \mathfrak{u} which is generated by the set $\log \theta$. Moreover, $N_{\text{Aut}(\mathfrak{u})}(\log \theta)$ is arithmetic in $\text{Aut}(\mathfrak{u})$, see [40, Chapter 6]. It follows that $\Lambda_{\theta} \leq \text{Aut}_a(\mathbf{H})_{\mathbf{S}}$ corresponds to

$$N_{\text{Aut}(\mathfrak{u})}(\text{Ad}(\mathbf{S})) \cap N_{\text{Aut}(\mathfrak{u})}(\log \theta(\Gamma)).$$

Thus, under the natural linear representation of $\text{Aut}_a(\mathbf{H})_{\mathbf{S}}$ in $\text{GL}(\mathfrak{u})$, Λ_{θ} is commensurable to the stabilizer of a lattice $L \subset \mathfrak{u}(\mathbf{Q})$. Therefore, Λ_{θ} is arithmetic in $\text{Aut}_a(\mathbf{H})_{\mathbf{S}}$.

Using AR4 in Section 2 we deduce that $\theta \rtimes \Lambda_\theta$ is arithmetic in $\mathbf{U} \rtimes \text{Aut}_a(\mathbf{H})_{\mathbf{S}}$. Since Λ_θ is the image of the arithmetic group $\theta \rtimes \Lambda_\theta$ under the \mathbf{Q} -defined surjective homomorphism Θ , Λ_θ is arithmetic in $\text{Aut}_a(\mathbf{H})$. This proves:

Proposition 3.9. — *Let $\theta \leq \mathbf{U}(\mathbf{Q})$ be a finitely generated subgroup which is Zariski-dense in \mathbf{U} . Then the group Λ_θ is an arithmetic subgroup of $\text{Aut}_a(\mathbf{H})$.*

3.3. Some closed subgroups of $\text{Aut}_a(\mathbf{H})$. — We stick to the conventions about \mathbf{H} , \mathbf{U} , \mathbf{S} . In addition, we introduce $\mathbf{F} \leq \mathbf{U}$ to be a \mathbf{Q} -closed normal subgroup of \mathbf{H} which contains the commutator group $[\mathbf{H}^\circ, \mathbf{H}^\circ]$. By SG7 of Section 2, we have $\mathbf{H} = \mathbf{F} \cdot \mathbf{D}$, for every \mathbf{Q} -closed Cartan subgroup of \mathbf{H} . As before \mathbf{S} denotes a maximal d-subgroup in \mathbf{H} .

Let $N_{\mathcal{A}}(\mathbf{F})$ denote the subgroup of elements in $\text{Aut}_a(\mathbf{H})$ which preserve \mathbf{F} . Additionally, we define:

$$\begin{aligned} \mathcal{A}_{\mathbf{H}|\mathbf{F}} &:= \{\Phi \in N_{\mathcal{A}}(\mathbf{F}) \mid \Phi|_{\mathbf{H}/\mathbf{F}} = \text{id}_{\mathbf{H}/\mathbf{F}}\}, \\ \mathcal{A}_{\mathbf{S}} &:= \{\Phi \in N_{\mathcal{A}}(\mathbf{F}) \mid \Phi|_{\mathbf{H}^\circ/\mathbf{F}} = \text{id}_{\mathbf{H}^\circ/\mathbf{F}}, \Phi(\mathbf{S}) = \mathbf{S}\}, \\ \mathcal{A}_{\mathbf{S}}^1 &:= \{\Phi \in \mathcal{A}_{\mathbf{S}} \mid \Phi|_{\mathbf{S}} = \text{id}_{\mathbf{S}}\}. \end{aligned}$$

It is easy to see that these groups are \mathbf{Q} -closed subgroups of $\text{Aut}_a(\mathbf{H})$.

Lemma 3.10. — *The following hold:*

- i) $\mathcal{A}_{\mathbf{S}}^1 \leq \mathcal{A}_{\mathbf{H}|\mathbf{F}}$.
- ii) Define $\mathcal{A}_{\mathbf{H}|\mathbf{U}} = \{\Phi \in \mathcal{A} \mid \Phi|_{\mathbf{H}/\mathbf{U}} = \text{id}_{\mathbf{H}/\mathbf{U}}\}$, then $\mathcal{A}_{\mathbf{H}|\mathbf{U}} \cap \mathcal{A}_{\mathbf{S}} = \mathcal{A}_{\mathbf{S}}^1$.
- iii) $\mathcal{A}_{\mathbf{H}|\mathbf{F}} \cap \mathcal{A}_{\mathbf{S}} = \mathcal{A}_{\mathbf{S}}^1$.

Proof. — Let $\Phi \in \mathcal{A}_{\mathbf{S}}^1$ and $u \in \mathbf{U}$. Since $\Phi|_{\mathbf{H}^\circ/\mathbf{F}} = \text{id}_{\mathbf{H}^\circ/\mathbf{F}}$, we have $\Phi(u) = uf_u$, where $f_u \in \mathbf{F}$. Now let $h \in \mathbf{H}$, and write $h = su$, where $s \in \mathbf{S}$, $u \in \mathbf{U}$. It follows that $\Phi(h) = hf_u$. Hence, $\Phi \in \mathcal{A}_{\mathbf{H}|\mathbf{F}}$. This proves i).

Let $\Phi \in \mathcal{A}_{\mathbf{H}|\mathbf{U}} \cap \mathcal{A}_{\mathbf{S}}$ and $s \in \mathbf{S}$. Then $\Phi(s) = su_s$, where $u_s \in \mathbf{U}$. Since $\Phi \in \mathcal{A}_{\mathbf{S}}$, $\Phi(s) \in \mathbf{S}$. It follows that $u_s = 1$. Therefore, $\Phi \in \mathcal{A}_{\mathbf{S}}^1$. This proves ii).

By i), $\mathcal{A}_{\mathbf{S}}^1 \subset \mathcal{A}_{\mathbf{H}|\mathbf{F}}$. Conversely, $\mathcal{A}_{\mathbf{H}|\mathbf{F}} \cap \mathcal{A}_{\mathbf{S}} \leq \mathcal{A}_{\mathbf{H}|\mathbf{U}} \cap \mathcal{A}_{\mathbf{S}}$, and hence by ii), $\mathcal{A}_{\mathbf{H}|\mathbf{F}} \cap \mathcal{A}_{\mathbf{S}} \leq \mathcal{A}_{\mathbf{S}}^1$. This proves iii).

Before we proceed let us introduce some notation concerning inner automorphisms.

Definition 3.11. — *Let A be a group and $B \leq A$ a subgroup. Let $a \in A$. We write $\text{Inn}_a \in \text{Aut}(A)$ for the inner automorphism of A defined by $g \mapsto aga^{-1}$, for all $g \in A$. Given an element $b \in B$, we set $\text{Inn}_b^A \in \text{Aut}(A)$ for the corresponding inner automorphism of A (to distinguish it from the induced inner automorphism of B). We write Inn_B^A for the subgroup of $\text{Aut}(A)$ consisting of all elements Inn_b^A , $b \in B$.*

Let \mathbf{D} be Cartan subgroup of \mathbf{H} which contains \mathbf{S} . Thus $\mathbf{D} = \mathbf{U}_{\mathbf{D}} \cdot \mathbf{S}$. We define

$$(22) \quad \mathcal{A}_{\mathbf{D}} := \{\Phi \in N_{\mathcal{A}}(\mathbf{F}) \mid \Phi|_{\mathbf{H}^{\circ}/\mathbf{F}} = \text{id}_{\mathbf{H}^{\circ}/\mathbf{F}}, \Phi(\mathbf{D}) \subset \mathbf{D}\}.$$

If \mathbf{D} is defined over \mathbf{Q} then $\mathcal{A}_{\mathbf{D}}$ is a \mathbf{Q} -defined subgroup of $\text{Aut}_a(\mathbf{H})$.

Lemma 3.12. — *The following hold:*

- i) Let $u \in \mathbf{U}_{\mathbf{D}}$ such that $\text{Inn}_u^{\mathbf{H}} \in \mathcal{A}_{\mathbf{H}|\mathbf{F}}$. Then there exists $v \in \mathbf{D} \cap \mathbf{F}$ such that, for all $s \in \mathbf{S}$, $\text{Inn}_u^{\mathbf{H}}(s) = \text{Inn}_v^{\mathbf{H}}(s)$.
- ii) $\mathcal{A}_{\mathbf{H}|\mathbf{F}} \cap \mathcal{A}_{\mathbf{D}} = \text{Inn}_{\mathbf{F} \cap \mathbf{D}}^{\mathbf{H}} \cdot \mathcal{A}_{\mathbf{S}}^1$.

Proof. — Let $s \in \mathbf{S}$. Then $\text{Inn}_u^{\mathbf{H}}(s) = sf_s$, where $f_s \in \mathbf{F} \cap \mathbf{D}$. Moreover, $f_s = 1$, for $s \in \mathbf{S}^{\circ}$ since u normalizes \mathbf{S}° . By a standard argument (compare for example [12]) the cocycle $s \mapsto f_s$ is of the form $f_s = s^{-1}vsv^{-1}$, for some $v \in \mathbf{F} \cap \mathbf{D}$. Thus, $\text{Inn}_u^{\mathbf{H}}(s) = \text{Inn}_v^{\mathbf{H}}(s)$, for all $s \in \mathbf{S}$.

Let $\Phi \in \mathcal{A}_{\mathbf{H}|\mathbf{F}} \cap \mathcal{A}_{\mathbf{D}}$. Since $\Phi(\mathbf{D}) \leq \mathbf{D}$, there exists $v \in \mathbf{U}_{\mathbf{D}}$ such that $v\mathbf{S}v^{-1} = \Phi(\mathbf{S})$. Then $\Psi = \text{Inn}_{v^{-1}}^{\mathbf{H}} \circ \Phi \in \mathcal{A}_{\mathbf{S}} \cap \mathcal{A}_{\mathbf{H}|\mathbf{U}}$ holds. By Lemma 3.10 (ii), $\Psi \in \mathcal{A}_{\mathbf{S}}^1$ follows. In particular, $\Psi(s) = s$, for all $s \in \mathbf{S}$. Hence, $\Phi(s) = \text{Inn}_v^{\mathbf{H}}(s)$, for all $s \in \mathbf{S}$. Since we have $\Phi \in \mathcal{A}_{\mathbf{H}|\mathbf{F}}$, this implies that $\text{Inn}_v^{\mathbf{H}} \in \mathcal{A}_{\mathbf{H}|\mathbf{F}}$ holds. By the first part, there exists $w \in \mathbf{D} \cap \mathbf{F}$ such that $\text{Inn}_w^{\mathbf{H}}(s) = \text{Inn}_v^{\mathbf{H}}(s)$ holds for all $s \in \mathbf{S}$. In particular we find $\text{Inn}_{w^{-1}}^{\mathbf{H}} \circ \Phi \in \mathcal{A}_{\mathbf{S}} \cap \mathcal{A}_{\mathbf{H}|\mathbf{F}} = \mathcal{A}_{\mathbf{S}}^1$. Hence, $\mathcal{A}_{\mathbf{H}|\mathbf{F}} \cap \mathcal{A}_{\mathbf{D}} \subset \text{Inn}_{\mathbf{F} \cap \mathbf{D}}^{\mathbf{H}} \cdot \mathcal{A}_{\mathbf{S}}^1$ holds. The lemma follows.

Proposition 3.13. — *Let \mathbf{H} be a \mathbf{Q} -defined solvable-by-finite linear algebraic group and $\mathbf{F} \leq \mathbf{U}_{\mathbf{H}}$ be a \mathbf{Q} -closed subgroup which contains $[\mathbf{H}^{\circ}, \mathbf{H}^{\circ}]$, then*

$$\mathcal{A}_{\mathbf{H}|\mathbf{F}} = \text{Inn}_{\mathbf{F}}^{\mathbf{H}} \cdot \mathcal{A}_{\mathbf{S}}^1$$

holds.

Proof. — It is clear that $\text{Inn}_{\mathbf{F}}^{\mathbf{H}} \cdot \mathcal{A}_{\mathbf{S}}^1 \leq \mathcal{A}_{\mathbf{H}|\mathbf{F}}$. Now let $\Phi \in \mathcal{A}_{\mathbf{H}|\mathbf{F}}$. Since \mathbf{F} contains $[\mathbf{H}^{\circ}, \mathbf{H}^{\circ}]$, there exists $v \in \mathbf{F}$ such that $v\mathbf{D}v^{-1} = \Phi(\mathbf{D})$. Therefore, $\text{Inn}_{v^{-1}}^{\mathbf{H}} \circ \Phi \in \mathcal{A}_{\mathbf{D}} \cap \mathcal{A}_{\mathbf{H}|\mathbf{F}}$ holds. The proposition follows from the previous lemma, part ii).

4. The algebraic hull of a polycyclic-by-finite wfn-group

Let Γ be a polycyclic-by-finite group. Its maximal nilpotent normal subgroup $\text{Fitt}(\Gamma)$ is called the Fitting subgroup of Γ . We assume that $\text{Fitt}(\Gamma)$ is torsion-free and $Z_{\Gamma}(\text{Fitt}(\Gamma)) \leq \text{Fitt}(\Gamma)$. These two conditions are equivalent to the requirement that Γ has no non-trivial finite normal subgroups. We call a group with this property a *wfn-group*. Proofs of the following results may be found in [8, Appendix A].

Theorem 4.1. — *Let Γ be a polycyclic-by-finite wfn-group. Then there exists a \mathbf{Q} -defined linear algebraic group \mathbf{H}_Γ and an injective group homomorphism $\psi : \Gamma \rightarrow \mathbf{H}_\Gamma(\mathbf{Q})$ such that:*

- i) $\psi(\Gamma)$ is Zariski-dense in \mathbf{H}_Γ ,*
- ii) \mathbf{H}_Γ has a strong unipotent radical $\mathbf{U} = \mathbf{U}_{\mathbf{H}_\Gamma}$,*
- iii) $\dim \mathbf{U} = \text{rank } \Gamma$.*

Moreover, $\psi(\Gamma) \cap \mathbf{H}_\Gamma(\mathbf{Z})$ is of finite index in Γ .

Here, $\text{rank } \Gamma$ denotes the number of infinite cyclic factors in a composition series of Γ . (This invariant is sometimes also called the Hirsch-rank of Γ .)

We remark that the group \mathbf{H}_Γ is determined by the conditions i)–iii) up to \mathbf{Q} -isomorphism of algebraic groups:

Proposition 4.2. — *Let Γ be a polycyclic-by-finite wfn-group. Let \mathbf{H}' be a \mathbf{Q} -defined linear algebraic group and $\psi' : \Gamma \rightarrow \mathbf{H}'(\mathbf{Q})$ an injective homomorphism which satisfies i) to iii) from above. Then there exists a \mathbf{Q} -defined isomorphism $\Phi : \mathbf{H}_\Gamma \rightarrow \mathbf{H}'$ such that $\psi' = \Phi \circ \psi$.*

Corollary 4.3. — *Let Γ be a polycyclic-by-finite wfn-group. The algebraic hull \mathbf{H}_Γ of Γ is unique up to \mathbf{Q} -isomorphism of algebraic groups. In particular, every automorphism ϕ of Γ extends uniquely to a \mathbf{Q} -defined automorphism Φ of \mathbf{H}_Γ .*

We call the \mathbf{Q} -defined linear algebraic group \mathbf{H}_Γ the *algebraic hull* for Γ . We shall identify Γ with the corresponding subgroup of its algebraic hull \mathbf{H}_Γ .

If Γ is finitely generated torsion-free nilpotent then \mathbf{H}_Γ is unipotent and Theorem 4.1 and Proposition 4.2 are essentially due to Malcev [29]. If Γ is torsion-free polycyclic, Theorem 4.1 is due to Mostow [31] (see also [36, §IV, p. 74] for a different proof).

Proposition 4.4. — *Let Γ be a polycyclic-by-finite wfn-group. Let \mathbf{H}_Γ be the algebraic hull for Γ . Then $\Gamma \cap \mathbf{U}_{\mathbf{H}_\Gamma} = \text{Fitt}(\Gamma)$ holds.*

Definition 4.5. — *We define $\mathbf{F} = \mathbf{F}_\Gamma := \overline{\text{Fitt}(\Gamma)} \leq \mathbf{H}_\Gamma$ as the Zariski-closure of the Fitting subgroup of Γ .*

Thus, in particular, \mathbf{F} is a connected unipotent normal subgroup of \mathbf{H}_Γ , and \mathbf{F} is defined over \mathbf{Q} . Moreover:

Proposition 4.6. — *The commutator subgroup $[\mathbf{H}_\Gamma^\circ, \mathbf{H}_\Gamma^\circ]$ is contained in \mathbf{F} . Let \mathbf{D} be a Cartan subgroup of \mathbf{H}_Γ . Then there is a decomposition*

$$(23) \quad \mathbf{H}_\Gamma = \mathbf{F} \cdot \mathbf{D}.$$

Proof. — Since Γ is Zariski-dense, we see that

$$[\mathbf{H}_\Gamma^\circ, \mathbf{H}_\Gamma^\circ] = \overline{[\Gamma \cap \mathbf{H}_\Gamma^\circ, \Gamma \cap \mathbf{H}_\Gamma^\circ]} \leq \overline{\text{Fitt}(\Gamma)} = \mathbf{F}.$$

The decomposition of \mathbf{H}_Γ follows (see SG7 of Section 2).

4.1. *A faithful rational representation of $\text{Aut}(\Gamma)$.* — Let $\text{Aut}_a(\mathbf{H}_\Gamma)$ be the group of algebraic automorphisms of \mathbf{H}_Γ with its natural structure of \mathbf{Q} -defined linear algebraic group. We view $\text{Aut}_a(\mathbf{H}_\Gamma)$ as a \mathbf{Q} -defined closed subgroup of $\text{GL}(n, \mathbf{C})$, for some $n \geq 0$. Then Proposition 3.6 shows that the extension

$$(24) \quad \text{Aut}(\Gamma) \ni \phi \mapsto \Phi \in \text{Aut}_{a, \mathbf{Q}}(\mathbf{H}_\Gamma) = \text{Aut}_a(\mathbf{H}_\Gamma)(\mathbf{Q})$$

which is defined in Corollary 4.3 gives rise to a faithful representation of $\text{Aut}(\Gamma)$ into $\text{GL}(n, \mathbf{Q})$:

Corollary 4.7. — *The extension homomorphism (24) is a faithful homomorphism of $\text{Aut}(\Gamma)$ into the group of \mathbf{Q} -points of the linear algebraic group $\text{Aut}_a(\mathbf{H}_\Gamma)$.*

Using this fact, if an embedding $\Gamma \leq \mathbf{H}_\Gamma$ is fixed we identify $\text{Aut}(\Gamma)$ with a subgroup of $\text{Aut}_a(\mathbf{H}_\Gamma)$. Remark though, that, in general, $\text{Aut}(\Gamma)$ is not Zariski-dense in $\text{Aut}_a(\mathbf{H}_\Gamma)$ because the elements of $\text{Aut}(\Gamma)$ preserve the Fitting subgroup F , and hence also the Zariski-closure \mathbf{F} of F . Thus, with the conventions from Section 3.3, $\text{Aut}(\Gamma)$ is contained in the subgroup $N_{\mathcal{A}}(\mathbf{F})$ of $\text{Aut}_a(\mathbf{H}_\Gamma)$.

Let $A_{\Gamma|F} \leq \text{Aut}(\Gamma)$ be the subgroup defined in (3). Since Γ is Zariski-dense in \mathbf{H}_Γ , the following is clear.

Lemma 4.8. — *We have $A_{\Gamma|F} \leq \mathcal{A}_{\mathbf{H}|F}(\mathbf{Q})$ under the extension homomorphism (24).*

4.2. *Thickenings of Γ in \mathbf{H}_Γ .* — Before introducing the thickening, we recall some results about finitely generated subgroups in unipotent algebraic groups. Let \mathbf{U} be a unipotent \mathbf{Q} -defined linear algebraic group, and let $F \leq \mathbf{U}(\mathbf{Q})$ be a finitely generated subgroup. Then F is a torsion-free nilpotent group. Let $\mathbf{F} := \bar{F} \leq \mathbf{U}$ be the Zariski-closure of F . Then \mathbf{F} is \mathbf{Q} -defined, and $\dim \mathbf{F} = \text{rank } F$. The group of \mathbf{Q} -points of \mathbf{F} is isomorphic to the Malcev radicable hull of F , i.e., $\mathbf{F}(\mathbf{Q})$ is radicable, and for every $x \in \mathbf{F}(\mathbf{Q})$ there exists $k \in \mathbf{N}$ such that $x^k \in F$. For $m \in \mathbf{N}$, we define

$$F_m^\perp := \langle x \in \mathbf{F}(\mathbf{Q}) \mid x^m \in F \rangle.$$

Then $F_m^\perp \leq \mathbf{F}(\mathbf{Q})$ is finitely generated, $F \leq F_m^\perp$ and $|F_m^\perp : F| < \infty$. Every finitely generated subgroup $G \leq \mathbf{F}(\mathbf{Q})$ is contained in F_m^\perp , for some $m \in \mathbf{N}$. See [40] for more details on all of this.

Now let Γ be a polycyclic-by-finite wfn-group, let F denote the Fitting subgroup of Γ , and let $\mathbf{F} \leq \mathbf{H}_\Gamma$ be the Zariski-closure of F in the algebraic hull of Γ . Since \mathbf{F} is unipotent, F_m^\perp is defined as a subgroup of $\mathbf{F}(\mathbf{Q})$.

Definition 4.9. — *A subgroup $\tilde{\Gamma}$ of \mathbf{H}_Γ which is of the form $\tilde{\Gamma} = F_m^\perp \cdot \Gamma$ is called a thickening of Γ .*

Clearly thickenings $\tilde{\Gamma}$ exist, for every $m \in \mathbf{N}$. A thickening $\tilde{\Gamma}$ of Γ is a finitely generated subgroup of $\mathbf{H}_\Gamma(\mathbf{Q})$ which is of finite index over Γ . We further remark that $\text{Fitt}(\tilde{\Gamma}) = \mathbf{F}_m^\perp$. The inclusion of $\tilde{\Gamma}$ into \mathbf{H}_Γ shows that \mathbf{H}_Γ is an algebraic hull also for the thickening $\tilde{\Gamma}$.

4.3. The automorphism group of the thickening. — Let Γ be a polycyclic-by-finite wfn-group and let $\tilde{\Gamma} \leq \mathbf{H}_\Gamma(\mathbf{Q})$ be a thickening of Γ . Let $\phi \in \text{Aut}(\Gamma)$, and let $\Phi : \mathbf{H}_\Gamma \rightarrow \mathbf{H}_\Gamma$ denote the extension of ϕ in $\text{Aut}_a(\mathbf{H}_\Gamma)$. Since the automorphism Φ preserves $\mathbf{H}_\Gamma(\mathbf{Q})$, it is clear that Φ preserves $\tilde{\Gamma} \leq \mathbf{H}_\Gamma(\mathbf{Q})$ as well. Restricting Φ to $\tilde{\Gamma}$ we thus obtain a natural inclusion $\text{Aut}(\Gamma) \hookrightarrow \text{Aut}(\tilde{\Gamma})$. This shows that we may identify $\text{Aut}(\Gamma)$ with a finite index subgroup of $\text{Aut}(\tilde{\Gamma})$ in a natural way:

Proposition 4.10. — *Let Γ be a polycyclic-by-finite wfn-group, and let $\tilde{\Gamma}$ be a thickening of Γ . Then the group $\text{Aut}(\Gamma) = \{\psi \in \text{Aut}(\tilde{\Gamma}) \mid \psi(\Gamma) = \Gamma\}$ is a subgroup of finite index in $\text{Aut}(\tilde{\Gamma})$.*

Proof. — Put $d = [\tilde{\Gamma} : \Gamma]$ for the index of Γ in $\tilde{\Gamma}$. Remark that there are only finitely many subgroups of $\tilde{\Gamma}$ with index d , since $\tilde{\Gamma}$ is a finitely generated group. The automorphism group $\text{Aut}(\tilde{\Gamma})$ acts on the set of such subgroups and the group $\text{Aut}(\Gamma)$ is the stabilizer of the subgroup Γ . Hence, we have $[\text{Aut}(\tilde{\Gamma}) : \text{Aut}(\Gamma)] \leq \ell$, where ℓ is the number of subgroups of index d .

5. Thickenings of Γ admit a supplement

We give here a short account of the construction of nilpotent-by-finite supplements in polycyclic-by-finite groups. Similar results are contained in the book [40] where nilpotent supplements in polycyclic groups are considered.

Definition 5.1. — *Let Γ be a polycyclic-by-finite group and let $C \leq \Gamma$ be a nilpotent-by-finite subgroup. We call C a nilpotent-by-finite supplement in Γ if $\Gamma = \text{Fitt}(\Gamma) \cdot C$.*

Nilpotent-by-finite supplements do not exist for general groups Γ . We will show below that a polycyclic-by-finite wfn-group Γ admits a thickening which has a nilpotent-by-finite supplement, a result originally due to Romankov (for polycyclic groups), see [38, Lemma 4.4].

As standing assumption for this section we have that Γ is a polycyclic-by-finite wfn-group, and we put $\mathbf{F} = \text{Fitt}(\Gamma)$. We fix an inclusion $\Gamma \leq \mathbf{H}_\Gamma(\mathbf{Q})$ of Γ into its algebraic hull \mathbf{H}_Γ . We put \mathbf{F} for the Zariski-closure of \mathbf{F} in \mathbf{H}_Γ , and put $\mathbf{N} = \mathbf{F}(\mathbf{Q})$.

Lemma 5.2. — *Let Γ be a polycyclic-by-finite wfn-group and let $\mathbf{D} \leq \mathbf{H}_\Gamma$ be a \mathbf{Q} -defined Cartan-subgroup. Then $\hat{C} = \Gamma\mathbf{N} \cap \mathbf{D}$ is a nilpotent-by-finite subgroup of $\Gamma \cdot \mathbf{N}$ such that $\Gamma \cdot \mathbf{N} = \hat{C} \cdot \mathbf{N}$ holds.*

Proof. — The decomposition (23) induces a corresponding decomposition for the group of \mathbf{Q} -points of \mathbf{H}_Γ , that is, $\mathbf{H}_\Gamma(\mathbf{Q}) = \mathbf{N} \cdot \mathbf{D}(\mathbf{Q})$.

Let $\gamma \in \Gamma$. Since $\Gamma \leq \mathbf{H}_\Gamma(\mathbf{Q})$ holds, it follows that $\gamma = n_\gamma c_\gamma$, where $n_\gamma \in \mathbf{N}$, $c_\gamma \in \mathbf{D}(\mathbf{Q}) \cap \Gamma\mathbf{N}$. Hence, $\Gamma \cdot \mathbf{N} = \hat{\mathbf{C}} \cdot \mathbf{N}$ holds.

We prove now the existence of supplements in a thickening $\tilde{\Gamma} = \mathbf{F}_m^\perp \cdot \Gamma$.

Proposition 5.3. — *Let Γ be a polycyclic-by-finite wfn-group and let $\mathbf{D} \leq \mathbf{H}_\Gamma$ be a \mathbf{Q} -defined Cartan subgroup. Then there exists $m \in \mathbf{N}$ such that*

$$\mathbf{F}_m^\perp \cdot \Gamma = \mathbf{F}_m^\perp \cdot \mathbf{C}, \quad \text{where } \mathbf{C} = (\mathbf{F}_m^\perp \cdot \Gamma) \cap \mathbf{D}.$$

Proof. — By the previous lemma, $\Gamma\mathbf{N} = \hat{\mathbf{C}}\mathbf{N}$, where $\hat{\mathbf{C}} = \Gamma\mathbf{N} \cap \mathbf{D}$. It follows that the natural map $\hat{\mathbf{C}} \rightarrow \Gamma\mathbf{N}/\mathbf{N}$ is surjective.

Since $\Gamma\mathbf{N}/\mathbf{N}$ is finitely generated, there exists a finitely generated group $\mathbf{C} \leq \hat{\mathbf{C}}$ so that $\mathbf{C} \rightarrow \Gamma\mathbf{N}/\mathbf{N}$ is surjective. Let c_1, \dots, c_k be generators for \mathbf{C} , $c_i = \gamma_i n_i$, where $\gamma_i \in \Gamma$ and $n_i \in \mathbf{N}$. Choose $m \in \mathbf{N}$ such that $n_i \in \mathbf{F}_m^\perp$, $i = 1 \dots k$. Then $\mathbf{C} \leq \tilde{\Gamma} = \Gamma\mathbf{F}_m^\perp$, and, in particular, $\mathbf{F}_m^\perp \mathbf{C} \leq \tilde{\Gamma}$.

The surjectivity of the map $\mathbf{C} \rightarrow \Gamma\mathbf{N}/\mathbf{N}$ shows that every $\gamma \in \Gamma$ is of the form $\gamma = cn$, where $c \in \mathbf{C}$, $n \in \mathbf{N} \cap \tilde{\Gamma} = \mathbf{F}_m^\perp$. This shows that $\tilde{\Gamma} = \mathbf{F}_m^\perp \cdot \mathbf{C}$.

A nilpotent-by-finite supplement is called *maximal* if it is a maximal element of the set of all nilpotent-by-finite supplements with respect to inclusion of subgroups. We show that the maximal supplements are those which arise by the construction of Proposition 5.3.

Proposition 5.4. — *Let Γ be a polycyclic-by-finite wfn-group. Let $\tilde{\Gamma} = \mathbf{F}_m^\perp \cdot \Gamma$ be a thickening which admits a maximal nilpotent-by-finite supplement \mathbf{C} . Then there exists a \mathbf{Q} -defined Cartan subgroup \mathbf{D} of \mathbf{H}_Γ such that $\mathbf{C} = \tilde{\Gamma} \cap \mathbf{D}$ and \mathbf{C} is Zariski-dense in \mathbf{D} .*

Proof. — Let \mathbf{C} be any nilpotent-by-finite supplement in $\tilde{\Gamma}$. Put $\mathcal{C} = \overline{\mathbf{C}}$ for the Zariski-closure of \mathbf{C} . Since $\mathbf{C} \leq \mathbf{H}_\Gamma(\mathbf{Q})$, \mathcal{C} is defined over \mathbf{Q} . Since $\tilde{\Gamma} = \mathbf{F}_m^\perp \cdot \mathbf{C}$ is Zariski-dense in \mathbf{H}_Γ , we have $\mathbf{H}_\Gamma = \mathbf{F} \cdot \mathcal{C}$. Let \mathbf{S} be a maximal d-subgroup of \mathcal{C} . Then \mathbf{S} is also maximal in \mathbf{H}_Γ . In particular, \mathcal{C} contains a maximal \mathbf{Q} -defined torus \mathbf{T} of \mathbf{H}_Γ . Since \mathcal{C} is nilpotent by-finite, \mathbf{T} is unique and normal in \mathcal{C} . We let \mathbf{D} denote the Cartan-subgroup corresponding to \mathbf{T} . Then $\mathcal{C} \leq \mathbf{D}$. It follows that $\mathbf{C} \leq \tilde{\Gamma} \cap \mathbf{D}$. In particular, if \mathbf{C} is maximal, then $\mathbf{C} = \tilde{\Gamma} \cap \mathbf{D}$.

Let us prove now that every maximal nilpotent by-finite supplement \mathbf{C} is Zariski-dense in the Cartan subgroup \mathbf{D} which contains \mathcal{C} . In fact, since $\mathbf{H} \leq \mathbf{F} \cdot \mathcal{C}$, $\mathbf{D} = \mathbf{U}_\mathbf{D} \cdot \mathbf{S}$, where $\mathbf{S} \leq \mathbf{D}$ is a maximal d-subgroup of \mathcal{C} , and $\mathbf{U}_\mathbf{D}$ is a unipotent normal subgroup. Hence, \mathbf{S} is a maximal d-subgroup of \mathbf{D} as well. Furthermore, we

can write $u \in \mathbf{U}_{\mathbf{D}}$ as $u = fu_1$, where $u_1 \in \mathbf{U}_{\mathcal{C}}$, $f \in \mathbf{F} \cap \mathbf{D}$. By the maximality of \mathbf{C} , $\mathbf{F} \cap \mathbf{D} \leq \mathbf{C}$. Therefore, $\mathbf{C} \cap \mathbf{F} = \mathbf{F} \cap \mathbf{D}$ is Zariski-dense in $\mathbf{F} \cap \mathbf{D}$. This means, $\mathbf{F} \cap \mathbf{D} \leq \mathcal{C}$. Hence, $f \in \mathcal{C}$ and $u \in \mathcal{C}$. It follows that $\mathcal{C} = \mathbf{D}$.

Proposition 5.5. — *Let Γ be a polycyclic-by-finite wfn-group, and $\tilde{\Gamma} = \mathbf{F}^{\perp} \cdot \Gamma$ a thickening of Γ . Then there are at most finitely many \mathbf{F}^{\perp} -conjugacy classes of maximal nilpotent by-finite supplements in $\tilde{\Gamma}$.*

Proof. — Let \mathbf{C} be a maximal nilpotent-by-finite supplement in $\tilde{\Gamma}$. By Proposition 5.4, $\mathbf{C} = \Gamma \cap \mathbf{D}$, where \mathbf{D} is a Cartan subgroup of \mathbf{H} . We consider $\tilde{\Gamma}_0 = \tilde{\Gamma} \cap \mathbf{H}^{\circ}$, $\mathbf{C}_0 = \mathbf{C} \cap \tilde{\Gamma}_0$. Then $\tilde{\Gamma}_0$ is a polycyclic normal subgroup of $\tilde{\Gamma}$ and $\mathbf{C}_0 \triangleleft \mathbf{C}$. Also $\mathbf{C}_0 = \mathbf{C} \cap \tilde{\Gamma}_0 = \mathbf{D} \cap \tilde{\Gamma}_0 = \mathbf{D}^{\circ} \cap \tilde{\Gamma}_0$. Since \mathbf{D}° is a Cartan subgroup in \mathbf{H}° , \mathbf{C}_0 is a maximal nilpotent supplement in $\tilde{\Gamma}_0$ (see Proposition 5.4). Note further that $\mathbf{C} = \tilde{\Gamma} \cap N_{\Gamma\mathbf{N}}(\mathbf{C}_0)$ is uniquely determined by \mathbf{C}_0 . By [40, Chapter 3, Theorem 4], there are only finitely many \mathbf{F}^{\perp} -conjugacy classes of maximal nilpotent supplements $\mathbf{C}_0 \leq \tilde{\Gamma}_0$. This also implies that there are only finitely many \mathbf{F}^{\perp} -conjugacy classes of maximal nilpotent by-finite supplements in $\tilde{\Gamma}$.

6. Lemmas from group theory

We provide here some simple facts which shall be needed later.

We start off with a few remarks on group cohomology with non-abelian coefficients. Let μ be a group, and let \mathbf{L} be a group on which μ acts by automorphisms. If $s \in \mu$ we write $v \mapsto v^s$, $v \in \mathbf{L}$, for the action of s on \mathbf{L} .

The set $Z^1(\mu, \mathbf{L}) = \{z : \mu \rightarrow \mathbf{L} \mid z(s_1 s_2) = z(s_1) z(s_2)^{s_1}\}$ is called the set of 1-cocycles. Two 1-cocycles z_1 and z_2 are cohomologous if and only if there exists $v \in \mathbf{L}$ such that $z_1(s) = v^{-1} z_2(s) v^s$. Let $H^1(\mu, \mathbf{L})$ denote the set of equivalence classes of cocycles. It is called the first cohomology set for μ with coefficients in \mathbf{L} . The following lemma is well known.

Lemma 6.1. — *Let μ be a finite group, and \mathbf{L} a finitely generated nilpotent group on which μ acts by automorphisms. Then the cohomology set $H^1(\mu, \mathbf{L})$ is finite.*

Proof. — Let $G = \mathbf{L} \rtimes \mu$ be the split extension corresponding to the given action of μ on \mathbf{L} . We consider \mathbf{L} as a normal subgroup in G . A 1-cocycle $z : \mu \rightarrow \mathbf{L}$ gives rise to the finite subgroup $\mu_z := \{(z(s), s) \mid s \in \mu\}$ of G . Two 1-cocycles z_1 and z_2 are cohomologous if and only if the corresponding subgroups μ_{z_1} and μ_{z_2} are conjugate by an element of \mathbf{L} . Since G is a finitely generated nilpotent-by-finite group we know (see [40], Chapter 8, Theorem 5) that G has only finitely many conjugacy classes of finite subgroups. Since \mathbf{L} has finite index in G , the lemma follows.

We also need the following lemma.

Lemma 6.2. — *Let N be a group and $M \leq N$ a finitely generated torsion-free abelian normal subgroup of finite index. Define*

$$\text{Aut}(N, M) = \{\phi \in \text{Aut}(N) \mid \phi(M) = M, \phi|_M = \text{id}_M\}.$$

then Inn_M^N (see Definition 3.11) is a subgroup of finite index in $\text{Aut}(N, M)$.

Proof. — We briefly sketch the argument. Remark first that it is sufficient to prove the lemma in the case that the extension $M \leq N$ is effective, that is, $Z_N(M) \leq M$. We set $\mu = N/M$. Assuming effectiveness, there exists a finite extension group $N \leq L$ of N which splits, that is, L is a semi-direct product $L = M_L \rtimes \mu$, where $M_L \geq M$ is a torsion-free abelian group which contains M as a subgroup of finite index. Every automorphism of N extends uniquely to an automorphism of L which preserves M_L . Therefore, it is enough to show the lemma for $\text{Aut}(L, M_L)$. Now let $\phi \in \text{Aut}(L, M_L)$, that is, $\phi|_{M_L} = \text{id}_{M_L}$, and assume additionally that ϕ is the identity on the finite quotient L/M_L . The group of all such ϕ is isomorphic to the group of 1-cocycles in $Z^1(\mu, M_L)$ with the inner automorphisms corresponding to 1-coboundaries. Since $H^1(\mu, M_L)$ (see Lemma 6.1) is finite, $\text{Inn}_{M_L}^L$ is of finite index in $\text{Aut}(L, M_L)$.

The following can be deduced from [40], Section 6, we skip the proof.

Lemma 6.3. — *Let \mathbf{U} be a unipotent \mathbf{Q} -defined linear algebraic group. The following hold:*

- i) Let $U_1 \leq U_2 \leq \mathbf{U}(\mathbf{Q})$ be two finitely generated subgroups and suppose that U_1 is Zariski-dense in \mathbf{U} then the index $[U_2 : U_1]$ is finite.*
- ii) Let $U \leq \mathbf{U}(\mathbf{Q})$ be a Zariski-dense finitely generated subgroup and let $d \in \mathbf{N}$. Let $V \leq \mathbf{U}(\mathbf{Q})$ be a subgroup which contains U and satisfies $[V : U] \leq d$. Then V is contained in $U^{\frac{1}{d}}$. In particular, the set of all such subgroups V is finite.*

7. Unipotent shadows of Γ

Let Γ be a polycyclic-by-finite wfn-group. We set $F = \text{Fitt}(\Gamma)$ and write \mathbf{F} for its Zariski-closure in \mathbf{H}_Γ . Furthermore, we choose a thickening $\tilde{\Gamma} = F^{\frac{1}{n}} \cdot \Gamma$ which has a (maximal) nilpotent-by-finite supplement $C \leq \tilde{\Gamma}$. We use this setup to construct (in a controlled way, depending on Γ) a finitely generated nilpotent group $\theta \leq \mathbf{U}(\mathbf{Q})$ which is Zariski-dense in \mathbf{U} . We shall later use θ to find arithmetic subgroups in $\text{Aut}(\Gamma)$.

Using the above data we start our construction. We shall use the results of Section 5. Let $\mathbf{D} = \overline{C} \leq \mathbf{H}_\Gamma$ denote the Zariski-closure of C . Then \mathbf{D} is a \mathbf{Q} -defined Cartan-subgroup of \mathbf{H}_Γ . We have $C = \tilde{\Gamma} \cap \mathbf{D}$, by Proposition 5.4. Let $\mathbf{S} \leq \mathbf{D}$ be

a maximal \mathbf{Q} -defined d -subgroup in \mathbf{D} . Then \mathbf{S} is a finite extension of the maximal torus \mathbf{S}° . The torus \mathbf{S}° is central in \mathbf{D}° , and $\mathbf{D} = N_{\mathbf{H}}(\mathbf{S}^\circ)$. We set $C_0 = C \cap \mathbf{D}^\circ$. Then C_0 is a nilpotent finite index normal subgroup of C .

We consider the split decompositions $\mathbf{D} = \mathbf{U}_{\mathbf{D}} \cdot \mathbf{S}$, $\mathbf{D}^\circ = \mathbf{U}_{\mathbf{D}} \cdot \mathbf{S}^\circ$, where $\mathbf{U}_{\mathbf{D}}$ is the unipotent radical of \mathbf{D} . Every $c \in \mathbf{D}(\mathbf{Q})$ can be written uniquely as

$$(25) \quad c = u_c \cdot s_c \quad \text{with } u_c \in \mathbf{U}_{\mathbf{D}}(\mathbf{Q}), s_c \in \mathbf{S}(\mathbf{Q}).$$

If $c \in \mathbf{D}^\circ(\mathbf{Q})$ holds, then $s_c \in \mathbf{S}^\circ(\mathbf{Q})$ follows. We define

$$(26) \quad U_{C,\mathbf{S}} = \langle u_c \mid c \in C \rangle, \quad U_{C_0} = \langle u_c \mid c \in C_0 \rangle.$$

Lemma 7.1. — *The groups $U_{C_0} \leq U_{C,\mathbf{S}}$ are finitely generated, Zariski-dense subgroups of $\mathbf{U}_{\mathbf{D}}$, and U_{C_0} is of finite index in $U_{C,\mathbf{S}}$. Moreover, $U_{C,\mathbf{S}}$ is normalized by C , and U_{C_0} is normalized by C_0 .*

Proof. — Since $\mathbf{S}^\circ \leq \mathbf{D}^\circ$ is central in \mathbf{D}° , the map $C_0 \ni c \mapsto u_c \in \mathbf{U}_{\mathbf{D}}$ is a homomorphism. Therefore, the group U_{C_0} is finitely generated. Moreover, since $\mathbf{U}_{\mathbf{D}} = \mathbf{U}_{C_0}$, the group U_{C_0} is Zariski-dense in $\mathbf{U}_{\mathbf{D}}$. Let $S \leq \mathbf{S}$ denote the image of the homomorphism $C \rightarrow \mathbf{S}$, $c \mapsto s_c$, and $S_0 \leq S$ the corresponding image of C_0 . The group C acts on C_0 and on $\mathbf{U}_{\mathbf{D}}$ by conjugation. Since \mathbf{S}° is central in \mathbf{D}° , this action factors over the finite group $\mu = S/S_0$. For all $c, d \in C$, we have the formula

$$(27) \quad u_{cd} = u_c s_c u_d s_c^{-1}.$$

This shows that the group $U_{C,\mathbf{S}}$ is generated by U_{C_0} and a finite set u_{c_1}, \dots, u_{c_l} , where the $c_i \in C$ represent generators for C/C_0 . Therefore, $U_{C,\mathbf{S}}$ is finitely generated. The statement about the finite index follows from Lemma 6.3.

Equation (27) also shows that the action of the finite group μ on $\mathbf{U}_{\mathbf{D}}$ preserves the subgroup $U_{C,\mathbf{S}}$. This implies that $U_{C,\mathbf{S}}$ is normalized by C . The second statement follows by similar reasoning.

Definition 7.2. — *We define*

$$(28) \quad \theta_{C,\mathbf{S}} := \langle \mathbf{F}_m^\perp, U_{C,\mathbf{S}} \rangle, \quad \theta_{C_0} := \langle \mathbf{F}_m^\perp, U_{C_0} \rangle.$$

The groups $\theta_{C,\mathbf{S}}$ are called unipotent shadows of Γ .

Since we have $\mathbf{U} = \overline{\mathbf{F}} \cdot \mathbf{D}_{\mathbf{U}}$ (see SG7 of Section 2), Lemma 7.1 shows that each unipotent shadow $\theta_{C,\mathbf{S}}$ is a finitely generated subgroup of $\mathbf{U}(\mathbf{Q})$ which is Zariski-dense in \mathbf{U} , and it contains the group θ_{C_0} as a normal subgroup of finite index.

Definition 7.3. — We call $\theta_{\mathbf{C},\mathbf{S}}$ a good unipotent shadow if the conditions

$$(29) \quad \theta_{\mathbf{C}_0} \cap \mathbf{F} = \theta_{\mathbf{C},\mathbf{S}} \cap \mathbf{F} = \mathbf{F}_m^\perp = \text{Fitt}(\tilde{\Gamma})$$

are satisfied.

Good shadows may be obtained by further thickening of the Fitting-subgroup.

Proposition 7.4. — Let Γ be a polycyclic-by-finite wfn-group. Then there is a thickening $\tilde{\Gamma} = \mathbf{F}_m^\perp \cdot \Gamma$ with a nilpotent-by-finite supplement $\mathbf{C} \leq \tilde{\Gamma}$, such that, for every maximal \mathbf{Q} -defined d -subgroup $\mathbf{S} \leq \overline{\mathbf{C}}$, $\theta_{\mathbf{C},\mathbf{S}}$ is a good unipotent shadow.

Proof. — Choose $\ell \in \mathbf{N}$ such that the thickening $\mathbf{F}_\ell^\perp \cdot \Gamma$ admits a nilpotent-by-finite supplement \mathbf{C} . Let \mathbf{S} be a maximal \mathbf{Q} -defined d -subgroup in the Zariski-closure \mathbf{D} of \mathbf{C} . Let $\theta_{\mathbf{C},\mathbf{S}}$ be defined as in Definition 7.2.

Since $\theta_{\mathbf{C},\mathbf{S}}$ is finitely generated, we may choose $m \in \mathbf{N}$, divisible by ℓ , such that $\theta_{\mathbf{C},\mathbf{S}} \cap \mathbf{F} \leq \mathbf{F}_m^\perp$. Now put $\tilde{\Gamma} = \mathbf{F}_m^\perp \cdot \Gamma$, and remark that \mathbf{C} is a nilpotent-by-finite supplement in $\tilde{\Gamma}$. By Proposition 5.4, $\mathbf{C}_1 = \mathbf{D} \cap \tilde{\Gamma}$ is a maximal nilpotent-by-finite supplement in $\tilde{\Gamma}$, and contains \mathbf{C} . Since every element $c_1 \in \mathbf{C}_1$ can be expressed as $c_1 = fc$ with $c \in \mathbf{C}$ and $f \in \mathbf{F}_m^\perp$, we find, going through the definitions of $\theta_{\mathbf{C},\mathbf{S}}$ and $\theta_{\mathbf{C}_1,\mathbf{S}}$, that $\theta_{\mathbf{C}_1,\mathbf{S}} \cap \mathbf{F} \leq \mathbf{F}_m^\perp \cdot (\theta_{\mathbf{C},\mathbf{S}} \cap \mathbf{F})$. This implies that $\theta_{\mathbf{C}_1,\mathbf{S}} \cap \mathbf{F} = \mathbf{F}_m^\perp = \text{Fitt}(\tilde{\Gamma})$. Hence the requirements of Definition 7.3 follow.

The following compatibility results are very important for our future constructions.

Proposition 7.5. — Let Γ be a polycyclic-by-finite wfn-group. Let $\tilde{\Gamma} = \mathbf{F}_m^\perp \cdot \Gamma$ be a thickening of Γ with a nilpotent-by-finite supplement $\mathbf{C} \leq \tilde{\Gamma}$. Let $\theta_{\mathbf{C},\mathbf{S}}$ be a corresponding unipotent shadow. Then the following hold:

- i) Let $\phi \in \text{Aut}(\tilde{\Gamma})$ be an automorphism which satisfies $\phi(\mathbf{C}) = \mathbf{C}$, and let Φ be its extension to an automorphism of \mathbf{H}_Γ . Then we have $\Phi(\theta_{\mathbf{C}_0}) = \theta_{\mathbf{C}_0}$.
- ii) For a finite index subgroup of the group of all automorphisms $\phi \in \text{Aut}(\tilde{\Gamma})$ with $\phi(\mathbf{C}) = \mathbf{C}$, the extension Φ satisfies $\Phi(\theta_{\mathbf{C},\mathbf{S}}) = \theta_{\mathbf{C},\mathbf{S}}$.
- iii) The group $\tilde{\Gamma}$ normalizes $\theta_{\mathbf{C},\mathbf{S}}$.

Proof. — Since $\phi(\mathbf{C}) = \mathbf{C}$, we have $\Phi(\mathbf{D}) = \mathbf{D}$, hence also $\Phi(\mathbf{D}^\circ) = \mathbf{D}^\circ$ and $\Phi(\mathbf{S}^\circ) = \mathbf{S}^\circ$. The definition of $\mathbf{U}_{\mathbf{C}_0}$ shows that $\Phi(\mathbf{U}_{\mathbf{C}_0}) = \mathbf{U}_{\mathbf{C}_0}$. Since Φ also stabilizes $\text{Fitt}(\tilde{\Gamma}) = \mathbf{F}_m^\perp$, it stabilizes $\theta_{\mathbf{C}_0}$. This proves i).

Since $\theta_{\mathbf{C}_0}$ is of finite index in $\theta_{\mathbf{C},\mathbf{S}}$, we can use i) together with part ii) of Lemma 6.3 to prove ii).

By Lemma 7.1, $\mathbf{U}_{\mathbf{C},\mathbf{S}}$ is normalized by \mathbf{C} . Since \mathbf{C} also normalizes $\text{Fitt}(\tilde{\Gamma}) = \mathbf{F}_m^\perp$ it normalizes $\theta_{\mathbf{C},\mathbf{S}} = \mathbf{F}_m^\perp \cdot \mathbf{U}_{\mathbf{C},\mathbf{S}}$. The Fitting subgroup \mathbf{F}_m^\perp normalizes $\theta_{\mathbf{C},\mathbf{S}}$ because it is contained in $\theta_{\mathbf{C},\mathbf{S}}$. Hence, $\tilde{\Gamma} = \mathbf{F}_m^\perp \cdot \mathbf{C}$ normalizes $\theta_{\mathbf{C},\mathbf{S}}$. Hence, iii) holds.

8. Arithmetic subgroups of $\text{Aut}(\Gamma)$

Let Γ be a polycyclic-by-finite wfn-group. As usually, the group Γ is considered as embedded in the \mathbf{Q} -points of its algebraic hull \mathbf{H}_Γ . This also fixes an embedding of $\text{Aut}(\Gamma)$ in the \mathbf{Q} -points of the \mathbf{Q} -defined linear algebraic group $\text{Aut}_a(\mathbf{H}_\Gamma)$.

We set $F = \text{Fitt}(\Gamma)$ and write \mathbf{F} for its Zariski-closure. We assume for this section that Γ admits a nilpotent-by-finite supplement. Thus we may choose a nilpotent-by-finite subgroup C of Γ such that $\Gamma = F \cdot C$ holds. We choose C maximal with these properties. We write $\mathbf{D} = \overline{C}$ for its Zariski-closure. Then $C = \Gamma \cap \mathbf{D}$. We further choose a \mathbf{Q} -defined d-subgroup $\mathbf{S} \leq \overline{C}$. Associated with these data comes a unipotent shadow $\theta = \theta_{C, \mathbf{S}}$, as constructed in the previous section. We make the additional assumption that θ is a good unipotent shadow (see Definition 7.3). Our general philosophy is that we always can replace a general wfn-group Γ by one of its thickenings to enforce these assumptions.

We define \mathbf{U} to be the unipotent radical of \mathbf{H}_Γ . The unipotent shadow $\theta \leq \mathbf{U}(\mathbf{Q})$ provides us with arithmetic subgroups of suitable \mathbf{Q} -closed subgroups of $\text{Aut}_a(\mathbf{H}_\Gamma)$ (compare Section 3.2). We then will find the position of $\text{Aut}(\Gamma) \leq \text{Aut}_a(\mathbf{H}_\Gamma)$ relative to them. Given a subgroup $B \leq \text{Aut}_a(\mathbf{H}_\Gamma)$ we define

$$(30) \quad B[\theta] := \{\Phi \in B \mid \Phi(\theta) = \theta\}$$

to be the stabilizer of θ in B . We have:

Lemma 8.1. — *Let $\mathcal{B} \leq \text{Aut}_a(\mathbf{H}_\Gamma)$ be a \mathbf{Q} -closed subgroup which acts faithfully on \mathbf{U} . Then $\mathcal{B}[\theta]$ is an arithmetic subgroup of \mathcal{B} .*

The lemma follows along the principles used in Section 3.2, that is, by linearizing the action on \mathbf{U} via the exponential function to a linear action on the Lie algebra of \mathbf{U} .

As a first application of Lemma 8.1, we obtain that $\mathcal{A}_\mathbf{S}^1[\theta]$ is arithmetic in $\mathcal{A}_\mathbf{S}^1$, (see Section 3.3 for the definition of $\mathcal{A}_\mathbf{S}^1$). We deduce:

Proposition 8.2. — *Given the data (Γ, C, \mathbf{S}) as described above. Then*

$$\text{Inn}_F^{\mathbf{H}_\Gamma} \cdot \mathcal{A}_\mathbf{S}^1[\theta] \leq \mathcal{A}_{\mathbf{H}_\Gamma | \mathbf{F}}(\mathbf{Q})$$

is an arithmetic subgroup of $\mathcal{A}_{\mathbf{H}_\Gamma | \mathbf{F}}$.

Proof. — By our definitions, $\mathcal{A}_\mathbf{S}^1[\theta]$ normalizes both \mathbf{F} and θ , and hence also $\mathbf{F} \cap \theta$. We have $\mathbf{F} \cap \theta = F$ since θ is a good shadow. It follows that

$$F \rtimes \mathcal{A}_\mathbf{S}^1[\theta] \leq \mathbf{F} \rtimes \mathcal{A}_\mathbf{S}^1$$

is an arithmetic subgroup. We consider the natural \mathbf{Q} -defined homomorphism $\mathbf{F} \rtimes \mathcal{A}_{\mathbf{S}}^1 \rightarrow \mathcal{A}_{\mathbf{H}_{\Gamma}\mathbf{F}}$ which is induced by (17). By Proposition 3.13, it is surjective. This implies the result. Of course we have also used that the image of an arithmetic group under a \mathbf{Q} -defined homomorphism is arithmetic (see AR1 of Section 2).

We turn now to the task of comparing $\text{Aut}(\Gamma)$ to the above arithmetic groups. We define, as in the introduction,

$$A_{\Gamma|\mathbf{F}} := \{\phi \in \text{Aut}(\Gamma) \mid \phi|_{\Gamma/\mathbf{F}} = \text{id}_{\Gamma/\mathbf{F}}\}.$$

Clearly, $A_{\Gamma|\mathbf{F}}$ is a characteristic subgroup of $\text{Aut}(\Gamma)$. Set also

$$A_{\Gamma|\mathbf{F}}^{\mathbf{C}} := \{\phi \in A_{\Gamma|\mathbf{F}} \mid \phi(\mathbf{C}) = \mathbf{C}\}.$$

We obtain from Proposition 5.5:

Lemma 8.3. — *Given the data $(\Gamma, \mathbf{C}, \mathbf{S})$ described above, $\text{Inn}_{\mathbf{F}}^{\Gamma} \cdot A_{\Gamma|\mathbf{F}}^{\mathbf{C}}$ has finite index in $A_{\Gamma|\mathbf{F}}$.*

Next we analyze the group $A_{\Gamma|\mathbf{F}}^{\mathbf{C}}[\theta]$. We obtain from Proposition 7.5 ii):

Lemma 8.4. — *Given the data $(\Gamma, \mathbf{C}, \mathbf{S})$ described above, $A_{\Gamma|\mathbf{F}}^{\mathbf{C}}[\theta]$ has finite index in $A_{\Gamma|\mathbf{F}}$.*

We have the semi-direct product decomposition $\mathbf{D} = \mathbf{U}_{\mathbf{D}} \cdot \mathbf{S}$. Relative to this decomposition we can consider the quotient homomorphism $\pi_{\mathbf{S}} : \mathbf{D} \rightarrow \mathbf{S}$, and define

$$(31) \quad \mathbf{S} := \pi_{\mathbf{S}}(\mathbf{C}), \quad \mathbf{S}_0 := \pi_{\mathbf{S}}(\mathbf{C} \cap \mathbf{D}^{\circ})$$

Notice that, by the constructions in Section 7, we have $c \cdot \pi_{\mathbf{S}}(c)^{-1} \in \theta$, for every $c \in \mathbf{C}$. We need the following technical observations:

Lemma 8.5. — *Given the data $(\Gamma, \mathbf{C}, \mathbf{S})$, described above and \mathbf{S}, \mathbf{S}_0 as defined in (31), we have*

- i) *The group \mathbf{S} normalizes $\mathbf{F} \cap \mathbf{C}$ and \mathbf{S}_0 centralizes it.*
- ii) *Let Φ be the extension of the automorphism $\phi \in A_{\Gamma|\mathbf{F}}^{\mathbf{C}}[\theta]$ to an automorphism of \mathbf{H}_{Γ} . Then $\Phi(s) \cdot s^{-1} \in \mathbf{F} \cap \mathbf{C}$ holds, for every $s \in \mathbf{S}$. If $s \in \mathbf{S}_0$ then $\Phi(s) \cdot s^{-1} = 1$.*

Proof. — i): Let s be in \mathbf{S} , choose $c \in \mathbf{C}$ with $\pi_{\mathbf{S}}(c) = s$ and define $v = c \cdot \pi_{\mathbf{S}}(c)^{-1}$. As remarked above, we have $v \in \theta$. Since c normalizes $\mathbf{F} \cap \mathbf{C}$, the following holds

$$s(\mathbf{F} \cap \mathbf{C})s^{-1} = v^{-1}(\mathbf{F} \cap \mathbf{C})v.$$

The right hand side is in θ and in \mathbf{F} , hence in $\theta \cap \mathbf{F} = \mathbf{F}$. This implies that the right hand side is in $\mathbf{D} = \mathbf{F} \cap \mathbf{D}$. The subgroup $\mathbf{D} \leq \Gamma$ is nilpotent-by-finite and

normalized by C . Both C and D are contained in \mathbf{D} . Hence $\langle C, D \rangle$ is a nilpotent-by-finite supplement in Γ . Since C is maximal, we have $D = F \cap \mathbf{D} \leq C$ and the first part of i) follows. For the second part, notice that $F \cap C \leq \mathbf{D}^\circ$.

ii): Let s be in S , choose $c \in C$ with $\pi_{\mathbf{S}}(c) = s$ and define $v = c \cdot \pi_{\mathbf{S}}(c)^{-1}$. Then

$$\phi(c)c^{-1} = \Phi(v)\Phi(s)s^{-1}v^{-1}.$$

Since Φ is the identity modulo \mathbf{F} , the right hand side is in \mathbf{U} , whereas the left hand side is in C . Hence, the right hand side is in $C \cap \mathbf{U} \leq \theta$. Our assumptions imply $\Phi(s)s^{-1} \in \theta$ and then $\Phi(s)s^{-1} \in \theta \cap \mathbf{F} = F$. Here we have used that θ is a good unipotent shadow. Furthermore the above equation shows that $\Phi(s)s^{-1} \in \mathbf{D}$ holds. As under i), we finish the proof of the first part of ii) by remarking that $F \cap \mathbf{D}$ is contained in C . For the second notice that if $s \in \mathbf{S}^\circ$ holds then $\Phi(s)s^{-1} \in \mathbf{F} \cap \mathbf{S}^\circ = \{1\}$ follows.

We proceed with the construction of subgroups in $\text{Aut}(\Gamma)$. We define

$$(32) \quad A_{\Gamma|\mathbf{F}}^C[\theta]^1 := \{\phi \in A_{\Gamma|\mathbf{F}}^C[\theta] \mid \Phi(\mathbf{S}) = \mathbf{S}, \Phi|_{\mathbf{S}} = \text{id}_{\mathbf{S}}\}.$$

Here Φ is as always the extension of the automorphism $\phi \in \text{Aut}(\Gamma)$ to an automorphism of \mathbf{H}_Γ . We have:

Lemma 8.6. — *Given the data (Γ, C, \mathbf{S}) , as above. Then $\text{Inn}_{F \cap C} \cdot A_{\Gamma|\mathbf{F}}^C[\theta]^1$ is of finite index in $A_{\Gamma|\mathbf{F}}^C[\theta]$.*

Proof. — Notice first that $\text{Inn}_{F \cap C}$ is contained in $A_{\Gamma|\mathbf{F}}^C[\theta]$, as follows from the definitions. Let $S, S_0 \subset \mathbf{S}$ be the subgroups defined in (31). The quotient group $\mu = S/S_0$ is finite and acts by conjugation on $F \cap C$ (see Lemma 8.5 i)). We let $Z^1(\mu, F \cap C)$ be the corresponding set of 1-cocycles and $H^1(\mu, F \cap C)$ the cohomology set (see Section 6 for definitions). This cohomology set is finite, by Lemma 6.1.

Given $\phi \in A_{\Gamma|\mathbf{F}}^C[\theta]$ with extension Φ , we obtain, using Lemma 8.5, a map $D_\phi : S \rightarrow F \cap C$ by setting $D_\phi(s) = \Phi(s) \cdot s^{-1}$. The verification of the following is straightforward:

– the above D induces a (well defined) map

$$D : A_{\Gamma|\mathbf{F}}^C[\theta] \rightarrow Z^1(\mu, F \cap C),$$

– the map D from the previous item induces a (well defined) map

$$\hat{D} : A_{\Gamma|\mathbf{F}}^C[\theta] / \text{Inn}_{F \cap C}^\Gamma \cdot A_{\Gamma|\mathbf{F}}^C[\theta]^1 \longrightarrow H^1(\mu, F \cap C),$$

– the map \hat{D} is injective.

As remarked before, $H^1(\mu, F \cap C)$ is finite and the lemma is proved.

We put now Lemmas 8.3, 8.4, and 8.6 together and obtain:

Lemma 8.7. — *Given the data $(\Gamma, \mathbf{C}, \mathbf{S})$, as above. Then the group $\text{Inn}_\Gamma^\Gamma \cdot A_{\Gamma|\mathbb{F}}^\mathbf{C}[\theta]^1$ is of finite index in $A_{\Gamma|\mathbb{F}}$.*

The link between Proposition 8.2 and Lemma 8.7 is given by:

Proposition 8.8. — *Given the data $(\Gamma, \mathbf{C}, \mathbf{S})$, as above. Then $A_{\Gamma|\mathbb{F}}^\mathbf{C}[\theta]^1 = \mathcal{A}_\mathbf{S}^1[\theta]$.*

Proof. — By the definitions, $A_{\Gamma|\mathbb{F}}^\mathbf{C}[\theta]^1 \leq \mathcal{A}_\mathbf{S}^1[\theta]$ holds. Now let Φ be an element of $\mathcal{A}_\mathbf{S}^1[\theta]$. We show that Φ is contained in $\text{Aut}(\Gamma)$. First of all, we have $\Phi(\mathbf{F}) = \Phi(\theta \cap \mathbf{F}) = \Phi(\theta) \cap \Phi(\mathbf{F}) = \mathbf{F}$. Here we used that θ is a good unipotent shadow. Since $\Phi|_{\mathbf{H}^\circ/\mathbf{F}} = \text{id}_{\mathbf{H}^\circ/\mathbf{F}}$, it follows that $\Phi|_{\theta/\mathbf{F}} = \text{id}_{\theta/\mathbf{F}}$. Let $c \in \mathbf{C}$, $c = us$ with $u \in \theta$ and $s \in \mathbf{S}$. By the above, $\Phi(u) = fu$, for some $f \in \mathbf{F}$. Therefore, we get

$$\Phi(c) = \Phi(u)\Phi(s) = \Phi(u)s = fus = fc \in \Gamma.$$

Since $\Phi(\mathbf{D}) = \mathbf{D}$, it follows that $\Phi(c) \in \Gamma \cap \mathbf{D} = \mathbf{C}$. This shows that Φ stabilizes \mathbf{C} . Hence, $\Phi(\Gamma) = \Phi(\mathbf{FC}) = \mathbf{FC} = \Gamma$. Thus, $\Phi \in \text{Aut}(\Gamma)$ holds. The lemma follows.

Putting together Lemma 8.7, Proposition 8.8, and Lemma 4.8, we obtain:

Corollary 8.9. — *The group $A_{\Gamma|\mathbb{F}}$ is an arithmetic subgroup of $\mathcal{A}_{\mathbf{H}_\Gamma|\mathbb{F}}$.*

Finally, let us consider the arithmetic subgroup A_θ (see Definition 3.7) of the group $\text{Aut}_a(\mathbf{H}_\Gamma)$. The group $A_{\tilde{\Gamma}|\tilde{\mathbb{F}}}[\theta]$ is defined as the stabilizer of θ in $A_{\Gamma|\mathbb{F}}$. We note:

Proposition 8.10. — *A finite index subgroup of $\text{Inn}_\Gamma \cdot A_{\Gamma|\mathbb{F}}[\theta]$ is contained in A_θ .*

Proof. — By construction, we have $A_{\Gamma|\mathbb{F}}^\mathbf{C}[\theta]^1 \leq A_\theta$. The group Inn_Γ stabilizes θ , by Proposition 7.5 iii). Obviously $\text{Inn}_{\tilde{\mathbb{F}}}$ is contained in A_θ . We have $\Gamma = \tilde{\mathbb{F}} \cdot \mathbf{C}$. Let c be an element of \mathbf{C} , we write $c = v \cdot s$ with $v \in \theta$, $s \in \mathbf{S}$. Then $\text{Inn}_v \in A_\theta$, and hence Inn_s stabilizes θ . Clearly, $\text{Inn}_s \in \text{Aut}_a(\mathbf{H}_\Gamma)_\mathbf{S}$, therefore Inn_s is also in A_θ . This shows that Inn_Γ is contained in A_θ . Now $\text{Inn}_\Gamma \cdot A_{\Gamma|\mathbb{F}}[\theta]^1$ is a finite index subgroup of $\text{Inn}_\Gamma \cdot A_{\Gamma|\mathbb{F}}[\theta]$, by Lemma 8.7.

9. The automorphism group of Γ as a subgroup of $\text{Aut}_a(\mathbf{H}_\Gamma)$

This section contains the final proof of Theorem 1.4. We also provide the input for the proofs of Theorem 1.3 and Theorem 1.1. (These proofs will be given in Section 11.)

Let Γ be a polycyclic-by-finite wfn-group. We stick to our usual conventions. Namely, Γ is embedded in the \mathbf{Q} -points of its algebraic hull \mathbf{H}_Γ , $\text{Aut}(\Gamma) \leq \text{Aut}_a(\mathbf{H}_\Gamma)(\mathbf{Q})$, and \mathbf{U} is the unipotent radical of \mathbf{H}_Γ . We also fix, as a reference, a thickening $\tilde{\Gamma} = F_m^\perp \cdot \Gamma$ of Γ in \mathbf{H}_Γ . We choose a thickening which satisfies the assumptions of Section 8 on the data $(\tilde{\Gamma}, C, \mathbf{S})$. In particular, $C \leq \tilde{\Gamma}$ is a maximal nilpotent-by-finite supplement, $\mathbf{D} = \overline{C}$ its Zariski-closure, $\mathbf{S} \leq \mathbf{D}$ a maximal \mathbf{Q} -defined d -subgroup, and $\theta = \theta_{\tilde{\Gamma}}$ is a good unipotent shadow for $\tilde{\Gamma}$. Such a thickening exists, by Proposition 7.4.

Proposition 9.1. — *Let Γ be a polycyclic-by-finite wfn-group. Then the subgroup $\text{Inn}_\Gamma \cdot A_{\Gamma|F}$ has finite index in $\text{Aut}(\Gamma)$.*

Proof. — Recall that $A_{\Gamma|F} = \{\phi \in \text{Aut}(\Gamma) \mid \phi|_{\Gamma/F} = \text{id}_{\Gamma/F}\}$. We shall use that

$$(33) \quad A_{\Gamma|F} = \text{Aut}(\Gamma) \cap \mathcal{A}_{\mathbf{H}_\Gamma|F} = \text{Aut}(\Gamma) \cap \mathcal{A}_{\mathbf{H}_\Gamma|\mathbf{U}}.$$

This is a straightforward consequence of Proposition 4.4.

Let us put $N = \pi_{\mathbf{U}}(\Gamma)$, and $M = \pi_{\mathbf{U}}(\Gamma_0)$, where $\Gamma_0 = \Gamma \cap \mathbf{H}^\circ$. Define $\hat{\mathbf{S}} = \mathbf{H}_\Gamma/\mathbf{U}$ and note that $\hat{\mathbf{S}}$ is a \mathbf{Q} -defined d -group.

Let $\phi \in \text{Aut}(\Gamma)$ and let $\Phi \in \text{Aut}_a(\mathbf{H}_\Gamma)$ be its extension to \mathbf{H}_Γ . The \mathbf{Q} -defined automorphism Φ induces a \mathbf{Q} -isomorphism $\Phi_{\hat{\mathbf{S}}}$ of $\hat{\mathbf{S}}$, which preserves N and M . The restriction of $\Phi_{\hat{\mathbf{S}}}$ to N will be denoted by ϕ_N . By the rigidity of tori (AG7), for all ϕ in a finite index subgroup of $\text{Aut}(\Gamma)$, ϕ_N is the identity on M , that is $\phi_N \in \text{Aut}(N, M)$. Thus, by Lemma 6.2, $\phi_N \in \text{Inn}_M^N$ holds in a finite index subgroup of $\text{Aut}(\Gamma)$. If $\phi_N \in \text{Inn}_M^N$, there exists $c \in \Gamma_0$ such that $(\text{Inn}_c^\Gamma \circ \phi)_N = \text{id}_N$. Since N is Zariski-dense in $\hat{\mathbf{S}}$, this implies $\text{Inn}_c^\Gamma \circ \phi \in \text{Aut}(\Gamma) \cap \mathcal{A}_{\mathbf{H}_\Gamma|\mathbf{U}} = A_{\Gamma|F}$. Therefore, $\text{Inn}_{\Gamma_0}^\Gamma \cdot A_{\Gamma|F}$ is of finite index in $\text{Aut}(\Gamma)$.

Proof of Theorem 1.4. — By the results proved in Section 4.1, $\text{Aut}(\Gamma)$ is contained in the \mathbf{Q} -points of $\text{Aut}_a(\mathbf{H}_\Gamma)$. We come now to the statement about $A_{\Gamma|F}$. We use that $\text{Aut}(\Gamma)$ is naturally contained in $\text{Aut}(\tilde{\Gamma})$ as a subgroup of finite index, see Proposition 4.10. The obvious fact that $\tilde{F} = \text{Fitt}(\tilde{\Gamma})$ satisfies $\tilde{F} \cap \Gamma = F$ implies that $A_{\Gamma|F} = A_{\tilde{\Gamma}|\tilde{F}} \cap \text{Aut}(\Gamma)$ is of finite index in $A_{\tilde{\Gamma}|\tilde{F}}$. Hence, by Corollary 8.9, $A_{\Gamma|F}$ is an arithmetic subgroup of $\mathcal{A}_{\mathbf{H}_\Gamma|F}$.

Now we prove that $\text{Aut}(\Gamma)$ is, up to finite index, contained in the arithmetic group A_θ defined relative to the good unipotent shadow $\theta = \theta_{\tilde{\Gamma}}$. (For the construction of A_θ refer to Definition 3.7.) Since $\text{Aut}(\Gamma)$ is with finite index naturally contained in $\text{Aut}(\tilde{\Gamma})$ (see Proposition 4.10) it is enough to prove that a finite index subgroup of $\text{Aut}(\tilde{\Gamma})$ is contained in A_θ . The latter is implied by Proposition 8.10 and Proposition 9.1.

Proof of Theorem 1.3 for wfn-groups. — By Proposition 9.1, we have that $\text{Inn}_\Gamma \cdot \text{A}_{\Gamma|\mathbb{F}}$ is of finite index in $\text{Aut}(\Gamma)$. Moreover, $\tilde{\Gamma} = \tilde{\mathbb{F}} \cdot \mathbb{C}$ contains Γ as a subgroup of finite index. Hence, $\mathbb{F} \cdot (\Gamma \cap \mathbb{C})$ is of finite index in Γ . We have $\text{Inn}_\mathbb{F}$ is in $\text{A}_{\Gamma|\mathbb{F}}$. Therefore, $\text{Inn}_\mathbb{C}^\Gamma \cdot \text{A}_{\Gamma|\mathbb{F}}$ is of finite index in $\text{Aut}(\Gamma)$. Now choose a finite index invariant nilpotent subgroup \mathbb{B} of $\text{Inn}_\mathbb{C}^\Gamma$ to obtain the result.

The usual induction procedure gives the following immediate corollary of Theorem 1.4:

Corollary 9.2. — *Let Γ be a polycyclic-by-finite wfn-group. Then there exists a faithful representation of $\text{Aut}(\Gamma)$ into $\text{GL}(n, \mathbf{Z})$, for some $n \in \mathbf{N}$.*

10. Extensions and quotients of arithmetic groups

In the following we shall accumulate some results about extensions and quotients of arithmetic groups which will be necessary for the proofs in the next subsection. To formulate these results we need the following concepts.

Definition 10.1. — *Let \mathcal{A} be a \mathbf{Q} -defined linear algebraic group and $\mathbb{A} \leq \mathcal{A}$ a subgroup. An automorphism ϕ of \mathbb{A} is said to be \mathcal{A} -rational if there is a \mathbf{Q} -defined automorphism of \mathcal{A} which normalizes \mathbb{A} and coincides with ϕ on \mathbb{A} . Moreover, a homomorphism $\rho : \mathbb{A} \rightarrow \mathcal{G}(\mathbf{Q})$ into a \mathbf{Q} -defined linear algebraic group \mathcal{G} is called \mathcal{A} -rational if ρ extends to a \mathbf{Q} -homomorphism $\rho_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{G}$.*

Definition 10.2. — *Let \mathbb{G} be a group of automorphisms of \mathbb{A} . If the action of \mathbb{G} on \mathbb{A} extends to an algebraic group of automorphisms on \mathcal{A} (see Definition 2.1) then \mathbb{G} is said to be an \mathcal{A} -algebraic group of automorphisms of \mathbb{A} .*

As explained in the introduction, a finite extension group of an arithmetic group need not be arithmetic. We shall give now a slight generalization of a criterion from [21] which allows to show that certain finite extension groups of arithmetic groups are again arithmetic.

Lemma 10.3. — *Let \mathcal{A} be a \mathbf{Q} -defined linear algebraic group and $\mathbb{A} \leq \mathcal{A}$ a Zariski-dense arithmetic subgroup. Let $\mathbb{B} \geq \mathbb{A}$ be a group containing \mathbb{A} as a normal subgroup of finite index. Suppose that conjugation of \mathbb{B} on \mathbb{A} is \mathcal{A} -rational. Then the following hold:*

- i) *The inclusion of \mathbb{A} into \mathcal{A} can be extended to an embedding of the group \mathbb{B} as an arithmetic subgroup into a \mathbf{Q} -defined linear algebraic group \mathcal{B} which contains \mathcal{A} as a subgroup of finite index.*
- ii) *Every automorphism of \mathbb{B} which normalizes \mathbb{A} and which induces an \mathcal{A} -rational automorphism of \mathbb{A} is \mathcal{B} -rational.*

- iii) Let G be a group which acts by automorphisms on B which normalize A . If G acts as an \mathcal{A} -algebraic group of automorphisms on A then G is a \mathcal{B} -algebraic group of automorphisms of B .
- iv) Let $\rho : B \rightarrow \mathcal{G}(\mathbf{Q})$ be a representation of B which restricts to an \mathcal{A} -rational representation of A . Then ρ is \mathcal{B} -rational.

Proof. — Statement i) is an application of [21, Proposition 2.2]. The \mathbf{Q} -defined linear algebraic group \mathcal{B} is constructed through the usual induction procedure, and if $R = \{r_1, \dots, r_n\} \subset B$ is a complete set of coset representatives for A in B then R also forms a complete set of coset representatives for \mathcal{A} in \mathcal{B} . There exists thus $r_{ij} \in R$ and $a_{ij} \in A$ such that $r_i r_j = r_{ij} a_{ij}$.

We show iv). Let $\rho : B \rightarrow \mathcal{G}$ be a homomorphism of B whose restriction to A is \mathcal{A} -rational. That is, there is a \mathbf{Q} -defined homomorphism $\rho_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{G}$ with $\rho_{\mathcal{A}}(a) = \rho(a)$ for all $a \in A$. We shall show now that ρ is \mathcal{B} -rational. Note first that, for all $b \in B$ and all $a \in \mathcal{A}$, we have

$$\rho_{\mathcal{A}}(b^{-1}ab) = \rho(b)^{-1} \rho_{\mathcal{A}}(a) \rho(b),$$

since this identity is valid on the Zariski-dense subgroup A of \mathcal{A} . Define now a map $f : \mathcal{B} \rightarrow \mathcal{G}$ by

$$f(r_i a) := \rho(r_i) \rho_{\mathcal{A}}(a) \quad (i = 1, \dots, n, a \in \mathcal{A}).$$

Clearly, f is a \mathbf{Q} -defined morphism of varieties. A straightforward computation using the above mentioned identity shows that f is a homomorphism of groups. It follows that f is a \mathbf{Q} -defined homomorphism of linear algebraic groups. It is clear that f coincides with ρ on B . This proves iv).

Note that ii) is an immediate consequence of iv).

To prove iii) use ii), and note that the condition of being an algebraic group of automorphisms on \mathcal{B} depends only on the connected component $\mathcal{B}^\circ = \mathcal{A}^\circ$.

The following remark is evident from the definition of an arithmetic group.

Lemma 10.4. — *Let \mathcal{B} be a \mathbf{Q} -defined linear algebraic group and $A \leq B \leq \mathcal{B}(\mathbf{Q})$ be subgroups. Assume that A is of finite index in B and is an arithmetic subgroup of its Zariski-closure. Then B is an arithmetic subgroup of its Zariski-closure in \mathcal{B} .*

Next we study certain arithmetic quotients of arithmetic groups.

Proposition 10.5. — *Let \mathcal{A} be a \mathbf{Q} -defined linear algebraic group and $A \leq \mathcal{A}$ a Zariski-dense arithmetic subgroup. Let $N \leq A$ be a normal subgroup of A and let \mathcal{N} denote the Zariski-closure of N in \mathcal{A} . Assume furthermore that N has finite index in $\mathcal{N} \cap A$. Then:*

- i) The group A/N embeds as an arithmetic subgroup into a \mathbf{Q} -defined linear algebraic group \mathcal{D} .
- ii) Every \mathcal{A} -rational automorphism of A which normalizes N induces a \mathcal{D} -rational automorphism of A/N .
- iii) Every \mathcal{A} -algebraic group of automorphisms on A which normalizes N induces a \mathcal{D} -algebraic group of automorphisms of A/N .
- iv) Let ρ be an \mathcal{A} -rational representation of A with $N \leq \ker \rho$. Then the induced quotient representation of A/N is \mathcal{D} -rational.

For the proof of Proposition 10.5 we need the following lemma.

Lemma 10.6. — *Under the assumptions of Proposition 10.5 there exists a finite index subgroup $C \leq A$ such that $\mathcal{N} \cap C \leq N$.*

Proof. — To prove the lemma, we may clearly assume that the group \mathcal{A} is connected. Let $\mathbf{U}_{\mathcal{A}}$ be the unipotent radical of \mathcal{A} . The unipotent radical $\mathbf{U}_{\mathcal{N}}$ of \mathcal{N} is contained as a normal subgroup in $\mathbf{U}_{\mathcal{A}}$. We may now choose a reductive complement \mathcal{A}^{red} for $\mathbf{U}_{\mathcal{A}}$, such that $\mathcal{N}^{\text{red}} = \mathcal{N} \cap \mathcal{A}^{\text{red}}$ is a reductive complement for $\mathbf{U}_{\mathcal{N}}$ in \mathcal{N} . In particular, \mathcal{N}^{red} is normal in \mathcal{A}^{red} . Thus, by AG5, we may choose an almost direct complement \mathcal{H} for \mathcal{N}^{red} in \mathcal{A}^{red} . That is, \mathcal{H} is a \mathbf{Q} -defined subgroup of \mathcal{A}^{red} which centralizes \mathcal{N}^{red} , satisfies $\mathcal{A}^{\text{red}} = \mathcal{N}^{\text{red}} \cdot \mathcal{H}$ and has finite intersection $\mathcal{N}^{\text{red}} \cap \mathcal{H}$.

We put $N_{\mathbf{U}} = \mathbf{U}_{\mathcal{N}} \cap N$, and $N_1 = \mathcal{N}^{\text{red}} \cap N$. Since N is arithmetic in \mathcal{N} , $N_{\mathbf{U}} \cdot N_1$ has finite index in N . Since $N_{\mathbf{U}}$ is arithmetic in the unipotent group $\mathbf{U}_{\mathcal{N}}$, there is a congruence subgroup G of A with the property $\mathbf{U}_{\mathcal{N}} \cap G \leq N_{\mathbf{U}}$ (see [40], Chapter 4, Theorem 5). This congruence subgroup may be chosen torsion-free as well. We now set

$$G_{\mathbf{U}} = \mathbf{U}_{\mathcal{A}} \cap G, \quad G_1 = \mathcal{N}^{\text{red}} \cap G, \quad G_{\mathcal{H}} = \mathcal{H} \cap G.$$

Since G is an arithmetic subgroup of \mathcal{A} , the product $G_{\mathbf{U}} \cdot G_1 \cdot G_{\mathcal{H}}$ is of finite index in G . Both N_1 and G_1 are arithmetic subgroups of \mathcal{N}^{red} . Hence, $C_1 = N_1 \cap G_1$ has finite index in N_1 and G_1 . Therefore,

$$C = G_{\mathbf{U}} \cdot C_1 \cdot G_{\mathcal{H}}$$

has finite index in G and A . Since G is torsion-free, $\mathcal{N} \cap C$ is contained in $G_{\mathbf{U}} \cdot C_1$. Now we find that

$$\mathcal{N} \cap C \leq G_{\mathbf{U}} \cdot C_1 \leq (\mathbf{U}_{\mathcal{A}} \cap C) \cdot (\mathcal{N}^{\text{red}} \cap G) \leq N_{\mathbf{U}} \cdot N_1 \leq N.$$

This finishes the proof of the lemma.

Proof of Proposition 10.5. — Replacing the subgroup C constructed in Lemma 10.6 above by one of its subgroups of finite index and also possibly by $C \cdot N$, the following can be arranged:

- C is a normal subgroup of finite index in A ,
- $C \cap \mathcal{N} = N$,
- C is normalized by every automorphism of A which normalizes N .

Let \mathcal{C} denote the Zariski-closure of C . This is a \mathbf{Q} -closed subgroup (of finite index) in \mathcal{A} which contains \mathcal{N} . Also C/N is contained as a Zariski-dense subgroup in the \mathbf{Q} -defined linear algebraic group \mathcal{C}/\mathcal{N} . Moreover, C/N is an arithmetic subgroup of \mathcal{C}/\mathcal{N} , by AR1.

Now $B = A/N$ is a finite extension group of C/N . Clearly, conjugations by elements of A give rise to \mathcal{C}/\mathcal{N} -rational automorphisms of C/N and so does every \mathcal{A} -rational automorphism of A which normalizes N . Therefore, the group \mathcal{D} may be constructed by application of Lemma 10.3, thus proving i), ii), iii).

To prove iv), let $\rho_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{G}$ denote the algebraic extension of ρ , and let $\rho_{A/N} : A/N \rightarrow \mathcal{G}(\mathbf{Q})$ be the quotient representation induced by ρ . Clearly, its restriction to C/N is \mathcal{C}/\mathcal{N} -rational, since $\rho_{\mathcal{A}}$ factors over \mathcal{C}/\mathcal{N} . Thus, part iv) of Lemma 10.3 shows that $\rho_{A/N}$ is \mathcal{D} -rational.

In the following we deal with the fact that a group which is isomorphic to an arithmetic group may admit essentially different arithmetic embeddings into linear algebraic groups. This phenomenon plays a role in our arithmeticity proofs. At this point we will also need the full strength of the assumption that G acts as an algebraic group of automorphisms on $A \leq \mathcal{A}$, in order to extend the action of G to a modification of the ambient group \mathcal{A} .

Proposition 10.7. — *Let A be an arithmetic subgroup of a \mathbf{Q} -defined linear algebraic group \mathcal{A} and G an \mathcal{A} -algebraic group of automorphisms of A . Let D be a normal subgroup of A which is contained in the center of A and which is normalized by G . Then the group A can be embedded as an arithmetic subgroup into a \mathbf{Q} -defined linear algebraic group \mathcal{E} such that:*

- i) *The subgroup D of A is unipotent-by-finite in \mathcal{E} .*
- ii) *The group G acts as an \mathcal{E} -algebraic group of automorphisms of A .*
- iii) *If $\rho : A \rightarrow \mathcal{G}(\mathbf{Q})$ is an \mathcal{A} -rational representation which satisfies $D \leq \ker \rho$ then ρ is \mathcal{E} -rational.*

Proof. — Clearly, there is no harm in assuming that A is Zariski-dense in \mathcal{A} . We choose a subgroup $C \leq A$, subject to the following conditions:

- C is torsion-free and a normal subgroup of finite index in A ,
- C is contained in the connected component \mathcal{A}° of \mathcal{A} ,
- C is normalized by G .

Note that C is Zariski-dense in \mathcal{A}° . We define $G_1 = \text{Inn}_A \cdot G$ to be the subgroup of the automorphism group of A which is generated by Inn_A and G . Then G_1 acts by \mathbf{Q} -defined automorphisms on \mathcal{A} . Now Lemma 2.2 and Lemma 2.6 show that G_1 is an \mathcal{A} -algebraic group of automorphisms of A (and also of C). Hence, according to Proposition 2.7, there exists a G_1 -invariant almost direct product decomposition

$$\mathcal{A}^\circ = \mathcal{Z}_1 \cdot \mathcal{A}_1,$$

where \mathcal{Z}_1 is the \mathbf{Q} -closed central d -subgroup consisting precisely of the semisimple elements contained in the center of \mathcal{A}° , and \mathcal{A}_1 is a \mathbf{Q} -closed normal subgroup of \mathcal{A} with unipotent by-finite center.

Next we define

$$Z_1 = C \cap \mathcal{Z}_1, \quad C_1 = C \cap \mathcal{A}_1, \quad C_2 = Z_1 \cdot C_1.$$

Observe that Z_1 is an arithmetic subgroup of \mathcal{Z}_1 and that C_1 is arithmetic in \mathcal{A}_1 . It follows that C_2 is arithmetic in \mathcal{A} and of finite index in C . Since C is torsion-free, we have $Z_1 \cap C_1 = \{1\}$, and therefore C_2 is isomorphic to the direct product $Z_1 \times C_1$. The action of G_1 as an \mathcal{A} -algebraic group of automorphisms of C stabilizes the factors Z_1 and C_1 , and hence also C_2 .

Now put \mathcal{D} for the Zariski-closure of D . Since D is central in A , $\mathcal{D} \leq Z(\mathcal{A})$. It follows that the maximal d -subgroup $\mathcal{S}_\mathcal{D}$ of \mathcal{D} is contained in \mathcal{Z}_1 . Since $\mathcal{S}_\mathcal{D}$ is invariant in \mathcal{D} , there exists, by virtue of AG6, an almost direct product decomposition $\mathcal{Z}_1 = \mathcal{S}_\mathcal{D} \cdot \mathcal{S}_2$ which is respected by G_1 . We define $Z_\mathcal{D} = Z_1 \cap \mathcal{S}_\mathcal{D}$ and $Z_2 = Z_1 \cap \mathcal{S}_2$. By the arithmeticity of the factors Z_1 and Z_2 , the product $Z_\mathcal{D} \cdot Z_2$ is of finite index in Z_1 . Also this decomposition is preserved by G_1 .

Now define $C_3 = Z_\mathcal{D} \cdot Z_2 \cdot C_1 \leq \mathcal{S}_\mathcal{D} \cdot \mathcal{S}_2 \cdot \mathcal{A}_1$. This group is an arithmetic subgroup, G_1 -invariant and of finite index in A . Since it is torsion-free it is also a direct product of its factors. Let us put $\mathcal{A}_2 = \mathcal{S}_2 \cdot \mathcal{A}_1$ and $\bar{\mathcal{A}}_2 = \mathcal{A}_2 / (\mathcal{A}_2 \cap \mathcal{S}_\mathcal{D})$. Then we have an induced arithmetic and Zariski-dense embedding

$$C_3 = Z_\mathcal{D} \times (Z_2 \times C_1) \leq \mathcal{S}_\mathcal{D} \times \bar{\mathcal{A}}_2.$$

Since $Z_\mathcal{D}$ is isomorphic to \mathbf{Z}^n , for some $n \geq 0$, we may embed this group as an arithmetic subgroup into \mathbf{G}_a^n . This gives rise to an arithmetic and Zariski-dense embedding

$$C_3 = Z_\mathcal{D} \times (Z_2 \times C_1) \leq \mathbf{G}_a^n \times \bar{\mathcal{A}}_2$$

which has the property that $D \cap C_3$ is unipotent by-finite in $\mathbf{G}_a^n \times \bar{\mathcal{A}}_2$.

Note that G_1 induces an $\bar{\mathcal{A}}_2$ -algebraic group of automorphisms of C_1 . We consider now the arithmetic embedding of $Z_\mathcal{D}$ into the unipotent group \mathbf{G}_a^n . The action of G_1 on $Z_\mathcal{D}$ extends to an action by \mathbf{Q} -defined automorphisms of \mathbf{G}_a^n . This turns G_1

into a \mathbf{G}_a^n -algebraic group of automorphisms of $Z_{\mathcal{D}}$. We infer from Lemma 2.5 that the product action of G_1 on $\mathbf{G}_a^n \times \bar{\mathcal{A}}_2$ turns G_1 into an algebraic group of automorphisms of $\mathbf{G}_a^n \times \bar{\mathcal{A}}_2$. This, in particular, turns G_1 into an $\mathbf{G}_a^n \times \bar{\mathcal{A}}_2$ -algebraic group of automorphisms of C_2 .

Now let ρ be an \mathcal{A} -rational representation of A which contains D in its kernel. In particular, it satisfies $\mathcal{D} \leq \ker \rho_{\mathcal{A}}$, where $\rho_{\mathcal{A}}$ denotes the extension of ρ to \mathcal{A} . The restriction of $\rho_{\mathcal{A}}$ to \mathcal{A}_2 gives rise to a \mathbf{Q} -defined homomorphism $\bar{\rho} : \mathbf{G}_a^n \times \bar{\mathcal{A}}_2 \rightarrow \mathcal{G}$ which has the subgroup \mathbf{G}_a^n in its kernel. We contend that $\bar{\rho}$ extends the representation ρ on C_3 . This is easily verified. Thus $\rho : C_3 \rightarrow \mathcal{G}$ is $\mathbf{G}_a^n \times \bar{\mathcal{A}}_2$ -rational.

Since C_3 is normal and of finite index in A , this allows, by application of Lemma 10.3, to embed the group A as an arithmetic subgroup in a finite extension \mathcal{E} of $\mathbf{G}_a^n \times \bar{\mathcal{A}}_2$ such that G acts as a \mathcal{E} -algebraic group of automorphisms of A . This embedding has the property that the finite index subgroup $C_3 \cap D \leq D$ is unipotent. Hence, D is unipotent-by-finite under the embedding of A into \mathcal{E} . This proves i) and ii). The last statement of Lemma 10.3 asserts that ρ is \mathcal{E} -rational, since the restriction of ρ to C_3 is $\mathbf{G}_a^n \times \bar{\mathcal{A}}_2$ -rational.

Part of Proposition 10.7 is reminiscent of Corollary 3.5 from [19] and of Proposition 3.3 from [21], but it is stronger since no passages to subgroups of finite index are required. Our ultimate arithmeticity result is contained in the next proposition.

Proposition 10.8. — *Let \mathcal{A} be a \mathbf{Q} -defined linear algebraic group and let $A \leq \mathcal{A}(\mathbf{Q})$ be a Zariski-dense subgroup. Assume N, B, C are normal subgroups of A such that the following hold:*

- i) $N \cdot B$ has finite index in A ,
- ii) B is an arithmetic subgroup in its Zariski-closure \mathcal{B} ,
- iii) C is an arithmetic subgroup in its Zariski-closure \mathcal{C} ,
- iv) $C \leq N \cap B$ and $D = (N \cap B)/C$ is in the center of B/C .

Then A/N is an arithmetic group.

Moreover, there exists an arithmetic embedding of A/N into a \mathbf{Q} -defined linear algebraic group \mathcal{A}_N which has the following property: For any \mathcal{A} -rational representation of A with $N \leq \ker \rho$, the induced quotient representation of A/N is an \mathcal{A}_N -rational representation.

Proof. — The group A induces by conjugation an \mathcal{B} -algebraic group of automorphisms of B which we call G . Note that \mathcal{B} and \mathcal{C} are normal in \mathcal{A} , and preserved by G as well. Since B and C are arithmetic subgroups of their respective Zariski-closures, we find that C has finite index in $\mathcal{C} \cap B$. We may hence use Proposition 10.5 to embed the group B/C as an arithmetic subgroup into a \mathbf{Q} -defined linear algebraic group \mathcal{D} . This embedding has the property that the group G of automorphisms of B/C is \mathcal{D} -algebraic.

We consider now the subgroup $D = (N \cap B)/C$ in B/C . By our assumption iv), D is central in B/C . Since G is \mathcal{D} -algebraic, we may, by Proposition 10.7, change the

arithmetic embedding of B/C in \mathcal{D} to an arithmetic embedding of B/C into a \mathbf{Q} -defined linear algebraic group \mathcal{E} such that D is unipotent-by-finite in \mathcal{E} . Moreover, the group G acts as an \mathcal{E} -algebraic group of automorphisms.

Let us consider now the Zariski-closure \mathcal{D}_1 of D in \mathcal{E} . Since D has a unipotent finite index subgroup, D is an arithmetic subgroup of the \mathbf{Q} -defined algebraic group \mathcal{D}_1 . (For a proof consult [40, Chapter 8]). Since D is an arithmetic subgroup it has finite index in $\mathcal{D}_1 \cap B/C$. Therefore, we may apply Proposition 10.5 to embed the quotient $(B/C)/D$ as an arithmetic subgroup into a \mathbf{Q} -defined linear algebraic group \mathcal{B}_1 such that G acts by \mathcal{B}_1 -rational automorphisms on $(B/C)/D$.

By assumption i), A/N is isomorphic to a finite extension group of

$$(B/C)/D \cong B/B \cap N \cong (N \cdot B)/N.$$

The elements of A/N act on $(B/C)/D$ as \mathcal{B}_1 -rational automorphisms since the elements of G have this property. Finally, we apply Lemma 10.3 to find that A/N is arithmetic in a \mathbf{Q} -defined linear algebraic group.

To prove that the restriction of ρ to A/N is \mathcal{A}_N -rational, we have to carry over the rationality of ρ in each of the construction steps above. The details are easily verified.

11. The arithmeticity of $\text{Out}(\Gamma)$

This section contains the complete proof of Theorem 1.1 which proceeds in two steps. These steps are carried out in Section 11.1 and in Section 11.2. On our way, we provide (respectively, finish) the proof of Theorem 1.5 in Section 11.1, as well as the proofs of Theorem 1.3 and of Theorem 1.9 in Section 11.2.

Throughout this section, Γ denotes a polycyclic-by-finite group. We also stick to the notation introduced in Sections 2 to 6. In particular, $F \leq \Gamma$ denotes the Fitting subgroup of Γ . If in addition Γ is a wfn-group, \mathbf{H}_Γ denotes the algebraic hull of Γ , and \mathbf{F} the Zariski-closure of F .

11.1. *The case of polycyclic-by-finite wfn-groups.* — The purpose of this subsection is to prove Theorem 1.5 of the introduction. Let us therefore assume here that Γ is a wfn-group. The arithmeticity of $\text{Out}(\Gamma)$, in the case that Γ is a wfn-group, is an immediate consequence of the structural properties of the embedding $\text{Aut}(\Gamma) \leq \text{Aut}_a(\mathbf{H})(\mathbf{Q})$ together with Proposition 10.8. To see this let us put now

$$(34) \quad A = \text{Aut}(\Gamma), \quad N = \text{Inn}_\Gamma, \quad B = A_{\Gamma|F}, \quad C = \text{Inn}_F.$$

Then we have:

Proposition 11.1. — *Let Γ be a wfn-group. Then the subgroups $N, B, C \leq A$ defined in (34) satisfy the hypotheses of Proposition 10.8 with respect to the Zariski-closure \mathcal{A} of $\text{Aut}(\Gamma)$ in $\text{Aut}_a(\mathbf{H})$.*

Proof. — Condition i) requires that $N \cdot B = \text{Inn}_\Gamma \cdot A_{\Gamma|F}$ has finite index in $A = \text{Aut}(\Gamma)$. This is contained in Proposition 9.1. Theorem 1.4 says that $B = A_{\Gamma|F}$ is arithmetic in its Zariski-closure \mathcal{B} in \mathcal{A} . This implies condition ii).

The construction of the algebraic structure on $\text{Aut}_a(\mathbf{H})$ (see Subsection 3.1) shows that the group $\text{Inn}_{\mathbf{F}}$ is a Zariski-closed subgroup of the unipotent radical of $\text{Aut}_a(\mathbf{H})$. Moreover, $\text{Inn}_{\mathbf{F}}$ contains the finitely generated group Inn_F as a Zariski-dense subgroup of rational points. In particular, $C = \text{Inn}_F$ is arithmetic in its Zariski-closure. This implies condition iii).

We shall finally verify the conditions iv) of Proposition 10.8. Clearly we have $\text{Inn}_F \leq \text{Inn}_\Gamma \cap A_{\Gamma|F}$, that is, $C \leq N \cap B$. Now let $\Phi \in A_{\Gamma|F}$ and $\gamma \in \Gamma$ with $\text{Inn}_\gamma \in A_{\Gamma|F}$. Then $\Phi(\gamma) = \gamma f$, where $f \in F$. It follows that

$$\Phi \circ \text{Inn}_\gamma \circ \Phi^{-1} = \text{Inn}_{\Phi(\gamma)} = \text{Inn}_\gamma \text{Inn}_f.$$

This shows that $\text{Inn}_\Gamma \cap A_{\Gamma|F} = N \cap B$ projects onto a central subgroup of $B/C = A_{\Gamma|F}/\text{Inn}_F$. Hence, iv) holds.

Proof of Theorem 1.5. — We may now apply Proposition 10.8 which asserts that there exists a \mathbf{Q} -defined linear algebraic group

$$\mathcal{O}_\Gamma = \mathcal{A}_N = \mathcal{A}_{\text{Inn}_\Gamma}$$

which contains an isomorphic copy of the group $\text{Out}(\Gamma) = A/N$ as an arithmetic subgroup. This already establishes the arithmeticity of $\text{Out}(\Gamma)$.

Consider next the algebraic outer automorphism group

$$\text{Out}_a(\mathbf{H}_\Gamma) = \text{Aut}_a(\mathbf{H}_\Gamma)/\text{Inn}_{\mathbf{H}_\Gamma},$$

and let $\pi_\Gamma : \text{Out}(\Gamma) \rightarrow \text{Out}_a(\mathbf{H}_\Gamma)$ be the homomorphism induced on $\text{Out}(\Gamma)$. Since the natural map $\text{Aut}_a(\mathbf{H}_\Gamma) \rightarrow \text{Out}_a(\mathbf{H}_\Gamma)$ is a \mathbf{Q} -defined homomorphism, it induces a \mathbf{Q} -defined homomorphism $\mathcal{A} \rightarrow \text{Out}_a(\mathbf{H}_\Gamma)$. Since π_Γ contains $N = \text{Inn}_\Gamma$ in its kernel, Proposition 10.8 asserts that the homomorphism $\pi_\Gamma : \text{Out}(\Gamma) \rightarrow \text{Out}_a(\mathbf{H}_\Gamma)$ can be extended to a \mathbf{Q} -defined homomorphism

$$\pi_{\mathcal{O}_\Gamma} : \mathcal{O}_\Gamma \rightarrow \text{Out}_a(\mathbf{H}_\Gamma).$$

This proves the first part of Theorem 1.5.

To show the statements about the kernel of π_Γ , we define:

$$\mathbf{K} = \text{Inn}_{\mathbf{H}} \cap \text{Aut}(\Gamma), \quad \mathbf{K}_F = \text{Inn}_{\mathbf{H}} \cap A_{\Gamma|F}, \quad \mathbf{E}_F = \text{Inn}_{\mathbf{F}}^{\mathbf{H}_\Gamma} \cap A_{\Gamma|F}.$$

Lemma 11.2. — *With the above notation the following hold:*

- i) Inn_F^Γ has finite index in E_F .
- ii) There is a finite index normal subgroup $T \leq A_{\Gamma|F}$ such that $T \cap E_F \leq \text{Inn}_F^\Gamma$.
- iii) The commutator group $[A_{\Gamma|F}, K_F]$ is contained in E_F .

Proof. — Note that $\text{Inn}_F^{\mathbf{H}}$ is Zariski-dense and arithmetic in the unipotent group $\text{Inn}_F^{\mathbf{H}}$. Since $A_{\Gamma|F}$ is an arithmetic subgroup of its Zariski-closure in \mathcal{A} , E_F is arithmetic in $\text{Inn}_F^{\mathbf{H}}$ as well. This implies i).

Now ii) follows from i) together with the congruence subgroup property for Inn_F^Γ . (Compare the proof of Lemma 10.6.)

For iii), let Φ be the extension of $\phi \in A_{\Gamma|F}$ to an automorphism of \mathbf{H}_Γ . Let $h \in \mathbf{H}$ such that $\text{Inn}_h = \psi \in K_F$. Since $\Phi \in \mathcal{A}_{\mathbf{H}_\Gamma|F}$

$$\Phi \circ \text{Inn}_h \circ \Phi^{-1} = \text{Inn}_{\Phi(h)} = \text{Inn}_h \circ \text{Inn}_{f_h}.$$

This in turn gives $\psi^{-1} \circ \phi \circ \psi \circ \phi^{-1} \in \text{Inn}_F^{\mathbf{H}_\Gamma} \cap A_{\Gamma|F} = E_F$, proving iii).

The kernel of π_Γ is the image of $K = \text{Inn}_{\mathbf{H}_\Gamma} \cap \text{Aut}(\Gamma)$ in $\text{Out}(\Gamma)$. Let \bar{K}_F and \bar{E}_F be the images of K_F, E_F in $A_{\Gamma|F}/\text{Inn}_F^\Gamma$. Let \bar{T} be the corresponding image of the finite index subgroup $T \leq A_{\Gamma|F}$ as in Lemma 11.2 ii). Lemma 11.2 i) shows that \bar{E}_F is finite. By ii) and iii) of the same lemma, \bar{K}_F is centralized by the finite index subgroup $\bar{T} \leq A_{\Gamma|F}/\text{Inn}_F^\Gamma$. In particular, \bar{K}_F is abelian-by-finite. Consider now the commutative diagram

$$\begin{array}{ccc} A_{\Gamma|F}/\text{Inn}_F^\Gamma & \longrightarrow & \text{Out}(\Gamma) = \text{Aut}(\Gamma)/\text{Inn}_\Gamma \\ \downarrow & \swarrow & \\ \text{Out}_d(\mathbf{H}_\Gamma) & & \end{array}$$

of natural homomorphisms. By AR3, every abelian subgroup of an arithmetic group is finitely generated. Hence, the image of \bar{K}_F in $\text{Out}(\Gamma)$ is so. We may also infer that \bar{K}_F is finitely generated.

Since $\text{Inn}_\Gamma \cdot A_{\Gamma|F}$ has finite index in $\text{Aut}(\Gamma)$, the normal subgroup \bar{K}_F maps onto a finite index subgroup of the image of K in $\text{Aut}(\Gamma)/\text{Inn}_\Gamma$. This proves that $\ker \pi_\Gamma$ is finitely generated, abelian-by-finite and centralized by a finite index subgroup of $\text{Out}(\Gamma)$. If Γ is nilpotent-by-finite then E_F is of finite index in K_F , and hence $\ker \pi_\Gamma$ is finite. This finishes the proof of Theorem 1.5.

11.2. *The case of a general polycyclic-by-finite group.* — In this subsection, we explain the transfer of our arithmeticity results from the case of wfn-groups to general polycyclic-by-finite groups. Thereby, we provide the final step in the proofs of Theorem 1.1 and Theorem 1.3. We also prove Proposition 1.6 and Theorem 1.9.

Let Γ be a polycyclic-by-finite group. Let τ_Γ denote the maximal finite normal subgroup of Γ . Note that τ_Γ is characteristic in Γ that is, it is normalized by every automorphism of Γ . The quotient group

$$\tilde{\Gamma} := \Gamma/\tau_\Gamma$$

is a wfn-group. We let $j : \Gamma \rightarrow \tilde{\Gamma}$ denote the quotient homomorphism. By Theorem 1.5, the group $\text{Out}(\tilde{\Gamma})$ is an arithmetic group. We shall show that $\text{Out}(\Gamma)$ has the same property.

Let $\Gamma_0 \leq \Gamma$ be a characteristic finite index subgroup with $\Gamma_0 \cap \tau_\Gamma = \{1\}$. We may suppose that the image $\tilde{\Gamma}_0 \leq \tilde{\Gamma}$ of Γ_0 in $\tilde{\Gamma}$ is also characteristic. (To obtain such a subgroup, let $n \in \mathbf{N}$ be the index of a torsion-free subgroup of finite index in Γ . Take Γ_0 to be the subgroup generated by all γ^n , $\gamma \in \Gamma$).

Let us put

$$\mu := \Gamma/\Gamma_0.$$

The quotient homomorphism j and the projection to μ induce injective homomorphisms

$$j_\mu : \Gamma \rightarrow \tilde{\Gamma} \times \mu, \quad \text{and} \quad k_\mu : \text{Aut}(\Gamma) \rightarrow \text{Aut}(\tilde{\Gamma}) \times \text{Aut}(\mu).$$

We define finite index subgroups of $\text{Aut}(\Gamma)$ and $\text{Aut}(\tilde{\Gamma})$, respectively:

$$\begin{aligned} A_0 &:= \{\phi \in \text{Aut}(\Gamma) \mid \phi_{\Gamma/\Gamma_0} = \text{id}_{\Gamma/\Gamma_0}\} \leq \text{Aut}(\Gamma), \\ \tilde{A}_0 &:= \{\phi \in \text{Aut}(\tilde{\Gamma}) \mid \phi_{\tilde{\Gamma}/\tilde{\Gamma}_0} = \text{id}_{\tilde{\Gamma}/\tilde{\Gamma}_0}\} \leq \text{Aut}(\tilde{\Gamma}). \end{aligned}$$

Lemma 11.3. — *With the above notation the following hold:*

- i) *The group $j_\mu(\Gamma)$ is of finite index in $\tilde{\Gamma} \times \mu$.*
- ii) *Let $F = \text{Fitt}(\Gamma)$. Then $j(F)$ is of finite index in $\text{Fitt}(\tilde{\Gamma})$.*
- iii) *The induced homomorphism $k : \text{Aut}(\Gamma) \rightarrow \text{Aut}(\tilde{\Gamma})$ maps the group A_0 isomorphically onto \tilde{A}_0 . In particular, $k(\text{Aut}(\Gamma))$ is of finite index in $\text{Aut}(\tilde{\Gamma})$.*
- iv) *The subgroup $A_{\Gamma|F} \leq \text{Aut}(\Gamma)$ is mapped by k onto a finite index subgroup of $A_{\tilde{\Gamma}|\tilde{F}}$.*

Proof. — Part i) is clear. For ii), note first that $j(F)$ is a nilpotent normal subgroup in $\tilde{\Gamma}$, and hence $j(F) \leq \text{Fitt}(\tilde{\Gamma})$. Then $F_0 = j^{-1}(\text{Fitt}(\tilde{\Gamma}) \cap \Gamma_0)$ is a nilpotent normal subgroup of Γ which is of finite index in the preimage $j^{-1}(\text{Fitt}(\tilde{\Gamma}))$. Moreover, $F_0 \leq F$. Hence F is of finite index in this preimage. This implies ii).

To prove iii), we show that $k_\mu(A_0) = \tilde{A}_0 \times \{1\}$. Clearly, $k_\mu(A_0)$ is contained in $\tilde{A}_0 \times \{1\}$. Let $\psi \in \tilde{A}_0$. We show that there exists $\phi \in A_0$ such that $\psi = k(\phi)$ is induced by ϕ . If $\gamma \in \Gamma$, we let $\tilde{\gamma} = j(\gamma)$ denote its projection into $\tilde{\Gamma}$. Since the projection j

maps Γ_0 isomorphically onto the invariant subgroup $\tilde{\Gamma}_0$, the automorphism ψ defines $\psi_0 \in \text{Aut}(\Gamma_0)$ uniquely with the property that

$$j(\phi_0(\gamma)) = \psi(\tilde{\gamma}) \quad (\gamma \in \Gamma_0).$$

Let $\Gamma/\Gamma_0 = \bigcup_i \gamma_i \Gamma_0$ be the coset decomposition. Now we can write $\psi(\tilde{\gamma}_i) = \tilde{\gamma}_i \epsilon_i$, $\epsilon_i \in \tilde{\Gamma}_0$. There exist unique $\delta_i \in \Gamma_0$ such that $\epsilon_i = \tilde{\delta}_i$. We declare now

$$\phi(\gamma_i s) = \tilde{\gamma}_i \delta_i \phi_0(s) \quad (s \in \Gamma_0).$$

It is easy to verify that this actually defines an automorphism $\phi \in \text{Aut}(\Gamma)$. This ϕ is clearly a lift of ψ .

For iv) consider the quotient homomorphism $\Gamma/F \rightarrow \tilde{\Gamma}/\tilde{F}$. By ii), this homomorphism has finite kernel. Since it is surjective, the group $A_{\Gamma|F}$ is mapped into $A_{\tilde{\Gamma}|\tilde{F}}$ by k . Let A_1 be the preimage of $A_{\tilde{\Gamma}|\tilde{F}}$ in $\text{Aut}(\Gamma)$. Now applying the reasoning of iii) to the above quotient homomorphism with finite kernel, we deduce that finite index subgroup of A_1 acts as the identity on $\tilde{\Gamma}/\tilde{F}$. That is, $A_{\Gamma|F}$ has finite index in A_1 . This shows iv).

Part iii) of the above lemma immediately implies Proposition 1.6:

Proof of Proposition 1.6. — The groups $\text{Aut}(\Gamma)$ and $\text{Aut}(\tilde{\Gamma})$ are commensurable, since they have isomorphic finite index subgroups A_0 and \tilde{A}_0 . If $\text{Aut}(\tilde{\Gamma})$ is arithmetic, then the product $\text{Aut}(\tilde{\Gamma}) \times \text{Aut}(\mu)$ is arithmetic. Since $\text{Aut}(\Gamma)$ embeds as a subgroup of finite index in the latter product, $\text{Aut}(\Gamma)$ is an arithmetic group as well. Conversely, if $\text{Aut}(\Gamma)$ is arithmetic, the finite index subgroup $A_0 \leq \text{Aut}(\Gamma)$ is arithmetic too. Therefore, the subgroup $\tilde{A}_0 \leq \text{Aut}(\tilde{\Gamma})$ is arithmetic.

A related result is:

Proposition 11.4. — *The subgroup $A_{\Gamma|F}$ of $\text{Aut}(\Gamma)$ is arithmetic.*

Proof. — By iv) of Lemma 11.3, the injection j_μ maps $A_{\Gamma|F}$ onto a finite index subgroup of $A_{\tilde{\Gamma}|\tilde{F}} \times \text{Aut}(\mu)$. Since $A_{\tilde{\Gamma}|\tilde{F}}$ is arithmetic, by Corollary 8.9, we can infer that $A_{\Gamma|F}$ is arithmetic.

Proof of Theorem 1.3 in the general case. — As remarked above, the projection k maps $A_{\Gamma|F}$ onto a finite index subgroup of $A_{\tilde{\Gamma}|\tilde{F}}$. Let $B \leq \Gamma \cap \Gamma_0$ be a nilpotent subgroup such that $A_{\tilde{\Gamma}|\tilde{F}} \cdot j(B)$ is of finite index in $\text{Aut}(\tilde{\Gamma})$ (see Section 9). Then $A_{\Gamma|F} \cdot B$ is of finite index in $\text{Aut}(\Gamma)$. Together with Proposition 11.4, this proves the required decomposition of $\text{Aut}(\Gamma)$.

As another consequence of Lemma 11.3, we infer that $\text{Out}(\Gamma)$ and $\text{Out}(\tilde{\Gamma})$ are S-commensurable:

Proof of Proposition 1.7. — Since $\mathbf{k}(\text{Inn}_{\Gamma_0}) \leq \text{Inn}_{\tilde{\Gamma}}$ is of finite index in $\text{Inn}_{\tilde{\Gamma}}$ and $\mathbf{k} : A_0 \rightarrow \tilde{A}_0$ is an isomorphism, $A_0 \cap \text{Inn}_{\Gamma}$ is a subgroup of finite index in $\mathbf{k}^{-1}(\tilde{A}_0 \cap \text{Inn}_{\tilde{\Gamma}})$. We consider the map on quotients

$$A_0/A_0 \cap \text{Inn}_{\Gamma} \xrightarrow{\mathbf{k}^*} \tilde{A}_0/\tilde{A}_0 \cap \text{Inn}_{\tilde{\Gamma}}$$

which is induced by \mathbf{k} . The above implies that \mathbf{k}^* has finite kernel. Since the left hand side is a finite index subgroup of $\text{Out}(\Gamma)$, the corollary follows.

Proof of Theorem 1.1 in the general case. — Let Γ be a polycyclic-by-finite group. Then we know that $\text{Out}(\Gamma)$ is residually finite (by [46]). By Proposition 1.7, it projects with finite kernel onto a finite index subgroup of the arithmetic group $\text{Out}(\tilde{\Gamma})$. Thus Proposition 1.8 implies that $\text{Out}(\Gamma)$ is arithmetic.

Proof of Theorem 1.9. — By Proposition 1.6 we reduce to the case of $\tilde{\Gamma}$. Using Proposition 9.1 we infer that $A_{\tilde{\Gamma}|\mathbb{F}}$ is of finite index in $\text{Aut}(\tilde{\Gamma})$. We finally use Theorem 1.4.

12. Polycyclic groups with non-arithmetic automorphism groups

We present examples of polycyclic groups whose automorphism groups are not isomorphic to any arithmetic group. In particular, we shall prove Theorem 1.2.

12.1. Automorphism groups of semi-direct products. — Here are some remarks concerning the automorphism group of groups Γ which are semi-direct products $F \rtimes D$ where D is a group and F is a (commutative) D -module.

We write the group product in F additively, and for $h \in D$, we write $f \mapsto h \cdot f$, $f \in F$, to denote the action of the element h on F . Let Θ be a subgroup of Γ . We write Inn_{Θ} for the subgroup of $\text{Aut}(\Gamma)$ consisting of the inner automorphisms defined by the elements of Θ . Similarly as before, we put

$$A_{\Gamma|\mathbb{F}} := \{\phi \in \text{Aut}(\Gamma) \mid \phi(F) = F, \phi|_{\Gamma/\mathbb{F}} = \text{id}_{\Gamma/\mathbb{F}}\}.$$

There are two constructions for automorphisms in $A_{\Gamma|\mathbb{F}}$. For the first, let $\text{Der}(D, F) = \{d : D \rightarrow F \mid d(h_1 h_2) = z(h_1) + h_1 \cdot z(h_2)\}$ be the group of derivations from D into F . The group of derivations naturally obtains the structure of a D -module by setting

$$g * d(h) := g \cdot d(g^{-1} h g) \quad (g, h \in D, d \in \text{Der}(D, F)).$$

A derivation $d \in \text{Der}(D, F)$ gives rise to an automorphism $\phi_d : \Gamma \rightarrow \Gamma$ by

$$\phi_d((m, g)) := (m + d(g), g) \quad (m \in F, g \in D).$$

We write $\text{Aut}^d(\Gamma)$ for the (abelian) subgroup of $A_{\Gamma|F}$ consisting of these automorphisms. We remark that the homomorphism

$$\text{Der}(D, F) \rightarrow A_{\Gamma|F}, \quad d \mapsto \phi_d \quad (d \in \text{Der}(D, F))$$

is D -equivariant with respect to the above D -action on $\text{Der}(D, F)$ and conjugation by elements of Inn_D on $A_{\Gamma|F}$.

Let us define $\text{Aut}_D(F)$ to be the group of D -equivariant automorphisms of F . Given a D -equivariant automorphism $\rho : F \rightarrow F$, we define an automorphism $\phi_\rho : \Gamma \rightarrow \Gamma$ by

$$\phi_\rho((m, g)) := (\rho(m), g) \quad (m \in F, g \in D).$$

We write $\text{Aut}^a(\Gamma)$ for the subgroup of $A_{\Gamma|F}$ consisting of these automorphisms.

Proposition 12.1. — *Let $\Gamma = F \rtimes D$ be a semi-direct product of a abelian D -module F by the group D . We then have:*

- i) $A_{\Gamma|F} = \text{Aut}^d(\Gamma) \cdot \text{Aut}^a(\Gamma)$.
- ii) $\text{Inn}_\Gamma \cdot A_{\Gamma|F} = (\text{Aut}^d(\Gamma) \cdot \text{Aut}^a(\Gamma)) \cdot \text{Inn}_D$.
- iii) $\text{Aut}^a(\Gamma)$ centralizes Inn_D .
- iv) $\text{Aut}^d(\Gamma) \cap \text{Aut}^a(\Gamma) = \{1\}$.
- v) $\text{Aut}^d(\Gamma)$ is an abelian normal subgroup in $\text{Inn}_\Gamma \cdot A_{\Gamma|F}$.

The proof of this proposition is straightforward, we skip it.

12.2. Examples. — In order to show that certain groups are not arithmetic we use the following simple criterion.

Proposition 12.2. — *For a matrix $A \in \text{GL}(n, \mathbf{Z})$, let $\Gamma(A) = \mathbf{Z}^n \rtimes \langle A \rangle$ be the split extension of \mathbf{Z}^n by the cyclic group generated by A . If $\Gamma(A)$ is an arithmetic group then either A is of finite order or a power of A is unipotent or A is semisimple.*

In the first two cases of Proposition 12.2 that is if either A is of finite order or a power of A is unipotent the group $\Gamma(A)$ is arithmetic. In case A is semisimple $\Gamma(A)$ can be arithmetic but examples in [22] show that it need not have this property.

Proof. — Suppose that $\Gamma(A)$ is an arithmetic group and that A is not of finite order nor a power of A is unipotent. In this case, we have $\text{Fitt}(\Gamma(A)) = \mathbf{Z}^n$. Assume further that $\Gamma(A)$ is an arithmetic group. We can find (compare [19, Theorem 3.4]) a solvable \mathbf{Q} -defined linear algebraic group \mathbf{H} , having a strong unipotent radical so that there is an isomorphism $\psi : \Gamma(A) \rightarrow \Gamma$ where Γ is a Zariski-dense arithmetic subgroup of $\mathbf{H}(\mathbf{Q})$. Let \mathbf{U} be the unipotent radical of \mathbf{H} and let \mathfrak{u} denote

the Lie algebra of \mathbf{U} . The exponential map $\exp : \mathfrak{u} \rightarrow \mathbf{U}$ is a \mathbf{Q} -defined isomorphism of varieties. The adjoint representation leads to a \mathbf{Q} -defined rational representation $\alpha_{\mathbf{H}} : \mathbf{H} \rightarrow \text{Aut}(\mathfrak{u})$ which is defined by

$$\alpha_{\mathbf{H}}(g)(x) = \exp^{-1}(g \exp(x) g^{-1}) \quad (g \in \mathbf{H}, x \in \mathfrak{u}).$$

Since $\alpha_{\mathbf{H}}$ is \mathbf{Q} -defined, the image $\alpha_{\mathbf{H}}(\Gamma)$ is a Zariski-dense and arithmetic (by AR1) subgroup of $\alpha_{\mathbf{H}}(\mathbf{H})$. The kernel of $\alpha_{\mathbf{H}}$ is equal to \mathbf{U} . Taking the image under \exp^{-1} of the standard basis in \mathbf{Z}^n we obtain a \mathbf{Q} -basis of $\mathfrak{u}(\mathbf{Q})$. Expressed in this basis $\psi(A)$ acts by the matrix A on the subspace \mathfrak{u}_0 spanned by these elements. Also \mathbf{H}_0 stabilizes \mathfrak{u}_0 . Hence the cyclic subgroup generated by A is arithmetic and Zariski-dense in $\mathbf{H}_1 = \alpha_{\mathbf{H}}(\mathbf{H})$.

Let $A = SJ$ be the Jordan-decomposition of A , that is $J \in \mathbf{H}_1(\mathbf{Q})$ is unipotent and $S \in \mathbf{H}_1(\mathbf{Q})$ is semisimple and $JS = SJ$ holds. There is a $n \in \mathbf{N}$ such that $J^n \in \mathbf{H}_1(\mathbf{Z})$, and hence a $m \in \mathbf{N}$ such that $J^m \in \langle A \rangle$. This implies $S^m \in \langle A \rangle$. We infer that $J = 1$.

We shall discuss now the example from the introduction. That is, we choose $d \in \mathbf{N}$ not a square, set $\omega = \sqrt{d}$ and let $\mathbf{K} = \mathbf{Q}(\omega)$ be the corresponding real quadratic number field. We write $x \mapsto \bar{x}$ for the non-trivial element of the Galois group of \mathbf{K} over \mathbf{Q} . We consider the subring $\mathcal{O} = \mathbf{Z} + \mathbf{Z}\omega \subset \mathbf{K}$ and choose a unit $\epsilon = a + b\omega$ of \mathcal{O} which is of infinite order and satisfies $\epsilon\bar{\epsilon} = 1$.

Let D_{∞} be the infinite dihedral group as in (1). We further take $F = \mathcal{O} \times \mathbf{Z}$ with the D_{∞} -module structure defined as in (2). As done in the introduction, we put

$$\Gamma(\epsilon) := F \rtimes D_{\infty}.$$

We describe four derivations d_1, \dots, d_4 in $\text{Der}(D_{\infty}, F)$ by specifying their values on the generators A, τ of D_{∞} . We define l to be the greatest common factor of $a+1$ and bd . Now put:

$$\begin{aligned} d_1(A) &= (0, 1), & d_1(\tau) &= (0, 0); \\ d_2(A) &= (0, 0), & d_2(\tau) &= (0, 1); \\ d_3(A) &= (\omega, 0), & d_3(\tau) &= (\omega, 0); \\ d_4(A) &= \left(\frac{(\epsilon+1)\omega}{l}, 0 \right), & d_4(\tau) &= (0, 0). \end{aligned}$$

Each of the above pairs of values defines a derivation by extension.

We also define

$$(35) \quad \hat{A} := \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & \frac{-2(a+1)}{l} \\ 0 & 0 & g & 2a+1 \end{pmatrix}.$$

The structure of $\text{Aut}(\Gamma(\epsilon))$ is described in the following proposition.

Proposition 12.3. — *The following hold in $\text{Aut}(\Gamma(\epsilon))$:*

- i) $\text{Aut}_{D_\infty}(\mathbf{F})$ is finite.*
- ii) The derivations d_1, \dots, d_4 are a \mathbf{Z} -basis of $\text{Der}(D_\infty, \mathbf{F})$.*
- iii) The action of Inn_Λ on $\text{Der}(D_\infty, \mathbf{F})$ expressed relative to the basis d_1, \dots, d_4 is given by the matrix \hat{A} .*
- iv) $\text{Aut}(\Gamma(\epsilon))$ contains a subgroup of finite index which is isomorphic to $\Gamma(\hat{A})$.*

Proof. — Items i), ii), iii) are proved by some straightforward computations which we skip. Remark that \mathbf{F} is the Fitting-subgroup of $\Gamma(\epsilon)$. Setting $\Gamma = \Gamma(\epsilon)$ we know from Proposition 9.1 that $\text{Inn}_\Gamma \cdot A_{\Gamma/\mathbf{F}}$ has finite index in $\text{Aut}(\Gamma(\epsilon))$. The rest follows from Proposition 12.1.

We are now ready for the proof of Theorem 1.2.

Proof of Theorem 1.2. — Suppose that $\text{Aut}(\Gamma(\epsilon))$ contains a subgroup of finite index which is an arithmetic group. We infer from iv) in Proposition 12.3 that $\Gamma(\hat{A})$ is an arithmetic group, where \hat{A} is as in (35). We finish by the remark that \hat{A} does not satisfy the necessary properties in Proposition 12.2.

Building on the above method it is possible to construct many more examples of polycyclic groups Γ with an automorphism group $\text{Aut}(\Gamma)$ which does not contain an arithmetic subgroup of finite index. For example, as a slight variation of the above groups $\Gamma(\epsilon)$, we may replace the dihedral group D_∞ by the non-trivial semi-direct product D_1 of \mathbf{Z} with itself, and let D_1 act on $\mathbf{F} = \mathcal{O} \times \mathbf{Z}$ via its natural homomorphism to D_∞ . We thus obtain a torsion-free, arithmetic polycyclic group $\Gamma_1(\epsilon)$ of rank five with non-arithmetic automorphism group. Another interesting class of examples may be constructed by starting with the (non-arithmetic) polycyclic groups constructed in [22]. For these examples the failure of arithmeticity is of rather different nature than in the groups $\Gamma(\epsilon)$.

13. Cohomology representations of $\text{Out}(\Gamma)$

In this section we study the representation of $\text{Aut}(\Gamma)$, Γ a torsion-free polycyclic-by-finite group, on the cohomology groups $H^*(\Gamma, \mathbf{R})$, where $\mathbf{R} = \mathbf{Z}, \mathbf{Q}, \mathbf{C}$. Since inner automorphisms act trivially on the cohomology of Γ , the outer automorphism group $\text{Out}(\Gamma)$ is represented on the cohomology ring $H^*(\Gamma, \mathbf{R})$. Considering the special case $\mathbf{R} = \mathbf{C}$, we find that the complex vector space $H^*(\Gamma, \mathbf{C})$ comes with a natural \mathbf{Z} -structure which is given by the image of the base change homomorphism $H^*(\Gamma, \mathbf{Z}) \rightarrow H^*(\Gamma, \mathbf{C})$. Recall that this image is a finitely generated subgroup containing a basis of $H^*(\Gamma, \mathbf{C})$. We fix here this \mathbf{Z} -structure and its resulting \mathbf{Q} -structure on $H^*(\Gamma, \mathbf{C})$. The

representation of $\text{Out}(\Gamma)$ is an *integral representation* on $H^*(\Gamma, \mathbf{C})$, that is, $\text{Out}(\Gamma)$ normalizes the \mathbf{Z} -lattice in $H^*(\Gamma, \mathbf{C})$ just described. The \mathbf{Q} -structure on $H^*(\Gamma, \mathbf{C})$ allows us to identify the group of invertible linear maps $\text{GL}(H^*(\Gamma, \mathbf{C}))$ with a \mathbf{Q} -defined linear algebraic group. The Zariski-closure of the image of $\text{Out}(\Gamma)$ in $\text{GL}(H^*(\Gamma, \mathbf{C}))$ is a \mathbf{Q} -closed subgroup. We will show that the representation of $\text{Out}(\Gamma)$ on $H^*(\Gamma, \mathbf{C})$ is an *arithmetic representation*, that is, the image of $\text{Out}(\Gamma)$ in $\text{GL}(H^*(\Gamma, \mathbf{C}))$ is an arithmetic subgroup in its Zariski-closure. In particular, this establishes our main results of Section 1.3.

To carry over the information from the embedding of $\text{Out}(\Gamma)$ into a linear algebraic group to topology and to the study of the cohomology $H^*(\Gamma, \mathbf{R})$, we apply geometric methods originating from [8].

13.1. Automorphisms of Lie algebra cohomology. — An important special case in our theory is that of a finitely generated torsion-free nilpotent group Γ . In this case, the cohomology of Γ is intimately related to the Lie algebra cohomology of the Lie algebra of the Malcev completion of Γ , see Section 13.2. We add here some well known facts about Lie algebra cohomology.

Let \mathfrak{g} denote a Lie algebra. The Lie product of \mathfrak{g} is expressed by a map $\varphi : \mathfrak{g} \wedge \mathfrak{g} \rightarrow \mathfrak{g}$ which satisfies the Jacobi-identity. The cohomology ring $H(\mathfrak{g})$ of \mathfrak{g} is defined as the cohomology of the *Koszul-complex* \mathcal{K} of \mathfrak{g} , cf. [26]. The complex \mathcal{K} has the structure of a differential graded algebra. As a graded algebra $\mathcal{K} = \bigwedge \mathfrak{g}^*$ is the exterior algebra of the dual of \mathfrak{g} . The differential d of \mathcal{K} is determined in degree one, where $d : \mathfrak{g}^* \rightarrow \bigwedge^2 \mathfrak{g}^*$ is defined as the dual of the Lie product φ . In particular, the cohomology of \mathfrak{g} in degree one is computed as $H^1(\mathfrak{g}) = Z^1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]^\perp$. Note furthermore that, via the duality, the automorphism group of the differential graded algebra \mathcal{K} identifies with the group of Lie algebra automorphisms $\text{Aut}(\mathfrak{g})$. The automorphism group $\text{Aut}(\mathfrak{g})$ acts on the cohomology $H(\mathfrak{g})$ with the inner automorphisms, generated by the exponentials of inner derivations of \mathfrak{g} acting trivially.

Assume now that \mathfrak{g} is nilpotent. We consider the descending central series of \mathfrak{g} which is defined by $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$. Since \mathfrak{g} is nilpotent, $\mathfrak{g}^k = \{0\}$, for some (minimal) $k \in \mathbf{N}$. Dualizing the descending central series, we obtain a filtration $\mathfrak{g}_0 = \{0\} \subset \mathfrak{g}_1 \subset \dots \subset \mathfrak{g}_k = \mathfrak{g}^*$, where $\mathfrak{g}_i = (\mathfrak{g}^i)^\perp$, and $d\mathfrak{g}_i \subset \bigwedge^2 \mathfrak{g}_{i-1}$.

Lemma 13.1. — *Let Φ be a semi-simple automorphism of the Koszul-complex of the nilpotent Lie algebra \mathfrak{g} . If Φ induces the identity on $H^1(\mathfrak{g})$ then $\Phi = \text{id}$.*

Proof. — Since Φ is the identity on H^1 , it is the identity on \mathfrak{g}_1 . We prove by induction that Φ is the identity on the subalgebra \mathcal{K}_j , generated by \mathfrak{g}_j , $j > 1$. Now \mathfrak{g}_j is obtained from \mathfrak{g}_{j-1} by adding finitely many generators $x \in \mathfrak{g}_j$. Since Φ is semisimple, x may be chosen in a Φ -invariant complement W of \mathfrak{g}_{j-1} in \mathfrak{g}_j . Since $dx \in \mathcal{K}_{j-1}$, $d\Phi x = \Phi dx = dx$ and $d(\Phi x - x) = 0$. Since W has no intersection with $\ker d^1 = \mathfrak{g}_1$, this implies $\Phi x = x$. Therefore, Φ is the identity on \mathfrak{g}_j , and hence on \mathcal{K}_j .

We thus obtain the following result:

Proposition 13.2. — *Let \mathfrak{g} be a nilpotent Lie algebra. Then the kernel of the natural representation of $\text{Aut}(\mathfrak{g})$ on the cohomology $H(\mathfrak{g})$ is unipotent. In particular, any reductive subgroup of $\text{Aut}(\mathfrak{g})$ acts faithfully on $H(\mathfrak{g})$, even on $H^1(\mathfrak{g})$.*

13.2. Computation of $H^*(\Gamma, \mathbf{C})$ via geometry and $\text{Aut}(\Gamma)$ -actions. — Before treating the general case, we start by recalling some known facts which allow us to compute the complex cohomology of a finitely generated torsion-free nilpotent group in terms of Lie algebra cohomology. Let Θ be a finitely generated torsion-free nilpotent group, and \mathbf{U} the complex Malcev-completion of Θ , \mathfrak{u} the Lie algebra of \mathbf{U} . Thus $\Theta \leq \mathbf{U}(\mathbf{Q})$ and $M_\Theta = \Theta \backslash \mathbf{U}(\mathbf{R})$ is a smooth manifold which is an Eilenberg–Mac Lane space of type $\mathbf{K}(\Theta, 1)$. Here, \mathbf{R} denotes the field of real numbers. In particular, there is a natural identification of $H^*(\Theta, \mathbf{Z})$ with the singular cohomology group $H^*(M_\Theta, \mathbf{Z})$, see the discussion later in this section. By de Rham’s theorem, the singular cohomology ring $H^*(M_\Theta, \mathbf{C})$ of the smooth manifold M_Θ is isomorphic to the cohomology $H_{\text{DR}}^*(M_\Theta, \mathbf{C})$ of complex valued C^∞ -differential forms on M_Θ . In this situation, Nomizu [33] proved that the natural map from \mathfrak{u} into the differential forms on $\mathbf{U}(\mathbf{R})$ induces an isomorphism of cohomology rings

$$(36) \quad n : H^*(\mathfrak{u}, \mathbf{C}) \xrightarrow{\cong} H_{\text{DR}}^*(M_\Theta, \mathbf{C}).$$

Composing this map with the natural isomorphisms

$$H_{\text{DR}}^*(M_\Theta, \mathbf{C}) \rightarrow H^*(M_\Theta, \mathbf{C}) \rightarrow H^*(\Theta, \mathbf{C})$$

gives thus a linear isomorphism

$$(37) \quad n_\Theta : H^*(\mathfrak{u}, \mathbf{C}) \xrightarrow{\cong} H^*(\Theta, \mathbf{C}).$$

Since $\text{Aut}(\Theta)$ acts on $\mathbf{U}(\mathbf{R})$ by algebraic automorphisms, it also acts through smooth maps on M_Θ . Moreover, the isomorphisms n and n_Θ are compatible with the induced cohomology actions of $\text{Aut}(\Theta)$ on $H^*(\mathfrak{u}, \mathbf{C})$, $H_{\text{DR}}^*(M_\Theta, \mathbf{C})$ and $H^*(\Theta, \mathbf{C})$.

A similar picture carries over to our general situation where we start with a torsion-free polycyclic-by-finite group Γ . We explain now some geometric constructions which extend the above picture from the case of torsion-free nilpotent groups to the more general situation. These constructions are closely connected with the algebraic setup discussed so far in this paper.

Let \mathbf{H}_Γ be the algebraic hull of Γ , \mathbf{S} a maximal \mathbf{Q} -closed d-subgroup, and \mathbf{U} the unipotent radical of \mathbf{H}_Γ . We have $\mathbf{H}_\Gamma = \mathbf{U} \cdot \mathbf{S}$. We report from [8] the construction of the standard Γ -manifold M_Γ . To construct this manifold we write a $\gamma \in \Gamma$ (uniquely) as $\gamma = us$ with $u \in \mathbf{U}(\mathbf{Q})$, $s \in \mathbf{S}(\mathbf{Q})$ and set

$$(38) \quad \gamma * x := usxs^{-1} = \gamma xs^{-1} \quad (x \in \mathbf{U}(\mathbf{R})).$$

As noted in [8], this establishes a fixed-point-free, differentiable and properly discontinuous action of Γ on $\mathbf{U}(\mathbf{R})$. Moreover, the quotient space

$$M_\Gamma = \Gamma \backslash \mathbf{U}(\mathbf{R})$$

is a compact C^∞ -manifold and an Eilenberg–Mac Lane space of type $\mathbf{K}(\Gamma, 1)$.

We now explain how to calculate the complex cohomology of M_Γ , and hence the cohomology of Γ . Let \mathfrak{u} denote the Lie algebra of \mathbf{U} , and let $\mathcal{K}_\mathfrak{u}$ be the Koszul-complex of \mathfrak{u} (see Section 13.1). We let \mathbf{S} act by conjugation on \mathbf{U} and by the adjoint action on \mathfrak{u} and $\mathcal{K}_\mathfrak{u}$. Let $\mathcal{K}_\mathfrak{u}^\mathbf{S} \subset \mathcal{K}_\mathfrak{u}$ denote the differential subcomplex of invariants for \mathbf{S} . Now, as is proved [8, §3], the obvious map

$$(39) \quad \mathfrak{n} : H^*(\mathcal{K}_\mathfrak{u}, \mathbf{C})^\mathbf{S} \rightarrow H_{\text{DR}}^*(M_\Gamma, \mathbf{C})$$

is an isomorphism of cohomology rings.

We explain next how $\text{Aut}(\Gamma)$ acts on the above cohomology spaces. Let $\phi \in \text{Aut}(\Gamma)$ and Φ be its extension to an algebraic automorphism of \mathbf{H}_Γ . We choose a $w_\phi \in \mathbf{U}(\mathbf{Q})$ with the property that $\Phi(\mathbf{S}) = w_\phi \mathbf{S} w_\phi^{-1}$ and set

$$(40) \quad X_\phi(x) = \Phi(x)w_\phi \quad (x \in \mathbf{U}(\mathbf{R})).$$

This defines a C^∞ -map $X_\phi : \mathbf{U}(\mathbf{R}) \rightarrow \mathbf{U}(\mathbf{R})$. A straightforward computation yields that

$$(41) \quad X_\phi(\gamma * x) = \phi(\gamma) * X_\phi(x).$$

This shows that the map X_ϕ descends to a map $\bar{X}_\phi : M_\Gamma \rightarrow M_\Gamma$.

Let

$$Z_{\mathbf{U}}(\mathbf{S}) := \{v \in \mathbf{U} \mid vsv^{-1} = s \text{ for all } s \in \mathbf{S}\}$$

be the centralizer of \mathbf{S} in \mathbf{U} . This is a \mathbf{Q} -closed subgroup of \mathbf{U} . Note that the similarly defined normalizer of \mathbf{S} in \mathbf{U} is in fact equal to $Z_{\mathbf{U}}(\mathbf{S})$. Multiplication from the right defines an action of $Z_{\mathbf{U}}(\mathbf{S})(\mathbf{R})$ on $\mathbf{U}(\mathbf{R})$ which commutes with the action of Γ on $\mathbf{U}(\mathbf{R})$ defined in (38). Since $Z_{\mathbf{U}}(\mathbf{S})(\mathbf{R})$ is connected, this action is homotopically trivial, and so is the induced action on M_Γ .

Let $\phi, \psi \in \text{Aut}(\Gamma)$ be automorphisms. A straightforward computation shows that $X_\phi \circ X_\psi$ differs from $X_{\phi \circ \psi}$ by an element of $Z_{\mathbf{U}}(\mathbf{S})(\mathbf{R})$ acting on $\mathbf{U}(\mathbf{R})$. This shows that $X_\phi \circ X_\psi$ and $X_{\phi \circ \psi}$ are homotopic maps, as well as the maps $\bar{X}_\phi \circ \bar{X}_\psi$ and $\bar{X}_{\phi \circ \psi}$. In particular, this implies that, via the maps \bar{X}_ϕ , $\phi \in \text{Aut}(\Gamma)$, we obtain an action of the group $\text{Aut}(\Gamma)$ on the cohomology spaces $H_{\text{DR}}^*(M_\Gamma, \mathbf{C})$ and $H^*(M_\Gamma, \mathbf{C})$. The de Rham isomorphism

$$(42) \quad \mathfrak{l}^* : H_{\text{DR}}^*(M_\Gamma, \mathbf{C}) \rightarrow H^*(M_\Gamma, \mathbf{C})$$

is obviously equivariant. Since M_Γ is a $K(\Gamma, 1)$, there is an isomorphism (compare [28, Theorem 11.5]),

$$(43) \quad J^* : H^*(M_\Gamma, \mathbf{C}) \rightarrow H^*(\Gamma, \mathbf{C}).$$

The isomorphism J^* is natural with respect to the pairs (\bar{X}_ϕ, ϕ) . (For this, property (41) is essential, as is explained in [28].) In particular, J^* is $\text{Aut}(\Gamma)$ -equivariant.

The group $\text{Aut}_a(\mathbf{H}_\Gamma)$ of algebraic automorphisms of \mathbf{H}_Γ stabilizes the unipotent radical \mathbf{U} , and hence it acts on \mathfrak{u} and $\mathcal{K}_\mathfrak{u}$. Since the group of inner automorphisms $\text{Inn}_\mathbf{U}$ acts trivially on $H^*(\mathcal{K}_\mathfrak{u}, \mathbf{C})$, we obtain an action of $\text{Aut}_a(\mathbf{H}_\Gamma) = \text{Inn}_\mathbf{U} \cdot \text{Aut}_a(\mathbf{H}_\Gamma)_\mathbf{S}$ on $H^*(\mathcal{K}_\mathfrak{u}, \mathbf{C})^\mathbf{S}$. Here $\text{Aut}_a(\mathbf{H}_\Gamma)_\mathbf{S}$ stands for the stabilizer of \mathbf{S} in $\text{Aut}_a(\mathbf{H}_\Gamma)$. In particular, identifying $\text{Aut}(\Gamma)$ as usually with a subgroup of $\text{Aut}_a(\mathbf{H}_\Gamma)$, this constructs a representation of $\text{Aut}(\Gamma)$ on $H^*(\mathcal{K}_\mathfrak{u}, \mathbf{C})^\mathbf{S}$. The isomorphism (39) can then be easily seen to be $\text{Aut}(\Gamma)$ -equivariant.

Let us define

$$(44) \quad \eta_\Gamma = J^* \circ I^* \circ n : H^*(\mathcal{K}_\mathfrak{u}, \mathbf{C})^\mathbf{S} \rightarrow H^*(\Gamma, \mathbf{C}).$$

We have proved:

Proposition 13.3. — *Let Γ be a torsion-free polycyclic-by-finite group. The isomorphism $\eta_\Gamma : H^*(\mathcal{K}_\mathfrak{u}, \mathbf{C})^\mathbf{S} \rightarrow H^*(\Gamma, \mathbf{C})$ is equivariant with respect to the action of $\text{Aut}(\Gamma)$ on $H^*(\mathcal{K}_\mathfrak{u}, \mathbf{C})^\mathbf{S}$ (as defined above) and the natural action on $H^*(\Gamma, \mathbf{C})$.*

13.3. *Rational action of $\text{Out}_a(\mathbf{H}_\Gamma)$ on $H^*(\Gamma, \mathbf{C})$.* — As remarked before, the cohomology $H^*(\Gamma, \mathbf{C})$ carries a natural \mathbf{Q} -structure induced by the coefficient homomorphism $H^*(\Gamma, \mathbf{Q}) \rightarrow H^*(\Gamma, \mathbf{C})$. Thus, in particular, the group $\text{GL}(H^*(\Gamma, \mathbf{C}))$ attains the natural structure of a \mathbf{Q} -defined group. We discuss now the naturally defined \mathbf{Q} -structure on $H^*(\mathcal{K}_\mathfrak{u}, \mathbf{C})^\mathbf{S}$.

Note first that the Lie algebra \mathfrak{u} is defined over \mathbf{Q} . This means, there exists a Lie algebra $\mathfrak{u}_\mathbf{Q}$ over \mathbf{Q} such that $\mathfrak{u} = \mathfrak{u}_\mathbf{Q} \otimes \mathbf{C}$ is the scalar extension of $\mathfrak{u}_\mathbf{Q}$. The \mathbf{Q} -subalgebra $\mathfrak{u}_\mathbf{Q}$ is called a \mathbf{Q} -structure on \mathfrak{u} . It is induced by the \mathbf{Q} -structure on \mathbf{H}_Γ (or, equivalently, by the unipotent shadow Θ of Γ) on \mathfrak{u} . (For related details concerning \mathbf{Q} -structures on nilpotent Lie algebras and unipotent groups one may consult [40, 24].) Since \mathfrak{u} is defined over \mathbf{Q} , we obtain a rational structure for the vector space $\mathcal{K}_\mathfrak{u}$ of the Koszul-complex of \mathfrak{u} , and all differentials are defined over \mathbf{Q} . Since \mathbf{S} is \mathbf{Q} -closed in \mathbf{H}_Γ , it follows that the complex $\mathcal{K}_\mathfrak{u}^\mathbf{S}$ and its cohomology vector spaces $H^*(\mathcal{K}_\mathfrak{u}^\mathbf{S}, \mathbf{C})$ inherit a natural \mathbf{Q} -structure from $\mathcal{K}_\mathfrak{u}$, representing $\text{GL}(H^*(\Gamma, \mathbf{C}))$ as a \mathbf{Q} -defined linear algebraic group.

Recall the construction of the \mathbf{Q} -defined algebraic structure of $\text{Aut}_a(\mathbf{H}_\Gamma)$ which is discussed in Section 3.1. It is obtained by taking the natural quotient $\text{Inn}_\mathbf{U} \rtimes \text{Aut}_a(\mathbf{H}_\Gamma)_\mathbf{S} \rightarrow \text{Aut}_a(\mathbf{H}_\Gamma)$. Since, by definition of its algebraic structure, the natural

representation of $\text{Aut}_a(\mathbf{H}_\Gamma)_\mathbf{s}$ on \mathbf{u} is defined over \mathbf{Q} , the natural representation of $\text{Aut}_a(\mathbf{H}_\Gamma)$ on $\text{H}^*(\mathcal{K}_\mathbf{u}, \mathbf{C})^\mathbf{s}$ is \mathbf{Q} -defined as well. This also implies that the representation of $\text{Aut}_a(\mathbf{H}_\Gamma)$ factors via a \mathbf{Q} -defined representation η of $\text{Out}_a(\mathbf{H}_\Gamma)$. In particular, the induced representation of $\text{Aut}(\Gamma)$ on $\text{H}^*(\mathcal{K}_\mathbf{u}, \mathbf{C})^\mathbf{s}$ factors through $\text{Out}(\Gamma)$, and, taking a basis with respect to the above constructed \mathbf{Q} -structure, every element of $\text{Out}(\Gamma)$ acts by a matrix with rational entries on $\text{H}^*(\mathcal{K}_\mathbf{u}, \mathbf{C})^\mathbf{s}$.

Proposition 13.4. — *The natural representation of $\text{Out}(\Gamma)$ on $\text{H}^*(\Gamma, \mathbf{C})$ is arithmetic.*

Proof. — By construction, $\text{Out}(\Gamma)$ is contained in the \mathbf{Q} -defined linear algebraic group \mathcal{O}_Γ as a Zariski-dense arithmetic subgroup. Via the homomorphism $\pi_{\mathcal{O}_\Gamma} : \mathcal{O}_\Gamma \rightarrow \text{Out}_a(\mathbf{H}_\Gamma)$ the representation of $\text{Out}(\Gamma)$ on $\text{H}^*(\mathcal{K}_\mathbf{u}, \mathbf{C})^\mathbf{s}$ is induced by a \mathbf{Q} -defined representation of \mathcal{O}_Γ . Let $\rho_{\mathcal{O}_\Gamma}$ denote the representation of \mathcal{O}_Γ on $\text{H}^*(\Gamma, \mathbf{C})$ which is obtained by conjugating with the isomorphism \mathfrak{n}_Γ defined in (44). Let ρ denote the natural representation of $\text{Out}(\Gamma)$ on $\text{H}^*(\Gamma, \mathbf{C})$. Since \mathfrak{n}_Γ is $\text{Aut}(\Gamma)$ -equivariant, $\rho_{\mathcal{O}_\Gamma}(\text{Out}(\Gamma)) = \rho(\text{Out}(\Gamma))$ consists of integral matrices in $\text{GL}(\text{H}^*(\Gamma, \mathbf{C}))$ (with respect to a basis of $\text{H}^*(\Gamma, \mathbf{C})$ taken in the image of $\text{H}^*(\Gamma, \mathbf{Z})$).

We show that $\rho_{\mathcal{O}_\Gamma}$ is a \mathbf{Q} -defined representation for the natural \mathbf{Q} -structure on $\text{H}^*(\Gamma, \mathbf{C})$. As $\text{Out}(\Gamma)$ is Zariski-dense in \mathcal{O}_Γ and consists of rational points, it follows that the homomorphism $\rho_{\mathcal{O}_\Gamma}$ maps a Zariski-dense subset of rational points of \mathcal{O}_Γ to rational points of $\text{GL}(\text{H}^*(\Gamma, \mathbf{C}))$. The Galois-criterion for rationality applied to the case of the extension \mathbf{C} over \mathbf{Q} implies that $\rho_{\mathcal{O}_\Gamma}$ is defined over \mathbf{Q} . By AR1, we infer that the image $\rho(\text{Out}(\Gamma))$ in $\text{GL}(\text{H}^*(\Gamma, \mathbf{C}))$ is arithmetic in $\rho_{\mathcal{O}_\Gamma}(\mathcal{O}_\Gamma)$.

Proposition 13.4 proves Theorem 1.13 of the introduction. Remark that the kernel of ρ is a finitely generated group since it is an arithmetic subgroup of the kernel of $\rho_{\mathcal{O}_\Gamma}$.

Remark 13.5. — The proof of Proposition 13.4 gives no information about the rationality of the isomorphism \mathfrak{n}_Γ . In fact, the representation of \mathcal{O}_Γ might be trivial.

For $\Gamma = \Theta$ a nilpotent group, some considerations on rationality questions for the isomorphism \mathfrak{n}_Θ may be found in [27]. We state here:

Proposition 13.6. — *The isomorphism \mathfrak{n}_Γ is defined over \mathbf{Q} .*

Sketch of proof. — We show that the map $\mathfrak{n} : \text{H}^*(\mathcal{K}_\mathbf{u}, \mathbf{C})^\mathbf{s} \rightarrow \text{H}_{\text{DR}}^*(M_\Gamma, \mathbf{C})$ is defined over \mathbf{Q} . This can be seen as follows. In exponential coordinates for $\mathbf{U}(\mathbf{R})$, the forms in $\mathcal{K}_\mathbf{u}(\mathbf{Q})$ define polynomial differential forms with rational coefficients. The de Rham isomorphism l^* factorizes over the cohomology of the piecewise differential forms, see [18, VIII]. The natural map from differential forms to piecewise-forms, maps the forms corresponding to elements of $\mathcal{K}_\mathbf{u}(\mathbf{Q})$ into piecewise linear forms with

rational coefficients. By Sullivan's *P. L. de Rham theorem* (see [44, 18]) these are mapped into $H^*(M_\Gamma, \mathbf{Q})$.

REFERENCES

1. L. AUSLANDER, The automorphism group of a polycyclic group, *Ann. Math. (2)*, **89** (1969), 314–322.
2. L. AUSLANDER, On a problem of Philip Hall, *Ann. Math. (2)*, **86** (1967), 112–116.
3. L. AUSLANDER and F. E. A. JOHNSON, On a conjecture of C. T. C. Wall, *J. Lond. Math. Soc., II. Ser.*, **14** (1976), no. 2, 331–332.
4. L. AUSLANDER and G. BAUMSLAG, Automorphism groups of finitely generated nilpotent groups, *Bull. Am. Math. Soc.*, **73** (1967), 716–717.
5. G. BAUMSLAG, Automorphism groups of nilpotent groups, *Am. J. Math.*, **91** (1969), 1003–1011.
6. G. BAUMSLAG, *Lectures on Nilpotent Groups*, C.B.M.S. Regional Conference Series, vol. 2, A.M.S., Providence, R.I., 1971.
7. O. BAUES, Finite extensions and unipotent shadows of affine crystallographic groups, *C. R. Acad. Sci., Paris, Sér. I, Math.*, **335** (2002), 785–788.
8. O. BAUES, Infrapolymanifolds and rigidity of subgroups in solvable linear algebraic groups, *Topology*, **43** (2004), 903–924.
9. A. BOREL, *Arithmetic properties of linear algebraic groups*, Proc. I.C.M. Stockholm (1962), 10–22.
10. A. BOREL, Density and maximality of arithmetic subgroups, *J. Reine Angew. Math.*, **224** (1966), 78–89.
11. A. BOREL, *Linear algebraic groups*, second edn., Graduate Texts in Mathematics, vol. 126, Springer, New York, 1991.
12. A. BOREL and J.-P. SERRE, Théorèmes de finitude en cohomologie galoisienne, *Comment. Math. Helv.* **39** (1964), 111–164.
13. A. BOREL and J. TITS, Groupes réductifs, *Publ. Math., Inst. Hautes Étud. Sci.*, **27** (1965), 55–150.
14. K. S. BROWN, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer, New York–Berlin, 1982.
15. R. M. BRYANT and J. R. J. GROVES, Algebraic groups of automorphisms of nilpotent groups and Lie algebras, *J. Lond. Math. Soc., II. Ser.*, **33** (1986), no. 3, 453–466.
16. P. DELIGNE, Extensions centrales non résiduellement finies de groupes arithmétiques, *C. R. Acad. Sci., Paris, Sér. A-B*, **287** (1978), no. 4, A203–A208.
17. M. DU SAUTOY, Polycyclic groups, analytic groups and algebraic groups, *Proc. Lond. Math. Soc., III. Ser.*, **85** (2002), no. 1, 62–92.
18. P. A. GRIFFITHS and J. W. MORGAN, *Rational homotopy theory and differential forms*, Progress in Mathematics, vol. 16, Birkhäuser, Boston, Mass., 1981.
19. F. GRUNEWALD and V. PLATONOV, Solvable arithmetic groups and arithmeticity problems, *Int. J. Math.*, **10** (1999), no. 3, 327–366.
20. F. GRUNEWALD and V. PLATONOV, On finite extensions of arithmetic groups, *C. R. Acad. Sci. Paris*, **325** (1997), 1153–1158.
21. F. GRUNEWALD and V. PLATONOV, Rigidity results for groups with radical, cohomology of finite groups and arithmeticity problems, *Duke Math. J.*, **100** (1999), 321–358.
22. F. GRUNEWALD and V. PLATONOV, Non-arithmetic polycyclic groups, *C. R. Acad. Sci. Paris*, **326** (1998), 1359–1364.
23. F. GRUNEWALD and V. PLATONOV, Rigidity and automorphism groups of solvable arithmetic groups, *C. R. Acad. Sci. Paris*, **327** (1998), 427–432.
24. F. GRUNEWALD and J. O'HALLORAN, Nilpotent groups and unipotent algebraic groups, *J. Pure Appl. Algebra*, **37** (1985), no. 3, 299–313.
25. F. GRUNEWALD and D. SEGAL, On affine crystallographic groups, *J. Differ. Geom.*, **40** (1994), no. 3, 563–594.
26. J.-L. KOSZUL, Homologie et cohomologie des algèbres de Lie, *Bull. Soc. Math. Fr.*, **78** (1950), 65–127.
27. L. A. LAMBE and S. B. PRIDY, Cohomology of nilmanifolds and torsion-free, nilpotent groups, *Trans. Am. Math. Soc.*, **273** (1982), no. 1, 39–55.
28. S. MACLANE, *Homology*, Grundlehren der Mathematischen Wissenschaften, vol. 114, Springer, Berlin–Göttingen–Heidelberg, 1963.

29. A. I. MAL'CEV, On a class of homogeneous spaces, *Am. Math. Soc. Transl.*, **39** (1951), 1–33.
30. Ju. I. MERZLJAKOV, Integer representation of the holomorphs of polycyclic groups, *Algebra Log.*, **9** (1970), 539–558 (in Russian).
31. G. D. MOSTOW, Representative functions on discrete groups and solvable arithmetic subgroups, *Am. J. Math.*, **92** (1970), 1–32.
32. G. D. MOSTOW, Some applications of representative functions to solvmanifolds, *Am. J. Math.*, **93** (1971), 11–32.
33. K. NOMIZU, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, *Ann. Math. (2)*, **59** (1954), 531–538.
34. P. F. PICKEL and J. ROITBERG, Automorphism groups of nilpotent groups and spaces, *J. Pure Appl. Algebra*, **150** (2000), no. 3, 307–319.
35. V. PLATONOV and A. RAPINCHUK, *Algebraic Groups and Number Theory*, Academic Press, Boston, MA, 1994.
36. M. S. RAGHUNATHAN, *Discrete subgroups of Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 68, Springer, New York–Heidelberg, 1972.
37. J. ROITBERG, Genus and symmetry in homotopy theory, *Math. Ann.*, **305** (1996), no. 2, 381–386.
38. V. A. ROMANKOV, Width of verbal subgroups in solvable groups, *Algebra Log.*, **21** (1982), 60–72 (in Russian).
39. J. W. RUTTER, *Spaces of homotopy self-equivalences*, Lect. Notes Math., vol. 1662, Springer, Berlin, 1997.
40. D. SEGAL, *Polycyclic groups*, Cambridge Univ. Press, London, 1983.
41. D. SEGAL, On the outer automorphism group of a polycyclic group, Proceedings of the Second International Group Theory Conference (Bressanone, 1989), *Rend. Circ. Mat. Palermo, II. Ser.* (1990), Suppl. no. 23, 265–278.
42. J.-P. SERRE, Arithmetic groups, *Lond. Math. Soc. Lect. Notes*, **36** (1979), 105–135.
43. J.-P. SERRE, Cohomologie des groupes discrets, Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970), pp. 77–169, *Ann. Math. Stud.*, no. 70, Princeton Univ. Press, Princeton, N.J., 1971.
44. D. SULLIVAN, Infinitesimal computations in topology, *Publ. Math., Inst. Hautes Étud. Sci.*, **47** (1977), 269–331.
45. D. SULLIVAN, Genetics of homotopy theory and the Adams conjecture, *Ann. Math. (2)*, **100** (1974), 1–79.
46. B. A. F. WEHRFRITZ, Two remarks on polycyclic groups, *Bull. Lond. Math. Soc.*, **26** (1994), 543–548.
47. B. A. F. WEHRFRITZ, On the holomorphs of soluble groups of finite rank, *J. Pure Appl. Algebra*, **4** (1974), 55–69.

O. B.

Institut für Algebra und Geometrie,
 Universität Karlsruhe,
 D-76128 Karlsruhe, Germany
 baues@math.uni-karlsruhe.de

F. G.

Mathematisches Institut,
 Heinrich-Heine-Universität,
 D-40225 Düsseldorf, Germany
 fritz@math.uni-duesseldorf.de

*Manuscrit reçu le 19 mai 2005
 publié en ligne le 17 novembre 2006.*