THE HOCHSCHILD COHOMOLOGY OF A CLOSED MANIFOLD

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ABSTRACT

Let M be a closed orientable manifold of dimension d and $\mathscr{C}^*(M)$ be the usual cochain algebra on M with coefficients in a field $\boldsymbol{k}$. The Hochschild cohomology of M, $HH^*(\mathscr{C}^*(M);\mathscr{C}^*(M))$ is a graded commutative and associative algebra. The augmentation map $\varepsilon : \mathscr{C}^*(M) \to \boldsymbol{k}$ induces a morphism of algebras $I : HH^*(\mathscr{C}^*(M);\mathscr{C}^*(M)) \to HH^*(\mathscr{C}^*(M);\boldsymbol{k})$. In this paper we produce a chain model for the morphism I. We show that the kernel of I is a nilpotent ideal and that the image of I is contained in the center of $HH^*(\mathscr{C}^*(M);\boldsymbol{k})$, which is in general quite small. The algebra $HH^*(\mathscr{C}^*(M);\mathscr{C}^*(M))$ is expected to be isomorphic to the loop homology constructed by Chas and Sullivan. Thus our results would be translated in terms of string homology.

1. Introduction

Let M be a simply connected closed oriented d-dimensional (smooth) manifold and $\boldsymbol{k}$ be a field. We denote by $\mathscr{C}^*(M)$ the cochain algebra of M with coefficients in $\boldsymbol{k}$ and by $HH^*(\mathscr{C}^*(M);\mathscr{C}^*(M))$ the Hochschild cohomology algebra of $\mathscr{C}^*(M)$ ([11]) with coefficients in itself. The augmentation $\varepsilon : \mathscr{C}^*(M) \to \boldsymbol{k}$ corresponding to the inclusion of a base point induces a morphism of graded algebras $I : HH^*(\mathscr{C}^*(M);\mathscr{C}^*(M)) \to HH^*(\mathscr{C}^*(M);\boldsymbol{k})$.

In this paper we give a model for the algebra $HH^*(\mathscr{C}^*(M);\mathscr{C}^*(M))$ and for the morphism I (3.5). From the model we directly deduce:

1. *Theorem (4.1-Theorem 7). — For any field* $\boldsymbol{k}$,

a) the kernel of I *is a nilpotent ideal of nilpotency index less than or equal to* $d/2$,
b) the image of I *lies in the center of* $HH^*(\mathscr{C}^*(M);\boldsymbol{k})$.

The morphism I connects in fact two well known homotopy invariants of the manifold. First of all, by the Adams Cobar construction ([1], [8]): there is an isomorphism of graded algebras

$$\theta : HH^*(\mathscr{C}^*(M);\boldsymbol{k}) \xrightarrow{\cong} H_*(\Omega M;\boldsymbol{k}) .$$

On the other hand, Jones has established an isomorphism of graded vector spaces ([12])

$$H_*(LM;\boldsymbol{k}) \xrightarrow{\cong} HH^*(\mathscr{C}^*(M);\mathscr{C}_*(M)) ,$$

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where $LM = M^{S^1}$ denotes the free loop space on M. Finally, the Poincaré duality of the manifold yields an isomorphism of graded vectors spaces $D : HH^*(\mathscr{C}^*(M); \mathscr{C}_*(M)) \stackrel{\cong}{\to} HH^{*-d}(\mathscr{C}^*(M); \mathscr{C}^*(M))$ (7.2-Theorem 13), and by composition an isomorphism of graded vector spaces

$$H_*(LM; \boldsymbol{k}) \stackrel{\cong}{\to} HH^{*-d}(\mathscr{C}^*(M); \mathscr{C}^*(M)) .$$

Using those isomorphisms, we can replace I by

$$\theta \circ I : HH^*(\mathscr{C}^*(M); \mathscr{C}^*(M)) \to H_*(\Omega M) .$$

Theorem 1 shows that the image of I is in general very small comparatively to the expected growth of $HH^*(\mathscr{C}^*(M); \mathscr{C}^*(M)) \cong H_*(M^{S^1}; \boldsymbol{k})$.

When $\boldsymbol{k}$ is a field of characteristic zero, Theorem 1 becomes more precise. Let us recall that an element $x \in \pi_q(M)$ is called a *Gottlieb element* ([10]-p. 377), if the map $x \vee id_M : S^q \vee M \to M$ extends to the product $S^q \times M$. These elements generate a subgroup $G_*(M)$ of $\pi_*(\Omega M)$ via the isomorphism $\pi_*(\Omega M) \cong \pi_{*+1}(M)$. Finally, we denote by cat M the Lusternik-Schnirelmann category of M normalized so that cat S^n=1.

2. *Theorem (5.2-Theorem 9). — If* $\boldsymbol{k}$ *is a field of characteristic zero then*

a) the kernel of I *is a nilpotent ideal of nilpotency index less than or equal to cat* M.
b) $(Im\ \theta \circ I) \cap (\pi_*(\Omega M) \otimes \boldsymbol{k}) = G_*(M) \otimes \boldsymbol{k}$.
c) $\sum_{i=0}^{n} dim\,(Im\,\theta \circ I \cap H_i(\Omega M; \boldsymbol{k})) \leq Cn^k$, *some constant* $C > 0$ *and* $k \leq$ *cat* M.

With our model we characterize when I is a surjective morphism:

3. *Theorem (6.3-Theorem 10). — The morphism* I *is surjective if and only if* M *has the rational homotopy type of a product of odd dimensional spheres.*

In ([3]), Chas and Sullivan construct a product on the desuspension,

$$\mathbf{H}_*(LM; \boldsymbol{k}) = H_{*+d}(LM; \boldsymbol{k}) ,$$

of the free loop space homology of M. This product, called the *loop product*, is defined at the chain level using both intersection product on the chains on M and loop composition. The homology $\mathbf{H}_*(LM; \boldsymbol{k})$ is a graded commutative and associative graded algebra. They refer to $\mathbf{H}_*(LM; \boldsymbol{k})$ endowed with the loop product as the *loop homology* of M.

For an open set $N \hookrightarrow M$ containing the base point we denote by $L_N M$ the space of loops that originate in N. By restriction, the loop product induces a product on $\mathbf{H}_*(L_N M; \boldsymbol{k})$ so that the induced map $\mathbf{H}_*(L_N M; \boldsymbol{k}) \to \mathbf{H}_*(LM; \boldsymbol{k})$ becomes a multiplicative morphism. Now the transversal intersection with ΩM defines a morphism $I_N : \mathbf{H}_*(L_N M; \boldsymbol{k}) \to H_*(\Omega M; \boldsymbol{k})$. The Chas-Sullivan loop homology and the Hochschild cohomology of M are related by a conjecture that extends the previous works of Adams and Jones:

1. *Conjecture. — There exist isomorphisms* Φ *and* Ψ_N *of graded algebras making commutative the following diagram*

$$\begin{array}{ccc} \mathbf{H}_*(L_N M; \boldsymbol{k}) & \stackrel{\Psi_N}{\to} & HH^*(\mathscr{C}^*(M); \mathscr{C}^*(N)) \\ \downarrow I_N & & \downarrow HH^*(\mathscr{C}^*(M); \varepsilon) \\ H_*(\Omega M; \boldsymbol{k}) & \stackrel{\Phi}{\to} & HH^*(\mathscr{C}^*(M); \boldsymbol{k}) \end{array} .$$

where ϵ *denotes the augmentation associated to the base point of* N.

There is no complete written proof of this conjecture in the litterature; however Tradler, Cohen, Jones have already understood the situation and have given substantial parts of a proof ([4], [6]). If we assume this result, our computations give a model for the loop product on $\mathbf{H}_*(LM)$ and for the homomorphism $I_M : \mathbf{H}_*(LM; \boldsymbol{k}) \to H_*(\Omega M; \boldsymbol{k})$.

To prove Theorem 1 we also use the following algebraic result concerning the center of the enveloping algebra of a graded Lie algebra.

4. *Theorem (5.1-Theorem 8). — Let* L *be a finite type graded Lie algebra defined on a field of characteristic zero, then the center of* UL *is contained in the enveloping algebra on the radical of* L.

The paper is organized as follows:

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2. Hochschild cohomology and Gerstenhaber product

In this section we fix some notations and recall the standard definitions of Hochschild cohomology and of Gerstenhaber product.

2.1. Let $\boldsymbol{k}$ be a principal ideal domain; modules, tensor product, linear map, ... are defined over $\boldsymbol{k}$. For notational simplicity, we avoid to mention $\boldsymbol{k}$. If V is a lower or upper graded module ($V_i = V^{-i}$) the suspension s is defined by $(sV)_n = V_{n+1}$, $(sV)^n = V^{n-1}$.

2.2. Let (A, d) be a differential graded augmented cochain algebra and (N, d) be a differential graded A-bimodule, $A = \{A^i\}_{i\geq 0}$, $N = \{N^j\}_{j\in\mathbf{Z}}$ and $\overline{A} = \ker(\varepsilon : A \to \boldsymbol{k})$. The *two-sided normalized bar construction*,

$$\overline{\mathbf{B}}(N; A; N) = N \otimes T(s\overline{A}) \otimes N\,, \quad \overline{\mathbf{B}}_k(N; A; N) = N \otimes T^k(s\overline{A}) \otimes N\,,$$

is defined as follows: For $k \geq 1$, a generic element $m[a_1|a_2|\dots|a_k]n \in \overline{\mathbf{B}}_k(N; A; N)$ has degree $|m| + |n| + \sum_{i=1}^k (|sa_i|)$. If $k = 0$, we write $m[\,]n = m \otimes 1 \otimes n \in N \otimes T^0(s\overline{A}) \otimes N$. The differential $d = d_0 + d_1$ is defined by:

$$d_0 : \overline{\mathbf{B}}_k(N; A; N) \to \overline{\mathbf{B}}_k(N; A; N)\,, \quad d_1 : \overline{\mathbf{B}}_k(N; A; N) \to \overline{\mathbf{B}}_{k-1}(N; A; N)\,,$$

with

$$\begin{aligned} d_0(m[a_1|a_2|\dots|a_k]n) = \; & d(m)[a_1|a_2|\dots|a_k]n \\ & - \sum_{i=1}^{k} (-1)^{\epsilon_i} m[a_1|a_2|\dots|d(a_i)|\dots|a_k]n \\ & + (-1)^{\epsilon_{k+1}} m[a_1|a_2|\dots|a_k]d(n) \end{aligned}$$

$$\begin{aligned} d_1(m[a_1|a_2|\dots|a_k]n) = \; & (-1)^{|m|} ma_1[a_2|\dots|a_k]n \\ & + \sum_{i=2}^{k} (-1)^{\epsilon_i} m[a_1|a_2|\dots|a_{i-1}a_i|\dots|a_k]n \\ & - (-1)^{\epsilon_k} m[a_1|a_2|\dots|a_{k-1}]a_k n\,. \end{aligned}$$

Here $\epsilon_i = |m| + \sum_{j<i}(|sa_j|)$.

2.3. For any differential graded algebra A, let A^{op} be the opposite graded algebra, $a \cdot^{op} b = (-1)^{|a|\cdot|b|} b \cdot a$, and $A^e = A \otimes A^{op}$ be the enveloping algebra. Any differential graded A-bimodule N is a differential graded A^e-module. Let A and N as in 2.2. The *Hochschild cochain complex* $\mathbf{C}^*(A; N)$ of A with coefficients in N is the differential graded module ([11], [13]):

$$\mathbf{C}^*(A; N) = \mathrm{Hom}_{A^e}(\overline{\mathbf{B}}(A; A; A), N)\,,$$
$$\mathbf{C}^n(A, M) = \prod_{p-q=n} \mathrm{Hom}_{A^e}(\overline{\mathbf{B}}(A, A, A)^p, N^q)\,,$$

equipped with the standard differential D defined by $Df = d \circ f - (-1)^{|f|} f \circ d$. The homology of the complex $\mathbf{C}^*(A; N)$ is called the *Hochschild cohomology* of A with values in N, and is denoted $HH^*(A; N)$.

This definition extends the classical one since:

1. *Lemma ([9]-Lemma 4.3)). — If* A *is a differential graded algebra such that* A *is a* $\boldsymbol{k}$*-free graded module then the multiplication in* A *extends in a semi-free resolution of* A^e*-modules*

$$m : \overline{\mathbf{B}}(A, A, A) \longrightarrow A\,.$$

This means that m is a quasi-isomorphism of differential graded A-bimodules which well behaves with quasi-isomorphisms of differential graded A-bimodules. In particular, we have the following lifting lemma:

2. *Lemma (***Lifting Homotopy Lemma***). — For any quasi-isomorphism* $\varphi : A' \to A$ *there exists a unique (up to homotopy in the category of differential graded bimodules) quasi-isomorphism* $\hat{m} : \overline{\mathbf{B}}(A, A, A) \longrightarrow A'$ *such that* $m \simeq \varphi \circ \hat{m}$.

2.4. Recall that $\overline{\mathbf{B}}(A) = \overline{\mathbf{B}}(\boldsymbol{k}; A; \boldsymbol{k}) := \big(T(s\overline{A}), d\big)$ is a differential graded coalgebra with

$$d([a_1|a_2|\cdots|a_k]) = -\sum_{i=1}^{k}(-1)^{\epsilon_i}[a_1|a_2|\ldots|d(a_i)|\ldots|a_k]$$
$$+\sum_{i=2}^{k}(-1)^{\epsilon_i}[a_1|a_2|\ldots|a_{i-1}a_i|\ldots|a_k]\,.$$

The canonical isomorphism of graded modules

$$\mathrm{Hom}_{A^e}(\overline{\mathbf{B}}(A; A; A), N) = \mathrm{Hom}(T(s\overline{A}), N)\,,$$

carries on $\mathrm{Hom}(T(s\overline{A}), N)$ a differential D'. Observe that the differential D' is not the canonical differential D of $\mathrm{Hom}(\overline{\mathbf{B}}(A), N)$ except when N is the trivial bimodule. If $N = A$, Gerstenhaber ([11]) has proved that the usual cup product on $\mathrm{Hom}(T(s\overline{A}), A)$

makes $(\mathrm{Hom}(\mathrm{T}(s\overline{\mathrm{A}}), \mathrm{A}), \mathrm{D}')$ a differential graded algebra such that the induced product on $\mathrm{HH}^*(\mathrm{A}; \mathrm{A})$, called the *Gerstenhaber product* ([11]) is commutative.

3. A chain model for $\mathrm{I} : \mathrm{HH}^*(\mathscr{C}^*(\mathrm{M}); \mathscr{C}^*(\mathrm{M})) \to \mathrm{HH}^*(\mathscr{C}^*(\mathrm{M}); \boldsymbol{k})$

In this section we construct, for any field of coefficients $\boldsymbol{k}$, an explicit model for the Hochschild cohomology algebra at the chain level.

3.1. Recall the Adams Cobar construction $\Omega\mathrm{C}$ on a coaugmented differential graded coalgebra $\mathrm{C} = \boldsymbol{k} \oplus \overline{\mathrm{C}}$. This is the differential graded algebra $(\mathrm{T}(s^{-1}\overline{\mathrm{C}}), d)$, where $d = d_1 + d_2$ is the unique derivation determined by:

$$d_1 s^{-1}c = -s^{-1}dc\,, \text{ and } d_2 s^{-1}c = \sum_i (-1)^{|c_i|} s^{-1}c_i \otimes s^{-1}c'_i\,, \quad c \in \overline{\mathrm{C}}\,,$$

where the reduced coproduct of $c \in \overline{\mathrm{C}}$ is written $\bar{\Delta}c = \sum_i c_i \otimes c'_i$. For sake of simplicity we put $\langle x_1|x_2|\cdots|x_n\rangle := s^{-1}x_1 \otimes \cdots \otimes s^{-1}x_n$.

3.2. Assume $\boldsymbol{k}$ is a field, and M is a 1-connected compact d-dimensional manifold. Denote by $f : (\mathrm{T}(\mathrm{V}), d) \to \mathscr{C}^*(\mathrm{M})$ a free minimal model for the singular cochain algebra on M ([9]), i.e. $(\mathrm{T}(\mathrm{V}), d)$ is a differential graded algebra, f is a quasi-isomorphism of differential graded algebras, and $d(\mathrm{V}) \subset \mathrm{T}^{\geq 2}(\mathrm{V})$. The differential graded algebra $(\mathrm{T}(\mathrm{V}), d)$ is uniquely defined, up to isomorphism, by the above properties. Moreover, $\mathrm{V}^p \cong \mathrm{H}^{p-1}(\Omega\mathrm{M})$, ([9]). Denote by S a complement of the vector space generated by the cocycles of degree d. The differential graded ideal $\mathrm{J} = (\mathrm{T}(\mathrm{V}))^{>d} \oplus \mathrm{S}$ is acyclic and the quotient algebra $\mathrm{A} = \mathrm{T}(\mathrm{V})/\mathrm{J}$ is a finite dimensional graded differential algebra.

3.3. Since A is finite dimensional, the graded dual $\mathrm{A}^\vee$ is a differential graded coalgebra and we consider the differential graded algebra $\Omega\mathrm{A}^\vee = (\mathrm{T}(\mathrm{W}), d)$, with in particular, $\mathrm{W} \cong \mathrm{Hom}(s\overline{\mathrm{A}}, \boldsymbol{k})$, and $\Omega\mathrm{A}^\vee = \mathrm{Hom}(\overline{\mathbf{B}}\mathrm{A}, \boldsymbol{k}) = (\mathrm{Hom}(\mathrm{T}(s\overline{\mathrm{A}}), \boldsymbol{k}), \mathrm{D})$ (2.4). We choose a homogeneous linear basis e_i for $\overline{\mathrm{A}}$, and its dual basis w_i for W. This determines the constants of structure α_{ij}^k and ρ_i^j:

$$\begin{gathered}\langle w_i, se_k\rangle = -(-1)^{|w_i|}\delta_{ik}\,, \quad e_i \cdot e_j = \textstyle\sum_k \alpha_{ij}^k e_k\,, \quad d(e_i) = \sum_j \rho_i^j e_j \\ d(w_i) = \textstyle\sum_{jk} a_i^{jk} w_j w_k + \sum_j \beta_i^j w_j\,, \quad a_i^{jk} = (-1)^{|e_j|+|e_j e_k|}\alpha_{jk}^i\,, \\ \beta_i^j = (-1)^{|w_j|}\rho_j^i\,.\end{gathered}$$

5. *Theorem. — Let $\boldsymbol{k}$ be a field and* M *be a 1-connected closed oriented manifold of dimension d. With notation introduced above:*

a) the derivation D *uniquely defined on the tensor product of graded algebras* $A \otimes T(W)$ *by*

$$\begin{cases} D(a \otimes 1) = d(a) \otimes 1 + \sum_j (-1)^{|a|+|e_j|}[a, e_j] \otimes w_j\,, & a \in A\,, \\ D(1 \otimes b) = 1 \otimes d(b) - \sum_j (-1)^{|e_j|} e_j \otimes [w_j, b]\,, & b \in TW\,, \end{cases}$$

is a differential. Here [,] *denotes the Lie bracket in the graded algebras* A *and* T(W).

b) the graded algebras $H_*(A \otimes T(W), D)$ *and* $HH^*(\mathscr{C}^*(M); \mathscr{C}^*(M))$ *are isomorphic.*

Proof. — a) is proved by a direct but laborious computation.

b) is a direct consequence of the definition. □

Observe that this model is dual to those constructed by one of us ([13]).

1. *Proposition.* — *Let* $\boldsymbol{k}$ *be a field and* M *be a 1-connected closed oriented manifold of dimension* d. *There is a cohomology spectral sequence of graded algebras such that*

$$E_2 = HH^*(H^*(M), H^*(M)) \Rightarrow HH^*(\mathscr{C}^*(M); \mathscr{C}^*(M))\,.$$

Proof. — The spectral sequence is obtained by filtering the complex $(\mathrm{Hom}\,(T(s\overline{A}, A), D'))$ by the differential ideals $\mathrm{Hom}\,(T^{\leq p}(s\overline{A}), A)$ (2.4). Since $H^*(A) = H^*(M)$, it follows that $E_1 = \mathrm{Hom}(\overline{\mathbf{B}}(H^*(M)), \boldsymbol{k}) \otimes H^*(M)$ and $E_2 = H^*(\mathrm{Hom}\,(\overline{\mathbf{B}}(H^*(M)), \boldsymbol{k}) \otimes H^*(M), D)$. □

1. *Example.* — If M is a formal space, (for instance M is a simply connected compact Kähler manifold for $\boldsymbol{k} = \mathbf{Q}$ ([7]) one can choose $A = H^*(M)$ and thus the algebras $HH^*(H^*(M); H^*(M))$ and $HH^*(\mathscr{C}^*(M); \mathscr{C}^*(M))$ are isomorphic graded vector spaces. If we put $H_* = H_*(M)$ the algebra $HH^*(\mathscr{C}^*(M); \mathscr{C}^*(M))$ is isomorphic to the graded algebra $H(A \otimes T(s\overline{H}_*), D)$ with $D(a \otimes 1) = 0$, $a \in A$ and $D(1 \otimes b) = -\sum_J (-i)^{|e_j|} e_j \otimes [w_j, b]$, $b \in \overline{H}_*$.

3.4. *The commutative case.* — Suppose that the algebra $\mathscr{C}^*(M)$ is connected by a sequence of quasi-isomorphisms to a commutative differential graded algebra (A, d). This is the case if either $\boldsymbol{k}$ is of characteristic zero, or else if $\boldsymbol{k}$ is a field of characteristic $p > d$ ([2], Proposition 8.7). We can also suppose that A is finite dimensional, $A^0 = \boldsymbol{k}$, $A^1 = 0$, $A^{>d} = 0$ and $A^d = \boldsymbol{k}\omega$. Then formulas of 3.3-Theorem 5 simplify as:

$$\begin{cases} D(a \otimes 1) = d(a) \otimes 1\,, \\ D(1 \otimes b) = 1 \otimes d(b) - \sum_j (-1)^{e_j} e_j \otimes [w_j, b]\,. \end{cases}$$

3.5. We can now interpret the intersection morphism in terms of our model:

6. *Theorem. — Let* $\boldsymbol{k}$ *be a field and* M *be a 1-connected closed oriented manifold of dimension* d. *There is a commutative diagram of algebras*

$$\begin{array}{ccc} HH^*(\mathscr{C}^*(M);\mathscr{C}^*(M)) & \xrightarrow{\cong} & H_*(A\otimes T(W), D) \\ \theta\circ I \downarrow & & \downarrow H(\varepsilon_A\otimes 1) \\ H_*(\Omega M) & \xrightarrow{\cong} & H_*(T(W), d)\,. \end{array}$$

Proof. — Recall that Hochschild cohomology $HH^*(A;N)$ is covariant in N and contravariant in A. Moreover, if $f : A \to B$ is a quasi-isomorphism of differential graded algebras and $g : N \to N'$ is a quasi-isomorphism of A-bimodules, we have isomorphisms

$$HH^*(B;N) \xrightarrow{\cong} HH^*(A;N) \xrightarrow{\cong} HH^*(A;N')\,.$$

We obtain therefore the following commutative diagram

$$\begin{array}{ccccc} HH^*(\mathscr{C}^*(M);\mathscr{C}^*(M)) & \xrightarrow{\cong} & HH^*(A;A) & \xrightarrow{\cong} & H_*(A\otimes T(W), D) \\ \downarrow HH^*(\mathscr{C}^*(M),\varepsilon) & & \downarrow HH^*(A,\varepsilon_A) & & \downarrow H(\varepsilon_A\otimes 1) \\ HH^*(\mathscr{C}^*(M);\boldsymbol{k}) & \xrightarrow{\cong} & HH^*(A;\boldsymbol{k}) & \xrightarrow{\cong} & H_*(T(W), d) \end{array}$$

□

4. The kernel and the image of I

4.1. If J is an ideal of an algebra A, we put $J^1 = J$ and $J^{n+1} = JJ^n$, $n \geq 1$. In the case J is nilpotent, we define

$$\mathrm{Nil}\,(J) = \sup\{n \,|\, J^n \neq 0\}\,.$$

7. *Theorem. — Let* $\boldsymbol{k}$ *be a field and* M *be a simply connected closed oriented* d*-dimensional manifold.*

a) The kernel of the intersection morphism I *is nilpotent and* $Nil\,(Ker\,I) \leq d/2$.

b) The image of $\theta \circ I$ *is contained in the center of* $H_*(\Omega M)$.

Proof. — a) By 3.5-Theorem 6, the kernel of I is generated by the classes of cocycles in $\overline{A}\otimes T(W)$. Since $A^1 = 0$ and $A^{>d} = 0$, the nilpotency of the kernel of I is less than or equal to $d/2$.

b) Let e_i and w_i be the elements defined in 3.3 and $[\alpha]$ be an element in the image of $H(\epsilon_A \otimes id)$. Then α is a cocycle in $T(W)$ and there exist elements α_i in

T(W) such that $\bar{\alpha} = 1 \otimes \alpha + \sum_i e_i \otimes \alpha_i$ is a cycle in $A \otimes T(W)$. A short calculation shows that the component of e_i in $d(\bar{\alpha})$ is

$$(-1)^{|e_i|}\Big(d(\alpha_i) - [w_i, \alpha] + \sum_j \beta_i^j \alpha_j + \sum_{j,k} a_i^{j,k}(-1)^{|u||w_k|}\alpha_j w_k + \sum_{j,k} a_i^{kj}(-1)^{|w_k|} w_k \alpha_j\Big).$$

Since this component must be 0, by Lemma 3 below there exists a surjective morphism

$$H(T(W), d) \otimes \boldsymbol{k}[u] \to H(T(W), d)$$

that maps u to $[\alpha]$. This implies that $[\alpha]$ is in the center of $H(T(W), d) \cong H_*(\Omega M)$. □

3. *Lemma.* — *Assume* $\boldsymbol{k}$ *is a field. Let* α *be a cycle in* $(T(W), d)$ *and let* u *be a variable in the same degree. Then with the notation of 3.3:*

1. There exists a surjective quasi-isomorphism

$$\varphi : \big(T\big(w_i, u, w_i'\big), D\big) \to (T(W), d) \otimes (\boldsymbol{k}[u], 0)\,, \quad |w_i'| = |u| + |w_i| + 1,$$

such that $\varphi(u) = u$, $\varphi(w_i) = w_i$ *and* $\varphi(w_i') = 0$, *and with* D *defined by*

$$D(w_i') = [w_i, u] - \sum_j \beta_i^j w_j' - \sum_{j,k} a_i^{j,k}(-1)^{|u||w_k|} w_j' w_k - \sum_{j,k} a_i^{kj}(-1)^{|w_k|} w_k w_j'\,.$$

2. There exists a morphism of differential graded algebras

$$\rho : \big(T\big(w_i, u, w_i'\big), D\big) \to (T(W), d)$$

such that $\rho(u) = \alpha$ *and* $\rho(w_i) = w_i$ *if and only if there are elements* $\alpha_i \in T(W)$ *satisfying*

$$d(\alpha_i) = [w_i, \alpha] - \sum_j \beta_i^j \alpha_j - \sum_{j,k} a_i^{j,k}(-1)^{|u||w_k|}\alpha_j w_k - \sum_{j,k} a_i^{kj}(-1)^{|w_k|} w_k \alpha_j\,.$$

Proof. — We define $D(w'_i)$ by the above formula. Proving that $D^2 = 0$ is an easy and standard computation. The morphism

$$\varphi : \big(T\big(w_i, u, w'_i\big), D\big) \to (T(w_i), d) \otimes (\boldsymbol{k}[u], 0)$$

defined by $\varphi(w_i) = w_i$, $\varphi(u) = u$ and $\varphi(w'_i) = 0$ is a surjective homomorphism of differential graded algebras. To prove that φ is a quasi-isomorphism, we filter each differential graded algebra by putting u in filtration degree 0 and the other variables in filtration degree one. We are then reduced to prove that

$$\begin{aligned}&\bar\varphi : \big(T\big(w_i, u, w'_i\big), D\big) \to (T(w_i), 0) \otimes (\boldsymbol{k}[u], 0)\,,\\ &d(w_i) = 0,\, d\big(w'_i\big) = [w_i, u]\,,\end{aligned}$$

is a quasi-isomorphism. Denote by K the kernel of $\bar\varphi$ and consider the short exact sequence of complexes

$$\begin{aligned}0 \to (K \otimes E, D) \to &\big(T\big(w_i, u, w'_i\big) \otimes E, D\big)\\ &\xrightarrow{\bar\varphi\otimes 1} ((T(w_i) \otimes \boldsymbol{k}[u]) \otimes E, D) \to 0\,,\end{aligned}$$

where E is the linear span of the elements $1, sw_i, su$ and sw'_i, and where D is defined by

$$\begin{aligned}&D(sw_i) = w_i \otimes 1\,,\, D(su) = u \otimes 1\,,\\ &D\big(sw'_i\big) = w'_i - (-1)^{|w_i|} w_i \otimes su + (-1)^{|u||w'_i|+|u|} u \otimes sw_i\,.\end{aligned}$$

By construction, $(T(w_i, u, w'_i) \otimes E, D)$ and $(T(w_i) \otimes \boldsymbol{k}[u] \otimes E, D)$ are contractible and therefore quasi-isomorphic. Now a non-zero cocycle of lowest degree in K remains a non-trivial cocycle in the complex $(K \otimes E, D)$. Therefore $H_*(K) = 0$ and φ is a quasi-isomorphism. Part 2. of Lemma 3 follows directly from the expression of D. □

5. Determination of I when $\boldsymbol{k}$ is a field of characteristic zero

In this section $\boldsymbol{k}$ is a field of characteristic zero.

5.1. By 4.1-Theorem 7, the image of I is contained in the center of $H_*(\Omega M)$. On the other hand, by the Milnor-Moore theorem (e.g [10]-Theorem 21.5), $H_*(\Omega M)$ is the universal enveloping algebra of the homotopy Lie algebra $L_M = \pi_*(\Omega M) \otimes \boldsymbol{k}$ ([10]-p. 294).

Let L be any graded algebra. The center, Z(UL), of the universal enveloping algebra UL contains the universal enveloping algebra of the center of the Lie algebra, UZ(L). However the inclusion can be strict. Consider for instance the Lie algebra $L = \mathbf{L}(a, b)/([b, b], [a, [a, b]])$, with $|a| = 2$ and $|b| = 1$. The element $(ab - ba)b$ is in the center of UL, but not in UZ(L). We denote by R(L) the sum of all solvable ideals in L, ([10]-p. 495).

8. *Theorem.* — *If* $L = \{L_i\}_{i\geq 1}$ *is a graded Lie algebra over a field of characteristic zero satisfying* $\dim L_i < \infty$ *then* $Z(UL) \subset UR(L)$.

Proof. — It is well known that in characteristic zero, UL decomposes into a direct sum

$$UL = \bigoplus_{k\geq 0} \Gamma^k(L)$$

where the $\Gamma^k(L)$ are sub-vector spaces that are stable for the adjoint representation of L on UL: $\Gamma^0(L) = \boldsymbol{k}$, $\Gamma^1(L) = L$, and $\Gamma^n(L)$ is the sub-vector space generated by the elements $\varphi(x_1, ..., x_n) = \sum_{\sigma\in\Sigma_n} \varepsilon_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$, $x_i \in L$. The coproduct Δ of UL respects the decomposition, i.e.

$$\Delta : \Gamma^n(L) \to \bigoplus_{p+q=n} \Gamma^p(L) \otimes \Gamma^q(L) .$$

If we denote by Δ_p the component of Δ in $\Gamma^p(L) \otimes \Gamma^{n-p}(L)$ then

$$\Delta_p(\varphi(x_1, ..., x_n)) = \sum_{\tau\in Sh_p} \varepsilon_\tau \binom{n}{p} \varphi(x_{\tau(1)}, ..., x_{\tau(p)}) \otimes \varphi(x_{\tau(p+1)}, ..., x_{\tau(n)}) ,$$

where Sh_p denotes the set of p-shuffles of the set $\{1, 2, ..., n\}$. This implies that the composition $\Gamma^n(L) \xrightarrow{\Delta_p} \Gamma^p(L) \otimes \Gamma^{n-p}(L) \xrightarrow{\text{multiplication}} UL$ is the multiplication by $\binom{n}{p}$. We then consider the composite

$$c : \Gamma^n(L) \xrightarrow{\Delta_1} L \otimes \Gamma^{n-1}(L) \xrightarrow{1\otimes\Delta_1} L \otimes L \otimes \Gamma^{n-2}(L) \to \cdots \to L^{\otimes^n} .$$

Let $\alpha \in UL$ be an element in the center of UL, $\alpha = \sum_{i=1}^n \alpha_i$ with $\alpha_i \in \Gamma^i(L)$. Since $\Gamma^i(L)$ is stable by adjunction, each α_i is in the center of UL. Therefore we can assume that $\alpha \in \Gamma^n(L)$. We write $c(\alpha)$ as a sum of monomials $x_{i_1} \otimes ... \otimes x_{i_n}$. Since $\text{mult} \circ c : \Gamma^n(L) \to UL$ is the multiplication by $n!$, the element α belongs to the Lie algebra generated by the x_{i_j}. Suppose that in the decomposition of $c(\alpha)$ the number of monomials is minimal, then for each r, $1 \leq r \leq n$, the elements $x_{i_1} \otimes ... \otimes x_{i_{r-1}} \otimes x_{i_{r+1}} ... \otimes x_{i_n}$ are linearly independent. Since $[\alpha, x] = 0$, $x \in L$, we obtain the equation:

$$0 = \sum_{k=1}^n \left(\sum_i (-1)^{|x|\cdot(|x_{i_1}|+...+|x_{i_{k-1}}|)} x_{i_1} \otimes ... \otimes [x, x_{i_k}] \otimes ... \otimes x_{i_n} \right) .$$

Let us assume that the x_{i_k} are ordered by increasing degrees then the elements x_{i_k} with maximal degree belong to Z(L). The above equation shows also that $[x_{i_k}, x]$ belongs to the subvector space generated by the elements x_{i_l} with higher degree. A decreasing induction on the degree shows that all the x_{i_k} belong to R(L). □

5.2. Denote by X_0 the 0-localization of a simply connected space X. The Lusternik-Schnirelmann category of X_0, $\operatorname{cat} X_0$, is less than or equal to the Lusternik-Schnirelmann of X, cat X. Moreover the invariant $\operatorname{cat} X_0$ is easier to compute than $\operatorname{cat} X$, ([10]-§-27).

9. *Theorem. — Let* M *be a simply connected oriented closed manifold and* $\boldsymbol{k}$ *is a field of characteristic zero. Then*

a) The kernel of I *is a nilpotent ideal and* $Nil\,(Ker\,(\mathrm{I})) \leq cat\,\mathrm{M}_0$.

b) $(Im\ \theta \circ \mathrm{I}) \cap (\pi_*(\Omega \mathrm{M}) \otimes \boldsymbol{k}) = \mathrm{G}_*(\mathrm{M}) \otimes \boldsymbol{k}$.

c) $\sum_{i=0}^{n} dim\ (Im\ \theta \circ \mathrm{I} \cap \mathrm{H}_i(\Omega \mathrm{M}; \boldsymbol{k})) \leq \mathrm{C}n^k$, *some constant* $\mathrm{C} > 0$ *and* $k \leq cat\ \mathrm{M}_0$.

Proof. — a) By ([10]-Theorems 29.1 and 28.5), $\mathscr{C}^*(\mathrm{M}; \mathbf{Q})$ is connected by a sequence of quasi-isomorphisms to a connected finite dimensional commutative differential graded algebra (A, d) satisfying $\mathrm{Nil}\ (\overline{\mathrm{A}}) \leq n$ for $n > \operatorname{cat} \mathrm{M}_0$. Thus we conclude as in 4.1-proof of Theorem 7.

b) The differential graded algebra $\Omega(\mathrm{A}^\vee) = (\mathrm{T}(\mathrm{W}), d)$ is the universal enveloping algebra on the graded Lie algebra $\mathscr{L}_\mathrm{M} = (\mathbf{L}(\mathrm{W}), d)$, and the differential graded algebra $(\mathrm{T}(\mathrm{W} \oplus \boldsymbol{k}u \oplus \mathrm{W}'), \mathrm{D})$ is the universal enveloping algebra of the differential graded Lie algebra $\mathscr{L}^1_\mathrm{M} = (\mathbf{L}(\mathrm{W} \oplus ku \oplus \mathrm{W}'), \mathrm{D})$, (e.g [10]-p. 289), with

$$\begin{cases} d(w_i) = \sum_j \beta_i^j w_j + \sum_{j,k} \frac{1}{2} a_i^{jk}[w_j, w_k]\,, \\ \mathrm{D}(w_i') = [w_i, u] - \sum_j \beta_i^j w_j' - \sum_{j,k} a_i^{kj}(-1)^{|w_k|}\big[w_k, w_j'\big]\,. \end{cases}$$

By construction $\mathscr{L}_\mathrm{M}$ is a free Lie model for M and $\mathscr{L}^1_\mathrm{M}$ is a free Lie model for $\mathrm{M} \times \mathrm{S}^n$ with $n = |u|+1$, ([10]-§24). Moreover there exists a bijection between homotopy classes of maps:

$$[\mathrm{X} \times \mathrm{S}^n, \mathrm{X}] \cong [(\mathbf{L}(\mathrm{W} \oplus \boldsymbol{k}u \oplus \mathrm{W}'), \mathrm{D}), (\mathbf{L}(\mathrm{W}), d)]\,.$$

Therefore a homomorphism $\varphi : (\mathbf{L}(\mathrm{W} \oplus \boldsymbol{k}u \oplus \mathrm{W}'), \mathrm{D}) \to (\mathbf{L}(\mathrm{W}), d)$ such that $\varphi(u) = \alpha$ and $\varphi(w) = w$, $w \in \mathrm{W}$, corresponds to a map $f : \mathrm{M} \times \mathrm{S}^n \to \mathrm{M}$ which extends $id_\mathrm{M} \vee g : \mathrm{M} \times \mathrm{S}^n \to \mathrm{M}$, such that $[g] = \alpha$ modulo the identifications $\pi_n(\mathrm{M}) \otimes \boldsymbol{k} \cong \pi_{n-1}(\Omega \mathrm{M}) \otimes \boldsymbol{k} \cong \mathrm{H}_{n-1}(\mathbf{L}(\mathrm{W}), d)$. This means exactly that $\mathrm{Image}\,\mathrm{I} \cap (\pi_*(\Omega \mathrm{M}) \otimes \boldsymbol{k}) = \mathrm{G}_*(\mathrm{M}) \otimes \boldsymbol{k}$.

c) By Theorems 36.4, 36.5 and 35.10 of [10] we know that if $\mathrm{L} = \pi_*(\Omega \mathrm{M}) \otimes \boldsymbol{k}$ then R(L) is finite dimensional and $\dim \mathrm{R}(\mathrm{L})_{\mathrm{even}} \leq \operatorname{cat} \mathrm{M}_0$. We conclude using the graded Poincaré-Birkhoff-Witt theorem ([10]-Theorem 21.1): $Z(UL) \subset UR(L) \cong \Lambda(R(L)_{odd}) \otimes \boldsymbol{k}[(R(L)_{even}]$. □

6. Examples and applications

In this section we assume that $\boldsymbol{k}$ is a field.

6.1. *The spheres* S^n. — Since the differential graded algebra $\mathscr{C}^*(S^n)$ is quasi-isomorphic to $(H^*(S^n), 0) = (\wedge u/u^2, 0)$, $|u| = n$, by 3.3-Example 1, $\mathbb{H}\mathbb{H}^*(\mathscr{C}^*(S^n); \mathscr{C}^*(S^n))$ is isomorphic as an algebra to

$$\begin{aligned} &H^*(\wedge u \otimes T(v), D)\,, \quad |v| = n-1\,, \quad |u| = -n\,, \\ &D(u) = 0\,, \quad D(v) = u \otimes [v, v]\,. \end{aligned}$$

When n is odd, $D = 0$, $\mathbb{H}\mathbb{H}^*(\mathscr{C}^*(S^n); \mathscr{C}^*(S^n)) \cong \wedge u \otimes T(v)$ and $I = \varepsilon \otimes 1 : \wedge u \otimes T(v) \to T(v)$. When n is even, $D(v^{2n}) = 0$, $D(v^{2n+1}) = 2u \otimes v^{2n+2}$. Therefore a set of generators is given by the elements $c = 1 \otimes v^2$, $b = u \otimes v$, $a = u \otimes 1$, $|a| = -n$, $|b| = -1$, $|c| = 2n - 2$ and,

$$\mathbb{H}\mathbb{H}^*(\mathscr{C}^*(S^n); \mathscr{C}^*(S^n)) \cong \wedge(b) \otimes \boldsymbol{k}[a, c]/(2ac, a^2, ab) \text{ (see also [5])}\,.$$

The homomorphism $\theta \circ I : \mathbb{H}\mathbb{H}^*(\mathscr{C}^*(S^n); \mathscr{C}^*(S^n)) \to H_*(\Omega S^n) = T(v)$ is given by: $I(c) = v^2$, $I(a) = I(b) = 0$.

6.2. *An example where* I *is the trivial homomorphism.* — Let M be the connected sum $M = (S^3 \times S^3 \times S^3) \# (S^3 \times S^3 \times S^3)$. The wedge $N = (S^3 \times S^3 \times S^3) \vee (S^3 \times S^3 \times S^3)$ is then obtained by attaching a 9-dimensional cell to M along the homotopy class determined by the collar between the two components of M. Recall that

$$\pi_*(\Omega N) \otimes \mathbf{Q} \cong Ab(a, b, c) \coprod Ab(e, f, g)\,,$$

where $Ab(u, v, w)$ means the abelian Lie algebra generated by u, v and w considered in degree 2. The inclusion $i : M \to N$ induces a surjective map $\pi_*(\Omega M) \otimes \mathbf{Q} \to \pi_*(\Omega N) \otimes \mathbf{Q}$, This means that the attachment of the cell is inert in the sense of [10]-p. 503. Therefore, ([10]-Theorem 38.5),

$$\pi_*(\Omega M) \otimes \mathbf{Q} \cong Ab(a, b, c) \coprod Ab(e, f, g) \coprod \mathbf{L}(x)$$

with $|x| = 7$. In particular R(L) is zero, and by 4.1-Theorems 7 and 5.1-Theorem 8, when $\boldsymbol{k}$ is of characteristic zero, the homomorphism I is trivial.

6.3. *Lie groups.* — Let $\boldsymbol{k}$ be a field of characteristic zero and G be a connected Lie group. Since G has the rational homotopy type of a product of odd dimensional spheres, we obtain

$$\mathrm{HH}^*(\mathscr{C}^*(\mathrm{G}); \mathscr{C}^*(\mathrm{G})) \cong \wedge(u_1, \ldots, u_n) \otimes \mathbf{T}(v_1, \ldots, v_n)\,,$$

and I_{G} is onto. This example generalizes in:

10. *Theorem.* — *Let* $\boldsymbol{k}$ *be a field of characteristic zero and* M *be a simply connected closed oriented d-dimensional manifold. The morphism* $\theta \circ \mathrm{I} : \mathrm{HH}^*(\mathscr{C}^*(\mathrm{M}); \mathscr{C}^*(\mathrm{M})) \to \mathrm{H}_*(\Omega \mathrm{M})$ *is surjective if and only if* M *has the rational homotopy type of a product of odd dimensional spheres.*

Proof. — When M has the rational homotopy type of the product of odd dimensional spheres, then I is clearly surjective. Conversely, if I is surjective, then $\pi_*(\Omega\mathrm{M}) \otimes \mathbf{Q} = \mathrm{G}_*(\mathrm{M}) \otimes \mathbf{Q}$. Thus, $\pi_*(\mathrm{M}) \otimes \mathbf{Q} = \mathrm{G}_{\mathrm{odd}} \otimes \mathbf{Q}$, ([10], Proposition 29.8). Let $\{f_i : \mathrm{S}^{n_i} \to \mathrm{M},\ i = 1, \cdots, r\}$ represent a given linear basis of $\pi_*(\mathrm{M}) \otimes \mathbf{Q}$, and let $\varphi_i : \mathrm{S}^{n_i} \times \mathrm{M} \to \mathrm{M}$ be maps that restrict to $f_i \vee id_{\mathrm{M}}$ on $\mathrm{S}^{n_i} \vee \mathrm{M}$. Then the composition

$$\mathrm{S}^{n_1} \times \ldots \times \mathrm{S}^{n_r} \hookrightarrow \mathrm{S}^{n_1} \times \ldots \times \mathrm{S}^{n_r} \times \mathrm{M} \xrightarrow{1\times\varphi_r} \mathrm{S}^{n_1} \times \ldots \times \mathrm{S}^{n_{r-1}} \times \mathrm{M} \xrightarrow{1\times\varphi_{r-1}} \ldots \xrightarrow{1\times\varphi_1} \mathrm{M}$$

induces an isomorphism on the homotopy groups. Therefore, M has the rational homotopy type of a product of odd dimensional spheres. □

7. Hochschild cohomology and Poincaré duality

When two A-bimodules M and N are quasi-isomorphic as bimodules, then the Hochschild cohomologies $\mathrm{HH}^*(\mathrm{A}; \mathrm{M})$ and $\mathrm{HH}^*(\mathrm{A}; \mathrm{N})$ are isomorphic. In this section we relate the Hochschild cohomology of the singular cochains algebra on X with coefficients in itself and with coefficients in the singular chains on X when X is a Poincaré duality space. The usual cap product with the fundamental class is not a bimodule morphism. However the vector spaces $\mathrm{HH}^n(\mathscr{C}^*(\mathrm{M}); \mathscr{C}_*(\mathrm{M}))$ and $\mathrm{HH}^{n-d}(\mathscr{C}^*(\mathrm{M}); \mathscr{C}^*(\mathrm{M}))$ are isomorphic.

7.1. Let V be a graded module, then $\mathrm{V}^\vee$ denotes the graded dual, $\mathrm{V}^\vee = \mathrm{Hom}_{\boldsymbol{k}}(\mathrm{V}, \boldsymbol{k})$, and $\langle -; - \rangle : \mathrm{V}^\vee \otimes \mathrm{V} \to \boldsymbol{k}$ denotes the duality pairing. We denote by $\lambda_{\mathrm{V}} : \mathrm{V} \to \mathrm{V}^{\vee\vee}$ the natural inclusion defined by $\langle \lambda_{\mathrm{V}}(v), \xi \rangle = (-1)^{|\xi|} \langle \xi, v \rangle$.

7.2. Let X be topological space. The $\mathscr{C}^*(X)$-bimodule structures on $\mathscr{C}_*(X)$ and $\mathscr{C}^*(X)^\vee$ are explicitly defined by:

$$f \cdot c \cdot g := (-1)^{|c|(|f|+|g|)+|f|+|f|\,|g|}(g \otimes id \otimes f)(\Delta_X \otimes id) \circ \Delta_X(c)\,, \qquad c \in \mathscr{C}_*(X)\,,$$

$$\langle f \cdot \alpha \cdot g; h\rangle := (-1)^{|f|}\langle \alpha; g \cup h \cup f\rangle\,, \quad f, g, h \in \mathscr{C}^*(X), \alpha \in \mathscr{C}^*(X)^\vee\,.$$

Remark that the associativity properties of AW and of Δ_X imply directly that $\mathscr{C}_*(X)$ is a graded $\mathscr{C}^*(X)$-bimodule.

Let $1 \in \mathscr{C}^0(X)$ be the 0-cochain which value is 1 on the points of X. The usual cap product is then defined by

$$\mathscr{C}^p(X) \otimes \mathscr{C}_k(X) \longrightarrow \mathscr{C}_{k-p}(X)\,,$$
$$f \otimes c \mapsto f \cap c = f \cdot c \cdot 1 = \sum_i (-1)^{|c_i|\cdot|f|} c_i f\left(c_i'\right).$$

The cap product with a cycle $x \in \mathscr{C}_k(X)$ is a well defined homomorphism of differential graded modules, but is not a "degree k homomorphism" of $\mathscr{C}^*(X)$-bimodules. However,

11. *Theorem.* — *Let* X *be a path connected space and* $c \in \mathscr{C}_*(X)$ *be a cycle of degree* $k > 0$. *Then there exists a (degree k) morphism of* $\mathscr{C}^*(X)$*-bimodules*

$$\gamma_c : \overline{\mathbf{B}}(\mathscr{C}^*(X), \mathscr{C}^*(X), \mathscr{C}^*(X)) \to \mathscr{C}_*(X)$$

such that

- $\gamma_c(1[\,]1) = c$,
- $H(\gamma_c) \circ H(m)^{-1} : H^*(X) \to H_*(X)$ *is the cap product by* $[c]$, m *is the quasi-isomorphism of* $\mathscr{C}^*(X)$*-modules defined in 2.3-Lemma 1.*

Recall that γ_c is a degree k morphism of $\mathscr{C}^*(X)$-bimodules means that the following two properties are satisfied:

a) $d \circ \gamma_c = (-1)^k \gamma_c \circ d$,
b) $\gamma_c(f \cdot \alpha \cdot g) = (-1)^{|f|k} f \cdot \gamma_c(\alpha) \cdot g$,

for $f, g \in \mathscr{C}^*(X)$ and $\alpha \in \overline{\mathbf{B}}(\mathscr{C}^*(X), \mathscr{C}^*(X), \mathscr{C}^*(X))$.

Proof. — For simplicity we denote by A^e the enveloping algebra of $A = \mathscr{C}^*(X)$ and by B the differential graded $\mathscr{C}^*(X)$-bimodule $\overline{\mathbf{B}}(\mathscr{C}^*(X), \mathscr{C}^*(X), \mathscr{C}^*(X))$.

Recall the loop space fibration $ev : X^{S^1} \to X$, $\gamma \mapsto \gamma(0) = \gamma(1)$ with the canonical section $\sigma : X \to X^{S^1}$, $x \mapsto$ the constant loop at x. Jones defined a quasi-isomorphism of differential graded modules ([4]-Theorem 8),

$$J_* : B \otimes_{A^e} A \to \mathscr{C}^*(X^{S^1})$$

making commutative the following diagram of differential graded modules

$$\begin{array}{ccc} B \otimes_{A^e} A & \stackrel{J_*}{\to} & \mathscr{C}^*(X^{S^1}) \\ {\scriptstyle i}\nwarrow & & \nearrow {\scriptstyle \mathscr{C}^*(ev)} \\ & \mathscr{C}^*(X) & \end{array}$$

where $i : \mathscr{C}^*(X) \to B \otimes_{A^e} \mathscr{C}^*(X)$, $f \mapsto 1[\,]1 \otimes f$, denotes the canonical inclusion. Let ρ be the composite $\mathscr{C}^*(\sigma) \circ J_*$ then ρ is a retraction of i: $\rho \circ i = id$.

Let $u \in \mathscr{C}^k(X)^\vee$, $k > 0$, be a cycle. Using the canonical isomorphism of differential graded modules

$$\Psi : \mathrm{Hom}(B \otimes_{A^e} A, \boldsymbol{k}) \to \mathrm{Hom}_{A^e}(B, A^\vee)\,, \quad (\Psi(\theta)(\alpha))\,(f) = \theta(\alpha \otimes f)\,,$$

we define the map

$$\theta_u : \overline{\mathbf{B}}(\mathscr{C}^*(X), \mathscr{C}^*(X), \mathscr{C}^*(X)) \to (\mathscr{C}^*(X))^\vee\,, \qquad \theta_u = \Psi(u \circ \rho)\,.$$

The element θ_u is a k-cycle in $\mathrm{Hom}_{A^e}(B, A^\vee)$ and for any $f \in A$, $\theta_u(1[\,]1)(f) = u \circ \rho(1[\,]1 \otimes f) = u \circ \rho \circ i(f) = u(f)$.

Since the linear map

$$\lambda : \mathscr{C}_*(X) \to \mathscr{C}^*(X)^\vee$$

is a morphism of differential graded $\mathscr{C}^*(X)$-bimodules, for a cycle $c \in \mathscr{C}_k(X)$, we have a morphism

$$\theta_{\lambda(c)} : \overline{\mathbf{B}}(\mathscr{C}^*(X), \mathscr{C}^*(X), \mathscr{C}^*(X)) \to (\mathscr{C}^*(X))^\vee$$

with $\theta_{\lambda(c)}(1[\,]1) = \lambda(c)$.

Since $\overline{\mathbf{B}}(\mathscr{C}^*(X), \mathscr{C}^*(X), \mathscr{C}^*(X))$ is semifree, we deduce from the lifting homotopy property (2.3-Lemma 2) a morphism of $\mathscr{C}^*(X)$-bimodules

$$\gamma_c : \overline{\mathbf{B}}(\mathscr{C}^*(X), \mathscr{C}^*(X), \mathscr{C}^*(X)) \to \mathscr{C}_*(X)$$

making commutative, up to homotopy, the diagram

$$\begin{array}{ccc} \overline{\mathbf{B}}(\mathscr{C}^*(X), \mathscr{C}^*(X), \mathscr{C}^*(X)) & \xrightarrow{\theta_{\lambda(c)}} & \mathscr{C}^*(X)^\vee \\ \| & & \uparrow \lambda \\ \overline{\mathbf{B}}(\mathscr{C}^*(X), \mathscr{C}^*(X), \mathscr{C}^*(X)) & \xrightarrow{\gamma_c} & \mathscr{C}_*(X) \end{array}$$

and such that $\gamma_c(1[\,]1) = c$. The equality $\mathrm{H}(\gamma_c) \circ \mathrm{H}(m)^{-1} = -\cap [c]$ comes from the commutativity of the diagram

$$\begin{array}{ccc} \overline{\mathbf{B}}_0(\mathscr{C}^*(\mathrm{X}), \mathscr{C}^*(\mathrm{X}), \mathscr{C}^*(\mathrm{X})) & \stackrel{\theta_{\lambda(c)}}{\rightarrow} & \mathscr{C}^*(\mathrm{X})^\vee \\ m \downarrow & & \uparrow \lambda \\ \mathscr{C}^*(\mathrm{X}) & \stackrel{-\cap c}{\longrightarrow} & \mathscr{C}_*(\mathrm{X}) \end{array}$$

i.e., for any $f, g, h \in \mathscr{C}^*(\mathrm{X})$, we have $\langle \theta_{\lambda(c)}(f[\,]g), h\rangle = \langle \lambda \circ (-\cap c) \circ m(f[\,]g), h\rangle$. □

As a special case, we deduce:

12. *Theorem. — Let* M *be a 1-connected* ***k****-Poincaré duality space of formal dimension* d. *Then there are quasi-isomorphisms of* $\mathscr{C}^*(\mathrm{M})$*-bimodules*

$$\mathscr{C}^*(\mathrm{M}) \stackrel{m}{\leftarrow} \overline{\mathbf{B}}(\mathscr{C}^*(\mathrm{M}), \mathscr{C}^*(\mathrm{M}), \mathscr{C}^*(\mathrm{M})) \stackrel{\gamma}{\rightarrow} \mathscr{C}_*(\mathrm{M})$$

where m *is defined in 2.3-Lemma 1 and* $\gamma = \gamma_{[\mathrm{M}]}$ *with* $[\mathrm{M}] \in \mathrm{H}_d(\mathrm{M})$ *a fundamental class of* M. *In particular, the composite,* $\mathrm{H}(m) \circ \mathrm{H}(\gamma)^{-1}$ *is the Poincaré isomorphism* $\mathscr{P} : \mathrm{H}_*(\mathrm{M}) \to \mathrm{H}^{d-*}(\mathrm{M})$.

Applying Hochschild cohomology, we obtain:

13. *Theorem. — Let* M *be a 1-connected* ***k****-Poincaré duality space of formal dimension* d *then there exist natural linear isomorphisms*

$$\mathrm{D} : \mathbb{HH}^n(\mathscr{C}^*(\mathrm{M}); \mathscr{C}_*(\mathrm{M})) \xrightarrow{\cong} \mathbb{HH}^{n-d}(\mathscr{C}^*(\mathrm{M}); \mathscr{C}^*(\mathrm{M}))\,.$$

Proof. — Let $\varphi : \mathrm{N} \to \mathrm{N}'$ be a homomorphism of differential graded A-bimodules and assume that A is a ***k***-module. Then we deduce from 2.3-Lemma 1 (see [9] for more details) that φ induces an isomorphism of graded modules

$$\mathbb{HH}^*(\mathrm{A}; \mathrm{N}) \to \mathbb{HH}^*(\mathrm{A}; \mathrm{N}')\,.$$

Theorem 13 follows directly from Theorem 12 when one observes that the suspended map $s^d\gamma$ is a quasi-isomorphism of differential graded $\mathscr{C}^*(\mathrm{X})$-bimodules. □

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