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THE WALL OF THE CAVE

by ALEXANDER M. POLYAKOV

Abstract. – In this article old and new relations between gauge fields and strings are discussed. We add new arguments that the Yang-Mills theories must be described by non-critical strings in a five dimensional curved space. The physical meaning of the fifth dimension is that of the renormalization scale represented by the Liouville field. We analyze the meaning of the zigzag symmetry and show that it is likely to be present if there is a minimal supersymmetry on the world sheet. We also present the new string backgrounds which may be relevant for the description of the ordinary bosonic Yang-Mills theories. The article is written on the occasion of the 40-th anniversary of the IHÉS.

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1. Introduction

Very often a source of strong poetry and strong science is a good metaphor. My favorite one is Plato's cave: the parable of the men sitting in a dark cave, watching the moving shadows on its wall. They think that the shadows are "real" and not just projections of the outside world.

It seems to me that the latest stages of the ongoing struggle to understand interactions of elementary particles create a picture stunningly close to this parable.

In this article I will try to summarize these latest developments, adding some new conjectures and results. Of course it should be remembered that the work is far from finished and many surprises lie ahead.

It was understood long ago that compactness of the gauge group is likely to lead to the collimation of the flux lines and thus to quark confinement. This was first discovered by K. Wilson [1] in the strong coupling limit of lattice gauge theories. The next step was the proof of quark confinement for compact abelian gauge groups based on the instanton mechanism [2,3]. These theories arise from the non-abelian ones after partial symmetry breaking. Random fields of instantons (which are magnetic monopoles in 3d and closed rings of monopole trajectories in 4d) prevent propagation of charged objects and collimate the flux lines. This picture received a physical interpretation in terms of the "dual superconductors" [4, 5].

This approach is not directly extendable to nonabelian theories and only partial progress was made in this field until recently. The new development is based on the old idea that gauge theories must allow an exact description in terms of strings representing the Faraday flux lines and also on a new concept of D-branes.

Here is a brief recent history of these developments. Some years ago I became convinced that the string theory describing gauge fields is a Liouville theory with a curved fifth dimension. The guiding principle corresponding to the gauge symmetry was claimed to be the zigzag invariance of the boundary action. That insured that only vector states survive at the open string boundary. Thus the problem was reduced to the solution of a nonlinear 2d sigma model in which the string tension may become either zero or infinite (these cases are related by T-duality and equally well satisfy the zigzag symmetry; the choice between them requires more dynamical information and will be discussed below). This problem still remains central. These results were summarized in [6].

Meanwhile there has been an apparently unrelated development. In a seminal paper [30] Polchinski introduced the idea that D-branes in the type II critical strings can be regarded as the gravitational solitons previously discovered by Horowitz and Strominger [24]. After that in a remarkable paper [7] Ig. Klebanov compared the absorption of external particles by the 3-branes, viewed as a maximally supersymmetric Yang-Mills theory, with the picture of 3-branes, viewed as a gravitational soliton. He noticed that a correspondence should exist in the large N limit and for a large t'Hooft coupling (since in this case the sigma model corrections are small), and confirmed this by a explicit computations. In [31] some of the Yang-Mills correlators were computed by this methods. Later in an insightful work J. Maldacena [8] noticed that the metric relevant to these calculations is that of $AdS_5 \times S^5$, conjectured that the super Y-M theory corresponds to the full string theory in this background and pointed out that the distance to the D-brane can be identified with the Liouville field of the previous approach [6]. This synthetic point of view was used in [9, 10] to formulate the rules for calculating correlators and anomalous dimensions in the above theory. After that this side of the subject received enormous development.

In this paper we concentrate on more difficult but physically important aspects, not related to supersymmetry. We shall begin our discussion by reviewing the basic properties of the non-critical strings and random surfaces.

2. The non-critical string and the Liouville dimension

The propagating strings sweep out world surfaces. The quantum string theory is essentially the theory of these random surfaces. The classical action for the purely bosonic string is given by [11, 12]

$$S = \int \sqrt{g}(g^{ab}\partial_a x^\mu \partial_b x^\mu + \lambda)\, d^2\xi \qquad (1)$$

with $g_{ab}(\xi)$ being an independent metric. Here we parametrize the random surface by $x^\mu = x^\mu(\xi^1, \xi^2), \mu = 1, \dots D$, where D is the dimension of the ambient space. This is the most general action which is parametrically invariant and contains the minimal number of derivatives. The metric seems to disappear from (1) if we chose the conformal gauge, $g_{ab} = e^{\varphi}\delta_{ab}$. However this is not so because of a quantum anomaly, and the effective action describing the string in the conformal gauge was shown to be [13]

$$S = \int d^2\xi [\frac{26 - D}{96\pi}(\partial\varphi)^2 + \lambda e^{\varphi} + \frac{1}{2}(\partial x^{\mu})^2] . \tag{2}$$

Here the constant λ must be adjusted in order to reach the continuum limit.

Still this is not the end of the story. One has to find a way of quantizing the φ-field consistently with the conformal invariance. The reason why the conformal symmetry on the world sheet is sacred is that it is a remnant of the general covariance in the conformal gauge, and thus any quantization scheme preserving the above covariance must be conformally invariant. On the other hand, any standard cut-off in the ξ-space would destroy this property. The way to overcome this difficulty suggested in [14, 15] was essentially to retain the standard cut-off but to modify the action in such a way that the conformal symmetry is intact. In the simplified case of models with the central charge $c \leqslant 1$ one has to consider an action of the form

$$S = \int d^2\xi \{(\partial\varphi)^2 + (\partial x)^2 + T(\varphi) + \Phi(\varphi)R_2\} \tag{3}$$

where R_2 is the fixed curvature of the world sheet and the functions T and Φ are determined by conformal invariance. It has been shown in [16] along the general lines of [17] that the above invariance requires minimization of the effective action

$$W[T] = \int_0^{\infty} d\varphi\, e^{-2\Phi}[(\frac{dT}{d\varphi})^2 - 2T^2 + V(T)] \tag{4}$$

where $\Phi = b\varphi$ and $1 + 12b^2 = 26 - c$; V(T) is some potential. More generally one may need other fields to be included into the action (this does not happen for the minimal models).

The partition function $Z(\lambda)$ of the string is given by the value of this action on the classical solution

$$Z(\lambda) = W[T_{cl}(\varphi)]; T_{cl}(0) = \lambda . \tag{5}$$

In general the partition function satisfies the Hamilton-Jacobi equations which can be interpreted as a non-linear renormalization group [18]. The partition function has a singularity in λ which is related to the tachyonic nature of the field T. Indeed if we replace T by some massive field Ψ in (4) and neglect the potential the classical solution will have the following asymptotic behaviour

$$\Psi(\varphi) \sim e^{b\varphi - \sqrt{b^2 + M^2}\varphi}; \varphi \rightarrow \infty . \tag{6}$$

We conclude that there are three types of large φ behaviour. First of all there are stable massive modes. The action is analytic in their initial data. Next there could be "good" tachyons with $-b^2 \leqslant M^2 \leqslant 0$. These tachyons begin to grow and then condense after being stopped by the potential $V(T)$. This is what happens in the models with $c \leqslant 1$ and generates the singularity of $Z(\lambda)$. These models are perfectly healthy in spite of the tachyon. Finally, if $M^2 \leqslant -b^2$ we have a case of "bad" tachyon. The theory becomes unstable and the effective action looses its positivity because of the oscillations of the corresponding modes. This happens when $c > 1$.

It was suggested in [6] that this problem can be overcome if we assume that the (φ, x) geometry for $c > 1$ is not flat anymore. Since the (x) geometry must remain flat, the most general form of action takes the form

$$S = \int \left[(\partial\varphi)^2 + a^2(\varphi)(\partial x)^2 + \sum_n \lambda_n(\varphi(\xi))V_n(\xi) \right] . \tag{7}$$

Here the vertex operators V_n are describing the modes with $M^2 \leqslant 0$, since only these modes have the tendency to condense. The function $a(\varphi)$ representing the running string tension and the couplings $\lambda_n(\varphi)$ must be determined from the condition of conformal invariance on the world sheet. The effective action for the tachyon is now modified due to the non-flatness of the (φ, x) geometry. It has the form

$$W = \int d\varphi e^{-2\Phi} a^D(\varphi)[(\frac{dT}{d\varphi})^2 - 2T^2 + ...] . \tag{8}$$

The dots include the action for a and Φ which we will discuss later and also possible higher derivative terms. The evolution of the tachyon now takes place in an "expanding universe" (with φ playing the role of the euclidean time). It is well known that expansion tends to moderate instabilities. For example Jeans instability in flat space is exponential, while in a Friedmann universe it is power-like. Something similar may happen in our case. If we assume that there is a solution, minimizing the above action, of the "big bang" type

$$a(\varphi) \sim \varphi^\alpha , \tag{9}$$

$$\Phi(\varphi) \sim \log\varphi . \tag{10}$$

The tachyon will behave as $T(\varphi) \sim \varphi^\gamma$ with some real γ. This type of solutions exist in the one loop approximation. Whether they can be promoted to exact conformal field theories is a difficult open question (which is somewhat more tractable in supersymmetric cases as we will discuss later). We use it here as an illustration of a possible scenario. Namely, the tachyon grows and tends to a constant, as it does at $c = 1$. As a result we obtain a theory of random surfaces in which the string tension $\sigma = a^2$ and the cosmological constant $\lambda = T$ are related by the scaling law $\sigma \sim \lambda^{\frac{2\alpha}{\gamma}}$. Of course this illustration must be taken with a grain of salt since it is based on the low energy effective action. It well may happen that in order to stabilize a random surface one needs supersymmetry on the world sheet. At least, as we will argue in the next sections, this supersymmetry is needed to describe gauge theories. Nevertheless the bosonic theory is an important preliminary step.

Let us summarize what we have learned. First, due to quantum anomalies, the target space of the D-dimensional random surface has D + 1 dimensions. This happens because renormalization changes the classical action of the random surface in the following way

$$\sqrt{g}g^{ab}\partial_a x\partial_b x \Rightarrow (\partial\varphi)^2 + a^2(\varphi)(\partial x)^2 + \ldots \tag{11}$$

All the background fields must be determined from the condition of conformal invariance. The extra dimension, grown by the quantum effects, comes out of the conformal factor of the world sheet metric. The background fields of the original, D-dimensional random surface, play the role of the initial (or, better to say, boundary) data for the D + 1 dimensional effective action. This action, considered as a function of the initial data is equal to the partition function of the non-critical string.

3. Gauge fields and zigzag symmetry

The loop equations provide a bridge between gauge theories and strings although this bridge is not completely safe. They are written for the functional W(C) which is the expectation value of the parallel transport around the loop C, calculated with the Yang-Mills action. Their role is to translate the Schwinger-Dyson equations for the Yang-Mills fields into the language of loops. The role of string theory is to solve these equations in terms of random surfaces. The equations have the form [19, 20]

$$\frac{\partial^2 W(C)}{\partial x^2(s)} = \int \delta(x(s) - x(u))\frac{dx}{ds}\frac{dx}{du}W(C_1)W(C_2)\,. \tag{12}$$

Here the operator on the left hand side is defined as a coefficient in front of the δ-function (the contact term) in the second variational derivative, the δ-function on the right hand side is nonzero if the disc bounded by C is pinched, and the contour splits into two parts, C_1 and C_2. The gauge field-string equivalence means that there is a string action, containing the Liouville field and maybe other fields, which solves this equation in the following sense

$$W(C) = \int Dx(\xi)D\varphi(\xi)e^{-S(\varphi,\,x)}\,. \tag{13}$$

Here the ξ space is assumed to be a disk or half-plane. At the boundary of this disc, we impose the Dirichlet conditions

$$x\,|_{\partial D} = x(s);\, \varphi\,|_{\partial D} = \varphi_*\,. \tag{14}$$

The first part of this choice is obvious – we just map the world sheet disk into a disk in the target space, bounded by our loop. The boundary conditions for the Liouville field are much more subtle. It is important to realize that they can not be chosen arbitrarily. The conformal invariance of the world sheet imposes very strong constraints and in particular fixes the value of φ_* if the allowed value exists at all. The second strong requirement is the zigzag symmetry. Let us discuss this requirement. The Wilson loop is invariant under reparametrizations of the contour

$$x(s) \Rightarrow x(\alpha(s)) . \qquad (15)$$

Since the string theory is invariant under diffeomorphisms of the world sheet, the above symmetry is automatic if $\alpha(s)$ is a diffeomorphism, which means that $\frac{d\alpha}{ds} > 0$. However the basic feature of the Wilson loop is the invariance under zigzags, when the last condition is dropped. We shall see now that in string theory the necessary condition for such an extended reparametrization invariance is $a(\varphi_*) = 0$ or ∞.

Let us discuss the first requirement. The reason why the generic Dirichlet condition is impossible in a curved space is that the D-brane defined by it will be subjected to gravitational forces tending to change its shape. This can be seen as following. Suppose that we have a 2d sigma model with the target space metric $G_{MN}(y)$ where $y = (x, \varphi)$. Let us try to impose the condition at the boundary of the world sheet disc $\varphi\,|_{\partial D} = \varphi_*(x)$. The effective action acquires a boundary term

$$W_B = \int d^4x e^{-\Phi(x,\varphi_*(x))} \sqrt{G(x, \varphi_*(x))} . \qquad (16)$$

The function $\varphi_*(x)$ must be chosen so as to minimize this action (or the modified action when the higher derivative terms are added). This condition defines a possible shape of the D-brane. In our case it gives

$$\frac{\partial}{\partial\varphi}(e^{-\Phi}a^4) = 0 . \qquad (17)$$

If, as it often happens, the particular solution does not satisfy this condition, the D-brane will be carried either to zero or to infinity where a separate consideration is needed. Namely, the zigzag invariance of the Wilson loop is reflected in the properties of the algebra of boundary operators. As usual, the open string amplitudes can be expressed as

$$A(p_1 \dots p_n) = \int ds_1 \dots ds_n \int Dx(s) W[x(s)] V_{p_1}(x(s_1)) \dots V_{p_n}(x(s_n)) . \qquad (18)$$

The vertex operators $V_p(x)$ are in general exponentials e^{ipx} multiplied by the product of different derivatives of $x(s)$. The usual reparametrization invariance is translated into the Virasoro conditions, satisfied by those vertex operators. That still leaves an infinite number of possible vertex operators, representing the states of the open string. When $W[x(s)]$ is zigzag invariant the situation is different. In this case the only allowed vertex operator is the vector one

$$V_\alpha(p) = \frac{dx_\alpha}{ds} e^{ipx(s)} \qquad (19)$$

since otherwise the integrand in (18) will loose its zigzag symmetry. That was our main conclusion in [6]: *the zigzag symmetry is equivalent to the finiteness of the algebra of boundary operators which contains only vector vertices.*

The other way to put it is to say that with the above boundary conditions the open string contains only gluons in its spectrum. If there are other fields in addition to the Yang-Millson, their vertex operators must be added at the boundary. In general *the spectrum of the boundary states is the spectrum of the corresponding field theory.*

It would be nice to be able to check directly that the loop equations are satisfied by our ansatz. Some relevant methods were discussed in [6, 21, 22] but they are still insufficient. However this check may be not so vital. The loop equations are bound to be satisfied by the following argument. Our open string theory has only vector particles and only positive norms. Therefore it must be the Yang-Mills theory. This reminds one of the standard deduction of the Einstein theory from the existence of massless tensor particles.

If we try to impose the Dirichlet boundary conditions for the φ-field at a finite value of the string tension $a^2(\varphi_*)$ the whole tower of the open string states will be present and the zigzag symmetry will not be there. That leaves us with two options, either to place the gauge theory at the horizon, $a(\varphi_*) = 0$, or at infinity, $a(\varphi_*) = \infty$. It seems that both choices will provide us with the zigzag symmetry and thus are equivalent. The first one, preferred in [6], eliminates the $(\partial x)^2$ term from the action. However the presence of the Wilson loop generates the term $B_{\mu\nu}(x)\varepsilon^{ab}\partial_a x^\mu \partial_b x^\nu$ in the string action which becomes dominant in this case. This term has the interpretation of a color electric flux [6]. However, it is difficult to find its explicit form and for that reason in the papers [9, 10, 23] a simpler choice was suggested, placing the gauge theory at infinity, also leading to the zigzag symmetry, has been. In this case the $B_{\mu\nu}$-term is negligible since $a(\varphi_*) = \infty$ and we can concentrate on a simpler set of background fields. Let us give a simple although non-rigorous argument which demonstrates the presence of the zigzag symmetry in the case of infinite tension. Consider some massive mode of the open string, $\Psi(\varphi_*, x)$. The quadratic part of the boundary effective action has the form

$$W_B \sim \int dx [a^{-2}(\varphi_*)(\frac{d\Psi}{dx})^2 + M^2\Psi^2]\,. \tag{20}$$

If $a(\varphi_*) = \infty$ the first term drops out and we conclude that minimization of the effective action requires $\Psi = 0$.

On the other hand the massless mode can have an arbitrary x dependence. This is just what is desirable for the description of the gauge theory, since we do not want to restrict the momentum carried by the vertex operators. It must be said, however, that the choice of boundary conditions requires further clarification which must be based on dynamical arguments. This has not been done.

Another important conclusion is that the tachyons will condense and, generally speaking, will be present at the boundary thus violating the zigzag symmetry. We shall get rid of them in the next section. Of course these arguments must be improved by considering the conformal field theory represented by the above non-linear sigma model. This also has not been done so far.

4. Elimination of the boundary tachyon

We saw that in the purely bosonic Liouville theory we expect that some scalar tachyonic modes will penetrate the boundary. Also, the Wilson loops and related quantities have some awkward divergences which made me suspect in [6] that supersymmetry on the world sheet may be helpful in comparing gauge fields and strings. In this section we will present a simple argument showing that in the Liouville theory with minimal supersymmetry on the world sheet and non-chiral GSO projection the boundary tachyon is absent and the zigzag symmetry is to be expected. This theory is purely bosonic in the target space and, according to the above arguments, must describe the pure Yang-Mills theory. In the theories of this kind the world sheet is parametrized by the coordinates $(\xi^1, \xi^2, \theta^1, \theta^2)$, where θ are anticommuting. The boundary of the world sheet is parametrized by one bosonic and one fermionic variable, (s, ϑ). It is well known that the formulas of the bosonic non-critical string theory are almost unchanged in the fermionic case if written in this superspace. In particular the loops and the loop equations are changed as follows

$$\int A_\mu \frac{dx^\mu}{ds} ds \Rightarrow \int ds d\vartheta A_\mu (x(s, \vartheta)) D_\vartheta x^\mu, \tag{21}$$

$$D_\vartheta = \frac{\partial}{\partial \vartheta} + \vartheta \frac{\partial}{\partial s}, \tag{22}$$

and the only change in the loop equation is the replacement of the s-derivatives by D_ϑ. The same minimal change takes place in the non-critical string action (11). It retains the same form in terms of superfields, with the ξ-derivatives being replaced by θ-derivatives. The above generalized Wilson loop describes the spin one half test particle.

There are, however, some important differences from the bosonic case. First of all the critical dimension in this string is $D_{cr} = 10$ and not 26. A more subtle difference is that to be consistent the theory requires the GSO projection. This projection is a summation over the spin structures, that is as we go around a cycle or an injection point, the world sheet fermions may or may not change sign. We must sum over the possibilities since otherwise the modular invariance and unitarity will be broken. In the critical superstrings the standard prescription is to sum over the signs of the left and right fermions independently. That adds to the operator algebra left and right spin operators which transform as space-time fermions and generate 10d supersymmetry. It also truncates the algebra of bosonic vertices by dropping those which contain odd number of either left or right world sheet fermions.

In the non-critical strings with variable string tension there is a slightly different way to make the GSO projection. Namely it is consistent to assume that the projection is non-chiral or that, as we go around the cycle, left and right fermions change in the same way. This will again introduce the spin operators but this time they will transform as a product of the space time fermions and will correspond to the Ramond-Ramond bosons. There also will be a truncation of the algebra of vertex operators, but this time only the total number of the world sheet fermions will have to be even. As a result in the closed string sector we have a tachyon described by a vertex operator

$$V(p) = \int d^2\xi d^2\theta \chi_p(\varphi(\xi, \theta)) e^{ipx(\xi, \theta)} \tag{23}$$

where the function $\chi_p(\varphi)$ is determined from the condition of conformal invariance on the world sheet (which reduces to the Laplace equation in the metric (7) in the weak coupling limit). Since this expression is even in θ it will be also even in the world sheet fermions after the θ-integration is performed. Thus this vertex is allowed by the non-chiral GSO projection. Presumably this tachyon should be of the "good" variety and peacefully condense in the bulk. The important thing is not to have a boundary tachyon which would spoil the loop equation. The GSO projection takes care of that. The boundary tachyon vertex has the form

$$V_B(p) = \int ds d\vartheta \chi_p^{(B)}(\varphi(s, \vartheta)) e^{ipx} \tag{24}$$

and clearly contains an odd number of fermions. It is eliminated by the GSO projection and we are left with the purely vector states surviving at the boundary.

We conclude that the supersymmetric Liouville theory with the non-chiral GSO projection is likely to describe the bosonic Yang-Mills theory. An important feature of this theory is the presence of bosonic Ramond-Ramond fields. These fields can create new conformal points. This happens in statistical mechanics where the imaginary magnetic field in the 2d Ising model (which is the simplest example of a RR-field) drives the theory to a new so-called Lee-Yang critical point. We shall see that something similar happens in our case. In the next section we discuss the extent to which we can believe in the existence of conformal theories with a curved geometry in the 5d Liouville space.

5. Cosmology inside hadrons (conformal symmetry on the world sheet)

As we already discussed, the conformal symmetry of our sigma model is absolutely necessary for the consistency of the whole approach. According to [17], the background fields providing conformal invariance must satisfy the so-called β-function equations. We know their explicit form only in the low energy approximation, which generally speaking is not sufficient for our purposes. This is the main obstacle for our approach. Still in some cases (which we discuss below) this approximation is good enough and also it often gives a correct qualitative picture. When RR-fields are present the β-function equations take the form

$$R_{AB} = \nabla_A \nabla_B \Phi + e^{-\Phi} \Theta_{AB} \tag{25}$$

where Θ_{AB} is the energy-momentum tensor for the RR-fields, indices A and B include the Liouville direction and we changed the normalization of the dilaton

$$-2\Phi \Rightarrow \Phi \tag{26}$$

There is also the condition balancing the central charge

$$R - 2\nabla^2\Phi - (\nabla\Phi)^2 + \frac{10 - D}{2} = 0 \tag{27}$$

where D is the total number of dimensions. The RR-fields appear as antisymmetric tensors of different ranks.

If we denote the rank $p + 1$ field by C_{p+1} the energy-momentum tensor is expressed in terms of the field strength $F_{p+2} = dC_{p+1}$. The RR fields satisfy the equation

$$d^{+}F_{p+2} = 0\,. \tag{28}$$

Their energy momentum tensor is given by

$$\Theta_{AB} = F_{A\ldots}F_{B\ldots} - \frac{1}{2(p+2)}G_{AB}F_{\ldots}F_{\ldots} \tag{29}$$

where the dots mean the indices contracted with the metric G. Let us look for a solution which leaves the x-space flat. That means that the metric takes the form (11). To incorporate some interesting examples in which extra fields are added to the Yang-Mills we shall consider the following ansatz

$$ds^2 = (d\varphi)^2 + a^2(\varphi)\,(dx)^2_{p+1} + b^2(\varphi)\,(dn)^2_{q+1} \tag{30}$$

$$C_{p+1} = c(\varphi)\,dx^1 \ldots dx^{p+1} \tag{31}$$

where n represents the additional $q + 1$ dimensional sphere and only the $p + 1$ form was assumed to be non-zero. This is the only possibility consistent with the symmetries of the x-space apart from switching also the zero form, corresponding to the theta terms. This class of solutions has been considered in the past as a description of the p-branes [24] (on which we will comment in the next section) and in the case of the Minkowskian signature as cosmological models (see e.g. [25]). However we have to concentrate on some special cases. The equation (28) gives

$$\frac{dc}{d\varphi} = N\frac{a^{p+1}}{b^{q+1}} \tag{32}$$

where N is the integration constant. The equations (25) and (27) in this ansatz take the form

$$\ddot{\Phi} + (p+1)\,a^{-1}\,\ddot{a} + (q+1)\,b^{-1}\,\ddot{b} = N^2 e^{-\Phi}\frac{1}{b^{2q+2}} \tag{33}$$

$$a^{-1}\,\ddot{a} + pa^{-2}\,\dot{a}^2 + a^{-1}\,\dot{a}\dot{\Phi} + (q+1)\,a^{-1}b^{-1}\,\dot{a}\dot{b} = N^2 e^{-\Phi}\frac{1}{b^{2q+2}} \tag{34}$$

$$b^{-1}\,\ddot{b} + qb^{-2}\,\dot{b}^2 + b^{-1}\,\dot{b}\dot{\Phi} + (p+1)\,a^{-1}b^{-1}\,\dot{a}\dot{b} - qb^{-2} = -N^2 e^{-\Phi}\frac{1}{b^{2q+2}} \tag{35}$$

$$\ddot{\Phi} + (p+1)\,a^{-1}\,\dot{a}\dot{\Phi} + (q+1)\,b^{-1}\,\dot{b}\dot{\Phi} + \dot{\Phi}^2 = \frac{10 - d}{2} - N^2(p - q + 1)\,e^{-\Phi}\frac{1}{b^{2q+2}}\,. \tag{36}$$

Here d is the total dimensionality, $d = p + q + 3$. In general these equations have power-like ("big bang") solutions. At best they can be taken as a hint that there exist a corresponding nonlinear sigma model with conformal invariance. There are however some cases in which the equations can be taken more seriously. These are the cases in which the curvature of the 5d Liouville space is constant. According to [8] the constant curvature solution correspond to the conformal symmetry in the target space and thus are more tractable. A look at the above equations shows that there are at least two possibilities to have such solutions. First let us take $d = p+q+3 = 10$ and $p - q + 1 = 0$. According to (36) the driving force for the dilaton vanishes and it can be set to a constant. We then easily solve the remaining equations by setting $b(\varphi)$ = const and $a(\varphi) = e^{\alpha\varphi}$. In this case $p = 3$, $q = 4$ and thus we are dealing with the space $L_5 \times S^5$ where L_5 is the 5d Lobachevsky space (the Minkowskian version of which is called AdS) and S^5 is a 5d sphere. All parameters are fixed by the above equation and have the following orders of magnitude

$$e^{-\Phi} \sim g_s^2; \frac{1}{b^8} \sim (g_s\mathrm{N})^2; \alpha^2 \sim \frac{(g_s\mathrm{N})^2}{b^{10}} \sim (g_s\mathrm{N})^{-\frac{1}{4}}; \tag{37}$$

where g_s is the closed string coupling constant.

This is of course the standard 3-brane solution [24] rescaled near the brane and written in the Liouville gauge. It describes the maximally supersymmetric Yang-Mills theory [7, 8]. We see that if we take $\mathrm{N} \to \infty$, while keeping $g_s\mathrm{N}$ fixed we can neglect the closed string loops. It is also possible to neglect the sigma model corrections if the curvature of this solution, which is $\sim \alpha^2$ is small. Hence the whole construction is trustworthy provided that $g_s\mathrm{N} >> 1$. This is the much explored case first analyzed in [7]. We shall not discuss it further except noticing that the real remaining challenge in this case, just as in the other theories we are discussing here, is to promote the above solution to a complete conformal field theory.

Let us turn to another constant curvature case which is less reliable than the above but much more physically interesting (the usual dichotomy). Namely let us consider the pure non-supersymmetric Yang-Mills theory. That means that this time we have no n-field and have to set $q = -1$. We can keep the dilaton from running if we chose the string coupling so as to set the right hand side of eq. (36) to zero. In this case

$$g_s\mathrm{N} = \sqrt{\frac{8-p}{2(p+1)}}. \tag{38}$$

After that the equations are easily solved by $a(\varphi) = e^{\alpha\varphi}$ and we have a constant curvature solution once again. It describes a conformal fixed point of the Yang-Mills theory. Unfortunately we can't be certain that the sigma model corrections (which are of the order of unity) will not destroy this solution. Usually however they don't, just renormalizing parameters, but not spoiling conformal symmetry. If this is the case, we come to the conclusion that strong interactions at small distances are described not by asymptotic freedom but by a conformal field theory, a possibility dreamt of in [26, 27]. Once again, the exact solution of the sigma model is needed to prove or disprove this drastic statement.

If there is no fixed point we must recover the asymptotic freedom from our string theory. That means that the dilaton and the curvature will be functions of φ. The "conformal time" $\tau = \int \frac{d\varphi}{a(\varphi)}$ defines an effective scale (perhaps related to the comments in [28]). Therefore the asymptotic freedom will manifest itself in the following relation

$$g_s N \sim g_{YM}^2 N \sim e^{-\frac{\Phi}{2}} \sim \frac{1}{\log(\tau)} . \tag{39}$$

In checking this relation the one loop approximation in the sigma model is even less adequate than in the previous case, since the YM coupling is small and the curvature of the Liouville space is large. However the problem is not hopeless. We need a solution of the sigma model with a large but slowly varying curvature. That means that we have to account the curvature terms in all orders but can neglect their derivatives (analogously to what is done in the case of the Born-Infeld action for the open strings [29]). There are reasons to believe that the above sigma models with constant curvature are completely integrable. Thus we may hope to find the complete solution of the gauge fields-strings problem and perhaps even to discover experimental manifestations of the fifth (Liouville) dimension.

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REFERENCES

[1] K. Wilson, *Phys. Rev.*, **D10** (1974), 2445.
[2] A. Polyakov, *Phys. Lett.*, **59B** (1975), 82.
[3] A. Polyakov, *Nucl. Phys.*, **B120** (1977), 429,
[4] S. Mandelstam, *Phys. Rep.*, **23C** (1976), 245.
[5] G. t'Hooft, in *High Energy Phys.*, Zichichi editor, Bolognia (1976).
[6] A. Polyakov, HEP-TH/9711002.
[7] I. Klebanov, *Nucl. Phys.*, **B496** (1997), 231.
[8] J. Maldacena, HEP-TH/9711200.
[9] S. Gubser, I. Klebanov, A. Polyakov, HEP-TH/9802109.
[10] E. Witten, HEP-TH/9802150.
[11] L. Brink, P. di Vecchia, P. Howe, *Phys. Lett.*, **63B** (1976), 471.
[12] S. Deser, B. Zumino, *Phys. Lett.*, **65B** (1976), 369.
[13] A. Polyakov, *Phys. Lett.*, **103B** (1981), 207.
[14] T. Curtright, C. Thorn, *Phys. Rev. Lett.*, **48** (1982), 1309.

[15] J. GERVAIS, A. NEVEU, *Nucl. Phys.*, **B199** (1982), 59.
[16] J. POLCHINSKI, *Nucl. Phys.*, **B346** (1990), 253.
[17] C. CALLAN, E. MARTINEC, M. PERRY, D. FRIEDAN, *Nucl. Phys.*, **B262** (1985), 593.
[18] A. POLYAKOV, Proceedings of Les Houches (1992).
[19] A. POLYAKOV, *Nucl. Phys.*, **B164** (1980), 171.
[20] Y. MAKEENKO, A. MIGDAL, *Nucl. Phys.*, **B188** (1981), 269.
[21] H. VERLINDE, HEP-TH/9705029.
[22] A. MIGDAL, *Nucl. Phys. Proc. Suppl.*, **41** (1995), 51.
[23] J. MALDACENA, HEP-TH/9803002.
[24] G. HOROWITZ, A. STROMINGER, *Nucl. Phys.*, **B360** (1991), 197.
[25] A. LUKAS, B. OVRUT, D. WALDRAM, HEP-TH/9802041.
[26] K. WILSON, *Phys. Rev.*, **179** (1969), 1499.
[27] A. POLYAKOV, ZHETF, **59** (1970), 542 ; Pisma v ZHETF **12** (1970), 538.
[28] A. PEET, J. POLCHINSKI, HEP-TH/9809022.
[29] E. FRADKIN, A. TSEYTLIN, *Phys. Lett.*, **B178** (1986), 34.
[30] S. GUBSER, I. KLEBANOV, HEP-TH/9708005.

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