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# THE INTERTWINING OF AFFINE KAC-MOODY AND CURRENT ALGEBRAS

*by* Lochlainn O'RAIFEARTAIGH

## 1. Introduction

Il faut féliciter l'IHÉS non seulement du grand succès de ses quarante années, mais aussi de sa fidélité au rêve d'être un lieu de rencontre amicale entre mathématiciens et physiciens. A ce propos, je voudrais présenter ici une brève histoire d'un développement récent, la découverte des algèbres de Kac-Moody affines, où mathématiques et physique ont bénéficié de leurs développements parallèles, et si enchevêtrés, qu'il est souvent difficile de les démêler.

## 2. The Mathematical Path

The mathematical origins of affine Kac-Moody (AKM) algebras go back [1] to the work of Killing, who first classified the simple Lie algebras. As was natural for a student of Weierstrass, Killing considered the *spectra* of elements with respect to adjoint action, and was thereby led to the well-known Cartan-Killing relations

$$[H_i, H_j] = 0 \qquad [H_i, E_\alpha] = \alpha_i E_\alpha \qquad [E_\alpha, E_{-\alpha}] = \vec{\alpha} \cdot \vec{H} \equiv \sum_{i=1}^{i=l} \alpha_i H_i \tag{1}$$

where $\alpha_i$ for $i = 1 \ldots l$ are the roots, and to the Cartan-Killing matrices

$$A_{ij} = 2\, \frac{\vec{\alpha}^{(i)} \cdot \vec{\alpha}^{(j)}}{\vec{\alpha}^{(i)} \cdot \vec{\alpha}^{(i)}} \in Z \tag{2}$$

where $\alpha_j^{(i)}$ for $i = 1 \ldots l$ are the *simple* roots (generators, with respect to addition, of the positive root lattice). The elements of the matrices A are integer-valued, as indicated, and satisfy the three conditions

$$(i) \quad A_{ii} = 2 \qquad (ii) \quad A_{ij} \leqslant 0 \quad i \neq j \qquad (iii) \quad A_{ij} = 0 \leftrightarrow A_{ji} = 0 \tag{3}$$

Coleman [1] describes the discovery of these matrices and their properties as one of the great mathematical achievements. The matrices A are also positive definite in the sense that

$$(f, \mathrm{A}f) \equiv f_i \mathrm{A}_{ij} f_j > 0 \tag{4}$$

for any real vectors $f_i$. Following earlier work by Chevalley [2], it was shown by Serre [3] that the Cartan-Killing matrices satisfying (3) and (4) actually *define* the semi-simple Lie algebras, which could be reconstructed from (3) (4) and the nilpotency condition

$$\left(\mathrm{Ad}(\mathrm{E}_{\alpha(i)})\right)^{(1-\mathrm{A}_{ij})} \mathrm{E}_{\alpha(j)} = 0 \qquad i \neq j \tag{5}$$

*Kac and Moody*

The Kac-Moody (KM) algebras were obtained [4] [5] by removing the positive-definite restriction on A i.e. requiring that it (be indecomposable) and satisfy only (3), and then applying the Serre construction. The resultant algebras fall into three classes, namely, those for which there exists a vector $u$ with only positive components such that

$$\mathrm{A}u > 0 \qquad \mathrm{A}u = 0 \qquad \mathrm{A}u < 0\,.$$

These classes correspond to finite-dimensional Lie algebras, affine Kac-Moody (AKM) algebras, and non-affine Kac-Moody (NAKM) algebras respectively. The non-affine algebras are largely unexplored and, to my knowledge, no natural physical applications have been found so far. The affine Kac-Moody (AKM) algebras are those of interest here. Remarkably, they may be characterized by the fact that A is positive semi-definite and has exactly one zero eigenvalue i.e.

$$(f, \mathrm{A}f) \geqslant 0 \quad \text{and} \quad \dim \ker(\mathrm{A}) = 1\,. \tag{6}$$

The procedures adopted by Kac and Moody to obtain these algebras were rather different. Moody considered the degeneracies of A, and in his first paper arrived at the AKM algebras, including the twisted (tiered) ones. Kac considered the growth properties of the algebras obtained by relaxing the positivity condition on A, using as measure $\ln(d(s,n))/\ln(n)$ where $s$ is the dimension of any finite subspace S of the algebra, and $d(s,n)$ the dimension of the subspace spanned by the commutators of any $m$ elements of S for $m \leqslant n$. He found that the three classes corresponded to finite, polynomial and exponential growth respectively.

The major breakthrough for AKM algebras came with the discovery that they have many of the properties of finite-dimensional simple Lie algebras [4]. In particular, they can be graded, the roots can be ordered, the Weyl group and Weyl character formula generalize in a natural way, and the concepts of highest and lowest weight representations and of Casimir operators carry through with only technical modifications. The Serre construction for the AKM's leads to algebras of the form

$$[\mathrm{D}, \mathrm{J}_n^a] = n\mathrm{J}_n^a \qquad [\mathrm{J}_n^a, \mathrm{J}_m^a] = f_{ab}^c \mathrm{J}_{m+n}^c + \mathrm{K}ng^{ab}\delta_{m+n,0} \tag{7}$$

where D is a grading element, $f^c_{ab}$ are the structure constants of a semi-simple Lie group $G_o$, with Cartan metric $g^{ab}$, K is a constant, and $n$, $m$ are integers (multiples of $\frac{1}{2}$, $\frac{1}{3}$ in the twisted cases). The $J^a_o$ are the generators of the finite-dimensional Lie algebra $G_o$. In physical applications the value of K depends on the model.

*The Loop Group*

By considering the Fourier transform $J_a(\phi)$ for $0 \leqslant \phi < 2\pi$ of $J^a_n$ the AKM algebra (7) may be written in the form

$$[J_a(\phi), J_b(\phi')] = f^c_{ab} J_c(\phi)\delta(\phi - \phi') + K g_{ab}\delta'(\phi - \phi') . \qquad (8)$$

This form shows that for K = 0 the AKM algebras are the Lie algebras of the *loop-groups*, defined as groups whose elements lie in $G_o$ and whose parameters are functions $f_a(\phi)$ on the circle i.e.

$$g(f) = e^{\int d\phi f_a(\phi) J_a(\phi)} \in G_o \qquad J^a(\phi) = \frac{\delta g(f)}{\delta f_a(\phi)} , \qquad (9)$$

the group multiplication being the usual $G_o$ multiplication with $f$ and $g$ treated as parameters. Equation (9) also shows that for the opposite, abelian, case, when $f^c_{ab} = 0$ and $K \neq 0$, the AKM algebra reduces to the canonical commutation algebras (CCA) of free quantum fields. Thus the AKM algebras have a dual interpretation as central extensions of loop-group Lie algebras and as non-abelian extensions of CCAs. This is important for the representation theory because the lowest weight concept for AKM algebras is a combination of the lowest weights concept for finite-dimensional Lie groups and the vacuum for free quantum fields.

## 3. Automorphisms of the AKM algebras

The automorphism groups $\mathcal{A}$ of an algebra are often of great importance, and the AKM algebras are no exception, admitting at least two important automorphism groups, namely, the Virasoro and Weyl groups.

*Virasoro Automorphism*

The loop-group formulation of AKM's shows that they are diffeomorphic-invariant and since the group-elements are diffeomorphic scalars the AKM generators $J^a(\phi)$ are vectors and thus under infinitesimal diffeomorphisms $\delta\phi = \varepsilon(\phi)$ have the variations $\delta J^a(\phi) = \partial_\phi(\varepsilon(\phi) J^a(\phi))$. These variations are generated by the charges $Q_f = \int d\phi\, \varepsilon(\phi) L(\phi)$, provided the adjoint action of $L(\phi)$ on the $J_a(\phi)$ is

$$[L(\phi), J^a(\phi')] = \partial_{\phi'}\Big(J_a(\phi)\delta(\phi - \phi')\Big) . \qquad (10)$$

The integrability (Jacobi self-consistency) relations for (10) are

$$[L(\phi), L(\eta)] = \delta(\phi - \eta)L'(\phi) + 2\delta'(\phi - \eta)L(\phi) + c\delta'''(\phi - \eta) \qquad (11)$$

where $c$ is a constant. The algebra (11) is called the Virasoro algebra and is clearly a central extension of the Lie algebra of the diffeomorphic group. By multiplying (11) by $\varepsilon(\phi)$ and integrating one sees that the Virasoro generators $L(\phi)$ are diffeomorphic connections of rank two. We shall see later that the Virasoro automorphisms are inner in the sense that the $L(\phi)$ can be constructed from the $J_a(\phi)$. The Virasoro algebra first appeared in physics in the context of string theory [6], and it is of central importance in all two-dimensional conformal theories [7], where the Virasoro generators are actually the components of the energy-momentum tensor density. It follows from (10) that the AKM grading element D can be identified with $\int d\phi L(\phi)$.

*The Weyl Group*

The second important subgroup of AKM automorphisms is the Weyl subgroup, which is defined in the same way as for finite-dimensional Lie algebras, namely as the subgroup generated by reflections in the roots, or as the normalizer in $\mathcal{A}$ of the Cartan subalgebra. The difference is that the AKM Cartan subalgebra is defined as the linear span of $\{D, H_o, K\}$, where $H_o$ is the Cartan of $G_o$ and as this $l+2$-dimensional root-space has an indefinite metric the Weyl group is infinite-dimensional. In fact it is the semi-direct product of the Weyl group for $G_o$ and the Galilean-type transformations

$$L_n \rightarrow L_n + \vec{v}\cdot\vec{H}_n + \delta_{no}\frac{1}{2}v^2 k \qquad \vec{H}_n \rightarrow \vec{H}_n + \delta_{no}k\vec{v} \qquad E^\alpha_n \rightarrow E^\alpha_{n+\vec{v}\cdot\vec{\alpha}} \qquad n \neq 0 \tag{12}$$

where the paramaters $v_i$ are discrete, being restricted to the co-weight lattice in general, and to the co-root lattice in the case of inner automorphisms.

An interesting feature of the transformations (12) is that they can be implemented by the adjoint action of $e^{i\vec{v}\cdot\vec{X}}$ where the $X^i$ are new operators defined so that they commute with all the AKM generators except

$$[X^i, D] = H^i_o \quad \text{and} \quad [X^i, H^j_o] = K\delta_{ij}. \tag{13}$$

In string theory the operators $\vec{X}$ and $\vec{H}_o$ have a direct physical meaning as centre-of-mass coordinates and total momentum of the string respectively. They also play an important role in the so-called vertex construction of the $E^\alpha_n$ namely,

$$E^\alpha_n = \frac{\gamma^\alpha}{2\pi i}\oint \frac{dz}{z^{n+1}} V^\alpha_o(z)\Pi\left[V^\alpha_n(z)V^{-\alpha}_{-n}(z)\right]_{n>0} \tag{14}$$

where the integration is around the origin,

$$V^\alpha_o(z) = e^{i\tilde{\alpha}\cdot\vec{X}}\, z^{i\tilde{\alpha}\cdot\vec{H}_o} \qquad\qquad V^\alpha_n = e^{i\frac{z^n}{n}\tilde{\alpha}\cdot\vec{H}_n} \tag{15}$$

and the $\gamma^\alpha$ are elements of a Clifford algebra which generate the standard off-diagonal Chevalley structure constants $N_{\alpha\beta}$ of $G_o$. In string theory these vertex operators are used to construct the vertices of Feynmann graphs.

## 4. Lowest Weight Representations

The representation theory of the AKM and Virasoro algebras has been studied intensively in recent years but any detail [4] is far beyond the scope of this article. What I should like to stress here is the interplay between physics and mathematics in their study. For physics, because of the existence of the vacuum and the link between probabilities and inner-products, the interesting representations are the lowest weight, unitary representations. But these are the ones that most resemble the representations of finite Lie algebras and are therefore also the ones of primary interest for mathematics. Their natural modules are Fock spaces, obtained by defining a vacuum state $|\Lambda>$ by

$$J_o^{\alpha}|\Lambda> = 0 \quad \alpha > 0 \qquad \text{and} \qquad J_n^a|\Lambda> = 0 \qquad L_n|\Lambda> = 0 \qquad n > 0 . \tag{16}$$

and acting an it with monomials in $L_n$ and $J_n^a$ for $n < 0$.

Apart from the existence of null vectors, and the technical difficulty of the algebraic manipulations, the theory proceeds very much as it does for the highest and lowest weight unitary representations of non-compact Lie groups.

*Weyl Character Formula*

It will be recalled that for the unitary irreducible representations of highest weight $j_o$ of compact simple algebras the Weyl character formula is

$$\chi(\phi) = \mathrm{tr} e^{(\phi, H_o)} = \frac{\sum \varepsilon(w_o) e^{i\left(\vec{\phi},\, \vec{w}_o(j_o+\rho_o)\right)}}{\sum \varepsilon(w_o) e^{i\left(\vec{\phi},\, \vec{w}_o(\rho_o)\right)}} \tag{17}$$

where $j_o$ is the highest weight, $\phi$ are the parameters of the Cartan, the sum is over all Weyl reflections $w_o$ of parity $\varepsilon(w_o)$, and $\rho_o$ is half the sum of the positive roots. This formula generalizes almost without change to lowest weight AKM representations [4], the differences being that the Weyl reflections are in the $l+2$ dimensional (indefinite-metric) AKM root-space and that the $\exp(i\phi, w_o(\lambda_o))$ are replaced by formal exponentials $e(\lambda)$ satisfying $e(\lambda)e(\mu) = e(\lambda+\mu)$ that are invertible on the AKM Cartan.

## 5. Physical Path: Current Algebras

The physical path to AKM algebras originated in the discovery of the (flavour) SU(3) symmetry of the strong nuclear interactions. A major part of the success of SU(3)-flavour came from the fact that the weak and electromagnetic currents were linear combinations of the SU(3) Noether currents $j_\mu^a(x)$, which are vectors with respect to space-time (indices $\mu = 0, 1, 2, 3$) and with respect to the Lie algebra of SU(3) (indices $a = 1 \ldots 8$). Thus

$$[Q^a, j_\mu^b(x)] = f_{ab}^c j_\mu^c(x) \quad \text{where} \quad Q^a = \int d^3x j_o^a(x) \tag{18}$$

where the $Q^a$ and the $F^c_{ab}$ are the generators and structure constants of SU(3). Motivated by this success, it was proposed [8] by Gell-Mann that the relations (18) be generalized to local relations of the form

$$\begin{aligned}[j^a_o(x), j^b_o(y)] &= f^c_{ab} j^c_o(x)\delta^3(x-y) \\ [j^a_o(x), j^b_i(y)] &= f^c_{ab} j^c_i(x)\delta(x-y) + k\partial_i\delta(x-y)\end{aligned} \qquad x^o = y^o \tag{19}$$

where the $j_o$ and $j_i$ denote the time- and space-like components of the currents and $k$ is a constant. Integrations of (19) reproduce (18) and the SU(3) algebra. An interesting feature of (19) is the presence of a central term, called a Schwinger term [9]. This is a quantum effect due to the normal-ordering of the fields (whose spins determine the value of $k$).

In contrast to (18), in which the charges $Q^a$ are conserved, the relations in (19) are restricted to equal times and in this respect are quite limited. But, being phase-space rather than dynamical relations, they are universal and reliable.

Although the algebra (19) is defined on four-dimensional Minkowski space, the resemblance between it and the AKM algebras (8) is startling. In fact, when the dimensions are reduced to two, so that the current has only one space-like component $j_1$ in addition to its time-like component $j_o$, the light-like combinations $j_o \pm j_1$ of these currents become AKM algebras with opposite values $\pm k$ of the central charge.

*Testing Current Algebra: PCAC*

Relations similar to (19) are postulated for axial currents $j^5_\mu(x)$ also, and, in fact, the best tests of current-algebra have come from the chiral-isospin, or SU(2) × SU(2), version, which consists of the restriction of (18) to SU(2) plus the relations

$$[j^a_o(x), j^{5b}_o(y)] = \varepsilon_{abc} j^{5c}(x)\delta(x-y) \quad \text{and} \quad [j^{5a}_o(x), j^{5b}_o(y)] = \varepsilon_{abc} j^c(x)\delta(x-y). \tag{20}$$

The advantage of axial currents is that, at low energies, they can be approximated by the neutral pion fields $\pi(x)$, according to the partially conserved axial current (PCAC) relation $\partial_\mu j^5_\mu(x) = f\pi(x)$ where $f$ is a known constant. The decisive breakthrough for current algebra came in 1965 when it was shown [8] that an experimentally satisfactory value for the axial weak coupling constant $g_A$, could be obtained by sandwiching the second relation in (20) between proton states, using PCAC, and evaluating the resultant sum over intermediate states from known pion-nucleon scattering results. This breakthrough initiated a huge industry in which current algebra, the PCAC relation and S-matrix theory were combined to obtain many sum rules and predictions for low energy nuclear and hadronic scattering, most of them in excellent agreement with experiment.

*The Success of Failure: Anomalies*

There was one PCAC-current algebra prediction, however, that was in total disagreement with experiment, namely the prediction of a zero rate for the decay of a neutral pi-meson into two photons, $\pi^0 \to 2\gamma$. The resolution of this discrepancy led to a major discovery in quantum field theory, namely the existence of anomalies [9]. A careful analysis showed that axial vector current conservation was broken not only by PCAC but even by the electromagnetic field $F_{\mu\nu}$, through a divergence of the form

$$\partial_\mu j^5_\mu(x) = f\pi(x) + \hbar e^2 \varepsilon_{\mu\nu\tau\sigma} F^{\mu\nu} F^{\tau\sigma} \tag{21}$$

where $\hbar$ is Planck's constant and $e$ is the electromagnetic charge. The use of (21) in the computation of the pion decay-rate led to the correct experimental value. This result was surprising because (21) is purely quantum-mechanical and thus the resolution of the pion-decay problem provided direct experimental evidence that classical symmetries do not necessarily hold at the quantum level. The anomaly in (21), called the axial anomaly, solved some other problems also, concerned with the $\eta \to 3\pi$ decay rate and the origin of the U(1) axial symmetry violation in strong interactions. The use of the anomaly also supported the quark-based prediction of colour-SU(3) symmetry and predicted three colours, in agreement with present experiments.

It soon became clear that the axial anomaly was only the first of many, and that the origin of the anomalies lay in the quantization procedure itself. This was particularly evident in the path-integral formalism, where the appearance of anomalies could be traced to the measure. They occurred typically when no measure existed that respected all the classical symmetries. The details may be found in most modern text-books, but there are two points that should be emphasized here. The first point is that by 1969 current algebra had become so well-established that its failure was regarded as a major problem. The second point is that there is a feed-back in the sense that anomalies are very relevant for determining the centres of AKM and Virasoro algebras. For example, the Schwinger term in (19), which becomes an AKM centre in two dimensions, is an anomaly, and as we shall see, at least part of the Virasoro centre for conformal field theories is due to anomalies.

### *The Sommerfield Sugawara (SS) Construction*

Before leaving the subject of current algebra in four dimensions there is one other development worth mentioning, namely the construction of the energy-momentum tensor $T_{\mu\nu}$ from the currents. In most field theories $T_{\mu\nu}$ and the other Noether currents, such as the electromagnetic current, are local functions of the fields but are not local functions of each other. Inspired by current algebra, Sommerfield and Sugawara [10] proposed that in some circumstances $T_{\mu\nu}$ might be a local function of the currents, namely

$$T_{\mu\nu}(x) = \kappa J_\mu(x) J_\nu(x) \tag{22}$$

where $\kappa$ is a constant. The importance of this proposal was not immediately appreciated, particularly when it turned out that for (22) the usual four-dimensional QFT short-distance singularities are even more severe than usual [11]. Sommerfeld, in particular, was so aware of these difficulties that he delayed publication and lost some priority.

## 6. Down-Sizing to Two Dimensions

Had current algebras remained four-dimensional their intertwining with AKM algebras might have remained marginal. However, some parallel developments in physics led to a reduction from four to two dimensions. The first such development was string-theory [6], in which the world-line of a particle is replaced by a two-dimensional world-sheet, with a conformally-invariant Action. The second was the discovery of the importance of two-dimensional conformal theories for phase transitions [12]. These discoveries, together with advances in specific areas, such as statistical and integrable models, and the fact that two-dimensional conformal field theories provided a reliable testing-ground for the various Ansatze of the four-dimensional quantum field theory, (such as the existence of operator-product expansions) made two-dimensional conformal field theories a mainstream subject of investigation. As the relevant two-dimensional current algebras corresponded to conformally invariant theories, they split into the direct sums of one-dimensional (light-like) AKM algebras, as described on page 157.

*The Virasoro Algebra*

One of the most striking properties of two-dimensional conformal field theories is that the generators of the Virasoro algebra coincide with the components of the energy-momentum tensor density $T_{\mu\nu}(x)$. This happens because, in two-dimensional conformal field theories, the component $T^{\mu}_{\mu}$ of $T_{\mu\nu}$ is zero and the components $T_{++}$ and $T_{--}$ are chiral i.e. are functions of the light-like variables $x_{\pm} = x_o \pm x_1$ respectively. These chiral components are the Virasoro generators. Thus, in strong contrast to higher dimensions, where only a (small) finite number of moments of the energy-momentum tensor $T_{\mu\nu}$ generate space-time symmetries, here every component of $T_{\mu\nu}$ generates a space-time symmetry, and together they generate the whole conformal group.

The Sommerfield Sugawara construction of $T_{\mu\nu}$ was a particular beneficiary of the down-sizing to two dimensions because in two dimensions the short-distance singularities disappear. Thus the four-dimensional SS ugly-duckling became a two-dimensional swan, and today the SS construction is used extensively in both the physical and mathematical literature.

*Weyl Anomaly and Virasoro centre*

It was mentioned in the previous section that, for two-dimensional conformal field theories $T^{\mu}_{\mu} = 0$. Strictly speaking, this is true only at the classical level and in a flat background. In the quantum theory the diffeomorphic-invariant path-integral measures for fields of spin other than one-half are not conformally-invariant and this introduces an anomaly that violates the corresponding quantum relation $< T^{\mu}_{\mu} > \neq 0$, where the bracket denotes quantum mechanical expectation value. The effects of the anomaly can be cancelled, and the relation $< T^{\mu}_{\mu} > = 0$ recovered, by making some some subtle adjustments, called improvements, to the energy-momentum tensor. An elegant way to make the adjustments is to embed the theory in a curved background. Then at the classical level $T^{\mu}_{\mu} = 0$ is replaced by $T^{\mu}_{\mu} = c\mathrm{R}$, where $c$ is a constant and R is the scalar curvature, and at the quantum level the effects of the anomaly may be cancelled by a coupling the fields to the background metric in a suitable manner. When this is done, not only is a relation $< T^{\mu}_{\mu} > = c\mathrm{R}$ recovered but the constant $c$ may be identified as the central constant of the Virasoro algebra!

*Wess-Zumino-Witten (WZW) Actions and AKM Algebras*

The archetypal class of two-dimensional conformal field-theories is the Wess-Zumino-Witten (WZW) class [7] [13]. Indeed practically all two-dimensional conformal field theories are special cases of the WZW class or can be derived from it. The interesting property of the WZW theories is that *they have AKM algebras as symmetry algebras and their energy-momentum tensors are of the SS form.* Thus they are physical embodiments of the AKM algebras and the SS construction. The WZW Actions are

$$\mathcal{A} = \frac{k}{2}\int d^2\sigma\sqrt{g}g^{\mu\nu}\mathrm{tr}\left(J_\mu J_\nu\right) + \eta\frac{k}{3}\int d^3x\varepsilon^{\mu\nu\tau}\mathrm{tr}\left(J_\mu J_\nu J_\tau\right) \qquad J_\mu = h^{-1}\partial_\mu h \tag{23}$$

where $\sigma$ are the coordinates, and $g_{\mu\nu}(\sigma)$ is the metric, of the background Minkowskian two-space, $\eta = \pm 1$, $k$ is an integer-valued coupling constant and the fields $h(\sigma)$ take their values in a Lie group $G_o$. An unusual feature is the appearance of the three-dimensional integral, where the 3-space is understood to have the 2-space of interest as boundary. This integral is the winding number for the map $h(x)$ of the 3-space into $G_o$ and its variation is a total divergence that can be converted into a boundary term. The boundary term contributes to the two-dimensional field equations, and simplifies them to

$$\partial_+ J = 0 \qquad \partial_- \tilde{J}_+ = 0 \quad \text{where} \quad \tilde{J} = hJh^{-1} \qquad \sigma_\pm = \sigma_o \pm \sigma_1 \tag{24}$$

for $\eta = 1$ (and the same equations with J and $\tilde{J}$ interchanged for $\eta = -1$). Thus the field equations reduce to the statement that the currents are chiral i.e. depend only on $\sigma_\pm$ respectively, and their general solution is easily seen to be $h_s(\sigma) = l(\sigma_+)r(\sigma_-)$ where $l$ and $r$ are arbitrary matrices in $G_o$. The Action (23) is invariant under the transformations $h(\sigma) \to h_o h(\sigma)$ and $h(\sigma) \to h(\sigma)h_o$ where $h_o \in G_o$ is a constant matrix, and the currents J and $\tilde{J}$ are just the Noether currents for these symmetries. They commute, and, being Noether currents, they satisfy closed algebras, which turn out to be AKM algebras with central constants $\pm k$ respectively. The system is conformally invariant and the energy momentum tensor is given by the SS construction. In the path-integral formulation of the quantum version the diffeomorphic-invariant measure is $d(g^{\frac{1}{4}}h)$ where $g$ is the determinant of the background metric, and the factor $g^{\frac{1}{4}}$ produces a Weyl anomaly. This, in turn, produces the Virasoro central constant

$$c = \left(\frac{k}{k_\gamma}\right)\dim(G_o) \qquad k_\gamma = k + \hbar\gamma \tag{25}$$

where $\gamma$ is the Coxeter number of $G_o$. The renormalization of $k \rightarrow k + \hbar\gamma$ for non-abelian groups is due to a further anomaly, called the WZW anomaly.

*String Theory*

The first widely studied example of a two-dimensional conformal field theory was bosonic string theory [6]. As this theory is widely described in the literature we shall present here only two particularly relevant features. The path-integral takes the form

$$Z(j) = \int d\mu(g)\, d(g^{\frac{1}{4}} X)\, e^{\int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X_\mu(\sigma) \cdot \partial_\beta X_\mu(\sigma) + j_\mu(\sigma) X_\mu(\sigma)} \tag{26}$$

where the $\sigma$'s are coordinates in the two-dimensional space of the Action, the $X_\mu$ are a set of N scalar fields which are identified as the coordinates in an N-dimensional Minkowskian target space and the $j_\mu(\sigma)$ are external currents. The Action is an abelian version of a WZW Action but the theory differs from WZW in that the components of the metric are allowed to vary, and act as Lagrange multipliers. The Action is diffeomorphic and Weyl invariant but, as in the WZW theory, the factor $g^{\frac{1}{4}}$ in the measure produces a Weyl anomaly. This produces a Virasoro centre proportional to $(N - 26)$, where the 26 comes from two ghost fields associated with the Lagrange multipliers, and the existence of this anomaly means that the theory is conformally-invariant only in 26-dimensions. If the theory is made supersymmetric by introducing fermionic fields to match the bosonic ones the number of dimensions required to make it conformally-invariant reduces to 10. But in no case does the number reduce to the familiar 4 dimensions of everyday physics.

The other relevant feature of string theory is that the modern supersymmetric versions, in particular the heterotic (chirally asymmetric) string version, have AKM algebras as symmetry algebras. What happens in these theories is that 16 of the 26 bosonic coordinates are compactified to a torus, $X_i(\sigma) \rightarrow \phi_i(\sigma)$ for $0 \leqslant \phi_i(\sigma) < 2\pi$ and $i = 1 \ldots 16$, and the 32 Majorana-Weyl fermions which can be constructed from their normal-ordered exponentials, $\psi_a(\sigma) =: \exp(\phi_i(\sigma)):$, form the supersymmetric partners of the remaining ten bosons. The Action of the supersymmetric theory may be written as the sum of the bosonic Action (26) restricted to ten fields $X_i$ and the fermionic Action

$$\int d^2\sigma \bar{\psi}(\sigma)_a \gamma^\mu \partial_\mu \psi_a(\sigma) \qquad a = 1 \ldots 32 \tag{27}$$

It turns out, however, that the compactification and conversion to fermions is consistent only if the lattice that defines the 16-dimensional torus is even and self-dual. It is well-known that there are only two such lattices, called $D_{16}^+$ and $E_8 + E_8$, with automorphism groups $SO(32)$ and $E_8 \times E_8$ respectively. Accordingly, it is not surprising that these two groups are the Lie groups associated with the AKM algebras. The $SO(32)$ group corresponds to the case in which all the fermions have the same periodicity conditions and the AKM algebra is just the algebra generated by the Noether currents $\bar{\psi}(\sigma)\gamma_\mu \tau \psi(\sigma)$, where the $\tau$'s are the generators of $SO(32)$. The $E_8 \times E_8$ algebra corresponds to the case in which the fermions are divided into two sets of sixteen fermions with different periodicity conditions. In this case only the $SO(16) \times SO(16)$ subalgebra of $E_8 \times E_8$ is represented by Noether currents of the form just shown. The remaining currents are obtained from these by commutation with the generators of the rigid coset $E_8 \times E_8/SO(16) \times SO(16)$ group, the rigid generators being constructed in a rather complicated manner by a vertex construction similar to that given in [15].

*Statistical Mechanics*

Many of the standard models of 2-dimensional statistical mechanics such as the Ising model, the Pott's model and various tri-critical models, owe their solvability to the fact that they are conformally invariant, and in some cases AKM invariant. The advent of Virasoro and AKM algebras has allowed the treatment of such models to be simplified and extended.

A particularly interesting feature of these models is that the Virasoro central constant $c$ lies in the range $0 \leqslant c \leqslant 1$. It is known that for unitary representations of the Virasoro algebra with $c$ in this range, the values of $c$ are quantized in the sense that they are limited to certain national values. For this reason the models in question are called rational models.

An interesting feature is that although these values of $c$ seem to be unrelated to the values $c \geqslant 1$ obtained from WZW models and their AKM algebras in (25), they can, in fact, be related to them by the so-called *coset construction* [13]. This consists of taking a subalgebra H of an AKM algebra G and considering the differences $V_G - V_H$ of the Virasoro generators for G and H. Remarkably, these differences generate Virasoro algebras, whose central constants $c$ are just the *differences* of the constants for $V_G$ and $V_H$, and it turns out that all the rational constants $0 \leqslant c \leqslant 1$ can be constructed from such differences.

*W-Algebras as Constrained AKM Algebras*

Natural generalizations of [14] of the Virasoro algebras are the W-algebras, which are graded, differential polynomial algebras of the form

$$\{W_n(x), W_m(y)\} = \sum P_s\big(W(x)\big)(\partial_x)^r \delta(x-y) \tag{28}$$

where the base elements $W_n$ (apart from the Virasoro, $W_2$) are supposed to be primary fields of conformal weight $n$, $\partial_x$ and $\delta(x)$ each have weight 1, and $P_s$ is a polynomial in the W's of weight $s = m + n - r - 1$. Thus the W-algebras are grade-preserving. In general, W-algebras are difficult to find, but a large class of them may be obtained by constraining AKM algebras in the following way: Consider any SL(2,R), with standard generators $\{M_o, M_\pm\}$, embedded in the rigid Lie algebra $G_o$ of an AKM algebra, and apply the linear constraints $J_+(\phi) = M_+$ and $\tilde{J}_-(\phi) = M_-$ to the AKM generators, where the grading is with respect to $M_o$. These constraints are first-class in the sense of Dirac and thus generate a gauge-group. The W-algebras are then the Poisson-bracket (or commutator) algebras of the gauge-invariants induced by the AKM algebras.

## 7. Integrable systems

The WZW model is a trivial example of an integrable system, as is evidenced by the simple solution of the field equations. However, two-dimensional conformal field theory contains a much larger variety of integrable systems, and many of these, such as (abelian and non-abelian) Toda systems [15] and KdV hierarchies [16], are intimately connected with AKM algebras. For example, the Toda systems may be obtained from the WZW systems by applying the constraints described in the previous section to the AKM generators of the WZW systems. Thus the Toda systems may be regarded as constrained AKM systems and there are many advantages of regarding them in this way. For example, the reason that Toda theories are associated with Cartan-Killing matrices becomes obvious, the general Toda solutions are easily obtained by algebraic reduction of the (trivial) WZW solutions, and the symmetry algebras of the Toda systems are immediately seen to be the W-algebras.

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