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DIOPHANTINE APPROXIMATIONS AND FOLIATIONS

by Michael McQUILLAN

Abstract. — In this paper we indicate the proof of an effective version of the Green-Griffiths conjecture for surfaces of general type and positive second Segre class (i.e. $c_1^2 > c_2$). Naturally this effective version is stronger than the Green-Griffiths conjecture itself.

0. Introduction

Let X be a regular projective scheme over a base S, and H an ample divisor on X. Ultimately we will only be interested in S = Spec **C**, but for the moment we allow the possibility that S = Spec $\mathcal{O}$ where $\mathcal{O}$ is the ring of integers in an algebraic number field or indeed a function field. Our interest will concern maps $f: Y \to X$ where Y has several possible flavours. If S = Spec **C** then Y may be either a finite ramified cover of $\mathbf{P}^1$ or even a finite ramified cover $p: Y \to \mathbf{C}$, while if S = Spec $\mathcal{O}$ then of course Y = Spec $\mathcal{O}'$ where $\mathcal{O}'$ is a finite integrally closed extension of $\mathcal{O}$. In any case if $\omega_{X/S}$ is the dualising sheaf then subject to suitable metricisation, and according to the flavour we may measure the heights $h_{\omega_{X/S}}(f)$, $h_H(f)$ of f with respect to $\omega_{X/S}$ and H respectively (1). Equally we may also measure the ramification of f over a suitable base, i.e. its so called discriminant $d(f)$, in the case where Y is a smooth curve this is just the degree of the canonical bundle of Y plus twice the degree of the cover, while in the arithmetic case it would be the logarithm of the discriminant of the appropriate number field (2).

With these preliminaries in mind we may state the relevant conjecture of Lang, Vojta et al. (cf. [VI]) regardless of the flavour:

Conjecture **0.0.** — *There exists a constant* α *and a proper subscheme* Z *of* X *such that for any* f *whose generic point does not lie in* Z *we have*

$$h_{\omega_{X/S}}(f) \leqslant \alpha d(f) + o(h_H(f)),$$

where the " little o *" term is used to indicate that the error term tends to zero as the height of* f *goes to infinity.*

(1) Note when we are in the case of a finite ramified cover over **C**, this must all be done at a fixed radius $r > 0$, i.e. if $Y(r) = p^{-1}\{z : |z| \leqslant r\}$ then what we measure is the height of $f: Y(r) \to X$. The appropriate definitions will be given in I, § 0.2.

(2) The same caveat as previously applies for ramified covers of the complex line.

The said conjecture is therefore the so-called effective or hard version of what are termed problems of Mordell type (arithmetic case) or Green-Griffiths type (incomplete curve case), i.e. observe that if $X \otimes k(S)$ has general type then the conjecture implies that the rational points of X cannot be Zariski dense or in its analytic flavour that no holomorphic map $f: \mathbf{C} \to X$ can have a Zariski dense image.

Let us recall that if $S = \operatorname{Spec} \mathbf{C}$ and X, Y are both smooth curves then the conjecture is a trivial exercise (i.e. the Riemann-Hurwitz theorem). Equally for X still a curve but Y now a ramified cover of $\mathbf{C}$ then the conjecture is basically Nevanlinna's Second Main Theorem (cf. [L-C]), and is already significantly less trivial, while in the arithmetic case, i.e. $X \otimes k(S)$ a curve, an elegant proof for $k(S)$ a field of functions may be found in the work of Vojta, cf. [V2], however for $\mathcal{O}$ a ring of algebraic integers there are no known non-trivial cases, and indeed if it were known for even one curve of genus at least two then it would imply the a, b, c-conjecture. In higher dimensions we may certainly note that for Y a complete or incomplete curve the conjecture is known if X has ample cotangent bundle, or if X is a subvariety of an abelian variety (1). Otherwise the single non-trivial higher dimensional case which is known for any of the flavours is the following theorem of Bogomolov (cf. [B1] or [D]) viz:

Theorem **0.1** (Bogomolov). — *Let* $X/\mathbf{C}$ *be a smooth projective surface of general type with* $s_2(X) > 0$ (i.e. $c_1(X)^2 > c_2(X)$); *then there exist constants* α *and* β *such that for any map* $f: Y \to X$, *where* Y *is a smooth complete curve, we have*

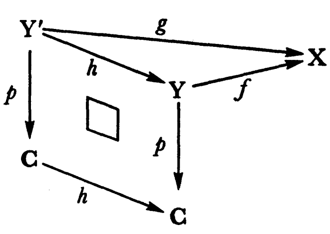
$$h_{K_X}(f) \leqslant \alpha\, d(f) + \beta.$$

Our goal will be to generalise this to the case of Nevanlinna theory and to prove a semi-effective version of the Green-Griffiths conjecture for such surfaces, i.e.

Theorem **0.2.** — Let $X/\mathbf{C}$ *be a smooth projective surface of general type with* $s_2(X) > 0$; *then there exists a proper subvariety* Z *of* X *such that if* $Y \xrightarrow{f} X$ (2) *is a finite ramified cover of* $\mathbf{C}$

$$\begin{array}{c} Y \xrightarrow{f} X \\ \downarrow p \\ \mathbf{C} \end{array}$$

mapping to X, *but not contained in* Z, *then after a possible base extension,*



(1) We are not sure that this is actually proved elsewhere in the case of Nevanlinna theory, but we give a proof in the spirit of this article in [M3].

(2) Throughout this introduction, and indeed throughout the manuscript, this is to be understood wherever it is written under the assumption that the ramification of f and p are mutually disjoint, and indeed both disjoint from the fibre over zero. This evidently involves little or no loss of generality.

where $h \in \Gamma(\mathcal{O}_{\mathbf{C}})$ *is étale, and* $g = h^* f$, *there is a constant* α, *depending on* g, *such that*

$$h_{\mathrm{K}_\mathrm{X}}(g(r)) \leqslant \alpha\, d(g(r)) + o(h_{\mathrm{H}}(g(r)))$$

for all $r \in \mathbf{R}$, *outside a set of finite Lebesgue measure.*

We make a couple of remarks. In the first place observe that the Green-Griffiths conjecture for such surfaces (i.e. there is no holomorphic map $f : \mathbf{C} \to \mathrm{X}$ with Zariski dense image) is a trivial corollary, but the theorem is strictly stronger than this; indeed it is stronger than the analogue in Nevanlinna theory of asking that the curves of a fixed genus on such surfaces are bounded in moduli (which is how Bogomolov's theorem is stated, although in turn it is stronger than this) as opposed to simply knowing that there are finitely many rational or elliptic curves on the surface. The étale base extension in the theorem is necessitated by the methodology, since without it we could do no better than an error term of the form $O(rh_{\mathrm{H}}^{1/2+\varepsilon}(f(r)))$ although if the surface has positive index, i.e. $\tau(\mathrm{X}) = \dfrac{c_1(\mathrm{X})^2 - 2c_2(\mathrm{X})}{3} > 0$, then we have

Theorem **0.3.** — *Let* $\mathrm{X}/\mathbf{C}$ *be a smooth projective surface of general type and positive index then there exists a proper subvariety* Z *and* $\alpha > 0$ *such that, if* $\begin{array}{ccc} \mathrm{Y} & \xrightarrow{f} & \mathrm{X} \\ \downarrow{\scriptstyle p} & & \\ \mathbf{C} & & \end{array}$ *is a finite ramified cover of* $\mathbf{C}$ *mapping to* X *but not contained in* Z, *then*

$$h_{\mathrm{K}_\mathrm{X}}(f(r)) \leqslant \alpha\, d(f(r)) + O(\log r + \log h_{\mathrm{H}}(f(r))).$$

The stated inequality being valid for all r outside a set of finite measure depending on f, and all implied constants being effective.

This latter theorem is in fact a trivial consequence (modulo work of Bogomolov and Miyaoka, cf. [B2] and [Mi1]) of our tautological inequality which as well as being a key step towards Theorem 0.2 is of independent interest, viz

Theorem A. — *Let* X *be a smooth projective variety,* $\begin{array}{ccc} \mathrm{Y} & \xrightarrow{f} & \mathrm{X} \\ \downarrow{\scriptstyle p} & & \\ \mathbf{C} & & \end{array}$ *a map from a ramified cover of the complex line and* $f' : \mathrm{X} \to \mathbf{P}(\Omega^1_{\mathrm{X}})$ (¹) *it's derivative with* $\mathcal{O}_{\mathbf{P}(\Omega^1_\mathrm{X})}(1)$ *the tautological bundle, then*

$$h_{\mathcal{O}_{\mathbf{P}(\Omega^1_\mathrm{X})}(1)}(f'(r)) \leqslant d(f(r)) + O(\log r + \log h_{\mathrm{H}}(f(r))),$$

where again the stated inequality is valid for all r outside a set of finite measure depending on f, and all the implied constants are effective.

(¹) Throughout we employ the conventions of E.G.A. on projective tangent bundles, as they seem to us technically advantageous.

Now the method of proof of Theorem A [1] is our method of dynamic diophantine approximation in Nevanlinna theory as introduced in [M1], and based on ideas in Faltings, [F], and Vojta, [V3]. Its proof will be the object of Chapter I, wherein some further introduction will be found. Although to our knowledge Theorem A implies all known " Main Theorem Type " results for ramified covers of the complex line, it is still far from proving Theorem 0.2, for which we need

Theorem B. — *Let* X *be a surface of general type and* $\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow{\scriptstyle p} & & \\ \mathbf{C} & & \end{array}$ *a map from a finite ramified cover of the complex line with Zariski dense image, which is a leaf of a foliation on* X. *Then after a possible étale base change à la Theorem* 0.2, *there is a constant* α, *depending on* g, *such that*

$$h_{K_X}(g(r)) \leqslant \alpha\, d(g(r)) + o(h_H(g(r))),$$

where again $r \in \mathbf{R}_{\geqslant 0}$ *is excluded from a set of finite measure depending on* f.

We remark that the constant α is shown to exist by the Hodge-Index Theorem, and therefore seems by this method to be hopelessly ineffective [2]. In any case along with the aforesaid method of dynamic diophantine approximation, the essential ingredients are the " semi-positivity " of the cotangent bundle as demonstrated by Miyaoka, and a theory of " residual heights " along foliations. More will be said in the introduction to Chapter II, wherein Theorem B will be proved.

It remains to thank Dan Abramovich, Jean-Benoît Bost, Fabrizio Catanese, Alain Connes, Ofer Gabber, Nick Shepherd-Baron and Paul Vojta for their help, and of course Cécile for her typing. We apologize in advance to those who expected positive characteristic arguments in this paper, but they are implicit in the use of the above theorem of Miyaoka—which we failed to improve on.

I. DYNAMIC DIOPHANTINE APPROXIMATION

0. Introduction

0.1. *Relation with arithmetic*

Classically in arithmetic the method of diophantine approximation was considered as proceeding along the following lines, viz

Suppose a variety X has more rational points than expected, then one finds some sort of polynomial F and points $x_1, \ldots, x_m$ such that $F(x_1, \ldots, x_m) = 0$ for some purely

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(1) Noguchi has brought to our attention that a weak version of Theorem A was already proved by himself in [No].

(2) Added in proof: M. Brunella, *Courbes entières et feuilletages holomorphes*, has communicated to us some simplifications of our argument which in fact permit a best possible determination of α, i.e. $1 + \varepsilon$.

formal reason, then do a lot of hard work to show that this does not happen. Viewed from the perspective of Arakelov theory the process is clearer, the choosing of the polynomial F corresponds to a line bundle L, say, on some auxiliary variety W associated to X (e.g. in the work of Faltings and Vojta, cf. [F], [V3] and [V4], $W = X^m$, and L is cooked up with the help of the so-called Poincaré bundles), further one has a point $w \in W$ by virtue of the points on X, and for purely formal reasons one knows that the intersection $\hat{c}_1(L) \cap w$ is negative. Subsequently one then does the hard work to produce sections on L or something close to L which do not vanish at w, whence obtaining a contradiction.

Ultimately the diophantine approximation is just an application of the celebrated principle of calculating the same thing in two different ways, with a view to proving a non-trivial theorem (or proving that Wittgenstein was right all along, depending on your point of view). The particular thing which we are calculating being an Arakelov intersection number.

On the other hand, given a line bundle L on a variety X and a curve Y one might hope to calculate the intersection number $c_1(L) \cap Y$ by any manner of means, e.g. a formal calculation such as adjunction, a dynamic intersection (cf. [Fu])—i.e. one finds a family of curves $Y_t \to Y$ and proceeds to justify a " principle of continuity ", viz: $c_1(L) \cap Y_t \to c_1(L) \cap Y$—or some mélange of these such as deformation to the normal cone.

Now it would be nice to believe that dynamic intersection may be possible in Arakelov theory, though currently the theory is purely static, i.e. there is no moving lemma, and naive efforts at a " principle of continuity " will certainly flounder due to the obstruction posed by Ullmo's Theorem, cf. [U]. However if we instead turn our attention to Nevanlinna theory, then it is not unreasonable to believe that a " principle of continuity " will be justifiable, and indeed we find that it is, though how universal it may be we are not sure since we tend to justify it as needs require. Evidently due to the presence of " metricised terms at infinity ", justifying the said principle will be less straightforward than that for complete curves. Ultimately, however, this method appears to be at its most powerfull in studying problems on defect relations, cf. [M2], while for the purposes of proving Theorem A we have a quasi-formal calculation based on the deformation to the normal cone. Nevertheless, we present both approaches.

0.2. *Notations and generalities*

Throughout, $X/\mathbf{C}$ will be a smooth projective complex variety, H an ample line bundle on X, $p : Y \to \mathbf{C}$ a finite ramified cover and $f : Y \to X$ a holomorphic map. For purely technical reasons (cf. 1.1) we impose the condition that the ramification loci of f and p are pairwise disjoint, and indeed both disjoint from $p^{-1}(0)$.

Now let $\overline{\mathrm{Pic}}(X)$ be the group of metricised Cartier divisors, i.e. Cartier divisors with metric, while for $r \in \mathbf{R}_+$, let $Y(r) := p^{-1}\{\Delta_r\}$, where $\Delta_r \subset \mathbf{C}$ is the disc of radius r. We have:

Definition **0.2.1.** — *Let* $\overline{\mathrm{D}} \in \overline{\mathrm{Pic}}(\mathrm{X})$. *Then we define the* height $h_{\overline{\mathrm{D}}}(f(r))$ *of* f *at* r *to be,*

$$h_{\overline{\mathrm{D}}}(f(r)) := \int_0^r \frac{dt}{t} \int_{\mathrm{Y}(t)} f^* \, c_1(\overline{\mathrm{D}}),$$

where $c_1(\overline{\mathrm{D}}) \in \mathrm{A}^{1,1}(\mathrm{X})$ *is the usual chern class. Equally we will employ the classical notation* $\mathrm{T}_{f,\overline{\mathrm{D}}}(r) = h_{\overline{\mathrm{D}}}(f(r))$ *if it is computationally convenient, or even* $\mathrm{Y}(r) \cdot_f \overline{\mathrm{D}}$, *should the desire to be more geometrical take us.*

Observe that the height is trivially additive with respect to metricised divisors and that we have the so called Green-Jensen formula of Nevanlinna, i.e. if s is a meromorphic section of $\mathcal{O}_{\mathrm{X}}(\mathrm{D})$, and η indicates a local coordinate about any $y \in p^{-1}(0)$, then, provided that f is contained in neither the zeros or poles of s,

$$\textbf{(0.2.2)} \qquad h_{\overline{\mathrm{D}}}(f(r)) = \sum_{0<|p(y)|<r} \mathrm{ord}_y(f^* s) \log \left|\frac{r}{p(y)}\right| - \frac{1}{2}\int \log \|f^* s\|^2 p^*\left(\frac{d\theta}{2\pi}\right)$$
$$+ \log r \sum_{y \in p^{-1}(0)} \mathrm{ord}_y(f^* s) + \frac{1}{2} \sum_{y \in p^{-1}(0)} \lim_{\eta \to 0} \log \left\{ \frac{\|f^* s\|^2}{|\eta|^{2\,\mathrm{ord}_y(f^* s)}} \right\}.$$

In particular if D is effective then $h_{\overline{\mathrm{D}}}(f(r)) \geqslant \mathrm{O}_{\overline{\mathrm{D}}}(1)$, i.e. a constant depending on $\overline{\mathrm{D}}$, while if $\overline{\mathrm{D}}$, $\overline{\overline{\mathrm{D}}}$ correspond to different metrics on the same underlying Cartier divisor then in fact, $|h_{\overline{\mathrm{D}}}(f(r)) - h_{\overline{\overline{\mathrm{D}}}}(f(r))| \leqslant \mathrm{O}(1)$, the implied constant being dependent on $\overline{\mathrm{D}}$ and $\overline{\overline{\mathrm{D}}}$. Whence unless there is cause to worry about it, any dependence on the metric will be notationally ignored.

Similarly we associate a discriminant to f as follows:

Definition **0.2.3.** — *Notations as above; then*

(i) $\mathrm{N}_{f,\mathrm{Ram}}(r) := \sum_{0<|p(y)|<r} \mathrm{ord}_y(\mathrm{R}_f) \log \left|\frac{r}{p(y)}\right|,$

(ii) $\mathrm{N}_{p,\mathrm{Ram}}(r) := \sum_{0<|p(y)|<r} \mathrm{ord}_y(\mathrm{R}_p) \log \left|\frac{r}{p(y)}\right|,$

(iii) $d(f(r)) := \mathrm{N}_{p,\mathrm{Ram}}(r) - \mathrm{N}_{f,\mathrm{Ram}}(r),$

where R_f, R_p *are the ramification divisors of* f *and* p *respectively.*

Remark **0.2.4.** — One ought to think of $f(\mathrm{Y}(r)) \subset \mathrm{X}$ as the object of our study, hence the appearance of the f ramification in our discriminant, rather than just the p-ramification as found in arithmetic.

Now observe that if $\mathbf{P}(\Omega^1_{\mathrm{X}})$ is the projective tangent bundle of X, then we have the derivative, $f' : \mathrm{Y} \to \mathbf{P}(\Omega^1_{\mathrm{X}})$ lifting f, thus we may state:

Theorem **0.2.5** (Tautological Inequality). — *For* $r \in \mathbf{R}$ *outside a set of finite measure,*

$$h_{\mathcal{O}_{\mathbf{P}(\Omega^1_{\mathrm{X}})}(1)}(f'(r)) \leqslant d(f(r)) + \mathrm{O}(\log h_{\mathrm{H}}(f(r)) + \log r).$$

Remark **0.2.6.** — The implied constant is effective, about 4 will do, plus some constant terms which may be evaluated as the leading terms of appropriate Taylor series.

0.3. *Some estimates*

Although the following will play no role in this chapter—rather they will arise naturally in the context of foliations—this seems the appropriate place to consider height estimates of boundary forms on X. We begin with a definition.

Definition **0.3.1.** — *For* $r > 0$ *and* $\eta \in A^2(X)$ *we define the* height transform:

$$T_f(r)\,(\eta) := \int_0^r \frac{dt}{t} \int_{Y(t)} f^*\,\eta.$$

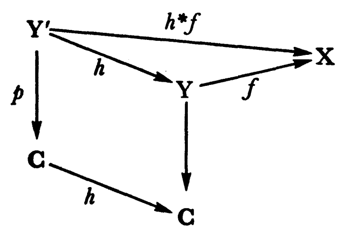
Remarks **0.3.2.** — (i) Evidently $T_f(r)$ may be viewed as a current on X.

(ii) We will employ the same notations and conventions even if η is simply l^2, provided that the height transform exists.

Now unlike the current of integration associated to a complete curve, $T_f(r)$ is certainly not closed, so its value on boundaries may be a priori quite big. However observe that if η is $(1, 1)$, d-closed, and a ∂ or $\bar{\partial}$-boundary, then in fact $\eta = dd^c\varphi$ for some smooth φ on X (cf. [So]), so that $|\,T_f(r)\,(\eta)\,|$ is bounded by a constant independent of r. The goal of this section will be to show that—under suitable hypothesis—such a result holds more generally if we only know that η is a ∂ or $\bar{\partial}$ boundary.

We suppose in what follows that f does not factor through a complete curve on X; then our estimate is

Proposition **0.3.3.** — *There exists an étale map* $h \in \Gamma(\mathcal{O}_{\mathbf{C}})$ *and a set* $E \subset \mathbf{R}$ *of finite measure such that, if* $\varphi \in A^{1,0}(X)$ *and*



is the corresponding base change, then

$$|\,T_{h^*f}(r)\,(\bar{\partial}\varphi)\,| \leqslant o(T_{h^*f,H}(r)), \quad \forall\, r \notin E.$$

Remarks **0.3.4.** — (i) Although h and E are independent of φ, it is clear that the "little o" term depends on φ.

(ii) Equally we have the same thing for $\partial\psi$, where $\psi \in A^{0,1}(X)$.

Proof. — We confine ourselves to the case $Y = \mathbf{C}$ for simplicity of notation, and begin by calculating $T_f(r)\ (\bar{\partial}\varphi)$ in order to establish what sort of h we ought to look for.

Since $\mathbf{C}$ is of dimension 1, Stokes' theorem gives:

$$T_f(r)\ (\bar{\partial}\varphi) = \int_0^r \frac{dt}{t} \int_{\partial D_t} f^* \varphi.$$

Now if we write $f^* \varphi = \Phi . dz$ for some smooth Φ on $\mathbf{C}$ and further choose a metric on H such that $\bar{c}_1(H)$ is a positive (1, 1)-form, then we may define $|f'|^2$ by the formula

$$f^* \bar{c}_1(H) = |f'|^2 \left\{ \frac{dz \wedge d\bar{z}}{-2\pi i} \right\}$$

and we conclude by the compactness of X that

$$|\Phi| \ll |f'|,$$

from which we compute that

$$|T_f(r)\ (\bar{\partial}\varphi)| \ll \int_{D_r} |f'|\ dt . d\theta;$$

on the other hand we observe that

$$T'_{f,H}(r) = \frac{1}{r} \int_{D_r} |f'|^2 \frac{t\, dt\, d\theta}{\pi}.$$

So that by Cauchy-Schwarz we have

(0.3.5) $$|T_f(r)\ (\bar{\partial}\varphi)| \ll \{ r T'_{f,H}(r) \}^{1/2} . (\log r)^{1/2} + O(1),$$

where all the implied constants depend on φ. On the other hand $T_{f,H}(r)$ is increasing, so by [L-C] (I.3.1) (or cf. 2.3.5-2.3.6) we know that there is a set of finite measure $E \subset \mathbf{R}$ such that

$$T'_{f,H}(r) \leqslant T_{f,H}(r) \{ \log T_{f,H}(r) \}^2, \quad \forall\, r \notin E,$$

so we obtain

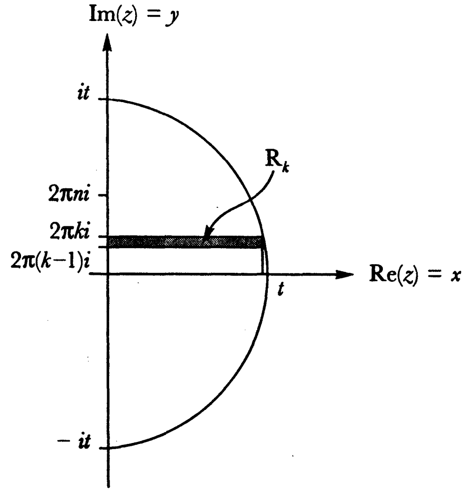
$$|T_f(r)\ (\bar{\partial}\varphi)| \ll T_{f,H}(r)^{1/2} \log T_{f,H}(r)\ (r \log r)^{1/2},$$

for all r outside a set of finite measure depending only on f. Consequently the proposition will follow on showing that we may make a good choice of h such that the order of growth of $T_{h^* f, H}(r)$ is sufficiently large. Intuitively $h = \exp(\exp(z))$ ought to be more than adequate, and this is indeed true. We give a few details:

Fact **0.3.6.** — *Hypothesis on f as given, then*

$$T_{\exp^* f, H}(r) \gg T_{f,H}(\exp(r/2)) + O(1).$$

Proof. — Put $V(r) := \int_{D_r} f^* \bar{c}_1(H)$, then we seek to estimate $\int_{D_t} \exp^* f^* \bar{c}_1(H)$ in terms of $V(t)$ for t sufficiently large; about bigger than 8π will be perfectly adequate. Let us put $n = \left[\frac{t}{4\pi}\right]$ and, for $1 \leqslant k \leqslant n$, consider the following diagram which ought to illuminate our strategy:



On the boundary of the region R_k as indicated in the diagram we have that

$$x \geqslant \sqrt{\{16\pi^2 n^2 - 4\pi^2 n^2\}} > t/2$$

whence
$$\int_{R_k} \exp^* f^* \bar{c}_1(H) \geqslant \int_{A(\exp(t/2),\,1)} f^* \omega,$$

where $A(\exp(t/2), 1)$ is the annulus $\{ z : 1 \leqslant |z| \leqslant \exp(t/2) \}$. Consequently,

$$\int_{D_t} \exp^* f^* \bar{c}_1(H) \geqslant 2 \sum_{k=1}^{n} \int_{R_k} \exp^* f^* \bar{c}_1(H) \geqslant t \int_{A(\exp(t/2),\,1)} f^* \omega$$
$$= t\{V(\exp(t/2)) - V(1)\}.$$

However by hypothesis f does not factor through a complete curve and so $V(t) \to \infty$ as $t \to \infty$, whence for t sufficiently large $\frac{1}{2} V(\exp(t/2)) > V(1)$ and we obtain

$$T_{\exp^* f, H}(r) \geqslant T_{f, H}(\exp(r/2)) + O(1),$$

as required.

Necessarily, $h(z) = \exp(\exp(z))$ is more than adequate for our purposes, since by hypothesis no multiple of $\log r$ can bound $T_{f,H}(r)$. Consequently we could even take $\exp(z^2)$, and allow for a little extra-ramification.

Remarks **0.3.7.** — (i) The proof works equally well as an estimate for integrals of the form

$$\int_0^r \frac{dt}{t} \int_{\partial D_t} f^* \varphi,$$

where φ is a 1-form on X whose coefficients may be bounded on a finite number of polydiscs, and will indeed be so employed.

(ii) A priori one might express philosophical doubts about using such a cheap trick as reparametrisation by double exponential on the left in order to deduce anything remotely useful in the context of the Green-Griffiths conjecture. More precisely if we were to believe the analogue in Nevanlinna theory of Manin's conjectures on the growth rates of rational points (cf. [L1]), then the growth rates of incomplete curves— "suitably normalised "—ought to give us much information about the effective cone in $NS_1(X)$. However from the latter point of view we are interested in deducing algebraic structure from the growth rates, while what actually concerns us is deducing something about rates of growth from algebraic structure. Consequently there is nothing paradoxical about re-parametrising in this way. Indeed the arithmetic analogue would be simply to take a sparse subset of an infinite set of rational points.

0.4. *Lelong numbers, currents and cohomology*

We proceed to refine some of the notations introduced in the previous section, with a view to their eventual application in the chapter on foliations. In particular $f: Y \to X$ will be assumed to be suitably re-parametrized in order that 0.3.3 is satisfied, and any subsets of the real line appearing will be supposed to exclude a specified set of finite measure, certainly containing that indicated in the said proposition. Our starting point is:

Definition **0.4.1.** — *Let* $z \in X(\mathbf{C})$ *with* $f^{-1}(z) \cap p^{-1}(0) = \emptyset$, $\{ x_i \}$ *a local coordinate system at* z, *and* $\varepsilon > 0$. *Then we define*

$$\mathscr{V}_{f,z}(\varepsilon, r) = \frac{1}{\varepsilon^2} T_f(r) \{ \mathbf{1}_\varepsilon \, dd^c \, \| x \|^2 \},$$

where $\| x \|^2 = \sum_i | x_i |^2$, *and* $\mathbf{1}_\varepsilon$ *is the characteristic function of the disc* $\{ \| x \| < \varepsilon \}$.

Remark **0.4.2.** — Now although a priori the definition appears to depend on the choice of local coordinates the standard comparison theorems for Lelong numbers, cf. [De1], will subsequently justify this omission from the notations.

In any case if we define a counting function à la 0.2.3 for f at z by taking the corresponding order function as that for the exceptional divisor E of the blow-up $\pi : \widetilde{X} \to X$ of X in z, then a standard integration by parts yields

$$(0.4.3) \qquad \mathscr{V}_{f,z}(\varepsilon, r) = N_{f,z}(r) - \frac{1}{2}\int_{\partial Y(r)} f^*\left\{ \mathbf{1}_\varepsilon \log \frac{\|x\|^2}{\varepsilon^2}\right\} p^*\left(\frac{d\theta}{2\pi}\right)$$
$$- \frac{1}{2}\int_{\partial Y(r)} f^*\left\{ \mathbf{1}_\varepsilon \left(1 - \frac{\|x\|^2}{\varepsilon^2}\right)\right\} p^*\left(\frac{d\theta}{2\pi}\right) + T_f(r)\{\mathbf{1}_\varepsilon\, dd^c \log \|x\|^2\}.$$

Now let us step back a little, and view the $T_f(r)$ (resp. $T_{\tilde f}(r)$, where $\tilde f$ lifts f to $\widetilde{X}$) as currents on X (resp. $\widetilde{X}$). Moreover in the weak topology $\Phi(r) := \dfrac{1}{T_{f,H}(r)}\, T_f(r)$ $\Big($resp. $\widetilde{\Phi}(r) := \dfrac{1}{T_{f,H}(r)}\, T_{\tilde f}(r)\Big)$ is a family of bounded currents, so that for any sequence $r_n \to \infty$, we may find a subsequence $R = (r_m)$ such that $\Phi(r_m)$ (resp. $\widetilde{\Phi}(r_m)$) converges weakly to Φ_R (resp. $\widetilde{\Phi}_R$), say. In addition 0.3.3 being satisfied implies that Φ_R (resp. $\widetilde{\Phi}_R$) is a closed positive current, and we have:

Proposition **0.4.4.** — *If* $\nu(z, \Phi_R)$ *denotes the Lelong number of* Φ_R *at* z, *then*

$$\nu(z, \Phi_R) = \lim_{r_m \to \infty} \frac{T_{\tilde f}(r_m)\,(\bar{c}_1(E))}{T_f(r_m)\,(\bar{c}_1(H))} + \lim_{\varepsilon \to 0} \lim_{r_m \to \infty} \frac{T_f(r_m)}{T_{f,H}(r_m)} (\mathbf{1}_\varepsilon\, dd^c \log \|x\|^2).$$

Proof. — Immediate from 0.4.3. □

Remarks **0.4.5.** — (i) The quantity $\lim \dfrac{T_{\tilde f}(r_m)\,(\bar{c}_1(E))}{T_f(r_m)\,(\bar{c}_1(H))}$ deserves, from an intersection theoretic point of view, to be called the *multiplicity* of f at z. Unfortunately, in this more general context, we cannot seems to prove an analogue of the classical " Lelong number = multiplicity " type theorem. This inevitably leads to complications.

(ii) In the Lelong number minus multiplicity error term, it is intended that it should be understood naively as a height transform, not as a current.

The next step is to specialize to X a surface, and to give a more geometric interpretation of the error term. In particular we suppose that H is in fact very ample, so that there is a natural semi-positive metric on $\pi^* H \otimes E^\vee$ deduced from a Fubini-Study type metric on H, and $A = \pi^* H^{\otimes 2} \otimes E^\vee$ is very ample. Whence, on using the above recipe to metricize A we may consider the measure $\|\widetilde{\Phi}_R\| := \widetilde{\Phi}_R \wedge \bar{c}_1(A)$ on $\widetilde{X}$ and deduce:

Proposition **0.4.6.** — *Notations and suppositions as given, then*

$$\lim_{\varepsilon \to 0} \lim_{r_m \to \infty} \frac{T_f(r_m)}{T_{f,H}(r_m)} (\mathbf{1}_\varepsilon\, dd^c \log \|x\|^2) = \int_{\widetilde{X}} \mathbf{1}_E\, d\|\widetilde{\Phi}_R\| = \int_E \nu(t, \widetilde{\Phi}_R)\, \mu(dt)$$

where the implied measure on E *is precisely that induced by* $\bar{c}_1(A)$.

Proof. — The first equality is basically the definition, while the second is a straightforward application of the dominated convergence theorem and Fubini's theorem. □

Finally we close this section by making some simple remarks on the functoriality of these definitions. To achieve this we will assume that f has Zariski dense image, and denote by $\beta \in \mathrm{NS}^1(\mathrm{X}; \mathbf{Q})$ the class of a big rational divisor, whence we define:

Definition **0.4.7.** — *A sequence* $\mathrm{R} = (r_n) \subset \mathbf{R}_{\geqslant 0}$ *is said to be convergent for the pair* (f, β) *if the weak limit of currents* $\dfrac{1}{\mathrm{T}_f(r_n)(\beta)} \mathrm{T}_f(r_n)$ *exists.*

Remarks **0.4.8.** — (i) In light of our assumptions regarding the height of f along the boundary of 1-forms, the normalizing factor $\mathrm{T}_f(r_n)(\beta)$ is essentially independent of our choice of lifting of β to a (1, 1)-form.

(ii) The height of f along β certainly dominates a multiple of the height along an ample bundle, by virtue of the assumption that f is non-degenerate.

Now all this being so, consider the following data:

(i) A proper generically finite morphism $\pi : \widetilde{\mathrm{X}} \to \mathrm{X}$, together with a lifting $\tilde{f} : \mathrm{Y} \to \widetilde{\mathrm{X}}$ of f.

(ii) A big class $\beta \in \mathrm{NS}^1(\mathrm{X})_{\mathbf{Q}}$, whence necessarily $\pi^* \beta$ is also big.

(iii) A sequence $\mathrm{R} = (r_n) \subset \mathbf{R}_{\geqslant 0}$ which is convergent for (f, β) and $(\tilde{f}, \pi^* \beta)$—necessarily any unbounded sequence in $\mathbf{R}_{\geqslant 0}$ contains such a sequence.

Then, with this data:

Proposition **0.4.9.** — *We have the equality* $\pi^*(\widetilde{\Phi}_{\mathrm{R}}) = \Phi_{\mathrm{R}}$.

Proof. — Yet another direct application of the definitions. □

In this way we can therefore use the harmonic projection on X to define for any suitable R, convergent for (f, β), a cohomology class φ_{R}, or just φ should there be no confusion, associated—all be it rather non-canonically—to f, which satisfies a not unreasonable type of functoriality. In any case a final, and trivial, observation is:

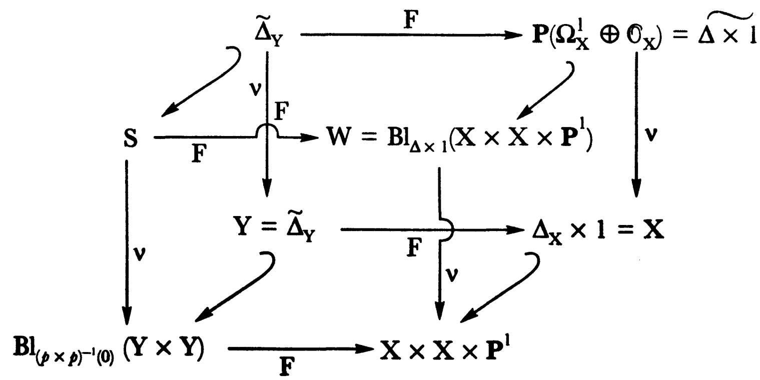
Proposition **0.4.10.** — *Notations as above (so that in particular f has Zariski dense image), then for* X *a surface* φ_{R} *is nef, and consequently* $\varphi_{\mathrm{R}}^2 \geqslant 0$.

1. Deformation to the normal cone of the diagonal

1.1. *Basic construction*

This will of course follow Fulton, cf. [Fu]. Philosophically speaking if we wish to estimate the derivative of something, then not only should we use the diagonal, but an infinitesimal neighbourhood of it, i.e. Fulton's deformation to the normal cone. The construction is best seen by a diagram, and our notations will be fixed as those which are found therein.

(1.1.1)



The map F is initially constructed as the rational map

$$f \times f \times [p : p] : Y \times Y \dashrightarrow X \times X \times \mathbf{P}^1,$$

which is resolved by blowing up in $(p \times p)^{-1}(0)$, S is some further resolution, which may involve an infinite sequence of blow-ups since f is a priori holomorphic; however the essential point is that this resolution does not involve blowing up any point on the diagonal, since the condition that f and p have disjoint ramifications ensures that the excess intersection $F^{-1}(\Delta \times 1) - \widetilde{\Delta}_Y$ contains no double points on $\widetilde{\Delta}_Y$ itself. Obviously all the other arrows denoted by F naturally derive themselves from this one.

Note in addition that the maps from S (resp. $\mathrm{Bl}_{(p \times p)^{-1}(0)}(Y \times Y)$) to $Y \times Y$ will also be denoted by ν, while the i-th-projection, $i = 1$ or 2, be it from W, $X \times X \times \mathbf{P}^1$, or $X \times X$ (resp. $Y \times Y$) to X (resp. Y) will be denoted by π_i while the projection from W or $X \times X \times \mathbf{P}^1$ to $\mathbf{P}^1$ will be denoted by π.

1.2. *Why bother?*

Observe that $\widetilde{\Delta \times 1}\,|_{\widetilde{\Delta \times 1}} = \mathcal{O}_{\mathbf{P}(\Omega^1_X \oplus \mathcal{O}_X)}(-1)$ so if we can deform suitably $\widetilde{\Delta}_Y$ we will obtain a dynamic intersection estimate for $-h_{\mathcal{O}_{\mathbf{P}(\Omega^1_X \oplus \mathcal{O}_X)}(1)}(F\,|_{\widetilde{\Delta}_Y}(r))$, at any $r \in \mathbf{R}_+$, and since anything off the diagonal ought to have positive height with respect to $\widetilde{\Delta \times 1}$, we ought to deduce an upper bound for $h_{\mathcal{O}_{\mathbf{P}(\Omega^1_X \oplus \mathcal{O}_X)}(1)}(F\,|_{\widetilde{\Delta}_Y}(r))$. Let us therefore consider the map, $F : Y \to \mathbf{P}(\Omega^1_X \oplus \mathcal{O}_X)$ in more detail. In local coordinates it is given by $y \mapsto \left[f'(y), \dfrac{p'(y)}{p(y)}\right]$, so that blowing up in the zero section,

$$[0] := \mathbf{P}(\mathcal{O}_X) \hookrightarrow \mathbf{P}(\Omega^1_X \oplus \mathcal{O}_X)$$

ought to give us a measure of the ramification.

For the moment let us denote $\mathbf{P}(\Omega^1_X \oplus \mathcal{O}_X)$ by P, and note that the short exact sequence of sheaves of graded algebras,

$$0 \to \mathrm{Sym}(\Omega^1_X \oplus \mathcal{O}_X)(-1) \otimes_{\mathcal{O}_X} \Omega^1_X \to \mathrm{Sym}(\Omega^1_X \oplus \mathcal{O}_X) \to \mathrm{Sym}(\mathcal{O}_X) \to 0,$$

gives rise to a surjection, $\mathcal{O}_{\mathrm{P}}(-1) \otimes_{\mathcal{O}_{\mathrm{P}}} \nu^* \Omega^1_{\mathrm{X}} \to \mathcal{I}_{[0]}$ and whence a diagram,

$$
\begin{array}{ccc}
\mathrm{V} := \mathrm{Bl}_{[0]}(\mathrm{P}) & \overset{i}{\hookrightarrow} & \mathbf{P}(\mathcal{O}_{\mathrm{P}}(-1) \otimes_{\mathcal{O}_{\mathrm{P}}} \nu^* \Omega^1_{\mathrm{X}}) \\
\downarrow{\scriptstyle \nu} & & \wr\downarrow{\scriptstyle \nu} \\
\mathrm{P} & \longleftarrow & \mathrm{P} \times \mathbf{P}(\Omega^1_{\mathrm{X}}) \\
\downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} \\
\mathrm{X} & \longleftarrow & \mathbf{P}(\Omega^1_{\mathrm{X}})
\end{array}
\tag{1.2.1}
$$

So that,

$$
\nu^* \mathcal{O}_{\mathrm{P}}(1) = \nu^* \mathcal{O}_{\mathbf{P}(\Omega^1_{\mathrm{X}})}(1) \otimes_{\mathcal{O}_{\mathrm{V}}} \mathcal{O}_{\mathrm{V}}([\widetilde{0}]),
\tag{1.2.2}
$$

and consequently an upper bound for the height with respect to $\mathcal{O}_{\mathrm{P}}(1)$ is exactly what we need to prove Theorem A, in the case of the complex line. Equally for arbitrary ramified covers, we need only observe that $\mathcal{O}_{\mathrm{P}}(1)$ is represented by the hyperplane at ∞, viz: $[\infty] := \mathbf{P}(\Omega^1_{\mathrm{X}}) \hookrightarrow \mathbf{P}(\Omega^1_{\mathrm{X}} \oplus \mathcal{O}_{\mathrm{X}})$, whose finite intersection with F is certainly the discriminant, and whence we need only control the so-called proximity term in the intersection, which turns out to require little other than some basic measure theory.

2. Dynamic approximation

2.1. *Notations and remarks*

Throughout this section we will confine ourselves to specific dynamic intersection estimates for the situation of § 1, with our notations fixed as those therein. In addition we will only give the details for $\mathrm{Y} = \mathbf{C}$, since the method is based on integral representations of holomorphic functions, which do not generalize well, while the deformations of $\Delta_{\mathrm{Y}} \hookrightarrow \mathrm{Y} \times \mathrm{Y}$ will in general have essential singularities, thus adding insult to injury, although when Y admits a compactification an alternative approach is provided to a dynamic intersection estimate by taking so-called Vojta divisors (cf. [V3] or [Bo]) on $\mathrm{Y} \times \mathrm{Y}$.

Nevertheless it will be clear that there is a certain amount of generality in the techniques which we will employ, though as previously remarked they are by no stretch of the imagination universal. We note in addition that the ideas proposed are fundamental to the more sophisticated and aforesaid proximity estimates in [M2].

2.2. *The deformation*

From now on $\mathrm{Y} = \mathbf{C}$, and taking $\mathbf{C} \times \mathbf{C} = \Delta \times \Delta^{\perp}$, where the perpendicular is taken with respect to the standard inner product, gives us our rather trivial deformation of the diagonal in this case. In addition we denote by (ξ, η) the natural coordinates on $\Delta \times \Delta^{\perp}$ and by $\Delta_{\xi,\eta}$ the twisted diagonal map, $\Delta_{\xi,\eta} : \mathbf{C} \to \mathbf{C} \times \mathbf{C} : z \mapsto (\xi z, \eta z)$. Equally there is of course a lifting $\widetilde{\Delta}_{\xi,\eta} : \mathbf{C} \to \mathrm{S}$.

2.3. *The estimates*

Observe that if H is an ample line bundle on X, then for suitable $m, n \in \mathbf{N}$ we have that $\mathrm{G} := \pi_1^* \mathrm{H}^m \otimes \pi_2^* \mathrm{H}^m \otimes \pi^* \mathcal{O}_{\mathbf{P}^1}(m) \otimes \mathcal{O}_{\mathrm{W}}(-n\,\widetilde{\Delta \times 1})$ is very ample on W, and consequently gives an embedding, $\mathrm{W} \hookrightarrow \mathbf{P}^{\mathrm{N}} := \mathbf{P}(\Gamma(\mathrm{W}, \mathrm{G})^{\vee})$, as usual.

Note further that if we consider the composite, $\mathrm{S} \xrightarrow{\mathrm{F}} \mathrm{W} \hookrightarrow \mathbf{P}^{\mathrm{N}}$, and denote by S′ the open subset of S obtained by removing the exceptional divisors arising from the double point locus of f, together with the point in the exceptional divisor over zero corresponding to $\xi = 0$, $\eta = 1$, then we may write, $\mathrm{F} = [\mathrm{F}_0, \ldots, \mathrm{F}_{\mathrm{N}}] : \mathrm{S}' \to \mathbf{P}$, where the F_i are holomorphic functions on S′, without common zero.

Consequently equipping G with the Fubini-Study metric we obtain,

$$h_{\mathrm{G}}(\widetilde{\Delta}^*_{\xi,\eta}\,\mathrm{F}(1)) = \frac{1}{2}\int_{|z|=1} \log \|\,\widetilde{\Delta}^*_{\xi,\eta}\,\mathrm{F}(z)\,\|^2\,\mu(dz) - \frac{1}{2}\log \|\,\widetilde{\Delta}^*_{\xi,\eta}\,\mathrm{F}(0)\,\|^2 \tag{2.3.1}$$

where $\|\,\mathrm{F}\,\|^2 := \sum_{i=0}^{\mathrm{N}} |\,\mathrm{F}_i\,|^2$. Now firstly observe that the variation of the term $\log \|\,\Delta^*_{\xi,\eta}\,\mathrm{F}(0)\,\|^2$ with ξ and η is simply that of a rational curve while for $|\,z\,| = 1$, and almost all $\xi, \eta \mapsto \log \|\,\widetilde{\Delta}^*_{\xi,\eta}\,\mathrm{F}(z)\,\|^2$ is a well defined pluri-subharmonic function of η, so that for all $|\,\xi\,| > \varepsilon > 0$, we obtain,

$$\int_{|\eta|=\varepsilon} h_{\mathrm{G}}(\widetilde{\Delta}^*_{\xi,\eta}\,\mathrm{F}(1))\,\mu(d\eta) \geqslant h_{\mathrm{G}}(\widetilde{\Delta}^*_{\xi,0}\,\mathrm{F}(1)) + \mathrm{o}(\log|\,\xi\,|). \tag{2.3.2}$$

So that on using the additivity of the height, we have

Proposition **2.3.3.** — *For* $|\,\xi\,| > \varepsilon > 0$,

$$\int_{|\eta|=\varepsilon} \{\, h_{\widetilde{\Delta \times 1}}(\widetilde{\Delta}^*_{\xi,\eta}\,\mathrm{F}(1)) - h_{\widetilde{\Delta \times 1}}(\widetilde{\Delta}^*_{\xi,0}\,\mathrm{F}(1))\,\}\,\mu(d\eta)$$
$$\leqslant \frac{2m}{n}\{\, h_{\mathrm{H}}(f(|\,\xi\,| + \varepsilon)) - h_{\mathrm{H}}(f(|\,\xi\,|))\,\} + \mathrm{o}(\log|\,\xi\,|).$$

Remarks **2.3.4.** — (1) As noted in the above considerations, a discrete set of possible values of ξ are excluded a priori from the statement of the proposition, while the integrand on the left is defined for almost all η.

(2) The heights with respect to the ample bundle H are taken with respect to a positive metric to guarantee that they are increasing functions of $|\,\xi\,|$, whence the inequality on combining (2.3.2) with the definition of G.

Consequently it is natural to seek a bound for $h_H(f(|\xi| + \varepsilon)) - h_H(f(|\xi|))$, though in what follows the only property of the height that we will make use of is that with respect to a positive metric on H it is an increasing, unbounded function of $|\xi|$. Whence let $T(x)$ be such a function of a real variable $x > 0$; then the following lemma sets the scene:

Lemma **2.3.5.** — *Let* $\alpha > 0$; *then for* x *outside a set of finite measure (depending on* α*) we have*

$$T\left(x + \frac{1}{T(x)}\right) \leqslant (1 + \alpha)\, T(x).$$

Proof. — Cf. [L2], Lemma 3.7, p. 177. □

Actually, although this would suffice for many of the applications that we have in mind, it is not, however, solely for the sake of aesthetics that we try a bit harder. Let us fix some $X > 0$ to be decided and choose a positive increasing differentiable function, $\psi : I := [X, \infty) \to \mathbf{R}_{>0}$ such that:

a) $b(\psi) := \displaystyle\int_I \frac{dx}{x\psi(x)} < \infty$;

b) $\exists\, c > 0$ such that $\psi'(x) \leqslant c\,\dfrac{\psi(x)}{x}$, where the former is decreasing and the latter tends to zero as $x \to \infty$.

Whence for example $\psi(x) = \{\log(x)\}^{1+\beta}$, $\beta > 0$, together with a suitable choice of X will suit our purposes admirably. Such, so called, Khintchine functions appear in the work of Lang and Cherry, cf. [L-C], and generalizing their thoughts a little we obtain:

Lemma **2.3.6.** — *Let* $S : I \to \mathbf{R}_{>0}$ *be any increasing differentiable function such that* $S(x) \geqslant T(x)$ *for all* $x \in I$; *then putting* $\varepsilon(x) = 1/S(x)$, *we have, for* x *outside a set of finite Lebesgue measure,*

$$\delta(x, \varepsilon(x)) := T(x + \varepsilon(x)) - T(x) \leqslant O(\varepsilon(x)\, \psi(T(x))\, T(x)) + O(1),$$

where the implied constant may be taken to be $1 + \gamma$, *for any* $\gamma > 0$.

Proof. — Let us take some $\alpha > 0$ and define

$$E := \{ x \in I : T(x + \varepsilon(x)) - T(x) > \varepsilon(x)\, T(x + \varepsilon(x))\, \psi(T(x + \varepsilon(x))) \}$$

and

$$F := \left\{ x \in I : S\left(x + \frac{1}{S(x)}\right) > (1 + \alpha)\, S(x) \right\}.$$

We will bound the measure of E\F; observe that

$$\mu(E\backslash F) \leqslant \int_I \frac{T(x+\varepsilon(x)) - T(x)}{\varepsilon(x)\, T(x+\varepsilon(x))\, \psi(T(x+\varepsilon(x)))} \cdot (1-\chi_F(x))\, \mu(dx)$$

$$= \int_I \frac{(1-\chi_F(x))\, \mu(dx)}{\varepsilon(x)\, T(x+\varepsilon(x))\, \psi(T(x+\varepsilon(x)))} \int_x^{x+\varepsilon(x)} T'(y)\, \mu(dy)$$

$$\leqslant \int_I \frac{T'(y)}{T(y)\, \psi(T(y))}\, \mu(dy) \int_{x \leqslant y \leqslant x+\varepsilon(x)} \frac{\{1-\chi_F(x)\}}{\varepsilon(x)}\, \mu(dx)$$

$$\leqslant \int_I \frac{T'(y)}{T(y)\, \psi(T'(y))}\, \mu(dy) \int_{y-(1+\alpha)\,\varepsilon(y)}^{y} S(x)\, dx < \infty.$$

Consequently applying Lemma 2.3.5, we conclude that E has finite measure, and a further application of the said lemma, together with our assumptions about ψ concludes the proof. □

Thus combining Proposition 2.3.3 and Lemma 2.3.6 gives

Proposition **2.3.7.** — *Let* $\varepsilon(|\,\xi\,|)$ *be a function satisfying the constraints of* 2.3.3 *and* 2.3.6. *Then,*

$$\int_{|\eta| = \varepsilon(|\xi|)} \{ h_{\widetilde{\Delta \times 1}}(\widetilde{\Delta}^*_{\xi,\eta}\, F(1)) - h_{\widetilde{\Delta \times 1}}(\widetilde{\Delta}^*_{\xi,0}\, F(1)) \}\, \mu(d\eta)$$
$$\leqslant O\{ \varepsilon(|\,\xi\,|)\, h_H(f(|\,\xi\,|))\, \psi(h_H(f(|\,\xi\,|))) \} + O(\log |\,\xi\,|)$$

for all $|\,\xi\,|$ *outside a set of finite measure.*

Remark **2.3.8.** — As will be seen in the following section this is precisely the shape of estimate which we need to justify a " dynamic intersection principle " for incomplete curves. Needless to say the above method worked because of the peculiar shape of the very ample bundle G, and examples indicate that perfectly natural constructions may fail to admit such an estimate should the aforesaid peculiarities of G not be present.

2.4. *Control of the proximity function, and end of demonstration*

As noted in the introduction the additional difficulty in utilising a dynamic intersection principle for incomplete curves is presented by the metricised term at infinity, or what is classically termed the proximity function. Now in particular if $s \in \Gamma(W, \mathcal{O}_W(\widetilde{\Delta \times 1}))$ and $\mathrm{Im}(\widetilde{\Delta}^*_{\xi,\eta}\, F) \not\subset \mathrm{div}(s) = \widetilde{\Delta \times 1}$, then:

$$\textbf{(2.4.1)} \qquad h_{\widetilde{\Delta \times 1}}(\widetilde{\Delta}^*_{\xi,\eta}\, F(1)) = \sum_{z \in D^\times} - \mathrm{ord}_z(\widetilde{\Delta}^*_{\xi,\eta}\, F^* s) \log |\,z\,| + m(\xi, \eta)$$

where $D^\times$ is the punctured unit disc, and if σ is the order of $\widetilde{\Delta}^*_{\xi,\eta}\, F^* s$ at zero then the proximity function $m(\xi, \eta)$ is given by,

$$(2.4.2) \qquad m(\xi,\eta) = -1/2\int_{|z|=1} \log \| \widetilde{\Delta}^*_{\xi,\eta}\, F^* s \|^2 \mu(dz) + \frac{1}{2}\log\left\{\lim_{|z|\to 0} \frac{\| \widetilde{\Delta}^*_{\xi,\eta}\, F^* s\|^2}{|z|^{2\sigma}}\right\}.$$

The compactness of W assures that the integral term appearing in (2.4.2) is bounded below by a constant depending only on the metricised divisor $\widetilde{\Delta \times 1}$, which only leaves what we may term the residue at zero to take care of, which we will denote by $r(\xi, \eta)$.

Denote by $1 \in \widetilde{0} \xrightarrow{\sim} \mathbf{P}^1$ (the exceptional divisor over 0 in s), the point $\widetilde{\Delta}_{\xi,0}(0)$, and choose γ a local generator of $\mathcal{O}_W(\widetilde{\Delta \times 1})$ in a neighbourhood of $F(1)$ with $\| \gamma(F(1)) \| = 1$. Now observe that on some fixed small neighbourhood of 1 we may write $F^* s = \varphi . F^* \gamma$, where φ is holomorphic on the said neighbourhood and the variation of $\| F^* \gamma \|$ across the neighbourhood is as small as we may subsequently choose to decide. Equally near to 1 we may take local coordinates ξ, t for some suitable parameter t such that $\eta = t\xi$, whence,

$$\varphi(\xi, t) = \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} a_{mn}\, \xi^m\, t^n$$

for some $m_0, n_0 \in \mathbf{N}$, and for a generic choice of ξ and η, $| r(\xi,\eta) - \log | \sum_{n=n_0}^{\infty} a_{m_0 n}\, \xi^{m_0 - n}\, \eta^n \| |$ is arbitrarily small, provided that $| \eta/\xi |$ is smaller than some fixed a priori constant.

However applying the Green-Jensen formula for sufficiently small positive ε gives

$$(2.4.3) \qquad \int_{|\eta|=\varepsilon} \log \Big| \sum_{n=n_0}^{\infty} a_{m_0 n}\, \xi^{m_0-n}\, \eta^n \Big| . \mu(d\eta) \geqslant n_0 \log \varepsilon + \log | a_{m_0 n_0}\, \xi^{m_0 - n_0} |.$$

Whence combining (2.4.2) and (2.4.3) we obtain that for ε and $| \eta/\xi |$ less than some suitable constant,

$$(2.4.4) \qquad \int_{|\eta|=\varepsilon} h_{\widetilde{\Delta \times 1}}(\widetilde{\Delta}^*_{\xi,\eta}\, F(1))\, \mu(d\eta) \geqslant O(\log \varepsilon + \log |\xi|).$$

So that utilising Proposition 2.3.7 with a choice of

$$\varepsilon(|\xi|) = \frac{1}{h_H(f(|\xi|))\, \psi(h_H(f(|\xi|)))}$$

and $\psi(x) = \{ \log(x) \}^2$ gives, on combining with (2.4.4),

$$(2.4.5) \qquad h_{\widetilde{\Delta \times 1}}(\Delta^*_{\xi,0}\, F(1)) \geqslant O(\log h_H(|\xi|) + \log(|\xi|)).$$

From which the tautological inequality immediately follows in light of (1.2.1). □

3. A quasi-formal approach

3.1. *Some observations*

The space Y is necessarily Stein with a trivial Picard group and so we may choose some global generator ∂ of the tangent bundle, and whence the map

$$F : Y \to \mathbf{P}(\Omega^1_X \oplus \mathcal{O}_X) := P,$$

of 1.2, should be more correctly written as $y \mapsto [p(y) f_*(\partial), p_*(\partial)]$. Further on representing the tautological bundle $\mathcal{O}_P(1)$ by the divisor at ∞ it is naturally metricised by a choice of metric on X together with the canonical metric on the trivial bundle, and in consequence, up to some constant, the height of F with respect to this divisor is given by

(3.1.1)
$$h_{\mathcal{O}_{P(1)}}(F(r)) = N_{p,\mathrm{Ram}}(r) - \frac{1}{2}\int_{\partial Y(r)} \log\left\{\frac{|p_*(\partial)|^2}{\|p(y) f_*(\partial)\|^2 + |p_*(\partial)|^2}\right\} p^*\left(\frac{d\theta}{2\pi}\right) + o(1).$$

3.2. *The calculation*

As remarked in 1.2, Theorem A must certainly follow on controlling the metricised term in 3.1.1. This is however extremely easy. To begin with we may evidently choose the metric on X to be induced by a positive metric on some ample line bundle H. For convenience of calculation we will denote the height of f with respect to this metric by $T(r)$, and note that

$$T(r) = \int_0^r \frac{dt}{t} \int_{Y(t)} \frac{\|f_*(\partial)\|^2}{|p_*(\partial)|^2} p^*\left(\frac{dz\, d\bar{z}}{-2\pi\sqrt{-1}}\right),$$

and in consequence,

$$\frac{(rT'(r))'}{r} = 2\int_{\partial Y(r)} \frac{\|f_*(\partial)\|^2}{|p_*(\partial)|^2} p^*\left(\frac{d\theta}{2\pi}\right).$$

Furthermore $T(r)$ and $rT'(r)$ are both increasing functions of r, so à la [L-C], a couple of applications of 2.3.6 with $\psi = \log^2$, or in fact a simpler variant of it, give

Lemma **3.2.1.** — *For all r outside a set of finite measure (depending on f) we have*

$$\int_{\partial Y(r)} \frac{\|f_*(\partial)\|^2}{|p_*(\partial)|^2} p^*\left(\frac{d\theta}{2\pi}\right) \leqslant O(T\psi(T))\, \psi(r\, O(\psi(T)\, T)).$$

The concavity of the logarithm, together with an immediate application of 3.2.1 then bounds the metricised term in 3.1.1 by an estimate of the form $O(\log r + \log T(r))$, for r outwith a given set of finite measure, from which we conclude.

II. FOLIATIONS

0. Introduction

0.1. *Notation and Generalities*

Henceforth unless stated otherwise X will denote a smooth complex surface, and $\mathscr{F}$ a foliation on X, i.e. a short exact sequence of sheaves

(0.1.1) $$0 \to N \to \Omega^1_X \to L.I_Z \to 0$$

where L, N are line bundles on X, and I_Z is the ideal of a subscheme Z of X supported in dimension zero, which may very well fail to be reduced.

Observe that the sequence (0.1.1) gives rise to a short exact sequence of sheaves of graded algebras

(0.1.2) $$0 \to N \otimes_{\mathcal{O}_X} \operatorname{Sym} \Omega^1_X(-1) \to \operatorname{Sym} \Omega^1_X \to \bigoplus_{n=0}^{\infty} L^n.I^n_Z \to 0.$$

So that to the foliation $\mathscr{F}$ we may associate the divisor

$$D := \operatorname{Proj} \{ \bigoplus_{n=0}^{\infty} L^n.I^n_Z \} \hookrightarrow \mathbf{P}(\Omega^1_X).$$

Thus $$\mathcal{O}_{\mathbf{P}(\Omega^1_X)}(D) = \mathcal{O}_{\mathbf{P}(\Omega^1_X)}(1) \otimes \pi^* N^\vee,$$

where $\pi : \mathbf{P}(\Omega^1_X) \to X$ is the projection, and $D \xrightarrow{\sim} \mathrm{Bl}_Z(X)$, the exceptional divisor on D being thus $\pi^* L \otimes \mathcal{O}_{\mathbf{P}(\Omega^1_X)}(-1)|_D$. In particular therefore $D \hookrightarrow \mathbf{P}(\Omega^1_X)$ is a quasi-section of π in the sense of Reid (cf. [R]), i.e. the closure of a section $s : X \dashrightarrow \mathbf{P}(\Omega^1_X)$ defined everywhere except on a subset of dimension zero. Indeed there is a one-to-one correspondence between quasi-sections of π and foliations, since given a quasi-section $D \hookrightarrow \mathbf{P}(\Omega^1_X)$ with $\mathcal{O}_{\mathbf{P}(\Omega^1_X)}(D) = \mathcal{O}_{\mathbf{P}(\Omega^1_X)}(1) \otimes \pi^* N^\vee$, for some line bundle N on X, we obtain a short exact sequence of the form (0.1.1) by applying π_* to the injection of sheaves

(0.1.3) $$0 \to \pi^* N \to \mathcal{O}_{\mathbf{P}(\Omega^1_X)}(1).$$

For the details of this equivalence, one is again referred to the aforesaid article of Reid.

In any case our interest will be in leaves of the foliation $\mathscr{F}$, i.e.

Definition **0.1.4.** — *Let* Y *be a smooth curve (complete or incomplete) and* $f : Y \to X$ *a morphism; then* f*—or simply* Y *if there is no confusion—is said to be a* leaf *of* $\mathscr{F}$ *if* $f^* N$ *is contained in the kernel of the natural map* $f^* \Omega^1_X \to \Omega^1_Y$.

It is therefore natural to ask whether Conjecture 0.0 holds in this situation, i.e. a so-called second main theorem for incomplete curves on surfaces which are themselves leaves of some a priori fixed foliation. For ease of exposition we re-work Theorem B as stated in the introduction a little. We begin with a definition.

Definition **0.1.5.** — *Given a ramified cover* $\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow p & & \\ \mathbf{C} & & \end{array}$ *which is a leaf of a foliation* $\mathcal{F}$ *on* X, *we will call it* large *if*:

a) f *has Zariski dense image.*

b) There exists $\varepsilon > 0$ *such that,* $h_{K_X}(f(r)) \geqslant \varepsilon h_H(f(r)) + o(h_H(f(r)))$, *where as usual* H *is ample on* X.

c) f *satisfies the conclusions of* I.0.3.3.

If in addition, we have:

d) There does not exist a set of finite measure F *and* $\alpha > 0$ *such that*

$$h_{K_X}(f(r)) \leqslant \alpha\, d(f(r)) + o(h_H(f(r))), \quad \forall\, r \notin \mathbf{F}.$$

Then we will call the leaf very large.

Our re-statement of Theorem B, and the goal of this chapter is then given by

Theorem **0.1.6.** — *Let* X *be a surface which admits a very large leaf of some foliation* $\mathcal{F}$, *then some (possibly singular) bi-rational model of* X *is dominated by a normal surface* S *on which there is a global vector field.*

We remark that Theorem 0.1.6 is equivalent to Theorem B, by the following observation [1], viz: even if S is singular we still have that the tangent space to the automorphism scheme of S is the global derivations of S. Consequently, since we are in characteristic zero, a positive dimensional group scheme must act on S. Whence X is either fibered by rational or elliptic curves, or is itself an abelian surface. We even a priori know however that Conjecture 0.0 holds for maps from ramified covers of the complex line to abelian surfaces (it is a trivial deduction from Theorem A) and so we certainly have the desired equivalence. It is also worth noting that the above re-formulation is consistent with what appears to be Lang's intuition vis-à-vis a proof of the Green-Griffiths conjecture, [N]; i.e. given a holomorphic map $f: \mathbf{C} \to X$ with Zariski dense image one should try and show the existence of a global vector field on X.

Now although thanks to some elegant jet spaces constructed by Arrondo, Sols, and Speiser (cf. [A-S-S]) the Green-Griffiths conjecture would follow for arbitrary

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[1] For which I thank Dan Abramovich.

surfaces on knowing a suitable analogue of Theorem 0.1.6 for foliations by curves on essentially a tower of $\mathbf{P}^1$-bundles over X [1], we choose not to discuss the consequences of Theorem A in this generality (however cf. [De2]) and proceed to motivate the theorem by that which is directly relevant (and indeed necessary to recall) i.e. Bogomolov's proof of Theorem 0.1 of the introduction and Miyaoka's almost ampleness theorem.

0.2. *Summary of Bogomolov-Miyaoka*

Throughout this section X will be a minimal surface of general type with $s_2(X) > 0$, $P := \mathbf{P}(\Omega^1_X) \xrightarrow{\pi} X$ will be the projectivised tangent space, and $\mathcal{O}(1)$ the tautological bundle on P. We will follow the succinct presentation of Lu and Yau, cf. [L-Y].

Theorem **0.2.1** (Bogomolov). — *Notations as given; then $\mathcal{O}(1)$ is big.*

Proof. — By assumption $c_1(\mathcal{O}(1))^3 = s_2(X) > 0$ and so by Riemann-Roch, $\chi(P, \mathcal{O}(m)) > cm^3$ for some suitable positive constant c, and all sufficiently large integers m. On the other hand, $h^i(P, \mathcal{O}(m)) = h^i(X, \mathrm{Sym}^m \Omega^1_X) = 0$, for all $i \geqslant 3$. Moreover, by Serre-duality, $h^2(X, \mathrm{Sym}^m \Omega^1_X) = h^0(X, \mathrm{Sym}^m \Omega^1_X \otimes K_X^{-(m-1)})$, and K_X is effective by assumption, whence the result. □

Now let us apply this to the study of curves on X in the style of Bogomolov, i.e. let

$$\begin{array}{ccc} Y & \xrightarrow[f]{} & X \\ \big\downarrow{\scriptstyle p} & & \\ \mathbf{C} & & \end{array}$$

be a ramified cover of $\mathbf{C}$ mapping to X, and let us suppose that $\mathrm{Im}(f)$ is Zariski dense in X; then by Theorem A, we have the tautological inequality, viz

$$h_{\mathcal{O}(1)}(f'(r)) \leqslant d(f(r)) + \mathrm{o}(\log r + \log T_{f,H}(r)),$$

where H is an ample line bundle on X, and r is excluded from a set of finite measure.

On the other hand since $\mathcal{O}(1)$ is big, then for m sufficiently large $m\mathcal{O}(1) - \pi^* H \geqslant 0$, whence if the image of the derivative, f', is not contained in the base locus of the linear system $| \mathcal{O}(m) \otimes \pi^* H^\vee |$, we obtain

$$h_H(f(r)) \ll d(f(r)) + \mathrm{o}(\log r + \log T_{f,H}(r)),$$

the implied constant certainly being effectively computable by the Riemann-Roch theorem, and in particular the conjectured second main theorem holds. Thus we are done unless f' lies in some irreducible divisor $D \hookrightarrow P$, and evidently if we suppose that f is Zariski dense then this divisor must dominate X. In this context Miyaoka refined Bogomolov's argument to prove,

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[1] Although in [G-G] it is shown that a non-degenerate $f : \mathbf{C} \to X$, X a surface of general type, is a leaf of a foliation on some variety dominating X, it is not shown (as a result of the jet spaces used) that this foliation has dimension 1.

Theorem **0.2.2** (Miyaoka's Almost Ampleness Theorem). — *Let* $h : \widetilde{D} \to D$ *be a resolution of the singularities of* D; *then if* X *has positive index* $\left(\textit{i.e.}\ \tau(X) := \dfrac{c_1^2 - 2c_2}{3} > 0\right)$, *the bundle* $h^* \mathcal{O}_D(1)$ *is big.*

Proof. — Let $d := \deg(D/X)$; then for some line bundle M on X we have $\mathcal{O}_P(D) = \mathcal{O}(d) \otimes \pi^* M^\vee$, and so applying π_* to the map $\pi^* M \to \mathcal{O}(d)$ yields a non-trivial homomorphism $M \to \mathrm{Sym}^d \Omega^1_X$. Thus by the semi-stability of Ω^1_X (and whence of $\mathrm{Sym}^d \Omega^1_X$) with respect to K_X we have

$$\int_X c_1(M) \cap c_1(K_X) \leqslant \int_X \frac{c_1(\mathrm{Sym}^d \Omega^1_X) \,.\, c_1(K_X)}{rk(\mathrm{Sym}^d \Omega^1_X)} = \frac{d}{2} c_1(X)^2.$$

On the other hand we have the equalities

$$\int_D c_1(\mathcal{O}_D(1)) \cap c_1(K_X) = \int_P c_1(\mathcal{O}(1)) \cap c_1(K_X) \cap D$$
$$= dc_1(X)^2 - \int_X c_1(K_X) \cap M$$

and

$$\int_D c_1(\mathcal{O}_D(1))^2 = \int_P c_1(\mathcal{O}(1))^2 \cap D = ds_2(X) - \int_X c_1(K_X) \cap M$$

so that

$$\int_D c_1(\mathcal{O}_D(1)) \cap c_1(K_X) \geqslant \frac{d}{2} c_1(X)^2 \quad \text{and} \quad \int_D c_1(\mathcal{O}_D(1))^2 \geqslant \frac{3d}{2} \tau(X).$$

Thus the theorem, by Riemann-Roch and Serre duality, applied to $h^* \mathcal{O}_D(1)$ on $\widetilde{D}$. □

Now applying this to the situation of our incomplete curve f, given the assumption of Zariski-dense image, we see that again we have the second main theorem since f necessarily lifts to $\widetilde{D}$ and the height inequality is functorial.

However there is still a long way to go when we do not have the condition of positive index. Nevertheless we may observe that the kernel F of the canonical map, $\Omega^1_X|_D \to \mathcal{O}_D(1) \to 0$, admits a map, $h^* F \to \Omega^1_{\widetilde{D}}$, which is generically an injection of line bundles since $\widetilde{D} \to X$ is generically étale. Moreover taking the saturation of $h^* F$ in $\Omega^1_{\widetilde{D}}$ we obtain some foliation on $\widetilde{D}$ of which the lifting of f to $\widetilde{D}$ is a leaf. Consequently it is clear that Theorem 0.1.6 must imply the second main theorem in this context.

0.3. *Strategy*

Returning then to the notations of § 0.1 our set up will be that we suppose there exists a very large leaf $\begin{matrix} Y \xrightarrow{f} X \\ \downarrow p \\ C \end{matrix}$ of a given foliation $\mathcal{F}$ on X. Now since $K_X = L + N$, and the height is additive, we propose to proceed as follows:

(i) Show that the height of f with respect to N is not too big.

(ii) Deduce from this that in fact the height with respect to L dominates the height with respect to some ample bundle.

(iii) Apply the method of diophantine approximations, once more, in order to refine the tautological inequality sufficiently so that in fact (ii) will imply Theorem 0.1.6.

Part (i) will be achieved by a study of what may be termed residual heights along foliations, while the tools of part (ii) will be those of the bi-rational geometry of surfaces, specifically Miyaoka's semi-positivity theorem for the cotangent bundle, and Zariski decomposition.

1. Residual heights

1.1. *An observation*

To begin with, let us permit X to be a projective complex variety of arbitrary dimension n and $\mathscr{F}$ a smooth foliation by curves on X, i.e. a short exact sequence

$$0 \to \mathrm{N} \to \Omega^1_{\mathrm{X}} \to \mathrm{L} \to 0,$$

where L is a line bundle and N a vector bundle of rank $(n-1)$; then we have

Proposition **1.1.2.**

$$c_1(\mathrm{N}) \in \mathrm{Im}\{ \mathrm{H}^1(\mathrm{X}, \mathrm{N}) \to \mathrm{H}^1(\mathrm{X}, \Omega^1_{\mathrm{X}}) \}.$$

Proof. — Since $\mathscr{F}$ is a foliation by curves, we may apply the Frobenius theorem to the foliation ($\mathrm{L}^\vee \hookrightarrow \mathrm{T}_{\mathrm{X}}$ being automatically closed under the Lie bracket) to obtain a cover $\{ \Delta_\alpha \mid \alpha \in \mathrm{A} \}$ by polydiscs of X such that $\mathrm{N}|_{\Delta_\alpha}$ is generated by $dz_1^\alpha, \ldots, dz_{n-1}^\alpha$ for some $z_1^\alpha, \ldots, z_{n-1}^\alpha \in \Gamma(\Delta_\alpha)$. Whence $\Lambda^{n-1}\mathrm{N}|_{\Delta_\alpha} = \mathcal{O}_{\Delta_\alpha}\, dz_1^\alpha \wedge \ldots \wedge dz_{n-1}^\alpha$ with corresponding transition functions $g_{\alpha\beta} \in \Gamma(\Delta_\alpha \cap \Delta_\beta)^\times$ satisfying

$$dz_1^\alpha \wedge \ldots \wedge dz_{n-1}^\alpha = g_{\alpha\beta}\, dz_1^\beta \wedge \ldots \wedge dz_{n-1}^\beta,$$

from whence we obtain, $d\log(g_{\alpha\beta}) \in \Gamma(\Delta_\alpha \cap \Delta_\beta, \mathrm{N})$, so that calculating Čech cohomology with respect to the given cover yields the proposition. □

Whence, modulo $\bar\partial$ of smooth forms, $c_1(\mathrm{N})$ is represented by a $(1,1)$-form ω in $\Gamma(\bar\partial\{ \mathscr{A}_{\mathrm{X}} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathrm{N} \})$, where $\mathscr{A}_{\mathrm{X}}$ is the sheaf of smooth functions. On the other hand if $\mathrm{Y} \xrightarrow{f} \mathrm{X}$ is a leaf of $\mathscr{F}$ then a priori $f^*\omega = 0$, and so if Y is complete $\int_{\mathrm{Y}} f^* c_1(\mathrm{N}) = 0$, while if Y is a ramified cover of $\mathbf{C}$ then, should it factor through a complete curve, $| h_{\mathrm{N}}(f(r)) |$ is, by the above, bounded by an absolute constant independent of r; otherwise I.0.3.3 applies, to give that $| h_{\mathrm{H}}(f(r)) | \leqslant o(h_{\mathrm{N}}(f(r)))$ after a possible reparametrisation of f by composition with a global holomorphic function on the left, and of course for r

outside a set of finite measure. We also remark that we have a derivation, $\partial_{\mathscr{F}} : \mathscr{A}_X \to \mathscr{A}_X \otimes_{\mathscr{O}_X} L$, defined in the natural way, and one may hope to construct a suitable Hodge theory to give in fact that, $f^* \bar{\partial}\varphi = dd^c \psi$ for ψ smooth on X, independent of any re-parametrisation of f, and consequently an amelioration of the above; however for foliations which are not ergodic, continuity of the corresponding harmonic forms in the direction normal to the foliation is problematic while in the case of singular foliations there appears to be a fairly significant problem in trying to prove even a " smoothing lemma " for solutions of the appropriate Laplacian in the tangential direction, and so we confine ourselves to using I.0.3.3, as indicated. In any case it is clear that in the case of smooth foliations, on combining these observations with Theorem A, Conjecture 0.0 is proved for leaves of such. Unfortunately there is absolutely no reason to believe that the foliations arising in the context of the Green-Griffiths conjecture will be smooth, but the above observation certainly indicates that the height of a leaf of $\mathscr{F}$ with respect to N in the notation of (0.1.1) ought not to be too big.

1.2. *Complete residue formula*

As a guide to what will follow for incomplete curves, and out of a certain intrinsic interest we will compute a residue formula for the degrees of complete leaves of a foliation with respect to the determinant of the corresponding subsheaves of Ω^1_X. For the final time we continue to allow X to have arbitrary dimension n, X of course being smooth and projective.

From the point of view of obtaining a nice residue formula it is convenient to view our foliation by curves as an injection of a line bundle F into T_X, where the given injection of sheaves fails to be an injection of line bundles on a closed subscheme Z of X of codimension at least two, and $\mathscr{N}$ will denote the annihilator of $F \hookrightarrow T_X$ in Ω^1_X.

Now we will suppose that Y is a smooth complete curve and $Y \xrightarrow{f} X$ a leaf of the foliation with $f(Y) \not\subset Z$, i.e. $f^{-1}(Z)\,(\mathbf{C})$ is finite. Given a point $y \in f^{-1}(Z)\,(\mathbf{C})$, choose a polydisc about $f(y)$ with coordinates $(x_1, \ldots, x_n)$ and $\partial = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$ a local generator of F, for some holomorphic functions $a_1, \ldots, a_n$ on the polydisc. Moreover if we fix a local coordinate z on Y in a neighbourhood of y then we may define a meromorphic function $\dfrac{f_*\left(\frac{\partial}{\partial z}\right)}{f^*\partial}$ by the formula,

$$f_*\left(\frac{\partial}{\partial z}\right) = \left\{\frac{f_*\left(\frac{\partial}{\partial z}\right)}{f^*\partial}\right\} f^*\partial \in f^* T_X.$$

Understanding these notations in the natural way as y varies we have therefore

Proposition **1.2.1** (Residue Formula).

$$\int_Y f^* c_1(\mathcal{N}) = - \sum_{y \in f^{-1}(Z)(C)} \mathrm{Re} \left\{ \mathrm{Res}_y \left(\left[\frac{f_*\left(\frac{\partial}{\partial z}\right)}{f^* \partial} . f^* \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} \right] . dz \right) \right\}.$$

Remarks **1.2.2.** — *a)* Re of course denotes the real part.

b) It is clear from considerations in local cohomology that the residue formula exists, indeed such considerations show that the right hand side would be zero for singularities of codimension at least three, however this explicit formula compares favourably with those calculated for the higher Chern classes of $\mathcal{N}$ by Baum and Bott (cf. [B-B]).

c) The residue formula together with Seidenberg's theorem—to be discussed in the next section—permits an effective calculation of the constant β appearing in Bogomolov's theorem (cf. Theorem **0**.0.1) as demanded by Lang (cf. [L1]).

Proof. — Notations as given, one finds that a local generator for $\det(\mathcal{N})$ in a neighbourhood Δ of $f(y)$ is given by $\omega = \sum_{i=1}^{n} (-1)^i a_i\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$. On the other hand for any $x \in f^{-1}(\Delta)$ distinct from y—supposing that a priori Δ is sufficiently small—we may choose a neighbourhood Δ_x of $f(x)$ such that ω_x generates $\det(\mathcal{N})|_{\Delta_x}$ and ω_x is closed (cf. proof of Proposition 1.1.2). Consequently, if we write $\omega_x = g_x\, \omega$ then we find that $d \log(g_x) \wedge \omega = - d\omega$, from which we obtain

$$f^* d \log(g_x) = - \frac{f_*\left(\frac{\partial}{\partial z}\right)}{f^* \partial} \left(\sum_{i=1}^{n} f^* \frac{\partial a_i}{\partial x_i} \right) . dz,$$

and the formula follows in view of the considerations already made away from the singularities as found in Proposition 1.1.2. □

1.3. *Seidenberg's Theorem*

From now on our notations will be those of § 0. Clearly as we have seen in § 1 our goal would be achieved if we were studying smooth foliations, and an obvious initial thought might be that after a finite number of monoidal transformations we could desingularise our initial foliation. This however is false (consider the foliation defined by $x\, dy + y\, dx$ on $\mathbf{A}^2$) though what can be achieved after a finite number of monoidal transformations is not actually too bad as was shown by Seidenberg (cf. [S]). Given then that the intersection theory of incomplete curves is significantly more complicated than that for complete curves we will be obliged to work out the theory of residual heights for the former on foliations which are as good as possible, i.e. utilising Seidenberg's theorem. Noting that the language employed in the context of this theorem seems maladapted to algebraic geometry, and consequently to our purposes, we will employ a different terminology, indicating where it exists the existing terminology.

Definition **1.3.1.** — *A singularity* $z \in \mathrm{Z}(\mathbf{C})$ *of a foliation* $\mathcal{F}$ *is said to be* of Seidenberg type *if* $\mathrm{N}_z = \mathcal{O}_{\mathrm{X},z} . \omega$, *where for some local coordinates* x, y *(either formal or holomorphic)* $\omega = x\,dy + \lambda y\,dx \pmod{m^2_{\mathrm{X},z} \otimes_{\mathcal{O}_\mathrm{X}} \Omega^1_\mathrm{X}}$, *and* $\lambda \in \mathbf{C}$ *is not a strictly negative rational number.*

Remark **1.3.2.** — Such singularities are normally called *reduced*, however with such notation $x\,dy + y^2\,dx$ would be reduced, yet the singularity is not just the point (0, 0), but the scheme with reduced structure defined by the ideal (x, y^2). There is therefore a real and important sense—as we shall see—in which such a singularity ought not to be called reduced.

Now the theorem of Seidenberg as one might imagine is

Theorem **1.3.3** (Seidenberg). — *Given a foliation* $\mathcal{F}$ *on* X, *there exists a finite sequence of monoidal transformations* $\mathrm{X}_d \to \mathrm{X}_{d-1} \to \dots \to \mathrm{X}_1 \to \mathrm{X}_0$ *(each* $\mathrm{X}_i \to \mathrm{X}_{i-1}$ *being a monoidal transformation in a geometric point of the singular locus of the induced foliation on* X_{i-1}*) such that the singularities of the induced foliation* $\mathcal{F}_d$ *on* X_d *are of Seidenberg type.*

Proof. — (Cf. [S].) □

In order to have a little more geometric feeling for this work of Seidenberg, observe that, for all $x \in \mathrm{X} - \mathrm{Z}$, there is a unique germ of an integral analytic curve through x, which is a local leaf of the foliation by the Frobenius theorem. On the other hand interpreting the statement that a germ of an integral analytic curve is a local leaf of the foliation in a natural way, we have:

Theorem **1.3.4** (Seidenberg). — *Given a foliation* $\mathcal{F}$ *on* X, *there exists a sequence of monoidal transformations, of the type one finds in Theorem* 1.3.3, *such that through any point* $z \in \mathrm{Z}_d(\mathbf{C})$ *of the singular locus of the induced foliation* $\mathcal{F}_d$ *on* X_d *there are at most two germs of integral analytic curves which are local leaves of the foliation.*

Remark **1.3.5.** — Integral is intended in the sense of reduced and irreducible.

Proof. — (Idem.) □

Let us now consider more precisely the result of a monoidal transformation in a point $z \in \mathrm{Z}(\mathbf{C})$—i.e. the blow-up in the reduced structure—on the foliation $\mathcal{F}$. Put $\tilde{\mathrm{X}} = \mathrm{Bl}_z(\mathrm{X}) \xrightarrow{\pi} \mathrm{X}$; then the induced foliation $\tilde{\mathcal{F}}$ on $\tilde{\mathrm{X}}$ must be of the form

$$0 \to \pi^* \mathrm{N} \otimes_{\mathcal{O}_{\tilde{\mathrm{X}}}} \mathcal{O}_{\tilde{\mathrm{X}}}(\nu \mathrm{E}) \to \Omega^1_{\tilde{\mathrm{X}}} \to \pi^* \mathrm{L} \otimes \mathcal{O}_{\tilde{\mathrm{X}}}((1 - \nu)\,\mathrm{E}) . \mathrm{I}_{\tilde{\mathrm{Z}}} \to 0,$$

where E is the exceptional divisor, $\nu \in \mathbf{N}$, and outside the exceptional divisor $\tilde{\mathrm{Z}}$ is isomorphic to Z.

Calculating $c_2(\widetilde{X})$ then yields:

$$c_1(L) \cap c_1(N) + \nu(\nu - 1) + \deg(\widetilde{Z}) = c_2(\widetilde{X}) = c_2(X) + 1$$
$$= c_1(L) \cap c_1(N) + \deg(Z) + 1.$$

Whence we obtain

(1.3.6) $$\deg(\widetilde{Z}) = \deg(Z) + 1 - \nu(\nu - 1).$$

Consequently we observe that, if $\nu \geqslant 2$, the degree of the singular locus must decrease, whereas if $\nu = 1$, then in fact its degree must increase by one. In particular if $\nu = 1$ then the first Chern class of the quotient of $\Omega^1_{\widetilde{X}}$ occuring in $\widetilde{\mathscr{F}}$ is simply the pull back of that of Ω^1_X occuring in $\mathscr{F}$. With this in mind we make the following definitions.

Definition **1.3.7.** — *A foliation $\mathscr{F}$ on X is said to be* stable *under monoidal transformations if for all monoidal transformations in all points of* $Z(\mathbf{C})$ *the corresponding ν of* (1.3.6) *is equal to one.*

Definition **1.3.8.** — *A foliation $\mathscr{F}$ on X is said to be* very stable *under monoidal transformations if for any sequence of monoidal transformations,* $X_d \to X_{d-1} \to \ldots \to X_1 \to X_0 = X$ *where $X_i \to X_{i-1}$ is a monoidal transformation in a geometric point of the singular locus of the, induced foliation on X_{i-1}, the induced foliation on X_d is stable under monoidal transformations.*

Remark **1.3.9.** — Provided that there is unlikely to be any confusion with the stability, or otherwise, of Ω^1, we will often drop the precision under monoidal transformations.

Now clearly after a suitable sequence of monoidal transformations we may assume that the foliation is stable, whether or not however it is very stable is a more subtle question which is answered by the following.

Theorem **1.3.10.** — *If a foliation $\mathscr{F}$ on X has singularities of Seidenberg type then it is very stable under monoidal transformations.*

Corollary **1.3.11.** — *Given a foliation $\mathscr{F}$ on X there exists a suitable sequence of monoidal transformations $X_d \to \ldots \to X_0 = X$, such that the induced foliation $\mathscr{F}_d$ on X_d is very stable under monoidal transformations.*

Proof. — Immediate by Theorems 1.3.3 and 1.3.10. □

The proof of Theorem 1.3.10 will be a similarly straightforward deduction from the following two lemmas.

Lemma **1.3.12.** — *If a foliation $\mathscr{F}$ has singularities of Seidenberg type then it is stable under monoidal transformations.*

Lemma **1.3.13.** — *If* $\tilde{X} \xrightarrow{\pi} X$ *is a monoidal transformation in a geometric point of the singular locus of a foliation* $\mathcal{F}$ *on* X *which has singularities of Seidenberg type, then the induced foliation* $\tilde{\mathcal{F}}$ *on* $\tilde{X}$ *also has singularities of Seidenberg type.*

Proof of Lemmas **1.3.12** *and* **1.3.13.** — It will suffice to consider what happens to a singularity of Seidenberg type under monoidal transformations, our notation will be that of Definition 1.3.1. Further it will clearly be sufficient to view the problem on an infinitesimal neighbourhood of the singularity in which we will blow up, thus since X is smooth we may without loss of generality assume that it is Spec $\mathbf{C}[[x, y]]$, with 0 as the singularity of $\mathcal{F}$. In any case put $\tilde{X} = \mathrm{Bl}_0(X) \hookrightarrow X \times \mathbf{P}^1 \xrightarrow{\pi} X$; then viewing $\mathbf{P}^1$ as having projective coordinates [S, T], $\tilde{X}$ is covered by the standard open neighbourhoods $S \neq 0$, and $T \neq 0$ on which we have coordinates x, t (with $y = xt$) and s, y (with $x = sy$) respectively.

Case 1. — *The singularity has non-trivial reduced structure, i.e.* $\lambda = 0$.

In this case we may assume that a priori x, y were chosen so that $\omega = x\,dy + \ell(y)\,dx$ where $\ell(y) = cy^d + \dots$, $d \geq 2$, $\in \mathbf{C}[[y]]$, (cf. [C-M], p. 125-127). Then,

$$\pi^* \omega = x(t\,dx + x\,dt) + \ell(xt)\,dx$$

on the $S \neq 0$ piece which gives the generator of the induced foliation, i.e. the saturation $\tilde{N}$ of $\pi^* N$ in $\Omega^1_{\tilde{X}}$, as being $t\,dx + x\,dt + \dfrac{\ell(xt)}{x}\,dx$. Whence the foliation is certainly stable, and the induced foliation on this piece has a singularity of Seidenberg type at $x = 0$, $t = 0$. Moreover on the $T \neq 0$ piece, the induced foliation has generator $s\left(1 + \dfrac{\ell(y)}{y}\right) dy + \ell(y)\,ds$, which again has a singularity of Seidenberg type at $s = 0$, $y = 0$, as required.

Case 2. — *One has* $\lambda \neq 0$.

Write $\omega = x\,dy + \lambda y\,dx + \tau$, where $\tau \in m^2_{X,0} \otimes_{\mathcal{O}_X} \Omega^1_X$, and let us fix attention on the $S \neq 0$ part. Then, $\pi^*(x\,dy + \lambda y\,dx) = x\{(1 + \lambda)\,t\,dx + x\,dt\}$. On the other hand let us write $\tau = a_2\,dx + b_2\,dy + a\,dx + b\,dy$, where a_2, b_2 are quadratic and the terms occurring in a, b are of degree at least three. Now certainly $\pi^* \tau = x\tau_0$ for some regular differential τ_0 and since λ is not strictly negative rational then a fortiori $(1 + \lambda) \neq 0$, and so the foliation is certainly stable. Whence in light of (1.3.6) the singularities of the induced foliation will necessarily occur at $p : x = t = 0$ on the $S \neq 0$ piece and $q : y = s = 0$ on $T \neq 0$. Let us therefore restrict our attention to the germ of the induced foliation at p where we find that $\dfrac{1}{x}\,\pi^*(b_2\,dy + a\,dx + b\,dy) \in m^2_{\tilde{X},p} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega^1_{\tilde{X}}$. On the other hand writing $a_2 = \alpha x^2 + \beta xy + \gamma y^2$ we see that $\dfrac{1}{x}\,\pi^*(\{\beta xy + \gamma y^2\}\,dx) \in m^2_{\tilde{X},p} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega^1_{\tilde{X}}$.

Whence there is a local generator $\widetilde{\omega}$ of the induced foliation with,

$$\widetilde{\omega} = \{ (1+\lambda)\, t + \alpha x \}\, dx + x\, dt (\mathrm{mod}\ m^2_{\widetilde{X}, p} \otimes_{\mathcal{O}_{\widetilde{X}}} \Omega^1_{\widetilde{X}}).$$

Now if $\widetilde{\partial}$ is the corresponding derivation annihilating ω_0 in $T_{\widetilde{X}}$ then likewise

$$\widetilde{\partial} \equiv \{ (1+\lambda)\, t + \alpha x \} \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} (\mathrm{mod}\ m^2_{\widetilde{X}, p} \otimes_{\mathcal{O}_{\widetilde{X}}} T_{\widetilde{X}}),$$

to which we associate the matrix $\begin{bmatrix} (1+\lambda) & \alpha \\ 0 & -1 \end{bmatrix} \in GL_2(\mathbf{C})$. However $\lambda + 1 \neq -1$, since λ is not strictly negative rational, whence the matrix is diagonalisable, so that after a linear change of coordinates we may write

$$\widetilde{\partial} \equiv (1+\lambda)\, t . \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} (\mathrm{mod}\ m^2_{\widetilde{X}, p} \otimes_{\mathcal{O}_{\widetilde{X}}} T_{\widetilde{X}}),$$

which implies $\widetilde{\omega} = (1+\lambda)\, t\, dx + x\, dt (\mathrm{mod}\ m^2_{\widetilde{X}, p} \otimes_{\mathcal{O}_{\widetilde{X}}} T_{\widetilde{X}})$; i.e. the singularity of the induced foliation at p is of Seidenberg type, and since equally $1/\lambda$ is not strictly negative rational the induced foliation also has a singularity of Seidenberg type at q. □

1.4. *Incomplete residue estimate*

In this section X will be a smooth projective complex surface and $\mathcal{F}$ a foliation with singularities of Seidenberg type, while $\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow{\scriptstyle p} & & \\ \mathbf{C} & & \end{array}$ will be an incomplete dense leaf of the foliation, suitably re-parametrised by a possible composition on the left with a global holomorphic function in order to ensure that the conclusion of I.0.3.3 is valid. In addition $\pi : \widetilde{X} \to X$ will be the blow up of X in $Z(\mathbf{C})$, with E the total exceptional divisor, $\widetilde{f}$ the lifting of f, and Z' the set of reduced singularities on the induced foliation on $\widetilde{X}$. Recalling the notations on Lelong numbers, currents and cohomology classes introduced in I.0.4 our goal is to show:

Theorem **1.4.1** (Incomplete residue estimate, weak version). — *There exists a set* $F \subset \mathbf{R}$ *of finite measure and constants* ε_0, $c > 0$ *such that for all* $\varepsilon < \varepsilon_0$ *there is a constant* $C(\varepsilon)$ *for which given any* $R = (r_n) \subset \mathbf{R}_{\geqslant 0} \backslash F$ *tending to infinity and convergent for* (f, H) *and* $(\widetilde{f}, \pi^* H)$ *we have:*

$$N . \varphi \leqslant C(\varepsilon) \{ E . \widetilde{\varphi}_R + \sum_{z \in Z'} \nu(z, \widetilde{\Phi}_R) \} + c\varepsilon$$

where of course H *is an ample divisor on* X.

" *Theorem* " **1.4.1** *bis* (Incomplete residue estimate, hard version). — *There exists a set* $F \subset \mathbf{R}$ *of finite measure and constants* ε_0, $c > 0$ *such that for all* $\varepsilon < \varepsilon_0$ *there is a constant* $C(\varepsilon)$ *for which we have:*

$$h_N(f(r)) \leqslant C(\varepsilon)\, h_E(\widetilde{f}(r)) + c\varepsilon h_H(f(r)), \quad \forall\, r \notin F.$$

Remarks **1.4.2.** — (i) The importance of the precise order of quantifiers in the theorem is non-trivial for applications.

(ii) We may certainly take $C(\varepsilon) = 0(\varepsilon^{-\deg(Z)})$, as will be seen in the course of the proof.

(iii) As the quotation marks indicate 1.4.1 *bis* is not really a theorem, since the proof flounders on our inability to do a seemingly inocuous integral (Problem 1.4.7). Nevertheless the exposition of this section is geared towards proving 1.4.1 *bis*, and from what we do prove it is an easy exercise to obtain Theorem 1.4.1, on which little further comment will be offered.

(iv) Whether in 1.4.1 or 1.4.1 *bis*, should all the singularities be reduced we could replace the $C(\varepsilon)$ by basically the maximum over all the local residues appearing in the complete residue theorem, cf. 1.2.1.

Unfortunately the proof is complicated by the possible presence of singularities with reduced structure, which will necessarily entail a division into cases. However for simplicity let us begin by assuming that all the singularities are reduced and of Seidenberg type, so that the graph (cf. 3.1.1) of the foliation is just the blow up $\widetilde{X}$ of X in $Z(\mathbf{C})$. The induced foliation on $\widetilde{X}$ is of the form

$$\textbf{(1.4.3)} \qquad 0 \to \pi^* N \otimes \mathcal{O}_{\widetilde{X}}(E) \to \Omega^1_{\widetilde{X}} \to \pi^* L . I_{\widetilde{Z}} \to 0$$

where of course $\pi : \widetilde{X} \to X$ is the projection and $\widetilde{Z}$ the induced singular locus. On the other hand we may consider the bundle $\Omega^1_{\widetilde{X}}(\log E)$ of 1-forms with logarithmic poles. We have a natural map, $\Omega^1_{\widetilde{X}} \to \Omega^1_{\widetilde{X}}(\log E)$, and the saturation of $\pi^* N \otimes \mathcal{O}_{\widetilde{X}}(E)$ with respect to this map is actually a sub-line bundle of $\Omega^1_{\widetilde{X}}(\log E)$, i.e. we get an exact sequence of line bundles

$$\textbf{(1.4.4)} \qquad 0 \to \pi^* N \otimes \mathcal{O}_{\widetilde{X}}(2E) \to \Omega^1_{\widetilde{X}}(\log E) \to \pi^* L \to 0.$$

Let us put $U = \widetilde{X} - \widetilde{Z}$, and consider the commutative diagram with exact rows and columns arising from taking local cohomology with coefficients in (1.4.4), viz:

$$\begin{array}{ccccc}
H^1_{\widetilde{Z}}(\widetilde{X}, \pi^* N \otimes \mathcal{O}_{\widetilde{X}}(2E)) & \longrightarrow & H^1(\widetilde{X}, \pi^* N \otimes \mathcal{O}_{\widetilde{X}}(2E)) & \longrightarrow & H^1(U, \pi^* N \otimes \mathcal{O}_{\widetilde{X}}(2E)) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{\widetilde{Z}}(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log E)) & \longrightarrow & H^1(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log E)) & \longrightarrow & H^1(U, \Omega^1_U(\log E)) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{\widetilde{Z}}(\widetilde{X}, \pi^* L) & \longrightarrow & H^1(\widetilde{X}, \pi^* L) & \longrightarrow & H^1(U, \pi^* L).
\end{array}$$

Using the natural map $H^1(\widetilde{X}, \Omega^1_{\widetilde{X}}) \to H^1(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log E))$, we may consider the images of Chern classes of line bundles in the latter group. Observe of course that the image of $c_1(E)$ under this natural map is zero. Whence by the considerations of 1.1.2, $c_1(N)$ must map to zero in $H^1(U, \pi^* L)$. Moreover since everything appearing in (1.4.4) is a line bundle the local cohomology in the left hand column vanishes, and we conclude that $c_1(N)$ lies in the image of natural map $H^1(\widetilde{X}, \pi^* N \otimes \mathcal{O}_{\widetilde{X}}(2E)) \to H^1(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log E))$. Intuitively we may think of this as proving that $c_1(N)$ is zero on the foliation modulo $\bar{\partial}$ of a 1-form with logarithmic poles, which is clearly very close to proving the incomplete residue theorem. However rather than computing the above groups using a log-Dolbeault complex, it seems better to start again and compute the local cohomology of (1.4.3).

Let us compute therefore the obstruction to extending the principle enunciated in Proposition 1.1.2 across the singularities of a foliation. We will continue to assume that the singularities are reduced and we will work locally about a neighbourhood Δ_z of our singularity z. Consequently we may assume that our foliation on Δ_z is defined by a 1-form $\omega_z = x\alpha\, dy - \lambda y\beta . dx$, where x, y are some local coordinates, $x = 0$ being the equation of the exceptional divisor through x, α, β are units on Δ_z, with $\alpha(0) = \beta(0) = 1$, and $0 \neq \lambda \in \mathbf{C}$, not a strictly positive rational number.

Firstly let us put $\Delta_z^* = \Delta_z \backslash \{ z \}$, and take $\{ \Delta_\alpha : \alpha \in A \}$ to be a cover of Δ_z^* such that over Δ_α the foliation is given by dz_α for some local coordinate $z_\alpha \in \Gamma(\Delta_\alpha)$—whence of course the cover may be infinite. In any case we have that: $dz_\alpha = h_\alpha\, \omega_z$, for some $h_\alpha \in \Gamma(\Delta_\alpha)^\times$, and if $g_{\alpha\beta} = h_\alpha / h_\beta$ then $d \log(g_{\alpha\beta}) \in \Gamma(\Delta_{\alpha\beta}, \pi^* N \otimes \mathcal{O}_{\widetilde{X}}(E))$.

Consider now the restriction of (1.4.3) to Δ_z^*, i.e.

$$0 \to \pi^* N \otimes \mathcal{O}_{\widetilde{X}}(E) \,|_{\Delta_z^*} \to \Omega^1_{\Delta_z^*} \xrightarrow{\theta} L \,|_{\Delta_z^*} \to 0.$$

Then necessarily $\theta(d \log h_\alpha) \,|_{\Delta_\beta} = \theta(d \log h_\beta) \,|_{\Delta_\alpha}$ and so patch to give a section L, say, of $\Gamma(\Delta_z^*, L)$. On the other hand L is a line bundle and $\widetilde{X}$ is smooth, so $\Gamma(\Delta_z^*, L) = \Gamma(\Delta_z, L)$.

We compute the section $\ell \in \Gamma(\Delta_z, L)$ by viewing $\Gamma(\Delta_z, L)$ as $\Gamma(\Delta_z, L^\vee)^\vee$, where $L^\vee \,|_{\Delta_z} = \mathcal{O}_{\Delta_z}\, \partial$ and $\partial = x\alpha . \dfrac{\partial}{\partial x} + \lambda y \beta . \dfrac{\partial}{\partial y}$. Using, as before, $d \log(h_\alpha) \wedge \omega_z = - d\omega_z$, we find that $\ell(\partial) = - \left\{ \dfrac{\partial}{\partial x}(x\alpha) + \dfrac{\partial}{\partial y}(\lambda y \beta) \right\}$. Equally a priori $\theta\left(\dfrac{dx}{x}\right)$ defines a meromorphic section of $\Gamma(\Delta_z, L)$; however $\theta\left(\dfrac{dx}{x}\right)(\partial) = \alpha \in \Gamma(\Delta_z)^\times$ so that in fact $\theta\left(\dfrac{dx}{x}\right)$ is actually a holomorphic section, and indeed is everywhere non-zero. Whence: $(1 + \lambda)\, \theta\left(\dfrac{dx}{x}\right) + \ell \in \Gamma(\Delta_z, L . I_z)$. Consequently there exists a holomorphic form $\tau_z \in \Gamma(\Delta_z, \Omega^1_{\Delta_z})$ such that $\nu_{\alpha_z} := \tau_z - (1 + \lambda)\, \dfrac{dx}{x} - d \log(h_\alpha)$ is a section of $N \otimes \mathcal{O}_X(E)$ over Δ_α with possibly meromorphic poles along x.

Now let us examine the global consequences of this local observation. We will return to arbitrary singularities, but continue to give precisions as to which case we are in respectively. Further we will take a finite cover of $\tilde{X}$ by polydiscs $\{ \Delta_\alpha \mid \alpha \in A \} \cup \{ \Delta_z \mid z \in Z \}$. A priori we demand that Δ_z is the only element of the cover containing z, that, for $z \neq w$, Δ_z, Δ_w are far apart, and certainly therefore disjoint, while $\Delta_\alpha \cap \Delta_z \neq \varphi$ implies that Δ_α is contained in some slightly bigger polydiscs about z, finally we demand that the Frobenius theorem holds on Δ_α and of course $\{ \rho_\alpha \mid \alpha \in A \} \cup \{ \rho_z \mid z \in Z \}$ is a partition of unity subordinate to this cover. Other precisions will appear in the course of what follows.

Firstly for notational simplicity we put $\tilde{N} := \pi^* N \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{\tilde{X}}(E)$ and it will clearly suffice to prove Theorem 1.4.1 with $\tilde{N}$ instead of N. We may also write $E = \sum_{z \in Z(\mathbf{C})} E_z$, where E_z denotes the fibre of the exceptional divisor over $z \in Z(\mathbf{C})$. Let us observe:

Lemma **1.4.5.** — (i) *There exists* $\kappa_z \in \Gamma(f^{-1}(\Delta_z^*), \Omega_Y^1)$ *such that*

$$T_{f,\tilde{N}}(r) = -\frac{1}{4\pi i} \sum_{z \in \tilde{Z}(\mathbf{C})} T_{\tilde{f}}(r) \{ d(\rho_z(\kappa_z - \bar{\kappa}_z)) \} + o(T_{f,H}(r)).$$

(ii) *If* E_z *is an irreducible component of the exceptional divisor with* $\{0, \infty\}$ *the two singularities in* $\tilde{Z}(\mathbf{C}) \cap E_z$, *then for* x_0, x_∞ *local equations for* E_z *in the appropriate neighbourhoods of* 0 *and* ∞ *respectively we have*:

$$T_{f,E_z}(t) = -T_f(r) \{ d(\rho_0 \, d^c \log |x_0|^2) + d(\rho_\infty \, d^c \log |x_\infty|^2) \} + o(T_{f,H}(r)).$$

Proof. — (i) This is just a standard re-writing of the above considerations on putting $\kappa_z = f^* d\log(g_{\alpha z})$, where $(g_{\alpha\beta})$ are the transition functions for $\tilde{N}$, together with I.0.3.3.

(ii) Idem on using that E_z is a leaf of the foliation. □

Remarks **1.4.6.** — (i) This is equally true regardless of whether or not the singularities are reduced.

(ii) From now until the end of the chapter we employ the notation $T_f(r)$ in place of our previous height notation, in light of the rather computational nature of what follows.

Consequently to complete the proof of the incomplete residue estimate, at least for reduced singularities, it would suffice to show that the modulus of the height transforms of $\partial(\rho_0 \bar{\partial} \log |x_0|^2)$ and $\bar{\partial}(\rho_0 \partial \log |x_0|^2)$ are bounded by the height along the exceptional divisor. The trouble lies in their respective failures to be $\bar{\partial}$ or ∂ closed.

In any case we will be done if we can prove:

Problem **1.4.7.** — Notations being as above there is a constant $C \geqslant 0$ such that:

$$| T_f(r) \{ \partial(\rho_0 \bar{\partial} \log |x_0|^2) \} | \leqslant C . T_{f,E_z}(r) + o(T_{f,H}(r)).$$

Remark **1.4.8.** — A solution to the problem for reduced singularities, implies its solution for non-reduced singularities.

Let us now therefore proceed to consider the case of reduced singularities. We begin with a lemma:

Lemma **1.4.9.** — *For any* $W \subset \tilde{Z}$ *and* $n \in \mathbf{N}$ *we have*:

(i)
$$T_{f,\tilde{N}}(r) = -\frac{1}{4\pi i}\sum_{w \in W(C)} \lim_{n\to\infty} T_f(r)\{d(\rho_w^n(\kappa_w - \bar{\kappa}_w))\}$$
$$-\frac{1}{4\pi i}\sum_{z \in \tilde{Z}(C)-W(C)} \lim_{n\to\infty} T_f(r)\{d(\rho_z(\kappa_z - \bar{\kappa}_z))\} + o(T_{f,H}(r)).$$

(ii)
$$T_{f,E_z}(r) = -\sum_{w \in W\cap\{0,\infty\}} \lim_{n\to\infty} T_f(r)\{d(\rho_w^n\, d^c \log |x_w|^2)\}$$
$$-\sum_{w' \in (Z-W)\cap\{0,\infty\}} T_f(r)\{d(\rho_{w'}\, d^c \log |x_{w'}|^2)\} + o(T_{f,H}(r)).$$

Proof. — This is precisely the same as the method for 1.4.5 together with an observation. Namely if we put for $w \in W$, $A_w = \{\alpha \in A : \Delta_\alpha \cap \Delta_w \neq \varphi\} \subset A$, the set parametrising our cover, then for any $n \in \mathbf{N}$,

$$\rho_w = \rho_w^n + (\rho_w^{n-1} + \dots + \rho_w)\sum_{\alpha \in A_w} \rho_\alpha$$

while
$$|\rho_w + \dots + \rho_w^{n-1}|\,|\sum_{\alpha \in A_w} \rho_\alpha| \leqslant 1;$$

consequently we use the foliation hypothesis together with I.0.3.1 to obtain the lemma. □

Remark **1.4.10.** — We will use 1.4.9 (ii) as follows, i.e. if a priori z is a non-reduced singularity then on E_z we have a reduced singularity at ∞, say, and a non-reduced singularity at 0; then

$$T_{f,E_z}(r) = -\lim_{n\to\infty} T_f(r)\{d(\rho_0^n\, d^c \log |x_0|^2)\}$$
$$- T_f(r)\{d(\rho_\infty\, d^c \log |x_\infty|^2)\} + o(T_{f,H}(r)).$$

Should 1.4.7 hold, then the modulus of the latter is bounded by the height along the exceptional divisor, and whence consequently the former. In any case we can certainly bound the latter by a Lelong number at a reduced singularity, and so obtain the residue estimate in its weak form.

Now to begin our calculation. Let us first note that if $x = 0$ is the equation of the exceptional divisor through our non-reduced singularity z, and e the degree of the singularity, then we may choose a local generator of $\tilde{N}$ on a suitably small neighbourhood Δ_z of z to be of the form

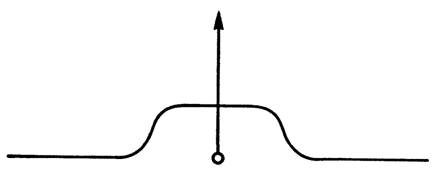
$$\omega_z = \beta\, dx - x\gamma\, dy,$$

where x is an algebraic equation defining the exceptional divisor, and y is an algebraic approximation to the branch of the foliation transverse to the exceptional divisor. Further, modulo some arbitrarily large power N of the maximal ideal of $\mathcal{O}_{X,z}$ we may a priori choose x and y so that, in addition,

$$\beta \equiv y (\mathrm{mod}\ m_{X,z}^N) \qquad \gamma \equiv x^{e-1}\, h(x)\ (\mathrm{mod}\ m_{X,z}^N),$$

where $h(x)$ is a polynomial in x with $h(0) \neq 0$. We fix a priori some sufficiently large N, about $4e$ would be more than adequate.

Finally we will make a specific choice of bump function, viz: $\rho_z = \rho(|x|^2)\,\rho(|y|^2)$ where ρ is some symmetric bump function of the form



The width of the support will be chosen as we require, but the important thing is that $\rho'(|x|^2)$ is thus non-positive. Put $\rho_x = \rho(|x|^2)$ and $\rho_y = \rho(|y|^2)$ and define, so to speak, local Chern characters, on Y (note henceforward we drop f^*)

$$c_1(\mathrm{E})_x := -f^*\,\rho_y\, d(\rho_x\, d^c \log |x|^2)$$

$$c_1(\mathrm{E})_y := -f^*\,\rho_x\, d(\rho_y\, d^c \log |x|^2)$$

and similarly for $c_1(\mathrm{N})_x$, and $c_1(\mathrm{N})_y$ with $\dfrac{\kappa_z - \bar{\kappa}_z}{4\pi i}$ replacing the $d^c \log |x|^2$.

Now if $\partial := x\gamma \dfrac{\partial}{\partial x} + \beta \dfrac{\partial}{\partial y}$ we may of course find a rational function φ defined on $f^{-1}(\Delta_z)$ such that $f_*\left(\dfrac{\partial}{\partial \zeta}\right) = \varphi(\zeta)\,(f^*\,\partial) \in f^*\,\mathrm{T_X}$; $d\mu$ will denote the usual measure on Y $\left(\text{e.g. } -\dfrac{dz \wedge d\bar{z}}{2\pi i} \text{ if } \mathrm{Y} = \mathbf{C}\right)$.

Observe then the following formulae for our so-called local Chern classes, i.e.

$$c_1(\mathrm{E})_x = \{\,\rho_y\,|\,\rho_x'\,|\,.\,|\,x\,|^2\,|\,\gamma\,|^2\,\}\,|\,\varphi\,|^2\, d\mu$$

$$c_1(\mathrm{E})_y = \{\,\rho_x\,|\,\rho_y'\,|\,.\,y\ \mathrm{Re}(\gamma\bar{\beta})\,\}\,|\,\varphi\,|^2\, d\mu$$

$$c_1(\mathrm{N})_x = -\{\,\rho_y\,|\,\rho_x'\,|\,.\,|\,x\,|^2\ \mathrm{Re}(\bar{\gamma}u)\,\}\,|\,\varphi\,|^2\, d\mu$$

$$c_1(\mathrm{N})_y = -\{\,\rho_x\,|\,\rho_y'\,|\,.\,y\ \mathrm{Re}(u\bar{\beta})\,\}\,|\,\varphi\,|^2\, d\mu$$

where $u := \dfrac{\partial}{\partial x}(\gamma) + \dfrac{\partial}{\partial y}(\beta)$, cf. 1.2.1. Now both u and "β/y" are very close to one, so we have that $c_1(\mathrm{E})_{\mathrm{loc}}$ is very positive in the horizontal direction and $c_1(\mathrm{N})_{\mathrm{loc}}$ very negative in the vertical direction (where of course $c_1(\mathrm{N})_{\mathrm{loc}} = c_1(\mathrm{N})_x + c_1(\mathrm{N})_y$, etc.) so the game is to consider a suitable constant $\mathrm{C} > 0$ such that $\mathrm{C}.c_1(\mathrm{E})_{\mathrm{loc}} - c_1(\mathrm{N})_{\mathrm{loc}}$ is positive. If we put $\dfrac{1}{\mathrm{C}+e} := \inf_{|x|=\varepsilon} \gamma$ to define C and, $c := \mathrm{C} + e - \dfrac{1}{\sup_{|x|\leqslant\varepsilon} |\gamma|}$ then using that the maximum modulus theorem is true for γ (mod $\mathrm{O}(\varepsilon^{\mathrm{N}})$) we find that $c = \mathrm{O}(\varepsilon^{2-e}) > 0$.

Further if Δ_ε is the polydisc $\{ |x| < \varepsilon, |y| < \varepsilon \}$, then on $\partial \Delta_\varepsilon$ we obtain

$$\mathrm{C}c_1(\mathrm{E})_x - c_1(\mathrm{N})_x \geqslant - \mathrm{O}(\varepsilon)\, c_1(\mathrm{E})_x$$

and

$$\mathrm{C}c_1(\mathrm{E})_y - c_1(\mathrm{N})_y \geqslant - \mathrm{O}(\varepsilon)\, \rho_x\, d(\rho_y\, d^c \log |y|^2). \tag{1.4.11}$$

Equally there is a finite thickening, all be it a small one, on which this holds, so we naturally take $\rho(t) \equiv 1$, $t \leqslant \varepsilon^2$, and $\rho(t) < 1$ for $t > \varepsilon^2$, and collapse to the boundary by considering $\rho_z^n = \rho_x^n \rho_y^n$ as $n \to \infty$.

We thus obtain:

$$\lim_{n\to\infty} \mathrm{T}_f(r) \left\{ - \mathrm{C}\, d(\rho_z^n\, d^c \log |x|^2) + d\left(\rho_z^n \wedge \frac{\kappa_z - \bar{\kappa}_z}{4\pi i}\right)\right\}$$
$$\geqslant - \mathrm{O}(\varepsilon) \lim_{n\to\infty} \mathrm{T}_f(r) \{ - d(\rho_x^n\, d^c \log |x|^2) - d(\rho_y^n\, d^c \log |y|^2) \}.$$

While we note that x, for example, is well-defined as a meromorphic function on $\tilde{\tilde{\mathrm{X}}}$, and ρ is a function with compact support on $\mathbf{P}^1$ so that the height transform of $d(\rho_x^n\, d^c \log |x|^2)$ makes sense.

However if we consider the map $f^* x : \mathrm{Y} \dashrightarrow \mathbf{P}^1$, we see that the calculation of

$$\lim_{n\to\infty} \mathrm{T}_f(r) \{ - d(\rho_x^n\, d^c \log |x|^2) \}$$

reduces to the calculation of the height with respect to the tautological bundle on $\mathbf{P}^1$ by an easier version of the methods employed previously (i.e. just use the partition of unity trick), so we conclude in particular that

$$\lim_{n\to\infty} \mathrm{T}_f(r) \{ - d(\rho_x^n\, d^c \log |x|^2) \} = \mathrm{O}(\mathrm{T}_{f,\mathrm{H}}(r))$$

and similarly for y, as required. □

2. Positivity of the cotangent bundle along a leaf

2.1. *Quick introduction*

Given a smooth foliation $\mathscr{F}$, say on a surface X, and a complete leaf $f : \mathrm{Y} \to \mathrm{X}$, with Y smooth, then we immediately have,

$$\int_{\mathrm{Y}} f^* c_1(\mathrm{L}) = 2g(\mathrm{Y}) - 2 \tag{2.1.1}$$

where $g(Y)$ is the genus of Y; consequently if Y is neither rational nor elliptic, then $\int_Y f^* c_1(L) > 0$, *i.e.* L is positive along such leaves. The goal of this section will be to prove that this continues to hold for large incomplete leaves of a singular foliation, cf. Definition 0.1.5. The singularities of any foliation appearing in this section will of course be assumed to be of Seidenberg type.

2.2. *Cohomological consequences of the residue estimate*

As always our set-up is that $p : Y \to \mathbf{C}$ is a finite ramified cover, and $f: Y \to X$ is a leaf of a given foliation $\mathscr{F}$, with $\mathrm{Im}(f)$ Zariski dense. The notations for cohomology classes associated to f are those of I.0.4, whose functoriality we will use along the map $\pi : \tilde{X} = \mathrm{Bl}_{Z(\mathbf{C})}(X) \to X$, while $\tilde{f}$ will be used undiscriminately for a lifting of f. In any case if E_z is the exceptional divisor over $z \in Z(\mathbf{C})$, then of course $\mathrm{NS}(\tilde{X})_{\mathbf{R}} = \mathrm{NS}(X)_{\mathbf{R}} \oplus \{ \bigoplus_{z \in Z} \mathbf{R}E_z \}$, and the pull back of any big divisor is again big.

Fixing an ample divisor H, recalling our definitions of a large leaf (i.e. Definition 0.1.5) and returning to our height notation, we find that the following is immediate:

Lemma **2.2.1.** — *If f is a large leaf of a foliation $\mathscr{F}$ and* $\lim_n \inf_{r \geqslant n} \dfrac{h_L(f(r))}{h_H(f(r))} \leqslant 0$, *$r$ being outside a given set of finite measure, then there exists a sequence* $R = (r_n) \subset \mathbf{R}_{\geqslant 0}$ *tending to infinity such that*:

(i) R *is convergent for* (f, H) *and* $(\tilde{f}, \pi^* H)$.

(ii) $\varphi_R . L \leqslant 0$.

(iii) $\varphi_R . N > 0$.

From now on we will assume that indeed the leaf f is large. In any case, the functoriality of our cohomology classes guarantees that, if $n_z := \tilde{\varphi} . E_z \geqslant 0$, then

$$\pi^* \varphi_R = \tilde{\varphi}_R + \sum_{z \in Z(C)} n_z E_z .$$

Whence on making a suitable choice of ε (i.e. $c\varepsilon < \min\{ \varphi_R . N, \varepsilon_0 \}$, in the notation of 1.4.1) and applying the residue estimate in its weak version we obtain:

$$\sum_{z \in Z} n_z + \sum_{w \in Z'} \nu(w, \tilde{\Phi}_R) > 0.$$

Now observe that $\tilde{\varphi}_R$ is nef, so if some n_z were strictly positive then φ_R would be not only nef, but nef and big (i.e. $\varphi_R^2 > 0$). Equally the height along any exceptional curve obtained by blowing up $\tilde{X}$ in some $w \in Z'$ is certainly bounded by some n_z, so a " Lelong number = multiplicity " type theorem would ensure this; on the other hand we cannot pove this, so dropping the dependence on R we go after:

Proposition **2.2.2.** — *If some multiple of φ is in fact in* $\mathrm{NS}(X; \mathbf{Q})$ then

$$\nu(w, \tilde{\Phi}_R) > 0 \text{ for some } w \in Z' \Rightarrow \varphi^2 > 0.$$

Proof. — We of course suppose that $\nu(w, \widetilde{\Phi}_R) > 0$ and $\varphi^2 = 0$, and proceed to obtain a contradiction. In particular the above discussion demands that all the n_z are zero, so that $\widetilde{\varphi} = \pi^* \varphi$. Whence on multiplying by a constant we may assume without loss of generality that $\widetilde{\varphi}$ is rational, and $\widetilde{\varphi}^2 = 0$, and replace the original X by $\widetilde{X}$, and similarly $\widetilde{X}$ by the blowing up of the old $\widetilde{X}$ in w, with E being the exceptional curve. Next, we apply the celebrated theorem of Siu, cf. [Si], that the Lelong numbers are upper semi-continuous in the Zariski topology, in conjunction with Proposition I.0.4.6, to obtain:

$$\nu(E) := \inf\{ \nu(x, \widetilde{\Phi}) : x \in E \} > 0.$$

Equally following Siu we may naturally extend the definition of $\nu(C)$ to arbitrary curves in $\widetilde{X}$, and use the said theorem once more (1) to obtain a closed positive (1, 1) current Ψ such that:

$$\textbf{(2.2.3)} \qquad \widetilde{\Phi} = \sum_{i=1}^{n} \nu(C_i)\, \delta_{C_i} + \Psi$$

where needless to say δ_C denotes integration over C, and the C_i are the irreducible components of the set $\{ x \in \widetilde{X} : \nu(x, \widetilde{\Phi}) \geqslant \nu(E) \} \supset E$. Consequently on taking some positive rational numbers $q_i < \nu(C_i)$ we obtain a pseudo-effective class $\psi \in NS(\widetilde{X}; \mathbf{Q})$ such that

$$\widetilde{\varphi} = \sum_{i=1}^{n} q_i\, C_i + \psi.$$

In order to proceed further we recall:

Definition **2.2.4.** — *An effective divisor* $F = \Sigma\, \delta_i F_i$, $\delta_i > 0$, F_i *integral curves, is said to be* contractible, *if the intersection matrix* $F_i . F_j$ *is negative definite.*

Along with,

Theorem **2.2.5** (Zariski decomposition) (cf. [Fuj]). — *If* $D \in NS(X; \mathbf{Q})$ *is pseudo-effective, there exists a contractible divisor* $F \in NS(X; \mathbf{Q})$ *and a nef divisor* $P \in NS(X; \mathbf{Q})$ *such that,*

$$D = P + F, \quad \textit{and} \quad P . F = 0.$$

Evidently we apply this by letting $\psi = P + F$ be the Zariski decomposition of ψ, and obtain, $\widetilde{\varphi} . P = 0$. On the other hand the Hodge index theorem leads us to conclude that P is in fact a multiple of $\widetilde{\varphi}$, and whence that $\widetilde{\varphi}$ is actually an effective class satisfying

$$\widetilde{\varphi} \geqslant \lambda . \sum_{i=1}^{n} q_i\, C_i$$

for some $\lambda > 0$, where the inequality sign indicates that the difference is an effective divisor.

Armed with all this additional information we may now prove

(1) Properly speaking we are using not the theorem quoted but Siu's decomposition theorem, cf. op. cit.

Lemma **2.2.6.** — *If* $\nu(C) > 0$, *then* C *is a leaf of the foliation.*

Proof. — Suppose indeed $\nu(C) > 0$ and C is not a leaf of the foliation, then the above discussion (or in fact a minor variant of it) shows that we can find a Cartier divisor D on $\tilde{X}$ such that C is contained in the support of D, D is nef, and $\tilde{\varphi}.D = 0$. Whence at a generic point of C we may calculate the Lelong number with a coordinate system consisting of a local equation for D and a coordinate ζ normal to the foliation, i.e. $d\zeta$ vanishes on leaves. It is then an easy exercise to see that D being nef allows us to conclude that the Lelong number is bounded by $\varphi.D$, and whence zero, which is absurd. □

On the other hand we know, cf. [J], that if the foliation has infinitely many complete leaves then it is a pencil, which is contradictory to the assumption that f has Zariski dense image, and so we may refine (2.2.3) to obtain,

$$\tilde{\Phi} = \sum_{i=1}^{n} \nu(C_i)\, \delta_{C_i} + \Psi,$$

where now the C_i are the finitely many components of the set, $\{ x \in \tilde{X} : \nu(x, \tilde{\Phi}) > 0 \}$, and Ψ is a closed positive (1, 1)-current whose Lelong numbers occur on a discrete subset of $\tilde{X}$. However for such a current Ψ Demailly, cf. [De3], has shown that its harmonic projection is actually nef and so another application of the Hodge index theorem brings us to conclude that there exists $a_i > 0$ for which

$$\varphi = \sum_{i=1}^{n} a_i\, C_i.$$

Nevertheless by hypothesis $\varphi^2 = 0$, and so, for $1 \leqslant i \leqslant n$, we must have

$$- a_i\, C_i^2 = \sum_{j \neq i} a_j\, C_i . C_j .$$

At the same time the C_i are leaves of the foliation which has singularities of Seidenberg type, and so we have that $\sum_i C_i$ is a divisor with simple normal crossings. In particular the C_i are smooth, from which we deduce:

$$N . C_i = - C_i^2 - s_Z(C_i),$$

where $s_Z(C_i)$ is the Segre class of C_i along the singular loci, Z, of the foliation. On the other hand the C_i's can only cross at the singularities of the foliation and so,

$$s_Z(C_i) \geqslant \sum_{j \neq i} C_i . C_j .$$

Combining the above observations therefore yields, for each $1 \leqslant i \leqslant n$,

$$a_i\, N . C_i \leqslant \sum_{j \neq i} a_j\, C_i . C_j - \sum_{j \neq i} a_i\, C_i . C_j .$$

Summing over i then gives $N.\varphi \leqslant 0$, in contradiction to 2.2.1. □

2.3. *Positivity or else*

Our object is to prove

Theorem **2.3.1.** — *If f is a large leaf of a foliation $\mathcal{F}$, which is not a ruling by rational curves and* $\lim_n \inf_{r \geqslant n} \frac{h_L(f(r))}{h_H(f(r))} \leqslant 0$, *$r$ being excluded from a given set of finite measure, then some bi-rational model* X *is dominated by a surface* S *on which there is a global vector field.*

Remark **2.3.2.** — The model in question may be singular.

Now modulo a rationality assumption, Lemma 2.2.1 and Proposition 2.2.2 yield a big and nef class $\varphi \in NS(X)_{\mathbf{R}}$ satisfying $L.\varphi \leqslant 0$ and $\varphi^2 > 0$. Since the reduction of a big and nef class is in fact nef in almost all finite characteristics the ideas of Ekedhal (cf. [E] or [S-B]) apply and allow us to conclude that the foliation would be a schematic equivalence relation (i.e. has a "global algebraic solution") in almost all positive characteristics. However we were able to make no additional use of this beyond what Miyaoka had already proved, by reduction to finite characteristic, viz:

Theorem **2.3.3** (Miyaoka's semi-positivity theorem for surfaces). — *If $\mathcal{F}$ is not a foliation by rational curves then* L *is pseudo effective, i.e.* $L.H \geqslant 0$ *for all ample divisors* H *on* X.

Proof. — Cf [Mi2] or [S-B].

In any case we conclude that $L.\varphi \geqslant 0$ and whence in fact $L.\varphi = 0$. On the other hand, we may let $P + F$ be the Zariski decomposition of L in the above notation, and whence: $0 = L.\varphi = P.\varphi + F.\varphi$. However φ is nef, so that in particular $P.\varphi = 0$. Now consider the possible cases: either $\varphi^2 > 0$ in which case the Hodge-index theorem guarantees that $P = 0$, or $\varphi^2 = 0$. In the latter case, however, we know $\varphi \neq 0$ so $P^2 = 0$, and again by Hodge φ is a multiple of P. This however guarantees the rationality assumption of Proposition 2.2.2, and so indeed we conclude that L is contractible, in $NS(X; \mathbf{Q})$.

Note that as Definition 2.2.4 suggests we have:

Theorem **2.3.4.** — *Let* $F_1, \ldots, F_n$ *be irreducible curves on a normal surface* X, *then the intersection matrix* $F_i.F_j$ *is negative definite if and only if the union* $\bigcup_{i=1}^{n} F_i$ *can be contracted to a finite number of normal points.*

Proof. — Cf. [Sa].

In particular let $X \xrightarrow{\pi} S$ be a normal contraction, of the contractible divisor representing L in $NS(X)_{\mathbf{Q}}$, so that outside the finite set $W \subset S$ to which we contract, S is smooth. Let us put $U := S\backslash W$. Observe that we have of course an induced foliation on S, viz:

(2.3.5) $$0 \to \mathcal{N} \to \Omega^1_S \to \mathcal{L} \to 0$$

where $\mathcal{N}$ and $\mathcal{L}$ are simply coherent rank one sheaves. Now we would like to believe that this foliation is " simpler " than the foliation on X. To be precise simpler ought to mean that, $\mathcal{L}|_U \xrightarrow{\sim} \mathcal{O}_U$, where for purely notational simplicity we suppose that $\pi(Z) \subset W$. In this context then we note the following:

Proposition **2.3.6.** — *If* $\mathcal{L}|_U \xrightarrow{\sim} \mathcal{O}_U$ *then* $\Gamma(S, \mathcal{E}_S) \neq 0$, *i.e.* S *admits a global vector field.*

Obviously then if we can reduce to the case of this Proposition we will be done. In any case let us give a proof. We begin with a lemma.

Lemma **2.3.7.** — *We have* $H^q_W(S, \mathcal{L}^\vee) = 0$ *for* $q = 0$ *or* 1.

Proof. — The question is local so we may assume that $S = \operatorname{Spec} \mathcal{O}$, and $W = \{ w \}$ is a single point, and all that is necessary to show is that $\mathcal{L}^\vee_w$ has depth at least two (there being no confusion in our notation since localisation and taking duals commute for coherent sheaves). Certainly $\mathcal{L}^\vee_w$ is torsion free, so the depth is at least one.

Furthermore S is normal and so there exists a regular sequence $\{ x, y \}$ of length two in $\mathcal{O}_w$. So suppose that y is a zero divisor in $\mathcal{L}^\vee_w / x\mathcal{L}^\vee_w$. Then there exist $\varphi, \psi \in \mathcal{L}^\vee_w$ such that: (i) $y.\varphi = x.\psi$ and (ii) $\varphi \not\equiv 0 \pmod{x\mathcal{L}^\vee_w}$. However for any $t \in \mathcal{L}_w$ we have that: $y.\varphi(t) = x\psi(t)$ and a priori y is not a zero divisor in $\mathcal{O}_w/(x)$, so we conclude that $\frac{1}{x}\varphi(t) \in \mathcal{O}_w$, which is absurd. □

Going from the lemma to the Proposition is now immediate. The long exact sequence in local cohomology gives an exact sequence:

$$0 \to \Gamma_W(S, \mathcal{L}^\vee) \to \Gamma(S, \mathcal{L}^\vee) \to \Gamma(U, \mathcal{O}_U) \to H^1_W(S, \mathcal{L}^\vee)$$

so we conclude to an isomorphism, $\Gamma(S, \mathcal{L}^\vee) \xrightarrow{\sim} \Gamma(U, \mathcal{O}_U) \xrightarrow{\sim} \mathbf{C}$, and of course we have the embedding $\Gamma(S, \mathcal{L}^\vee) \hookrightarrow \Gamma(S, \mathcal{E}_S)$, and so we are done. □

To reduce ourselves to the above proposition we divide ourselves into cases according to the irregularity of X.

Case 1, $q(X) \geqslant 2$ (2.3.8)

We have an exact sequence:

$$0 \to \Gamma(X, N) \to \Gamma(X, \Omega^1_X) \to \Gamma(X, L).$$

So either $\Gamma(X, L) \neq 0$ or $h^0(X, N) \geqslant 2$. Let us suppose the latter, i.e. that $\exists$ linearly independent, $\omega_1, \omega_2 \in \Gamma(X, N)$.

Consider then the rational map, $\varphi = \dfrac{\omega_1}{\omega_2} : X \dashrightarrow \mathbf{P}^1$. Now we know that ω_1, ω_2 are closed, and whence: $d\varphi \wedge \omega_2 = 0$, i.e. $d\varphi$ is a meromorphic section of N, whence the foliation is just the pencil $\varphi =$ constant, and we conclude that f cannot be Zariski dense, which is absurd. Thus L is isomorphic to some Cartier divisor $\mathcal{O}_X(D)$. However by the information that we have already calculated in $NS(X)_{\mathbf{Q}}$, we conclude that $f_*(D) \approxeq 0$ in $NS(S)_{\mathbf{Q}}$, but D is effective, and so this can only happen if D is supported on the contractible curves, and whence $L|_U \xrightarrow{\sim} \mathscr{L}|_U \xrightarrow{\sim} \mathcal{O}_U$.

Case 2, $q(X) = 1$ (2.3.9)

In this case the albanese of X is an elliptic curve E, and we have a map, $\pi : X \to E$. Consequently if ω is the global holomorphic differential on E either $\pi^* \omega$ restricts to a global section of L, or $\pi^* \omega \in \Gamma(X, N)$. However the latter implies that the lifting of $f^* \pi$ to the universal cover of E is constant, which is absurd, and so we are reduced to the considerations of the previous case.

Case 3, $q(X) = 0$ (2.3.10)

In this case $\mathrm{Pic}^0(X) = 0$. However the Zariski decomposition theorem is only a priori valid with rational coefficients, and so we conclude that for some $m \in \mathbf{N}$, $L^{\otimes m} \xrightarrow{\sim} \mathcal{O}_X(D)$ where D is an effective Cartier divisor supported on the contractible curves, and so: $\mathscr{L}_U^{\otimes m} \xrightarrow{\sim} \mathcal{O}_U$, i.e. we are close, but not quite done.

Evidently however we are reduced to Proposition 2.3.8 if we can prove:

Proposition **2.3.11.** — *There exists a normal space* Z *and a flat map* $p : Z \to S$ *such that the restriction,* $p^{-1}(U) \to U$ *is étale and* $\mathscr{L}|_{p^{-1}(U)}$ *is trivial.*

Proof. — Take an affine cover $\{ V_\alpha \mid \alpha \in A \}$ of U such that $(\mathscr{L}_U)^{\otimes m}$ is represented by the Cartier divisor (V_α, h_α) where $h_\alpha \in \Gamma(V_\alpha)^\times$, and m is the smallest positive integer for which this holds. Consequently if $g_{\alpha\beta}$ are the transition functions for $\mathscr{L}_U^\vee$ with respect to this cover, things may be arranged so that: $g_{\alpha\beta}^m = \dfrac{h_\alpha}{h_\beta}$.

Continuing to restrict our attention to U we define a subspace, Z_0, of $\mathbf{V}(\mathscr{L}_U^\vee)$ as the space cut out by a sheaf of ideals $\mathscr{I}$ of Sym $\mathscr{L}_U^\vee$, defined as follows. Over V_α write; $\mathscr{L}_{V_\alpha}^\vee = \mathcal{O}_{V_\alpha} T_\alpha$ so that, $T_\alpha = g_{\alpha\beta} T_\beta$. Then Sym $\mathscr{L}_{V_\alpha}^\vee = \mathcal{O}_{V_\alpha}[T_\alpha]$ and we put $\mathscr{I}_{V_\alpha} = (T_\alpha^m - h_\alpha)$. By construction then, $p : Z_0 \to U$ is étale and $\mathscr{L}|_{Z_0}$ is isomorphic to the trivial line bundle.

To complete over the singularities observe that $k(Z_0)/k(S)$ is galois, and that $\mathcal{O}_{Z_0}$ is the integral closure of $\mathcal{O}_U$ in $k(Z_0)$. Define Z therefore to be the integral closure of $\mathcal{O}_S$ in $k(Z_0)$, then since the extension is galois the fibres of Z over S are conjugate under galois so we get the extra bonus that $Z \to S$ is not only finite, but flat. □

3. Refined tautological class inequality

3.1. *What is left to do?*

Clearly a combination of the tautological inequality and Theorem 2.3.1 is very close to completing the demonstration of Theorem B. However for a singular foliation $\mathscr{F}$

there will certainly be a difference between L and the tautological bundle. To facilitate examining this difference let us make the following definition,

Definition **3.1.1.** — *The graph* $\tilde{\mathrm{X}}$ *of a foliation* $\mathcal{F}$ *on* X *is the corresponding quasi-section* (*cf.* II.0.1) *in* $\mathbf{P}(\Omega^1_{\mathrm{X}})$ [1].

Remark **3.1.2.** — Even if $\mathcal{F}$ has singularities of Seidenberg type, since $\tilde{\mathrm{X}} \xrightarrow{\sim} \mathrm{Bl}_{\mathrm{Z}}(\mathrm{X})$, and Z may carry reduced structure, $\tilde{\mathrm{X}}$ will not in general be smooth.

Now if $\mathcal{O}_{\tilde{\mathrm{X}}}(1)$ denotes the restriction of the tautological bundle to the graph of the foliation and E is the total exceptional divisor then, $\nu^* \mathrm{L} = \mathcal{O}_{\tilde{\mathrm{X}}}(1) \otimes_{\mathcal{O}_{\tilde{\mathrm{X}}}} \mathcal{O}_{\tilde{\mathrm{X}}}(\mathrm{E})$ where $\nu : \tilde{\mathrm{X}} \to \mathrm{X}$ is the projection (retaking the notations of I.1.1.1). At this point it may seem that the optimal course of action would be to refine our method of diophantine approximation so as to obtain an inequality for L, however for arbitrary X this is certainly false—e.g. consider a suitable foliation by elliptic curves on $\mathbf{P}^2$—but what we will do is to prove a height inequality for the tautological class plus "enough" of the exceptional divisor.

3.2. *Graphs of foliations of Seidenberg type*

In this paragraph, and until the end of the paper $\mathcal{F}$ will be a foliation of Seidenberg type. Now let $\tilde{\mathrm{X}}$ be its graph, then the singularities of $\tilde{\mathrm{X}}$ appear by blowing up in any reduced structure that may arise in the singularities of $\mathcal{F}$. However we have already observed that a non-reduced singularity of Seidenberg type and degree $d \geqslant 2$ is equivalent to writing a local generator for N, in some formal coordinates x, y about the singularity, as $x\,dy + \ell(y)\,dx$, where $\ell(y) = cy^d + \ldots \in \mathbf{C}[[y]]$, $c \neq 0$. Consequently if $\mathrm{X}_1 \to \tilde{\mathrm{X}}$ is a minimal smooth resolution of $\tilde{\mathrm{X}}$ then its local description in a neighbourhood of a singularity is very simple, viz:

a) Evidently if the singularity is reduced then locally about the singularity $\mathrm{X}_1 \to \mathrm{X}$, is just the blow up in the singularity, and it is locally isomorphic to $\tilde{\mathrm{X}}$.

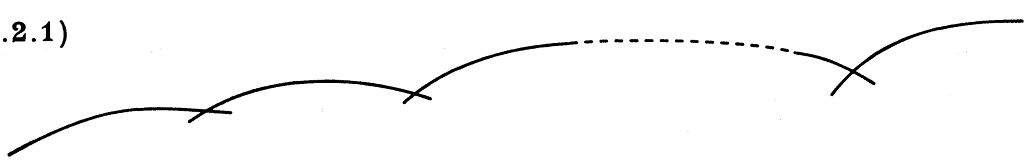
b) If the singularity is reduced of degree $d \geqslant 2$ then locally about the singularity $\mathrm{X}_1 \to \mathrm{X}$ is obtained by a sequence of monoidal transformations,

$$\mathrm{X}_1 = \mathrm{V}_d \to \mathrm{V}_{d-1} \to \ldots \to \mathrm{V}_1 \to \mathrm{V}_0 = \mathrm{X},$$

where $\mathrm{V}_i \to \mathrm{V}_{i-1}$ is the blow up of V_{i-1} in the reduced structure at the point where the induced foliation has a singularity of degree d.

Whence the fibre of $\mathrm{X}_1 \to \mathrm{X}$ over a singularity of degree d is of the form,

(3.2.1)



i.e. a chain of d rational curves.

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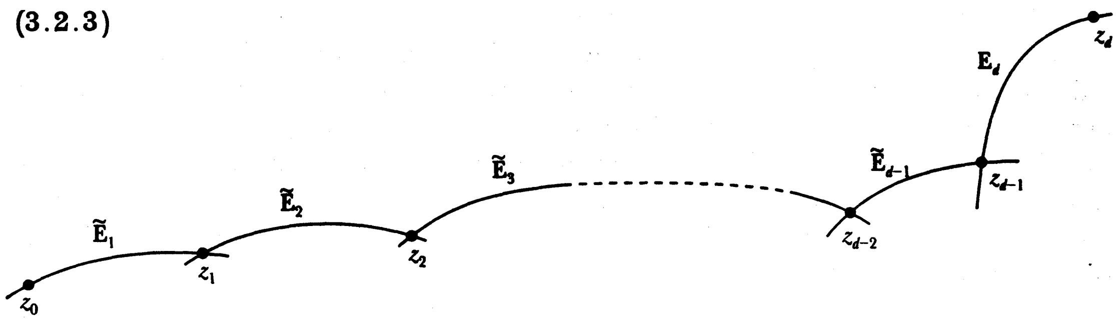
[1] Given the context, no confusion is anticipated with the use of the words graph of a foliation as employed by Connes, et al., cf. [C].

Now let us introduce some notation, i.e. if $X_1 \to \widetilde{X} \to X$, is a minimal smooth resolution of the graph then we know that $X_1 \to X$ is obtained via a sequence of monoidal transformations, $X_1 = T_n \to T_{n-1} \to \dots \to T_1 \to T_0 = X$ where $T_i \to T_{i-1}$ is a blow up in the reduced structure of a singularity of the induced foliation on T_{i-1}. Denote therefore by E_i the pull-back to X_1 of the exceptional divisor arising from the monoidal transformation $T_i \to T_{i-1}$. Certainly therefore $E_i . E_j = -\delta_{ij}$ and since $\mathscr{F}$ is supposed to be of Seidenberg type, it is consequently very stable, whence if $\mathscr{F}_1$ is the induced foliation on X_1 then it is given by

(3.2.2) $$0 \to \nu^* N \otimes \mathcal{O}_{X_1}(E_1 + \dots + E_n) \to \Omega^1_{X_1} \to L I_{Z_1} \to 0$$

where of course we continue to denote the projection to X by ν, and Z_1 is the induced singular locus. Observe that we have shown that $n = \deg(Z)$ and so by (1.3.6) $\deg(Z_1) = 2 \deg(Z)$. In fact we may make our picture (3.2.1) of the fibre over a singularity $z \in Z(\mathbf{C})$ of degree d more precise, i.e.

(3.2.3)



The $\sim$ of course denotes proper transform, while $\forall\, 0 \leqslant i \leqslant d-1$, z_i is a singularity of degree 1, with z_d a singularity of degree d for the induced foliation $\mathscr{F}_1$.

In addition we have of course a short exact sequence of line bundles on X_1, viz

(3.2.4) $$0 \to \nu^* N \otimes \mathcal{O}_{X_1}(E_1 + \dots + E_n) \to \nu^* \Omega^1_X \to \nu^* L \otimes \mathcal{O}_{X_1}(-E_1 \dots - E_n) \to 0.$$

From which we conclude that $\mathcal{O}_{\widetilde{X}}(1)\,|_{X_1} = \nu^* L \otimes \mathcal{O}_{X_1}(-E_1 \dots - E_n)$. Now let us suppose that $\lim_n \inf_{r \leqslant n} \dfrac{h_{\mathcal{O}_{\widetilde{X}}(1)}(f'(r))}{h_H(f(r))} \leqslant 0$, $r \in \mathbf{R}_{\geqslant 0}$ being outside a given set of finite measure, and consider the consequences. As in § 2.2 and 2.3 we of course associate cohomology classes $\varphi_1 \in NS(X_1)_{\mathbf{R}}$ and $\varphi_0 \in NS(X)_{\mathbf{R}}$ to f for a suitable sequence $R \subset \mathbf{R}$ such that,

$$a)\ \nu_* \varphi_1 = \varphi_0 \qquad b)\ \int_X c_1(L) \cap \varphi_0 > 0 \qquad c)\ \int_{X_1} c_1(\mathcal{O}_{\widetilde{X}}(1)) \cap \varphi_1 \leqslant 0.$$

From which of course we conclude,

$$\text{(3.2.5)} \qquad \sum_{i=1}^{n} \int_{X_1} c_1(E_i) \cap \varphi_1 \geqslant \int_{X_0} c_1(L) \cap \varphi_0 > 0.$$

So that as previously observed Theorem B will certainly follow if we can refine the tautological inequality sufficiently by adding " enough " of the exceptional divisor term.

Consider therefore the following sequence of varieties, viz:

$$\text{(3.2.6)} \qquad \ldots \to X_{N+1} \to X_N \to X_{N-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_i \to X_{i-1}$ is the minimal smooth resolution of the graph of the induced foliation on X_{i-1}. Further let $\mathcal{O}_{X_i}(1)$ denote the restriction of the tautological bundle on the graph to X_i, and $E_{i1}, \ldots, E_{ik(i)}$ the exceptional divisors appearing in the sequence of monoidal transformations that give rise to X_i from X_{i-1}, precisely as we defined the $E_1, \ldots, E_n$ above. Observe that if $\nu_i : X_i \to X$ is the projection then,

$$\mathcal{O}_{X_i}(1) = \nu_i^* L \otimes \mathcal{O}_{X_i}(\sum_{j=1}^{k(i)} - E_{ij}).$$

Yet on the other hand, $\mathcal{O}_{X_{i-1}}(1)\,|_{X_i} = \nu_i^* L \otimes \mathcal{O}_{X_i}(\sum_{j=1}^{k(i-1)} - \lambda_i^* E_{i-1,\,j})$, where $\lambda_i : X_i \to X_{i-1}$ is the projection. However: $\sum_{j=1}^{k(i)} E_{ij} < \sum_{j=1}^{k(i-1)} \lambda_i^* E_{i-1,\,j}$, and so at each stage we certainly obtain a refinement of the tautological inequality on X, so let us suppose that this is still not good enough to prove Theorem B, i.e. if $f_i : Y \to X_i$ is the i-th lifting of f then we suppose that $\lim_n \inf_{r \geqslant n} \dfrac{h_{\mathcal{O}_{X_i(1)}}(f_i(r))}{h_H(f(r))} \leqslant 0$ being outside a given set of finite measure.

In any case let us denote by $F_i \subset \mathbf{R}_+$ the set of finite measures outside which the tautological inequality holds for f lifted to X_i; then there exists some large $t_i \in \mathbf{R}_+$ such that if $F_i' := F_i \cap [0, t_i]$, and $F_i'' := F_i - F_i'$, the measure of F_i'' is bounded by $1/i^2$. Whence if we put $F'' := \bigcup_{i=1}^{\infty} F_i''$, F'' is of bounded measure. Consequently we obtain a sequence of classes $\varphi_i \in NS(X_i, \mathbf{R})$ for which the tautological inequality holds, and such that:

$$\text{(3.2.7)} \qquad a)\ (\lambda_i)_*(\varphi_i) = (\lambda_{i-1})_*(\varphi_{i-1}) = \ldots = (\lambda_1)_*(\varphi_1) = \varphi_0$$

$$b)\ \int_X c_1(L) \cap \varphi_0 > 0$$

$$c)\ \int_{X_i} c_1(\mathcal{O}_{X_i}(1)) \cap \varphi_i \leqslant 0.$$

Now let us define non-negative real numbers α_{ij} by the formula,

$$\text{(3.2.8)} \qquad \alpha_{ij} := \int_{X_i} c_1(\mathcal{O}_{X_i}(E_{ij})) \cap \varphi_i.$$

We must have

$$(3.2.9)\qquad \varphi_i = \lambda_i^* \varphi_{i-1} - \sum_{j=1}^{k(i)} \alpha_{ij} E_{ij}.$$

From which we obtain,

$$(3.2.10)\qquad \varphi_N = \nu_N^* \varphi_0 - \sum_{i=1}^{N} \sum_{j=1}^{k(i)} \alpha_{ij} \nu_{Ni}^* E_{ij},$$

where $\nu_{Ni} : X_N \to X_i$ is the natural map. On the other hand $f_N : Y \to X_N$ continues to have Zariski-dense image, so that II.2.2.5 applies to give $\varphi_N^2 \geqslant 0$, i.e.

$$(3.2.11)\qquad \varphi_0^2 \geqslant \sum_{i=1}^{N} \sum_{j=1}^{k(i)} \alpha_{ij}^2.$$

However by (3.2.5) we have

$$(3.2.12)\qquad \sum_{j=1}^{k(i)} \alpha_{ij} \geqslant \int_X c_1(L) \cap \varphi_0 := \varepsilon > 0, \quad \forall\, i \in \mathbf{N}.$$

Initially one might hope that (3.2.11) and (3.2.12) are mutually contradictory, however one has the basic estimate

$$(3.2.13)\qquad \sum_{j=1}^{k(i)} \alpha_{ij}^2 \geqslant \frac{1}{k(i)} \{\sum_{j=1}^{k(i)} \alpha_{ij}\}^2,$$

while unfortunately $k(i) = 2^i \deg(Z)$, so we can only obtain a contradiction by reducing the number of singularities that we need to count at the i-th stage. This will be done in the next section, the singularities in question being those through which there are two complete leaves of the foliation.

3.3. *Proof of refined tautological inequality, and end of demonstration*

Let us begin with a definition.

Definition **3.3.1.** — *A singularity $z \in Z(\mathbf{C})$ of a foliation $\mathcal{F}$ will be called* small *if*:

(i) *The scheme structure of Z supported at z is reduced.*

(ii) *There exist precisely two leaves of the foliation through z, both complete, whose intersection—scheme-theoretically—is precisely z.*

Furthermore we will denote by $Z' \subset Z$ the set of all small singularities (so that in particular $Z' = Z'(\mathbf{C})$) and $Z'' := Z - Z'$.

As in the previous section we continue to denote by $\widetilde{X} \hookrightarrow \mathbf{P}(\Omega_X^1)$ the graph of the foliation, and $\mathcal{O}_{\widetilde{X}}(1)$ the restriction to $\widetilde{X}$ of the tautological bundle. The fibre over a small singularity z will of course be a smooth rational curve, E_z, which is in fact a Cartier divisor since the singularity is reduced. This being so, our refined tautological inequality is

Theorem **3.3.2.** — *Let* H *be an ample bundle on* X; *then*

$$h_{\mathcal{O}_{\widetilde{X}}(1)}(f'(r)) + \sum_{z \in Z'} h_{\mathcal{O}_{\widetilde{X}}(E_Z)}(f'(r)) \leqslant d(f(r)) + O(\log r + \log h_H(f(r))).$$

In fact when $Y = \mathbf{C}$ we can prove a much better inequality, valid for a perfectly general $f: \mathbf{C} \to X$ which omits both the hypothesis on the foliation, and indeed the dimension, viz:

Theorem **3.3.2** *bis.* — *Let* X *be a smooth projective variety, with* H *ample on* X, Z′ *a reduced set of dimension zero,* $f: \mathbf{C} \to X$ *a holomorphic map and* $\delta > 0$; *then, if* $f': \mathbf{C} \to \mathbf{P}(\Omega^1_X)$ *is the lifting to the projective tangent space, we have*:

$$h_{\mathcal{O}_{\mathbf{P}(\Omega^1_X)}(1)}(f'(r)) + m_{f,Z'}(r) \leqslant \delta h_H(f(r)) + O(\log r + \log h_H(f(r))),$$

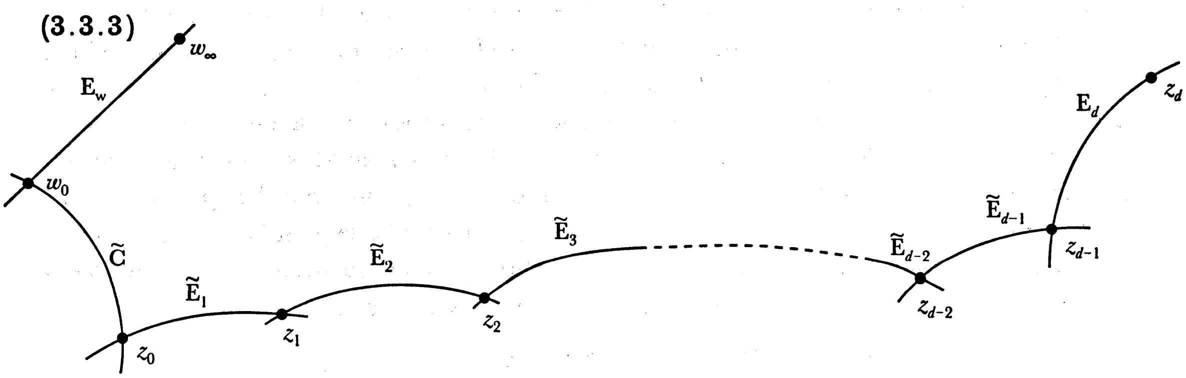
where $m_{f,Z}(r)$ *is the defect of f with respect to* Z, *i.e. the defect with respect to the exceptional divisor on the blowing up of* X *in* Z.

The inequality being valid off a set of finite measure which depends on H, but not δ. The implied constant in the error term depending on δ only in so much as $O_\delta(\log r + \log T_{f,H}(r)) = O(\log r + \log T_{f,H}(r)) + O_\delta(1)$, where the subscript, or lack of it, denotes dependence or non-dependence on δ respectively. Evidently the previous conditions on small singularities are such that they allow us to conclude a weak version of Theorem 3.3.2 from Theorem 3.3.2 *bis*.

Bearing in mind the considerations of the previous section, let us proceed to complete the proof of Theorem B, while assuming Theorem 3.3.2. In fact let us assume the slightly worse error term of Theorem 3.3.2 *bis*, for fun.

Observe that if our goal is to obtain a contradiction from the convergence of the series appearing in (3.2.11) we may without loss of generality replace $X_0 = X$ by X_1, i.e. we may assume that through each singularity of the foliation there is a smooth rational curve of the form indicated in (3.2.3). Whence if z is a singularity of degree d, and C the aforesaid complete rational leaf of the foliation, then there is a singularity $w \in C$, distinct from z, of degree 1, and the fibre over C in X_1 may be represented by a diagram of the form:

(3.3.3)



From which we see that $\deg(Z_1'') \leqq \deg(Z)$, and indeed for any $i \in \mathbf{N}$, $\deg(Z_i'') \leqslant \deg(Z)$, since a monoidal transformation in a small singularity must lead to two new small singularities, given that a priori we are assuming $\mathscr{F}$ to be of Seidenberg type with at most two leaves through any point.

Let us therefore, for each $i \in \mathbf{N}$, denote by E_{ij}, $1 \leqslant j \leqslant k(i)''$, those divisors among our previous set E_{ij}, $1 \leqslant j \leqslant k(i)$, which arise on X_i from monoidal transformations in singularities which are not small; of course $k(i)'' = \deg(Z_i'') \leqslant \deg(Z)$.

Now this time we will assume that the height of f_i with respect to $\mathcal{O}_{X_i}(1) \otimes \mathcal{O}_{X_i}(\Sigma_{z \in Z'} E_z)$ does not dominate a suitable multiple of the ample height outside a given set of finite measure (i.e. as previously

$$\lim_n \inf_{r \geqslant n} \frac{1}{h_H(f(r))} \cdot h_{\mathcal{O}_{X_i}(1) \otimes \mathcal{O}_{X_i}(\sum_{z \in Z'} E_z)}(f_i(r)) \leqslant 0);$$

then proceeding as in the previous section and with the same notations we obtain a sequence of classes $\varphi_i \in \mathrm{NS}(X_i; \mathbf{R})$ such that

(3.3.4)
$$a)\ (\lambda_i)_*(\varphi_i) = (\lambda_{i-1})_*(\varphi_{i-1}) = \ldots = (\lambda_1)_*(\varphi_1) = \varphi_0,$$

$$b)\ \int_X c_1(L) \cap \varphi_0 > 0,$$

$$c)\ \int_{X_i} \{c_1(\mathcal{O}_{X_i}(1)) + \sum_{z \in Z_i'} c_1(\mathcal{O}_{X_i}(E_z))\} \cap \varphi_i \leqslant 0.$$

Consequently arguing as in (3.2.8)-(3.2.13) with the same notations we obtain,

(3.3.5)
$$\sum_{j=1}^{k(i)''} \alpha_{ij} \geqslant \int_X c_1(L) \cap \varphi_0 := \varepsilon > 0.$$

Whence we obtain,

(3.3.6)
$$\varphi_0^2 \geqslant \frac{N}{\deg(Z)} \varepsilon^2, \text{ for any suitably large } N.$$

An evident contradiction from which we conclude that there exists some suitable constant $\alpha(f)$ depending on f, for which Theorem B holds in the form claimed.

Let us now retake the notations of I.1.1.1 and proceed to the proof of the refined tautological inequality in the harder form of Theorem 3.3.2 *bis*. As noted previously the dynamic approach appears to be more adapt to producing inequalities involving proximity functions. To begin with, observe that through a small singularity there are precisely two leaves, both complete, whence since we suppose that $\mathrm{Im}(f)$ is Zariski-dense we conclude that f cannot meet a small singularity. Consequently we will suppose for simplicity that f does not meet Z'; the additional complication is not a problem as exemplified in [M2]; in any case if we were to blow up W in

$$\nu^{-1}(z \times z \times 1) = \mathbf{P}(\Omega_X^1 \otimes k(z) \oplus \mathbf{1})$$

then F would lift from S to this blowing up, and a dynamic intersection estimate with respect to the proper transform of $\widetilde{\Delta \times 1}$ ought to give us what we are looking for. Unfortunately a dynamic intersection estimate of a suitable form appears to be false for this naive construction whence we proceed as follows, notations being fixed once more according to those in the diagram;

(3.3.7)
$$\begin{array}{ccc}
W_3 := \mathrm{Bl}_{\widetilde{P}_{Z'}}(W_2) & & \\
\downarrow \nu & \searrow \nu & \\
W_2 := \mathrm{Bl}_{\widetilde{\mathbf{P}}^1_{Z'}}(W_1) & & \\
\downarrow \nu & & \mathrm{Bl}_{Z' \times Z' \times \mathbf{P}^1}(W_0) \\
W_1 := W = \mathrm{Bl}_{\Delta \times 1}(X \times X \times \mathbf{P}^1) & & \\
\downarrow \nu & \swarrow \nu & \\
W_0 := X \times X \times \mathbf{P}^1 & &
\end{array}$$

Note that if $z \in Z'$ then $\nu^{-1}(z \times z \times \mathbf{P}^1)$ is the union of the two subvarieties $P_z := \mathbf{P}(\Omega^1_X \otimes k(z) \oplus \mathbf{1}) \hookrightarrow \widetilde{\Delta \times 1}$, and the proper transform $\widetilde{\mathbf{P}}^1_z$ of $z \times z \times \mathbf{P}^1$. Naturally then, $\widetilde{\mathbf{P}}^1_{Z'} := \bigcup_{z \in Z'} \widetilde{\mathbf{P}}^1_z$, and $P_{Z'} = \bigcup_{z \in Z'} P_z$, the two unions being disjoint. While, moreover, P_z and $\widetilde{\mathbf{P}}^1_z$ meet in the zero section $0_z = [0, 0, 1]$ over z. Equally therefore we denote $0_{Z'} := \bigcup_{z \in Z'} 0_z$. Naturally then $\widetilde{P}_{Z'}$ is just the proper transform of $P_{Z'}$ in W_2, i.e. $\widetilde{P}_{Z'} = \bigcup_{z \in Z'} \mathrm{Bl}_{0_z}(P_z)$. Now let us denote by ${}^2\widetilde{\mathbf{P}}^1_{Z'}$ the total transform of $\widetilde{\mathbf{P}}^1_{Z'}$ in W_2; the total transform of ${}^2\widetilde{\mathbf{P}}^1_{Z'}$ in W_3 is again some irreducible divisor denoted ${}^3\widetilde{\mathbf{P}}^1_{Z'}$, while similarly we write ${}^3\widetilde{P}_{Z'}$ for the total transform of $\widetilde{P}_{Z'}$ in W_3. In addition one sees, on explicit computation, that with respect to $W_3 \to W_0$ the inverse of the ideal defining $Z' \times Z' \times \mathbf{P}^1$ is precisely $\mathcal{O}_{W_3}(-{}^3\widetilde{P}_{Z'}) . \mathcal{O}_{W_3}(-{}^3\widetilde{\mathbf{P}}^1_{Z'})$. Denoting then by D the total exceptional divisor on $\mathrm{Bl}_{Z' \times Z' \times \mathbf{P}^1}(W_0)$, we have

(3.3.8) $$\nu^* D = {}^3\widetilde{P}_{Z'} + {}^3\widetilde{\mathbf{P}}^1_{Z'}.$$

The reason for all this being, as we shall see, that a good dynamic intersection estimate with respect to D is very straightforward. In any case let us note further that the proper transform ${}^2\widetilde{\Delta \times 1}$ of $\widetilde{\Delta \times 1}$ in W_2 is isomorphic to $\mathrm{Bl}_{0_{Z'}}(\mathbf{P}(\Omega^1_X \oplus \mathcal{O}_X))$, while the proper transform ${}^3\widetilde{\Delta \times 1}$ of ${}^2\widetilde{\Delta \times 1}$ in W_3 is isomorphic to $\mathrm{Bl}_{\widetilde{P}_{Z'}}({}^2\widetilde{\Delta \times 1})$ and that the total transform of $\widetilde{\Delta \times 1}$ with respect to $W_3 \to W_1$ is given by

(3.3.9) $$\nu^*(\widetilde{\Delta \times 1}) = {}^3\widetilde{\Delta \times 1} + {}^3\widetilde{P}_{Z'}.$$

As indicated previously, then, our plan will be to carry out the dynamic intersection procedure with respect to ${}^3\widetilde{\Delta \times 1}$. Before proceding, let us consider what answer we may expect. As in § 1.2 let us denote $\widetilde{\Delta \times 1} \xrightarrow{\sim} \mathbf{P}(\Omega^1_X \oplus \mathcal{O}_X)$ by P; then as previously indicated we have the tower of blow-ups arising from the construction of (3.3.7) restricted to $\widetilde{\Delta \times 1}$, viz:

$$\textbf{(3.3.10)} \qquad \begin{array}{ccccccc} {}^3\widetilde{0}_{Z'} & \hookrightarrow & {}^3[\widetilde{0}] \cup {}^3\widetilde{0}_{Z'} & \hookrightarrow & {}^3\widetilde{\Delta \times 1} = \mathrm{Bl}_{\widetilde{P}_{Z'}}({}^2\widetilde{\Delta \times 1}) & \hookleftarrow & {}^3\widetilde{P}_{Z'} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ {}^2\widetilde{0}_{Z'} & \hookrightarrow & {}^2[\widetilde{0}] \cup {}^2\widetilde{0}_{Z'} & \hookrightarrow & {}^2\widetilde{\Delta \times 1} = \mathrm{Bl}_{0_{Z'}}(P) & \hookleftarrow & \widetilde{P}_{z'} = \mathrm{Bl}_{0_{Z'}}(P_{z'}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0_{Z'} & \hookrightarrow & [0] & \hookrightarrow & \widetilde{\Delta \times 1} = \mathbf{P}(\Omega'_X \oplus \mathcal{O}_X) = P & \hookleftarrow & P_{Z'}. \end{array}$$

Those objects not already defined are defined by the condition that all the squares in the diagram are cartesian; note in addition that the fibre of [0] over ${}^2\widetilde{\Delta \times 1} \to \widetilde{\Delta \times 1}$ is indeed scheme-theoretically as indicated. Consequently we obtain a map

$$\nu : V_1 := \mathrm{Bl}_{{}^3[\widetilde{0}]}({}^3\widetilde{\Delta \times 1}) \to V_0 := \mathrm{Bl}_{[0]}(P)$$

with

$$\textbf{(3.3.11)} \qquad \nu^* \mathcal{O}_{V_0}({}^2[\widetilde{0}]) = \mathcal{O}_{V_1}({}^3\widetilde{0}^{\#}_{Z'} + {}^3[\widetilde{0}]^{\#}),$$

where # denotes taking the fibre with respect to the map, $V_1 \to {}^3\widetilde{\Delta \times 1}$. Necessarily therefore by I.1.2.1 we have a map $\nu : V_1 \to \mathbf{P}(\Omega^1_X)$ and the formula

$$\textbf{(3.3.12)} \qquad \mathcal{O}_P(1)\,|_{V_1} = \nu^* \mathcal{O}_{\mathbf{P}(\Omega_X)}(1) \otimes_{\mathcal{O}_{V_1}} \mathcal{O}_{V_1}({}^3\widetilde{0}^{\#}_{Z'} + {}^3[\widetilde{0}]^{\#}).$$

Moreover if for $z \in Z'$ we consider $E_z := \mathbf{P}(\Omega^1_X \otimes k(z)) \hookrightarrow \mathbf{P}(\Omega^1_X)$, and of course $E_{Z'} := \bigcup_{z \in Z'} E_z$, then the scheme-theoretic pre-image $\nu^{-1}(E_{Z'})$ is just ${}^3\widetilde{0}^{\#}_{Z'} \cup {}^3\widetilde{P}^{\#}_{Z'}$ in V_1 so that we obtain a map $\nu : V_1 \to \widetilde{\mathbf{P}(\Omega^1_X)} := \mathrm{Bl}_{E_{Z'}}(\mathbf{P}(\Omega^1_X))$; consequently if $\widetilde{E}_{Z'}$ is the total exceptional divisor we have the formula,

$$\textbf{(3.3.13)} \qquad \nu^* \widetilde{E}_{Z'} = {}^3\widetilde{0}^{\#}_{Z'} + {}^3\widetilde{P}^{\#}_{Z'}.$$

Whence combining (3.3.9), (3.3.12) and (3.3.13) we obtain,

$$\textbf{(3.3.14)} \qquad \mathcal{O}_{{}^3\widetilde{\Delta \times 1}}(-\,{}^3\widetilde{\Delta \times 1})\,|_{V_1} = \nu^*\{\, \mathcal{O}_{\mathbf{P}(\Omega^1_X)}(1) \otimes \mathcal{O}_{\widetilde{\mathbf{P}(\Omega^1_X)}}(\widetilde{E}_{Z'})\,\} \otimes_{\mathcal{O}_{V_1}} \mathcal{O}_{V_1}({}^3[\widetilde{0}]).$$

An estimate of the form already obtained in the dynamic intersection procedure of Chapter I with respect to ${}^3\widetilde{\Delta \times 1}$ will therefore yield precisely what we are looking for.

Turning then to the question of justifying a dynamic intersection estimate in this case, and denoting by π_i (respectively π) the projection from W_3 to the i-th X factor (respectively $\mathbf{P}^1$), let us put $G = \pi_1^* H^m \otimes \pi_2^* H^m \otimes \pi^* \mathcal{O}_{\mathbf{P}^1}(m) \otimes \mathcal{O}_{W_3}(-\widetilde{{}^3\Delta \times 1})$ for some sufficiently large $m \in \mathbf{N}$ to be decided. Then we have

Lemma **3.3.15.** — *There exists a nef bundle* B *on* W_3 *such that* $G = B + \nu^* D$, *where* D *is as in* (3.3.8).

Proof. — Observe that,

$$B := G - \nu^* D = \pi_1^* H^m \otimes \pi_2^* H^m \otimes \pi^* \mathcal{O}_{\mathbf{P}^1}(m) \otimes \nu^* \mathcal{O}_{W_2}(-\widetilde{\Delta \times 1} - {}^2\widetilde{\mathbf{P}}^1_{Z'})$$

is just the pull back of some bundle on W_2. On the other hand for $m \gg 0$, $\pi_1^* H^m \otimes \pi_2^* H^m \otimes \pi^* \mathcal{O}_{\mathbf{P}^1}(m) \otimes \mathcal{O}_{W_1}(-\widetilde{\Delta \times 1})$ is certainly ample on W_1, while taking m sufficiently large guarantees that the intersection of B with an integral curve on W_2 is positive unless the curve is the proper transform of a curve on W_1 contracted by $W_1 \to W$. Consequently the lemma reduces to the observation that if $h : \mathrm{Bl}_0(\mathbf{P}^n) \to \mathbf{P}^n$, is the blowing up of $\mathbf{P}^n$ in a point, and E the exceptional divisor then $h^* \mathcal{O}_{\mathbf{P}^2}(1) \otimes \mathcal{O}(-E)$ is nef on $\mathrm{Bl}_0(\mathbf{P}^n)$. □

We now fix $\delta > 0$ a small rational number, and M an ample bundle on W_3 so that for some sufficiently large $n \in \mathbf{N}$ we have that: $A := n(B + \delta M)$ is very ample on W_3. Moreover since f misses Z' we have that there is a lifting of $F : S \to W_1$ to a map $F : S \to W_3$ (cf. I, § 1 and § 2), so that repeating the arguments—and notations—of § 2 we find, for almost all ξ and all $|\xi| > \varepsilon > 0$ that,

$$\textbf{(3.3.16)} \qquad \delta \int_{|\eta| = \varepsilon} h_M(\widetilde{\Delta}^*_{\xi,\eta} F(1))\, \mu(d\eta) + \int_{|\eta| = \varepsilon} h_B(\widetilde{\Delta}^*_{\xi,\eta} F(1))\, \mu(d\eta) \geqslant h_B(\widetilde{\Delta}^*_{\xi,0} F(1)) + O(\log |\xi|).$$

On the other hand M must necessarily be of the form $\pi_1^* H \otimes \pi_2^* H \otimes \pi^* \mathcal{O}_{\mathbf{P}^1}(m)$ minus some suitably weighted sum of the exceptional divisors arising in the blowing up process to obtain W_3, whence on controlling the proximity functions à la I.2.4, we obtain for ε and $|\eta/\xi|$ bounded by some appropriate a priori constant,

$$\textbf{(3.3.17)} \qquad \int_{|\eta| = \varepsilon} h_B(\widetilde{\Delta}^*_{\xi,\eta} F(1))\, \mu(d\eta) + 2\,\delta h_H\,(f(|\xi| + |\varepsilon|)) \geqslant h_B(\widetilde{\Delta}^*_{\xi,0} F(1)) + O\,(\log |\xi| + |\log \varepsilon|).$$

Evidently then we are very close to producing the necessary analogue of Proposition 2.3.3, and hence Proposition 2.3.7, provided that we can get a good dynamic intersection type estimate with respect to D. However, this is reasonably straightforward, in fact we have:

Proposition **3.3.18.** — *Let* $\varepsilon(|\xi|)$ *be a function bounded by* $1/r^2\, T^3_{f,\mathrm{H}}(|\xi|)\log^6(T_{f,\mathrm{H}}(|\xi|))$, *and also by a suitable constant as indicated above, then for all* $|\xi|$ *outside a set of finite measure,*

$$\int_{|\eta|=\varepsilon(|\xi|)} \{h_{\nu^*\mathrm{D}}(\widetilde{\Delta}^*_{\xi,\eta}\,\mathrm{F}(1)) - h_{\nu^*\mathrm{D}}(\widetilde{\Delta}^*_{\xi,0}\,\mathrm{F}(1))\}\,\mu(d\eta)\,| \leqslant \mathrm{O}(1).$$

Remark **3.3.19.** — It is evident from the discussion of I, § 2 that a bound of the type indicated holds without the absolute value sign on the left hand side; needless to say we require the stronger estimate as given above.

Proof. — Let us denote by $\widetilde{\mathrm{W}}$ the blow up of $\mathrm{X}\times\mathrm{X}$ in $\mathrm{Z}'\times\mathrm{Z}'$; then, since f misses Z', $f\times f:\mathrm{Y}\times\mathrm{Y}\to\mathrm{X}\times\mathrm{X}$ lifts to $\widetilde{\mathrm{W}}$ and we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{S} & \xrightarrow{\ \mathrm{F}\ } & \mathrm{W}_3 & & \\ & & & \searrow & \\ \downarrow & & \downarrow & & \mathrm{Bl}_{z'\times z'\times\mathbf{P}^1}(\mathrm{W}_0) \\ & & & \swarrow & \\ \mathrm{Y}\times\mathrm{Y} & \xrightarrow[f\times f]{} & \widetilde{\mathrm{W}}. & & \end{array} \qquad (3.3.20)$$

Consequently by the functoriality of the height we are reduced to proving our estimate for the map $\mathrm{F}=f\times f:\mathrm{Y}\times\mathrm{Y}\to\widetilde{\mathrm{W}}$, with respect to the total exceptional divisor on $\widetilde{\mathrm{W}}$, also denoted D by abuse of notation. This is however a rather special case of [M2], Proposition 6, from which we conclude. □

Remark **3.3.25.** — This method of proof fails to justify the naive plan of obtaining a dynamic intersection estimate for the exceptional divisor on W_1 blown up in $z\times z\times 1$, since a similar estimate for $\mathrm{X}\times\mathbf{P}^1$ blown up in $z\times 1$ appears false—essentially because f may get closer to z arbitrarily quicker than a rational function approaches 1.

Combining then (3.3.17) with Proposition 3.3.18 and Lemma 2.3.6 we see that we have proved

Proposition **3.3.26.** — *Let* $\varepsilon(|\xi|)$ *satisfy the constraints of Proposition* 3.3.18, *then, for* $|\xi|$ *outside a set of finite measure,*

$$\begin{aligned}\int_{|\eta|=\varepsilon(|\xi|)} \{h_{3\widetilde{\Delta\times 1}}(\widetilde{\Delta}^*_{\xi,\eta}\,\mathrm{F}(1)) - h_{3\widetilde{\Delta\times 1}}(\widetilde{\Delta}^*_{\xi,0}\,\mathrm{F}(1))\}\,\mu(d\eta) &\leqslant 2\,\delta h_{\mathrm{H}}(f(|\xi|+\varepsilon)) \\ &+ \mathrm{O}(\varepsilon(|\xi|)\,h_{\mathrm{H}}(f(|\xi|))\,\psi(h_{\mathrm{H}}(f(|\xi|))) \\ &+ \mathrm{O}(\log|\xi| + |\log\varepsilon(|\xi|)|).\end{aligned}$$

Remark **3.3.27.** — We see from the proof that the set of finite measure that it was necessary to exclude in the above proposition had no dependence on δ, and indeed that the variation of the implied constants on the right with δ is simply that of the type indicated in Theorem 2.3.2.

Consequently therefore we have obtained precisely the analogue of Proposition I. 2.3.7 we require, and the refined tautological inequality follows immediately via the arguments employed in I, § 2.4.

Should we content ourselves with supposing that f is a dense leaf of a foliation on a surface then of course f misses all the small singularities, so that if $E_{Z'}$ is the total sum of the exceptional curves on X_1, the minimal smooth model of the graph, contracting to Z′ then the map F of I.1.2 arising from the deformation to the normal cone of the diagonal on X naturally gives rise to a map $F : Y \to \mathbf{P}(\mathcal{O}_{X_1}(1)\ (E_{Z'}) \oplus \mathcal{O}_{X_1})$. Everything is now almost formally identical, for an arbitrary ramified cover to I.3 with the Ω^1_X in the general case now being replaced by the $\mathcal{O}_{X_1}(1)\ (E_{Z'})$, the only problem being that there is a slightly more difficult metricised term at infinity. This term, however may be handled by a suitable variant of the Ahlfors-Wong method (which incidentally would not be the case for non-reduced singularities) as may be found for example in [M3]. □

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