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COLIN J. BUSHNELL

GUY HENNIART

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LOCAL TAME LIFTING FOR GL(N) I: SIMPLE CHARACTERS

by Colin J. BUSHNELL *and* Guy HENNIART

CONTENTS

Let F be a non-Archimedean local field (of any characteristic whatsoever) and $N \geqslant 1$ an integer. Let K/F be a finite separable field extension. The local Langlands conjectures demand the existence of a process $\pi \mapsto \ell_{K/F}(\pi)$ which associates to an irreducible smooth representation π of GL(N, F) an irreducible smooth representation $\ell_{K/F}(\pi)$ of GL(N, K). This is to be strictly analogous to the more obvious one of restricting a Frobenius-semisimple representation of the Weil-Deligne group $\mathcal{W}\mathcal{D}_F$ of F to its subgroup $\mathcal{W}\mathcal{D}_K$.

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Our aim here is to make the first substantial step towards defining $\ell_{K/F}$ *in explicit local terms* when the extension K/F is tamely ramified, but not necessarily Galois. We start from the classification of the irreducible smooth representations of the group GL(N, F), in terms of explicit local data, given in [BK1]. Our approach is thus in the spirit of [K], [KM], [Pa], and very different from the globally-derived " base change " methods of, for example, [L], [AC]. This makes our task here a dual one. We must first give a rigorous (albeit partial) definition of local tame lift, and then connect it with the theory of base change. This is reflected in the structure of the paper.

The central concept of [BK1] is that of *simple type*. This is constructed in three stages, starting with a *simple stratum*, which is basically a field-theoretic object. The second step is a *simple character*, which is an arithmetically defined abelian character of a certain compact open subgroup of GL(N, F) determined by the underlying simple stratum. The final step will not concern us in this paper. An irreducible supercuspidal representation π of GL(N, F) must contain a simple character θ_F, say. Its lift $\ell_{K/F}(\pi)$, whatever that may be, is not necessarily supercuspidal. However, it will be built, via a familiar process of parabolic induction [Ze], from a uniquely determined collection of irreducible supercuspidal representations ρ_i of groups $GL(N_i, K)$, with $\Sigma_i N_i = N$. Each of these ρ_i will contain a simple character θ^i_K. We proceed on the tentative hypothesis that the collection $\{ \theta^i_K \}$ is in some way determined by the original character θ_F (and, we might add, conversely).

We therefore seek a way of lifting simple characters. There are, however, a number of other factors which need to be taken into account. First, our original supercuspidal representation π will contain many different simple characters. Any two of these will, of necessity, intertwine in GL(N, F). A fundamental result of [BK1] then shows that they will be conjugate in GL(N, F). However, they may arise from quite different constructions. In particular, they can be attached to distinct simple strata, and it is not straightforward, given two explicitly defined simple characters, to determine whether or not they are conjugate. See [BK3], [BK2], [KP] for some discussion of this matter. Further, we have families of relations between simple characters in GL(N, F) and simple characters in GL(N′, F) for any integer N′. These relations reflect, among other things, the connections between simple types and parabolic induction. They must therefore be respected by any lifting process. There is a further complication. If we have two conjugate simple characters in GL(N, F), it is not obvious *a priori* that the related characters in some GL(N′, F) will be conjugate. This problem has to be resolved first, and then we have to show that our definition of lift respects all of these relations. To do this, we must first invent an object which encapsulates all the relevant relations.

We describe this briefly in the language of [BK1]. To define a simple character, we first need a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $\mathrm{End}_F(V)$, for some finite-dimensional F-vector space V. Thus $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(V)$, n is a positive integer determined by $\mathfrak{A}$ and the element $\beta \in \mathrm{Aut}_F(V)$, and β is such that the algebra $F[\beta]$ is a field whose multiplicative group normalises $\mathfrak{A}$. There is also the technical condition

" $k_0(\beta, \mathfrak{A}) < 0$ " of [BK1] (1.4) which we ignore in this overview. These data give rise to a pair of open subgroups $H^1(\beta, \mathfrak{A}) \subset J^1(\beta, \mathfrak{A})$ of the parahoric subgroup $\mathbf{U}(\mathfrak{A})$. (Unexplained notations have their standard meanings, summarised below.) There is then a distinguished finite set $\mathscr{C}(\mathfrak{A}, 0, \beta)$ of abelian characters of $H^1(\beta, \mathfrak{A})$, and these are what we call simple characters.

The constructions of [BK1] (3.6) tell us how to assemble these simple characters into what we clumsily delineate *potential simple characters*, or *ps-characters* for short. A ps-character is a pair (Θ, β). The component β is an element of some finite field extension of F, of negative valuation and subject to a technical restriction " $k_F(\beta) < 0$ ". The other component Θ is a simple-character-valued function as follows. Let $\mathfrak{B}$ be a hereditary $\mathfrak{o}_{F[\beta]}$-order in $\mathrm{End}_{F[\beta]}(V)$, for some finite-dimensional $F[\beta]$-vector space V. The lattice chain in V which defines $\mathfrak{B}$ also determines a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $\mathrm{End}_F(V)$, and there is a unique integer $n_{\mathfrak{A}} > 0$ such that $[\mathfrak{A}, n_{\mathfrak{A}}, 0, \beta]$ is a simple stratum. The function Θ then gives a simple character $\Theta(\mathfrak{A}) \in \mathscr{C}(\mathfrak{A}, 0, \beta)$ and the characters $\Theta(\mathfrak{A})$, as $\mathfrak{B}$ varies, are subject to a strict coherence condition. We call $\Theta(\mathfrak{A})$ the *realization of* (Θ, β) *on* $\mathfrak{A}$. Two ps-characters (Θ^i, β_i) are then *endo-equivalent* if there exist realizations on the same order $\mathfrak{A}$ in some $\mathrm{End}_F(V)$ such that the characters $\Theta^i(\mathfrak{A})$ are conjugate in $\mathrm{Aut}_F(V)$. This notion of an endo-class of simple characters (i.e., endo-equivalence class of ps-characters) is formally set up in § 8, and relies heavily on the counting results of [BK3]. A more straightforward version of it applies to simple strata (under the appellation " equivalence class of simple pairs ") and this is described in § 1.

It is time to introduce our concept of lifting. The underlying idea is extremely simple. We are given a fixed, finite tamely ramified field extension K/F. If we have a finite field extension $F[\beta]/F$, we can form the algebra $K \otimes_F F[\beta]$. This is a product of fields E_i, and $E_i = K[\beta_i]$, where β_i is the canonical projection of β into the i-th factor. The β_i are what we call the K/F-lifts of β. If β satisfies the crucial condition $k_F(\beta) < 0$, we then have $k_K(\beta_i) < 0$ as a consequence of our tameness hypothesis on K/F. This trivial notion of lift, $\beta \mapsto \{\beta_i\}$, is the foundation on which we erect our theory of lifting. However, it takes § 2-6 to establish that it has the qualities necessary for this role. Note, however, that it is self-evidently transitive in the field extension K/F.

For simple characters too, the basic idea behind the definition of lift is very easy. We start with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ over F and a simple character $\theta_F \in \mathscr{C}(\mathfrak{A}, 0, \beta)$. This is the realization $\Theta(\mathfrak{A})$ of some ps-character (Θ, β). Let $\tilde{\beta}$ be some K/F-lift of β, and let V be a $K[\tilde{\beta}]$-vector space. In particular, V is an $F[\beta]$-vector space. We choose a hereditary $\mathfrak{o}_K$-order $\mathfrak{C}$ in $\mathrm{End}_K(V)$ which is normalized by $K[\tilde{\beta}]^\times$. We then get a simple stratum $[\mathfrak{C}, m, 0, \tilde{\beta}]$. Let $\mathfrak{A}'$ be the hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(V)$ defined by the same lattice chain as $\mathfrak{C}$. The stratum $[\mathfrak{A}', m, 0, \beta]$ is then simple (we need not distinguish between β and $\tilde{\beta}$ when working over F) and (Θ, β) determines a character $\theta' = \Theta(\mathfrak{A}') \in \mathscr{C}(\mathfrak{A}', 0, \beta)$. Two remarkable facts now reveal themselves. First, we have

$$H^1(\beta, \mathfrak{A}') \cap \mathrm{Aut}_K(V) = H^1(\tilde{\beta}, \mathfrak{C}).$$

We can thus restrict the character θ' to a character $\tilde{\theta}$ of $H^1(\tilde{\beta}, \mathfrak{C})$, and we find

$$\tilde{\theta} \in \mathscr{C}(\mathfrak{C}, 0, \tilde{\beta}).$$

This is the starting point of § 7, where we investigate the correspondence $\theta_F \mapsto \tilde{\theta}$ in detail. The character $\tilde{\theta}$ defines a ps-character $(\tilde{\Theta}, \tilde{\beta})$, and the set of endo-classes of these, as $\tilde{\beta}$ ranges over the K/F-lifts of β, is defined to be the set of K/F-lifts of the endo-class of (Θ, β). Of course, the first main result is that this definition does not depend on the many choices made in the construction. The other properties are quick to state:

(i) *the endo-classes* $(\tilde{\Theta}, \tilde{\beta})$ *are distinct (as* $\tilde{\beta}$ *ranges over the* K/F*-lifts of* β*)*;
(ii) *the endo-class of any one of the* $(\tilde{\Theta}, \tilde{\beta})$ *determines the endo-class of* (Θ, β) *uniquely*;
(iii) *any endo-class over* K *arises as a lift of some (uniquely determined) endo-class over* F.

The lifting process is again transitive in K/F. All of this is proved in § 9.

This concludes the first half of the paper. At this stage, we have a coherent method for lifting simple characters which respects the manifold relations between them. Our next task must be to connect this abstract lifting with an operation on the set of irreducible representations of some $GL(N, F) \cong \mathrm{Aut}_F(V)$ containing some realization of a given endo-class over F. This is the subject matter of § 10, 11. We take a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $\mathrm{End}_F(V)$ and a field extension $E/F[\beta]$ such that E/F is a maximal subfield of $\mathrm{End}_F(V)$. This allows us to identify $V = E$. There is then a canonical choice of hereditary $\mathfrak{o}_F$-order $\mathfrak{A}_M$ in $\mathrm{End}_F(E \otimes_F K)$ and this gives rise to a simple stratum $[\mathfrak{A}_M, n_M, 0, \beta]$. We fix a simple character $\theta_F \in \mathscr{C}(\mathfrak{A}, 0, \beta)$. This defines, in an explicit manner via an Iwahori decomposition, a character $\theta_M \in \mathscr{C}(\mathfrak{A}_M, 0, \beta)$. (The characters θ_F, θ_M are realizations of the same ps-character.) The order $\mathfrak{A}_M$ is constructed to be normalized by $K^\times$, so $\mathfrak{C} = \mathfrak{A}_M \cap \mathrm{End}_K(V \otimes K)$ is a hereditary $\mathfrak{o}_K$-order. We form the group

$$\mathbf{H}^1_K = H^1(\beta, \mathfrak{A}_M) \cap \mathrm{Aut}_K(V \otimes K)$$

and the character

$$\boldsymbol{\theta}_K = \theta_M \mid \mathbf{H}^1_K.$$

The pair $(\mathbf{H}^1_K, \boldsymbol{\theta}_K)$ is not, in general, a simple character: this happens if and only if $K \otimes_F E$ is a field. In the general case, it should be thought of as a " semisimple character ", in the manner of the semisimple types of [BK4]. The group $\mathbf{H}^1_K$ admits an Iwahori decomposition, and the restrictions of $\boldsymbol{\theta}_K$ to the diagonal blocks of this decomposition are just the K/F-lifts (up to endo-equivalence and predictable multiplicity) of the original θ_F. This setup gives us a framework in which we can directly compare the character θ_F and its various lifts.

The situation becomes particularly interesting when we restrict to the case in which K/F is *cyclic*. We fix a generator σ of Gal(K/F). In this case, we show in § 12 that the characters θ_F, $\boldsymbol{\theta}_K$ are intimately related via the " twisted norm " $x \mapsto \mathcal{N}_\sigma x = x . \sigma(x) . \sigma^2(x) \ldots \sigma^{d-1}(x)$, $x \in GL(N, K)$, where $d = [K : F]$. More than this is true; we get a formal term-by-term comparison between certain " character-like " sums of values of θ_F, $\boldsymbol{\theta}_K$ over appropriate conjugacy classes. (In fact, we only treat unramified and totally tamely ramified extensions K/F in § 12, in order to confine the burgeoning technicality. Theses cases are adequate for our purposes, but it looks likely that the results hold in greater generality.)

In the standard theory of base change, as developed in [Sa], [Sh], [L], [AC], it is the map $\mathcal{N}_\sigma$ (on appropriate conjugacy classes) which provides the link between a representation of GL(N, F) and its base change lift. Thus the results of § 12 give us the starting point for a comparison between our naïve local (and partial) definition of lifting and that given by base change. To make such a comparison, it is enough to treat *supercuspidal* representations of GL(N, F): one knows that base change respects parabolic induction and the corresponding feature has been built into the local definition of lift. An irreducible supercuspidal representation of GL(N, F) must contain a simple character $\theta_F \in \mathscr{C}(\mathfrak{A}, 0, \beta)$, for some simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $\mathbf{M}(N, F)$. The incompleteness of our local definitions now imposes a restriction: *we consider only supercuspidal representations of* GL(N, F) *containing a simple character* $\theta_F \in \mathscr{C}(\mathfrak{A}, 0, \beta)$ *for which the field degree* $[F[\beta]: F]$ *is equal to* N. The general case requires further investigation and must be postponed to another occasion.

Our main result comes in § 16. There we assume that K/F is a finite, tamely ramified, Galois extension. We check that base change unambiguously defines a lifting process relative to such extensions K/F, and then we prove:

An irreducible smooth representation π *of* GL(N, F) *contains the simple character* θ_F *if and only if its* K/F*-base change contains the character* $\boldsymbol{\theta}_K$.

One *caveat* is needed here: we assume without formal justification an algebraic property of the character $\boldsymbol{\theta}_K$ defined above. This property (16.10) is a special case of the more general considerations of [BK4], so it is appropriate to give the proof elsewhere.

This result is approached via a couple of special cases, treated (without recourse to the unproven (16.10)) in § 14 and § 15. The arguments of § 14 apply to cyclic extensions K/F which are either unramified or totally tamely ramified, and deal with the case where the algebra $K \otimes_F F[\beta]$ is a field. This amounts to saying that our element β has a unique K/F-lift, and that the character $\boldsymbol{\theta}_K$ is in fact a simple character over K. Moreover, any irreducible representation of GL(N, K) containing $\boldsymbol{\theta}_K$ is supercuspidal. The technique of § 14 is to compare the character relation defining base change with the relations given by § 12. We get an equality, indeed a term-by-term comparison, between a finite sum of characters of representations over F and a finite sum of twisted characters over K (and hence a partial answer to a question of L. Clozel). In fact, the relations of § 12 imply more general results of this kind: see especially the intriguing identity (14.5). However, it is only in the present case, where $F[\beta]/F$ has degree N, that we can interpret (14.5) directly as a character relation.

A different special case is treated in § 15. There, we assume that the extension K/F embeds in F[β]/F: one can think of this as the case where θ_F " splits completely " over K. Owing to the presently opaque nature of the relation (14.5), we cannot directly compare the local lift with base change in this case. We therefore compare it with the " dual " process of *automorphic induction*, as treated in [HH]. We are able to show, in a sufficient number of special cases, that our local lift is compatible with automorphic induction. The two cases of § 14, 15 then combine with formal properties of base change and automorphic induction to give the main theorem quoted above.

Some other points need to be made. First, given a little more work (largely omitted here), the processes of base change and automorphic induction can be regarded as different sides of the same coin. Thus our results show equally that local lifting is compatible with automorphic induction. Further, the indirectly achieved comparison with base change enables us, at this stage, to interpret (14.5) in more general cases.

Next, we have to recall that both base change and automorphic induction are presently only available in characteristic zero, so our comparison results can only be valid with that restriction. However, all our local arguments are entirely characteristic-free. The restriction to characteristic zero is only ever invoked at the last stage of the proofs, alongside the fact that characters of distinct irreducible supercuspidal representations remain linearly independent on restriction to the elliptic regular set. This is again only known in characteristic zero. However, when this result, base change and automorphic induction become available in positive characteristic, along with some unsurprising formal properties, our comparison results will become valid there. We give a more precise description of the situation at the end of § 16.

We conclude with an Appendix on basic properties of characters. We need these rather standard results in arbitrary characteristic not only for GL(N, F) but also for open finite-index subgroups of GL(N, F) and for groups of the form GL(N, K) $\rtimes$ Gal(K/F), where K/F is a finite Galois extension. The required combination of hypotheses rarely occurs in the literature. We found it more satisfactory to write these few pages than to endlessly insert lame and automatically suspect statements along the lines of " arguing as in [Xx] (x.y.z) (which does not actually require the hypothesis...)... ". Apart from a couple of minor observations (see especially the finiteness properties in (A.14)), there is nothing really new here. Rather similar comments apply to our § 13. We hope the reader will indulge us in this small matter.

Notation. — Throughout, F denotes a non-Archimedean local field. We write

$\mathfrak{o}_F$ = the discrete valuation ring in F;
$\mathfrak{p}_F$ = the maximal ideal of $\mathfrak{o}_F$;
$k_F = \mathfrak{o}_F/\mathfrak{p}_F$ = the residue field of F;
$v_F : F \twoheadrightarrow \mathbf{Z} \cup \{\infty\}$ is the normalised additive valuation on F.

If E/F is a finite field extension, we use similar notations relative to E. We also write $e(E \mid F)$, $f(E \mid F)$ for the ramification index and residue class degree of the extension E/F.

Let V be a finite-dimensional F-vector space, and $\mathscr{L}$ an $\mathfrak{o}_F$-lattice chain in V. Thus $\mathscr{L}$ defines a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $A = \mathrm{End}_F(V)$: we have $\mathfrak{A} = \mathrm{End}^0_{\mathfrak{o}_F}(\mathscr{L})$ in the notation of [BK1]. The Jacobson radical of $\mathfrak{A}$ is invariably denoted $\mathfrak{P}$, and we put
$\mathbf{U}(\mathfrak{A}) = \mathfrak{A}^\times$;
$\mathbf{U}^n(\mathfrak{A}) = 1 + \mathfrak{P}^n$, for $n \geqslant 1$;
$\mathfrak{K}(\mathfrak{A}) = \{ x \in \mathrm{Aut}_F(V) : x^{-1}\mathfrak{A}x = \mathfrak{A} \}$.

There is a special case we shall frequently use. Let E/F be a finite field extension. The set $\mathscr{L} = \{ \mathfrak{p}_E^j : j \in \mathbf{Z} \}$ is an $\mathfrak{o}_F$-lattice chain in E, and we usually write $\mathfrak{A}(E)$ for the order $\mathrm{End}^0_{\mathfrak{o}_F}(\mathscr{L})$. Alternatively, $\mathfrak{A}(E)$ can be described as the unique hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $\mathrm{End}_F(E)$ such that $E^\times \subset \mathfrak{K}(\mathfrak{A})$.

Finally, if $x \in \mathbf{R}$, we write $[x]$ for the greatest integer $\leqslant x$.

1. Simple pairs

The simple pairs of the title of the section amount to an abstraction of the simple strata of [BK1]. We start by recalling some of the salient features of simple strata, and establishing a system of notation more convenient for our present purposes. Let V be a finite-dimensional vector space over our non-Archimedean local field F, and let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in $A = \mathrm{End}_F(V)$, with Jacobson radical $\mathfrak{P}$, attached to the lattice chain $\mathscr{L}$ in V. As usual, we write $e = e(\mathfrak{A} \mid \mathfrak{o}_F)$ for the $\mathfrak{o}_F$-period of the lattice chain $\mathscr{L}$.

Let $E \supset F$ be a subfield of A, so that we may view V as an E-vector space. We write $B = \mathrm{End}_E(V)$. The $\mathfrak{o}_F$-lattice chain $\mathscr{L}$ is then an $\mathfrak{o}_E$-lattice chain if and only if $E^\times \subset \mathfrak{K}(\mathfrak{A})$. When this condition is satisfied, the ring

$$\mathfrak{B} = \mathfrak{A} \cap B = \mathrm{End}^0_{\mathfrak{o}_E}(\mathscr{L}),$$

is a hereditary $\mathfrak{o}_E$-order in B. Moreover, its radical is $\mathfrak{Q} = \mathfrak{P} \cap B$, and

$$e(\mathfrak{B} \mid \mathfrak{o}_E) = \frac{e(\mathfrak{A} \mid \mathfrak{o}_F)}{e(E \mid F)}.$$

(All of this can be found in the first two sections of [BK1] Ch. 1.)

If $\mathscr{L}$ is an $\mathfrak{o}_F$-lattice chain in V, say $\mathscr{L} = \{ L_j : j \in \mathbf{Z} \}$, $L_j \supsetneq L_{j+1}$, we write

$$d_i(\mathscr{L}) = d_i^F(\mathscr{L}) = \dim_{k_F}(L_i/L_{i+1}), \quad i \in \mathbf{Z}.$$

(1.1) *Proposition. — Let* V *be a finite-dimensional* F*-vector space, and* $\mathfrak{A}$ *a hereditary* $\mathfrak{o}_F$*-order in* $A = \mathrm{End}_F(V)$*, defined by the lattice chain* $\mathscr{L}$ *in* V*. Let* E/F *be a finite field extension. There exists an embedding* $\varphi : E \to A$ *of* F*-algebras such that* $\varphi(E^\times) \subset \mathfrak{K}(\mathfrak{A})$ *if and only if the following conditions are satisfied:*

a) $f(E \mid F)$ *divides* $d_j(\mathscr{L})$ *for all* $j \in \mathbf{Z}$;
b) $e(E \mid F)$ *divides* $e(\mathfrak{A} \mid \mathfrak{o}_F)$;
c) we have $d_j(\mathscr{L}) = d_{j+t}(\mathscr{L})$*, for all* $j \in \mathbf{Z}$*, where* $t = e(\mathfrak{A} \mid \mathfrak{o}_F)/e(E \mid F)$.

Proof. — This is just a restatement of [BK3] (1.2). □

In practice, our finite field extension E/F will always come with a distinguished element $\beta \in E^\times$ such that $E = F[\beta]$. If E/F is also a subfield of A which normalizes $\mathfrak{A}$, we then use the notation

$$(1.2) \qquad e_\beta(\mathfrak{A}) = e(\mathfrak{B} \mid \mathfrak{o}_E).$$

Since, in this situation, we have $\beta \in \mathfrak{K}(\mathfrak{A})$, there is an integer n such that $\beta\mathfrak{A} = \mathfrak{P}^{-n}$. Explicitly, this is given by

$$n = -\, e_\beta(\mathfrak{A})\, \nu_E(\beta).$$

It will here be more convenient to use the notation

$$(1.3) \qquad n_F(\beta) = -\, \nu_E(\beta),$$

so that $n = e_\beta(\mathfrak{A})\, n_F(\beta)$.

Now let a_β denote the adjoint map $A \to A$ given by $x \mapsto \beta x - x\beta$. Recall ([BK1] (1.4.5), (1.4.11)) the quantity $k_0(\beta, \mathfrak{A}) \in \mathbf{Z} \cup \{-\infty\}$, which can be defined as follows: if $\beta \in F$, then $k_0(\beta, \mathfrak{A}) = -\infty$, while, otherwise, $k_0(\beta, \mathfrak{A})$ is the least integer k for which

$$\mathfrak{P}^k \cap a_\beta(A) \subset a_\beta(\mathfrak{A}).$$

We mention a special case of this setup. We view E as a vector space over F. The set $\{ \mathfrak{p}_E^j : j \in \mathbf{Z} \}$ is then an $\mathfrak{o}_F$-lattice chain in E, giving rise to a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}(E)$ in $\mathrm{End}_F(E)$,

$$\mathfrak{A}(E) = \mathrm{End}^0_{\mathfrak{o}_F}(\{ \mathfrak{p}_E^j \}),$$

which is the unique hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(E)$ normalized by $E^\times$. We write

$$(1.4) \qquad k_F(\beta) = k_0(\beta, \mathfrak{A}(E)).$$

If V is any finite-dimensional E-vector space and $\mathfrak{A}$ is any hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(V)$ normalized by $E^\times$, we then have (see [BK1], (1.4.13))

$$k_0(\beta, \mathfrak{A}) = k_F(\beta)\, e_\beta(\mathfrak{A}).$$

We also recall that if $k_F(\beta)$ is finite, then (see [BK1] (1.4.15))

$$k_F(\beta) \geqslant -\, n_F(\beta).$$

We therefore make the following definition.

(1.5) *Definition.* — *A simple pair over* F *is a pair* $[m, \beta]$ *consisting of a nonzero element* β *of some finite field extension of* F *and an integer* m *such that*

$$m < \min\{ n_F(\beta), -\, k_F(\beta)\}.$$

Two simple pairs $[m_i, \beta_i]$, $i = 1, 2$, are *isomorphic* if $m_1 = m_2$ and there exists an F-isomorphism $\varphi : F[\beta_1] \to F[\beta_2]$ such that $\varphi(\beta_1) = \beta_2$.

Let $[m, \beta]$ be a simple pair over F, and set $E = F[\beta]$. Let V be some finite-dimensional E-vector space and $\mathfrak{B}$ a hereditary $\mathfrak{o}_E$-order in $\mathrm{End}_E(V)$. Let $\mathfrak{A}$ be the hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(V)$ defined by the same lattice chain as $\mathfrak{B}$. Set $n_{\mathfrak{A}} = n_F(\beta)\, e_\beta(\mathfrak{A})$, and let $m_{\mathfrak{A}}$ be some integer satisfying

$$\left[\frac{m_{\mathfrak{A}}}{e_\beta(\mathfrak{A})}\right] = m.$$

Observe that this implies $m_{\mathfrak{A}} < -k_0(\beta, \mathfrak{A})$ and $m_{\mathfrak{A}} < n_{\mathfrak{A}}$, so $[\mathfrak{A}, n_{\mathfrak{A}}, m_{\mathfrak{A}}, \beta]$ is a *simple stratum* in $\mathrm{End}_F(V)$, in the sense of [BK1] (1.5.5). We call this stratum the *realization* of the simple pair $[m, \beta]$ on the order $\mathfrak{A}$. Obviously, isomorphic simple pairs have effectively the same realizations.

Conversely, if we are given a simple stratum $[\mathfrak{A}, n, m, \beta]$ in $\mathrm{End}_F(V)$, for some vector space V, then $[[m/e_\beta(\mathfrak{A})], \beta]$ is a simple pair, of which $[\mathfrak{A}, n, m, \beta]$ is a realization.

Remark. — It is sometimes useful to view this situation slightly differently. Suppose we are given only a finite-dimensional F-vector space V and a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $\mathrm{End}_F(V)$. Let $[m, \beta]$ be a simple pair over F. A *realization of* $[m, \beta]$ *on* $\mathfrak{A}$ is then a simple stratum of the form $[\mathfrak{A}, n', m', \varphi(\beta)]$, where $\varphi : F[\beta] \to \mathrm{End}_F(V)$ is a homomorphism of F-algebras such that $\varphi(F[\beta]^\times) \subset \mathfrak{K}(\mathfrak{A})$ and m' is an integer such that $[m'/e_{\varphi(\beta)}(\mathfrak{A})] = m$. Of course, we can view V as an $F[\beta]$-vector space via the map φ, and we are in the same situation as before. The following elementary result shows that this concept is, in essence, independent of the embedding φ subject to the stated conditions:

(1.6) *Lemma.* — *Let* V *be a finite-dimensional* F*-vector space, and let* $\mathfrak{A}$ *be a hereditary* $\mathfrak{o}_F$*-order in* $A = \mathrm{End}_F(V)$. *Let* E/F *be a finite field extension and, for* $i = 1, 2$, *let* $\varphi_i : E \to A$ *be an* F*-embedding such that* $\varphi_i(E^\times) \subset \mathfrak{K}(\mathfrak{A})$. *There exists* $u \in \mathbf{U}(\mathfrak{A})$ *such that*

$$\varphi_2(x) = u^{-1}\, \varphi_1(x)\, u, \quad x \in E^\times.$$

Proof. — Let us write V^i for V viewed as an E-vector space via the embedding φ_i. Let $\mathscr{L} = \{ L_j : j \in \mathbf{Z} \}$ be the lattice chain which defines $\mathfrak{A}$. This determines an $\mathfrak{o}_E$-lattice chain $\mathscr{L}^i = \{ L^i_j \}$ in V^i, for each value of i. The lattice chain $\mathscr{L}^i$ is determined up to $\mathfrak{o}_E$-isomorphism by $\dim_E(V^i)$ and the sequence of integers

$$\delta^i_j = \dim_{k_E}(L^i_j / L^i_{j+1}).$$

Of course, $\dim_E(V^1) = \dim_E(V^2)$, while, in the notation of (1.1),

$$\delta^i_j = f(E \mid F)^{-1}\, d_j(\mathscr{L}).$$

Thus there exists an E-isomorphism $V^1 \to V^2$ which maps L^1_j to L^2_j, for each j. This isomorphism is given by an element $u \in \mathrm{Aut}_F(V)$ such that $\varphi_1(x) = u^{-1}\, \varphi_2(x)\, u$ and

$u\mathrm{L}_j = \mathrm{L}_j$, for all $x \in \mathrm{E}^\times$ and all $j \in \mathbf{Z}$. The second property here is equivalent to $u \in \mathbf{U}(\mathfrak{A})$, so this proves the lemma. □

We now recall some of the basic ideas associated with strata: see [BK1], especially (1.5). For $i = 1, 2$, let $[\mathfrak{A}, n, m, b_i]$ be a stratum in $\mathrm{End}_\mathrm{F}(\mathrm{V})$, for some finite-dimensional F-vector space V. Thus $m < n$ and $b_i \in \mathfrak{P}^{-n}$, where $\mathfrak{P}$ is the radical of $\mathfrak{A}$. We say these two strata are *equivalent*, denoted

$$[\mathfrak{A}, n, m, b_1] \sim [\mathfrak{A}, n, m, b_2],$$

if we have

$$b_1 + \mathfrak{P}^{-m} = b_2 + \mathfrak{P}^{-m}.$$

We now summarize the implications of [BK1] (2.4.1) (ii) in this situation:

(1.7) *For $i = 1, 2$, let $[\mathfrak{A}, n, m, \beta_i]$ be a simple stratum in* $\mathrm{End}_\mathrm{F}(\mathrm{V})$ *and suppose that* $[\mathfrak{A}, n, m, \beta_1] \sim [\mathfrak{A}, n, m, \beta_2]$. *We then have*

$$\begin{aligned} e(\mathrm{F}[\beta_1] \mid \mathrm{F}) &= e(\mathrm{F}[\beta_2] \mid \mathrm{F}),\\ f(\mathrm{F}[\beta_1] \mid \mathrm{F}) &= f(\mathrm{F}[\beta_2] \mid \mathrm{F}),\\ k_0(\beta_1, \mathfrak{A}) &= k_0(\beta_2, \mathfrak{A}), \end{aligned}$$

and therefore

$$\begin{aligned} e_{\beta_1}(\mathfrak{A}) &= e_{\beta_2}(\mathfrak{A}),\\ k_\mathrm{F}(\beta_1) &= k_\mathrm{F}(\beta_2),\\ n_\mathrm{F}(\beta_1) &= n_\mathrm{F}(\beta_2). \end{aligned}$$

On the other hand, we say that two strata $[\mathfrak{A}_i, n_i, m_i, b_i]$ in $\mathrm{End}_\mathrm{F}(\mathrm{V})$ *intertwine (formally) in* $\mathrm{End}_\mathrm{F}(\mathrm{V})$ if there exists $x \in \mathrm{Aut}_\mathrm{F}(\mathrm{V})$ such that

$$x^{-1}(b_1 + \mathfrak{P}_1^{-m_1})\, x \cap (b_2 + \mathfrak{P}_2^{-m_2}) \neq \emptyset.$$

Now we recall one of the main results (2.6.1) of [BK1].

(1.8) *Let $[\mathfrak{A}, n, m, \beta_1]$, $[\mathfrak{A}, n, m, \beta_2]$ be simple strata in* $\mathrm{End}_\mathrm{F}(\mathrm{V})$, *which intertwine. There exists $u \in \mathbf{U}(\mathfrak{A})$ such that*

$$[\mathfrak{A}, n, m, \beta_1] \sim [\mathfrak{A}, n, m, u^{-1}\beta_2 u].$$

In particular, the elements β_i have the same arithmetical invariants, as in (1.7).

Thus intertwining is an equivalence relation on the set of simple strata with given values of the parameters $\mathfrak{A}$, n, m. Our immediate task is to unify this family of equivalence relations into an equivalence relation on the set of simple pairs. First, however, we note that one can often " extend the range " of an intertwining relation between simple strata.

(1.9) *For $i = 1, 2$, let $[\mathfrak{A}, n, m, \beta_i]$ be a simple stratum, and suppose that these two strata intertwine. We then have $e_{\beta_1}(\mathfrak{A}) = e_{\beta_2}(\mathfrak{A})$. Let*

$$m_0 = e_{\beta_1}(\mathfrak{A}) \left[\frac{m}{e_{\beta_1}(\mathfrak{A})}\right] = e_{\beta_2}(\mathfrak{A}) \left[\frac{m}{e_{\beta_2}(\mathfrak{A})}\right].$$

The strata $[\mathfrak{A}, n, m_0, \beta_1]$, $[\mathfrak{A}, n, m_0, \beta_2]$ then intertwine.

Proof. — The first assertion follows from (1.7), (1.8), and the second from (1.8) and [BK1] (2.4.1), (2.2.1). □

We now investigate the intertwining relations between various realizations of given simple pairs.

(1.10) *Proposition. — Let $[m, \beta_1]$, $[m, \beta_2]$ be simple pairs over F, and suppose that $[F[\beta_1] : F] = [F[\beta_2] : F]$. Let V be a finite-dimensional F-vector space, and $\mathfrak{A}$ a hereditary $\mathfrak{o}_F$-order in $A = \mathrm{End}_F(V)$. For $i = 1, 2$, let $[\mathfrak{A}, n_i, m_i, \varphi_i(\beta_i)]$ be a realization of $[m, \beta_i]$ on $\mathfrak{A}$. Suppose that the strata $[\mathfrak{A}, n_1, m_1, \varphi_1(\beta_1)]$, $[\mathfrak{A}, n_2, m_2, \varphi_2(\beta_2)]$ intertwine in A. Then:*

(i) *We have*

$$\begin{aligned} e(F[\beta_1] \mid F) &= e(F[\beta_2] \mid F), \\ f(F[\beta_1] \mid F) &= f(F[\beta_2] \mid F), \\ n_F(\beta_1) &= n_F(\beta_2), \\ k_F(\beta_1) &= k_F(\beta_2). \end{aligned}$$

(ii) *Let V′ be some finite-dimensional F-vector space and $\mathfrak{A}'$ a hereditary $\mathfrak{o}_F$-order in $A' = \mathrm{End}_F(V')$. For $i = 1, 2$, let $[\mathfrak{A}', n_i', m_i', \varphi_i'(\beta_i)]$ be a realization of $[m, \beta_i]$ on $\mathfrak{A}'$. The strata $[\mathfrak{A}', n_i', m_i', \varphi_i'(\beta_i)]$ then intertwine in A′.*

Proof. — The strata $[\mathfrak{A}, n_i, n_i - 1, \varphi_i(\beta_i)]$ are each equivalent to a simple stratum, and certainly intertwine. It follows (see [BK1] (2.6.2)) that $n_1 = n_2 = n$, say. By symmetry, we can now assume that $m_2 \geqslant m_1$. We choose a simple stratum $[\mathfrak{A}, n, m_2, \gamma]$ equivalent to $[\mathfrak{A}, n, m_2, \varphi_1(\beta_1)]$, as we may by [BK1] (2.4.1). The simple strata $[\mathfrak{A}, n, m_2, \gamma]$, $[\mathfrak{A}, n, m_2, \varphi_2(\beta_2)]$ intertwine, so (1.7), (1.8) give us $[F[\gamma] : F] = [F[\beta_2] : F]$. By hypothesis, we have $[F[\beta_1] : F] = [F[\beta_2] : F]$, so $[F[\gamma] : F] = [F[\beta_1] : F]$. Appealing to [BK1] (2.4.1), we deduce that the stratum $[\mathfrak{A}, n, m_2, \varphi_1(\beta_1)]$ is simple. The desired equalities now follow from (1.8). This proves part (i) of the Proposition.

We turn to part (ii), the proof of which is considerably more intricate. We abbreviate $E_i = F[\beta_i]$, and consider the order

$$\mathfrak{A}(E_i) = \mathrm{End}^0_{\mathfrak{o}_F}(\{\, \mathfrak{p}^j_{E_i} : j \in \mathbf{Z} \,\}) \subset A(E_i) = \mathrm{End}_F(E_i).$$

This is a principal order, satisfying $e(\mathfrak{A}(E_i) \mid \mathfrak{o}_F) = e(E_i \mid F)$. A principal order $\mathfrak{A}$ in some $\mathrm{End}_F(V)$ is determined up to isomorphism by the quantities $\dim_F V$, $e(\mathfrak{A} \mid \mathfrak{o}_F)$. The relations $[E_1 : F] = [E_2 : F]$, $e(E_1 \mid F) = e(E_2 \mid F)$ thus imply $\mathfrak{A}(E_1) \cong \mathfrak{A}(E_2)$ as $\mathfrak{o}_F$-orders.

Therefore there exists an F-embedding ψ of E_2 in $A(E_1)$ such that $\psi(E_2^\times) \subset \mathfrak{K}(\mathfrak{A}(E_1))$. In particular, we get

$$e_{\beta_1}(\mathfrak{A}(E_1)) = 1 = e_{\beta_2}(\mathfrak{A}(E_1)),$$

whence we have simple strata

$$[\mathfrak{A}(E_1), n_0, m, \beta_1], \qquad [\mathfrak{A}(E_1), n_0, m, \psi(\beta_2)],$$

in which $n_0 = n_F(\beta_1) = n_F(\beta_2)$.

At this point, we break off to recall another structure, described fully in [BK1] (1.2). Let V_1 be an E_1-vector space and $\mathfrak{A}_1$ a hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(V_1)$ which is normalized by $E_1^\times$. Set $\mathfrak{B}_1 = \mathfrak{A}_1 \cap \mathrm{End}_{E_1}(V_1)$. By choosing an $\mathfrak{o}_{E_1}$-basis of the lattice chain defining $\mathfrak{A}_1$, we get a " (W_1, E_1)-decomposition " of $\mathfrak{A}_1$. This is, in particular, an isomorphism

$$\mathfrak{A}_1 = \mathfrak{A}(E_1) \otimes_{\mathfrak{o}_{E_1}} \mathfrak{B}_1$$

of $(\mathfrak{A}(E_1), \mathfrak{B}_1)$-bimodules. If $\mathfrak{P}_1$ is the radical of $\mathfrak{A}_1$ and $\mathfrak{B}(E_1)$ that of $\mathfrak{A}(E_1)$, this isomorphism identifies $\mathfrak{P}(E_1)^a \otimes \mathfrak{B}_1$ with $\mathfrak{P}_1^{ae_1}$, $a \in \mathbf{Z}$, where $e_1 = e_{\beta_1}(\mathfrak{A}_1)$. We also have the property

$$\mathfrak{K}(\mathfrak{A}(E_1)) \otimes 1 \subset \mathfrak{K}(\mathfrak{A}_1).$$

Thus a simple stratum $[\mathfrak{A}(E_1), r, s, \gamma]$ in $A(E_1)$ determines a simple stratum

$$[\mathfrak{A}_1, re_1, se_1, \gamma \otimes 1]$$

in $\mathrm{End}_F(V_1)$. Indeed, we get a family of simple strata

$$\textbf{(1.11)} \qquad [\mathfrak{A}_1, re_1, s_1, \gamma \otimes 1], \qquad se_1 \leqslant s_1 < (s+1)\, e_1.$$

It is easy to characterize the " image " of this inflation process.

(1.12) *Lemma. — Let* $[\mathfrak{A}_1, t, u, \delta]$ *be a simple stratum, and put* $s = [u/e_1]$, *where* $e_1 = e_{\beta_1}(\mathfrak{A}_1) = e(\mathfrak{A}_1 \mid \mathfrak{o}_F)/e(\mathfrak{A}(E_1) \mid \mathfrak{o}_F)$ *as above. The following are equivalent:*

(i) *there exists a simple stratum* $[\mathfrak{A}(E_1), r, s, \gamma]$ *in* $A(E_1)$ *such that* $[\mathfrak{A}_1, t, u, \delta]$ *intertwines with a stratum of the form* (1.11);

(ii) *we have:*

$$f(F[\delta] : F) \textit{ divides } f(E_1 : F), \qquad \textit{and} \qquad e(F[\delta] \mid F) \textit{ divides } e(E_1 \mid F);$$

(iii) *the simple pair* $[[u/e_\delta(\mathfrak{A}_1)], \delta]$ *admits a realization on* $\mathfrak{A}(E_1)$.

Proof. — The implication (iii) $\Rightarrow$ (i) follows from (1.6); (i) $\Rightarrow$ (ii) is given by (1.7) and (1.1), while (ii) $\Rightarrow$ (iii) is given by (1.1). □

(1.13) *Lemma. — For* $i = 1, 2$, *let* $[\mathfrak{A}(E_1), r, s, \gamma_i]$ *be a simple stratum in* $A(E_1)$. *These two strata intertwine if and only if the strata* $[\mathfrak{A}_1, re_1, se_1, \gamma_i \otimes 1]$ *in* $\mathrm{End}_F(V_1)$ *intertwine.*

Proof. — Suppose first that the $[\mathfrak{A}(E_1), r, s, \gamma_i]$ intertwine in $A(E_1)$. Appealing to (1.8), there exists $x \in \mathbf{U}(\mathfrak{A}(E_1))$ such that

$$[\mathfrak{A}(E_1), r, s, x^{-1}\gamma_1 x] \sim [\mathfrak{A}(E_1), r, s, \gamma_2].$$

This implies that $x^{-1}\gamma_1 x - \gamma_2 \in \mathfrak{P}(E_1)^{-s}$, where $\mathfrak{P}(E_1)$ denotes the radical of $\mathfrak{A}(E_1)$. We have $x \otimes 1 \in \mathbf{U}(\mathfrak{A}_1)$, and we have

$$x^{-1}\gamma_1 x \otimes 1 = (x \otimes 1)^{-1}(\gamma_1 \otimes 1)(x \otimes 1).$$

This gives us

$$x^{-1}\gamma_1 x \otimes 1 - \gamma_2 \otimes 1 \in \mathfrak{P}(E_1)^{-s} \otimes 1.$$

However, $\mathfrak{P}(E_1)^{-s} \otimes 1 \subset \mathfrak{P}(E_1)^{-s} \otimes \mathfrak{B}_1 = \mathfrak{B}_1^{-se_1}$, and this proves one implication of the lemma.

We can extract more from this implication: it shows that the inflation process

$$[\mathfrak{A}(E_1), r, s, \gamma] \mapsto [\mathfrak{A}_1, re_1, se_1, \gamma \otimes 1]$$

induces a well-defined map from *intertwining classes* of simple strata $[\mathfrak{A}(E_1), r, s, \gamma]$ in $A(E_1)$ to intertwining classes of strata $[\mathfrak{A}_1, t, u, \delta]$ in $\mathrm{End}_F(V_1)$ satisfying the equivalent conditions of (1.12). Moreover, by (1.12), this map is surjective. The opposite implication of the present lemma is equivalent to this map being injective. However, these two sets of intertwining classes of simple strata are finite and have the same numbers of elements, by [BK3] (1.15), and the lemma follows. □

Now we return to the proof of (1.10), and the strata $[\mathfrak{A}(E_1), n_0, m, \beta_1]$, $[\mathfrak{A}(E_1), n_0, m, \psi(\beta_2)]$ which realize our given simple pairs $[m, \beta_i]$ on $\mathfrak{A}(E_1)$. By (1.13), (1.9), these intertwine, and (1.10) (ii) now follows from (1.13). □

We now define our basic equivalence relation.

(1.14) *Definition. — For $i = 1, 2$, let $[k_i, \beta_i]$ be a simple pair over* F. *We say these pairs are* equivalent, *denoted*

$$[k_1, \beta_1] \approx [k_2, \beta_2],$$

if the following conditions are satisfied:

(i) $k_1 = k_2$;

(ii) $[F[\beta_1] : F] = [F[\beta_2] : F]$;

(iii) *there exists a finite-dimensional* F-*vector space* V *and a hereditary* $\mathfrak{o}_F$-*order* $\mathfrak{A}$ *in* $\mathrm{End}_F(V)$, *together with realizations* $[\mathfrak{A}, n_i, m_i, \varphi_i(\beta_i)]$ *of the pairs* $[k_i, \beta_i]$ *on* $\mathfrak{A}$ *which intertwine in* $\mathrm{End}_F(V)$.

(1.10) implies that $\approx$ is indeed an equivalence relation, and (1.9) says that the exact choices of the m_i are irrelevant. We write $\mathscr{SP}(F)$ for the set of these equivalence classes of simple pairs over F, and $(k, \beta) \in \mathscr{SP}(F)$ for the equivalence class of $[k, \beta]$.

Before proceeding, we record some immediate consequences of (1.7), (1.8) in this language.

(1.15) *For $i = 1, 2$, let $[k, \beta_i]$ be a simple pair over* F, *and suppose that* $[k, \beta_1] \approx [k, \beta_2]$. *Then*

$$\begin{aligned} n_F(\beta_1) &= n_F(\beta_2), \\ e(F[\beta_1] \mid F) &= e(F[\beta_2] \mid F), \\ f(F[\beta_1] \mid F) &= f(F[\beta_2] \mid F), \\ k_F(\beta_1) &= k_F(\beta_2). \end{aligned}$$

Temporarily write SP(F) for the set of all simple pairs over F. If we have a field F′ and an isomorphism $\varphi : F \xrightarrow{\approx} F'$, then φ induces a bijection SP(F) ≅ SP(F′). Explicitly, if $[m, \beta]$ is a simple pair over F, the composition

$$F' \xrightarrow{\varphi^{-1}} F \longrightarrow F[\beta]$$

defines a simple pair $[m, \varphi(\beta)]$ over F′. This preserves equivalence in the sense of (1.14), so we get a bijection $\varphi : \mathcal{SP}(F) \cong \mathcal{SP}(F')$.

As a particular case of this, suppose that F is a finite Galois extension of some field F_0, and write $\Sigma = \mathrm{Gal}(F/F_0)$. If we have a simple pair $[m, \beta]$ and $\sigma \in \Sigma$, the isomorphism $\sigma^{-1} : F \to F$ thus determines a simple pair $[m, \sigma^{-1}(\beta)]$, which we prefer to denote $[m, \beta^\sigma]$. We thus get an action

$$\textbf{(1.16)} \qquad \begin{aligned} &\mathcal{SP}(F) \times \Sigma \to \mathcal{SP}(F), \\ &((m, \beta), \sigma) \mapsto (m, \beta^\sigma). \end{aligned}$$

In more concrete terms, suppose we have a simple stratum $[\mathfrak{A}, n, m, \beta]$ in $A = \mathrm{End}_F(V)$, for some finite-dimensional F-vector space V. This determines a simple pair $[k, \beta]$, say. By choosing an F-basis of V, we identify A with $\mathbf{M}(N, F)$ for some N, and hence get an action of Σ on A. The stratum $[\mathfrak{A}^\sigma, n, m, \beta^\sigma]$ is still simple. The simple pair which it determines is then $[k, \beta^\sigma]$, in the sense above.

2. Interior tame lifting

We now take a finite, *tamely ramified* field extension K/F of our base field F. In this section, we make some preliminary investigations of the relations between simple strata over K and simple strata over F.

Let V be a finite-dimensional K-vector space. We write $C = \mathrm{End}_K(V)$ and $A = \mathrm{End}_F(V)$. Let $\mathfrak{C}$ be a hereditary $\mathfrak{o}_K$-order in C, and $\mathfrak{A}$ the hereditary $\mathfrak{o}_F$-order in A defined by the lattice chain in V which defines $\mathfrak{C}$. Thus $K^\times \subset \mathfrak{K}(\mathfrak{A})$ and $\mathfrak{C} = \mathfrak{A} \cap C$. We write $\mathfrak{R}$ for the radical of $\mathfrak{C}$, and $\mathfrak{P}$ for that of $\mathfrak{A}$, so we have

$$\textbf{(2.1)} \qquad \mathfrak{R}^n = \mathfrak{P}^n \cap C, \quad n \in \mathbf{Z}.$$

It follows that if $[\mathfrak{C}, n, m, c]$ is a stratum in C, then $[\mathfrak{A}, n, m, c]$ is a stratum in A, and the process $[\mathfrak{C}, n, m, c] \mapsto [\mathfrak{A}, n, m, c]$ respects equivalence of strata. Likewise, if we have a stratum $[\mathfrak{A}, n, m, b]$ in A such that b commutes with K, i.e., $b \in \mathrm{C}$, then $[\mathfrak{C}, n, m, b]$ is a stratum in C.

Recall that a stratum $[\mathfrak{A}, n, m, b]$ is called *pure* if the algebra $\mathrm{F}[b]$ is a field which normalizes $\mathfrak{A}$ and $b\mathfrak{A} = \mathfrak{P}^{-n}$.

(2.2) *Definition. — Let* $[\mathfrak{A}, n, m, b]$ *be a stratum in* A *such that* $\mathfrak{A}$ *is normalized by* $\mathrm{K}^\times$, *as above. The stratum* $[\mathfrak{A}, n, m, b]$ *is called* K-pure *if*:

(i) $b \in \mathrm{C}$;
(ii) $b\mathfrak{A} = \mathfrak{P}^{-n}$;
(iii) *the algebra* $\mathrm{K}[b]$ *is a field such that* $\mathrm{K}[b]^\times$ *normalizes* $\mathfrak{A}$.

Immediately, a K-pure stratum $[\mathfrak{A}, n, m, b]$ in A is pure, and the corresponding stratum $[\mathfrak{C}, n, m, b]$ in C is pure. By (2.1), we have:

(2.3) *In the situation above, the process* $[\mathfrak{C}, n, m, c] \mapsto [\mathfrak{A}, n, m, c]$ *gives a bijection, respecting equivalence, between the set of pure strata in* C *and the set of* K-*pure strata in* A.

The situation with regard to simple strata requires more investigation. This brings us to the main result of this section.

(2.4) *Theorem.* — Let K/F *be a finite, tamely ramified field extension, and let* V *be a finite-dimensional* K-*vector space. Write* $\mathrm{C} = \mathrm{End}_{\mathrm{K}}(\mathrm{V})$, $\mathrm{A} = \mathrm{End}_{\mathrm{F}}(\mathrm{V})$. *Let* $[\mathfrak{A}, n, m, \beta]$ *be a* K-*pure stratum in* A, *and set* $\mathfrak{C} = \mathfrak{A} \cap \mathrm{C}$. *Then*

$$k_0(\beta, \mathfrak{C}) \leqslant k_0(\beta, \mathfrak{A}).$$

In particular, if $[\mathfrak{A}, n, m, \beta]$ *is simple, then* $[\mathfrak{C}, n, m, \beta]$ *is simple.*

Proof. — We start by recalling, from [BK1] (1.3), the notion of *tame corestriction*. For the moment, let V be some finite-dimensional F-vector space and put $\mathrm{A} = \mathrm{End}_{\mathrm{F}}(\mathrm{V})$. Let E/F be some subfield of A and write $\mathrm{B} = \mathrm{End}_{\mathrm{E}}(\mathrm{V})$. A tame corestriction on A relative to E/F is then a (B, B)-bimodule homomorphism $s : \mathrm{A} \to \mathrm{B}$ with the property

$$s(\mathfrak{A}) = \mathfrak{A} \cap \mathrm{B},$$

for some (equivalently, any) hereditary $\mathfrak{o}_{\mathrm{F}}$-order $\mathfrak{A}$ in A such that $\mathfrak{K}(\mathfrak{A}) \supset \mathrm{E}^\times$. This condition does not determine s uniquely: if s is a tame corestriction as above, and $u \in \mathfrak{o}_{\mathrm{E}}^\times$, the map $a \mapsto us(a)$ is also a tame corestriction on A relative to E/F, and they all arise this way.

Immediately from the definition, we get

(2.5) *Let* $\mathrm{E}_1 \supset \mathrm{E}_2 \supset \mathrm{F}$ *be subfields of* $\mathrm{A} = \mathrm{End}_{\mathrm{F}}(\mathrm{V})$, *and write* $\mathrm{B}_i = \mathrm{End}_{\mathrm{E}_i}(\mathrm{V})$, $i = 1, 2$. *Let* $s_{\mathrm{E}_2/\mathrm{F}}$ *(resp.* $s_{\mathrm{E}_1/\mathrm{E}_2}$*) denote a tame corestriction on* A *(resp.* B_2*) relative to* E_2/F *(resp.* $\mathrm{E}_1/\mathrm{E}_2$*). Then* $s_{\mathrm{E}_1/\mathrm{E}_2} \circ s_{\mathrm{E}_2/\mathrm{F}}$ *is a tame corestriction on* A *relative to* E_1/F.

We now return to the situation of (2.4), so that, in particular, V is a K-vector space and $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in $A = \mathrm{End}_F(V)$ normalized by $K^\times$. We use the following notation throughout the proof:

$$\textbf{(2.6)}\qquad \begin{cases} C = \mathrm{End}_K(V), & \mathfrak{C} = \mathfrak{A} \cap C, \\ E = F[\beta], \quad B = \mathrm{End}_E(V), & \mathfrak{B} = \mathfrak{A} \cap B, \\ D = \mathrm{End}_{K[\beta]}(V), & \mathfrak{D} = \mathfrak{A} \cap D, \\ \mathfrak{P} = \mathrm{rad}(\mathfrak{A}), \quad \mathfrak{R} = \mathrm{rad}(\mathfrak{C}), & \mathfrak{S} = \mathrm{rad}(\mathfrak{D}). \end{cases}$$

(2.7) *Lemma. — Use the notation above, and let $s_{E/F}$ (resp. $s_{K/F}$) be a tame corestriction on* A *relative to* E/F *(resp.* K/F*). Then:*

(i) $s_{K/F} \mid B$ *is a tame corestriction on* B *relative to* $K[\beta]/E$;
(ii) $s_{E/F} \mid C$ *is a tame corestriction on* C *relative to* $K[\beta]/K$.

Proof. — We start with (i). Since any two choices of $s_{K/F}$ differ by a factor $u \in \mathfrak{o}_K^\times$, we see that (i) holds for one choice of $s_{K/F}$ if and only if it holds for all. This enables us to choose $s_{K/F}$ conveniently.

To do this, we write $C^\perp$ for the orthogonal complement of C in A relative to the (nondegenerate) symmetric F-bilinear form

$$(x, y) \mapsto \mathrm{tr}_{A/F}(xy), \quad x, y \in A,$$

where $\mathrm{tr}_{A/F}$ denotes the trace mapping $A \to F$. Thus $A = C \oplus C^\perp$, and the orthogonal projection $A \to C$ is a tame corestriction on A relative to K/F (see [BK1] (1.3.8)). This is our choice for $s_{K/F}$. It is characterized among the set of these tame corestrictions by the property

$$s_{K/F}(c) = c, \quad c \in C.$$

Since $D = B \cap C$, the restriction $s_{K/F} \mid B$ is indeed a (D, D)-bimodule homomorphism $B \to C$. We have to show that $s_{K/F}(B) \subset D$ and that $s_{K/F}(\mathfrak{B}) = \mathfrak{D}$. For $b \in B$, we have $\beta b - b\beta = 0$. Since $\beta \in D$, this implies

$$0 = s_{K/F}(\beta b - b\beta) = \beta s_{K/F}(b) - s_{K/F}(b)\,\beta, \quad b \in B.$$

This shows that $s_{K/F}(B)$ is contained in the C-centralizer of β, which is D. Thus we also have

$$s_{K/F}(\mathfrak{B}) \subset s_{K/F}(\mathfrak{A}) \cap D = \mathfrak{C} \cap D = \mathfrak{D}.$$

On the other hand, we have $\mathfrak{D} \subset C$, so $s_{K/F}(x) = x$ for all $x \in \mathfrak{D}$. Surely $\mathfrak{B} \supset \mathfrak{D}$, so $s_{K/F}(\mathfrak{B}) = \mathfrak{D}$, as desired.

To prove (ii), we proceed with the same $s_{K/F}$. Let $s_{K[\beta]/K}$ be some tame corestriction on C relative to $K[\beta]/K$. By the choice of $s_{K/F}$, we have

$$s_{K[\beta]/K} = s_{K[\beta]/K} \circ s_{K/F} \mid C.$$

However, by (2.5), $s_{K[\beta]/K} \circ s_{K/F}$ is a tame corestriction $s_{K[\beta]/F}$ on A relative to K[β]/F. Thus $s_{K[\beta]/K} = s_{K[\beta]/F} \mid C$. Property (2.5) further implies that $s_{K[\beta]/F} = s_{K[\beta]/E} \circ s_{E/F}$, for our given $s_{E/F}$ and some tame corestriction $s_{K[\beta]/E}$ on B relative to K[β]/E. This gives us

$$s_{K[\beta]/K} = s_{K[\beta]/E} \circ s_{E/F} \mid C.$$

The field extension K[β]/E is tamely ramified, so we can adjust our original choice of $s_{K[\beta]/K}$ (by a unit of $\mathfrak{o}_{K[\beta]}$) to arrange that $s_{K[\beta]/E} \mid D$ is the identity map.

The restriction $s_{E/F} \mid C$ is surely a (D, D)-bimodule homomorphism $C \to B$. We show next that $s_{E/F}(C) \subset D$. For $x \in K$ and $c \in C$, we have $xc - cx = 0$, so

$$0 = s_{E/F}(xc - cx) = xs_{E/F}(c) - s_{E/F}(c)\, x,$$

since $x \in K \subset B$. Thus $s_{E/F}(C)$ is contained in the B-centralizer D of K. This gives us

$$s_{K[\beta]/K}(c) = s_{K[\beta]/E}(s_{E/F}(c)) = s_{E/F}(c), \quad c \in C,$$

since we have arranged for $s_{K[\beta]/E}$ to be the identity map on D. We have therefore shown that $s_{E/F} \mid C = s_{K[\beta]/K}$, as required. □

Before proving the Theorem, we need to recall another piece of the machinery of [BK1]. Starting with our pure stratum $[\mathfrak{A}, n, m, \beta]$ and an integer k, we define

$$\mathfrak{N}_k(\beta, \mathfrak{A}) = \{ x \in \mathfrak{A} : a_\beta(x) \in \mathfrak{P}^k \}.$$

Here, a_β is the adjoint map $x \mapsto \beta x - x\beta$, as above. By [BK1] (1.4.5), we have

$$k > k_0(\beta, \mathfrak{A}) \Leftrightarrow \mathfrak{N}_k(\beta, \mathfrak{A}) \subset \mathfrak{B} + \mathfrak{P},$$

using the notation (2.6).

We take $k \in \mathbf{Z}$, $k > k_0(\beta, \mathfrak{A})$. We have to show that $k > k_0(\beta, \mathfrak{C})$, or, equivalently, that

$$\mathfrak{N}_k(\beta, \mathfrak{C}) \subset \mathfrak{D} + \mathfrak{R}.$$

It is immediate from the definition that $\mathfrak{N}_k(\beta, \mathfrak{C}) = \mathfrak{N}_k(\beta, \mathfrak{A}) \cap C$, so it is enough to show that

$$\textbf{(2.8)} \qquad (\mathfrak{B} + \mathfrak{P}) \cap C = \mathfrak{D} + \mathfrak{R}.$$

We certainly have

$$(\mathfrak{B} + \mathfrak{P}) \cap C \supset \mathfrak{B} \cap C + \mathfrak{P} \cap C = \mathfrak{D} + \mathfrak{R}.$$

On the other hand, let $s_{K/F}$ be the orthogonal projection $A \to C$, as in the proof of (2.7). Then for any additive subgroup M of A, we have $s_{K/F}(M) \supset M \cap C$. In particular, $s_{K/F}(\mathfrak{B}) = \mathfrak{D}$, by (2.7), while $s_{K/F}(\mathfrak{P}) = \mathfrak{R}$, by [BK1] (1.3.4). Thus

$$(\mathfrak{B} + \mathfrak{P}) \cap C \subset s_{K/F}(\mathfrak{B} + \mathfrak{P}) = s_{K/F}(\mathfrak{B}) + s_{K/F}(\mathfrak{P}) = \mathfrak{D} + \mathfrak{R}.$$

The desired equality (2.8) now follows. □

(2.9) *Remark.* — In (2.4), it is easy to produce examples in which $k_0(\beta, \mathfrak{C}) < k_0(\beta, \mathfrak{A})$, in other words, examples of simple strata $[\mathfrak{C}, n, m, \beta]$ such that $[\mathfrak{A}, n, m, \beta]$ is not simple. The generic example, in a certain sense, is given by taking $m = n - 1$, with $\beta \in K^\times$ such that β is not minimal over F. (We recall this concept below). However, one of our main results below will show that a simple stratum $[\mathfrak{C}, n, m, \alpha]$ in C is *equivalent* to some simple stratum $[\mathfrak{C}, n, m, \alpha']$ such that $[\mathfrak{A}, n, m, \alpha']$ is simple.

It is worthy of note that (2.4) depends crucially on the tameness of the ramification of the extension K/F. To illustrate this, write p for the residual characteristic of F, and let K/F, E/F be totally ramified extensions of degree p such that KE/F is totally ramified of degree p^2: this is not difficult to arrange. We then have $E = F[\alpha]$, for some element α which is minimal over F: it has only to satisfy the condition $\gcd(\nu_E(\alpha), e(E \mid F)) = 1$. However, $EK = K[\alpha]$, and $\nu_{EK}(\alpha) = p\nu_E(\alpha)$, which is certainly not relatively prime to $p = e(EK/K)$. Thus α is not minimal over K.

The property mentioned in (2.9) also fails for wildly ramified extensions K/F: a simple stratum over K need not be equivalent to one which is simple over F. An entertaining example of this is provided by taking $F = \mathbf{Q}_2$ (cf. [W]). This field has a quartic extension $K/\mathbf{Q}_2$ whose normal closure has Galois group A_4. In particular, $K/\mathbf{Q}_2$ has no quadratic subextension. We view K as embedded in $\mathrm{End}_{\mathbf{Q}_2}(K) \cong \mathbf{M}(4, \mathbf{Q}_2)$, and normalizing the principal order $\mathfrak{A} = \mathrm{End}^0_{\mathbf{Z}_2}(\{\mathfrak{p}_K^j\})$ (which has $e(\mathfrak{A} \mid \mathfrak{o}_F) = 4$). We take $\alpha \in K$ with $\nu_K(\alpha) = -n \equiv 2 \pmod 4$. The stratum $[\mathfrak{o}_K, n, n-1, \alpha]$ is simple in K. If it were equivalent to a simple stratum $[\mathfrak{o}_K, n, n-1, \beta]$ with $[\mathfrak{A}, n, n-1, \beta]$ simple, we would have:

a) $\beta \in K$, since β must commute with K;

b) $e(\mathbf{Q}_2(\beta) \mid \mathbf{Q}_2) = e(\mathfrak{A} \mid \mathfrak{o}_F)/\gcd(n, e(\mathfrak{A} \mid \mathfrak{o}_F)) = 2$.

Since $K/\mathbf{Q}_2$ is totally ramified, *b)* would imply $[\mathbf{Q}_2(\beta) : \mathbf{Q}_2] = 2$, which is impossible.

3. Tame lifting of simple pairs

We now come to our main results concerning tame lifting of simple strata. Thus we fix a finite, tamely ramified field extension K/F, and consider relations between the sets $\mathcal{SP}(K)$, $\mathcal{SP}(F)$ of equivalence classes of simple pairs induced by the inclusion $F \to K$.

We start with a little elementary field theory. This corresponds to lifting field elements, ignoring the metric considerations imposed by stratum structures. Let E/F be a finite field extension, and $\alpha \in E^\times$ such that $E = F[\alpha]$. We can form the K-algebra

$$\mathscr{E} = E \otimes_F K.$$

Simply because K/F is finite separable, we get a canonical decomposition

$$\mathscr{E} = \prod_{i=1}^{r} E_i$$

of $\mathscr{E}$ as a direct product of field extensions E_i/K. We write π_i for the canonical projection

$$\pi_i : \mathscr{E} \to E_i.$$

We then get elements

$$\alpha_i = \pi_i(\alpha \otimes 1) \in \mathrm{E}_i$$

such that $\mathrm{E}_i = \mathrm{K}[\alpha_i]$, $1 \leqslant i \leqslant r$. These α_i we call the K/F-*lifts* of α. They are distinct over K: if $i \neq j$, there exists no K-isomorphism $\mathrm{K}[\alpha_i] \cong \mathrm{K}[\alpha_j]$ carrying α_i to α_j. One can, of course, phrase this in terms of polynomials: if $\Phi_i(\mathrm{X}) \in \mathrm{K}[\mathrm{X}]$ is the minimal polynomial of α_i over K, then $\prod_i \Phi_i(\mathrm{X})$ lies in $\mathrm{F}[\mathrm{X}]$ and is the minimal polynomial $\varphi_\alpha(\mathrm{X})$ of α over F. Thus the K/F-lifts of α are given precisely by the K-irreducible factors of $\varphi_\alpha(\mathrm{X})$.

This lifting process $\alpha \mapsto \{ \alpha_i : 1 \leqslant i \leqslant r \}$ is *transitive* in the field extension K/F. Indeed, if L/K is a finite tamely ramified extension, and if $\{ \beta_{ij} : 1 \leqslant j \leqslant r_i \}$ is the set of L/K-lifts of α_i, then $\{ \beta_{ij} : 1 \leqslant j \leqslant r_i, 1 \leqslant i \leqslant r \}$ is the set of L/F-lifts of α.

(3.1) *Let* K/F *be a finite Galois extension with* $\Gamma = \mathrm{Gal}(\mathrm{K}/\mathrm{F})$, *and use the notation above. We have*:

(i) Γ *permutes the factors* E_i *transitively, and the stabilizer of* E_i *is the canonical image of* $\mathrm{Gal}(\mathrm{E}_i/\mathrm{E})$ *in* Γ;
(ii) Γ *acts transitively on the set* $\{ \alpha_1, \alpha_2, \ldots, \alpha_r \}$, *and the stabilizer of* α_i *is* $\mathrm{Gal}(\mathrm{E}_i/\mathrm{E})$;
(iii) $e(\mathrm{E}_i \mid \mathrm{K}) = e(\mathrm{E}_j \mid \mathrm{K})$ *and* $f(\mathrm{E}_i \mid \mathrm{K}) = f(\mathrm{E}_j \mid \mathrm{K})$, *for all* i, j.

This is standard. The situation is not so tidy when K/F is not Galois, but we do get a useful property concerning ramification in the fields E_i.

(3.2) *Proposition. — Let* K/F *be a finite, tamely ramified, field extension, and use the other notation above. The field extensions* E_i/F *then all have the same ramification index, namely*

$$e(\mathrm{E}_i \mid \mathrm{F}) = \mathrm{lcm}(e(\mathrm{E} \mid \mathrm{F}), e(\mathrm{K} \mid \mathrm{F})), \quad 1 \leqslant i \leqslant r.$$

Proof. — This follows readily from the standard structure theory of tamely ramified extensions: see, for example, [F] § 8. □

In the situation of (3.2), where K/F is tamely ramified, it will be convenient to have the notation

$$\begin{aligned} &e_{\mathrm{F}}(\alpha) = e(\mathrm{E} \mid \mathrm{F}), \\ \textbf{(3.3)} \qquad &e_\alpha(\mathrm{K} \mid \mathrm{F}) = \frac{e(\mathrm{K} \mid \mathrm{F})}{\gcd(e(\mathrm{E} \mid \mathrm{F}), e(\mathrm{K} \mid \mathrm{F}))} = e(\mathrm{E}_i \mid \mathrm{E}), \\ &n_{\mathrm{K}}(\alpha) = n_{\mathrm{F}}(\alpha)\, e_\alpha(\mathrm{K} \mid \mathrm{F}). \end{aligned}$$

Here, $n_{\mathrm{K}}(\alpha)$ is just $n_{\mathrm{K}}(\alpha_i) = -\nu_{\mathrm{E}_i}(\alpha_i)$ in our earlier notation: it is in particular independent of i.

In general, the residue class degree $f(\mathrm{E}_i \mid \mathrm{K})$ will vary with i, at least when K/F and E/F are both ramified and K/F is not a Galois extension. For example, suppose that K/F is totally tamely ramified of prime degree ℓ and not Galois. Thus F contains no primitive ℓ-th root of unity. Suppose also that ℓ divides $e(\mathrm{E} \mid \mathrm{F})$. Then one of the factors E_i has degree 1 over E, and the others are all isomorphic (over E) to the field $\mathrm{E}[\zeta]$, where ζ is a primitive ℓ-th root of unity.

(3.4) *Proposition. — Let* $[m, \alpha]$ *be a simple pair over* F, *and let* K/F *be a finite, tamely ramified field extension. Let* α_i *be a* K/F*-lift of* α, *and* M *an integer satisfying*

$$\left[\frac{\mathrm{M}}{e_\alpha(\mathrm{K} \mid \mathrm{F})}\right] \leqslant m.$$

Then $[\mathrm{M}, \alpha_i]$ *is a simple pair over* K.

Proof. — We have $n_{\mathrm{F}}(\alpha) > m$, so $n_{\mathrm{K}}(\alpha_i) = n_{\mathrm{K}}(\alpha) < \mathrm{M}$, as required by the definition of a simple pair.

As above, write E_i for the factor $\mathrm{F}[\alpha_i]$ of $\mathscr{E} = \mathrm{K} \otimes_{\mathrm{F}} \mathrm{E}$. We view E_i as a K-vector space, so that we can form the algebras

$$\mathrm{A}(\mathrm{E}_i) = \mathrm{End}_{\mathrm{F}}(\mathrm{E}_i),$$

$$\mathrm{C}(\mathrm{E}_i) = \mathrm{End}_{\mathrm{K}}(\mathrm{E}_i).$$

We have a natural embedding $\mathrm{F}[\alpha] \to \mathrm{E}_i$, given by $\alpha \mapsto \alpha_i$, which we may use here to identify α_i with α. Let $\mathfrak{A}(\mathrm{E}_i)$ denote the order

$$\mathrm{End}^0_{\mathfrak{o}_{\mathrm{F}}}(\{\, \mathfrak{p}^j_{\mathrm{E}_i} : j \in \mathbf{Z} \,\}).$$

We then have $e(\mathfrak{A}(\mathrm{E}_i) \mid \mathfrak{o}_{\mathrm{F}}) = e(\mathrm{E}_i \mid \mathrm{F})$, so that $e_\alpha(\mathfrak{A}(\mathrm{E}_i)) = e_\alpha(\mathrm{K} \mid \mathrm{F})$. Thus $[\mathfrak{A}(\mathrm{E}_i), n_{\mathrm{F}}(\alpha)\, e_\alpha(\mathrm{K} \mid \mathrm{F}), m e_\alpha(\mathrm{K} \mid \mathrm{F}), \alpha]$ is a realization of $[m, \alpha]$ on $\mathfrak{A}(\mathrm{E}_i)$. Further, this stratum is K-pure, in the sense of (2.2). The intersection $\mathfrak{A}(\mathrm{E}_i) \cap \mathrm{C}(\mathrm{E}_i)$ is just

$$\mathfrak{C}(\mathrm{E}_i) = \mathrm{End}^0_{\mathfrak{o}_{\mathrm{K}}}(\{\, \mathfrak{p}^j_{\mathrm{E}_i} : j \in \mathbf{Z} \,\}).$$

Theorem (2.4) now tells us

$$k_0(\alpha, \mathfrak{C}(\mathrm{E}_i)) \leqslant k_0(\alpha, \mathfrak{A}(\mathrm{E}_i)),$$

whence

$$k_{\mathrm{K}}(\alpha) \leqslant k_{\mathrm{F}}(\alpha)\, e_\alpha(\mathrm{K} \mid \mathrm{F})$$

(see (1.4) *et seq.*). We are given the relation $(m+1) \leqslant -k_{\mathrm{F}}(\alpha)$, and the pair $[\mathrm{M}, \alpha]$ is simple over K provided $(\mathrm{M}+1) \leqslant -k_{\mathrm{K}}(\alpha)$. This will hold provided $(\mathrm{M}+1) \leqslant (m+1)\, e_\alpha(\mathrm{K} \mid \mathrm{F})$, which is equivalent to $[\mathrm{M}/e_\alpha(\mathrm{K} \mid \mathrm{F})] \leqslant m$, as required. □

Thus the simple pair $[m, \alpha]$, together with a choice of integer M such that $[\mathrm{M}/e_\alpha(\mathrm{K}/\mathrm{F})] \leqslant m$, gives rise to a finite set $[\mathrm{M}, \alpha_i]$ of simple pairs over K. We refer to these as the K/F-lifts of $[m, \alpha]$ (relative to the choice of M). This dependence on M is somewhat spurious: see (6.1) below. The process $[m, \alpha] \mapsto \{[M, \alpha_i]\}$ is again transitive in K/F: this follows from the relation $e_\alpha(L \mid F) = e_\alpha(K \mid F)\, e_{\alpha_i}(L \mid K)$.

Our first result asserts that this lifting process preserves the relation of equivalence between simple pairs, as in (1.14).

(3.5) *Theorem. — Let* K/F *be a finite, tamely ramified field extension, and let* $[m, \alpha]$ *be a simple pair over* F. *Let* $\alpha_1, \ldots, \alpha_r$ *be the* K/F*-lifts of* α, *and let* M *denote the integer*

$$M = (m+1)\, e_\alpha(K \mid F) - 1.$$

The set

$$\mathbf{L}_{K/F}(m, \alpha) = \{(M, \alpha_i) : 1 \leqslant i \leqslant r\}$$

of equivalence classes of simple pairs over K *depends only on the equivalence class* (m, α): *if* $[m, \alpha']$ *is a simple pair equivalent to* $[m, \alpha]$, *then* $\mathbf{L}_{K/F}(m, \alpha') = \mathbf{L}_{K/F}(m, \alpha)$. *Moreover, we have* $(M, \alpha_i) = (M, \alpha_j)$ *if and only if* $i = j$.

Thus the process

$$(m, \alpha) \mapsto \mathbf{L}_{K/F}(m, \alpha) = \{(M, \alpha_i)\}$$

(in the notation of (3.5)) gives us a well-defined map from $\mathscr{SP}(F)$ to finite subsets of $\mathscr{SP}(K)$. This process is, moreover, injective in the following sense:

(3.6) *Theorem. — Let* $[m, \alpha]$, $[k, \beta]$ *be simple pairs over* F. *Suppose there exist* K/F*-lifts* $\tilde{\alpha}$, $\tilde{\beta}$ *of* α, β *respectively, and an integer* M *satisfying*

$$M + 1 \leqslant \min\{(m+1)\, e_\alpha(K \mid F),\ (k+1)\, e_\beta(K \mid F)\},$$

such that $[M, \tilde{\alpha}] \approx [M, \tilde{\beta}]$. *We then have* $[\ell, \alpha] \approx [\ell, \beta]$, *where* $\ell = \min\{m, k\}$.

We also have a surjectivity property:

(3.7) *Theorem. — Let* $[k, \beta]$ *be a simple pair over* K. *There exists a simple pair* $[m, \alpha]$ *over* F *and a* K/F*-lift* α_i *of* α *such that*

$$\left[\frac{k}{e_\alpha(K \mid F)}\right] = m,$$

and

$$[k, \beta] \approx [k, \alpha_i].$$

The proofs of these theorems occupy the next three sections. In the remainder of this section, we present some corollaries. First, we answer the question left open by (2.4) (see also (2.9)).

(3.8) *Corollary. — Let* $[\mathfrak{C}, n, m, \beta]$ *be a simple stratum over* K. *Let* $\mathfrak{A}$ *be the hereditary* $\mathfrak{o}_F$*-order defined by the same lattice chain as* $\mathfrak{C}$. *There exists a simple stratum* $[\mathfrak{C}, n, m, \beta']$, *which is equivalent to* $[\mathfrak{C}, n, m, \beta]$, *such that* $[\mathfrak{A}, n, m, \beta']$ *is simple.*

We will be able to deduce (3.8) from (3.7) once we have proved (4.3) below.

We can summarize our lifting theorems as a list of properties of a certain base-field restriction or " induction " map as follows.

(3.9) *Corollary. — Let* K/F *be a finite, tamely ramified field extension. There exists a unique map*

$$\mathrm{Res}_{K/F} : \mathcal{SP}(K) \to \mathcal{SP}(F)$$

with the following property: if $(k, \beta) \in \mathcal{SP}(K)$, *then* $\mathrm{Res}_{K/F}(k, \beta) = (m, \alpha)$, *where* $[k/e_\alpha(K \mid F)] = m$ *and* $[k, \beta] \approx [k, \tilde{\alpha}]$, *for some* K/F-*lift* $\tilde{\alpha}$ *of the element* α. *Moreover,*

(i) *the map* $\mathrm{Res}_{K/F}$ *is surjective;*

(ii) *for* $(m, \alpha) \in \mathcal{SP}(F)$, *the fibre of* $\mathrm{Res}_{K/F}$ *over* (m, α) *is the set* $\{(k, \alpha_i)\}$, *where* k *ranges over all integers satisfying*

$$\left[\frac{k}{e_\alpha(K \mid F)}\right] = m$$

and $\{\alpha_i : 1 \leqslant i \leqslant r\}$ *is the set of* K/F-*lifts of the element* α;

(iii) *in* (ii), *we have* $(k, \alpha_i) = (k, \alpha_j)$ *if and only if* $\alpha_i = \alpha_j$, $1 \leqslant i, j \leqslant r$.

If K/F is also Galois, with $\Gamma = \mathrm{Gal}(K/F)$, the fibre of $\mathrm{Res}_{K/F}$ over (m, α) is the union of Galois orbits $\{(k, \tilde{\alpha}^\sigma) : \sigma \in \Gamma\}$, for a fixed lift $\tilde{\alpha}$ of α.

The transitivity property of the lifting process for simple pairs is a direct consequence of the same property for lifting of field elements. However, it can be expressed very tidily in terms of the map Res:

(3.10) *Let* L/K, K/F *be finite tamely ramified field extensions. Then*

$$\mathrm{Res}_{L/F} = \mathrm{Res}_{K/F} \circ \mathrm{Res}_{L/K}.$$

Remark. — One can simplify the statements of the lifting theorems by specializing to the following case. Write $\mathcal{SP}^0(F)$ for the set of equivalence classes of simple pairs o the form $[0, \alpha]$. Lifting gives us a map from $\mathcal{SP}^0(F)$ to finite subsets of $\mathcal{SP}^0(K)$, and base-field restriction a surjective map $\mathcal{SP}^0(K) \to \mathcal{SP}^0(F)$ whose fibre above a given $(0, \alpha)$ is the set $\{(0, \alpha_i)\}$, with α_i ranging over the K/F-lifts of the element α. Once the basic theory is established, this is the only case which will interest us. However, the extra generality is essential, both for the proofs of the lifting theorems here and for the explicit constructions of § 7.

The proofs of the results stated here occupy the next three sections. We prove (3.6) in § 4, along with some general preliminary results. We also show how to deduce (3.8) from the theorems. The theorems themselves are proved inductively. The first step is given in § 5, and the general case of the induction occupies § 6.

4. Preliminary reductions

We start with:

Proof of (3.6). — We use the notation of the statement. Also, for a finite field extension L/K, we write

$$\mathfrak{C}(\mathrm{L}) = \mathrm{End}^0_{\mathfrak{o}_\mathrm{K}}(\{\mathfrak{p}^j_\mathrm{L}\}).$$

The hypothesis $[\mathrm{M}, \tilde{\alpha}] \approx [\mathrm{M}, \tilde{\beta}]$ implies (see (1.15)) that $n_\mathrm{K}(\tilde{\alpha}) = n_\mathrm{K}(\tilde{\beta})$ and $\mathfrak{C}(\mathrm{K}[\tilde{\alpha}]) \cong \mathfrak{C}(\mathrm{K}[\tilde{\beta}])$. We can therefore find a K-vector space V of dimension $[\mathrm{K}[\tilde{\alpha}] : \mathrm{K}] = [\mathrm{K}[\tilde{\beta}] : \mathrm{K}]$, and a hereditary $\mathfrak{o}_\mathrm{K}$-order $\mathfrak{C}$ in $\mathrm{End}_\mathrm{K}(\mathrm{V})$, together with K-embeddings $\varphi : \mathrm{K}[\tilde{\alpha}] \to \mathrm{End}_\mathrm{K}(\mathrm{V})$, $\psi : \mathrm{K}[\tilde{\beta}] \to \mathrm{End}_\mathrm{K}(\mathrm{V})$ whose images normalize $\mathfrak{C}$. Note that $\mathfrak{C} \cong \mathfrak{C}(\mathrm{K}[\tilde{\alpha}]) \cong \mathfrak{C}(\mathrm{K}[\tilde{\beta}])$ as $\mathfrak{o}_\mathrm{K}$-orders. Thus we have realizations $[\mathfrak{C}, n, \mathrm{M}, \varphi(\tilde{\alpha})]$, $[\mathfrak{C}, n, \mathrm{M}, \psi(\tilde{\beta})]$ of our simple pairs, with $n = n_\mathrm{K}(\tilde{\alpha}) = n_\mathrm{K}(\tilde{\beta})$. By (1.10) and hypothesis, these realizations must intertwine in $\mathrm{End}_\mathrm{K}(\mathrm{V})$. By (1.8), we can adjust φ, say, by a $\mathbf{U}(\mathfrak{C})$-conjugation and assume we have an equivalence

$$[\mathfrak{C}, n, \mathrm{M}, \varphi(\tilde{\alpha})] \sim [\mathfrak{C}, n, \mathrm{M}, \psi(\tilde{\beta})].$$

Let $\mathfrak{A}$ be the hereditary $\mathfrak{o}_\mathrm{F}$-order defined by the same lattice chain as $\mathfrak{C}$. This means we have an equivalence of simple strata

$$[\mathfrak{A}, n, \mathrm{M}, \varphi(\tilde{\alpha})] \sim [\mathfrak{A}, n, \mathrm{M}, \psi(\tilde{\beta})].$$

Of course, when working over F, we do not need to distinguish between α, $\tilde{\alpha}$ and likewise for β. This last equivalence of simple strata implies $n_\mathrm{F}(\alpha) = n_\mathrm{F}(\beta)$. Combining this with the equation $n_\mathrm{K}(\tilde{\alpha}) = n_\mathrm{K}(\tilde{\beta})$, we get

$$e_\alpha(\mathrm{K} \mid \mathrm{F}) = e_\beta(\mathrm{K} \mid \mathrm{F}).$$

The above equivalence of strata now further implies that $[\ell, \alpha] \approx [\ell, \beta]$, for any integer $\ell \geqslant [\mathrm{M}/e_\alpha(\mathrm{K} \mid \mathrm{F})]$.

This completes the proof of (3.6). □

Having proved (3.6), Theorems (3.5) and (3.7) when taken together are transitive in the field extension K/F in the following very strong sense.

(4.1) *Lemma. — Let* L/K, K/F *be finite tamely ramified field extensions. Suppose that* (3.5), (3.7) *hold for two of the extensions* L/K, K/F, L/F. *They then hold for the third.*

Proof. — We only prove one of the three assertions of the Lemma: of the others, one is very similar and the remaining one easy. We assume that (3.5) and (3.7) hold for the extensions L/F, L/K, and deduce them for the extension K/F. Let $[m, \alpha]$ be a simple pair over F, and let $\alpha_1, \ldots, \alpha_r$ be the K/F-lifts of α. Define

$$\begin{aligned} \mathrm{M} &= (m+1)\, e_\alpha(\mathrm{K} \mid \mathrm{F}) - 1, \\ \mathrm{M}' &= (\mathrm{M}+1)\, e_{\alpha_i}(\mathrm{L} \mid \mathrm{K}) - 1, \\ \mathrm{M}'' &= (m+1)\, e_\alpha(\mathrm{L} \mid \mathrm{F}) - 1. \end{aligned}$$

Note that the definition of $e_{\alpha_i}(\mathrm{L} \mid \mathrm{K})$ is independent of the choice of i, and indeed

$$e_\alpha(\mathrm{L} \mid \mathrm{F}) = e_\alpha(\mathrm{K} \mid \mathrm{F})\, e_{\alpha_i}(\mathrm{L} \mid \mathrm{K}).$$

Let α_{ij}, $1 \leqslant j \leqslant s_i$ be the set of L/K-lifts of α_i, so that $\{\alpha_{ij}\}$ is the set of L/F-lifts of α. The classes $(\mathrm{M}'', \alpha_{ij})$ are all distinct, by (3.5) applied to L/F. Applying the first statement of (3.5) to L/K, we see that the (M', α_i) are distinct.

Next, we take a simple pair $[m, \alpha']$ over F equivalent to $[m, \alpha]$. Applying (3.5) to L/F, these pairs have the same sets of equivalence classes of L/F-lifts. Theorem (3.6), applied to L/K, now shows that they have the same sets of equivalence classes of K/F-lifts. This proves (3.5) for K/F.

To prove (3.7) for K/F, let $[k, \beta]$ be a simple pair over K. Let $\tilde{\beta}$ be an L/K-lift of β, and let k' be the least integer for which $[k'/e_\beta(\mathrm{L} \mid \mathrm{K})] = k$. Applying (3.7) to L/F, there exists a simple pair $[m, \alpha]$ over F and an L/F-lift $\tilde{\alpha}$ of α such that $[k'/e_\alpha(\mathrm{L} \mid \mathrm{F})] = m$ and $[k', \tilde{\alpha}] \approx [k', \tilde{\beta}]$. We finish the proof by applying (3.6) to the extension L/K. □

Our next result is a conditional one, relating the assertion of (3.7) to the more concrete considerations of § 2. As always, K/F denotes a finite tamely ramified field extension.

(4.2) *Proposition.* — *Let* V *be a finite-dimensional* K-*vector space, and* $[\mathfrak{C}, n, m, \beta]$ *a simple stratum in* $\mathrm{C} = \mathrm{End}_{\mathrm{K}}(\mathrm{V})$. *Let* $\mathfrak{A}$ *be the hereditary* $\mathfrak{o}_{\mathrm{F}}$-*order in* $\mathrm{A} = \mathrm{End}_{\mathrm{F}}(\mathrm{V})$ *defined by the same lattice chain as* $\mathfrak{C}$. *Suppose there exists a simple stratum* $[\mathfrak{C}, n, m, \delta]$ *in* C *such that*

a) $[\mathfrak{C}, n, m, \delta] \sim [\mathfrak{C}, n, m, \beta]$, *and*
b) $[\mathfrak{A}, n, m, \delta]$ *is simple.*

There then exists a K/F-*lift* $\tilde{\delta}$ *of* δ *such that the simple pairs* $[[m/e_\beta(\mathfrak{C})], \tilde{\delta}]$, $[[m/e_\beta(\mathfrak{C})], \beta]$ *are equivalent.*

Proof. — The obvious embedding $\mathrm{F}[\delta] \to \mathrm{End}_{\mathrm{K}}(\mathrm{V})$ extends uniquely to a K-algebra homomorphism $\mathrm{F}[\delta] \otimes_{\mathrm{F}} \mathrm{K} \to \mathrm{End}_{\mathrm{K}}(\mathrm{V})$. The image here is a field (namely the field $\mathrm{K}[\delta]$), so this map factors through the canonical projection of $\mathrm{F}[\delta] \otimes \mathrm{K}$ to one of its field factors. This factor is of the form $\mathrm{K}[\tilde{\delta}]$, for some K/F-lift $\tilde{\delta}$ of δ. Our given map $\mathrm{F}[\delta] \to \mathrm{End}_{\mathrm{K}}(\mathrm{V})$ therefore extends to a K-embedding $\mathrm{K}[\tilde{\delta}] \to \mathrm{End}_{\mathrm{K}}(\mathrm{V})$ which maps $\tilde{\delta}$ to our original element δ. Because of the equivalence $[\mathfrak{C}, n, m, \beta] \sim [\mathfrak{C}, n, m, \delta]$, we have $e_\delta(\mathfrak{C}) = e_\beta(\mathfrak{C})$. Thus $[\mathfrak{C}, n, m, \delta]$ is a realization of the simple pair $[[m/e_\beta(\mathfrak{C})], \tilde{\delta}]$, which intertwines with the given realization $[\mathfrak{C}, n, m, \beta]$ of $[[m/e_\beta(\mathfrak{C})], \beta]$. In other words, $[[m/e_\beta(\mathfrak{C})], \widetilde{\delta}] \approx [[m/e_\beta(\mathfrak{C})], \beta]$, as required. □

There is also a conditional result converse to this.

(4.3) *Proposition. — Let* $[k, \beta]$ *be a simple pair over* K. *Suppose there exists a simple pair* $[m, \alpha]$ *over* F *and a* K/F*-lift* $\widetilde{\alpha}$ *of* α *such that*

a) $[k/e_\alpha(\mathrm{K} \mid \mathrm{F})] = m$ *and*

b) $[k, \beta] \approx [k, \widetilde{\alpha}]$.

Let $[\mathfrak{C}, n, q, \beta]$ *be some realization of* $[k, \beta]$, *and let* $\mathfrak{A}$ *be the hereditary* $\mathfrak{o}_\mathrm{F}$*-order defined by the same lattice chain as* $\mathfrak{C}$. *There then exists a simple stratum* $[\mathfrak{C}, n, q, \beta']$, *equivalent to* $[\mathfrak{C}, n, q, \beta]$, *such that* $[\mathfrak{A}, n, q, \beta']$ *is simple.*

Proof. — The equivalence $[k, \widetilde{\alpha}] \approx [k, \beta]$ implies, via (1.15), that

$$e(\mathrm{K}[\widetilde{\alpha}] \mid \mathrm{K}) = e(\mathrm{K}[\beta] \mid \mathrm{K}),$$

and likewise for residue class degrees. Therefore there exists a realization $[\mathfrak{C}, n, q, \varphi(\widetilde{\alpha})]$ of $[k, \widetilde{\alpha}]$ on $\mathfrak{C}$ by (1.1). Further, this must intertwine with $[\mathfrak{C}, n, q, \beta]$ by (1.10). Thus, by (1.8), we can replace φ by some $\mathbf{U}(\mathfrak{C})$-conjugate and assume that

$$[\mathfrak{C}, n, q, \varphi(\widetilde{\alpha})] \sim [\mathfrak{C}, n, q, \beta].$$

We have

$$\begin{aligned} q < (k+1)\, e_\beta(\mathfrak{C}) &= (k+1)\, e_{\widetilde{\alpha}}(\mathfrak{C}) \\ &= (k+1)\, e(\mathfrak{C} \mid \mathfrak{o}_\mathrm{K})/e(\mathrm{K}[\widetilde{\alpha}] \mid \mathrm{K}) \\ &= (k+1)\, e(\mathfrak{A} \mid \mathfrak{o}_\mathrm{F})/e(\mathrm{K}[\widetilde{\alpha}] \mid \mathrm{F}) \\ &= (k+1)\, e_\alpha(\mathfrak{A})/e_\alpha(\mathrm{K} \mid \mathrm{F}) \\ &\leqslant (m+1)\, e_\alpha(\mathrm{K} \mid \mathrm{F})\, e_\alpha(\mathfrak{A})/e_\alpha(\mathrm{K} \mid \mathrm{F}) \end{aligned}$$

and this is $\leqslant -k_0(\alpha, \mathfrak{A})$ because $[m, \alpha]$ is simple. Thus $[\mathfrak{A}, n, q, \varphi(\widetilde{\alpha})]$ is simple, and the result follows. □

Theorem (3.7) asserts that the hypotheses of (4.3) are satisfied for any simple pair over K. Thus (3.8) is a consequence of (3.7).

We now need some technical results concerning the extrastructure available when the lifting extension K/F is Galois.

(4.4) *Proposition. — Let* K/F *be a finite Galois extension, and* $\mathrm{F}[\alpha]/\mathrm{F}$ *a finite field extension. Let* $\alpha_1, \ldots, \alpha_r$ *be the* K/F*-lifts of* α.

Let V *be a finite-dimensional* K*-vector space, and* $\mathfrak{C}$ *a hereditary* $\mathfrak{o}_\mathrm{K}$*-order in* $\mathrm{C} = \mathrm{End}_\mathrm{K}(\mathrm{V})$. *Suppose there exists an embedding* $\varphi_1 : \mathrm{K}[\alpha_1] \to \mathrm{C}$ *of* K*-algebras such that* $\varphi_1(\mathrm{K}[\alpha_1]^\times)$ *normalizes* $\mathfrak{C}$. *Then, for each* i, *there is a* K*-embedding* $\varphi_i : \mathrm{K}[\alpha_i] \to \mathrm{C}$ *such that* $\varphi_i(\mathrm{K}[\alpha_i]^\times)$ *normalizes* $\mathfrak{C}$.

Proof. — This follows from (1.1) and (3.1). □

Of course, one can vary the embedding of K in $\mathfrak{C}$ by composing with the action of the Galois group $\Gamma = \mathrm{Gal}(K/F)$. The next two results describe the effect of this on the elements α_i.

(4.5) *Proposition. — Let* K/F *be a finite Galois extension with* $\Gamma = \mathrm{Gal}(K/F)$. *Let* V *be a finite-dimensional* K*-vector space, and let* $\mathfrak{C}$ *be a hereditary* $\mathfrak{o}_K$*-order in* $C = \mathrm{End}_K(V)$. *Let* $\mathfrak{A}$ *be the hereditary* $\mathfrak{o}_F$*-order in* $A = \mathrm{End}_F(V)$ *defined by the same lattice chain as* $\mathfrak{C}$. *Then:*

(i) *The* $\mathfrak{K}(\mathfrak{A})$*-centralizer of* $K^\times$ *is* $\mathfrak{K}(\mathfrak{C})$.

(ii) *Let* $\mathcal{N}_{\mathfrak{A}}(K)$ *denote the* $\mathfrak{K}(\mathfrak{A})$*-normalizer of* $K^\times$. *Then* $\mathcal{N}_{\mathfrak{A}}(K)$ *normalizes* $\mathfrak{K}(\mathfrak{C})$, *and restriction to* K *induces an isomorphism*

$$\mathcal{N}_{\mathfrak{A}}(K)/\mathfrak{K}(\mathfrak{C}) \cong \Gamma.$$

Proof. — This follows from (1.6) and the Skolem-Noether theorem. □

Remark. — In the situation of (4.4-5), we have $\mathfrak{K}(\mathfrak{A}) = \mathfrak{K}(\mathfrak{C})\,\mathbf{U}(\mathfrak{A})$. Thus here and in (4.6) below, we could replace $\mathfrak{K}(\mathfrak{A})$ by $\mathbf{U}(\mathfrak{A})$ without changing anything.

(4.6) *Proposition. — Let* K/F, α, $\mathfrak{C}$, φ_1 *be as in* (4.4), *and* $\mathfrak{A}$ *as in* (4.5). *For* $1 \leqslant i \leqslant r$, *let* Φ_i *be the set of* K*-embeddings* $\varphi_i : K[\alpha_i] \to C$ *with* $\varphi_i(K[\alpha_i]^\times) \subset \mathfrak{K}(\mathfrak{C})$, *and put* $\boldsymbol{\Phi} = \bigcup \Phi_i$.

(i) *Let* $x \in \mathcal{N}_{\mathfrak{A}}(K)$, *and let* $\sigma \in \Gamma$ *be the element satisfying* $\sigma(y) = xyx^{-1}$, $y \in K$. *Given* i, *there exists a unique* j *such that* σ *extends to an* F*-isomorphism* $\bar\sigma : K[\alpha_i] \to K[\alpha_j]$ *satisfying* $\bar\sigma(\alpha_i) = \alpha_j$. *The extension* $\bar\sigma$ *is uniquely determined. Moreover, if* $\varphi_i \in \Phi_i$, *then* $\mathrm{Ad}(x) \circ \varphi_i \circ \bar\sigma^{-1} \in \Phi_j$.

(ii) *The action of* $\mathcal{N}_{\mathfrak{A}}(K)$ *on* $\boldsymbol{\Phi}$ *given by* (i) *is transitive.*

(iii) *The stabilizer of* Φ_i *is* $\mathcal{N}_{\mathfrak{B}}(K[\varphi_i(\alpha_i)])\,\mathfrak{K}(\mathfrak{C})$, *where* φ_i *is any element of* Φ_i *and* $\mathfrak{B}$ *is the hereditary* $\mathfrak{o}_{F[\varphi_i(\alpha_i)]}$*-order* $\mathfrak{A} \cap \mathrm{End}_{F[\varphi_i(\alpha_i)]}(V)$.

Proof. — Part (i) is self-explanatory. Given any pair (i,j), together with $\varphi_i \in \Phi_i$ and $\varphi_j \in \Phi_j$, there is an F-isomorphism $\xi : K[\varphi_i(\alpha_i)] \to K[\varphi_j(\alpha_j)]$ with $\xi(\varphi_i(\alpha_i)) = \varphi_j(\alpha_j)$. We have $\xi(K) = K$, and $\xi \mid K \in \Gamma$. This isomorphism ξ is realized by conjugation by some element $u \in \mathbf{U}(\mathfrak{A})$, by (1.6). However, u conjugates K to itself so $u \in \mathcal{N}_{\mathfrak{A}}(K)$, as required for (ii).

In part (i), if we allow x to range over the stabilizer of Φ_i, the corresponding element $\sigma \in \Gamma$ ranges over $\Gamma_i = \mathrm{Gal}(K[\alpha_i]/F[\alpha])$. The inverse image of Γ_i in $\mathcal{N}_{\mathfrak{A}}(K)$ is the group $\mathcal{N}_{\mathfrak{B}}(K[\varphi_i(\alpha_i)])\,\mathfrak{K}(\mathfrak{C})$, and the result follows. □

5. Tame lifting for minimal pairs

In this section, we deal with the lifting theorems (3.5), (3.7) in a special case, which will form the first step of an inductive argument.

A simple pair $[m, \alpha]$ over F is called *minimal* if we have $m = n_F(\alpha) - 1$. Thus $k_F(\alpha)$ equals $-\infty$ or $-n_F(\alpha)$. This is equivalent to α being minimal over F, in the sense of [BK1] (1.4.14). Explicitly, this means

(5.1) (i) $\gcd(n_F(\alpha), e_F(\alpha)) = 1$, *and*

(ii) *if* π_F *is a prime element of* F, *the coset*

$$\pi_F^{n_F(\alpha)}\,\alpha^{e_F(\alpha)} + \mathfrak{p}_{F[\alpha]}(\subset \mathbf{U}(\mathfrak{o}_{F[\alpha]}))$$

generates the residue field extension $k_{F[\alpha]}/k_F$.

For our choice of prime element π_F, we write

$$\rho_F(\alpha) = \rho_F(\alpha; \pi_F) = (\pi_F^{n_F(\alpha)}\,\alpha^{e_F(\alpha)} + \mathfrak{p}_{F[\alpha]}) \in k_{F[\alpha]}^{\times}.$$

The minimal simple pair $[m, \alpha]$ then determines a triple of invariants, namely the integers $n_F(\alpha)$, $e_F(\alpha)$ and the (monic) minimal polynomial $f_\alpha(X) \in k_F[X]$ of $\rho_F(\alpha)$ over k_F. The integers $e_F(\alpha)$, $n_F(\alpha)$ are subject to the conditions

$$e_F(\alpha) \geqslant 1,$$

$$\gcd(e_F(\alpha), n_F(\alpha)) = 1,$$

while the monic irreducible polynomial $f_\alpha(X)$ is subject only to

$$f_\alpha(X) \neq X.$$

Of course, the polynomial $f_\alpha(X)$ does depend on the initial choice of prime element π_F.

(5.2) *Proposition. — Fix a prime element* π_F *of* F. *The map*

$$[m, \alpha] \mapsto (n_F(\alpha), e_F(\alpha), f_\alpha(X))$$

establishes a bijection between the set of equivalence classes of minimal simple pairs $[m, \alpha]$ *over* F *and the set of triples* $(n, e, f(X))$ *consisting of an integer* n, *a positive integer* e *such that* $\gcd(e, n) = 1$, *and a monic irreducible polynomial* $f(X) \in k_F[X]$ *such that* $f(X) \neq X$.

Proof. — In view of (1.10), this is simply a restatement of [BK3] (1.4). □

This enables us to treat lifting of minimal simple pairs in terms of invariants.

(5.3) *Proposition. — Let* $[n-1, \alpha]$ *be a minimal simple pair over* F *(so that* $n = n_F(\alpha)$*). Let* K/F *be a finite tamely ramified field extension, and let* $\{\alpha_i : 1 \leqslant i \leqslant r\}$ *be the set of* K/F*-lifts of* α. *Then:*

(i) $[ne_\alpha(K \mid F) - 1, \alpha_i]$ *is a minimal simple pair over* K;

(ii) *the equivalence classes* $(ne_\alpha(K \mid F) - 1, \alpha_i) \in \mathcal{SP}(K)$, $1 \leqslant i \leqslant r$, *are distinct, and the set* $\mathbf{L}_{K/F}(n-1, \alpha) = \{(ne_\alpha(K \mid F) - 1, \alpha_i) : 1 \leqslant i \leqslant r\}$ *depends only on* $(n-1, \alpha) \in \mathcal{SP}(F)$.

(5.4) *Proposition. — Let* K/F *be as in* (5.3), *and let* $[m-1, \beta]$ *be a minimal simple pair over* K. *There exists a unique minimal* $(n-1, \alpha) \in \mathcal{SP}(F)$ *such that* $(m-1, \beta) \in \mathbf{L}_{K/F}(n-1, \alpha)$.

We note that Proposition (5.3) (i) is a special case of (3.4), while the uniqueness statement in (5.4) is a special case of (3.6) (which we have already proved in general).

(5.5) *Lemma. — Let* L/K, K/F *be finite tamely ramified extensions. Suppose that* (5.3), (5.4) *hold for two of the extensions* L/K, K/F, L/F. *They then hold for the third.*

This is just a special case of (4.1). We can therefore assume, when convenient, that K/F is either unramified or totally ramified (of prime degree).

We treat first the case where K/F is *unramified*. We take a minimal simple pair $[n-1, \alpha]$ over F. We fix a prime element π_F of F, so that, in addition, π_F will serve as a prime element of K. Let p denote the residual characteristic of F, and let $\tilde{\rho}_F$ be the unique p-prime root of unity in $F[\alpha]$ such that

$$\tilde{\rho}_F \equiv \rho_F(\alpha; \pi_F) \pmod{\mathfrak{p}_{F[\alpha]}}.$$

If we view $\tilde{\rho}_F$ as an element of $k_{F[\alpha]}$, then $f_\alpha(X)$ is the minimal polynomial of $\tilde{\rho}_F$ over k_F. Further, we have

$$\mathfrak{o}_{F[\alpha]} = \mathfrak{o}_F[\tilde{\rho}_F, \pi_\alpha],$$

where π_α denotes some prime element of $F[\alpha]$. Now write

$$\mathscr{E} = F[\alpha] \otimes_F K = \prod_{i=1}^{r} E_i,$$

as before, where E_i is the field $K[\alpha_i]$. Since K/F is unramified, the identification of $\mathscr{E}$ with $\prod E_i$ induces further identifications

$$\mathfrak{o}_{F[\alpha]} \otimes_{\mathfrak{o}_F} \mathfrak{o}_K = \prod_{i=1}^{r} \mathfrak{o}_{E_i},$$

$$\mathfrak{p}_{F[\alpha]} \otimes_{\mathfrak{o}_F} \mathfrak{o}_K = \prod_{i=1}^{r} \mathfrak{p}_{E_i},$$

$$k_{F[\alpha]} \otimes_{k_F} k_K = \prod_{i=1}^{r} k_{E_i}.$$

However, $k_{F[\alpha]} \cong k_F[X]/(f_\alpha(X))$. Since k_K/k_F is a finite separable extension, $f_\alpha(X)$ splits as a product $\varphi_1(X)\,\varphi_2(X)\,\ldots\,\varphi_r(X)$ of distinct irreducible factors in $k_K[X]$, which we may number so that

$$k_{E_i} \cong k_K[X]/(\varphi_i(X)).$$

However, we have $n_K(\alpha_i) = n_F(\alpha)$, and $e_K(\alpha_i) = e_F(\alpha)$, by (3.1) since K/F is unramified. If we write $x \mapsto x_i$ for the canonical projection $\mathscr{E} \to E_i$, we thus have

$$(\tilde{\rho}_F(\alpha; \pi_F))_i = \tilde{\rho}_K(\alpha_i; \pi_F).$$

It follows that the invariants of the simple pair $[ne_\alpha(K \mid F) - 1, \alpha_i]$ (which is minimal simple by (3.4)) are

$$(n_F(\alpha), e_F(\alpha), \varphi_i(X)).$$

These are distinct, and the set

$$\{(n_F(\alpha), e_F(\alpha), \varphi_i(X)) : 1 \leqslant i \leqslant r\}$$

depends only on $(n_F(\alpha), e_F(\alpha), f_\alpha(X))$. This proves (5.3), and (5.4) is now immediate (assuming that K/F is unramified).

We now have to treat the case in which K/F is totally tamely ramified. Indeed, appealing to transitivity, we can (and do) assume that K/F is of prime degree ℓ, say. There is an easy case of which we can dispose immediately, namely that where ℓ does not divide $e_F(\alpha) = e(F[\alpha] \mid F)$. Here, the algebra $\mathscr{E} = F[\alpha] \otimes K$ is a field, we have $r = 1$, and the first part of (5.3) (ii) is trivial. We may choose prime elements π_F, π_K so that

$$\text{(5.6)} \qquad \pi_K^\ell = \pi_F.$$

The extension $\mathscr{E}/F[\alpha]$ is totally ramified of degree ℓ, so $n_K(\alpha_1) = \ell n_F(\alpha)$, while $e_K(\alpha_1) = e_F(\alpha)$. This means that $\rho_K(\alpha_1; \pi_K) = \rho_F(\alpha; \pi_F)$. Hence the invariants of the lifted pair $[n\ell - 1, \alpha_1]$ are $(n\ell, e_F(\alpha), f_\alpha(X))$, and these depend only on the invariants $(n, e_F(\alpha), f_\alpha(X))$ of the given pair. This proves (5.3) in the case $\ell \nmid e_F(\alpha)$.

In the opposite direction, suppose we are given a minimal simple pair $[m - 1, \beta]$ over K, where K/F is totally ramified of degree ℓ, with invariants $(m, e, \varphi(X))$. We assume that φ has been calculated relative to a prime element satisfying (5.6). Suppose also that ℓ divides m. It follows that ℓ does not divide e. Then by the calculation above, $[m - 1, \beta]$ is equivalent to the unique lift of the simple pair over F with invariants $(m/\ell, e, \varphi(X))$. Thus (5.4) holds for such pairs $[m - 1, \beta]$.

We must next treat the case where K/F is totally ramified of prime degree ℓ and ℓ divides $e_F(\alpha)$. This has the effect that each extension $K[\alpha_i]/F[\alpha]$ is unramified. Let L/F be some finite unramified extension. We know that (5.3), (5.4) hold for the extensions L/F and KL/K. As in (5.5), they will therefore hold for K/F provided they hold for KL/L. The effect of this observation is that we can replace F by any convenient finite unramified extension of F.

First, replacing F by the maximal unramified subextension of $F[\alpha]/F$, we can reduce to the case in which $F[\alpha]/F$ is totally ramified. Since ℓ is not the residual characteristic, we can further enlarge F (by an unramified extension) and assume that it contains a primitive ℓ-th root of unity. Now abbreviate $E = F[\alpha]$. With these conditions, we have $r = \ell$, and each of the extensions $K[\alpha_i]/E$ is trivial.

Take some prime element π_E of E. Since E/F is totally ramified, we have $\pi_E^{[E:F]} = \pi_F u$, for some $u \in \mathbf{U}^1(\mathfrak{o}_E)$, and some prime element π_F of F. Replacing F by a finite unramified extension, we can now assume that we have a prime element π_K of K such that $\pi_K^\ell = \pi_F$. Let E^0/F denote the unique subextension of E/F of degree ℓ. There is then a prime element π_0 of E^0 which satisfies

$$\left.\begin{aligned} \pi_0 &\equiv \pi_E^{[E:F]/\ell} \\ \pi_0^\ell &\equiv \pi_F \end{aligned}\right\} \quad (\text{mod } \mathbf{U}^1(\mathfrak{o}_E)).$$

Our element α has the form $\zeta\pi_E^{-n} v$, where $v \in \mathbf{U}^1(\mathfrak{o}_E)$, $n = n_F(\alpha)$, and ζ is a root of unity in F of order prime to p. In particular, we have $f_\alpha(X) = X - \zeta^{e_F(\alpha)}$. Consider the element

$$\delta = \alpha^{e_F(\alpha)/\ell} \otimes \pi_K^n \in \mathscr{E}.$$

We have $n = n_F(\alpha) = n_K(\alpha)$, and so $\delta^\ell = \rho_F(\alpha; \pi_F)$. On the other hand, the image δ_i of δ in $K[\alpha_i]$ is just $\rho_K(\alpha_i; \pi_K)$. We can rewrite

$$\delta = \zeta^{e_F(\alpha)/\ell}\, \pi_0^{-n}\, w \otimes \pi_K^n,$$

for some $w \in \mathbf{U}^1(\mathfrak{o}_E)$. To compute the δ_i, we consider the subalgebra $E^0 \otimes K$ of $\mathscr{E}$. We have

$$E^0 \otimes_F K \cong K \times K \times \ldots \times K,$$

as K-algebra, with ℓ factors here. The projections $E^0 \to K$ are given by $\pi_0 \mapsto \eta_i\, \pi_K$ (mod $\mathbf{U}^1(\mathfrak{o}_K)$), where η_i, $1 \leqslant i \leqslant \ell$, ranges over the ℓ-th roots of unity in F. Thus

$$\delta_i \equiv \zeta^{e_F(\alpha)/\ell}\, \eta_i^{-n}$$

mod 1-units. These ℓ values are distinct, since n is prime to ℓ. This says that the invariants of the lifted pairs $[n-1, \alpha_i]$ are $(n, e_F(\alpha)/\ell, X - \eta_i^{-n}\zeta^{e_F(\alpha)/\ell})$. This proves (5.3).

Combining these calculations with the first part of the proof (where K/F was unramified), we have:

(5.7) *Let* $[n-1, \alpha]$ *be a minimal simple pair over* F, *let* K/F *be totally ramified of prime degree* ℓ. *Suppose that* ℓ *divides* $e_F(\alpha)$. *Calculating relative to primes* π_F, π_K *satisfying* $\pi_K^\ell = \pi_F$, *the invariants of the* K/F-*lifts of* $[n-1, \alpha]$ *are* $(n, e_F(\alpha)/\ell, \varphi_i(X))$, *where* $\varphi_i(X)$ *ranges over the irreducible factors of* $f_\alpha(X^\ell)$.

It remains only to finish the proof of (5.4). We are given a totally ramified extension K/F of prime degree ℓ and a minimal simple pair $[m-1, \beta]$ over K. We have to find a minimal simple pair $[n-1, \alpha]$ over F of which $[m-1, \beta]$ is a lift. The case where ℓ divides m has been dealt with above. We therefore assume that $\ell \nmid m$. Take prime elements π_K, π_F such that $\pi_K^\ell = \pi_F$. If, relative to this choice, $[m-1, \beta]$ has invariants $(m, e, \varphi(X))$, we take $[n-1, \alpha]$ to be the pair with invariants $(m, e\ell, f(X))$, where $f(X)$ is the minimal polynomial over k_F of δ^ℓ, for some root δ of $\varphi(X)$ in $k_{K[\beta]}$.

This completes the proofs of (3.5), (3.7) in the special case. □

6. Completion of the proofs

We now treat the general case of our "lifting theorems" (3.5), (3.7). We start, however, with a useful result which does not form part of the main sequence. Its attractive feature is that it gives us some latitude in the treatment of the "level" of lifted simple pairs.

(6.1) *Proposition. — Let* $[m, \alpha]$ *be a simple pair over* F, *and let* $\widetilde{\alpha}$ *be some* K/F-*lift of* α. *Let* $[m, \beta]$ *be another simple pair over* F, *and assume that*

a) $[m, \beta] \approx [m, \alpha]$, *and*
b) there exists a K/F-*lift* $\widetilde{\beta}$ *of* β *such that* $[\mathrm{M}, \widetilde{\beta}] \approx [\mathrm{M}, \widetilde{\alpha}]$, *where* $\mathrm{M} = (m+1)\, e_\alpha(\mathrm{K} \mid \mathrm{F}) - 1$.

Then

$$[q, \widetilde{\beta}] \approx [q, \widetilde{\alpha}]$$

for any integer q such that

$$\left[\frac{q}{e_\alpha(\mathrm{K} \mid \mathrm{F})}\right] = m.$$

Suppose further that we have $[m-1, \alpha] \approx [m-1, \beta]$. *Then* $[\mathrm{M}', \widetilde{\alpha}] \approx [\mathrm{M}', \widetilde{\beta}]$, *where* $\mathrm{M}' = m e_\alpha(\mathrm{K} \mid \mathrm{F}) - 1$.

Proof. — We choose convenient, equivalent, realizations of our simple pairs $[\mathrm{M}, \widetilde{\alpha}]$, $[\mathrm{M}, \widetilde{\beta}]$, as follows. Let $\mathrm{V} = \mathrm{K}(\widetilde{\alpha})$, viewed as a K-vector space, and let $\mathfrak{C}$ denote the hereditary order

$$\mathfrak{C} = \mathrm{End}^0_{\mathfrak{o}_\mathrm{K}}(\{\, \mathfrak{p}^k_{\mathrm{K}[\widetilde{\alpha}]} \,\})$$

in $\mathrm{C} = \mathrm{End}_\mathrm{K}(\mathrm{V})$. By (1.10), (1.15), (1.6), we can choose a K-embedding of $\mathrm{K}[\widetilde{\beta}]$ in C so that the stratum $[\mathfrak{C}, n, \mathrm{M}, \widetilde{\beta}]$ is simple and equivalent to $[\mathfrak{C}, n, \mathrm{M}, \widetilde{\alpha}]$, where $n = n_\mathrm{K}(\widetilde{\alpha}) = n_\mathrm{K}(\widetilde{\beta})$.

Now write $\mathrm{A} = \mathrm{End}_\mathrm{F}(\mathrm{V})$, and let $\mathfrak{A}$ denote the hereditary $\mathfrak{o}_\mathrm{F}$-order in A defined by the same lattice chain as $\mathfrak{C}$. Write $\mathrm{E} = \mathrm{F}[\alpha]$. The subfield $\mathrm{F}[\widetilde{\alpha}]$ of $\mathrm{K}[\widetilde{\alpha}]$ is isomorphic to E via $\widetilde{\alpha} \mapsto \alpha$. We therefore regard E as a subfield of $\mathrm{K}[\widetilde{\alpha}]$ such that $\mathrm{K}[\widetilde{\alpha}] = \mathrm{KE}$. Let s denote a tame corestriction on A relative to E/F. According to (2.7), the restriction of s to C is a tame corestriction relative to $\mathrm{K}[\widetilde{\alpha}]/\mathrm{K}$.

Now write $\mathrm{M}' = m e_\alpha(\mathrm{K}/\mathrm{F}) - 1$. Let q be the least integer, $\mathrm{M} \geqslant q \geqslant \mathrm{M}'$, such that the strata $[\mathfrak{C}, n, q, \widetilde{\alpha}]$, $[\mathfrak{C}, n, q, \widetilde{\beta}]$ intertwine in C. We assume that $q > \mathrm{M}'$, and there is no harm in changing our embedding of $\mathrm{K}[\widetilde{\beta}]$ in C to arrange that

$$[\mathfrak{C}, n, q, \widetilde{\alpha}] \sim [\mathfrak{C}, n, q, \widetilde{\beta}].$$

Write $\mathrm{B} = \mathrm{End}_\mathrm{E}(\mathrm{V})$, $\mathrm{D} = \mathrm{End}_{\mathrm{K}[\widetilde{\alpha}]}(\mathrm{V})$, $\mathfrak{B} = \mathfrak{A} \cap \mathrm{B}$, $\mathfrak{D} = \mathfrak{A} \cap \mathrm{D}$, and consider the derived stratum $[\mathfrak{D}, q, q-1, s(\widetilde{\beta} - \widetilde{\alpha})]$. By [BK1] (2.4.1), this is either equivalent to a simple stratum $[\mathfrak{D}, q, q-1, \delta]$, or it is the null stratum $[\mathfrak{D}, q, q-1, 0]$. We can exclude the latter case, since this would imply, via [BK1] (2.2.1), that the strata $[\mathfrak{C}, n, q-1, \widetilde{\alpha}]$, $[\mathfrak{C}, n, q-1, \widetilde{\beta}]$ intertwine, contrary to hypothesis. Further, by (5.4) and (4.3) applied to the extension $\mathrm{K}[\widetilde{\alpha}]/\mathrm{E}$, we can assume that the equivalent strata

$$[\mathfrak{B}, q, q-1, \delta], \qquad [\mathfrak{B}, q, q-1, s(\beta - \alpha)]$$

are simple. This implies that the strata $[\mathfrak{A}, n, q-1, \alpha]$, $[\mathfrak{A}, n, q-1, \beta]$ do not intertwine (see [BK3] (1.9)). However,

$$\left[\frac{q-1}{e_\alpha(\mathfrak{A})}\right] \leqslant m.$$

If we have equality here, this contradicts the assumption that $[m, \alpha] \approx [m, \beta]$. We deduce that $q - 1 = \mathrm{M}'$, and the argument above implies $[m-1, \alpha] \not\approx [m-1, \beta]$. Thus, if we do indeed have $[m-1, \alpha] \approx [m-1, \beta]$, we must have $q = \mathrm{M}'$.

This completes the proof of the Proposition. □

Now we start the proofs of the lifting theorems (3.5), (3.7). By (4.1), we can assume that our lifting extension K/F is Galois of prime degree ℓ. We prove the following statement, which is a rephrasing of the results we seek.

(6.2) *Let* K/F *be a tamely ramified, Galois field extension of prime degree* ℓ. *Let* $[\mathrm{M}, \beta]$ *be a simple pair over* K. *Then*:

(i) *there exists a simple pair* $[m, \alpha]$ *over* F *such that* $m = [\mathrm{M}/e_\alpha(\mathrm{K} \mid \mathrm{F})]$ *and a* K/F*-lift* $\widetilde{\alpha}$ *of* α *such that* $[\mathrm{M}, \widetilde{\alpha}] \approx [\mathrm{M}, \beta]$;

(ii) *if* $[m, \alpha]$ *is as in* (i), *and if* $\widehat{\alpha}$ *is a* K/F*-lift of* α, *then* $[\mathrm{M}, \widehat{\alpha}] \approx [\mathrm{M}, \beta]$ *if and only if* $\widehat{\alpha} = \widetilde{\alpha}$;

(iii) *in the same situation, if* $[m, \alpha']$ *is some simple pair over* F *equivalent to* $[m, \alpha]$, *there exists a* K/F*-lift* $\widetilde{\alpha}'$ *of* α' *such that* $[\mathrm{M}, \widetilde{\alpha}'] \approx [\mathrm{M}, \widetilde{\alpha}]$.

Here, (i) implies (3.7). By (2.4) and (3.4), any $(m, \alpha) \in \mathscr{SP}(\mathrm{F})$ and any K/F-lift $\widetilde{\alpha}$ of α arise from some $(\mathrm{M}, \beta) \in \mathscr{SP}(\mathrm{K})$ in the manner of (6.2) (i). Thus (3.5) is implied by (6.2) (ii) and (iii).

We prove (6.2) by induction on the positive integer $n_{\mathrm{K}}(\beta) - \mathrm{M}$. The case $n_{\mathrm{K}}(\beta) - \mathrm{M} = 1$ is covered by the arguments of § 5. We therefore assume that $n_{\mathrm{K}}(\beta) - \mathrm{M} \geqslant 2$, and that (6.2) holds for all simple pairs $[\mathrm{M}', \beta']$ over K with $n_{\mathrm{K}}(\beta') - \mathrm{M}' < n_{\mathrm{K}}(\beta) - \mathrm{M}$. Before proceeding, it will be useful to note one consequence of this inductive hypothesis.

(6.3) *Lemma. — Suppose that* (6.2) *holds for all simple pairs* $[\mathrm{M}, \beta]$ *over* K *such that* $n_{\mathrm{K}}(\beta) - \mathrm{M} \leqslant k$, *for some constant* k. *Let* V *be a* K*-vector space, and* $[\mathfrak{C}, n, m, \beta]$ *a simple stratum in* $\mathrm{C} = \mathrm{End}_{\mathrm{K}}(\mathrm{V})$ *defining a simple pair* $[\mathrm{M}, \beta]$ *with* $n_{\mathrm{K}}(\beta) - \mathrm{M} \leqslant k$. *Let* $\mathfrak{A}$ *denote the hereditary* $\mathfrak{o}_{\mathrm{F}}$*-order in* $\mathrm{A} = \mathrm{End}_{\mathrm{F}}(\mathrm{V})$ *defined by the same lattice chain as* $\mathfrak{C}$. *Let* $[\mathfrak{A}, n, m, \alpha]$ *be a simple stratum in* A *which intertwines with* $[\mathfrak{A}, n, m, \beta]$. *There then exists a* K/F*-lift* $\widetilde{\alpha}$ *of* α *and a* K*-embedding* φ *of* $\mathrm{K}[\widetilde{\alpha}]$ *in* C *with the following properties*:

(i) $\varphi(\mathrm{K}[\widetilde{\alpha}]^\times) \subset \mathfrak{K}(\mathfrak{C})$;

(ii) $[\mathfrak{C}, n, m, \varphi(\widetilde{\alpha})]$ *intertwines with* $[\mathfrak{C}, n, m, \beta]$.

Proof. — By inductive hypothesis and (4.3), we can replace $[\mathfrak{C}, n, m, \beta]$ by an equivalent stratum and assume that $[\mathfrak{A}, n, m, \beta]$ is simple. Since this intertwines with the simple stratum $[\mathfrak{A}, n, m, \alpha]$, the F-simple pairs defined by these strata are of the form $[h, \beta]$, $[h, \alpha]$, for some h, and $[h, \beta] \approx [h, \alpha]$. The lemma now follows from (6.1) and the part of the inductive hypothesis corresponding to (6.2) (iii). □

We now take $[M, \beta]$ as in (6.2), and start by proving (6.2) (i). Let V be some finite-dimensional $K[\beta]$-vector space, and take a realization $[\mathfrak{C}, n, M, \beta]$ of $[M, \beta]$ on V, where $\mathfrak{C}$ is a principal order in C with $e(\mathfrak{C} \mid \mathfrak{o}_K) = e(K[\beta] \mid K)$. Thus, in particular, $n = n_K(\beta)$. We write $C = \mathrm{End}_K(V)$, $A = \mathrm{End}_F(V)$, and we let $\mathfrak{A}$ denote the hereditary $\mathfrak{o}_F$-order in A defined by the same lattice chain as $\mathfrak{C}$. We next use [BK1] (2.4.1) to find a simple stratum $[\mathfrak{C}, n, M+1, \gamma]$ equivalent to $[\mathfrak{C}, n, M+1, \beta]$. The quotient $e_1 = e(K[\beta] \mid K)/e(K[\gamma] \mid K)$ is an integer, by [BK1] (2.4.1). Thus the simple pair defined by $[\mathfrak{C}, n, M+1, \gamma]$ is of the form $[M_1, \gamma]$, where

$$M_1 = \left[\frac{M+1}{e_1}\right].$$

We also have $n_K(\gamma) = n_K(\beta)/e_1$, so altogether

$$n_K(\gamma) - M_1 < n_K(\beta) - M.$$

We can therefore apply our inductive hypothesis to find a simple stratum $[\mathfrak{C}, n, M+1, \gamma']$ equivalent to $[\mathfrak{C}, n, M+1, \gamma]$ and such that $[\mathfrak{A}, n, M+1, \gamma']$ is simple. Indeed, we can now economize on notation and assume that $\gamma = \gamma'$, i.e. that $[\mathfrak{A}, n, M+1, \gamma]$ is simple.

Now let us write $E = F[\gamma]$, $B = \mathrm{End}_E(V)$, $\mathfrak{B} = \mathfrak{A} \cap B$. We also set $D = \mathrm{End}_{K[\gamma]}(V)$, $\mathfrak{D} = \mathfrak{A} \cap D$. We choose a tame corestriction s on A relative to E/F, so that, by (2.7), the restriction $s \mid C$ is a tame corestriction on C relative to $K[\gamma]/K$. We form the derived stratum $[\mathfrak{D}, M+1, M, s(\beta - \gamma)]$. This is either null or equivalent to a simple stratum $[\mathfrak{D}, M+1, M, \delta]$. Then, by (4.3) and (5.4) (or inductive hypothesis), we can assume that $[\mathfrak{B}, M+1, M, \delta]$ is simple or null.

By [BK1] (2.2.8), there exists a simple stratum $[\mathfrak{A}, n, M, \alpha]$ such that $[\mathfrak{A}, n, M+1, \alpha]$ is equivalent to $[\mathfrak{A}, n, M+1, \gamma]$ and $[\mathfrak{B}, M+1, M, s(\alpha - \gamma)]$ is equivalent to $[\mathfrak{B}, M+1, M, \delta]$.

(6.4) *Lemma. — In the situation above, let $\tilde{\alpha}$ be a K/F-lift of α. Then*

$$e(K[\tilde{\alpha}] \mid K) = e(K[\beta] \mid K).$$

Proof. — By [BK1] (2.4.1), we have $e(K[\beta] \mid K) = e(K[\gamma] \mid K)\, e(K[\gamma, \delta] \mid K[\gamma])$. Likewise, $e(F[\beta] \mid F) = e(F[\gamma] \mid F)\, e(F[\gamma, \delta] \mid F[\gamma])$. The lemma now follows from (3.2) via a straightforward computation. □

The situation for residue class degrees is a little more uncertain at this stage:

(6.5) *Lemma. — In the situation above, $f(K[\tilde{\alpha}] \mid K)$ divides $\ell f(K[\beta] \mid K)$.*

Proof. — When K/F is unramified, an argument analogous to that of (6.4) gives us $f(K[\tilde{\alpha}] \mid K) = f(K[\beta] \mid K)$. We therefore assume that K/F is totally ramified, as well as being Galois of prime degree ℓ. This means that, if L/F is any finite extension and L' is some field component of $L \otimes_F K$, then $f(L' \mid K)$ is either $f(L \mid F)$ or $\ell f(L \mid F)$. The latter can only occur if ℓ divides $e(L \mid F)$ and $L \otimes K$ is a field. We therefore have

$$\begin{aligned} f(K[\tilde{\alpha}] \mid K) &= \varepsilon_1 f(F[\alpha] \mid F) \\ &= \varepsilon_1 f(F[\gamma] \mid F) f(F[\gamma, \delta] \mid F[\gamma]) \\ &= \varepsilon_1 \varepsilon_2^{-1} \varepsilon_3^{-1} f(K[\gamma] \mid K) f(K[\gamma, \delta] \mid K[\gamma]) \\ &= \varepsilon_1 \varepsilon_2^{-1} \varepsilon_3^{-1} f(K[\beta] \mid K), \end{aligned}$$

where the ε_i are constants with value 1 or ℓ. The assertion follows. □

We now impose a further condition on our vector space V, and assume that its $K[\beta]$-dimension is divisible by the prime number $\ell = [K : F]$. By (1.1), (6.4), (6.5), our choice of $\mathfrak{C}$ gives us:

(6.6) *In the situation above, let $\tilde{\alpha}$ be any K/F-lift of α. There exists a K-embedding $\varphi : K[\tilde{\alpha}] \to C$ whose image normalizes $\mathfrak{C}$.*

We choose a lift $\tilde{\alpha}$ and an embedding φ as in (6.6), and abbreviate $\varepsilon = \varphi(\tilde{\alpha})$. The stratum $[\mathfrak{C}, n, M, \varepsilon]$ is then simple. By construction, its restriction $[\mathfrak{A}, n, M, \varepsilon]$ is also simple, and indeed $\mathbf{U}(\mathfrak{A})$-conjugate to $[\mathfrak{A}, n, M, \alpha]$. Now we apply our inductive hypothesis again to produce a simple stratum $[\mathfrak{C}, n, M+1, \varphi]$ equivalent to $[\mathfrak{C}, n, M+1, \varepsilon]$ and such that $[\mathfrak{A}, n, M+1, \varphi]$ is simple. It follows that $[\mathfrak{A}, n, M+1, \varphi]$ intertwines with $[\mathfrak{A}, n, M+1, \gamma]$. By inductive hypothesis and (6.3), there is a lift $\tilde{\varphi}$ of φ and a K-embedding Ψ of $K[\tilde{\varphi}]$ in C such that $\Psi(K[\tilde{\varphi}]^\times) \subset \mathfrak{K}(\mathfrak{C})$ and such that $[\mathfrak{C}, n, M+1, \Psi(\tilde{\varphi})] \sim [\mathfrak{C}, n, M+1, \gamma]$. Now we appeal to (4.6): we can replace φ by a $\mathcal{N}_{\mathfrak{A}}(K)$-conjugate to arrange $[\mathfrak{C}, n, M+1, \varepsilon] \sim [\mathfrak{C}, n, M+1, \gamma]$. In other words, at this stage, we may as well take $\varphi = \gamma$. Now we compare the derived strata $[\mathfrak{D}, M+1, M, s(\beta - \gamma)]$, $[\mathfrak{D}, M+1, M, s(\varepsilon - \gamma)]$. These are equivalent to simple (or null) strata $[\mathfrak{D}, M+1, M, \delta]$, $[\mathfrak{D}, M+1, M, \delta']$ respectively, such that (by inductive hypothesis) the restrictions $[\mathfrak{B}, M+1, M, \delta]$, $[\mathfrak{B}, M+1, M, \delta']$ are simple (or null). Moreover, by construction (and [BK3] (1.9)), these restrictions intertwine. By (6.3) and inductive hypothesis, we can now conjugate by an element of $\mathcal{N}_{\mathfrak{B}}(K[\gamma])$ to arrange $[\mathfrak{D}, M+1, M, \delta] \sim [\mathfrak{D}, M+1, M, \delta']$. It now follows that the strata $[\mathfrak{C}, n, M, \beta]$, $[\mathfrak{C}, n, M, \varepsilon]$ intertwine. This element ε has become an $\mathcal{N}_{\mathfrak{A}}(K)$-conjugate of the original $\varphi(\tilde{\alpha})$. By (3.4), the stratum $[\mathfrak{C}, n, M, \varepsilon]$ therefore defines a simple pair $[M, \hat{\alpha}]$, for some K/F-lift $\hat{\alpha}$ of α. We have shown that $[M, \hat{\alpha}] \approx [M, \beta]$, and this proves (6.2) (i).

It is worth recording the conclusion of this argument, in a more general context.

(6.7) *Proposition. — Let K/F be a finite, Galois, tamely ramified field extension. Let V be a finite-dimensional K-vector space, and $[\mathfrak{C}, n, m, \beta_i]$ simple strata in $\mathrm{End}_K(V)$, $i = 1, 2$. Let $\mathfrak{A}$ be the hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(V)$ defined by the same lattice chain as $\mathfrak{C}$. Suppose that* *the strata* $[\mathfrak{A}, n, m, \beta_i]$ *are both simple, and intertwine in* $\mathrm{End}_F(V)$. *Then there exists* $x \in \mathcal{N}_{\mathfrak{A}}(K)$ *such that*

$$[\mathfrak{C}, n, m, x^{-1}\beta_1 x] \sim [\mathfrak{C}, n, m, \beta_2].$$

Proof. — The assertion is transitive in the extension K/F, so we may assume that K/F has prime degree ℓ. When ℓ divides $\dim_K(V)$, the assertion is then given by the arguments above.

For the general case, we write $\mathcal{L} = \{L_j\}$ for the lattice chain defining $\mathfrak{A}$. We write $V' = V \oplus \ldots \oplus V$ (ℓ factors), and define a lattice chain $\mathcal{L}' = \{L'_j\}$ in V' by

$$L'_j = L_j \oplus L_j \oplus \ldots \oplus L_j, \quad j \in \mathbf{Z}.$$

We write $\mathfrak{A}' = \mathrm{End}^0_{\mathfrak{o}_F}(\mathcal{L}')$, and define $\mathfrak{C}'$ similarly. The stratum $[\mathfrak{A}', n, m, \alpha_i]$ defines the same simple pair as $[\mathfrak{A}, n, m, \alpha_i]$, so the strata $[\mathfrak{A}', n, m, \alpha_1]$, $[\mathfrak{A}', n, m, \alpha_2]$ intertwine. Likewise, the strata $[\mathfrak{C}', n, m, \beta_i]$, $[\mathfrak{C}, n, m, \beta_i]$ define the same simple pair, call it $[k, \beta_i]$, over K. The case above gives $x \in \mathcal{N}_{\mathfrak{A}'}(K)$ such that

$$[\mathfrak{C}', n, m, x^{-1}\beta_1 x] \sim [\mathfrak{C}', n, m, \beta_2].$$

This says, via (4.6) and the remarks concluding § 1, that there exists $\sigma \in \mathrm{Gal}(K/F)$ such that $[k, \beta_1^\sigma] \approx [k, \beta_2]$. The proposition now follows from (4.6). □

We can now deduce (6.2) (iii). We choose K/F-lifts $\tilde{\alpha}$, $\tilde{\alpha}'$ of α, α' respectively. By (1.15), we have $e_F(\alpha) = e_F(\alpha')$, so (by (3.2)) $e_\alpha(K \mid F) = e_{\alpha'}(K \mid F)$. We choose a K-vector space V and a hereditary $\mathfrak{o}_K$-order $\mathfrak{C}$ in $C = \mathrm{End}_K(V)$ for which there exist simultaneous realizations $[\mathfrak{C}, n, M, \tilde{\alpha}]$, $[\mathfrak{C}, n, M, \tilde{\alpha}']$ of the simple pairs $[m', \tilde{\alpha}]$, $[m', \tilde{\alpha}']$, where $m' = (m+1)\, e_\alpha(K \mid F) - 1$. If $\mathfrak{A}$ is the hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(V)$ defined by the same lattice chain as $\mathfrak{C}$, then by hypothesis, the strata $[\mathfrak{A}, n, M, \alpha]$, $[\mathfrak{A}, n, M, \alpha']$ intertwine. Now we apply (6.7), and the result follows.

This leaves us with proving (6.2) (ii). We start with a simple pair $[m, \alpha]$ over F, and choose some K/F-lift $\tilde{\alpha}$ of α. If $m + 1 < -k_F(\alpha)$, the assertion follows immediately from our inductive hypothesis. We therefore assume the contrary. We let $V = K[\tilde{\alpha}]$, viewed as K-vector space, and let $\mathfrak{C}$ be the unique hereditary $\mathfrak{o}_K$-order in $C = \mathrm{End}_K(V)$ which is normalized by $K[\tilde{\alpha}]^\times$. We set $M = (m+1)\, e_\alpha(K \mid F) - 1$, so that $[M, \tilde{\alpha}]$ is a simple pair over K, of which we have a realization $[\mathfrak{C}, n, M, \tilde{\alpha}]$ on $\mathfrak{C}$, for some integer n. Let $\mathfrak{A}$ be the hereditary $\mathfrak{o}_F$-order in $A = \mathrm{End}_F(V)$ defined by the same lattice chain as $\mathfrak{C}$, so that the " restriction " of $[\mathfrak{C}, n, M, \tilde{\alpha}]$ is $[\mathfrak{A}, n, M, \alpha]$.

We choose a simple stratum $[\mathfrak{C}, n, M+1, \gamma]$ equivalent to $[\mathfrak{C}, n, M+1, \tilde{\alpha}]$ such that $[\mathfrak{A}, n, M+1, \gamma]$ is simple. As usual, we write $B = \mathrm{End}_{F[\gamma]}(V)$, $D = \mathrm{End}_{K[\gamma]}(V)$, $\mathfrak{B} = \mathfrak{A} \cap B$, $\mathfrak{D} = \mathfrak{A} \cap D$. We choose a tame corestriction s on A relative to $F[\gamma]/F$. We choose a simple stratum $[\mathfrak{D}, M+1, M, \delta]$, equivalent to $[\mathfrak{D}, M+1, M, s(\tilde{\alpha} - \gamma)]$, and such that $[\mathfrak{B}, M+1, M, \delta]$ is simple. In particular, we have

$$[\mathfrak{B}, M+1, M, \delta] \sim [\mathfrak{B}, M+1, M, s(\alpha - \gamma)].$$

Of course, γ, viewed as an element of $K[\gamma]$, is just a K/F-lift of γ viewed as an element of $F[\gamma]$. A similar remark applies to δ. Thus, invoking (3.1), we have

(6.8) *Let $\tilde{\gamma}$ be some* K/F*-lift of* γ, *and* $\tilde{\delta}$ *some* $K[\tilde{\gamma}]/F[\gamma]$*-lift of* δ. *There exists a* K*-embedding* $K[\tilde{\gamma}, \tilde{\delta}] \to C$ *whose image normalizes* $\mathfrak{C}$.

Now we return to our simple stratum $[\mathfrak{A}, n, M, \alpha]$, and the simple stratum $[\mathfrak{A}, n, M+1, \gamma]$ equivalent to $[\mathfrak{A}, n, M+1, \alpha]$. Let $\gamma_1, \gamma_2, \dots, \gamma_r$ be the set of K/F-lifts of γ. For each i, we choose a K-embedding φ_i of $K[\gamma_i]$ in C whose image normalizes $\mathfrak{C}$. For each i, there exists $x_i \in \mathcal{N}_{\mathfrak{A}}(K)$ such that $\varphi_i(\gamma_i) = x_i^{-1}\gamma x_i$, and we use this element to transfer s to a tame corestriction s_i on A relative to $F[\varphi_i(\gamma_i)]/F$. By inductive hypothesis, no two of the simple strata $[\mathfrak{C}, n, M+1, \varphi_i(\gamma_i)]$ intertwine in C.

Next, we let $\delta_1, \delta_2, \dots, \delta_t$ be the set of $K[\gamma]/F[\gamma]$-lifts of δ. We set $B_i = \mathrm{End}_{F[\varphi_i(\gamma_i)]}(V)$, and define D_i, $\mathfrak{B}_i$, $\mathfrak{D}_i$ in the obvious way. For each pair i, j, we extend φ_i to a K-embedding $\varphi_{ij} : K[\gamma_i, \delta_j] \to C$ whose image normalizes $\mathfrak{C}$. Then, for given i, no two of the simple strata $[\mathfrak{D}_i, M+1, M, \varphi_{ij}(\delta_j)]$ intertwine in D_i. However, any two of them are conjugate under the group $\mathcal{N}_{\mathfrak{B}_i}(K[\varphi_i(\gamma_i)]) \subset \mathcal{N}_{\mathfrak{A}}(K)$. Now, for each pair i, j, we choose a simple stratum $[\mathfrak{C}, n, M, \beta_{ij}]$ such that

(6.9) (i) $[\mathfrak{C}, n, M+1, \beta_{ij}] \sim [\mathfrak{C}, n, M+1, \varphi_i(\gamma_i)]$;
(ii) $[\mathfrak{D}_i, M+1, M, s_i(\beta_{ij} - \varphi_i(\gamma_i))] \sim [\mathfrak{D}_i, M+1, M, \varphi_{ij}(\delta_j)]$;
(iii) *the stratum* $[\mathfrak{A}, n, M, \beta_{ij}]$ *is simple.*

By construction, no two of the strata $[\mathfrak{C}, n, M, \beta_{ij}]$ intertwine in C. However, all of the strata $[\mathfrak{A}, n, M, \beta_{ij}]$ intertwine with $[\mathfrak{A}, n, M, \alpha]$ in A. The construction also shows that any $[\mathfrak{C}, n, M, \beta_{ij}]$ intertwines with some conjugate of $[\mathfrak{C}, n, M, \tilde{\alpha}]$ under $\mathcal{N}_{\mathfrak{A}}(K)$. Thus, for each pair i, j, there exists a K/F-lift α_{ij} of α, and a K-embedding Ψ_{ij} of $K[\alpha_{ij}]$ in C whose image normalizes $\mathfrak{C}$, so that $[\mathfrak{C}, n, M, \Psi_{ij}(\alpha_{ij})]$ is equivalent to $[\mathfrak{C}, n, M, \beta_{ij}]$. It follows that the simple pairs (over K)

$$[M, \alpha_{ij}], \quad 1 \leqslant i \leqslant r, \quad 1 \leqslant j \leqslant t,$$

are mutually inequivalent. Moreover, we have the relation

$$[K[\beta_{ij}] : K] = [K[\alpha_{ij}] : K],$$

for all i and j. However,

$$\begin{aligned}[K[\beta_{ij}] : K] &= [K[\gamma_i, \delta_j] : K[\gamma_i]].[K[\gamma_i] : K] \\ &= t^{-1}[F[\gamma, \delta] : F[\gamma]].r^{-1}[F[\gamma] : F] \\ &= (rt)^{-1}[F[\alpha] : F].\end{aligned}$$

Thus $$\dim_K(\Pi_{i,j} K[\alpha_{ij}]) = [F[\alpha] : F] = \dim_K(F[\alpha] \otimes_F K).$$

The α_{ij} therefore exhaust the K/F-lifts of α, and the simple pairs defined by the lifts of α are mutually inequivalent.

This completes the proofs. □

7. Ring lifting and simple characters

In this and the following two sections, we extend the results of §§ 1-6 to the context of simple characters. In the present section, we work in the " interior " lifting situation of § 2, and give the explicit constructions on which our notion of lifting for simple characters is to be based.

To start with, let V be some finite-dimensional F-vector space, and $[\mathfrak{A}, n, 0, \beta]$ a simple stratum in $A = \mathrm{End}_F(V)$. In [BK1] (3.1), we attached to such a stratum a pair of $\mathfrak{o}_F$-orders in A,

$$\mathfrak{H}(\beta, \mathfrak{A}) \subset \mathfrak{J}(\beta, \mathfrak{A}) \subset \mathfrak{A}.$$

These depend only on the $\sim$-equivalence class of $[\mathfrak{A}, n, 0, \beta]$. Each comes equipped with a canonical filtration by two-sided ideals

$$\mathfrak{H}^k(\beta, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A}) \cap \mathfrak{P}^k,$$
$$\mathfrak{J}^k(\beta, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A}) \cap \mathfrak{P}^k,$$

where $k \geqslant 0$ and $\mathfrak{P}$ denotes the Jacobson radical of $\mathfrak{A}$. Likewise, we get subgroups of the unit groups of the rings $\mathfrak{H}$, $\mathfrak{J}$, by

$$J(\beta, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A})^{\times},$$
$$J^k(\beta, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A}) \cap \mathbf{U}^k(\mathfrak{A}),$$
$$H^k(\beta, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A}) \cap \mathbf{U}^k(\mathfrak{A}),$$

where $k \geqslant 1$.

(7.1) *Proposition.* — *Let* K/F *be a finite, tamely ramified, field extension. Let* V *be a finite-dimensional* K*-vector space, and let* $\mathfrak{C}$ *be a hereditary* $\mathfrak{o}_K$*-order in* $C = \mathrm{End}_K(V)$. *Let* $\mathfrak{A}$ *be the hereditary* $\mathfrak{o}_F$*-order in* $A = \mathrm{End}_F(V)$ *defined by the same lattice chain as* $\mathfrak{C}$.

Let $[\mathfrak{A}, n, 0, \beta]$ *be a* K*-pure simple stratum in* A. *Then*

$$\mathfrak{H}(\beta, \mathfrak{A}) \cap C = \mathfrak{H}(\beta, \mathfrak{C}),$$

and
$$\mathfrak{J}(\beta, \mathfrak{A}) \cap C = \mathfrak{J}(\beta, \mathfrak{C}).$$

Remark. — By (2.4), the stratum $[\mathfrak{C}, n, 0, \beta]$ in C is simple, so the objects $\mathfrak{H}(\beta, \mathfrak{C})$, $\mathfrak{J}(\beta, \mathfrak{C})$ are indeed defined.

Proof. — Our procedure will be " inductive along β ", in the manner of many of the proofs of [BK1]. However, it will be convenient to have a stronger inductive hypothesis, so we actually show:

(7.2) *In the situation of* (7.1), *let* $s_{K/F}$ *be a tame corestriction on* A *relative to* K/F. *We then have*

$$\mathfrak{H}(\beta, \mathfrak{A}) \cap C = \mathfrak{H}(\beta, \mathfrak{C}) = s_{K/F}(\mathfrak{H}(\beta, \mathfrak{A})),$$

and likewise for $\mathfrak{J}$.

We observe that this condition is independent of the choice of $s_{K/F}$, since $\mathfrak{H}(\beta, \mathfrak{A})$ is an $\mathfrak{o}_K$-lattice. We therefore take $s_{K/F}$ so that its restriction to C is the identity map, just as in the proof of (2.7).

We prove (7.2) only for $\mathfrak{H}$, since the argument for $\mathfrak{J}$ is virtually identical. We write $\mathfrak{P} = \mathrm{rad}(\mathfrak{A})$, $\mathfrak{R} = \mathrm{rad}(\mathfrak{C})$. We also set $E = F[\beta]$, $B = \mathrm{End}_E(V)$, $\mathfrak{B} = \mathfrak{A} \cap B$, $\mathfrak{Q} = \mathrm{rad}(\mathfrak{B})$.

We assume to start with that β is minimal over F. Then, by definition,

$$\mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{B} + \mathfrak{P}^{[n/2]+1}.$$

Now we put $D = \mathrm{End}_{K[\beta]}(V)$ and $\mathfrak{D} = \mathfrak{A} \cap D$. By (2.4), the element β is minimal over K, so we have

$$\mathfrak{H}(\beta, \mathfrak{C}) = \mathfrak{D} + \mathfrak{R}^{[n/2]+1}.$$

First we note that

$$(\mathfrak{B} + \mathfrak{P}^{[n/2]+1}) \cap C \supset (\mathfrak{B} \cap C) + (\mathfrak{P}^{[n/2]+1} \cap C) = \mathfrak{D} + \mathfrak{R}^{[n/2]+1}.$$

For the opposite containment, we recall that for any $\mathfrak{o}_K$-submodule L of A, $s_{K/F}(L)$ is the orthogonal projection (relative to the reduced trace pairing on A) of L into C, whence $s_{K/F}(L) \supset L \cap C$. Applying this to the $\mathfrak{o}_K$-lattice $\mathfrak{B} + \mathfrak{P}^{[n/2]+1}$, we get

$$(\mathfrak{B} + \mathfrak{P}^{[n/2]+1}) \cap C \subset s_{K/F}(\mathfrak{B} + \mathfrak{P}^{[n/2]+1}) = s_{K/F}(\mathfrak{B}) + s_{K/F}(\mathfrak{P}^{[n/2]+1}).$$

We have $s_{K/F}(\mathfrak{P}^t) = \mathfrak{R}^t$, for any $t \in \mathbf{Z}$, by [BK1] (1.3.4). On the other hand, (2.7) implies that $s_{K/F}(\mathfrak{B}) = \mathfrak{D}$. This gives

$$\mathfrak{H}(\beta, \mathfrak{A}) \cap C = \mathfrak{H}(\beta, \mathfrak{C}) = s_{K/F}(\mathfrak{H}(\beta, \mathfrak{A}))$$

in the present case, as required for (7.2).

We now assume that β is not minimal over F, and set $r = -k_0(\beta, \mathfrak{A}) < n$. We choose a simple stratum $[\mathfrak{C}, n, r, \gamma]$ equivalent to $[\mathfrak{C}, n, r, \beta]$ such that $[\mathfrak{A}, n, r, \gamma]$ is simple (as we may, by (3.8)). Using the notation above, we therefore have

$$\mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{B} + \mathfrak{H}^{[r/2]+1}(\gamma, \mathfrak{A}),$$

whence $\quad \mathfrak{H}(\beta, \mathfrak{A}) \cap C \supset \mathfrak{B} \cap C + \mathfrak{H}^{[r/2]+1}(\gamma, \mathfrak{A}) \cap C.$

As before, $\mathfrak{B} \cap C = \mathfrak{D}$, while, by inductive hypothesis, we have

$$\begin{aligned}\mathfrak{H}^{[r/2]+1}(\gamma, \mathfrak{A}) \cap C &= \mathfrak{P}^{[r/2]+1} \cap \mathfrak{H}(\beta, \mathfrak{A}) \cap C \\ &= \mathfrak{H}^{[r/2]+1}(\gamma, \mathfrak{C}).\end{aligned}$$

On the other hand, we have

$$\mathfrak{H}(\beta, \mathfrak{A}) \cap C \subset s_{K/F}(\mathfrak{H}(\beta, \mathfrak{A})) = s_{K/F}(\mathfrak{B}) + s_{K/F}(\mathfrak{H}^{[r/2]+1}(\gamma, \mathfrak{A})).$$

Again, $s_{K/F}(\mathfrak{B}) = \mathfrak{D}$. Further,

$$\begin{aligned} s_{K/F}(\mathfrak{H}^{[r/2]+1}(\gamma, \mathfrak{A})) &= s_{K/F}(\mathfrak{H}(\gamma, \mathfrak{A}) \cap \mathfrak{P}^{[r/2]+1}) \\ &\subset s_{K/F}(\mathfrak{H}(\gamma, \mathfrak{A})) \cap s_{K/F}(\mathfrak{P}^{[r/2]+1}) \\ &= \mathfrak{H}(\gamma, \mathfrak{C}) \cap \mathfrak{R}^{[r/2]+1} \\ &= \mathfrak{H}^{[r/2]+1}(\gamma, \mathfrak{C}). \end{aligned}$$

Altogether, this gives us

$$\mathfrak{D} + \mathfrak{H}^{[r/2]+1}(\gamma, \mathfrak{C}) \subset \mathfrak{H}(\beta, \mathfrak{A}) \cap C \subset s_{K/F}(\mathfrak{H}(\beta, \mathfrak{A})) \subset \mathfrak{D} + \mathfrak{H}^{[r/2]+1}(\gamma, \mathfrak{C}).$$

Therefore

$$\mathfrak{H}(\beta, \mathfrak{A}) \cap C = s_{K/F}(\mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{D} + \mathfrak{H}^{[r/2]+1}(\gamma, \mathfrak{C}).$$

It remains only to prove that

(7.3) $$\mathfrak{D} + \mathfrak{H}^{[r/2]+1}(\gamma, \mathfrak{C}) = \mathfrak{H}(\beta, \mathfrak{C}).$$

We know from (2.4) that $k_0(\beta, \mathfrak{C}) \leqslant k_0(\beta, \mathfrak{A})$. In the case where $k_0(\beta, \mathfrak{C}) = k_0(\beta, \mathfrak{A})$, (7.3) is the definition of $\mathfrak{H}(\beta, \mathfrak{C})$. We therefore assume $k_0(\beta, \mathfrak{C}) < k_0(\beta, \mathfrak{A}) = -r$. The stratum $[\mathfrak{C}, n, r, \beta]$ is therefore simple, and, by definition, equivalent to $[\mathfrak{C}, n, r, \gamma]$. By [BK1] (3.1.9), we have

$$\mathfrak{H}^{[r/2]+1}(\beta, \mathfrak{C}) = \mathfrak{H}^{[r/2]+1}(\gamma, \mathfrak{C}).$$

This gives us

$$\mathfrak{D} + \mathfrak{H}^{[r/2]+1}(\gamma, \mathfrak{C}) = \mathfrak{D} + \mathfrak{H}^{[r/2]+1}(\beta, \mathfrak{C}) \subset \mathfrak{H}(\beta, \mathfrak{C}).$$

On the other hand, if we put $t = -k_0(\beta, \mathfrak{C}) > r$, then [BK1] (3.1.9) (ii) gives us

$$\mathfrak{H}(\beta, \mathfrak{C}) = \mathfrak{D} + \mathfrak{H}^{[t/2]+1}(\beta, \mathfrak{C}) \subset \mathfrak{D} + \mathfrak{H}^{[r/2]+1}(\beta, \mathfrak{C}),$$

which implies (7.3). This completes the proof of the Proposition. □

Before passing on, we note that (7.1) implies immediately that

$$\mathfrak{H}^k(\beta, \mathfrak{A}) \cap C = \mathfrak{H}^k(\beta, \mathfrak{C}),$$
$$\mathfrak{J}^k(\beta, \mathfrak{A}) \cap C = \mathfrak{J}^k(\beta, \mathfrak{C}),$$

for all $k \geqslant 0$. It will sometimes be useful to have the following more detailed version of (7.2).

(7.4) *Lemma. — In the situation of* (7.2), *we have*

$$\mathfrak{H}^k(\beta, \mathfrak{A}) \cap C = \mathfrak{H}^k(\beta, \mathfrak{C}) = s_{K/F}(\mathfrak{H}^k(\beta, \mathfrak{A})),$$
$$\mathfrak{J}^k(\beta, \mathfrak{A}) \cap C = \mathfrak{J}^k(\beta, \mathfrak{C}) = s_{K/F}(\mathfrak{J}^k(\beta, \mathfrak{A})),$$

for all $k \geqslant 0$.

Proof. — Again we deal only with $\mathfrak{H}$. As we remarked above, the first equality is trivial and, since $s_{K/F}$ is effectively a projection, we have $\mathfrak{H}^k(\beta, \mathfrak{A}) \cap C \subset s_{K/F}(\mathfrak{H}^k(\beta, \mathfrak{A}))$. However,

$$\begin{aligned} s_{K/F}(\mathfrak{H}^k(\beta, \mathfrak{A})) &= s_{K/F}(\mathfrak{H}(\beta, \mathfrak{A}) \cap \mathfrak{P}^k) \\ &\subset s_{K/F}(\mathfrak{H}(\beta, \mathfrak{A})) \cap s_{K/F}(\mathfrak{P}^k) \\ &= \mathfrak{H}(\beta, \mathfrak{C}) \cap \mathfrak{R}^k \\ &= \mathfrak{H}^k(\beta, \mathfrak{C}). \end{aligned}$$

This proves the lemma. □

We now turn to simple characters. A simple stratum $[\mathfrak{A}, n, m, \beta]$, with $m \geqslant 0$, determines a finite set $\mathscr{C}(\mathfrak{A}, m, \beta)$ of distinguished abelian characters of the group $H^{m+1}(\beta, \mathfrak{A})$. These characters we call " simple ". They are defined and discussed at length in [BK1], Ch. 3. Here, we continue in the situation of (7.1), and consider the simple character sets defined by the simple strata $[\mathfrak{A}, n, 0, \beta]$, $[\mathfrak{C}, n, 0, \beta]$. The relation between simple characters and their underlying stratum depends on a choice of basic additive character. We therefore fix, once for all, a continuous character ψ^F of the additive group of F with conductor $\mathfrak{p}_F$, i.e. such that $\mathfrak{p}_F \subset \mathrm{Ker}(\psi^F)$, but $\mathfrak{o}_F \not\subset \mathrm{Ker}(\psi^F)$. We also define

$$\psi^K = \psi^F \circ \mathrm{Tr}_{K/F}, \tag{7.5}$$

where $\mathrm{Tr}_{K/F}$ denotes the field trace. Since K/F is tamely ramified, ψ^K has conductor $\mathfrak{p}_K$. When V is a finite-dimensional F-vector space and $A = \mathrm{End}_F(V)$, we write ψ^A for the additive character $\psi^F \circ \mathrm{tr}_{A/F}$ of A. Then, for $b \in A$, we write ψ_b^A for the function

$$\psi_b^A : x \mapsto \psi^A(b(x-1))$$

on A or various subsets of A. We use similar notations relative to K. In particular, if V is a K-vector space and $C = \mathrm{End}_K(V)$, we have the functions ψ_c^C, $c \in C$. These are related to the corresponding objects over F as follows. If $s_{K/F}$ is the tame corestriction on A relative to K/F which is the identity on C (as in § 2), we have

$$\begin{aligned} \psi_b^A \mid C &= \psi^C_{s_{K/F}(b)}, \quad && b \in A, \\ \psi_c^A \mid C &= \psi_c^C, && c \in C. \end{aligned} \tag{7.6}$$

(7.7) *Theorem.* — *Let* V *be a finite-dimensional* K*-vector space, and* $[\mathfrak{C}, n, m, \beta]$ *a simple stratum in* $C = \mathrm{End}_K(V)$ *with* $m \geqslant 0$. *Let* $\mathfrak{A}$ *denote the hereditary* $\mathfrak{o}_F$*-order in* $A = \mathrm{End}_F(V)$ *defined by the same lattice chain as* $\mathfrak{C}$, *and suppose that the stratum* $[\mathfrak{A}, n, m, \beta]$ *is simple. Let* $\theta \in \mathscr{C}(\mathfrak{A}, m, \beta)$. *Then*

$$\theta \mid H^{m+1}(\beta, \mathfrak{C}) \in \mathscr{C}(\mathfrak{C}, m, \beta).$$

Proof. — For this proof, we use the same system of notation as above, namely $E = F[\beta]$, $B = \mathrm{End}_E(V)$, $\mathfrak{B} = \mathfrak{A} \cap B$, $D = \mathrm{End}_{K[\beta]}(V)$, $\mathfrak{D} = \mathfrak{A} \cap D$.

Again we proceed by induction along β, viewed as an element of $F[\beta]$. We therefore assume first that β is minimal over F, and also that $m \geqslant \left[\frac{n}{2}\right]$. Thus $H^{m+1}(\beta, \mathfrak{A}) = \mathbf{U}^{m+1}(\mathfrak{A})$, and $\theta = \psi_\beta^A \mid \mathbf{U}^{m+1}(\mathfrak{A})$. On the other hand, we have $H^{m+1}(\beta, \mathfrak{C}) = \mathbf{U}^{m+1}(\mathfrak{C})$, and the unique element of $\mathscr{C}(\mathfrak{C}, m, \beta)$ is the character ψ_β^C. The assertion of the theorem therefore amount here to the identity

$$\psi_\beta^A \mid \mathbf{U}^{m+1}(\mathfrak{C}) = \psi_\beta^C,$$

and this is given by (7.6).

For the next step, we assume that β is minimal over F, but that $0 \leqslant m < \left[\frac{n}{2}\right]$. This means we have

$$H^{m+1}(\beta, \mathfrak{A}) = \mathbf{U}^{m+1}(\mathfrak{B})\, \mathbf{U}^{[n/2]+1}(\mathfrak{A}),$$

and, by definition, the character θ satisfies

(7.8) (i) $\theta \mid \mathbf{U}^{[n/2]+1}(\mathfrak{A}) = \psi_\beta^A$, *and*

(ii) $\theta \mid \mathbf{U}^{m+1}(\mathfrak{B})$ *factors through the determinant mapping* $\det_B : B^\times \to E^\times$.

On the other hand, we have $H^{m+1}(\beta, \mathfrak{C}) = \mathbf{U}^{m+1}(\mathfrak{D})\, \mathbf{U}^{[n/2]+1}(\mathfrak{C})$. As in the first case above, we have $\theta \mid \mathbf{U}^{[n/2]+1}(\mathfrak{C}) = \psi_\beta^C$. On the other factor, we note that

$$\det_B \mid D^\times = N_{K[\beta]/E} \circ \det_D,$$

where $N_{K[\beta]/E}$ denotes the field norm $K[\beta] \to E$. Thus $\theta \mid \mathbf{U}^{m+1}(\mathfrak{D})$ factors through $\det_D$. This is enough to imply $\theta \mid H^{m+1}(\beta, \mathfrak{C}) \in \mathscr{C}(\mathfrak{C}, m, \beta)$, as required.

(7.9) *Remark.* — In the notation of the last argument, let $\theta \in \mathscr{C}(\mathfrak{C}, m, \beta)$. We have $\det_D(\mathbf{U}^{m+1}(\mathfrak{D})) = \mathbf{U}^{m'+1}(\mathfrak{o}_{K[\beta]})$ for some $m' \geqslant 0$. Thus θ determines a character φ of $\mathbf{U}^{m'+1}(\mathfrak{o}_{K[\beta]})$ by the relation

$$\theta \mid \mathbf{U}^{m+1}(\mathfrak{D}) = \varphi \circ \det_D.$$

The character θ is then the restriction of a character from $\mathscr{C}(\mathfrak{A}, m, \beta)$ if and only if φ factors through the norm $N_{K[\beta]/E}$. This, of course, is not invariably the case. Thus the restriction map $\mathscr{C}(\mathfrak{A}, m, \beta) \to \mathscr{C}(\mathfrak{C}, m, \beta)$ *need not be surjective*. However, we will see below that it is always injective.

We return to the proof, under the assumption that β is not minimal over F. We put $r = -k_0(\beta, \mathfrak{A})$, and we only treat the case $r > m \geqslant \left[\frac{r}{2}\right]$: using (7.3), the case $0 \leqslant m < \left[\frac{r}{2}\right]$ reduces to this one exactly as before. We choose a simple stratum $[\mathfrak{C}, n, r, \gamma]$, equivalent to $[\mathfrak{C}, n, r, \beta]$, and such that $[\mathfrak{A}, n, r, \gamma]$ is simple. In particular, $[\mathfrak{A}, n, r, \gamma]$ is equivalent to $[\mathfrak{A}, n, r, \beta]$. Let $\theta \in \mathscr{C}(\mathfrak{A}, m, \beta)$. Then, by definition, $H^{m+1}(\beta, \mathfrak{A}) = H^{m+1}(\gamma, \mathfrak{A})$ and $\theta = \theta_0 . \psi_c^A$ for some $\theta_0 \in \mathscr{C}(\mathfrak{A}, m, \gamma)$, where $c = \beta - \gamma$.

The next step is to observe that $H^{m+1}(\beta, \mathfrak{C}) = H^{m+1}(\gamma, \mathfrak{C})$. This is the definition of H if $k_0(\beta, \mathfrak{C}) = -r$. Otherwise, we have $k_0(\beta, \mathfrak{C}) < -r$ by (2.4), so the strata $[\mathfrak{C}, n, r, \beta]$, $[\mathfrak{C}, n, r, \gamma]$ are both simple, and the equality of H^{m+1}-groups is given by [BK1] (3.1.9). By inductive hypothesis, we have

$$\theta_0 \mid H^{m+1}(\gamma, \mathfrak{C}) \in \mathscr{C}(\mathfrak{C}, m, \gamma),$$

while certainly $\psi_c^A \mid H^{m+1}(\beta, \mathfrak{C}) = \psi_c^C$. In the case where $k_0(\beta, \mathfrak{C}) = -r$, we are through: the character $(\theta_0 \mid H^{m+1}(\gamma, \mathfrak{C})) . \psi_c^C$ lies in $\mathscr{C}(\mathfrak{C}, m, \beta)$ by definition of simple character, and the injectivity of (7.8) follows by induction. We are therefore left with the case where $[\mathfrak{C}, n, r, \beta]$ is simple, and this follows from [BK1] (3.3.20). □

As we observed in (7.9), the restriction process in (7.7) is rarely surjective. However, the character set $\mathscr{C}(\mathfrak{C}, m, \beta)$ does not determine the simple stratum $[\mathfrak{C}, n, m, \beta]$, even up to equivalence. Thus, to recognize a given simple character over K as a " lift " of a simple character over F, we are at liberty to vary the underlying additive parameter β.

(7.10) *Theorem. — Let* V *be a finite-dimensional* K*-vector space, and let* $[\mathfrak{C}, n, m, \beta]$ *be a simple stratum in* $C = \mathrm{End}_K(V)$ *with* $m \geqslant 0$. *Let* $\mathfrak{A}$ *be the hereditary* $\mathfrak{o}_F$*-order in* $A = \mathrm{End}_F(V)$ *defined by the same lattice chain as* $\mathfrak{C}$. *Let* $\vartheta \in \mathscr{C}(\mathfrak{C}, m, \beta)$. *There exists a simple stratum* $[\mathfrak{C}, n, m, \beta_1]$ *with the following properties:*

(i) $\mathscr{C}(\mathfrak{C}, m, \beta_1) = \mathscr{C}(\mathfrak{C}, m, \beta)$;

(ii) $[\mathfrak{A}, n, m, \beta_1]$ *is simple;*

(iii) *there exists* $\theta \in \mathscr{C}(\mathfrak{A}, m, \beta_1)$ *such that* $\theta \mid H^{m+1}(\beta, \mathfrak{C}) = \vartheta$.

Moreover, the character θ *in* (iii) *is uniquely determined, i.e. the restriction map*

$$\mathscr{C}(\mathfrak{A}, m, \beta_1) \to \mathscr{C}(\mathfrak{C}, m, \beta_1)$$

is injective.

Remark. — In (iii), we have $H^{m+1}(\beta, \mathfrak{C}) = H^{m+1}(\beta_1, \mathfrak{C}) = H^{m+1}(\beta_1, \mathfrak{A}) \cap C$. We can assume that $[\mathfrak{A}, n, m, \beta]$ is simple, so that the group $H^{m+1}(\beta, \mathfrak{A})$ is defined. However, we need *not* have the equality $H^{m+1}(\beta, \mathfrak{A}) = H^{m+1}(\beta_1, \mathfrak{A})$. To get an example where these groups differ, one can take $m = 0$, n sufficiently large, $\beta \in F$, and $\vartheta \in \mathscr{C}(\mathfrak{C}, 0, \beta)$ not in the image of $\mathscr{C}(\mathfrak{A}, 0, \beta)$. Thus ϑ is the composition of the determinant $C^\times \to K^\times$ with a character χ of $\mathbf{U}^1(\mathfrak{o}_K)$ which does not factor through the field norm $N_{K/F}$. The element β_1 then cannot lie in F. If we further impose the condition that $\chi \mid \mathbf{U}^3(\mathfrak{o}_K)$ does not factor through $N_{K/F}$, we get $-k_F(\beta) \geqslant 2$. This implies $H^1(\beta_1, \mathfrak{A}) \subsetneq \mathbf{U}^1(\mathfrak{A}) = H^1(\beta, \mathfrak{A})$.

Proof of (7.10). — We proceed by induction on m. Suppose first that $m \geqslant [n/2]$, so that $H^{m+1}(\beta, \mathfrak{C}) = \mathbf{U}^{m+1}(\mathfrak{C})$, and $\mathscr{C}(\mathfrak{C}, m, \beta) = \{\psi_\beta^C\}$. The theorem then asserts that, given a simple stratum $[\mathfrak{C}, n, m, \beta]$ in C, there is a simple stratum $[\mathfrak{C}, n, m, \beta']$, equivalent to $[\mathfrak{C}, n, m, \beta]$, such that $[\mathfrak{A}, n, m, \beta']$ is simple. This, however, is given by (3.8). The uniqueness assertion is immediate.

We therefore assume that $m < [n/2]$. We start by choosing a simple stratum $[\mathfrak{C}, n, m+1, \gamma]$ equivalent to $[\mathfrak{C}, n, m+1, \beta]$ and such that $[\mathfrak{A}, n, m+1, \gamma]$ is simple. We then have $H^{m+1}(\beta, \mathfrak{C}) = H^{m+1}(\gamma, \mathfrak{C})$, and $\vartheta \mid H^{m+2}(\gamma, \mathfrak{C}) \in \mathscr{C}(\mathfrak{C}, m+1, \gamma)$. Inductively, there is a simple stratum $[\mathfrak{C}, n, m+1, \gamma_1]$ such that

(7.11) (i) $\mathscr{C}(\mathfrak{C}, m+1, \gamma_1) = \mathscr{C}(\mathfrak{C}, m+1, \gamma)$;

(ii) $[\mathfrak{A}, n, m+1, \gamma_1]$ *is simple;*

(iii) *there exists a unique* $\theta_0 \in \mathscr{C}(\mathfrak{A}, m+1, \gamma_1)$ *such that*

$$\theta_0 \mid H^{m+2}(\gamma_1, \mathfrak{C}) = \vartheta \mid H^{m+2}(\gamma_1, \mathfrak{C}).$$

Now we appeal to [BK3] (2.11): there exists a simple stratum $[\mathfrak{C}, n, m, \beta_1]$ such that

$$[\mathfrak{C}, n, m+1, \beta_1] \sim [\mathfrak{C}, n, m+1, \gamma_1],$$
$$\mathscr{C}(\mathfrak{C}, m, \beta_1) = \mathscr{C}(\mathfrak{C}, m, \beta).$$

Replacing $[\mathfrak{C}, n, m, \beta_1]$ by an equivalent simple stratum changes neither of these conditions, so we can assume further that $[\mathfrak{A}, n, m, \beta_1]$ is simple.

The next step is to choose a character $\widetilde{\theta}_0 \in \mathscr{C}(\mathfrak{A}, m, \gamma_1)$ which extends θ_0. The restriction

$$\vartheta' = \widetilde{\theta}_0 \mid H^{m+1}(\gamma_1, \mathfrak{C})$$

then lies in $\mathscr{C}(\mathfrak{C}, m, \gamma_1)$, by (7.7). The character ϑ' agrees with our given ϑ on $H^{m+2}(\gamma_1, \mathfrak{C})$, so there exists $c \in \mathfrak{R}^{-(1+m)}$ (where $\mathfrak{R} = \mathrm{rad}(\mathfrak{C})$) such that

$$\vartheta = \vartheta' . \psi_c^{C}.$$

We now choose a tame corestriction s on A relative to the field extension $F[\gamma_1]/F$. Thus, by (2.7), $s \mid C$ is a tame corestriction on C relative to $K[\gamma_1]/K$. We write $B_1 = \mathrm{End}_{F[\gamma_1]}(V)$, $\mathfrak{B}_1 = \mathfrak{A} \cap B_1$, $D_1 = \mathrm{End}_{K[\gamma_1]}(V)$, $\mathfrak{D}_1 = \mathfrak{A} \cap D_1$. By [BK3] (2.7), the stratum $[\mathfrak{D}_1, m+1, m, s(c)]$ is either null or is equivalent to a simple stratum $[\mathfrak{D}_1, m+1, m, \delta]$. In either case, $[\mathfrak{C}, n, m, \gamma_1 + c]$ is equivalent to a simple stratum $[\mathfrak{C}, n, m, \beta_2]$, and $\vartheta \in \mathscr{C}(\mathfrak{C}, m, \beta_2)$. We can choose β_2 so that $[\mathfrak{A}, n, m, \beta_2]$ is simple, and we again have $\widetilde{\theta}_0\, \psi_c^{A} \in \mathscr{C}(\mathfrak{A}, m, \beta_2)$. This proves the assertions (i)-(iii) of the theorem.

We now have to show that the condition $\theta \mid H^{m+1}(\beta_2, \mathfrak{C}) = \vartheta$ determines the character $\theta \in \mathscr{C}(\mathfrak{A}, m, \beta_2)$ uniquely. At this point, there is no harm in assuming that the element β_2 constructed above is equal to β. We now choose a core approximation to $[\mathfrak{A}, n, m, \beta]$, in the sense of [BK1] (3.5). By definition, this is a simple stratum $[\mathfrak{A}, n, \ell, \delta]$ such that

(7.12) (i) $\ell \geqslant m$;

(ii) $[\mathfrak{A}, n, \ell, \beta] \sim [\mathfrak{A}, n, \ell, \delta]$;

(iii) $m < [t/2]$, *where* $t = -k_0(\delta, \mathfrak{A})$;

(iv) *if* $[\mathfrak{A}, n, \ell', \delta']$ *is a simple stratum satisfying* (i)-(iii), *with* $t' = -k_0(\delta', \mathfrak{A})$, *then* $t' \geqslant t$ *and* $\ell' \geqslant \ell$.

(7.13) *Lemma.* — *Let* $[\mathfrak{A}, n, m, \beta]$ *be a* K*-pure simple stratum with* $m \geqslant 0$. *Then* $[\mathfrak{A}, n, m, \beta]$ *has a core approximation which is* K*-pure.*

Proof. — Choose a core approximation $[\mathfrak{A}, n, \ell, \delta]$ as in (7.12). With this value of ℓ, choose a simple stratum $[\mathfrak{C}, n, \ell, \delta']$ equivalent to $[\mathfrak{C}, n, \ell, \beta]$ such that $[\mathfrak{A}, n, \ell, \delta']$ is simple. This is the required core approximation. □

Now suppose that we have characters $\theta_1, \theta_2 \in \mathscr{C}(\mathfrak{A}, m, \beta)$ such that

$$\theta_1 \mid \mathrm{H}^{m+1}(\beta, \mathfrak{C}) = \theta_2 \mid \mathrm{H}^{m+1}(\beta, \mathfrak{C}).$$

Inductively, we have

$$\theta_1 \mid \mathrm{H}^{m+2}(\beta, \mathfrak{A}) = \theta_2 \mid \mathrm{H}^{m+2}(\beta, \mathfrak{A}).$$

We now take a K-pure core approximation $[\mathfrak{A}, n, \ell, \delta]$ to $[\mathfrak{A}, n, m, \beta]$, as in (7.13), and we choose a tame corestriction s_δ on A relative to $\mathrm{F}[\delta]/\mathrm{F}$. According to [BK1] (3.5.6), there exists $d \in \mathfrak{P}^{-(1+m)}$ such that

(7.14) (i) $\theta_2 = \theta_1 \psi_d^{\mathrm{A}}$, *and*
(ii) $s_\delta(d) \in \mathrm{F}[\delta] + \mathfrak{P}^{-m}$.

The definition of core approximation ensures that the character

$$\psi_d^{\mathrm{A}} \mid \mathrm{H}^{m+1}(\beta, \mathfrak{A})$$

actually only depends on the coset $s_\delta(d) + \mathfrak{P}^{-m}$. This follows from the identity

$$\mathrm{H}^{m+1}(\beta, \mathfrak{A}) = \mathbf{U}^{m+1}(\mathfrak{B}_\delta)\, \mathrm{H}^{m+2}(\beta, \mathfrak{A}),$$

where $\mathfrak{B}_\delta$ is the $\mathfrak{A}$-centralizer of δ; see [BK1] (3.5) for this. Choose an element $x \in \mathrm{F}[\delta]$ such that $x \equiv s_\delta(d) \pmod{\mathfrak{P}^{-m}}$. This element x lies in $\mathfrak{P}^{-(1+m)}$, and commutes with both K and δ. Thus there exists $d' \in \mathfrak{R}^{-(1+m)}$ such that

$$s_\delta(d') \equiv x \equiv s_\delta(d) \pmod{\mathfrak{P}^{-m}}.$$

Replacing d by d' does not change the character θ_2, so we may as well assume that $d \in \mathrm{C}$.

Our hypothesis implies that ψ_d^{A} is trivial on $\mathrm{H}^{m+1}(\beta, \mathfrak{C})$. In particular, if we write $\mathfrak{D}_\delta$ for the $\mathfrak{C}$-centralizer of δ, this character is trivial on

$$\mathbf{U}^{m+1}(\mathfrak{D}_\delta) = \mathbf{U}^{m+1}(\mathfrak{B}_\delta) \cap \mathrm{C}.$$

This implies $s_\delta(d) \in \mathfrak{S}_\delta^{-m}$, where $\mathfrak{S}_\delta$ denotes the radical of $\mathfrak{D}_\delta$. Of course, $\mathfrak{S}_\delta^{-m} \subset \mathfrak{Q}_\delta^{-m}$, where $\mathfrak{Q}_\delta$ is the radical of $\mathfrak{B}_\delta$. This implies that ψ_d^{A} is trivial on $\mathrm{H}^{m+1}(\beta, \mathfrak{A})$. In other words, we have shown $\theta_2 = \theta_1$, as required. □

Returning to the statement of (7.10), we observe that there will, in general, be many other simple strata $[\mathfrak{C}, n, m, \beta_1']$ satisfying the required conditions. The fact that the given ϑ lies in both $\mathscr{C}(\mathfrak{C}, m, \beta_1)$ and $\mathscr{C}(\mathfrak{C}, m, \beta_1')$ ensures that $\mathscr{C}(\mathfrak{C}, m, \beta_1) = \mathscr{C}(\mathfrak{C}, m, \beta_1')$

(see [BK1] (3.5.8)). However, we have as yet no relation between the sets $\mathscr{C}(\mathfrak{A}, m, \beta_1)$, $\mathscr{C}(\mathfrak{A}, m, \beta_1')$. This is provided by the following result.

(7.15) *Theorem. — Let* V *be a finite-dimensional* K*-vector space and* $\mathfrak{C}$ *a hereditary* $\mathfrak{o}_K$*-order in* $C = \mathrm{End}_K(V)$. *Let* $\mathfrak{A}$ *be the hereditary* $\mathfrak{o}_F$*-order in* $A = \mathrm{End}_F(V)$ *defined by the same lattice chain as* $\mathfrak{C}$.

For $i = 1, 2$, *let* $[\mathfrak{A}, n, m, \beta_i]$ *be a* K*-pure simple stratum in* A *with* $m \geqslant 0$, *and suppose that*

a) $\mathscr{C}(\mathfrak{C}, m, \beta_1) = \mathscr{C}(\mathfrak{C}, m, \beta_2)$, *and*

b) the images of the sets $\mathscr{C}(\mathfrak{A}, m, \beta_1)$, $\mathscr{C}(\mathfrak{A}, m, \beta_2)$ *in* $\mathscr{C}(\mathfrak{C}, m, \beta_i)$ *have non-empty intersection.*

We then have

$$\mathscr{C}(\mathfrak{A}, m, \beta_1) = \mathscr{C}(\mathfrak{A}, m, \beta_2).$$

Proof. — As before, there is nothing to prove if $m \geqslant [n/2]$, so we assume that $0 \leqslant m < [n/2]$. Again we proceed by induction on m. Thus we may assume that

$$\mathscr{C}(\mathfrak{A}, m+1, \beta_1) = \mathscr{C}(\mathfrak{A}, m+1, \beta_2).$$

This implies ([BK1] (3.5.9)) that

$$H^{m+1}(\beta_1, \mathfrak{A}) = H^{m+1}(\beta_2, \mathfrak{A}).$$

By inductive hypothesis and (7.10), we may choose $\theta_i \in \mathscr{C}(\mathfrak{A}, m, \beta_i)$ such that

$$\theta_1 \mid H^{m+1}(\beta_i, \mathfrak{C}) = \theta_2 \mid H^{m+1}(\beta_i, \mathfrak{C}),$$
$$\theta_1 \mid H^{m+2}(\beta_i, \mathfrak{A}) = \theta_2 \mid H^{m+2}(\beta_i, \mathfrak{A}).$$

Thus $\theta_2 = \theta_1 \psi_c^A$, for some $c \in \mathfrak{P}^{1-m}$. Consider the character ψ_c^A of $H^{m+1}(\beta_i, \mathfrak{A})$. In the notation of the last proof, this is effectively a character of $U^{m+1}(\mathfrak{B}_\delta)$, for a K-pure core approximation $[\mathfrak{A}, n, \ell, \delta]$ for $[\mathfrak{A}, n, m, \beta_1]$. Write $B_i = \mathrm{End}_{F[\beta_i]}(V)$. According to [BK1] (3.3.3), the character θ_i is intertwined by the whole of $B_i^\times$. In particular, it is intertwined by $K^\times$. Therefore, the character $\psi_c^A \mid U^{m+1}(\mathfrak{B}_\delta)$ is intertwined, indeed normalized, by the whole of $K^\times$. Now we need a lemma:

(7.16) *Lemma. — Let* $\mathfrak{A}$ *be a hereditary* $\mathfrak{o}_F$*-order in* $A = \mathrm{End}_F(V)$, *for some finite-dimensional* F*-vector space* V. *Let* E/F *be a subfield of* A *such that* $E^\times \subset \mathfrak{K}(\mathfrak{A})$, *and let* $b \in \mathfrak{P}^{-n}$, *where* $\mathfrak{P} = \mathrm{rad}(\mathfrak{A})$. *Suppose further that*

$$x^{-1}(b + \mathfrak{P}^{1-n})\, x = b + \mathfrak{P}^{1-n},$$

for all $x \in E^\times$. *Then there exists* $b' \in \mathfrak{P}^{-n}$ *such that*

(i) $b' + \mathfrak{P}^{1-n} = b + \mathfrak{P}^{1-n}$, *and*

(ii) $x^{-1}\, b'\, x = b'$, *for all* $x \in E^\times$.

Proof. — The assertion is effectively transitive in the field extension E/F, so we may assume that $E = F[\alpha]$, for some element α which is minimal over F. Let $\alpha\mathfrak{A} = \mathfrak{P}^t$. We then have

$$\alpha b - b\alpha \in \mathfrak{P}^{1+t-n}.$$

Now we recall that, since α is minimal over F, we have $k_0(\alpha, \mathfrak{A}) = t$ (or $-\infty$, this being the trivial case $E = F$). This now implies ([BK1] (1.4.7)) that there exists $a \in \mathfrak{P}^{-n}$ such that $s_\alpha(a) \equiv b \pmod{\mathfrak{P}^{1-n}}$, where s_α is a tame corestriction on A relative to E/F. We take $b' = s_\alpha(a)$, and this implies the assertion of the lemma. □

With δ as above, we choose a tame corestriction s_δ on A relative to $F[\delta]/F$. We apply (7.16) to the coset $s_\delta(c) + \mathfrak{Q}_\delta^{-m}$, with base field $F[\delta]$ and extension $K[\delta]/F[\delta]$. We thus find an element $\varepsilon \in \mathfrak{Q}_\delta^{-(1+m)}$ which commutes with $K[\delta]$, in particular with K, such that $\varepsilon \equiv s_\delta(c) \pmod{\mathfrak{P}^{-m}}$. Continuing with our earlier notation (as at the end of the proof of (7.10)), this means $\varepsilon \in \mathfrak{S}_\delta^{-(1+m)}$. However, the tame corestriction s_δ restricts to a tame corestriction on C relative to $K[\delta]/K$. Therefore there exists $c' \in \mathfrak{N}^{-(1+m)}$ such that $s_\delta(c') \equiv \varepsilon \pmod{\mathfrak{S}_\delta^{-m}}$. With this element c', we have

$$\psi_{c'}^{A} \mid H^{m+1}(\beta_i, \mathfrak{A}) = \psi_c^{A} \mid H^{m+1}(\beta_i, \mathfrak{A}).$$

In other words, we could have assumed at the beginning that $c \in C$. This implies that $s_\delta(c) \in C$, and that

$$\psi^{B_\delta}_{s_\delta(c)} \mid D_\delta = \psi^{D_\delta}_{s_\delta(c)}.$$

Here, the character ψ^{B_δ} is defined by $\psi_x^A \mid B_\delta = \psi^{B_\delta}_{s_\delta(x)}$, $x \in A$, and $\psi^{D_\delta} = \psi^{B_\delta} \mid D_\delta$. Moreover, $\psi^{B_\delta} = \psi^{F[\delta]} \circ \mathrm{tr}_{B_\delta/F[\delta]}$, for some character $\psi^{F[\delta]}$ of $F[\delta]$ with conductor $\mathfrak{p}_{F[\delta]}$, and likewise for ψ^{D_δ}.

Now we recall that our assumption implies $\psi_c^A \mid H^{m+1}(\beta_i, \mathfrak{C})$ is trivial. This just says, in the same notation, that $s_\delta(c) \in \mathfrak{S}_\delta^{-m}$, which implies $s_\delta(c) \in \mathfrak{Q}_\delta^{-m}$. In other words, ψ_c^A is trivial on $H^{m+1}(\beta_i, \mathfrak{A})$. This means $\theta_1 = \theta_2$, and hence (by [BK1] (3.5.8)) that $\mathscr{C}(\mathfrak{A}, m, \beta_1) = \mathscr{C}(\mathfrak{A}, m, \beta_2)$, as required. □

For application to representation theory, one needs to consider the way in which lifting interacts with intertwining between simple characters. This is a somewhat complex matter, which requires the more indirect approach of later sections. However, it is possible to say something useful at this stage. For this, we need to recall one of the fundamental results of [BK1].

(7.18) *Intertwining implies conjugacy.* — *For* $i = 1, 2$, *let* $[\mathfrak{A}, n, m, \beta_i]$ *be a simple stratum in* $\mathrm{End}_F(V)$, *with* $m \geqslant 0$. *Let* $\theta_i \in \mathscr{C}(\mathfrak{A}, m, \beta_i)$, *and assume that* θ_1, θ_2 *intertwine in* $\mathrm{Aut}_F(V)$. *Then there exists* $x \in \mathbf{U}(\mathfrak{A})$ *such that* $\mathscr{C}(\mathfrak{A}, m, \beta_1) = \mathscr{C}(\mathfrak{A}, m, x^{-1}\beta_2 x)$ *and* $\theta_1 = \theta_2^x$.

This is [BK1] (3.5.11).

(7.19) *Proposition. — Let* V *be a finite-dimensional* K*-vector space, and put* $C = \mathrm{End}_K(V)$. *Let* $[\mathfrak{A}, n, m, \beta]$ *be a* K*-pure simple stratum in* $A = \mathrm{End}_F(V)$, *with* $m \geqslant 0$. *Let* $\mathfrak{C} = \mathfrak{A} \cap C$, *and write*

$$\theta^K = \theta \mid H^{m+1}(\beta, \mathfrak{C}), \quad \theta \in \mathscr{C}(\mathfrak{A}, m, \beta).$$

Let $\theta_1, \theta_2 \in \mathscr{C}(\mathfrak{A}, m, \beta)$. *Suppose that the characters* θ_i^K *intertwine in* $C^\times$. *The characters* θ_i *then intertwine in* $A^\times$.

Proof. — By (7.18), there exists $x \in \mathbf{U}(\mathfrak{C})$ such that

$$\mathscr{C}(\mathfrak{C}, m, x^{-1}\beta x) = \mathscr{C}(\mathfrak{C}, m, \beta)$$

and

$$(\theta_2^K)^x = \theta_1^K.$$

Then by (7.15), we have $\mathscr{C}(\mathfrak{A}, m, x^{-1}\beta x) = \mathscr{C}(\mathfrak{A}, m, \beta)$ and, by (7.10), $\theta_2^x = \theta_1$. This proves the proposition. □

It is not hard, using the results of [BK3], to produce examples in the context of (7.19) where the θ_i intertwine, but the θ_i^K do not. As we shall see, this situation arises where a given lift θ_1^K of θ_1 intertwines with a different lift (i.e. one corresponding to a different K-embedding of β) of θ_2.

8. Endo-classes of simple characters

In § 1, we organized the set of simple strata over F first into simple pairs and then into equivalence classes of simple pairs. In this section, we give an analogous organization for simple *characters* over F. The process is somewhat more tortuous here.

We start with the notion of a *potential simple character*, or *ps-character*, for short. We fix a simple pair $[k, \beta]$ with $k \geqslant 0$. We let $[\mathfrak{A}, n, m, \beta]$ range over all realizations of $[k, \beta]$, so that, in particular, $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in the algebra of F-endomorphisms of some $F[\beta]$-vector space, $n = n_F(\beta)\, e_\beta(\mathfrak{A})$ and $[m/e_\beta(\mathfrak{A})] = k$. We set

$$\mathfrak{R}[k,\beta] = \bigcup_{[\mathfrak{A}, n, m, \beta]} \mathscr{C}(\mathfrak{A}, m, \beta).$$

Now we need to recall the main idea of [BK1] (3.6). For each pair of realizations $[\mathfrak{A}_i, n_i, m_i, \beta]$ of $[k, \beta]$, [BK1] (3.6.14) gives a canonical bijection

$$\tau_{\mathfrak{A}_1, \mathfrak{A}_2, \beta} : \mathscr{C}(\mathfrak{A}_1, m_1, \beta) \xrightarrow{\approx} \mathscr{C}(\mathfrak{A}_2, m_2, \beta).$$

(We omit the m_i from the notation here; if we have two realizations of $[k, \beta]$ on the same order $\mathfrak{A}$ in the F-endomorphisms of a given $F[\beta]$-vector space, the two simple character sets are in canonical bijection by [BK1] (3.6.7).) The uniqueness properties of this family of " transfer maps " τ imply the identities

$$\begin{aligned} \tau_{\mathfrak{A}_2, \mathfrak{A}_1, \beta} &= \tau_{\mathfrak{A}_1, \mathfrak{A}_2, \beta}^{-1}, \\ \tau_{\mathfrak{A}_1, \mathfrak{A}_3, \beta} &= \tau_{\mathfrak{A}_2, \mathfrak{A}_3, \beta} \circ \tau_{\mathfrak{A}_1, \mathfrak{A}_2, \beta}, \end{aligned} \tag{8.1}$$

in the obvious notation.

(8.2) *Definition.* — *Let* $[k, \beta]$ *be a simple pair over* F *with* $k \geqslant 0$. *For* $i = 1, 2$, *let* $\theta_i \in \mathscr{C}(\mathfrak{A}_i, m_i, \beta) \subset \mathfrak{R}[k, \beta]$. *We say that* θ_1 *is* β-equivalent to θ_2 *if* $\theta_2 = \tau_{\mathfrak{A}_1, \mathfrak{A}_2, \beta}\, \theta_1$.

The identities (8.1) show that β-equivalence is indeed an equivalence relation on $\mathfrak{R}[k, \beta]$. A *potential simple character over* F *supported by* $[k, \beta]$ (usually *ps-character* for short) is then an equivalence class for this relation. If Θ is a ps-character supported by $[k, \beta]$, and $\theta \in \Theta \subset \mathfrak{R}[k, \beta]$, we call θ the *realization of* Θ *on* $\mathfrak{A}$, where $\theta \in \mathscr{C}(\mathfrak{A}, m, \beta)$. It is sometimes useful to have the notation

$$\theta = \Theta(\mathfrak{A}) = \Theta(\mathfrak{A}, m)$$

in this situation.

It will sometimes be convenient to use the notation $[\Theta, k, \beta]$ for a ps-character supported by $[k, \beta]$. Likewise, if θ is a simple character, $\theta \in \Theta \subset \mathfrak{R}[k, \beta]$, we say that θ is *supported by* $[k, \beta]$.

As for simple pairs, we can define realization in terms of embeddings of $F[\beta]$ in $\mathrm{End}_F(V)$, for F-vector spaces V.

It is important to remember that one can have non-trivial identities $\mathscr{C}(\mathfrak{A}, m, \beta_1) = \mathscr{C}(\mathfrak{A}, m, \beta_2)$, where the strata $[\mathfrak{A}, n, m, \beta_i]$ define inequivalent simple pairs $[k, \beta_i]$. However, these pairs have much in common.

(8.3) *Proposition.* — *For* $i = 1, 2$, *let* $[\mathfrak{A}, n, m, \beta_i]$ *be a simple stratum defining a simple pair* $[k_i, \beta_i]$, *and suppose that* $\mathscr{C}(\mathfrak{A}, m, \beta_1) = \mathscr{C}(\mathfrak{A}, m, \beta_2)$. *Then*

$$\begin{aligned} k_1 &= k_2, \\ e(F[\beta_1] \mid F) &= e(F[\beta_2] \mid F), \\ f(F[\beta_1] \mid F) &= f(F[\beta_2] \mid F), \\ k_F(\beta_1) &= k_F(\beta_2). \end{aligned}$$

Proof. — All but the first of these statements are given by [BK1] (3.5.1), while the first follows from the second. □

(8.4) *Proposition.* — *For* $i = 1, 2$, *let* $[k_i, \beta_i]$ *be a simple pair and let* Θ_i *be a ps-character supported by* $[k_i, \beta_i]$. *Suppose that* $[F[\beta_1] : F] = [F[\beta_2] : F]$, *and that we have simultaneous realizations* $[\mathfrak{A}, n_i, m_i, \beta_i]$ *of the* $[k_i, \beta_i]$ *on the same order* $\mathfrak{A}$ *in some* $A = \mathrm{End}_F(V)$ *such that the corresponding characters* $\Theta_i(\mathfrak{A}) \in \mathscr{C}(\mathfrak{A}, m_i, \beta_i)$ *intertwine in* $\mathrm{Aut}_F(V)$. *Then*

$$\begin{aligned} n_1 &= n_2, \\ n_F(\beta_1) &= n_F(\beta_2), \\ e(F[\beta_1] \mid F) &= e(F[\beta_2] \mid F), \\ f(F[\beta_1] \mid F) &= f(F[\beta_2] \mid F), \\ k_F(\beta_1) &= k_F(\beta_2). \end{aligned}$$

Proof. — For each i, choose a simple stratum $[\mathfrak{A}, n_i, n_i - 1, \alpha_i]$ equivalent to $[\mathfrak{A}, n_i, n_i - 1, \beta_i]$. We have $\mathrm{H}^{n_i}(\beta_i, \mathfrak{A}) = \mathbf{U}^{n_i}(\mathfrak{A})$, and the restriction of $\Theta_i(\mathfrak{A})$ to $\mathbf{U}^{n_i}(\mathfrak{A})$ is equal to the character $\psi^{\mathrm{A}}_{\alpha_i}$. These characters $\psi^{\mathrm{A}}_{\alpha_i}$ intertwine, which forces $n_1 = n_2$. We now abbreviate $n_1 = n_2 = n$.

Now we consider the integers m_1, m_2. By symmetry, we can assume that $m_1 \geqslant m_2$. Choose a simple stratum $[\mathfrak{A}, n, m_1, \gamma]$ such that

$$[\mathfrak{A}, n, m_1, \gamma] \sim [\mathfrak{A}, n, m_1, \beta_2],$$

and set $\theta_0 = \Theta_2(\mathfrak{A}) \mid \mathrm{H}^{m_1+1}(\gamma, \mathfrak{A})$. The simple characters θ_0, $\Theta_1(\mathfrak{A})$ still intertwine. This implies (by (7.18)) the existence of $x \in \mathbf{U}(\mathfrak{A})$ such that $\mathscr{C}(\mathfrak{A}, m_1, x^{-1}\gamma x) = \mathscr{C}(\mathfrak{A}, m_1, \beta_1)$ and $\Theta_1(\mathfrak{A}) = \theta_0^x$. Proposition (8.3) then gives $[\mathrm{F}[\gamma] : \mathrm{F}] = [\mathrm{F}[\beta_1] : \mathrm{F}]$, and our hypothesis on field degrees implies that $[\mathfrak{A}, n, m_1, \beta_2]$ is simple. The result now follows from (8.3). $\square$

Our next result is an extension of (1.9) to the present situation.

(8.5) *Lemma.* — *For $i = 1, 2$, let $[\mathfrak{A}, n, m, \beta_i]$ be a simple stratum in $\mathrm{A} = \mathrm{End}_{\mathrm{F}}(\mathrm{V})$ and let $\theta_i \in \mathscr{C}(\mathfrak{A}, m, \beta_i)$. Suppose that the characters θ_i intertwine in* $\mathrm{Aut}_{\mathrm{F}}(\mathrm{V})$. *Then*

(i) $$e_{\beta_1}(\mathfrak{A}) = e_{\beta_2}(\mathfrak{A}).$$

Write $m_0 = e_{\beta_i}(\mathfrak{A})[m/e_{\beta_i}(\mathfrak{A})]$ (which is independent of i). We have:

(ii) *for each i, there is a unique character $\widetilde{\theta}_i \in \mathscr{C}(\mathfrak{A}, m_0, \beta_i)$ such that $\widetilde{\theta}_i \mid \mathrm{H}^{m+1}(\beta_i, \mathfrak{A}) = \theta_i$;*

(iii) *the characters $\widetilde{\theta}_1, \widetilde{\theta}_2$ intertwine in* $\mathrm{Aut}_{\mathrm{F}}(\mathrm{V})$.

Proof. — Part (i) is given by (7.18) and (8.4). Part (ii) is given by [BK1] (3.6.7), and also applies when m_0 is replaced by any integer m' such that $m_0 \leqslant m' \leqslant m$. We can therefore assume that m is the least integer $\geqslant m_0$ for which the characters

$$\theta_{i,m} = \widetilde{\theta}_i \mid \mathrm{H}^{m+1}(\beta_i, \mathfrak{A})$$

intertwine. If $m = m_0$, we are done, so we assume otherwise. Applying (7.18), we can replace β_2 by a $\mathbf{U}(\mathfrak{A})$-conjugate and assume that $\theta_{1,m} = \theta_{2,m}$, and hence ([BK1] (3.5.8)) that $\mathscr{C}(\mathfrak{A}, m, \beta_1) = \mathscr{C}(\mathfrak{A}, m, \beta_2)$. Now we use [BK1] (3.5.9): this gives us a simple stratum $[\mathfrak{A}, n, m, \beta_3]$ such that

$$[\mathfrak{A}, n, m, \beta_3] \sim [\mathfrak{A}, n, m, \beta_1],$$

and $$\mathscr{C}(\mathfrak{A}, m-1, \beta_3) = \mathscr{C}(\mathfrak{A}, m-1, \beta_2).$$

By (1.9), there exists $y \in \mathbf{U}(\mathfrak{A})$ such that $[\mathfrak{A}, n, m-1, y^{-1}\beta_3 y] \sim [\mathfrak{A}, n, m-1, \beta_1]$. In particular, we have $[\mathfrak{A}, n, m, y^{-1}\beta_1 y] \sim [\mathfrak{A}, n, m, \beta_1]$, so, by [BK1] (3.3.2), (1.5.8), $\theta^y = \theta$ for all $\theta \in \mathscr{C}(\mathfrak{A}, n, m, \beta_1)$. We now see that $\theta^y_{2,m-1} \in \mathscr{C}(\mathfrak{A}, m-1, \beta_1)$, and it extends $\theta_{1,m}$. By (ii), this means $\theta^y_{2,m-1} = \theta_{1,m-1}$, which contradicts the choice of m. $\square$

(8.6) *Definition. — For $i = 1, 2$, let $[k_i, \beta_i]$ be a simple pair over* F *with $k_i \geqslant 0$. Let Θ_i be a ps-character supported by $[k_i, \beta_i]$. The ps-characters Θ_i are* endo-equivalent, *denoted*

$$\Theta_1 \approx \Theta_2,$$

if $[F[\beta_1] : F] = [F[\beta_2] : F]$ and there exists a finite-dimensional F*-vector space* V *together with realizations $[\mathfrak{A}, n_i, m_i, \beta_i]$ of the pairs $[k_i, \beta_i]$ such that the characters $\Theta_i(\mathfrak{A})$ intertwine in* $\mathrm{Aut}_F(V)$.

Of course, we have yet to establish that the relation we have called endo-equivalence is in fact an equivalence relation. This is the point of the next result, which generalizes (1.10).

(8.7) *Theorem. — For $i = 1, 2$, let $[\Theta_i, k_i, \beta_i]$ be a ps-character over* F, *and suppose that $\Theta_1 \approx \Theta_2$.*

Let V′ *be a finite-dimensional* F*-vector space, and suppose we have simultaneous realizations $[\mathfrak{A}', n_i', m_i', \beta_i]$ of the simple pairs $[k_i, \beta_i]$ on some hereditary order $\mathfrak{A}'$ in $\mathrm{End}_F(V')$. The characters $\Theta_1(\mathfrak{A}')$, $\Theta_2(\mathfrak{A}')$ then intertwine in $\mathrm{Aut}_F(V')$.*

Proof. — By hypothesis, there is a vector space V and a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $\mathrm{End}_F(V)$, together with realizations $[\mathfrak{A}, n_i, m_i, \beta_i]$ of the $[k_i, \beta_i]$ such that the characters $\Theta_i(\mathfrak{A})$ intertwine. We can apply (8.4) to deduce $n_1 = n_2$, etc., from which it follows that $k_1 = k_2 = k$, say. Further, we can assume (by (8.5)) that $m_1 = m_2 = m$, say. In the context of V′ therefore, we have $n_1' = n_2'$ and we may as well take $m_1' = m_2'$.

Returning to the context of V, our first lemma will allow us to adjust the order $\mathfrak{A}$.

(8.8) *Lemma. — For $i = 1, 2$, let $[\mathfrak{A}, n, m, \beta_i]$ be a simple stratum in $A = \mathrm{End}_F(V)$ with $m \geqslant 0$, and suppose we have $e(F[\beta_1] \mid F) = e(F[\beta_2] \mid F)$, $f(F[\beta_1] \mid F) = f(F[\beta_2] \mid F)$. Let $\mathfrak{A}_0$ be a hereditary $\mathfrak{o}_F$-order in $A = \mathrm{End}_F(V)$ which contains $\mathfrak{A}$, is normalized by $F[\beta_1]^\times$, and is maximal for this property. Then*

(i) *$\mathfrak{A}_0$ is normalized by $F[\beta_2]^\times$ and is maximal for this property.*

Let $m_0 = [m/e_{\beta_i}(\mathfrak{A})]$. Then

(ii) *$[\mathfrak{A}_0, n_0, m_0, \beta_i]$ is a simple stratum, where $n_0 = n/e_{\beta_i}(\mathfrak{A})$.*

Let $\theta_i \in \mathscr{C}(\mathfrak{A}, m, \beta_i)$, and write $\widetilde{\theta}_i = \tau_{\mathfrak{A}, \mathfrak{A}_0, \beta_i}(\theta_i) \in \mathscr{C}(\mathfrak{A}_0, m_0, \beta_i)$. Then

(iii) *the characters θ_i intertwine in $\mathrm{Aut}_F(V)$ if and only if the characters $\widetilde{\theta}_i$ intertwine in $\mathrm{Aut}_F(V)$.*

Proof. — The maximality condition on $\mathfrak{A}_0$ is equivalent to $e_{\beta_1}(\mathfrak{A}_0) = 1$. The order $\mathfrak{A}_0$ is obtained as follows: there is an $\mathfrak{A}$-lattice L in V such that $\mathfrak{A}_0$ is defined by the lattice chain $F[\beta_1]^\times L$. However, the set of $\mathfrak{A}$-lattices is linearly ordered and the equality of ramification indices yields $F[\beta_1]^\times L = F[\beta_2]^\times L$, and this proves (i).

Part (ii) is immediate. In (iii), suppose first that the characters θ_i intertwine. Using (8.5), we can now assume that $m = m_0 e_{\beta_i}(\mathfrak{A})$. Further, using (7.18), we can conjugate by an element of $\mathbf{U}(\mathfrak{A}) \subset \mathbf{U}(\mathfrak{A}_0)$ and assume that $\theta_1 = \theta_2$. Our choice of m gives us

$$\mathbf{U}^{m+1}(\mathfrak{A}) \supset \mathbf{U}^{m_0+1}(\mathfrak{A}_0).$$

By [BK1] (8.1.8), there is a unique irreducible representation ρ of the group $\mathbf{U}^{m+1}(\mathfrak{A})$ which contains the character $\theta_1 = \theta_2$; indeed, the representation of $\mathbf{U}^{m+1}(\mathfrak{A})$ induced by θ_i is a multiple of ρ. This representation ρ thus contains any character of the group $H^{m_0+1}(\beta_i, \mathfrak{A}_0)$ which agrees with θ_i on $H^{m_0+1}(\beta_i, \mathfrak{A}_0) \cap H^{m+1}(\beta_i, \mathfrak{A})$. However, by [BK1] (3.6.1), the character $\widetilde{\theta}_i$ is the unique element of $\mathscr{C}(\mathfrak{A}_0, m_0, \beta_i)$ which agrees with θ_i on this intersection. The irreducible representation ρ therefore contains both characters $\widetilde{\theta}_i$, which therefore intertwine, as required.

For the converse, we define two sets. First, write $e = e(F[\beta_i] \mid F)$, $f = f(F[\beta_i] \mid F)$. Let $\mathscr{C}$ be the union of the sets $\mathscr{C}(\mathfrak{A}, m, \gamma)$, where $[\mathfrak{A}, n, m, \gamma]$ ranges over all simple strata with $e(F[\gamma] \mid F) = e$, $f(F[\gamma] \mid F) = f$.

Note that, for such γ, our order $\mathfrak{A}_0$ is normalized by $F[\gamma]^\times$ and is maximal for this property. Moreover, each $[\mathfrak{A}_0, n_0, m_0, \gamma]$ is a simple stratum. If $[\mathfrak{A}_0, n_0, m_0, \delta]$ is simple and $e(F[\delta] \mid F) = e$, $f(F[\delta] \mid F) = f$, then it intertwines with one of the form $[\mathfrak{A}_0, n_0, m_0, \gamma]$ as above. We now define a set $\mathscr{C}_0$ as above, using $(\mathfrak{A}_0, n_0, m_0)$ in place of $(\mathfrak{A}, n, m)$.

The group $\mathbf{U}(\mathfrak{A})$ acts on $\mathscr{C}$ by conjugation; we write $\overline{\mathscr{C}}$ for the set of orbits here. Likewise define $\overline{\mathscr{C}}_0$. The first part of the proof shows that $\theta \mapsto \boldsymbol{\tau}\theta$ gives a well-defined map $\overline{\mathscr{C}} \to \overline{\mathscr{C}}_0$, and this is moreover surjective. However, by [BK3] (2.6), the sets $\overline{\mathscr{C}}$, $\overline{\mathscr{C}}_0$ are finite with the same cardinality. This map is therefore a bijection. In the context of the lemma, this says that the characters θ_i intertwine if and only if the $\widetilde{\theta}_i$ intertwine, as required. □

Let us now prove (8.7) in the case where V and V′ have the same F-dimension. By (8.8), we can assume that both orders $\mathfrak{A}$, $\mathfrak{A}'$ are maximal for the property of being normalized by $F[\beta_i]^\times$. In particular, $\mathfrak{A} \cong \mathfrak{A}'$ as $\mathfrak{o}_F$-orders. Further, we can choose an $F[\beta_1]$-isomorphism $V \cong V'$ inducing an isomorphism $\varphi : \mathrm{End}_F(V') \to \mathrm{End}_F(V)$ such that $\varphi(\mathfrak{A}') = \mathfrak{A}$. This isomorphism φ thus induces an embedding of $F[\beta_2]^\times$ in $\mathfrak{K}(\mathfrak{A})$ which, by (1.6), is $\mathbf{U}(\mathfrak{A})$-conjugate to the given embedding (from the $F[\beta_2]$-vector space structure of V). In terms of characters, we have $\Theta_1(\mathfrak{A}') = \Theta_1(\mathfrak{A}) \circ \varphi^{-1}$; indeed, composition with φ^{-1} is exactly the map $\boldsymbol{\tau}_{\mathfrak{A}, \mathfrak{A}', \beta_1}$ in this situation. On the other hand, $\Theta_2(\mathfrak{A}')$ is the "pull-back" of a $\mathbf{U}(\mathfrak{A})$-conjugate of $\Theta_2(\mathfrak{A})$, that is, a $\mathbf{U}(\mathfrak{A}')$-conjugate of $\Theta_2(\mathfrak{A}) \circ \varphi^{-1}$. Since the $\Theta_i(\mathfrak{A})$ intertwine by hypothesis, so do the $\Theta_i(\mathfrak{A}')$, as required.

To deal with the general case, we have to delve into another case of the definition of the transfer maps $\boldsymbol{\tau}$. To this end, we fix positive integers e, f, n and an integer m, $0 \leqslant m < n$. We choose an F-vector space V_0 of dimension ef and a principal order $\mathfrak{A}_0$ in $\mathrm{End}_F(V_0)$ with $e(\mathfrak{A}_0 \mid \mathfrak{o}_F) = e$. We write

$$\mathscr{C}_0 = \bigcup \mathscr{C}(\mathfrak{A}_0, m, \gamma),$$

for simple strata $[\mathfrak{A}_0, n, m, \gamma]$ in $\mathrm{End}_F(V_0)$ with $e(F[\gamma] \mid F) = e$, $f(F[\gamma] \mid F) = f$. We write $\overline{\mathscr{C}}_0$ for the set of $\mathbf{U}(\mathfrak{A}_0)$-conjugacy classes of simple characters represented by elements of $\mathscr{C}_0$.

Now let t be a positive integer, and let $V = V_0 \oplus \ldots \oplus V_0$ be a direct sum of t copies of V_0. Let $\mathscr{L}_0$ be the lattice chain in V_0 which defines $\mathfrak{A}_0$. We let $\mathscr{L}$ be the lattice chain $\{L \oplus \ldots \oplus L : L \in \mathscr{L}_0\}$ in V, and write $\mathfrak{A}$ for the hereditary order in $\mathrm{End}_F(V)$ defined by $\mathscr{L}$. Thus $\mathfrak{A}$ is a principal order and $e(\mathfrak{A} \mid \mathfrak{o}_F) = e$. A simple stratum $[\mathfrak{A}_0, n, m, \gamma]$ gives rise in the obvious way to a simple stratum $[\mathfrak{A}, n, m, \gamma]$, and $\mathfrak{A}$ is maximal for the property of being normalized by $F[\gamma]^\times$. Further, any simple stratum $[\mathfrak{A}, n, m, \delta]$, with $e(F[\delta] \mid F) = e, f(F[\delta] \mid F) = f$, is $\mathbf{U}(\mathfrak{A})$-conjugate to one of this form. Write

$$\mathscr{C} = \bigcup \mathscr{C}(\mathfrak{A}, m, \delta)$$

for simple strata $[\mathfrak{A}, n, m, \delta]$ satisfying $e(F[\delta] \mid F) = e, f(F[\delta] \mid F) = f$. Likewise write $\overline{\mathscr{C}}$ for the set of $\mathbf{U}(\mathfrak{A})$-conjugacy classes of simple characters represented by elements of $\mathscr{C}$. If we take $\theta \in \mathscr{C}_0$ and choose a simple stratum such that $\theta \in \mathscr{C}(\mathfrak{A}_0, m, \gamma)$, we get an element $\widetilde{\theta} = \boldsymbol{\tau}_{\mathfrak{A}_0, \mathfrak{A}, \gamma}\, \theta \in \mathscr{C}$.

The construction of $\boldsymbol{\tau}$ in this case is particularly straightforward. (The following is a paraphrase of the proof of [BK1] (3.6.14).) We identify $\mathrm{Aut}_F(V_0)^t$ with a Levi subgroup M of $\mathrm{Aut}_F(V)$. We choose an opposite pair (P, P^-) of parabolic subgroups of $\mathrm{Aut}_F(V)$ with Levi component M, and write U, U^- for their unipotent radicals. We get $\mathbf{U}(\mathfrak{A}) \cap M = \mathbf{U}(\mathfrak{A}_0)^t$. If $[\mathfrak{A}_0, n, m, \gamma]$ is a simple stratum as above, we get an Iwahori decomposition

$$H^{m+1}(\gamma, \mathfrak{A}) = H \cap U^- . H \cap M . H \cap U,$$

with the obvious abbreviation. Moreover, $H \cap M$ is just $H^{m+1}(\gamma, \mathfrak{A}_0)^t$. For $\theta \in \mathscr{C}(\mathfrak{A}_0, m, \gamma)$, the character $\boldsymbol{\tau}\theta$ agrees with θ on the diagonal blocks $H^{m+1}(\gamma, \mathfrak{A}_0)$ and is trivial on $H \cap U^-$, $H \cap U$. The character $\boldsymbol{\tau}\theta$ is therefore independent of the choice of γ such that $\theta \in \mathscr{C}(\mathfrak{A}_0, m, \gamma)$. Further, if $\theta_1, \theta_2 \in \mathscr{C}_0$ intertwine, we have $x \in \mathbf{U}(\mathfrak{A}_0)$ such that $\theta_2 = \theta_1^x$, whence $\boldsymbol{\tau}\theta_2 = \boldsymbol{\tau}(\theta_1^x)$ intertwines with $\boldsymbol{\tau}(\theta_1)^x$ (where we view $\mathbf{U}(\mathfrak{A}_0)$ as embedded in $\mathbf{U}(\mathfrak{A}) \cap M$ on the diagonal). In all, we have a well-defined map $\overline{\mathscr{C}}_0 \to \overline{\mathscr{C}}$, which is moreover surjective. By [BK3] (2.6), the sets $\overline{\mathscr{C}}_0$, $\overline{\mathscr{C}}$ are finite with the same cardinality, so this map is bijective. Invoking (1.6), we have proved:

(8.9) *Lemma. — Let $\mathfrak{A}_0$ be a principal order in* $\mathrm{End}_F(V_0)$. *For $i = 1, 2$, let $[\mathfrak{A}_0, n, m, \beta_i]$ be a simple stratum such that* $F[\beta_i]$ *is a maximal subfield of* $\mathrm{End}_F(V_0)$. *Let $\theta_i \in \mathscr{C}(\mathfrak{A}_0, m, \beta_i)$.*

Let $\mathfrak{A}$ be a principal order in $\mathrm{End}_F(V)$ *with $e(\mathfrak{A} \mid \mathfrak{o}_F) = e(\mathfrak{A}_0 \mid \mathfrak{o}_F)$. Let $\varphi_i : F[\beta_i] \to \mathrm{End}_F(V)$ be an F-embedding such that $\varphi_i(F[\beta_i]^\times)$ normalizes $\mathfrak{A}$.*

The characters $\boldsymbol{\tau}_{\mathfrak{A}_0, \mathfrak{A}, \beta_i}\, \theta_i$ then intertwine in $\mathrm{Aut}_F(V)$ *if and only if the characters θ_i intertwine in* $\mathrm{Aut}_F(V_0)$.

Now we can complete the proof of (8.7). We are given realizations $\Theta_i(\mathfrak{A}) \in \mathscr{C}(\mathfrak{A}, m, \beta_i)$ in some $\mathrm{End}_F(V)$ which intertwine. We are also given realizations $\Theta_i(\mathfrak{A}') \in \mathscr{C}(\mathfrak{A}', m', \beta_i)$ in some $\mathrm{End}_F(V')$. We have to show that the latter intertwine.

By (8.8), we can assume that the order $\mathfrak{A}$ is maximal for the property of being normalized by $F[\beta_i]^\times$. By (8.9), we can assume that $\dim_F(V) = [F[\beta_i] : F]$. We can apply the same reduction procedure to $\mathfrak{A}'$. This reduces us to the case where $\dim_F(V) = \dim_F(V')$, with which we dealt above. □

(8.10) *Corollary. — The relation* $\approx$, *defined by* (8.6), *is an equivalence relation on the set of ps-characters over* F.

An *endo-class* over F is then an equivalence class of ps-characters over F for the relation $\approx$. Of course, we can equally regard an endo-class as a set of simple characters over F.

There are several invariants attached to an endo-class $\boldsymbol{\Theta}$ over F.

(8.11) *Proposition. — Let* $\boldsymbol{\Theta}$ *be an endo-class of simple characters over* F, *and let* $\theta \in \mathscr{C}(\mathfrak{A}, m, \beta)$ *be a realization of some ps-character belonging to* $\boldsymbol{\Theta}$. *The following quantities depend only on* $\boldsymbol{\Theta}$, *and not on the choice of* θ:

$$\begin{aligned} n_F(\boldsymbol{\Theta}) &= n_F(\beta), \\ k_F(\boldsymbol{\Theta}) &= k_F(\beta), \\ e_F(\boldsymbol{\Theta}) &= e(F(\beta) \mid F), \\ f_F(\boldsymbol{\Theta}) &= f(F(\beta) \mid F), \\ \ell_F(\boldsymbol{\Theta}) &= \left[\frac{m}{e_\beta(\mathfrak{A})}\right]. \end{aligned}$$

Proof. — The first four assertions are given by (8.3), while the remaining one comes from the definition of realization. □

For a given endo-class $\boldsymbol{\Theta}$, consider the set of all simple pairs $[k, \beta]$ which support some ps-character $\Theta \in \boldsymbol{\Theta}$. The integer k is the invariant $\ell_F(\boldsymbol{\Theta})$. These simple pairs $[k, \beta]$ fall into finitely many equivalence classes. We sometimes use the notation $(\boldsymbol{\Theta}, k, \beta)$ to indicate that $\boldsymbol{\Theta}$ contains a ps-character supported by an element of the class (k, β) of simple pairs.

We write $\mathscr{E}\mathscr{C}(F)$ for the set of endo-classes of simple characters over F. As for simple pairs, the association $F \mapsto \mathscr{E}\mathscr{C}(F)$ is functorial in that an isomorphism $F \cong F'$ of local fields induces a canonical bijection $\mathscr{E}\mathscr{C}(F) \cong \mathscr{E}\mathscr{C}(F')$. In particular, the group of continuous automorphisms of F acts on $\mathscr{E}\mathscr{C}(F)$ in a natural way.

9. Tame lifting of endo-classes

In this section, we finally reach the main definition. We are given an endo-class $(\boldsymbol{\Theta}, k, \beta)$ over F and a finite tamely ramified field extension K/F. We have to define the K/F-lifts of $\boldsymbol{\Theta}$.

As in the parallel situation of § 3, we have to start with a specific representative. Therefore, we take a ps-character $[\Theta, k, \beta]$ where, of course, $k \geqslant 0$. Let $\beta_1, \beta_2, \ldots, \beta_r$ be the K/F-lifts of the field element β. Let l be some integer satisfying

(9.1) $$\left[\frac{l}{e_\beta(\mathrm{K} \mid \mathrm{F})}\right] = k.$$

For each i, $1 \leqslant i \leqslant r$, we choose a realization $[\mathfrak{C}_i, n_i, m_i, \beta_i]$ of the simple pair $[l, \beta_i]$ on a finite-dimensional K-vector space V_i. We write $\mathrm{C}_i = \mathrm{End}_\mathrm{K}(\mathrm{V}_i)$, $\mathrm{A}_i = \mathrm{End}_\mathrm{F}(\mathrm{V}_i)$, and we let $\mathfrak{A}_i$ denote the hereditary $\mathfrak{o}_\mathrm{F}$-order in A_i defined by the same lattice chain as $\mathfrak{C}_i$. The stratum $[\mathfrak{A}_i, n_i, m_i, \beta_i]$ is then simple, and is a realization of $[k, \beta]$. Let $\theta_i \in \mathscr{C}(\mathfrak{A}_i, m_i, \beta_i)$ be the corresponding realization of Θ. We put

(9.2) $$\theta_i^\mathrm{K} = \theta \mid \mathrm{H}^{m_i+1}(\beta_i, \mathfrak{C}_i),$$

using, in particular, (7.1) and (7.7). This character θ_i^K lies in $\mathscr{C}(\mathfrak{C}_i, m_i, \beta_i)$, and is a realization of a ps-character $[\Theta_i^\mathrm{K}, l, \beta_i]$ over K.

(9.3) *Proposition. — The ps-character* $[\Theta_i^\mathrm{K}, l, \beta_i]$ *depends only on* $[\Theta, k, \beta]$ *and the lift* β_i *of* β, *but not on the specific realization chosen in the definition.*

Proof. — Let $[\mathfrak{C}_i, n_i, m_i, \beta_i]$ be as above. Suppose we have another realization $[\mathfrak{C}_i', n_i', m_i', \beta_i]$ of $[l, \beta_i]$ in the same $\mathrm{End}_\mathrm{K}(\mathrm{V}_i)$. This gives us a realization $[\mathfrak{A}_i', n_i', m_i', \beta_i]$ of $[k, \beta]$. Let $\theta_i' \in \mathscr{C}(\mathfrak{A}_i', m_i', \beta_i)$ be the corresponding realization of Θ. By the definition of ps-character and [BK1] (3.6.1), the characters θ_i, θ_i' agree on

$$\mathrm{H}^{m_i+1}(\beta_i, \mathfrak{A}_i) \cap \mathrm{H}^{m_i'+1}(\beta_i, \mathfrak{A}_i').$$

They then certainly agree on $\mathrm{H}^{m_i+1}(\beta_i, \mathfrak{C}_i) \cap \mathrm{H}^{m_i'+1}(\beta_i, \mathfrak{C}_i')$, and hence define the same ps-character.

In the general case, suppose we have a second realization $[\mathfrak{C}_i', n_i', m_i', \beta_i]$ of $[l, \beta_i]$ in $\mathrm{End}_\mathrm{K}(\mathrm{V}_i')$, and otherwise use the notation of the previous paragraph. By the first part, we can assume that each of the orders $\mathfrak{C}_i$, $\mathfrak{C}_i'$ are maximal for the property of being normalized by $\mathrm{K}[\beta_i]^\times$. We compare these via the standard realization of $[l, \beta_i]$ n $\mathrm{End}_\mathrm{K}(\mathrm{K}[\beta_i])$. This implies $m_i = m_i' = l$. By transitivity, we may as well take $\mathrm{V}_i = \mathrm{K}[\beta_i]$. We then use an Iwahori decomposition argument, as in the proof of (8.7). We construct a Levi subgroup M_K of $\mathrm{Aut}_\mathrm{K}(\mathrm{V}_i')$ along with unipotent radicals U_K, U_K^- to give us an Iwahori decomposition

$$\mathrm{H}_\mathrm{K} = \mathrm{H}_\mathrm{K} \cap \mathrm{U}_\mathrm{K}^- . \mathrm{H}_\mathrm{K} \cap \mathrm{M}_\mathrm{K} . \mathrm{H}_\mathrm{K} \cap \mathrm{U}_\mathrm{K},$$
$$\mathrm{H}_\mathrm{K} \cap \mathrm{M}_\mathrm{K} = \mathrm{H}^{l+1}(\beta_i, \mathfrak{C}_i)^t,$$

where $\mathrm{H}_\mathrm{K} = \mathrm{H}^{l+1}(\beta_i, \mathfrak{C}_i')$ and $t \geqslant 1$. The groups M_K, U_K, U_K^- determine analogous subgroups M, U, U^- of $\mathrm{Aut}_\mathrm{F}(\mathrm{V}_i')$, and we get a corresponding Iwahori decomposition of $H^{l+1}(\beta_i, \mathfrak{A}'_i)$. In particular, $H^{l+1}(\beta_i, \mathfrak{A}'_i) \cap M$ is the product of t copies of $H^{l+1}(\beta_i, \mathfrak{A}_i)$. The result will follow if we show that the restriction of θ'_i to any of these " blocks " $H^{l+1}(\beta_i, \mathfrak{A}_i)$ is equal to θ_i. However, this follows from the construction of the transfer maps τ, as reviewed in the proof of (8.7). □

(9.4) *Definition. — Let* $[\Theta, k, \beta]$ *be a ps-character over* F, *and let* K/F *be a finite tamely ramified field extension. Let* $\beta_1, \beta_2, \ldots, \beta_r$ *be the set of* K/F*-lifts of the field element* β. *Let* l *be an integer satisfying* (9.1), *and let* $[\Theta_i^{K}, l, \beta_i]$ *be the ps-character over* K *defined by the character* θ_i^{K} *of* (9.2). *Then*

$$\{\,[\Theta_i^{K}, l, \beta_i] : 1 \leqslant i \leqslant r\,\}$$

is the set of (K/F, l)-lifts of the ps-character $[\Theta, k, \beta]$.

As for simple pairs, the integer l plays virtually no role beyond having the property (9.1): see (9.10) below. We will therefore drop it from the notation as soon as possible.

We also have a transitivity property.

(9.5) *Proposition. — Let* $[\Theta, k, \beta]$ *be a ps-character over* F *and let* K/F, L/K *be finite tamely ramified field extensions. Let* $\{\,\beta_i : 1 \leqslant i \leqslant r\,\}$ *be the set of* K/F*-lifts of* β *and let* $\{\,[\Theta_i^{K}, l, \beta_i]\,\}$ *be the* (K/F, l)*-lifts of* $[\Theta, k, \beta]$, *where* l *is some integer satisfying* $[l/e_\beta(K \mid F)] = k$.

Let $\{\,\beta_{ij} : 1 \leqslant j \leqslant r_i\,\}$ *be the set of* L/K*-lifts of* β_i, $1 \leqslant i \leqslant r$, *and let* $\{\,[\Theta_{ij}^{L}, m, \beta_{ij}]\,\}$ *be the* (L/K, m)*-lifts of* $[\Theta_i^{K}, l, \beta_i]$, *where* m *is some integer satisfying* $[m/e_{\beta_i}(L \mid K)] = l$.

We then have

$$\left[\frac{m}{e_\beta(L \mid F)}\right] = k$$

and the set

$$\{\,[\Theta_{ij}^{L}, m, \beta_{ij}] : 1 \leqslant j \leqslant r_i,\ 1 \leqslant i \leqslant r\,\}$$

is the set of (L/F, m)*-lifts of* $[\Theta, k, \beta]$.

Proof. — We note first the identity $e_\beta(L \mid F) = e_\beta(K \mid F)\, e_{\beta_i}(L \mid K)$, so the definition of m is certainly independent of i. Moreover, it satisfies the required condition. In the construction above of Θ_i^{K}, we could have taken $\mathfrak{C}_i$ to be the hereditary $\mathfrak{o}_K$-order defined by the unique $\mathfrak{o}_{L[\beta_{ij}]}$-lattice chain in $L[\beta_{ij}]$, and the result follows immediately. □

We can now state our main results.

(9.6) *Theorem. — Let* K/F *be a finite, tamely ramified field extension.*

(i) *Let* Θ *be a ps-character over* F, *supported by a simple pair* $[k, \beta]$. *Let*

$$l = (k+1)\, e_\beta(K \mid F) - 1,$$

and let $\{\,[\Theta_i^{K}, l, \beta_i] : 1 \leqslant i \leqslant r\,\}$ *be the set of* (K/F, l)*-lifts of* Θ. *We have* $\Theta_i^{K} \approx \Theta_j^{K}$ *if and only if* $i = j$.

(ii) *For* $j = 1, 2$, *let* Θ_j *be a ps-character over* F, *supported by the simple pair* $[k, \beta_j]$. *Suppose we have* $\Theta_1 \approx \Theta_2$, *so that, in particular,* $e_{\beta_1}(\mathrm{K} \mid \mathrm{F}) = e_{\beta_2}(\mathrm{K} \mid \mathrm{F})$. *Let* l *be the integer*

$$l = (k+1)\, e_{\beta_1}(\mathrm{K} \mid \mathrm{F}) - 1 = (k+1)\, e_{\beta_2}(\mathrm{K} \mid \mathrm{F}) - 1,$$

and let $\{ \Theta^{\mathrm{K}}_{ji} : 1 \leqslant i \leqslant r_j \}$ *be the set of* $(\mathrm{K}/\mathrm{F}, l)$*-lifts of* Θ_j, $j = 1, 2$. *Then we have* $r_1 = r_2$ *and, after renumbering,*

$$\Theta^{\mathrm{K}}_{1i} \approx \Theta^{\mathrm{K}}_{2i}, \quad 1 \leqslant i \leqslant r_1.$$

In other words, equivalent ps-characters give rise to equivalent sets of lifts and, moreover, distinct lifts of a given ps-character are inequivalent. The first enables us to extend the definition of lifting to endo-classes of simple characters. Formally:

(9.7) *Definition. — Let* $\boldsymbol{\Theta}$ *be an endo-class of simple characters over* F, *and let* $[\Theta, k, \beta]$ *be a ps-character representing* $\boldsymbol{\Theta}$. *Define ps-characters* $[\Theta^{\mathrm{K}}_i, l, \beta_i]$ *over* K *as in* (9.4), *and let* $\boldsymbol{\Theta}^{\mathrm{K}}_i$ *denote the endo-class of* $[\Theta^{\mathrm{K}}_i, l, \beta_i]$. *Then*

$$\{ \boldsymbol{\Theta}^{\mathrm{K}}_i : 1 \leqslant i \leqslant r \}$$

is the set of $(\mathrm{K}/\mathrm{F}, l)$-lifts of $\boldsymbol{\Theta}$.

As before, the lifting process is injective:

(9.8) *Theorem. — Let* K/F *be a finite, tamely ramified field extension and, for* $j = 1, 2$, *let* Θ_j *be a ps-character over* F *supported by a simple pair* $[k_j, \beta_j]$. *Suppose there exists an integer* M *such that*

$$\mathrm{M} + 1 \leqslant \min_j \{ (k_j + 1)\, e_{\beta_j}(\mathrm{K} \mid \mathrm{F}) \}$$

and $(\mathrm{K}/\mathrm{F}, \mathrm{M})$*-lifts* Θ^{K}_j *of the* Θ_j *such that* $\Theta^{\mathrm{K}}_1 \approx \Theta^{\mathrm{K}}_2$. *Then* $\Theta_1 \approx \Theta_2$.

Likewise, there is a surjectivity property:

(9.9) *Theorem. — Let* K/F *be a finite, tamely ramified field extension, and let* Υ *be a ps-character over* K *supported by a simple pair* $[l, \delta]$. *There exist a ps-character* Θ *over* F, *supported by a simple pair* $[k, \beta]$, *and a* $(\mathrm{K}/\mathrm{F}, l)$*-lift* Θ^{K} *of* Θ *such that*

$$k = \left[\frac{l}{e_\beta(\mathrm{K} \mid \mathrm{F})} \right],$$

and

$$\Theta^{\mathrm{K}} \approx \Upsilon.$$

Of these theorems, (9.9) follows from (7.10), while (9.8) follows from (7.15) and the injectivity statement of (7.10). We have therefore only to prove (9.6). The first point to note is that, generalizing (6.1), the dependence on the parameter l in the definition of lift is spurious.

(9.10) *Lemma.* — *For $i = 1, 2$, let $[\Theta_i, k, \beta_i]$ be a ps-character over* F. *Let $\widetilde{\beta}_i$ be a* K/F-*lift of β_i, and Θ_i^K the corresponding lift of Θ_i. Suppose that we have:*

a) $\Theta_1 \approx \Theta_2$ *(so that $e_{\beta_1}(K \mid F) = e_{\beta_2}(K \mid F)$)*;

b) $[\Theta_1^K, l, \widetilde{\beta}_1] \approx [\Theta_2^K, l, \widetilde{\beta}_2]$, *for some integer l such that $[l/e_{\beta_i}(K \mid F)] = k$.*

Let l_0 be the integer $ke_{\beta_i}(K \mid F)$. Then

$$[\Theta_1^K, l_0, \widetilde{\beta}_1] \approx [\Theta_2^K, l_0, \widetilde{\beta}_2].$$

Proof. — We may as well start in the worst case

$$l = (k+1)\, e_{\beta_i}(K \mid F) - 1.$$

Let q be an integer such that $l_0 \leqslant q \leqslant l$, and choose realizations $[\mathfrak{C}, n, m, \widetilde{\beta}_i]$ of the simple pairs $[q, \widetilde{\beta}_i]$ on some K-space V. We set $C = \mathrm{End}_K(V)$, $A = \mathrm{End}_F(V)$. The ps-character Θ_i^K then determines a unique character $\vartheta_i^q \in \mathscr{C}(\mathfrak{C}, m, \widetilde{\beta}_i)$. Now choose q minimal for the property that the ϑ_i^q intertwine (and assume that $q > l_0$, since otherwise there is nothing to prove). This property of the ϑ_i^q is independent of the choice of realizations. In particular, it is independent of the choice of m subject to $[m/e_{\widetilde{\beta}_i}(\mathfrak{C})] = q$, by (8.5). We can therefore take $m = qe_{\widetilde{\beta}_i}(\mathfrak{C})$. By the fact that intertwining implies conjugacy (7.18), we can therefore arrange $\mathscr{C}(\mathfrak{C}, m, \widetilde{\beta}_1) = \mathscr{C}(\mathfrak{C}, m, \widetilde{\beta}_2)$, and $\vartheta_1^q = \vartheta_2^q$.

Now let $\mathfrak{A}$ be the hereditary $\mathfrak{o}_F$-order defined by the same lattice chain as $\mathfrak{C}$. The stratum $[\mathfrak{A}, n, m, \widetilde{\beta}_i]$ is then simple, and is a realization of $[k, \beta_i]$. Let θ_i^q be the corresponding realization of Θ_i. By definition, we have

$$\theta_i^q \mid H^{m+1}(\widetilde{\beta}_i, \mathfrak{C}) = \vartheta_i^q,$$

and, by (7.15), (7.7), also $\theta_1^q = \theta_2^q$. Further, by hypothesis, the characters θ_i^{q-1} (of $H^m(\widetilde{\beta}_i, \mathfrak{A})$) intertwine. Let us now interpret this condition. There is an element $c \in \mathfrak{P}^{-m}$, where $\mathfrak{P}$ is the radical of $\mathfrak{A}$, such that $\theta_2^{q-1} = \theta_1^{q-1}\, \psi_c^A$. For suitable choices of additive characters, this gives us $\vartheta_2^{q-1} = \vartheta_1^{q-1}\, \psi_c^C$. We choose a tame corestriction s on A relative to $F[\widetilde{\beta}_1]/F$. As in (2.7), this restricts to a tame corestriction on C relative to $K[\widetilde{\beta}_1]/K$. Write $B = \mathrm{End}_{F[\widetilde{\beta}_1]}(V)$, $\mathfrak{B} = \mathfrak{A} \cap B$. By [BK3] (2.7), the stratum $[\mathfrak{B}, m, m-1, s(c)]$ is either null or equivalent to a simple stratum. Since the θ_i^{q-1} intertwine, it is in fact null, by [BK3] (2.8). It follows that the stratum $[\mathfrak{B} \cap \mathfrak{C}, m, m-1, s(c)]$ is null, whence the ϑ_i^{q-1} also intertwine.

This contradicts the definition of q, and so proves the Lemma. □

To prove (9.6), it is enough to treat the case where the extension K/F is Galois: the justification for this is formally identical to the corresponding argument in (4.1). We first prove (9.6) (ii). We take our equivalent ps-characters $[\Theta_i, k, \beta_i]$, and choose a K/F-lift of each β_i (which we continue to denote by β_i). Set $l = (k+1)\, e_{\beta_i}(K \mid F) - 1$. We take realizations $[\mathfrak{C}, n, l, \beta_i]$ of minimal dimension (which saves notation), and let $\vartheta_i \in \mathscr{C}(\mathfrak{C}, l, \beta_i)$ be the character defined by the lift Θ_i^K corresponding to our chosen lift of β_i. Now let $\mathfrak{A}$ be the $\mathfrak{o}_F$-order defined by the same lattice chain as $\mathfrak{C}$, and let x range over the group $\mathscr{N}_{\mathfrak{A}}(K)/\mathfrak{K}(\mathfrak{C})$. The simple pairs defined by the strata $[\mathfrak{C}, n, l, \beta_i^x]$

then range over the lifts of $[k, \beta_i]$, and the ps-characters defined by the ϑ_i^x range over the lifts of Θ_i. To prove (9.6) (ii), we therefore have to show:

(9.11) *In the situation above, there exists* $x \in \mathcal{N}_{\mathfrak{A}}(\mathrm{K})$ *such that the characters* ϑ_1, ϑ_2^x *intertwine in* $\mathrm{Aut}_{\mathrm{K}}(\mathrm{V})$.

We proceed by induction. If $l \geqslant [n/2]$, we can appeal to (6.7). We therefore assume otherwise. We choose a simple stratum $[\mathfrak{C}, n, l+1, \gamma]$, equivalent to $[\mathfrak{C}, n, l+1, \beta_1]$ and such that $[\mathfrak{A}, n, l+1, \gamma]$ is simple. Inductively, we can assume that the ϑ_i agree on $\mathrm{H}^{l+2}(\beta_i, \mathfrak{C}) = \mathrm{H}^{l+2}(\gamma, \mathfrak{C})$. This implies, by (7.10), that the θ_i agree on $\mathrm{H}^{l+2}(\gamma, \mathfrak{A})$. Choose $\theta \in \mathscr{C}(\mathfrak{A}, l, \gamma)$ agreeing with the θ_i on H^{l+2}, and write ϑ for the restriction of θ to $\mathrm{H}^{l+1}(\gamma, \mathfrak{C})$. We now choose a tame corestriction s on A relative to $\mathrm{F}[\gamma]/\mathrm{F}$, and write $\mathfrak{B} = \mathfrak{A} \cap \mathrm{End}_{\mathrm{F}[\gamma]}(\mathrm{V})$, $\mathfrak{D} = \mathfrak{C} \cap \mathrm{End}_{\mathrm{K}[\gamma]}(\mathrm{V})$.

Now we write $\theta_i = \theta\psi_{c_i}^{\mathrm{A}}$, in the usual way. As in the proof of (7.15), we can take $c_i \in \mathrm{C}$, so that $\vartheta_i = \vartheta\psi_{c_i}^{\mathrm{C}}$. By [BK3] (2.7), the stratum $[\mathfrak{D}, l+1, l, s(c_i)]$ is either null or equivalent to a simple stratum. We choose a simple (or null) stratum $[\mathfrak{D}, l+1, l, \delta_i]$ equivalent to $[\mathfrak{D}, l+1, l, s(c_i)]$ such that $[\mathfrak{B}, l+1, l, \delta_i]$ is simple (or null). Since the θ_i intertwine, the strata $[\mathfrak{B}, l+1, l, \delta_i]$ intertwine. Now, by (6.7), there exists $y \in \mathcal{N}_{\mathfrak{B}}(\mathrm{K}[\gamma])$ such that $[\mathfrak{D}, l+1, l, \delta_2^y]$ intertwines with $[\mathfrak{D}, l+1, l, \delta_1]$. It follows that ϑ_2^y intertwines with ϑ_1. This proves (9.11), and hence (9.6) (ii).

Let us now prove (9.6) (i). We take a ps-character $[\Theta, k, \beta]$ over F, and show that the various lifts of this are distinct (modulo endo-equivalence). Again, we assume that K/F is Galois. We fix a lift of β, which we continue to call β. We set $l = (k+1)\, e_\beta(\mathrm{K} \mid \mathrm{F}) - 1$, and choose a realization $[\mathfrak{C}, n, l, \beta]$ of $[l, \beta]$ of minimal dimension. Let $\mathfrak{A}$ be the hereditary $\mathfrak{o}_{\mathrm{F}}$-order defined by the same lattice chain as $\mathfrak{C}$, and let $\theta \in \mathscr{C}(\mathfrak{A}, l, \beta)$ be the character defined by Θ. Put $\vartheta = \theta \mid \mathrm{H}^{l+1}(\beta, \mathfrak{C})$. We have to produce r non-intertwining characters of the form ϑ^x, where r is the number of K/F-lifts of β and x ranges over the group $\mathcal{N}_{\mathfrak{A}}(\mathrm{K})$. If $l \geqslant [n/2]$, this is given by (3.5). The assertion follows immediately from the inductive hypothesis if the stratum $[\mathfrak{A}, n, l+1, \beta]$ is simple. We therefore assume the contrary, and choose a simple stratum $[\mathfrak{C}, n, l+1, \gamma]$, equivalent to $[\mathfrak{C}, n, l+1, \beta]$ and such that $[\mathfrak{A}, n, l+1, \gamma]$ is simple. We write $\mathrm{B} = \mathrm{End}_{\mathrm{F}[\gamma]}(\mathrm{V})$, $\mathfrak{B} = \mathfrak{A} \cap \mathrm{B}$, $\mathrm{D} = \mathrm{End}_{\mathrm{K}[\gamma]}(\mathrm{V})$, $\mathfrak{D} = \mathfrak{C} \cap \mathrm{D}$. We choose a tame corestriction s on A relative to $\mathrm{F}[\gamma]/\mathrm{F}$. We then choose a simple stratum $[\mathfrak{D}, l+1, l, \delta]$, equivalent to $[\mathfrak{D}, l+1, l, s(\beta-\gamma)]$, such that $[\mathfrak{B}, l+1, l, \delta]$ is simple. As in the proof of (6.2) (ii), we have $r = r_\gamma r_\delta$, where r_γ is the number of distinct K/F-lifts of γ and r_δ is the number of $\mathrm{K}[\gamma]/\mathrm{F}[\gamma]$-lifts of δ. Inductively, there are r_γ non-intertwining characters of the form $\vartheta^x \mid \mathrm{H}^{l+2}(\gamma, \mathfrak{C})$, $x \in \mathcal{N}_{\mathfrak{A}}(\mathrm{K})$. Fix one of these, which may as well be $\vartheta \mid \mathrm{H}^{l+2}(\gamma, \mathfrak{C})$. There exists $\vartheta_0 \in \mathscr{C}(\mathfrak{C}, l, \gamma)$, agreeing with ϑ on H^{l+2}, such that $\vartheta = \vartheta_0\, \psi_{\beta-\gamma}^{\mathrm{C}}$. Let $y_1, y_2 \in \mathcal{N}_{\mathfrak{B}}(\mathrm{K}[\gamma])$. We have $\vartheta_0^{y_i} = \vartheta_0$, and the characters ϑ^{y_i} intertwine if and only if the strata $[\mathfrak{D}, l+1, l, \delta^{y_i}]$ intertwine. However, there are precisely r_δ non-intertwining strata of this form, and the result follows. □

This technique is worth recording explicitly. The arguments above assume that we have realizations of minimal dimension. However, that hypothesis plays no serious role, and we deduce.

(9.12) *Corollary.* — *Let* K/F *be a finite, tamely ramified Galois extension. Let* V *be a finite-dimensional* K*-vector space, and* $\mathfrak{C}$ *a hereditary* $\mathfrak{o}_K$*-order in* $C = \mathrm{End}_K(V)$. *Let* $\mathfrak{A}$ *be the hereditary* $\mathfrak{o}_F$*-order in* $A = \mathrm{End}_F(V)$ *defined by the same lattice chain as* $\mathfrak{C}$. *For* $i = 1, 2$, *let* $[\mathfrak{A}, n, m, \beta_i]$ *be a* K*-pure simple stratum with* $m \geqslant 0$, *and let* $\theta_i \in \mathscr{C}(\mathfrak{A}, m, \beta_i)$. *Define* $\vartheta_i \in \mathscr{C}(\mathfrak{C}, m, \beta_i)$ *by* $\vartheta_i = \theta_i \mid H^{m+1}(\beta_i, \mathfrak{C})$.

Suppose that the characters θ_1, θ_2 *intertwine in* $\mathrm{Aut}_F(V)$. *There exists* $x \in \mathscr{N}_{\mathfrak{A}}(K)$ *such that the characters* ϑ_1, ϑ_2^x *intertwine in* $\mathrm{Aut}_K(V)$.

As in § 3, we can summarize our lifting theorems in terms of a certain base-field restriction or induction map.

(9.13) *Corollary.* — *Let* K/F *be a finite tamely ramified field extension. There exists a unique map*

$$\mathrm{Res}_{K/F} : \mathscr{E}\mathscr{C}(K) \to \mathscr{E}\mathscr{C}(F)$$

with the following property: for $\mathbf{Y} \in \mathscr{E}\mathscr{C}(K)$ *represented by a ps-character* $[\Upsilon, l, \beta]$, *the endo-class* $\mathrm{Res}_{K/F}(\mathbf{Y}) \in \mathscr{E}\mathscr{C}(F)$ *is represented by* $[\Theta, k, \alpha]$, *where* $[l/e_\alpha(K \mid F)] = k$ *and* $[\Upsilon, l, \beta]$ *is endo-equivalent to some* K/F*-lift of* $[\Theta, k, \alpha]$.

Moreover, we have:

(i) *the map* $\mathrm{Res}_{K/F}$ *is surjective;*

(ii) *for a ps-character* $[\Theta, k, \alpha]$ *over* F, *the fibre of* $\mathrm{Res}_{K/F}$ *above the endo-class of* Θ *consists of the endo-classes of* $[\Theta_i^K, l, \alpha_i]$, *where* α_i *ranges over the* K/F*-lifts of* α *and* l *over integers such that* $[l/e_\alpha(K \mid F)] = k$;

(iii) *if* L/K *is another finite tamely ramified field extension, we have*

$$\mathrm{Res}_{L/F} = \mathrm{Res}_{K/F} \circ \mathrm{Res}_{L/K}.$$

In (ii) here, we have used the notation of (9.4). Also, (iii) follows from (9.5).

If K/F is Galois, the group $\Gamma = \mathrm{Gal}(K/F)$ acts on $\mathscr{E}\mathscr{C}(K)$ and stabilizes the fibres of $\mathrm{Res}_{K/F}$. Indeed, if in (9.13) (ii) we choose a lift $\tilde{\alpha}$ of α and write $\tilde{\Theta}$ for the corresponding lift of Θ, the fibre above the endo-class of $[\Theta, k, \alpha]$ consists of the (mutually distinct) endo-classes

$$[\tilde{\Theta}^\sigma, l, \tilde{\alpha}^\sigma], \quad \sigma \in \Gamma, \quad ke_\alpha(K \mid F) \leqslant l < (k+1)\, e_\alpha(K \mid F).$$

In practice, we shall usually only be interested in the set $\mathscr{E}\mathscr{C}^0(F)$ of endo-classes of ps-characters of the form $[\Theta, 0, \alpha]$. We then get a surjective restriction map $\mathscr{E}\mathscr{C}^0(K) \to \mathscr{E}\mathscr{C}^0(F)$ whose fibres are parametrized just by lifts of elements.

10. Iwahori decomposition and parabolic norm

In preparation for the next couple of sections, we need to examine with some care various " Iwahori decompositions " of simple characters. We will later have to compare a simple character θ_F in $GL(N, F)$ with a related object in $GL(N, K)$ made from the various K/F-lifts of θ_F. We approach this via simple characters in $GL(Nd, F) \supset GL(N, K)$, where $d = [K : F]$, the connection being made through various " diagonal " embeddings of $GL(N, F)$ in $GL(Nd, F)$. Here we examine how these embeddings interact with the arithmetic structures attached to simple characters.

To start with, V denotes some finite-dimensional F-vector space, and we put $A = \mathrm{End}_F(V)$, $G = \mathrm{Aut}_F(V)$. Let M be some Levi subgroup of G. Thus we have a decomposition

$$V = V^1 \oplus V^2 \oplus \ldots \oplus V^r \tag{10.1}$$

of V as a direct sum of (let us assume) non-zero subspaces V^i and

$$M = \prod_{i=1}^{r} \mathrm{Aut}_F(V^i). \tag{10.2}$$

Let P be some parabolic subgroup of G with Levi factor M. Thus there is a permutation π of the set $\{1, 2, \ldots, r\}$ such that $P = P_\pi$ is the G-stabilizer of the flag

$$\{0\} \subset V^{\pi(1)} \subset V^{\pi(1)} \oplus V^{\pi(2)} \subset \ldots \subset \coprod_{1 \leqslant i \leqslant r} V^{\pi(i)} = V \tag{10.3}$$

of subspaces of V. We have $P = MU$, where U is the unipotent radical of P. If $P = P_\pi$ as in (10.3), then the opposite permutation $\pi^- : i \mapsto \pi(r-i)$ gives rise to the parabolic subgroup P^- *opposite* to P. Again, $P^- = MU^-$, where U^- is the unipotent radical of P^-, and $P \cap P^- = M$.

Let P be a parabolic subgroup of G with Levi component M, and let $\mathscr{G}$ be some subgroup of G. We say that $\mathscr{G}$ has *Iwahori decomposition relative to the pair* (P, M) if

$$\mathscr{G} = \mathscr{G} \cap U^- . \mathscr{G} \cap M . \mathscr{G} \cap U$$

and

$$\mathscr{G} \cap M = \prod_{i=1}^{r} \mathscr{G} \cap \mathrm{Aut}_F(V^i),$$

where M is given by (10.2). The cases of concern to us all arise in the following manner.

(10.4) *Proposition. — Let* M *be a Levi subgroup of* G *defined by* (10.2) *and let* X *be an* $\mathfrak{o}_F$*-lattice in* A *such that* $1 + X$ *is a subgroup of* G. *Write* e_i *for the projection* $V \to V$ *with image* V^i *and kernel* $\Sigma_{j \neq i} V^j$. *Suppose that*

$$e_i X e_j \subset X$$

for all pairs i, j. *The group* $1 + X$ *then has Iwahori decomposition relative to* (P, M), *for any parabolic subgroup* P *of* G *with Levi component* M.

Proof. — The assertion is independent of the choice of P, so we may as well assume that P is given by (10.3) and the permutation π there is the identity. We choose a basis of V, consisting of a union of bases of the V^i, ordered in the obvious way so that P is a group of upper triangular block matrices. The space $e_i \mathrm{A} e_j$ is just $\mathrm{Hom}_F(V^j, V^i)$, so, put another way,

$$\mathrm{P} = \mathrm{G} \cap (\Pi_{i \leqslant j}\, e_i \mathrm{A} e_j).$$

The hypothesis on X implies $\mathrm{X} \cap e_i \mathrm{A} e_j = e_i \mathrm{X} e_j$ which in turn implies

$$\mathrm{X} = \coprod_{i,j} e_i \mathrm{X} e_j.$$

In particular, $1_{V^i} + e_i \mathrm{X} e_i$ is a compact open subgroup of $\mathrm{Aut}_F(V^i)$ for each i, and

$$(1 + \mathrm{X}) \cap \mathrm{M} = \Pi(1 + e_i \mathrm{X} e_i).$$

By hypothesis we have $e_i \mathrm{X} e_j . e_j \mathrm{X} e_k \subset e_i \mathrm{X} e_k$ for all i, j, k. The assertion now follows readily using elementary row and column operations on matrices. □

Thus, in the situation of (10.4), the *existence* of an Iwahori decomposition depends only on the Levi subgroup M. However, the components of the decomposition of a given element $x \in 1 + \mathrm{X}$ do depend on the choice of parabolic subgroup P.

We continue for the moment in the situation and with the notation of (10.4), and suppose we are given an $\mathfrak{o}_F$-lattice chain $\mathscr{L} = \{ \mathrm{L}_j : j \in \mathbf{Z} \}$ in V. Let $\mathfrak{A} = \mathrm{End}^0_{\mathfrak{o}_F}(\mathscr{L})$ be the hereditary order defined by $\mathscr{L}$, and write $\mathfrak{P}$ for the radical of $\mathfrak{A}$. We say that M *conforms to* $\mathscr{L}$ if

(10.5) $$\mathrm{L}_k = \sum_i \mathrm{L}_k \cap \mathrm{V}^i, \quad k \in \mathbf{Z}.$$

This condition implies that $e_i \mathrm{L}_k = \mathrm{L}_k \cap \mathrm{V}^i \subset \mathrm{L}_k$ for all i and k, whence $e_i \in \mathfrak{A}$. Thus $e_i \mathfrak{A} e_j \subset \mathfrak{A}$ for all i, j, and $\mathfrak{A}$ is the direct sum of the $e_i \mathfrak{A} e_j$. Likewise, $e_i \mathfrak{P} e_j \subset \mathfrak{P}$. We can therefore invoke (10.4) to establish the first assertion of:

(10.6) *Proposition.* — *Let* $\mathfrak{A} = \mathrm{End}^0_{\mathfrak{o}_F}(\mathscr{L})$ *be a hereditary* $\mathfrak{o}_F$*-order in* $\mathrm{A} = \mathrm{End}_F(\mathrm{V})$ *with radical* $\mathfrak{P}$. *Let* M *be a Levi subgroup of* $\mathrm{G} = \mathrm{Aut}_F(\mathrm{V})$, *defined by* (10.2), *which conforms to* $\mathscr{L}$. *We have*:

(i) *The group* $\mathbf{U}^1(\mathfrak{A})$ *has Iwahori decomposition relative to the pair* (P, M), *for any parabolic subgroup* P *of* G *with Levi component* M.

(ii) $\mathfrak{A}_i = e_i \mathfrak{A} e_i$ *is the hereditary* $\mathfrak{o}_F$*-order in* $e_i \mathrm{A} e_i = \mathrm{End}_F(\mathrm{V}^i)$ *defined by the lattice chain* $\{ \mathrm{L}_j \cap \mathrm{V}^i : j \in \mathbf{Z} \}$. *The radical of* $\mathfrak{A}_i$ *is* $e_i \mathfrak{P} e_i$.

(iii) *We have* $\mathbf{U}^1(\mathfrak{A}) \cap \mathrm{Aut}_F(\mathrm{V}^i) = \mathbf{U}^1(\mathfrak{A}_i)$, $1 \leqslant i \leqslant r$.

Proof. — To prove (ii), we write $\mathscr{L}^i$ for the lattice chain $\{ \mathrm{L}_j \cap \mathrm{V}^i : j \in \mathbf{Z} \}$, and define $\mathfrak{A}_i = \mathrm{End}^0_{\mathfrak{o}_F}(\mathscr{L}^i)$. For any $x \in \mathfrak{A}$, we have $e_i x e_i \mathrm{L}_j \subset \mathrm{L}_j \cap \mathrm{V}^i$, so $e_i x e_i \in \mathfrak{A}_i$. On the other hand, if $y \in \mathfrak{A}_i$ (regarded as an element of A), we have

$y\mathrm{L}_j = ye_i\,\mathrm{L}_j = y(\mathrm{L}_j \cap \mathrm{V}^i) \subset \mathrm{L}_j \cap \mathrm{V}^i \subset \mathrm{L}_j$, so $y \in \mathfrak{A} \cap e_i\,\mathrm{A}e_i = e_i\,\mathfrak{A}e_i$. Thus $e_i\,\mathfrak{A}e_i = \mathfrak{A}_i$, as required. For the second assertion, we use a duality argument (cf. [BK1] (1.1)). We choose a continuous character ψ^{F} of the additive group of F with conductor $\mathfrak{p}_{\mathrm{F}}$, and set $\psi^{\mathrm{A}} = \psi^{\mathrm{F}} \circ \mathrm{tr}_{\mathrm{A/F}}$. For a subset S of A, we put $\mathrm{S}^* = \{ x \in \mathrm{A} : \psi^{\mathrm{A}}(x\mathrm{S}) = \{1\}\}$. We then have $\mathfrak{A}^* = \mathfrak{P}$. Further, $\psi^{\mathrm{A}} \mid e_i\,\mathrm{A}e_i = \psi^{e_i\mathrm{A}e_i}$, in the obvious notation. We write $\mathfrak{P}_i$ for the radical of $\mathfrak{A}_i$, and then $\mathfrak{P}_i = \{ x \in e_i\,\mathrm{A}e_i : \psi^{e_i\mathrm{A}e_i}(x\mathfrak{A}_i) = \{1\}\}$. The inclusion $\mathfrak{P} \cap \mathfrak{A}_i \subset \mathfrak{P}_i$ is now immediate. On the other hand, if $y \in \mathfrak{P}_i \subset e_i\,\mathrm{A}e_i$, then $\psi^{\mathrm{A}}(y\mathfrak{A}) = \psi^{\mathrm{A}}(ye_i\,\mathfrak{A}e_i) = \psi^{e_i\mathrm{A}e_i}(y\mathfrak{A}_i) = \{1\}$. This proves (ii), and (iii) now follows. □

We now consider a more delicate situation. With M, $\mathfrak{A}$, $\mathscr{L}$ as before, suppose we are actually given a subfield E/F of A such that $\mathrm{E}^\times \subset \mathfrak{K}(\mathfrak{A})$, i.e. such that $\mathscr{L}$ is an $\mathfrak{o}_{\mathrm{E}}$-lattice chain. We say that M *conforms to* $\mathscr{L}$ *over* E if it conforms to $\mathscr{L}$ in the sense of (10.5) and the subspaces V^i of (10.1) which define M are E-subspaces of V.

(10.7) *Proposition. — Let* $[\mathfrak{A}, n, 0, \beta]$ *be a simple stratum in* A, *and write* $\mathrm{E} = \mathrm{F}[\beta]$, $\mathfrak{A} = \mathrm{End}^0_{\mathfrak{o}_{\mathrm{F}}}(\mathscr{L})$. *Let* M *be a Levi subgroup of* G *which conforms to* $\mathscr{L}$ *over* E. *The groups* $\mathrm{H}^1(\beta, \mathfrak{A})$, $\mathrm{J}^1(\beta, \mathfrak{A})$ *then have Iwahori decomposition relative to* (P, M), *for any parabolic subgroup* P *of* G *with Levi component* M.

Proof. — Using our earlier notation, the hypotheses say that the projections e_i all lie in the hereditary $\mathfrak{o}_{\mathrm{E}}$-order $\mathfrak{B} = \mathfrak{A} \cap \mathrm{End}_{\mathrm{E}}(\mathrm{V})$. Since $\mathfrak{H}^1(\beta, \mathfrak{A})$, $\mathfrak{J}^1(\beta, \mathfrak{A})$ are $\mathfrak{B}$-bimodules [BK1] (3.1.9), we are in the situation of (10.4), and this gives the result. □

In the degree of generality of (10.7), it can be difficult to describe the diagonal factors $\mathrm{H}^1(\beta, \mathfrak{A}) \cap \mathrm{Aut}_{\mathrm{F}}(\mathrm{V}^i)$. For each i, there is an integer n_i such that $[\mathfrak{A}_i, n_i, 0, \beta]$ is a simple stratum in $\mathrm{End}_{\mathrm{F}}(\mathrm{V}^i)$, but it often happens that $\mathrm{H}^1(\beta, \mathfrak{A}) \cap \mathrm{Aut}_{\mathrm{F}}(\mathrm{V}^i) \neq \mathrm{H}^1(\beta, \mathfrak{A}_i)$, and likewise for J^1. We therefore make a definition:

(10.8) *Definition. — Let* $[\mathfrak{A}, n, 0, \beta]$ *be a simple stratum in* $\mathrm{A} = \mathrm{End}_{\mathrm{F}}(\mathrm{V})$, *and write* $\mathrm{E} = \mathrm{F}[\beta]$, $\mathfrak{A} = \mathrm{End}^0_{\mathfrak{o}_{\mathrm{F}}}(\mathscr{L})$. *Let* M *be a Levi subgroup of* G, *defined by* (10.2), *and use the notation* e_i *as above. We say that* M conforms to $[\mathfrak{A}, n, 0, \beta]$ *if:*

(i) M *conforms to the lattice chain* $\mathscr{L}$ *over* E;

(ii) *for all* i, *we have* $\mathrm{H}^1(\beta, \mathfrak{A}) \cap \mathrm{Aut}_{\mathrm{F}}(\mathrm{V}^i) = \mathrm{H}^1(\beta, \mathfrak{A}_i)$ (*in the notation of* (10.6)) *and likewise for* J^1;

(iii) *for any* $\theta \in \mathscr{C}(\mathfrak{A}, 0, \beta)$, *the restriction* $\theta \mid \mathrm{H}^1(\beta, \mathfrak{A}) \cap \mathrm{U}$ *is null, where* U *is the unipotent radical of any parabolic subgroup* P *of* G *with Levi component* M.

There are two basic examples of the phenomenon (10.8).

(10.9) *Example. — Let* $[\mathfrak{A}, n, 0, \beta]$ *be a simple stratum in* A, *with* $\mathrm{E} = \mathrm{F}[\beta]$, *and let* $\mathscr{L} = \{ \mathrm{L}_j : j \in \mathbf{Z} \}$ *be the lattice chain which defines* $\mathfrak{A}$. *Let* M *be a Levi subgroup of* G *as in* (10.2), *and suppose it conforms to the lattice chain* $\mathscr{L}$ *over* E. *Suppose also that all of the lattice chains* $\mathscr{L}$ *and* $\{ \mathrm{L}_j \cap \mathrm{V}^i : j \in \mathbf{Z} \}$ *have the same period. Then* M *conforms to* $[\mathfrak{A}, n, 0, \beta]$.

At the other extreme we have:

(10.10) *Example.* — *Let* $[\mathfrak{A}, n, 0, \beta]$ *be a simple stratum in* A, *with* $E = F[\beta]$, *and let* $\mathscr{L} = \{ L_j : j \in \mathbf{Z} \}$ *be the lattice chain which defines* $\mathfrak{A}$. *Let* M *be a Levi subgroup of* G *as in* (10.2), *and suppose it conforms to the lattice chain* $\mathscr{L}$ *over* E. *Write* $L_j^i = L_j \cap V^i$. *Suppose that*

(i) r *divides the integer* $e(\mathfrak{A} \mid \mathfrak{o}_F)/e(E \mid F)$, *and*
(ii) *we may renumber the* V^i *so that*

$$L^i_{i+mr} = L^i_{i+mr+1} = \ldots = L^i_{i+(m+1)r-1} \neq L^i_{i+(m+1)r},$$

for $1 \leqslant i \leqslant r$ *and all* $m \in \mathbf{Z}$.

Then M *conforms to* $[\mathfrak{A}, n, 0, \beta]$.

This case is given by [BK1] (7.1.14). Note that, to verify (10.8) (iii), it is enough to check that θ is null on $1 + e_i \mathfrak{H}^1(\beta, \mathfrak{A})\, e_j$ whenever $i \neq j$, and this is certainly implied by [ibid.]. The proof of (10.9) is parallel, but rather easier so we omit the details. Note that we used a special case of (10.9) in the proof of (8.9).

In both cases, write $\mathfrak{A}_i = \mathrm{End}^0_{\mathfrak{o}_F}(\{ L_j \cap V^i \})$. If $\theta \in \mathscr{C}(\mathfrak{A}, 0, \beta)$, and we write $\theta^i = \theta \mid H^1(\beta, \mathfrak{A}_i)$, then in both cases (10.9), (10.10), we have $\theta^i \in \mathscr{C}(\mathfrak{A}_i, 0, \beta)$ and $\theta \mapsto \theta^i$ is the canonical bijection

$$\tau_{\mathfrak{A}, \mathfrak{A}_i, \beta} : \mathscr{C}(\mathfrak{A}, 0, \beta) \to \mathscr{C}(\mathfrak{A}_i, 0, \beta)$$

discussed in § 8. In the case (10.10), this is proved in [BK1] (7.1.19). The case (10.9) is parallel. It is worth observing that in the second case, we get an Iwahori decomposition for the group $J(\beta, \mathfrak{A})$, while this does not usually hold in the first example.

We need to combine these examples, as follows. It will also be convenient to introduce a little extra generality.

(10.11) *Proposition.* — *Let* $[\mathfrak{A}, n, 0, \beta]$ *be a simple stratum in* A, *and* M *a Levi subgroup of* G. *Let* $E/F[\beta]$ *be a field extension in* A *with* $E^\times \subset \mathfrak{K}(\mathfrak{A})$. *Let* $\mathscr{L}$ *be the lattice chain in* V *which defines* $\mathfrak{A}$. *Suppose that* M *is given by* (10.2), *that* M *conforms to* $\mathscr{L}$ *over* E, *and that each space* V^i *has* E*-dimension* 1. *Then* M *conforms to the simple stratum* $[\mathfrak{A}, n, 0, \beta]$.

Proof. — We first observe that, since M conforms to $\mathscr{L}$ over E, it must also conform to $\mathscr{L}$ over $F[\beta]$, so (10.8) (i) is satisfied.

We write $\mathscr{L} = \{ L_j : j \in \mathbf{Z} \}$ as usual. We treat first the case in which the lattice chain $\mathscr{L}$ has $\mathfrak{o}_E$-period 1. For each i, the chain $\{ L_j \cap V^i : j \in \mathbf{Z} \}$ then also has $\mathfrak{o}_E$-period 1, and the result follows from (10.9).

We now reduce to this case. For each i, we choose $v_i \in V$ such that $L_0 \cap V^i = \mathfrak{o}_E\, v_i$. Then, after a suitable renumbering, the set $\{ v_i : 1 \leqslant i \leqslant r \}$ is an $\mathfrak{o}_E$-basis of the lattice chain $\mathscr{L}$. More precisely, we have integers $1 \leqslant d(1) < d(2) < \ldots < d(e) = r$, where e denotes the $\mathfrak{o}_E$-period of $\mathscr{L}$, such that

$$L_0 = \sum_{i=1}^{r} \mathfrak{o}_E v_i,$$

$$L_j = \sum_{1 \leqslant i \leqslant d(j)} \mathfrak{p}_E v_i \oplus \sum_{i > d(j)} \mathfrak{o}_E v_i, \quad 1 \leqslant j \leqslant e-1.$$

We now let W^k denote the E-linear span of $v_{d(k-1)+1}, \ldots, v_{d(k)}$, $1 \leqslant k \leqslant e$ (with the convention $d(0) = 0$). This set of subspaces defines a Levi subgroup $M^0 \supset M$ of G, which conforms to $\mathscr{L}$ over E, and it conforms to $[\mathfrak{A}, n, 0, \beta]$ by (10.10). We can now work in the "blocks" $\mathrm{Aut}_F(W^k)$: set $\mathfrak{A}(k) = \mathfrak{A} \cap \mathrm{End}_F(W^k)$, $M(k) = M \cap \mathrm{Aut}_F(W^k)$. Thus $[\mathfrak{A}(k), n/r, 0, \beta]$ is a simple stratum in $\mathrm{End}_F(W^k)$ and $M(k)$ is a Levi subgroup of $\mathrm{Aut}_F(W^k)$ which conforms over E to the lattice chain $\{L_j \cap W^k\}$ defining $\mathfrak{A}(k)$. By (10.10), we have $H^1(\beta, \mathfrak{A}) \cap \mathrm{Aut}_F(W^k) = H^1(\beta, \mathfrak{A}(k))$, and likewise for J^1, so the result will follow if we prove that $M(k)$ conforms to the stratum $[\mathfrak{A}(k), n/r, 0, \beta]$. However, the lattice chain defining $\mathfrak{A}(k)$ has $\mathfrak{o}_E$-period 1, and we are in the first case. □

Remark. — One can obtain a Levi subgroup M as in (10.11) by taking an $\mathfrak{o}_E$-basis $\{w_i\}$ of $\mathscr{L}$ and setting $M = \prod_i \mathrm{Aut}_F(Ew_i)$. In fact, all M satisfying the hypotheses of (10.11) are of this form.

We continue in the situation of (10.11), and write $\mathfrak{A}_i = \mathfrak{A} \cap \mathrm{Aut}_F(V^i)$, as before. Thus, for $\theta \in \mathscr{C}(\mathfrak{A}, 0, \beta)$, the character $\theta^i = \theta \mid H^1(\beta, \mathfrak{A}_i)$ lies in $\mathscr{C}(\mathfrak{A}_i, 0, \beta)$ and

(10.12) *The map $\theta \mapsto \theta^i$ is the bijection $\tau_{\mathfrak{A}, \mathfrak{A}_i, \beta} : \mathscr{C}(\mathfrak{A}, 0, \beta) \to \mathscr{C}(\mathfrak{A}_i, 0, \beta)$.*

This follows from (8.1), the proof of (10.11) and the corresponding property in the cases (10.9), (10.10).

If we choose a parabolic subgroup $P = MU$ of G with Levi component M, an element $x \in H^1(\beta, \mathfrak{A})$ can be written in the form

(10.13) $$x = y.(x_1, \ldots, x_r).z,$$

with $y \in H^1(\beta, \mathfrak{A}) \cap U^-$, $z \in H^1(\beta, \mathfrak{A}) \cap U$, and $x_i \in H^1(\beta, \mathfrak{A}_i)$. We then have

(10.14) $$\theta(x) = \prod_{i=1}^{r} \theta^i(x_i).$$

We shall need to view (10.14) as a relation between elements. We start with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A = \mathrm{End}_F(V)$. We let $E/F[\beta]$ be a finite field extension with $E^\times \subset \mathfrak{K}(\mathfrak{A})$ and put

$$B = \mathrm{End}_E(V),$$

$$A(E) = \mathrm{End}_F(E),$$

$$\mathfrak{A}(E) = \mathrm{End}^0_{\mathfrak{o}_F}(\{\mathfrak{p}_E^j\}).$$

Let $\mathscr{B} = \{ w_1, \ldots, w_r \}$ be some E-basis of V, and write W for the F-linear span of $\mathscr{B}$. Thus the natural map $E \otimes_F W \to V$ is an isomorphism, and it induces an isomorphism $A(E) \otimes_F \mathrm{End}_F(W) \cong A$ of F-algebras. We regard this isomorphism as identifying A with the ring of $r \times r$ matrices over A(E). In particular, it induces an embedding

$$\iota_{\mathscr{B}} : A(E) \to A,$$

which in matrix terms is the obvious diagonal one. Note that the isomorphism $A(E) \otimes \mathrm{End}_F(W) \cong A$ and the embedding $\iota_{\mathscr{B}}$ actually only depend on the F-space W rather than the chosen basis $\mathscr{B}$.

(10.15) *Proposition. — Let* $[\mathfrak{A}, n, 0, \beta]$ *be a simple stratum in* $A = \mathrm{End}_F(V)$. *Let* $E/F[\beta]$ *be a field extension with* $E^\times \subset \mathfrak{K}(\mathfrak{A})$, *and use the other notation above. Let* $\mathscr{B} = \{ w_1, \ldots, w_r \}$ *be an* $\mathfrak{o}_E$*-basis of the lattice chain defining* $\mathfrak{A}$ *(up to ordering), and use this to identify* A *with* $\mathbf{M}(r, A(E))$ *as above. Let* M *be the Levi subgroup*

$$M = M_{\mathscr{B}} = \prod_{i=1}^{r} \mathrm{Aut}_F(Ew_i),$$

i.e. M *is the group of invertible diagonal matrices over* A(E). *The group* M *then conforms to the stratum* $[\mathfrak{A}, n, 0, \beta]$, *and* $M \cap H^1(\beta, \mathfrak{A})$ *is the group of diagonal matrices with entries in* $H^1(\beta, \mathfrak{A}(E))$. *Moreover,*

$$H^1(\beta, \mathfrak{A}) \cap \iota_{\mathscr{B}}(A(E)) = \iota_{\mathscr{B}}(H^1(\beta, \mathfrak{A}(E))).$$

The analogous properties hold for $J^1(\beta, \mathfrak{A})$.

Proof. — As above, let W denote the F-linear span of $\mathfrak{B}$. We thus have $E \otimes_F \mathrm{End}_F(W) = B = \mathrm{End}_E(V)$, and so we have canonical isomorphisms

$$\begin{aligned} A \cong A(E) \otimes_F \mathrm{End}_F(W) &\cong (A(E) \otimes_E E) \otimes_F \mathrm{End}_F(W) \\ &\cong A(E) \otimes_E (E \otimes_F \mathrm{End}_F(W)) \cong A(E) \otimes_E B, \end{aligned}$$

giving a canonical isomorphism $A \cong A(E) \otimes_E B$ *of* (A(E), B)*-bimodules.* (This is a " (W, E)-decomposition " in the sense of [BK1] (1.2).) However, when as here $\mathscr{B}$ is an $\mathfrak{o}_E$-basis of the chain defining $\mathfrak{A}$, this isomorphism restricts to an isomorphism $\mathfrak{A}(E) \otimes_{\mathfrak{o}_E} \mathfrak{B} \cong \mathfrak{A}$, where $\mathfrak{B}$ is the hereditary $\mathfrak{o}_E$-order $\mathfrak{A} \cap B$ (see [BK1] (1.2.8)). In matrix terms, $\mathfrak{A}$ becomes identified with an order of block matrices in $A = \mathbf{M}(r, A(E))$, in which the blocks below the diagonal have entries in $\mathfrak{P}(E) = \mathrm{rad}(\mathfrak{A}(E))$ while the others have entries in $\mathfrak{A}(E)$. In particular, $\mathrm{End}_F(Ew_i) \cap \mathfrak{A} = \mathfrak{A}(E)$ in these identifications. In the terms of (10.6), we thus have $\mathfrak{A}_i = \mathfrak{A}(E)$ for all i, and the first assertion follows from (10.11). The group $\iota_{\mathscr{B}}(H^1(\beta, \mathfrak{A}(E))$ is the group of " scalar " matrices $\mathrm{diag}(x, x, \ldots, x)$, $x \in H^1(\beta, \mathfrak{A}(E))$, and the second assertion again follows from (10.11). □

Remark. — It needs to be remembered that a basis of a lattice chain, for example $\mathfrak{B}$ in (10.15) above, is not just a set: its elements have to be ordered in a certain way.

The full definition is [BK1] (1.1.7), and it is effectively repeated in the proof of (10.11) above. For many purposes, including those of (10.15), this ordering is irrelevant.

(10.16) *Corollary. — In the situation of* (10.15), *let* $\theta \in \mathscr{C}(\mathfrak{A}, 0, \beta)$. *There is a unique character* $\theta_F \in \mathscr{C}(\mathfrak{A}(E), 0, \beta)$ *such that*

$$\theta \mid H^1(\beta, \mathfrak{A}) \cap M = \theta_F \otimes \theta_F \otimes \ldots \otimes \theta_F.$$

We continue with the notation of (10.15). To recapitulate, we have a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $\mathrm{End}_F(V)$, a finite field extension $E/F[\beta]$ with $E^\times \subset \mathfrak{K}(\mathfrak{A})$, and an $\mathfrak{o}_E$-basis $\mathscr{B} = \{ w_1, w_2, \ldots, w_r \}$ of the lattice chain $\mathscr{L}$ defining $\mathfrak{A}$. (For the moment, we need to assume that $\mathscr{B}$ is ordered properly.) We set $V^i = Ew_i$, and define a Levi subgroup $M_{\mathscr{B}}$ of $G = \mathrm{Aut}_F(V)$ by

$$M_{\mathscr{B}} = \prod_{i=1}^{r} \mathrm{Aut}_F(V^i).$$

For definiteness, let $P_{\mathscr{B}}$ be the parabolic subgroup of G with Levi component $M_{\mathscr{B}}$ defined by the flag $\{ \Sigma_{1 \leqslant i \leqslant j} V^i : 1 \leqslant j \leqslant r \}$. Write $U_{\mathscr{B}}$ for the unipotent radical of $P_{\mathscr{B}}$ and $P_{\mathscr{B}}^- = M_{\mathscr{B}} U_{\mathscr{B}}^-$ for the opposite of $P_{\mathscr{B}}$. For $x \in H^1(\beta, \mathfrak{A})$, we have a unique expression

$$x = y . (x_1, x_2, \ldots, x_r) . z,$$

with $y \in U_{\mathscr{B}}$, $z \in U_{\mathscr{B}}^-$ and $x_i \in H^1(\beta, \mathfrak{A}(E)) \subset \mathrm{Aut}_F(V^i)$. We define

$$\mathscr{N}_{\mathscr{B}}(x) = x_1 x_2 \ldots x_r \in H^1(\beta, \mathfrak{A}(E)). \tag{10.17}$$

We refer to the map $\mathscr{N}_{\mathscr{B}}$ as the *parabolic norm* defined by the basis $\mathfrak{B}$. We shall usually only be interested in the composite $\theta_F \circ \mathscr{N}_{\mathscr{B}}$, where (10.16) gives us the relation

$$\theta(x) = \theta_F(\mathscr{N}_{\mathscr{B}}(x)), \quad x \in H^1(\beta, \mathfrak{A}). \tag{10.18}$$

Immediately, we have the property

(10.19) *Proposition. — In the situation above, the map* $x \mapsto \mathscr{N}_{\mathscr{B}}(x)$ *induces an isomorphism of abelian groups*

$$\frac{H^1(\beta, \mathfrak{A})}{\mathrm{Ker}(\theta)} \cong \frac{H^1(\beta, \mathfrak{A}(E))}{\mathrm{Ker}(\theta_F)}.$$

It should be noted that the parabolic subgroup $P_{\mathscr{B}}$ plays little effective role in the definition. We could replace it by any parabolic with Levi component $M_{\mathscr{B}}$, and define a norm $\mathscr{N}_{\mathscr{B}}^P$. We would still have the relation $\theta(x) = \theta_F(\mathscr{N}_{\mathscr{B}}^P(x))$. Thus the ordering of the chain basis $\mathscr{B}$ is again irrelevant.

In practice, we will sometimes have to deal with a slight generalization of (10.15).

(10.20) *Proposition.* — *Let* $\mathscr{B}' = \{v_1, \ldots, v_r\}$ *be an* E-*basis of* V, *and assume there exist elements* $y_1, \ldots, y_r \in E^\times$ *such that* $\mathscr{B} = \{y_1 x_1, \ldots, y_r x_r\}$ *is an* $\mathfrak{o}_E$-*basis of the lattice chain defining* $\mathfrak{A}$, *up to order. We then have*

$$H^1(\beta, \mathfrak{A}) \cap \iota_{\mathscr{B}'}(A(E)) = \iota_{\mathscr{B}'}(H^1(\beta, \mathfrak{A}(E))).$$

Moreover, if $\theta \in \mathscr{C}(\mathfrak{A}, 0, \beta)$, *we have*

$$\theta(\iota_{\mathscr{B}'}(x)) = \theta(\iota_{\mathscr{B}}(x)), \quad x \in H^1(\beta, \mathfrak{A}(E)). \tag{10.21}$$

Proof. — The embedding $\iota_{\mathscr{B}'}$ is the composite of $\iota_{\mathscr{B}}$ with conjugation by the diagonal matrix $y = \operatorname{diag}(y_1, \ldots, y_r)$. This conjugation stabilizes the group $M_{\mathscr{B}} \cap H^1(\beta, \mathfrak{A})$, which gives the first assertion. It also fixes the character $\theta \mid M_{\mathscr{B}} \cap H^1(\beta, \mathfrak{A})$, since θ is intertwined by every element of $\operatorname{Aut}_E(V)$. This implies (10.21). □

11. The semisimple lift

We now take a simple stratum of the form $[\mathfrak{A}(E), n, 0, \beta]$, where E is a finite extension of the field $F[\beta]$ and $\mathfrak{A}(E)$ denotes the order $\operatorname{End}^0_{\mathfrak{o}_F}(\{\mathfrak{p}_E^j\})$, as usual. The case $E = F[\beta]$ is the most significant, but the extra generality costs nothing and is expected to be useful elsewhere. We fix a simple character $\theta_F \in \mathscr{C}(\mathfrak{A}(E), 0, \beta)$.

Let K/F be a finite, tamely ramified field extension. In this section, we construct a compact open subgroup $\mathbf{H}^1_K$ of $\operatorname{Aut}_K(K \otimes_F E)$ and a character $\boldsymbol{\theta}_K$ of $\mathbf{H}^1_K$. The pair $(\mathbf{H}^1_K, \boldsymbol{\theta}_K)$ will effectively incorporate simultaneous realizations of the various K/F-lifts of (the endoclass of) the simple character θ_F. It will, in certain circumstances, enable us later to give a direct connection between the irreducible representations of $\operatorname{Aut}_F(E)$ containing θ_F and the irreducible representations of $\operatorname{Aut}_F(K \otimes E)$ containing $\boldsymbol{\theta}_K$.

Parallel to the standard case, there is another compact open subgroup $\mathbf{J}^1_K \supset \mathbf{H}^1_K$. This carries a unique irreducible representation whose restriction to $\mathbf{H}^1_K$ contains $\boldsymbol{\theta}_K$. Many of the statements of this section come in two versions, one for **H** and one for **J**· The proofs are usually identical, so we only treat the first case (and sometimes forget to say so).

Let $\mathfrak{D}_{K \otimes E}$, or just $\mathfrak{D}$, denote the unique maximal $\mathfrak{o}_F$-order in $K \otimes_F E$, and write $\mathfrak{r}$ for its Jacobson radical. Explicitly, $\mathfrak{r} = \prod_i \mathfrak{p}_{E_i}$, where E_i runs over the field factors of $K \otimes_F E$. This gives us an $\mathfrak{o}_F$-lattice chain

$$\mathscr{L} = \mathscr{L}_{K \otimes E} = \{\mathfrak{r}^j : j \in \mathbf{Z}\}$$

in $K \otimes E$ which is also, of course, a lattice chain over $\mathfrak{o}_E$ and $\mathfrak{o}_K$. We set $\mathfrak{A}_M = \operatorname{End}^0_{\mathfrak{o}_F}(\mathscr{L})$, and write $\mathfrak{P}_M$ for the radical of $\mathfrak{A}_M$. The period of $\mathfrak{A}_M$ is given by $e(\mathfrak{A}_M \mid \mathfrak{o}_F) = e(E_i \mid F)$, where E_i is any of the field factors of $K \otimes_F E$. We have a simple stratum $[\mathfrak{A}_M, n_M, 0, \beta]$

in $\mathrm{End}_F(K \otimes E)$, for a certain integer n_M. We let $\Theta_M \in \mathscr{C}(\mathfrak{A}_M, 0, \beta)$ be the realization on $\mathfrak{A}_M$ of the ps-character defined by the pair (θ_F, β), as in § 8.

We now write:

$$(11.1)\qquad \begin{cases} C = C(K \otimes E) = \mathrm{End}_K(K \otimes E), \\ \mathfrak{C} = \mathfrak{C}(K \otimes E) = \mathfrak{A}_M \cap C, \\ G(F) = \mathrm{Aut}_F(E), \\ G(K) = \mathrm{Aut}_K(K \otimes E), \\ \mathfrak{H}_K^1 = \mathfrak{H}^1(\beta, \mathfrak{A}_M) \cap C, \\ \mathbf{H}_K^1 = H^1(\beta, \mathfrak{A}_M) \cap G(K), \\ \mathfrak{J}_K^1 = \mathfrak{J}^1(\beta, \mathfrak{A}_M) \cap C, \\ \mathbf{J}_K^1 = J^1(\beta, \mathfrak{A}_M) \cap G(K), \\ \boldsymbol{\theta}_K = \Theta_M \mid \mathbf{H}_K^1. \end{cases}$$

This notation will be used throughout. We also put

$$K \otimes_F E = \prod_{i=1}^{r} E_i,$$

in our standard way, where the E_i are fields. This is the unique decomposition of $K \otimes E$ as a direct sum of minimal ideals. Each E_i is, in particular, both a K-subspace and an E-subspace of $K \otimes E$. We have $\mathfrak{D} \cap E_i = \mathfrak{o}_{E_i}$, and more generally, $\mathfrak{r}^m \cap E_i = \mathfrak{p}_{E_i}^m$, $m \in \mathbf{Z}$. Further,

$$\mathfrak{r}^m = \prod_i \mathfrak{p}_{E_i}^m.$$

Choosing the numbering at random, the flag

$$E_1 \subset E_1 \oplus E_2 \subset \ldots \subset K \otimes E$$

defines a parabolic subgroup P_M of $\mathrm{Aut}_F(K \otimes E)$. We write U_M for its unipotent radical and

$$M_M = \prod \mathrm{Aut}_F(E_i)$$

for the obvious Levi factor. As usual, $P_M^- = M_M U_M^-$ denotes the opposite of P_M.

(11.2) *Proposition.* — (i) *The groups* $H^1(\beta, \mathfrak{A}_M)$, $J^1(\beta, \mathfrak{A}_M)$ *have Iwahori decomposition relative to* (M_M, P_M), *i.e.*,

$$H^1(\beta, \mathfrak{A}_M) = (H^1(\beta, \mathfrak{A}_M) \cap U_M^-) . (H^1(\beta, \mathfrak{A}_M) \cap M_M) . (H^1(\beta, \mathfrak{A}_M) \cap U_M),$$

$$J^1(\beta, \mathfrak{A}_M) = (J^1(\beta, \mathfrak{A}_M) \cap U_M^-) . (J^1(\beta, \mathfrak{A}_M) \cap M_M) . (J^1(\beta, \mathfrak{A}_M) \cap U_M).$$

Moreover, we have

$$\mathrm{H}^1(\beta, \mathfrak{A}_{\mathrm{M}}) \cap \mathrm{End}_{\mathrm{F}}(\mathrm{E}_i) = \mathrm{H}^1(\beta, \mathfrak{A}_i),$$

where $\mathfrak{A}_i = \mathfrak{A}(\mathrm{E}_i) = \mathrm{End}^0_{\mathfrak{o}_{\mathrm{F}}}(\{\mathfrak{p}^j_{\mathrm{E}_i}\})$. *A similar property holds for* $\mathrm{J}^1(\beta, \mathfrak{A}_{\mathrm{M}})$.

(ii) *The characters* $\Theta_{\mathrm{M}} \mid \mathrm{H}^1(\beta, \mathfrak{A}_{\mathrm{M}}) \cap \mathrm{U}^-_{\mathrm{M}}$ *and* $\Theta_{\mathrm{M}} \mid \mathrm{H}^1(\beta, \mathfrak{A}_{\mathrm{M}}) \cap \mathrm{U}_{\mathrm{M}}$ *are null, while*

$$\Theta_i = \Theta_{\mathrm{M}} \mid \mathrm{H}^1(\beta, \mathfrak{A}_i)$$

lies in $\mathscr{C}(\mathfrak{A}_i, 0, \beta)$. *The characters* Θ_{M}, Θ_i, θ_{F} *all have the same endoclass over* F.

(iii) *The group* $\mathbf{H}^1_{\mathrm{K}}$ *has Iwahori decomposition*

$$\mathbf{H}^1_{\mathrm{K}} = (\mathbf{H}^1_{\mathrm{K}} \cap \mathrm{U}^-_{\mathrm{M}}).(\mathbf{H}^1_{\mathrm{K}} \cap \mathrm{M}_{\mathrm{M}}).(\mathbf{H}^1_{\mathrm{K}} \cap \mathrm{U}_{\mathrm{M}}).$$

Moreover,

$$\mathbf{H}^1_{\mathrm{K}} \cap \mathrm{M}_{\mathrm{M}} = \prod_i \mathbf{H}^1_{\mathrm{K}} \cap \mathrm{Aut}_{\mathrm{F}}(\mathrm{E}_i),$$

$$\mathbf{H}^1_{\mathrm{K}} \cap \mathrm{Aut}_{\mathrm{F}}(\mathrm{E}_i) = \mathrm{H}^1(\beta_i, \mathfrak{C}_i),$$

where β_i *denotes the canonical image of* β *in* E_i *and*

$$\mathfrak{C}_i = \mathfrak{C}(\mathrm{E}_i) = \mathrm{End}^0_{\mathfrak{o}_{\mathrm{K}}}(\{\mathfrak{p}^j_{\mathrm{E}_i}\}).$$

The analogous statements hold for the group $\mathbf{J}^1_{\mathrm{K}}$.

(iv) *The set* $\{(0, \beta_i) : 1 \leqslant i \leqslant r\}$ *of equivalence classes of simple pairs over* K *is the set of* K/F-*lifts of the class* $(0, \beta)$.

(v) *The characters* $\boldsymbol{\theta}_{\mathrm{K}} \mid \mathbf{H}^1_{\mathrm{K}} \cap \mathrm{U}^-_{\mathrm{M}}$, $\boldsymbol{\theta} \mid \mathbf{H}^1_{\mathrm{K}} \cap \mathrm{U}_{\mathrm{M}}$ *are null, while* $\boldsymbol{\theta}_i = \boldsymbol{\theta}_{\mathrm{K}} \mid \mathrm{H}^1(\beta_i, \mathfrak{C}_i)$ *lies in* $\mathscr{C}(\mathfrak{C}_i, 0, \beta_i)$. *The endoclass of* $\boldsymbol{\theta}_i$ *is the* K/F-*lift of the endoclass of* θ_{F} *corresponding to the* K/F-*lift* β_i *of* β.

Proof. — Throughout, we only treat the H-groups: the proofs for the J-groups are essentially identical.

Parts (i) and (ii) are covered by (10.9), since all of the chains $\{\mathfrak{r}^m\}$, $\{\mathfrak{r}^m \cap \mathrm{E}_i\}$ have the same period $e(\mathrm{E}_i \mid \mathrm{F})$ by (3.2). This Iwahori decomposition is given by the method of (10.4): the lattice $\mathfrak{H}^1_{\mathrm{M}} = \mathfrak{H}^1(\beta, \mathfrak{A}_{\mathrm{M}})$ is the direct sum of the blocks $e_i \mathfrak{H}^1_{\mathrm{M}} e_j$, where e_i denotes the canonical projection $\mathrm{K} \otimes \mathrm{E} \to \mathrm{E}_i$. These projections commute with K, so $\mathfrak{H}^1_{\mathrm{K}}$ is the direct sum of the $e_i \mathfrak{H}^1_{\mathrm{M}} e_j \cap \mathrm{C} = e_i \mathfrak{H}^1_{\mathrm{K}} e_j$. The Iwahori decomposition for $\mathbf{H}^1_{\mathrm{K}}$ then follows immediately. Further, we have

$$\begin{aligned}\mathbf{H}^1_{\mathrm{K}} \cap \mathrm{Aut}_{\mathrm{F}}(\mathrm{E}_i) &= \mathrm{H}^1(\beta, \mathfrak{A}_{\mathrm{M}}) \cap \mathrm{G}(\mathrm{K}) \cap \mathrm{Aut}_{\mathrm{F}}(\mathrm{E}_i)\\ &= \mathrm{H}^1(\beta, \mathfrak{A}_i) \cap \mathrm{Aut}_{\mathrm{K}}(\mathrm{E}_i),\end{aligned}$$

which equals $\mathrm{H}^1(\beta_i, \mathfrak{C}_i)$ by (7.1) (note here that the action of β on E_i is given by the inclusion $\mathrm{E} \to \mathrm{E}_i$, $\beta \mapsto \beta_i$). This proves all the statements of (iii).

In (iv), we rewrite $K \otimes_F E = K \otimes_F F[\beta] \otimes_{F[\beta]} E$. If $\widetilde{\beta}_j$, $1 \leqslant j \leqslant s$, are the distinct K/F-lifts of β, we have

$$K \otimes_F F[\beta] = \prod_j K[\widetilde{\beta}_j].$$

The field extension $K[\widetilde{\beta}_j]/F[\beta]$ is tame, so the factor $K[\widetilde{\beta}_j] \otimes_{F[\beta]} E$ is a product of fields E_{jk}. Write δ_{jk} for the canonical projection to E_{jk}. Thus δ_{jk} induces a K-embedding $K[\widetilde{\beta}_j] \to E_{jk}$ and it follows that any two fields $\delta_{jk_a} K[\widetilde{\beta}_j]$, $a = 1, 2$, are K-isomorphic via an isomorphism taking $\delta_{jk_1}(\widetilde{\beta}_j)$ to $\delta_{jk_2}(\widetilde{\beta}_j)$. In other words, the elements $\delta_{jk_a}(\widetilde{\beta}_j)$ define the same K/F-lift of β. Allowing j to range from 1 to s, we get all such lifts. The elements $\delta_{jk}(\widetilde{\beta}_j)$ are exactly the β_i. This proves (iv).

In (v), the characters $\Theta_M \mid H^1(\beta, \mathfrak{A}_M) \cap U_M^-$, $\Theta_M \mid H^1(\beta, \mathfrak{A}_M) \cap U_M$ are both trivial by (10.9), and remain so when we restrict further. The next statement is given by (7.7) and the final one is the definition (9.4). □

Remark. — In the situation of (11.2), each K/F-lift of β occurs among the β_i with multiplicity equal to the number of field factors of $K[\widetilde{\beta}_j] \otimes_{F[\beta]} E$ (for any choice of j).

In particular, there is a unique irreducible representation η_i of $J^1(\beta_i, \mathfrak{C}_i)$ whose restriction to $H^1(\beta_i, \mathfrak{C}_i)$ contains (and is indeed a multiple of) $\boldsymbol{\theta}_i$. We now show that a similar property holds for the group $\mathbf{J}_K^1$.

(11.3) *Proposition.* — *There is a unique irreducible representation η_K of $\mathbf{J}_K^1$ whose restriction to $\mathbf{H}_K^1$ contains $\boldsymbol{\theta}_K$. Indeed, $\eta_K \mid \mathbf{H}_K^1$ is a multiple of $\boldsymbol{\theta}_K$.*

The representation η_K can be constructed as follows. Form the group

$$\mathscr{G} = (\mathbf{H}_K^1 \cap U_M^-)\,(\mathbf{J}_K^1 \cap M_M)\,(\mathbf{J}_K^1 \cap U_M).$$

Let ξ be the representation of $\mathscr{G}$ which is trivial on $\mathbf{H}_K^1 \cap U_M^-$ and $\mathbf{J}_K^1 \cap U_M$, while

$$\xi \mid \mathbf{J}_K^1 \cap M_M = \eta_1 \otimes \ldots \otimes \eta_r.$$

Then η_K is the representation of $\mathbf{J}_K^1$ induced by ξ.

Proof. — The process $(x, y) \mapsto \Theta_M[x, y]$ (where $[x, y] = x^{-1} y^{-1} xy$) induces a nondegenerate alternating pairing

$$k_M : \frac{J^1(\beta, \mathfrak{A}_M)}{H^1(\beta, \mathfrak{A}_M)} \times \frac{J^1(\beta, \mathfrak{A}_M)}{H^1(\beta, \mathfrak{A}_M)} \to \mathbf{C},$$

as in [BK1] (3.4). When convenient, we identify $J^1(\beta, \mathfrak{A}_M)/H^1(\beta, \mathfrak{A}_M)$ with $\mathfrak{J}^1(\beta, \mathfrak{A}_M)/\mathfrak{H}^1(\beta, \mathfrak{A}_M)$.

The first assertion of the Proposition will follow when we prove that the restriction of k_M to $(\mathbf{J}_K^1/\mathbf{H}_K^1)^2$ is also nondegenerate. Let p denote the residual characteristic of F. We can find a prime element π_K of K such that $\pi_K^{e(K \mid F)} = \pi_F\, \zeta_0$, for a prime element π_F

of F and a p-prime root of unity $\zeta_0 \in K$. We let $\zeta \in K$ be another p-prime root of unity in K such that $K_0 = F[\zeta]/F$ is the maximal unramified subextension of K/F. The (commuting) elements π_K, ζ act on the alternating $\mathbf{F}_p$-space $(J^1(\beta, \mathfrak{A}_M)/H^1(\beta, \mathfrak{A}_M), k_M)$ as symplectic automorphisms of order prime to p. The restriction of k_M to the space of fixed points of the group $\Delta = \langle \pi_K, \zeta \rangle$ is thus nondegenerate. We therefore have to show that this space is the same as $\mathbf{J}^1_K/\mathbf{H}^1_K$. We have $K = F[\Delta]$, so the group of fixed points of Δ in $\mathfrak{J}^1(\beta, \mathfrak{A}_M)$ is $\mathfrak{J}^1_K$, and likewise for $\mathfrak{H}^1(\beta, \mathfrak{A}_M)$. Further, the group of conjugations by Δ on $\mathrm{End}_F(K \otimes E)$ has order prime to p, so the cohomology group $H^1(\Delta, \mathfrak{H}^1(\beta, \mathfrak{A}_M))$ is trivial. The assertion follows.

To prove the second assertion, we note that k_M restricts to the canonical commutator pairing on $J^1(\beta_i, \mathfrak{C}_i)/H^1(\beta_i, \mathfrak{C}_i)$ and so is nondegenerate there. Just as in [BK1] (7.2.3), one shows that

$$\left(\frac{\mathbf{J}^1_K \cap U^-_M}{\mathbf{H}^1_K \cap U^-_M}, \frac{\mathbf{J}^1_K \cap U_M}{\mathbf{H}^1_K \cap U_M}\right)$$

is a pair of complementary totally isotropic subspaces of $\mathbf{J}^1_K/\mathbf{H}^1_K$, and the assertion follows. □

It follows that the restriction $\eta_M \mid \mathbf{J}^1_K$ is a multiple of $\boldsymbol{\eta}_K$, where η_M is the unique irreducible representation of $J^1(\beta, \mathfrak{A}_M)$ which contains Θ_M.

Remark. — Suppose for the moment that K/F is a Galois extension, and put $\Gamma = \mathrm{Gal}(K \mid F)$. Thus Γ acts on $K \otimes E$ via the first factor, so we may regard it as a subgroup of G(K), or even $\mathrm{Aut}_E(K \otimes E)$. As such, it stabilizes each lattice $\mathfrak{r}^m$ and so $\Gamma \subset \mathbf{U}(\mathfrak{A}_M)$. Since Γ also commutes with E, it normalizes $H^1(\beta, \mathfrak{A}_M)$ and fixes the character Θ_M. Restricting to the centralizer of K, which is also Γ-stable, we see that Γ acts on $\mathbf{H}^1_K$ (hence also on $\mathfrak{H}^1_K$) and fixes the character $\boldsymbol{\theta}_K$. Likewise, Γ acts on $\mathbf{J}^1_K$ and it follows that $\boldsymbol{\eta}^\sigma_K \cong \boldsymbol{\eta}_K$ for all $\sigma \in \Gamma$.

We have a canonical identification $\mathrm{End}_F(K \otimes E) = A(E) \otimes_F A(K)$, where $A(E) = \mathrm{End}_F(E)$, $A(K) = \mathrm{End}_F(K)$. In particular, we have a canonical algebra map

$$\iota_K : A(E) \to \mathrm{End}_F(K \otimes E)$$

whose image commutes with K. In the notation of § 10, we have $\iota_K = \iota_{\mathscr{B}'}$, where $\mathscr{B}'$ is any F-basis of K.

(11.4) *Proposition. — In the notation above, we have*

$$\begin{aligned} H^1(\beta, \mathfrak{A}_M) \cap \iota_K(A(E)) &= \mathbf{H}^1_K \cap \iota_K(A(E)) \\ &= \iota_K(H^1(\beta, \mathfrak{A}(E))), \\ J^1(\beta, \mathfrak{A}_M) \cap \iota_K(A(E)) &= \mathbf{J}^1_K \cap \iota_K(A(E)) \\ &= \iota_K(J^1(\beta, \mathfrak{A}(E))). \end{aligned}$$

Proof. — Again we only treat the H-groups. We prove that

$$H^1(\beta, \mathfrak{A}_M) \cap \iota_K(A(E)) = \iota_K(H^1(\beta, \mathfrak{A}(E))),$$

and the other equality is then immediate. This equality follows from (10.15), (10.20) once we have shown:

(11.5) *Lemma.* — *There exist elements $x_1, x_2, \ldots, x_d \in K, y_1, y_2, \ldots, y_d \in E^\times$ such that*
(i) $\{x_1, x_2, \ldots, x_d\}$ *is an F-basis of* K, *and*
(ii) $\mathscr{B} = \{x_1 \otimes y_1, x_2 \otimes y_2, \ldots, x_d \otimes y_d\}$ *is an* $\mathfrak{o}_E$*-basis of the lattice chain* $\mathscr{L}_{K\otimes E}$.
Moreover, the set $\{x_1, x_2, \ldots, x_d\}$ *may be chosen to be an* $\mathfrak{o}_F$*-basis of the lattice chain* $\{\mathfrak{p}_K^j : j \in Z\}$, *up to order.*

Proof. — Let us abbreviate $e(E) = e(E \mid F)$, $e(K) = e(K \mid F)$ and

$$e_0 = \gcd(e(E), e(K)).$$

The $\mathfrak{o}_E$-period of the chain $\mathscr{L}_{K\otimes E}$ is $e(E_i \mid E) = e(K)/e_0$, by (3.2). In other words, if π_K, π_E are prime elements of K and E respectively, we have $\pi_K \mathfrak{O} = \mathfrak{r}^{e(E)/e_0}$ and $\pi_E \mathfrak{D} = \mathfrak{r}^{e(K)/e_0}$. Choose integers a and b such that $ae(E) + be(K) = e_0$. We put $\psi = \pi_K^a \otimes \pi_E^b$, and we then have

$$\psi\mathfrak{O} = \mathfrak{r}.$$

The next step is to find suitable elements $b_i \in \mathfrak{O}$ such that the cosets $b_i + \mathfrak{r}$ form a k_E-basis of $\mathfrak{O}/\mathfrak{r}$. The elements $\psi^j b_i$, $0 \leqslant j \leqslant e(K)/e_0 - 1$, will then form an $\mathfrak{o}_E$-basis of $\mathscr{L}$.

Since K/F is tamely ramified, the k_E-dimension of the quotient $\mathfrak{O}/\mathfrak{r}$ is $f(K \mid F)\, e_0$. We choose $z_1, \ldots, z_f \in \mathfrak{o}_K$, $f = f(K \mid F)$, such that the cosets $z_j + \mathfrak{p}_K$ form a basis of k_K over k_F. The element $\xi = \pi_K^{e(K)/e_0} \otimes \pi_E^{-e(E)/e_0}$ lies in $\mathfrak{O}^\times$, and it is a pleasant exercise to show that the set

$$\{\xi^i z_j + \mathfrak{r} : 0 \leqslant i \leqslant e_0 - 1, 1 \leqslant j \leqslant f\}$$

is a k_E-basis of $\mathfrak{O}/\mathfrak{r}$. By the remark above, the set $\mathscr{B} = \{\psi^i \xi^j z_k\}$, where $0 \leqslant i \leqslant e(K)/e_0 - 1$, $0 \leqslant j \leqslant e_0 - 1$, $1 \leqslant k \leqslant f$ is then an $\mathfrak{o}_E$-basis of $\mathscr{L}$, at least when ordered suitably. Next, we observe that we could have chosen π_K so that $\pi_K^{e(K\mid F)} = \pi_F \zeta$, where π_F is a prime element of F and $\zeta \in K$ is a root of unity of order prime to the residual characteristic p. Every element of $\mathscr{B}$ can then be written in the form $\pi_K^j u \otimes \pi_E^r \pi_F^s$, where $u \in \mathbf{U}(\mathfrak{o}_K)$ and $0 \leqslant j \leqslant e(K \mid F) - 1$. By construction, the set of elements $\pi_K^j u$ so obtained form an $\mathfrak{o}_F$-basis of $\{\mathfrak{p}_K^k : k \in \mathbf{Z}\}$ (when suitably ordered). □

This completes the proof of (11.4). □

When the extension K/F is Galois, the canonical image $\iota_K(A(E))$ of A(E) in $\mathrm{End}_F(K \otimes E)$ is particularly easy to recognize: we have

$$\iota_K(A(E)) = \mathrm{End}_K(K \otimes E)^\Gamma,$$

where $\Gamma = \mathrm{Gal}(K/F)$. We deduce:

(11.6) *Corollary. — Suppose that the tamely ramified extension* K/F *is Galois, and put* $\mathrm{Gal}(K/F) = \Gamma$. *We then have*

$$(\mathbf{H}^1_K)^\Gamma = \iota_K(H^1(\beta, \mathfrak{A}(E))),$$
$$(\mathbf{J}^1_K)^\Gamma = \iota_K(J^1(\beta, \mathfrak{A}(E))).$$

Without any hypothesis on our tame extension K/F, the basis $\mathscr{B}$ constructed in (11.5) gives us a parabolic norm map $\mathscr{N}_{\mathscr{B}}$ as in (10.17). We restrict $\mathscr{N}_{\mathscr{B}}$ to a map

$$\mathscr{N}_{\mathscr{B}} : \mathbf{H}^1_K \to H^1(\beta, \mathfrak{A}(E)) \tag{11.7}$$

which satisfies

$$\theta_F(\mathscr{N}_{\mathscr{B}}\, x) = \boldsymbol{\theta}_K(x), \quad x \in \mathbf{H}^1_K, \tag{11.8}$$

by (10.18). In particular, we get a well-defined homomorphism of abelian groups

$$\mathscr{N}_{\mathscr{B}} : \frac{\mathbf{H}^1_K}{\mathrm{Ker}(\boldsymbol{\theta}_K)} \to \frac{H^1(\beta, \mathfrak{A}(E))}{\mathrm{Ker}(\theta_F)}.$$

This is certainly injective, and is in fact a bijection, although we will not need this. We can now prove a basic identity. (A much stronger version of this will be proved, for certain cyclic extensions, in § 12.)

(11.9) *Proposition. — Let* $x \in \mathbf{U}^1(\mathfrak{o}_K) = K^\times \cap \mathbf{H}^1_K$. *We have*

$$\boldsymbol{\theta}_K(x) = \theta_F(N_{K/F}(x)),$$

where $N_{K/F}$ *denotes the field norm.*

Proof. — We have to relate the field norm to the " parabolic norm " $\mathscr{N}_{\mathscr{B}}$. First, let $\{z_1, \ldots, z_d\}$ be an $\mathfrak{o}_F$-basis of the lattice chain of powers of $\mathfrak{p}_K$ (up to order), and use it to identify $\mathrm{Aut}_F(K)$ with $\mathbf{M}(d, F)$. Let M_K be the Levi subgroup of diagonal matrices in $\mathrm{Aut}_F(K)$, and P_K be the upper triangular matrices. We have $\mathbf{U}^1(\mathfrak{o}_K) \subset \mathbf{U}^1(\mathfrak{A}(K))$, so our element x has Iwahori decomposition relative to (P_K, M_K), say

$$x = y . \mathrm{diag}(x_1, \ldots, x_d) . z,$$

with y, z unipotent and $x_i \in F$. We then have $\prod x_i = \det x = N_{K/F}(x)$.

We can assume (by (11.5)) that there are elements $y_i \in E^\times$ such that $\mathscr{B} = \{ z_i \otimes y_i : 1 \leqslant i \leqslant d\}$ is an $\mathfrak{o}_E$-basis of $\mathscr{L}_{K \otimes E}$. Use this basis to identify $\mathrm{End}_E(K \otimes E)$ with $\mathbf{M}(d, E)$. The Iwahori decomposition of x, with respect to upper triangular and diagonal matrices, is again $x = y' \,\mathrm{diag}(x_1, \ldots, x_d)\, z'$, with y', z' unipotent (in fact conjugate to y, z by the diagonal matrix $\mathrm{diag}(y_1, \ldots, y_d)$) and the same x_i as before (except that they are now scalar matrices). Moreover, this is the Iwahori decomposition of x, viewed as an element of $\mathrm{Aut}_F(K \otimes E)$, with respect to $(M_{\mathscr{B}}, P_{\mathscr{B}})$. This tells us that $\mathscr{N}_{\mathscr{B}}(x) = N_{K/F}(x)$, and the result follows from (11.8). □

It will be convenient to have the abbreviations

$$\textbf{(11.10)} \qquad \mathfrak{H}^1_F = \mathfrak{H}^1(\beta, \mathfrak{A}(E)), \quad H^1_F = H^1(\beta, \mathfrak{A}(E)),$$

and likewise for J-groups. We henceforward identify $A(E)$ with $\iota_K(A(E)) \subset \mathrm{End}_F(K \otimes E)$, and so omit the notation ι_K. We thus identify $C = A(E) \otimes_F K$. We write $\mathrm{Tr}_{K/F}$ for the field trace $K \to F$. This extends in an obvious way to a map

$$\mathrm{Tr}_{K/F} : C \to A(E).$$

(11.11) *Proposition. — We have*

$$\mathrm{Tr}_{K/F}(\mathfrak{H}^1_K) = \mathfrak{H}^1_F,$$

$$\mathrm{Tr}_{K/F}(\mathfrak{J}^1_K) = \mathfrak{J}^1_F.$$

Proof. — The main step in the proof is to show that $\mathrm{Tr}_{K/F}(\mathfrak{H}^1_K) \subset \mathfrak{H}^1_F$. For then, if $x \in \mathfrak{H}^1_F$, we have $x = \mathrm{Tr}_{K/F}(x\alpha)$, for any $\alpha \in \mathfrak{o}_K$ which satisfies $\mathrm{Tr}_{K/F}(\alpha) = 1$.

In the case where K/F is Galois, with $\mathrm{Gal}(K \mid F) = \Gamma$, the trace function is $x \mapsto \sum_{\gamma \in \Gamma} \gamma(x)$. Its image thus lies in $(\mathfrak{H}^1_K)^\Gamma$, which equals $\mathfrak{H}^1_F$ by (11.6).

We must therefore assume that K/F is not Galois. There is then a finite unramified extension L/K such that L/F is Galois. Write $\Delta = \mathrm{Gal}(L \mid K)$, $\Gamma = \mathrm{Gal}(L \mid F)$. We have $\mathfrak{H}^1_L \subset \mathrm{End}_L(E \otimes_F L)$, and we can identify $\mathrm{End}_L(E \otimes L)$ with $\mathrm{End}_F(E) \otimes L$. The group Γ acts via the second factor, and the space of Δ-fixed points is $\mathrm{End}_F(E) \otimes K = \mathrm{End}_K(E \otimes K)$. We show that

$$\mathfrak{H}^1_K \subset (\mathfrak{H}^1_L)^\Delta.$$

Given this, we can choose $\alpha' \in \mathfrak{o}_L$ with $\mathrm{Tr}_{L/K}(\alpha') = 1$ and then, for $x \in \mathfrak{H}^1_K$, we have

$$\mathrm{Tr}_{K/F}(x) = \mathrm{Tr}_{K/F} \circ \mathrm{Tr}_{L/K}(x\alpha') = \mathrm{Tr}_{L/F}(x\alpha') \in \mathfrak{H}^1_F$$

by the Galois case.

We now choose an $\mathfrak{o}_K$-basis $\{y_1, \ldots, y_s\}$ of $\mathfrak{o}_L$. We identify $L \otimes_F E = L \otimes_K K \otimes_F E$. Write $\mathfrak{O}_L$ for the unique maximal $\mathfrak{o}_F$-order in $L \otimes_F E$ and $\mathfrak{r}_L$ for its radical. We then have

$$\textbf{(11.12)} \qquad \mathfrak{r}^m_L = \coprod_{j=1}^{s} y_j \otimes \mathfrak{r}^m, \quad m \in \mathbf{Z}.$$

We use the basis $\{y_j\}$ to identify $\mathrm{End}_F(L \otimes E)$ with the algebra of $s \times s$ matrices over $\mathrm{End}_F(K \otimes E)$. Let P_0 be the parabolic subgroup of invertible upper triangular matrices over $\mathrm{End}_F(K \otimes E)$ and M_0 the obvious block diagonal Levi component of P_0. Let

$\mathfrak{A}_0 = \mathrm{End}^0_{\mathfrak{o}_F}(\{ \mathfrak{r}^j_L \})$. The groups $\mathbf{U}^1(\mathfrak{A}_0)$, $\mathrm{H}^1(\beta, \mathfrak{A}_0)$ have Iwahori decomposition with respect to $(\mathrm{P}_0, \mathrm{M}_0)$ (this is a case of (10.9)) and, in particular,

$$\mathrm{H}^1(\beta, \mathfrak{A}_0) \cap \mathrm{M}_0 = \mathrm{H}^1(\beta, \mathfrak{A}_M) \times \ldots \times \mathrm{H}^1(\beta, \mathfrak{A}_M)$$

(with s factors in the product). Thus $\mathrm{H}^1(\beta, \mathfrak{A}_0)$ contains the natural diagonal embedding of $\mathrm{H}^1(\beta, \mathfrak{A}_M)$, and therefore also $\mathbf{H}^1_K$. However, this diagonal embedding of $\mathbf{H}^1_K$ commutes with L, so we have $\mathbf{H}^1_K \subset \mathbf{H}^1_L$. It is clearly fixed by Δ, and we have finished the proof. □

The immediate point of this sequence of results is that we can now compare conjugacy classes in the groups H^1_F, $\mathbf{H}^1_K$.

(11.13) *Corollary. — Let $x, y \in \mathrm{H}^1_F$ and suppose there exists $t \in \mathfrak{H}^1_K$ such that $(1 + t)^{-1} x(1 + t) = y$. There then exists $t_0 \in \mathfrak{H}^1_F$ such that $(1 + t_0)^{-1} x(1 + t_0) = y$. The same result holds for* J*-groups.*

Proof. — Choose $\alpha \in \mathfrak{o}_K$ with $\mathrm{Tr}_{K/F}(\alpha) = 1$. The given relation amounts to $x + xt = y + ty$, whence $\alpha x + x\alpha t = \alpha y + \alpha t y$ (noting that α commutes with x). This is a relation between elements in $\mathfrak{H}^1_K$. Applying $\mathrm{Tr}_{K/F}$, we get

$$x + xt_0 = y + t_0 y,$$

where $t_0 = \mathrm{Tr}_{K/F}(\alpha t) \in \mathfrak{H}^1_F$. □

We conclude with a more special result.

(11.14) *Proposition. — Suppose that the extension* K/F *is unramified. The canonical embedding $\iota_K : \mathfrak{H}^1(\beta, \mathfrak{A}(E)) \to \mathfrak{H}^1_K$ induces an isomorphism*

$$\mathfrak{H}^1_K \cong \mathfrak{H}^1(\beta, \mathfrak{A}(E)) \otimes_{\mathfrak{o}_F} \mathfrak{o}_K.$$

This isomorphism preserves the canonical filtrations of these lattices. Likewise,

$$\mathfrak{J}^1_K \cong \mathfrak{J}^1(\beta, \mathfrak{A}(E)) \otimes_{\mathfrak{o}_F} \mathfrak{o}_K.$$

Proof. — As usual, we only treat the H-groups, the other case being identical. We now omit the notation ι_K, and view A(E) as canonically embedded in $C = A(E) \otimes K$. Let us abbreviate $\mathfrak{H}^1(\beta, \mathfrak{A}(E)) = \mathfrak{H}^1_F$. We surely have $\mathfrak{H}^1_F \otimes \mathfrak{o}_K \subset \mathfrak{H}^1_K$ and these two lattices have the same Γ-fixed points. The same applies to $\mathfrak{H}^m_F$, $\mathfrak{H}^m_K$ for $m \geqslant 1$. The Proposition now follows from:

(11.15) *Lemma. — Let* K/F *be a finite unramified field extension and put* $\Gamma = \mathrm{Gal}(K \mid F)$. *Let* V *be a finite-dimensional* F*-vector space. The maps $L \mapsto L \otimes_{\mathfrak{o}_F} \mathfrak{o}_K$, $M \mapsto M^\Gamma$ are mutually inverse bijections between the set of $\mathfrak{o}_F$-lattices* L *in* V *and the set of Γ-stable $\mathfrak{o}_K$-lattices* M *in* $V \otimes_F K$.

Proof. — We surely have $(L \otimes \mathfrak{o}_K)^\Gamma = L$ and $M \supset M^\Gamma \otimes \mathfrak{o}_K$. It is therefore enough to prove that if $M_1 \supset M_2$ are Γ-stable $\mathfrak{o}_K$-lattices in $V \otimes K$ with $M_1^\Gamma = M_2^\Gamma$ then

$M_1 = M_2$. There is no harm in replacing M_2 by $M_2 + \mathfrak{p}_K M_1$ and assuming that $M_2 \supset \mathfrak{p}_K M_1$.

The first step is to show that $(M_1/M_2)^\Gamma = \{0\}$. To do this, we choose $\alpha \in \mathfrak{o}_K$ such that $\Sigma_{\gamma \in \Gamma} \gamma(\alpha) = 1$. Let $x \in (M_1/M_2)^\Gamma$ be the image of $m \in M_1$. Consider the element

$$m_0 = \sum_{\gamma \in \Gamma} \gamma(m\alpha).$$

We have $m_0 \in M_1^\Gamma$, hence $m_0 \in M_2$ and it has zero image in M_1/M_2. On the other hand, the image of m_0 in M_1/M_2 is $x\Sigma_\gamma \gamma(\alpha) = x$, whence $x = 0$, as desired.

We can view the quotient $X = M_1/M_2$ as a left vector space over k_K, with an action $(\gamma, x) \mapsto \gamma(x)$ of Γ such that $\gamma(x\alpha) = \gamma(x)\,\gamma(\alpha)$, $x \in X$, $\alpha \in k_K$. In other words, X is a module over the " twisted group algebra " $k_K\,\Gamma$. This is isomorphic to $\mathrm{End}_{k_F}(k_K)$ via the natural inclusion. In particular, it is a simple k_F-algebra, which has a unique simple module, namely k_K with the obvious Γ-action, and this module has nontrivial Γ-fixed points. It follows that $X = \{0\}$, as required. □

12. The cyclic norm

We now relate the material of § 11 with a structure originating in a different part of the subject, that of the " cyclic norm " $\mathcal{N}_\sigma$ of [AC] (and many predecessors). We continue with the notation of § 11 (especially (11.1)), but we now assume that our tamely ramified extension K/F is *cyclic*. We write $\Gamma = \mathrm{Gal}(K \mid F)$, and fix a generator σ of Γ.

To start with, let V be some finite-dimensional F-vector space, and write $G_F = \mathrm{Aut}_F(V)$. The group $G_K = \mathrm{Aut}_K(V \otimes_F K)$ inherits an action of Γ in the obvious way, which we denote $(\sigma, g) \mapsto \sigma(g)$. We can thus form the semidirect product $G_K \rtimes \Gamma$, in which the multiplication satisfies $\sigma . g . \sigma^{-1} = \sigma(g)$.

Two elements $g_1, g_2 \in G_K$ are *σ-conjugate* if there exists $h \in G_K$ such that $g_2 . \sigma = hg_1 . \sigma . h^{-1}$, or, equivalently, $g_2 = hg_1\,\sigma(h)^{-1}$. For $g \in G_K$, we put

$$\mathcal{N}_\sigma\, g = g . \sigma(g) . \sigma^2(g) \ldots \sigma^{d-1}(g) = (g . \sigma)^d,$$

where $d = [K : F]$. Let us summarize the main properties of this procedure.

(12.1) *Use the notation above.*

(i) *If* $g_1 . \sigma = h^{-1} g_2 . \sigma h$, $g_i, h \in G_K$, *then* $\mathcal{N}_\sigma\, g_1 = h^{-1} \mathcal{N}_\sigma\, g_2\, h$.

(ii) *For* $g \in G_K$, *there exists* $h \in G_F$ *such that* h *is* G_K*-conjugate to* $\mathcal{N}_\sigma\, g$. *The element* h *is uniquely determined up to* G_F*-conjugacy.*

(iii) *Let* $g_1, g_2 \in G_K$, *and suppose that the elements* $\mathcal{N}_\sigma\, g_i$ *are* G_K*-conjugate. The elements* g_i *are then* σ*-conjugate in* G_K.

This is taken from [AC] Ch. 1, Lemma 1.1. The assumption there that F has characteristic zero is not used in the proof. In all, the cyclic norm $\mathcal{N}_\sigma$ induces an injective map from the set of σ-conjugacy classes in G_K to the set of conjugacy classes of G_F. We sometimes also denote this map by $\mathcal{N}_\sigma$.

We will, of course, apply this in the context of our groups $G(F) = \mathrm{Aut}_F(E)$, $G(K) = \mathrm{Aut}_K(K \otimes E)$. The behaviour of the " cyclic norm " $\mathcal{N}_\sigma$ on the groups $\mathbf{H}^1_K$, $\mathbf{J}^1_K$ of (11.1) is both pleasant and illuminating. In particular, it gives a direct and canonical relation between the characters θ_F, $\boldsymbol{\theta}_K$.

As before, we abbreviate

$$\mathfrak{H}^1_F = \mathfrak{H}^1(\beta, \mathfrak{A}(E)), \qquad H^1_F = H^1(\beta, \mathfrak{A}(E)),$$

and similarly for J-groups.

(12.2) *Notation.* — For the rest of this section, K/F is a cyclic extension *which is either unramified or totally tamely ramified.* We put $\Gamma = \mathrm{Gal}(K \mid F)$, and fix a generator σ of Γ. Further, p denotes the characteristic of the residue field k_F.

The results of this section will only be used (in § 16) to compare two situations which are known to be transitive in the field extension K/F, so the restriction (12.2) will have no practical consequences. It seems likely that the assertions here are valid for arbitrary cyclic (tame) extensions K/F, but (12.2) usefully simplifies the proofs at certain points.

(12.3) *Proposition. — With the notation* (12.2), *we have:*

(i) *Let* $x \in \mathbf{H}^1_K$. *There exists* $u \in \mathbf{H}^1_K$ *such that* $y_x = ux\sigma(u)^{-1}$ *satisfies* $\mathcal{N}_\sigma y_x \in H^1_F$.

(ii) *The process* $x \mapsto \mathcal{N}_\sigma y_x$ *induces a bijection between* σ*-conjugacy classes in* $\mathbf{H}^1_K$ *and conjugacy classes in* H^1_F.

(iii) *The analogues of* (i) *and* (ii) *hold for* J*-groups.*

Proof. — It will be more convenient to work with an element of the form $1 + x$, $x \in \mathfrak{H}^1_K$.

(12.4) *Lemma. — Let* $x \in \mathfrak{H}^1_K$ *and let* $\alpha \in \mathfrak{o}_K$ *satisfy* $\mathrm{Tr}_{K/F}(\alpha) = 1$. *There then exists* $x_0 \in \mathfrak{H}^1_F$ *such that* $1 + x$ *is* σ*-conjugate in* $\mathbf{H}^1_K$ *to* $1 + x_0\alpha$. *In particular, if* $p \nmid [K : F]$, x *is* σ*-conjugate in* $\mathbf{H}^1_K$ *to an element of* H^1_F.

Proof. — As in (11.11), we extend the field trace to give us a map $\mathrm{Tr}_{K/F} : \mathfrak{H}^1_K \to \mathfrak{H}^1_F$. Set $x_1 = \mathrm{Tr}_{K/F}(x)$. We then have $x_1\alpha \in \mathfrak{H}^1_K$ and $\mathrm{Tr}_{K/F}(x - x_1\alpha) = 0$. The cohomology $H^1(\Gamma, \mathfrak{H}^1_K)$ is trivial, by (11.14) when K/F is unramified, or because $\# \Gamma$ is prime to p and $\mathfrak{H}^1_K$ is a pro-p-group when K/F is totally ramified. Hence there exists $y \in \mathfrak{H}^1_K$ such that

$$x_1\alpha = x + y - \sigma(y),$$

which implies

$$(1 + y)\,(1 + x)\,\sigma(1 + y)^{-1} \equiv 1 + x_1\alpha \pmod{\mathfrak{H}^2_K}.$$

We iterate this process in the obvious way to get the desired element x_0. $\square$

We continue with the notation of the last proof, and put

$$f_\alpha(t) = \prod_{\tau \in \Gamma} (1 + \tau(\alpha)\, t) \in \mathfrak{o}_F[t].$$

For $x_0 \in \mathfrak{H}_F^1$, we then have

$$\mathcal{N}_\sigma(1 + x_0\,\alpha) = f_\alpha(x_0) \in \mathrm{H}_F^1.$$

This proves part (i) of the Proposition. Moreover,

$$f_\alpha(x_0) \equiv 1 + x_0 \pmod{x_0\,\mathfrak{H}_F^1},$$

and $x_0\,\mathfrak{H}_F^1 \subset \mathfrak{H}_F^2$. Indeed, we have $f_\alpha(t) \equiv 1 + t$ modulo terms of degree $\geqslant 2$.

For an indeterminate X, we set $\mathrm{F}(\mathrm{X}) = f_\alpha(\mathrm{X}) - 1$. There then exists a (unique) formal power series $g_\alpha(\mathrm{X}) \in \mathfrak{o}_F[[\mathrm{X}]]$,

$$g_\alpha(\mathrm{X}) = \mathrm{X} + \sum_{n \geqslant 2} b_n\,\mathrm{X}^n,$$

with the properties

$$\mathrm{F} \circ g_\alpha(\mathrm{X}) = \mathrm{X},$$
$$g_\alpha \circ \mathrm{F}(\mathrm{X}) = \mathrm{X}.$$

Thus, if $x \in \mathfrak{H}_F^1$, the power series $g_\alpha(x)$ converges to an element $x_0 \in \mathfrak{H}_F^1$ such that

$$\mathcal{N}_\sigma(1 + x_0\,\alpha) = 1 + x.$$

Moreover, x_0 is the unique element with this property.

Now we prove part (ii) of the Proposition. Let $x_1, x_2 \in \mathfrak{H}_K^1$, and suppose that the elements $1 + x_i$ are σ-conjugate in $\mathbf{H}_K^1$. We may as well take $x_i = y_i\,\alpha$, with $y_i \in \mathfrak{H}_F^1$. The norms $\mathcal{N}_\sigma(1 + x_i) = f_\alpha(y_i) \in \mathrm{H}_F^1$ are then conjugate in $\mathbf{H}_K^1$, and hence also in H_F^1 by (11.13). Conversely, suppose that the $1 + y_i$ are conjugate in H_F^1,

$$1 + y_2 = u(1 + y_1)\,u^{-1},$$

say, for some $u \in \mathrm{H}_F^1$. But, in the notation above, we have

$$x_2 = g_\alpha(y_2) = g_\alpha(uy_1\,u^{-1}) = ug_\alpha(y_1)\,u^{-1} = ux_1\,u^{-1}.$$

Since $\sigma(u) = u$, this says that the $(1 + x_i)$ are σ-conjugate in $\mathbf{H}_K^1$, as required. □

We will need to refer back to the details of the constructions in (12.3), so we now exhibit them explicitly.

(12.5) *Corollary. — With the notation of* (12.2), *choose* $\alpha \in \mathfrak{o}_K$ *with* $\mathrm{Tr}_{K/F}(\alpha) = 1$. *There exists a formal power series* $g_\alpha(\mathrm{X}) \in \mathfrak{o}_F[[\mathrm{X}]]$, *with constant term* X *and the following property: if* $x, x_0 \in \mathfrak{H}_F^1$, *then* $\mathcal{N}_\sigma(1 + x_0\,\alpha) = 1 + x$ *if and only if* $x_0 = g_\alpha(x)$. *The maps*

$$1 + x \mapsto 1 + g_\alpha(x)\,\alpha, \quad x \in \mathfrak{H}_F^1,$$
$$1 + y\alpha \mapsto \mathcal{N}_\sigma(1 + y\alpha), \quad y \in \mathfrak{H}_F^1,$$

induce mutually inverse bijections between the following sets:

a) conjugacy classes in H^1_F;

b) σ*-conjugacy classes in* $\mathbf{H}^1_K$;

c) H^1_F*-conjugacy classes of elements* $1 + y\alpha \in \mathbf{H}^1_K$ *with* $y \in \mathfrak{H}^1_F$.

The analogous result, with the same series $g_\alpha(X)$, *holds for* J*-groups.*

We have already remarked (following the proof of (11.3)) that the character $\boldsymbol{\theta}_K$ is stabilized by Γ,

$$\boldsymbol{\theta}_K(\sigma(x)) = \boldsymbol{\theta}_K(x), \quad x \in \mathbf{H}^1_K.$$

It follows that $\boldsymbol{\theta}_K$ is constant on σ-conjugacy classes in $\mathbf{H}^1_K$. This brings us to the first substantial result of the section.

(12.6) *Theorem.* — *Use the notation* (12.2). *Let* $x \in \mathbf{H}^1_K$ *satisfy* $\mathcal{N}_\sigma(x) \in H^1_F$. *We then have*

$$\boldsymbol{\theta}_K(x) = \theta_F(\mathcal{N}_\sigma(x)).$$

Proof. — Put $d = [K : F]$ and assume that K/F is totally ramified. Thus $p \nmid d$ (and this is actually the only hypothesis we require for this argument). In (12.4), we can therefore take $\alpha = d^{-1} \in \mathfrak{o}_F$, and we may as well assume that $x = 1 + y$, for some $y \in \mathfrak{H}^1_F$. We then have $\mathcal{N}_\sigma(x) = x^d$, so we have to show that

$$\boldsymbol{\theta}_K(x) = \theta_F(x)^d, \quad x \in H^1_F.$$

We use the basis $\mathcal{B}$ of the chain $\mathcal{L}_{K \otimes E}$ provided by (11.5). We let M be the Levi subgroup of $\mathrm{Aut}_F(K \otimes E)$ defined by $\mathcal{B}$,

$$M = \prod_{b \in \mathcal{B}} \mathrm{Aut}_F(Eb).$$

Let $\iota_{\mathcal{B}}$ be the embedding $A(E) \to \mathrm{End}_F(K \otimes E)$ defined by the basis $\mathcal{B}$. We then have $\iota_{\mathcal{B}}(H^1_F) \subset M \cap H^1(\beta, \mathfrak{A}_M)$ by (10.15) and $\Theta_M(\iota_{\mathcal{B}}(x)) = \theta_F(x)^d$ by (10.17). However, $\Theta_M(\iota_{\mathcal{B}}(x)) = \Theta_M(x)$ by (10.21) (recall that we are identifying A(E) with $\iota_K(A(E)) \subset \mathrm{End}_K(K \otimes E)$, and the map ι_K plays the role of $\iota_{\mathcal{B}'}$ in (10.21)). However, by definition, $\Theta_M(x) = \boldsymbol{\theta}_K(x)$ when $x \in \mathbf{H}^1_K$, and we have finished the proof in this case.

We therefore assume that K/F is unramified. We can take $x = 1 + y\alpha$, where $y \in \mathfrak{H}^1_F$, as in (12.4). In the context of (11.5), an $\mathfrak{o}_F$-basis $\mathcal{B} = \{x_1, \ldots, x_d\}$ of $\mathfrak{o}_K$ is automatically an $\mathfrak{o}_E$-basis of the lattice chain $\mathcal{L}_{K \otimes E}$. The embedding $A(E) \to \mathrm{End}_F(K \otimes E)$ induced by $\mathcal{B}$ is the canonical embedding ι_K. We use this to identify $\mathrm{End}_F(K \otimes E)$ with the ring of $d \times d$ matrices over A(E). We write $\mathfrak{o}_F[[y]]$ for the image of the formal power series ring $\mathfrak{o}_F[[Y]]$ (where Y is an indeterminate) under the specialization $Y \mapsto y$. Our element x then belongs to the commutative subring

$$\mathfrak{o}_F[[y]] \otimes_{\mathfrak{o}_F} \mathfrak{o}_K \subset \mathfrak{o}_F[[y]] \otimes_{\mathfrak{o}_F} \mathrm{End}_{\mathfrak{o}_F}(\mathfrak{o}_K) \cong \mathbf{M}(d, \mathfrak{o}_F[[y]]).$$

Further, $\mathfrak{o}_F[[y]] \otimes \mathfrak{o}_K \cong \mathfrak{o}_K[[y]]$, and $\mathcal{N}_\sigma(x)$ is just the determinant of x viewed as a matrix over $\mathfrak{o}_F[[y]]$. On the other hand, the Levi subgroup M of $\mathrm{End}_F(K \otimes E)$ defined by the basis $\mathscr{B}$ is just the group of invertible diagonal matrices over A(E). We take for **P** the upper triangular group with Levi component M. We work out the Iwahori decomposition of the element x relative to (P, M):

$$x = u.(x_1, x_2, \ldots, x_d).z,$$

where u and z are the unipotent components and $(x_1, \ldots, x_d) \in \mathrm{M}$. However, the entries of the matrix x all lie in $\mathfrak{o}_F[[y]]$ and, when we use elementary row and column operations to compute the Iwahori components, we find that their entries also lie in $\mathfrak{o}_F[[y]]$. Thus $\mathcal{N}_{\mathscr{B}}(x) = x_1 x_2 \ldots x_d$ is just the determinant of x as a matrix over $\mathfrak{o}_F[[y]]$, i.e. $\mathcal{N}_{\mathscr{B}}(x) = \mathcal{N}_\sigma(x)$. We have $\boldsymbol{\theta}_K(x) = \Theta_M(x) = \theta_F(\mathcal{N}_{\mathscr{B}}(x))$ by (10.18), and we have proved the theorem. □

We can with advantage rearrange the conclusion of (12.6). As we have observed, the group Γ acts on the group $\mathbf{H}^1_K$ and this action fixes the character $\boldsymbol{\theta}_K$. We can therefore extend $\boldsymbol{\theta}_K$ to a character of $\mathbf{H}^1_K \rtimes \Gamma$ by, for example, making it trivial on the second factor. We then have

(12.7) $$\boldsymbol{\theta}_K(x.\sigma) = \theta_F(\mathcal{N}_\sigma\, x), \quad x \in \mathbf{H}^1_K,$$

where, in the right hand side, $\mathcal{N}_\sigma\, x$ is understood to be an element of H^1_F which is $\mathbf{H}^1_K$-conjugate to $x.\sigma(x) \ldots \sigma^{d-1}(x)$, $d = [\mathrm{K} : \mathrm{F}]$.

We now return to the unique irreducible representation $\boldsymbol{\eta}_K$ of $\mathbf{J}^1_K$ which contains $\boldsymbol{\theta}_K$. Again, the group Γ acts on $\mathbf{J}^1_K$ and, by the uniqueness property (11.3) of $\boldsymbol{\eta}_K$, we have

$$\boldsymbol{\eta}_K^\sigma \cong \boldsymbol{\eta}_K.$$

We can therefore extend $\boldsymbol{\eta}_K$ to a representation of $\mathbf{J}^1_K \rtimes \Gamma$.

If we have a finite-dimensional representation ρ of a group G, we now write $g \mapsto \mathrm{tr}(\rho(g))$, $g \in \mathrm{G}$, for the character of ρ.

(12.8) *Proposition. — Use the notation above and that of* (12.2). *We have*

$$\mathrm{tr}(\boldsymbol{\eta}_K(x.\sigma)) = c_\sigma . \mathrm{tr}(\eta_F(\mathcal{N}_\sigma\, x)),$$

where $x \in \mathbf{J}^1_K$ *and* $\mathcal{N}_\sigma\, x$ *denotes some element of* J^1_F *which is* $\mathbf{J}^1_K$*-conjugate to* $x.\sigma(x) \ldots \sigma^{d-1}(x)$, $d = [\mathrm{K} : \mathrm{F}]$. *The constant* c_σ *is given by*

$$c_\sigma = \frac{\mathrm{tr}(\boldsymbol{\eta}_K(\sigma))}{\dim \eta_F}.$$

Proof. — The left hand side of the desired inequality vanishes unless the $\mathbf{J}^1_K\,\Gamma$-conjugacy class of $x.\sigma$ meets $\mathbf{H}^1_K\,\Gamma$. (For this, see § 13 below.) This condition is equivalent to the σ-conjugacy class of x meeting $\mathbf{H}^1_K$. Assuming this to be the case, we can take

$x \in \mathbf{H}^1_{\mathrm{K}}$. We then get $\eta_{\mathrm{K}}(x.\sigma) = \boldsymbol{\theta}_{\mathrm{K}}(x)\,\eta_{\mathrm{K}}(\sigma)$, while $\eta_{\mathrm{F}}(\mathcal{N}_\sigma\, x) = \theta_{\mathrm{F}}(\mathcal{N}_\sigma\, x)\,\eta_{\mathrm{F}}(1)$, and the result follows from (12.6).

We therefore suppose that the σ-conjugacy class of x in $\mathbf{J}^1_{\mathrm{K}}$ does not meet $\mathbf{H}^1_{\mathrm{K}}$, so that $\mathrm{tr}(\eta_{\mathrm{K}}(x.\sigma)) = 0$. By (12.5), we can take $x = 1 + x_0\,\alpha$, with $x_0 \in \mathfrak{J}^1_{\mathrm{F}}$ and $\alpha \in \mathfrak{o}_{\mathrm{K}}$ of trace 1. In the notation of that proof, we have $\mathcal{N}_\sigma\, x = 1 + y$, and $x_0 = g_\alpha(y)$. If y were an element of $\mathfrak{H}^1_{\mathrm{F}}$, we would have $x_0 \in \mathfrak{H}^1_{\mathrm{F}}$ and $x \in \mathbf{H}^1_{\mathrm{K}}$, contrary to hypothesis. We deduce that $\mathcal{N}_\sigma\, x \notin \mathrm{H}^1_{\mathrm{F}}$ and that $\mathrm{tr}(\eta_{\mathrm{F}}(\mathcal{N}_\sigma\, x)) = 0$. □

We can assemble the identities (12.7) into a more informative result.

(12.9) *Theorem. — Use the notation* (12.2). *Let* $s \in \mathbf{H}^1_{\mathrm{K}}$, $t \in \mathrm{H}^1_{\mathrm{F}}$, *and suppose that* t *is* $\mathbf{H}^1_{\mathrm{K}}$*-conjugate to* $\mathcal{N}_\sigma\, s$. *Let* $\mathfrak{S}(t)$ *denote the set of cosets* $g \in \mathrm{G}(\mathrm{F})/\mathrm{H}^1_{\mathrm{F}}$ *with the property* $g^{-1}\,tg \in \mathrm{H}^1_{\mathrm{F}}$, *and likewise let* $\mathfrak{S}(s,\sigma)$ *be the set of* $h \in \mathrm{G}(\mathrm{K})/\mathbf{H}^1_{\mathrm{K}}$ *such that* $h^{-1}\,s.\sigma.h \in \mathbf{H}^1_{\mathrm{K}}.\sigma$. *There is then a bijection*

$$\varphi : \mathfrak{S}(t) \xrightarrow{\approx} \mathfrak{S}(s,\sigma)$$

with the property

$$\boldsymbol{\theta}_{\mathrm{K}}(\varphi(g)^{-1}\,s.\sigma.\varphi(g)) = \theta_{\mathrm{F}}(g^{-1}\,tg), \quad g \in \mathfrak{S}(t).$$

Proof. — We start with the case in which K/F *is unramified*. The machinations of this part of the proof are inspired by [Ko], but the situation here is simpler.

If L/F is an unramified extension, we can define

$$\text{(12.10)}\qquad \begin{aligned} &\mathrm{G}(\mathrm{L}) = \mathrm{Aut}_{\mathrm{L}}(\mathrm{L}\otimes_{\mathrm{F}}\mathrm{E}),\\ &\mathfrak{H}^1_{\mathrm{L}} = \mathfrak{o}_{\mathrm{L}}\otimes_{\mathfrak{o}_{\mathrm{F}}}\mathfrak{H}^1_{\mathrm{F}}, \quad \mathrm{H}^1_{\mathrm{L}} = 1 + \mathfrak{H}^1_{\mathrm{L}},\\ &\mathfrak{J}^1_{\mathrm{L}} = \mathfrak{o}_{\mathrm{L}}\otimes_{\mathfrak{o}_{\mathrm{F}}}\mathfrak{J}^1_{\mathrm{F}}, \quad \mathrm{J}^1_{\mathrm{L}} = 1 + \mathfrak{J}^1_{\mathrm{L}}. \end{aligned}$$

The lattices $\mathfrak{H}^1_{\mathrm{L}}$, and hence the groups $\mathrm{H}^1_{\mathrm{L}}$, come equipped with a filtration inherited from that of $\mathfrak{H}^1_{\mathrm{F}}$. By (11.14), we have $\mathfrak{H}^1_{\mathrm{K}} = \boldsymbol{\mathfrak{H}}^1_{\mathrm{K}}$. Write $\mathrm{F}_{\mathrm{nr}}/\mathrm{F}$ for a maximal unramified extension of F (assumed to contain K) and $\widetilde{\mathrm{F}}$ for the completion of F_{nr}. The definitions (12.10) then make sense with $\mathrm{L} = \mathrm{F}_{\mathrm{nr}}$ or $\mathrm{L} = \widetilde{\mathrm{F}}$. We choose a topological generator σ of the Galois group of $\mathrm{F}_{\mathrm{nr}}/\mathrm{F}$ whose restriction to K is our original element σ. The Galois group of $\mathrm{F}_{\mathrm{nr}}/\mathrm{F}$ acts on $\widetilde{\mathrm{F}}$ in a natural way, and hence on objects like $\mathfrak{H}^1_{\widetilde{\mathrm{F}}}$ via the second tensor factor in their definition.

(12.11) *Lemma. — The map* $x \mapsto x^{-1}\,\sigma(x)$ *induces surjections* $\mathrm{H}^1_{\widetilde{\mathrm{F}}} \to \mathrm{H}^1_{\widetilde{\mathrm{F}}}$ *and* $\mathrm{J}^1_{\widetilde{\mathrm{F}}} \to \mathrm{J}^1_{\widetilde{\mathrm{F}}}$.

Proof. — As usual, we only deal with the first case. The residue class field $\mathfrak{o}_{\widetilde{\mathrm{F}}}/\mathfrak{p}_{\mathrm{F}}\,\mathfrak{o}_{\widetilde{\mathrm{F}}}$ is the same as $\mathfrak{o}_{\mathrm{F}_{\mathrm{nr}}}/\mathfrak{p}_{\mathrm{F}}\,\mathfrak{o}_{\mathrm{F}_{\mathrm{nr}}}$, which we can identify with an algebraic closure $\widetilde{\mathrm{k}}_{\mathrm{F}}$ of k_{F}. The map $x \mapsto \sigma(x) - x$ is a surjection $\widetilde{\mathrm{k}}_{\mathrm{F}} \to \widetilde{\mathrm{k}}_{\mathrm{F}}$. Thus, if V is a finite-dimensional k_{F}-vector space and we make σ act on $\mathrm{V}\otimes_{\mathrm{k}_{\mathrm{F}}}\widetilde{\mathrm{k}}_{\mathrm{F}}$ via the second factor, we get a surjection of $\mathrm{V}\otimes_{\mathrm{k}_{\mathrm{F}}}\widetilde{\mathrm{k}}_{\mathrm{F}}$ to itself by $v \mapsto \sigma(v) - v$. In particular, $x \mapsto \sigma(x) - x$ gives a surjection of $\mathfrak{H}_{\tilde{F}}^m/\mathfrak{H}_{\tilde{F}}^{m+1}$ to itself. Put another way, given $x \in H_{\tilde{F}}^m$, there exists $y \in H_{\tilde{F}}^m$ such that $y^{-1}x\sigma(y) \in H_{\tilde{F}}^{m+1}$. Iterating and passing to the limit (as we may in $\widetilde{F}$), we see that, given $x \in H_{\tilde{F}}^1$, there exists $y \in H_{\tilde{F}}^1$ such that $y^{-1}x\sigma(y) = 1$, and we have proved the Lemma. □

We will need a sharper version of this formal result.

(12.12) *Lemma.* — *Let* $\alpha \in \mathfrak{o}_K$, *and suppose* $\mathrm{Tr}_{K/F}(\alpha) = 1$. *There exists a formal power series* $C_\alpha(X) \in \mathfrak{o}_{\tilde{F}}[[X]]$ *with constant term* 1 *and the following property: if* $x_0 \in \mathfrak{H}_F^1$ *and* $s = 1 + x_0\alpha \in \mathbf{H}_K^1$, *then* $s = c^{-1}\sigma(c)$, *where* $c = C_\alpha(x_0)$.

Proof. — Let K_p/K be the unramified extension of K in F_{nr} with $[K_p : K] = p$. There then exists $\alpha_1 \in \mathfrak{o}_{K_p}$ such that $\alpha \equiv \alpha_1 - \sigma(\alpha_1) \pmod{\mathfrak{p}_{F_{nr}}}$. The element $(1 - x_0\alpha_1)s\sigma(1 - x_0\alpha_1)^{-1}$ then lies in $H_{K_p}^2$ and is also an element of $\mathfrak{o}_{K_p}[[x_0]]$. It is of the form $1 + x_0\pi_F\gamma$, where π_F is a prime of F and $\gamma \in \mathfrak{o}_{K_p}$. We now find $\alpha_2 \in \mathfrak{o}_{K_{p^2}}$ (in the obvious notation) with $\gamma \equiv \alpha_2 - \sigma(\alpha_2) \pmod{\mathfrak{p}_{F_{nr}}}$, to get

$$(1 - x_0\pi_F\alpha_2)(1 - x_0\alpha_1)s\sigma((1 - x_0\alpha_1)(1 - x_0\pi_F\alpha_2))^{-1} \in H_{K_{p^2}}^3.$$

We iterate this process. The desired power series is therefore

$$C_\alpha(X) = \prod_{n=1}^{\infty}(1 - X\pi_F^{n-1}\alpha_n),$$

and the element α_n lies in $\mathfrak{o}_{K_{p^n}}$. □

(12.13) *Lemma.* — *The space*

$$(G(\widetilde{F})/H_{\tilde{F}}^1)^\sigma$$

of σ*-fixed points in* $G(\widetilde{F})/H_{\tilde{F}}^1$ *is equal to* $G(F)/H_F^1$.

Proof. — Let $gH_{\tilde{F}}^1$ be a fixed point, so that $\sigma(g) = gh$, for some $h \in H_{\tilde{F}}^1$. We use (12.11) to write $h = k^{-1}\sigma(k)$, for some $k \in H_{\tilde{F}}^1$. Thus $g^{-1}\sigma(g) = k^{-1}\sigma(k)$, whence $\sigma(gk^{-1}) = gk^{-1}$. This says that $gk^{-1} \in G(F)$, as required. □

We now consider the set $\mathfrak{S}(t)$, $t \in H_F^1$. For $g \in G(F)$, we have $g^{-1}tg \in H_F^1$ if and only if $tgH_F^1 = gH_F^1$, with the result that

$$\mathfrak{S}(t) = (G(\widetilde{F})/H_{\tilde{F}}^1)^{\langle\sigma, t\rangle}, \quad t \in H_F^1.$$

Likewise, we have

$$\mathfrak{S}(s, \sigma) = (G(\widetilde{F})/H_{\tilde{F}}^1)^{\langle\sigma^d, s.\sigma\rangle}, \quad s \in \mathbf{H}_K^1,$$

where $d = [K : F]$. The assertion of (12.9) is unchanged if we replace t by an H_F^1-conjugate and s by an $(\mathbf{H}_K^1, \sigma)$-conjugate. Therefore, if we choose $\alpha \in \mathfrak{o}_K$ with

$\mathrm{Tr}_{K/F}(\alpha) = 1$, we can take $s = 1 + x_0\,\alpha$, $x_0 \in \mathfrak{H}^1_F$, and $t = \mathcal{N}_\sigma\, x$ (equality of elements). We write $s = c^{-1}\,\sigma(c)$ according to (12.12). Thus

$$s.\sigma = c^{-1}.\sigma.c$$

as elements of $G(\widetilde{F}) \rtimes \langle\,\sigma\,\rangle$. This gives us

$$t = \mathcal{N}_\sigma\, s = (s.\sigma)^d.\sigma^{-d} = c^{-1}\,\sigma^d(c) = \sigma^d(c)\,c^{-1},$$

since, by construction (12.12), the elements c, $\sigma^d(c)$ commute. We can rewrite the relation $t = \sigma^d(c)\,c^{-1}$ as

$$c^{-1}\,t^{-1}.\sigma^d.c = \sigma^d.$$

Altogether, we now have

$$\langle\, s.\sigma, \sigma^d\,\rangle = c^{-1}\,\langle\, t, \sigma\,\rangle\, c$$

as subgroups of $G(\widetilde{F}) \rtimes \langle\,\sigma\,\rangle$. We therefore have a bijection

$$(G(\widetilde{F})/H^1_{\widetilde{F}})^{\langle\sigma,\,t\rangle} \xrightarrow{\approx} (G(\widetilde{F})/H^1_{\widetilde{F}})^{\langle\sigma^d,\,s.\sigma\rangle}$$

given by $x \mapsto c^{-1}x$. That is, given $x \in G(F)$ such that $xH^1_F \in \mathfrak{S}(t)$, there exists $y \in \mathfrak{S}(s, \sigma) \subset G(K)$, which is uniquely determined modulo $\mathbf{H}^1_K$, such that

(12.14) $\qquad yH^1_{\widetilde{F}} = c^{-1}\,xH^1_{\widetilde{F}}.$

This process $x \mapsto y$ gives a bijection

(12.15) $\qquad \mathfrak{S}(t) \xrightarrow{\approx} \mathfrak{S}(s, \sigma).$

We have to show that (12.15) has the property demanded by the theorem. We can write the relation (12.14) as $y = c^{-1}\,xj$, for some $j \in H^1_{\widetilde{F}}$. We evaluate $\boldsymbol{\theta}_K(y^{-1}\,s.\sigma y)$. Using (12.3), we can adjust y on the right by an element of $\mathbf{H}^1_K$ to ensure that $\mathcal{N}_\sigma(y^{-1}\,s\sigma(y)) \in H^1_F$. This done, we have the relation

$$\mathcal{N}_\sigma(y^{-1}\,s\sigma(y)) = y^{-1}\,ty.$$

In particular we have $\boldsymbol{\theta}_K(y^{-1}\,s\sigma(y)) = \theta_F(y^{-1}\,ty)$ by (12.6). Now we substitute $y = c^{-1}\,xj$ to get

$$y^{-1}\,ty = j^{-1}\,x^{-1}\,ctc^{-1}\,xj.$$

By construction, the elements c, t commute: they both lie in $\mathfrak{o}_{\widetilde{F}}[[x_0]]$, for some $x_0 \in \mathfrak{H}^1_F$. This leaves us with the relation $y^{-1}\,ty = j^{-1}\,x^{-1}\,txj$, both sides lying in H^1_F. By definition, we also have $x^{-1}\,tx \in H^1_F$.

(12.16) *Lemma. — In the situation above, there exists* $k \in H^1_F$ *such that* $k^{-1}\,x^{-1}\,txk = y^{-1}\,ty$.

Proof. — We have $j = x^{-1}cy$, where $x \in G(F)$, $y \in G(K)$, and $c = C_\alpha(x_0)$ for some $x_0 \in \mathfrak{H}^1_F$ and a formal power series $C_\alpha(X) \in \mathfrak{o}_{\tilde F}[[X]]$. The construction of C_α shows that, given an integer n, there exists a finite unramified extension K_n/K and a polynomial $C_n(X) \in \mathfrak{o}_{K_n}[X]$ such that $C_\alpha(X) \equiv C_n(X) \pmod{\mathfrak{p}^n_{\tilde F}}$. By continuity, if n is sufficiently large, there exists $m(n) \geqslant 1$ such that $x^{-1}\mathfrak{H}^n_{\tilde F}y \subset \mathfrak{H}^{m(n)}_{\tilde F}$. Put another way, given n, we can find a finite unramified extension K_n/K and an element $j_n \in H^1_{K_n}$ such that $j = j_n h$, $h \in H^{n+1}_{\tilde F}$. Write $j_n = 1 + a_n$, $h = 1 + b$. Abbreviate $t_1 = y^{-1}ty$, $t_2 = x^{-1}tx$. Our relation reads

$$(1 + a_n)(1 + b)\, t_1 = t_2(1 + a_n)(1 + b),$$

whence

$$t_1 + a_n t_1 \equiv t_2 + t_2 a_n \pmod{\mathfrak{H}^{n+2}_{\tilde F}}.$$

Multiplying this by $\beta \in K_n$ such that $\mathrm{Tr}_{K_n/F}(\beta) = 1$ and taking the trace from K_n to F, we get

$$t_1 + c_n t_1 \equiv t_2 + t_2 c_n \pmod{\mathfrak{H}^{n+2}_{\tilde F}},$$

where

$$c_n = \mathrm{Tr}_{K_n/F}(\beta a_n) = \sum_{i=0}^{d_n-1} \sigma^i(\beta a_n) \in \mathfrak{H}^1_F, \quad d_n = [K_n : F].$$

Setting $k_n = 1 + c_n \in H^1_F$, we have $k_n^{-1}x^{-1}txk_n \equiv y^{-1}ty \pmod{H^{n+2}_{\tilde F}}$. Passing to the limit, we get the result. □

We now have

$$\theta_K(y^{-1}s.\sigma.y) = \theta_F(y^{-1}ty) = \theta_F(k^{-1}x^{-1}txk) = \theta_F(x^{-1}tx).$$

This completes the proof of (12.9) in the case where K/F is unramified.

We now assume that our finite cyclic extension K/F *is totally ramified.* Again we put $d = [K : F]$, and note that $p \nmid d$. By (12.5) (with $\alpha = d^{-1}$) we can assume that $s = 1 + b$, $b \in \mathfrak{H}^1_F$, so that $t = s^d$. Indeed, if we write $t = 1 + c$, $c \in \mathfrak{H}^1_F$, we have $b = \Phi(c)$ for a formal power series $\Phi(X) \in \mathfrak{o}_F[[X]]$ with leading term $d^{-1}X$.

Now take $y \in G(F)$, with $gH^1_F \in \mathfrak{S}(t)$. We then have

$$y^{-1}s.\sigma.y = y^{-1}sy.\sigma = (1 + \Phi(y^{-1}ty)).\sigma.$$

This element lies in $H^1_F.\sigma \subset \mathbf{H}^1_K.\sigma$, whence $y \in \mathfrak{S}(s, \sigma)$. Thus we have a map $\mathfrak{S}(t) \to \mathfrak{S}(s, \sigma)$ which is injective by (11.6).

In the opposite direction, take $x \in \mathfrak{S}(s, \sigma)$ viewed as an element of G(K). We again adjust x on the right by an element of $\mathbf{H}^1_K$ to achieve $x^{-1}s\sigma(x) \in H^1_F$, by (12.4). This implies $x^{-1}s^d x = x^{-1}tx \in H^1_F$. Writing $t = 1 + c$ as before, we have $x^{-1}cx \in \mathfrak{H}^1_F$ whence $x^{-1}sx = 1 + \Phi(x^{-1}cx)$ also lies in $\mathfrak{H}^1_F$. Therefore

$$\sigma(x) = xh,$$

for some $h \in H^1_F$. We thus have $\sigma^2(x) = \sigma(x)\, h = xh^2$ and, iterating,

$$x = \sigma^d(x) = xh^d.$$

Thus $h^d = 1$ and, since H^1_F is a pro-p-group and $p \nmid d$, we have $h = 1$ and so $x \in G(F)$. We have already seen that $x^{-1}\, tx \in H^1_F$, so $x\mathbf{H}^1_K$ lies in the image of $\mathfrak{S}(t)$. Thus we have a bijection which surely satisfies the requirements of the theorem. □

There is of course an analogue of (12.9) using the groups J in place of H. We exhibit this explicitly.

(12.17) *Theorem. — Use the notation* (12.2), *and let* η_F *(resp.* $\boldsymbol{\eta}_K$*) be the unique irreducible representation of* J^1_F *(resp.* $\mathbf{J}^1_K$*) which contains* θ_F *(resp.* $\boldsymbol{\theta}_K$*). Extend* $\boldsymbol{\eta}_K$*, in some manner, to a representation of* $\mathbf{J}^1_K \rtimes \Gamma$*. Let* $s \in \mathbf{J}^1_K$*,* $t \in J^1_F$*, and suppose that* t *is* $\mathbf{J}^1_K$*-conjugate to* $\mathcal{N}_\sigma\, s$*. Write*

$$\mathfrak{S}'(t) = \{ g \in G(F)/J^1_F : g^{-1}\, tg \in J^1_F \},$$

$$\mathfrak{S}'(s, \sigma) = \{ h \in G(K)/\mathbf{J}^1_K : h^{-1}\, s.\sigma.h \in \mathbf{J}^1_K.\sigma \}.$$

There exists a bijection $\varphi : \mathfrak{S}'(t) \to \mathfrak{S}'(s, \sigma)$ *with the property*

$$\mathrm{tr}(\boldsymbol{\eta}_K(\varphi(g)^{-1}\, s.\sigma.\varphi(g))) = \mathrm{tr}(\eta_F(g^{-1}\, tg))\, \frac{\mathrm{tr}(\boldsymbol{\eta}_K(\sigma))}{\dim \eta_F}, \quad g \in \mathfrak{S}'(t).$$

The proof is identical to that of (12.9), except that it relies on (12.8) in place of (12.6).

It is striking that the groups G play very little role in the proof of (12.9) (and, by implication, in that of (12.17)). In the case where K/F is unramified, we simply need to be able to form $G(\widetilde{F}) \supset H^1_{\widetilde{F}}$ with the fixed-point properties $G(\widetilde{F})^\sigma = G(F)$, $G(\widetilde{F})^{\sigma^d} = G(K)$. In the ramified case, we only need $G(K)$ and the property $G(K)^\sigma = G(F)$. There are many " functorial groups " between H^1_F and $G(F)$ with analogous properties, hence many variations on the theme of (12.8), (12.17). We list some of these.

The first comes from the unit groups of hereditary orders. Let $\mathfrak{A}(E)$ be as before, and recall the notation $\mathfrak{C} = \mathrm{End}^0_{\mathfrak{o}_K}(\mathscr{L}_{K\otimes E})$. We can define a functor of unramified extensions L/F by $L \mapsto \mathbf{U}_L = (\mathfrak{A}(E) \otimes_{\mathfrak{o}_F} \mathfrak{o}_L)^\times$. This certainly has the right fixed-point properties. On the other hand, if K/F is totally ramified, we need to know that

(12.18) $$\mathbf{U}(\mathfrak{C})^\Gamma = \mathbf{U}(\mathfrak{A}(E)).$$

Here we observe that the lattices of fixed points $\mathfrak{r}^m \cap E = (\mathfrak{r}^m)^\Gamma$ form a lattice chain in E, which must be stabilized by the ring $\mathfrak{C}^\Gamma$ of fixed points in $\mathfrak{C}$. However, the $\mathfrak{o}_E$-lattice chain $\{ \mathfrak{r}^m \cap E \}$ is just that of powers of $\mathfrak{p}_E$. Thus $\mathfrak{C}^\Gamma \subset \mathfrak{A}(E)$. We surely have $\mathfrak{A}(E) \subset \mathfrak{C}^\Gamma$, (examine its effect on the basis $\mathscr{B}$ of (11.5)) so (12.18) holds. Writing $\mathfrak{K}$ for the normalizer of a principal order, we similarly obtain $\mathfrak{K}(\mathfrak{C})^\Gamma = \mathfrak{K}(\mathfrak{A}(E))$. We now have:

(12.19) *Corollary.* — *In the situation of* (12.8), *there is a bijection* φ *between the set of* $g \in \mathbf{U}(\mathfrak{A}(\mathrm{E}))/\mathrm{H}^1_{\mathrm{F}}$ *such that* $g^{-1}tg \in \mathrm{H}^1_{\mathrm{F}}$ *and the set of* $h \in \mathbf{U}(\mathfrak{C})/\mathbf{H}^1_{\mathrm{K}}$ *such that* $h^{-1}s\sigma(h) \in \mathbf{H}^1_{\mathrm{K}}$ *with the property*

$$\boldsymbol{\theta}_{\mathrm{K}}(\varphi(g)^{-1}s\sigma(\varphi(g))) = \theta_{\mathrm{F}}(g^{-1}tg).$$

The result also holds if we replace $\mathbf{U}$ *by* $\mathfrak{K}$ *or* $\mathbf{U}^1$ *throughout.*

The analogue of (12.17) *likewise holds here.*

In the general situation of § 11, we can define

$$\mathfrak{J}_{\mathrm{K}} = \mathfrak{J}(\beta, \mathfrak{A}_{\mathrm{M}}) \cap \mathfrak{C},$$
$$\mathbf{J}_{\mathrm{K}} = \mathfrak{J}_{\mathrm{K}}^{\times} = \mathrm{J}(\beta, \mathfrak{A}_{\mathrm{M}}) \cap \mathbf{U}(\mathfrak{C}).$$

In the unramified case we get $\mathfrak{J}_{\mathrm{K}} = \mathfrak{J}(\beta, \mathfrak{A}(\mathrm{E})) \otimes \mathfrak{o}_{\mathrm{K}}$, hence $\mathbf{J}_{\mathrm{K}}^{\Gamma} = \mathrm{J}(\beta, \mathfrak{A}(\mathrm{E}))$. In the totally ramified cyclic case we likewise have

$$\mathbf{J}_{\mathrm{K}}^{\Gamma} = \mathrm{J}(\beta, \mathfrak{A}_{\mathrm{M}}) \cap \mathbf{U}(\mathfrak{A}(\mathrm{E})) = \mathrm{J}(\beta, \mathfrak{A}(\mathrm{E})).$$

Now abbreviate $\mathrm{J}_{\mathrm{F}} = \mathrm{J}(\beta, \mathfrak{A}(\mathrm{E}))$. The analogue of (12.19) thus holds with $(\mathbf{J}_{\mathrm{K}}, \mathrm{J}_{\mathrm{F}})$ in place of $(\mathbf{U}(\mathfrak{C}), \mathbf{U}(\mathfrak{A}(\mathrm{E})))$. However, conjugation by the groups J stabilizes the characters θ, so it is the bijection which is the point here. If $s \in \mathbf{H}^1_{\mathrm{K}}$ and $h \in \mathbf{J}_{\mathrm{K}}$, we have $h^{-1}s\sigma(h) \in \mathbf{H}^1_{\mathrm{K}}$ if and only if $h^{-1}\sigma(h) \in \mathbf{H}^1_{\mathrm{K}}$. We thus retrieve the cohomological triviality statements:

$$\left(\frac{\mathbf{J}_{\mathrm{K}}}{\mathbf{H}^1_{\mathrm{K}}}\right)^{\Gamma} = \frac{\mathrm{J}_{\mathrm{F}}}{\mathrm{H}^1_{\mathrm{F}}}, \quad \left(\frac{\mathbf{J}_{\mathrm{K}}}{\mathbf{J}^1_{\mathrm{K}}}\right)^{\Gamma} = \frac{\mathrm{J}_{\mathrm{F}}}{\mathrm{J}^1_{\mathrm{F}}}. \tag{12.20}$$

13. Characters of some finite group extensions

We relax from the rigours of the preceding analysis to prove a simple result from the representation theory of finite groups. Very similar results can be found in [Ge] and [Ho1], but these are not quite what we need. *The notation introduced here is valid for this section only.*

Let p be a not necessarily odd prime number and G an " extra-special finite p-group of class 2 ". We write Z for the centre of G and assume we have a faithful character χ of Z. Thus Z is cyclic, and contains the commutator group of G. We set $\mathrm{V} = \mathrm{G}/\mathrm{Z}$, and this is an elementary abelian p-group. (The example we have in mind is $\mathrm{G} = \mathbf{J}^1_{\mathrm{K}}/\mathrm{Ker}(\boldsymbol{\theta}_{\mathrm{K}})$, which satisfies these conditions by (11.3).)

The commutator pairing

$$(x, y) \mapsto \chi[x, y], \quad x, y \in \mathrm{G},$$

where $[x, y] = x^{-1}y^{-1}xy$, takes values in the group μ_p of p-th roots of unity in $\mathbf{C}$: this follows from the elementary identity $[x, yz] = [x, z]\,[x, y]^z$. It thus defines a non-degenerate alternating form $\mathrm{V} \times \mathrm{V} \to \mu_p$, which we denote by $\langle\,,\rangle$.

Let ξ denote the unique irreducible representation of G whose restriction to Z contains χ. We have $\dim \xi = \# V^{1/2}$. We assume given a cyclic group Γ of automorphisms of G which fix the centre Z. They therefore fix the character χ and the representation ξ. For elementary reasons, the representation ξ admits extension to a representation, still denoted ξ, of the semidirect product $G \rtimes \Gamma$. For $\gamma \in \Gamma$, we write G^γ for the group of fixed points (i.e. centralizer) of γ in G, and likewise for V.

(13.1) *Proposition. — In the situation above, suppose that the natural map* $G^\gamma \to V^\gamma$ *is surjective for all* $\gamma \in \Gamma$. *The character* $\mathrm{tr}(\xi)$ *of the representation* ξ *of* $G \rtimes \Gamma$ *then satisfies the following, for all* $\gamma \in \Gamma$:

(i) $\mathrm{tr}(\xi(g\gamma)) \neq 0$ *if and only if* $g\gamma$ *is conjugate in* $G\Gamma$ *to an element of* $Z\Gamma$.

(ii) *For* $z \in Z$, *we have* $|\,\mathrm{tr}(\xi(z\gamma))\,| = \#(V^\gamma)^{1/2}$.

Proof. — Let us write $g \mapsto \bar{g}$ for the quotient map $G \to V$. For $\gamma \in \Gamma$, let $\mathscr{I}_\gamma$ denote the space of elements $\{ v^{-1}\gamma(v) : v \in V \}$. We first note that an element $g\gamma$, $g \in G$, is conjugate to one of the form $z\gamma'$, for $\gamma' \in \Gamma$ and $z \in Z$, if and only if $\gamma' = \gamma$ and $\bar{g} \in \mathscr{I}_\gamma$. The next point to note is that $\mathscr{I}_\gamma$ is orthogonal, under the pairing $\langle\, ,\, \rangle$, to the space V^γ of γ-fixed points in V. Comparing sizes, and recalling that our pairing is nondegenerate, we see that $V^\gamma = (\mathscr{I}_\gamma)^\perp$, the orthogonal complement of $\mathscr{I}_\gamma$.

Suppose first that $g\gamma$ is not conjugate to an element of $Z\gamma$. Thus there exists $h \in G$ such that $\bar{h} \in V^\gamma$ and $\langle \bar{g}, \bar{h} \rangle \neq 1$. We can (and do) further choose $h \in G^\gamma$. Consider the element $hg\gamma h^{-1}$. We can write this as

$$\textbf{(13.2)} \qquad hg\gamma h^{-1} = hgh^{-1}g^{-1}.g\gamma.\gamma^{-1}h\gamma h^{-1}.$$

The first factor here lies in Z and the choice of h implies $\chi(hgh^{-1}g^{-1}) \neq 1$. Since h is fixed by γ, we have $\gamma^{-1}h\gamma h^{-1} = 1$. Applying ξ to (13.2) and taking the trace, we therefore get

$$\begin{aligned} \mathrm{tr}(\xi(g\gamma)) &= \mathrm{tr}(\xi(hg\gamma h^{-1})) \\ &= \chi(hgh^{-1}g^{-1})\,\mathrm{tr}(\xi(g\gamma)) \end{aligned}$$

while $\chi(hgh^{-1}g^{-1}) \neq 1$. Thus $\mathrm{tr}(\xi(g\gamma)) = 0$, as required.

This proves one implication in (i), and the converse follows from (ii). We prove (ii) by an argument lifted virtually *verbatim* from [Ho1]. First we note that our fixed-point hypothesis implies:

(13.3) *For* $z_1, z_2 \in Z$ *and* $\gamma \in \Gamma$, *the elements* $z_1\gamma$, $z_2\gamma$ *are conjugate in* $G\Gamma$ *if and only if* $z_1 = z_2$.

It will now be easier to write $c = \#\Gamma$, $g = \#G$, and define $\rho_\xi = \mathrm{tr}(\xi)$ as a function on $G\Gamma$. We fix a generator σ of Γ. Let α range over the linear characters of the finite cyclic group Γ, viewed as characters of $G\Gamma$ via $G\Gamma/G = \Gamma$, when convenient. The representations $\alpha \otimes \xi$ are then irreducible and mutually inequivalent: this follows readily from Clifford theory. At the level of characters therefore, $\alpha \mapsto \alpha\rho_\xi$ gives an injective map from the space of (class) functions on Γ to the space of class functions on $G\Gamma$. Since it takes irreducible characters to irreducible characters, it must preserve the canonical inner products on these function spaces.

For $0 \leqslant i \leqslant c-1$, let C_{ij}, $1 \leqslant j \leqslant n_i$, be the conjugacy classes of $G\Gamma$ whose image in Γ is σ^i. Now using " bar " to denote complex conjugation, the fact that $\alpha \mapsto \alpha\rho_\xi$ is an isometric injection on class functions gives us the relation

$$c^{-1} \sum_{i=0}^{c-1} \alpha_1(\sigma^i)\, \overline{\alpha}_2(\sigma^i) = (cg)^{-1} \sum_{i=0}^{c-1} \alpha_1(\sigma^i)\, \overline{\alpha}_2(\sigma^i) \sum_{j=1}^{n_i} |\, \rho_\xi(C_{ij})\, |^2\, \# C_{ij},$$

where α_1, α_2 are linear characters of Γ. Writing t_i for the inner sum on the right hand side, this says that the element

$$x = \sum_{i=0}^{c-1} (g^{-1} t_i - 1)\, \sigma^i$$

of the complex group algebra $\mathbf{C}[\Gamma]$ satisfies $\beta(x) = 0$ for all linear characters β of Γ. This implies $x = 0$ or, equivalently,

$$\sum_{j=1}^{n_i} |\, \rho_\xi(C_{ij})\, |^2\, \# C_{ij} = g, \quad 0 \leqslant i \leqslant c-1.$$

By part (i) of the Proposition, only those C_{ij} which meet $Z\Gamma$ can contribute to this sum. We number the C_{ij} so that C_{i1} is the conjugacy class of σ^i in $G\Gamma$. The conjugacy classes zC_{i1}, $z \in Z$, $0 \leqslant i \leqslant c-1$ are distinct by (13.3). Further, $\rho_\xi(zC_{i1}) = \chi(z)\, \rho_\xi(C_{i1})$. Substituting in the sum above, we get

$$|\, \rho_\xi(C_{i1})\, |^2\, \# C_{i1} = (G : Z), \quad 0 \leqslant i \leqslant c-1.$$

However, $\# C_{i1}$ is just the cardinality of the set $\mathscr{I}_{\sigma^i} = \{\, v^{-1}\, \sigma^i(v) : v \in V\,\}$, as we saw above. Therefore

$$|\, \rho_\xi(C_{i1})\, |^2 = (V : \mathscr{I}_{\sigma^i}) = \# V^{\sigma^i},$$

as required. $\square$

We apply this in the proof of (12.8), where $G = \mathbf{J}^1_K/\mathrm{Ker}(\boldsymbol{\theta}_K)$, $Z = \mathbf{H}^1_K/\mathrm{Ker}(\boldsymbol{\theta}_K)$, $\chi = \boldsymbol{\theta}_K$, $\xi = \eta_K$ and $\Gamma = \mathrm{Gal}(K/F)$. Further, the quotient $V = G/Z$ is just $\mathfrak{J}^1_K/\mathfrak{H}^1_K$. We have to check that the fixed-point hypothesis of (13.1) is satisfied. This is trivial when Γ has order prime to p. We may therefore assume that the extension K/F is unramified. In this case, (11.10) implies that the 1-cohomology of Γ' in $\mathfrak{H}^1_K$ is trivial, for any subgroup Γ' of Γ. Thus $(\mathfrak{J}^1_K)^\gamma = \mathfrak{J}^1_F$ maps onto $(\mathfrak{J}^1_K/\mathfrak{H}^1_K)^\gamma$, for any $\gamma \in \Gamma$.

14. Comparison with base change I

We return to the situation of § 12. In particular, K/F is a cyclic, tamely ramified extension of degree d, which is either unramified or totally ramified. We set $\Gamma = \mathrm{Gal}(K/F)$ and fix a generator σ of Γ. We now make some preliminary comparisons between our local lifting procedure and the base-change lift of [AC]. Of course, one must remember that the latter is only available in characteristic 0. For the time being, however, the characteristic of F remains arbitrary, and we make extensive use of the basic properties of characters set out in the Appendix. The first step is to derive a " trace comparison " formula valid in full generality.

We are given a simple stratum of the form $[\mathfrak{A}(E), n, 0, \beta]$ as before, except that in this section we insist

$$E = F[\beta].$$

Set

$$G(F) = \mathrm{Aut}_F(E) \cong GL(N, F),$$

say. We are given a simple character $\theta_F \in \mathscr{C}(\mathfrak{A}, 0, \beta)$. Let η_F denote the unique irreducible representation of the group $J_F^1 = J^1(\beta, \mathfrak{A}(E))$ whose restriction to $H_F^1 = H^1(\beta, \mathfrak{A})$ contains θ_F. We also write

$$\mathfrak{N} = \mathfrak{N}_{K/F} = N_{K/F}(K^\times),$$

where N denotes the field norm. Thus $F^\times \supset \mathfrak{N} \supset \mathbf{U}^1(\mathfrak{o}_F)$ and $\mathfrak{N}$ is of index $d = [K : F]$ in $F^\times$. We fix a quasicharacter ω_F of $\mathfrak{N}$, and define a quasicharacter ω_K of $K^\times$ by

$$\omega_K = \omega_F \circ N_{K/F}.$$

We write η_F^ω for the representation of $\mathfrak{N}J_F^1$ which extends η_F and is a multiple of ω_F on $\mathfrak{N}$. It will be convenient to write $\mathbf{J} = E^\times J_F^1$. We can decompose the coset space $G(F)/\mathbf{J}$ into $(J_F^1, \mathbf{J})$-double cosets,

$$G(F)/\mathbf{J} = \bigcup_{x \in J_F^1\backslash G(F)/\mathbf{J}} \ \bigcup_{y \in J_F^1 x\mathbf{J}/\mathbf{J}} y\mathbf{J}. \tag{14.1}$$

We consider the series

$$X_F(t) = \sum_{x \in J_F^1\backslash G(F)/\mathbf{J}} \Big(\sum_{y \in J_F^1 x\mathbf{J}/\mathbf{J}} \mathrm{tr}\, \eta_F^\omega (y^{-1} t y)\Big). \tag{14.2}$$

Here, $t \in G(F)_{qr}$, the set of *quasi-regular* elements of G(F): see Appendix (A.2) *et seq.* for this term. We view $g \mapsto \mathrm{tr}\, \eta_F^\omega(g)$ as a function on G(F) which is zero outside $\mathfrak{N}J_F^1$. The group $\mathbf{J}$ normalizes J_F^1 and fixes the representation η_F, so the (extended) function $\mathrm{tr}\, \eta_F^\omega$ is invariant under conjugation by $\mathbf{J}$.

(14.3) *Proposition.* — *We use the notation given at the beginning of the section. Let*

$$r = e(\mathrm{E} \mid \mathrm{F})\ (\mathrm{k}_\mathrm{E}^\times : \mathrm{k}_\mathrm{F}^\times)\ [\mathrm{K} : \mathrm{F}].$$

Then:

(i) *there are precisely r inequivalent irreducible smooth representations $\pi_1, \pi_2, \ldots, \pi_r$ of* G(F) *which contain the representation η_F^ω;*
(ii) *each π_i is supercuspidal.*

Proof. — Our representation η_F^ω extends to a representation of the group $\mathbf{J} = \mathrm{E}^\times \mathrm{J}_\mathrm{F}^1$. There are $r = (\mathbf{J} : \mathrm{J}_\mathrm{F}^1 \mathfrak{N})$ such extensions Λ_i, $1 \leqslant i \leqslant r$, and no two of them intertwine, by [BK1] (5.2.2), (6.1.2). Moreover, the smooth representation π_i of G(F) induced by Λ_i is irreducible and supercuspidal, by [BK1] (6.2.2). This proves (i) and (ii). □

If π is an irreducible smooth representation of G(F), we write Θ_π for the function on $\mathrm{G(F)}_{\mathrm{qr}}$ which represents the character of π, in the sense of (A.11). Thus, in particular, Θ_π is a locally constant function on $\mathrm{G(F)}_{\mathrm{qr}}$.

(14.4) *Proposition.* — (i) *The series* (14.2) *converges absolutely and uniformly on compact subsets of* $\mathrm{G(F)}_{\mathrm{qr}}$, *and represents a locally constant function there. Indeed, the outer sum in* (14.2) *has only finitely many non-zero terms.*

(ii) *Let $\pi_1, \ldots, \pi_r$ be as in* (14.3). *We have*

$$\frac{1}{r}\sum_{i=1}^{r} \Theta_{\pi_i}(t) = \mathrm{X}_\mathrm{F}(t),$$

for every $t \in \mathrm{G(F)}_{\mathrm{qr}}$.

Proof. — Write $\pi_i = c\text{-Ind}(\Lambda_i)$, as in the proof of (14.3). We invoke (A.14) below, with J_F^1 playing the role of K_1 in the arguments leading to that result. Thus

$$\Theta_{\pi_i}(t) = \sum_{x \in \mathrm{J}_\mathrm{F}^1\backslash \mathrm{G(F)}/\mathbf{J}} \Big(\sum_{y \in \mathrm{J}_\mathrm{F}^1 x \mathbf{J}/\mathbf{J}} \operatorname{tr} \Lambda_i(y^{-1} ty)\Big)$$

for every element $t \in \mathrm{G(F)}_{\mathrm{qr}}$. The convergence is absolute and uniform on compact subsets of G_{qr}; indeed, there are only finitely many non-zero terms in the outer sum here. The summand $\operatorname{tr} \Lambda_i(y^{-1} ty)$ is zero unless $y^{-1} ty \in \mathbf{J}$. However, for $y^{-1} ty \in \mathbf{J}$, we have

$$\sum_{i=1}^{r} \operatorname{tr} \Lambda_i(y^{-1} ty) = \begin{cases} r \operatorname{tr} \eta_\mathrm{F}^\omega(y^{-1} ty) & \text{if } y^{-1} ty \in \mathfrak{N}\mathrm{J}_\mathrm{F}^1, \\ 0 & \text{otherwise} \end{cases}$$

We can now add the series Θ_{π_i} term by term to get the identity in (ii), and the resulting sum has the properties required by (i). □

Notice that (14.4) (ii) tells us that X_F is a *class function* on $\mathrm{G(F)}_{\mathrm{qr}}$, i.e.

$$\mathrm{X}_\mathrm{F}(g^{-1} tg) = \mathrm{X}_\mathrm{F}(t), \quad t \in \mathrm{G(F)}_{\mathrm{qr}}, \quad g \in \mathrm{G(F)}.$$

We now take $\mathbf{J}_K^1$, η_K as in (11.1), and we set $\mathbf{K}_K = K^\times \mathbf{J}_K^1$. By (11.9), there si a unique irreducible representation η_K^ω of $\mathbf{K}_K$ which extends η_K and is a multiple of ω_K on $K^\times$. We fix a representation $\tilde{\eta}_K^\omega$ of $\mathbf{K}_K \rtimes \Gamma$ which extends η_K^ω, and write $\tilde{\eta}_K$ for its restriction to $\mathbf{J}_K^1 \rtimes \Gamma$. We put (without yet worrying about convergence):

$$X_K(s\sigma) = \frac{\dim \eta_F}{\operatorname{tr} \tilde{\eta}_K(\sigma)} \sum_{x \in J_K^1 \backslash G(K)/\mathbf{K}_K} \Big(\sum_{y \in J_K^1 x \mathbf{K}_K/\mathbf{K}_K} \operatorname{tr} \tilde{\eta}_K^\omega(y^{-1} s\sigma y)\Big),$$

where $s \in G(K)$. We shall actually only be concerned here with a rather restricted class of elements s.

(14.5) *Theorem. — Let $s \in \mathbf{K}_K = K^\times \mathbf{J}_K^1$, and let $t \in \mathfrak{N} J_F^1$ be $\mathbf{J}_K^1$-conjugate to $\mathcal{N}_\sigma s$. Suppose also that t is a quasi-regular element of* $G(F)$. *Write $d = [K : F]$, $r = e(E \mid F)\,(k_E^\times : k_F^\times)\, d$ as before. We have*

$$X_F(t) = \frac{d}{r} X_K(s\sigma).$$

The series defining $X_K(s\sigma)$ converges absolutely and uniformly on compact sets of elements $s \in \mathbf{K}_K$ such that $\mathcal{N}_\sigma s$ is conjugate to a quasi-regular element of $G(F)$. *Indeed, for such elements s, there are only finitely many non-zero terms in the outer sum defining $X_K(s\sigma)$.*

Proof. — The contribution of the centres is effectively trivial here, so we may as well take $s \in \mathbf{J}_K^1$. The inner sum in the definition of X_K is constant on $\mathbf{J}_K^1$-conjugacy classes, so we may as well assume (using (12.3)) that $t = \mathcal{N}_\sigma s \in J_F^1$.

The next step is to re-arrange our definition of $X_F(t)$. For this paragraph, we only need $t \in G(F)_{qr}$. We put $\mathbf{K}_F = \mathfrak{N} J_F^1$ and try to sum according to $(J_F^1, \mathbf{K}_F)$-double cosets. We note to start with that any double coset $J_F^1 x \mathbf{J}$ is the union of exactly r distinct $(J_F^1, \mathbf{K}_F)$-double cosets. For, $J_F^1 x \mathbf{J}$ consists of exactly $(J_F^1 : J_F^1 \cap x \mathbf{J} x^{-1})$ cosets $y\mathbf{J}$. However, $\mathbf{J}$ has a unique maximal pro-p-subgroup, namely J_F^1, so $J_F^1 \cap x\mathbf{J}x^{-1} = J_F^1 \cap x J_F^1 x^{-1}$. Likewise, a double coset $J_F^1 x \mathbf{K}_F$ consists of $(J_F^1 : J_F^1 \cap x J_F^1 x^{-1})$ cosets $y\mathbf{K}_F$. Now we recall that $r = (\mathbf{J} : \mathbf{K}_F)$, and our assertion follows. More precisely, we have

$$J_F^1 x \mathbf{J} = \dot{\bigcup_{j \in \mathbf{J}/\mathbf{K}_F}} J_F^1 xj \mathbf{K}_F, \quad x \in G(F).$$

The sum

$$\sum_{y \in J_F^1 xj \mathbf{K}_F/\mathbf{K}_F} \operatorname{tr} \eta_F^\omega(y^{-1} ty)$$

is clearly independent of $j \in \mathbf{J}/\mathbf{K}_F$, so we can write

$$\textbf{(14.6)} \qquad X_F(t) = \frac{1}{r} \sum_{x \in J_F^1 \backslash G(F)/\mathbf{K}_F} \ \sum_{y \in J_F^1 x \mathbf{K}_F/\mathbf{K}_F} \operatorname{tr} \eta_F^\omega(y^{-1} ty),$$

where $t \in G(F)_{qr}$, without affecting the convergence properties.

Now we use the comparison theorem (12.17). Recall that we have arranged $t = \mathcal{N}_\sigma s \in \mathbf{J}^1_F$. Let $\mathfrak{S}(t)$ be the set of cosets $g\mathbf{J}^1_F \in G(F)/\mathbf{J}^1_F$ such that $g^{-1}tg \in \mathbf{J}^1_F$ (which, we note, is equivalent to $g^{-1}tg \in \mathbf{K}_F$). Similarly define $\mathfrak{S}(s, \sigma)$. Note here, however, that $s\sigma$ has conjugates in $\mathbf{K}_K\,\sigma$ which do not lie in $\mathbf{J}^1_K\,\sigma$ (see (14.8) below). Theorem (12.17) gives us a bijection $\varphi : \mathfrak{S}(t) \to \mathfrak{S}(s, \sigma)$ with the property (in our present notation)

$$\mathrm{tr}\, \eta^\omega_F(g^{-1}tg) = k_\sigma\, \mathrm{tr}\, \bar{\eta}^\omega_K(\varphi(g)^{-1}s\sigma\varphi(g)), \quad g\mathbf{J}^1_F \in \mathfrak{S}(t),$$

where k_σ denotes the constant $\dim \eta_F/\mathrm{tr}\, \bar{\eta}_K(\sigma)$.

(14.7) *Lemma. — In the situation above, we have*

$$\mathbf{J}^1_K x \mathbf{J}^1_K \cap G(F) = \mathbf{J}^1_F x \mathbf{J}^1_F, \quad x \in G(F),$$

and

$$\varphi(\mathbf{J}^1_F x \mathbf{J}^1_F \cap \mathfrak{S}(t)) = \mathbf{J}^1_K\, \varphi(x)\, \mathbf{J}^1_K \cap \mathfrak{S}(s, \sigma), \quad x\mathbf{J}^1_F \in \mathfrak{S}(t).$$

We prove this later. To proceed with the present argument, take $x\mathbf{J}^1_F \in \mathfrak{S}(t)$ and consider the sum

$$\sum_{y \in \mathbf{J}^1_F x \mathbf{K}_F/\mathbf{K}_F} \mathrm{tr}\, \eta^\omega_F(y^{-1}ty) = \sum_{y \in \mathbf{J}^1_F x \mathbf{J}^1_F/\mathbf{J}^1_F} \mathrm{tr}\, \eta^\omega_F(y^{-1}ty).$$

According to (12.17) and (14.7), this is equal to

$$k_\sigma \cdot \sum_{z \in \mathbf{J}^1_K\, \varphi(x)\, \mathbf{J}^1_K/\mathbf{J}^1_K} \mathrm{tr}\, \bar{\eta}^\omega_K(z^{-1}s\sigma z) = k_\sigma \cdot \sum_{z \in \mathbf{J}^1_K\, \varphi(x)\, \mathfrak{N}\mathbf{J}^1_K/\mathfrak{N}\mathbf{J}^1_K} \mathrm{tr}\, \bar{\eta}^\omega_K(z^{-1}s\sigma z).$$

We return to our expression (14.6) for $X_F(t)$, and note that we can replace $G(F)$ by $\mathbf{J}^1_F\mathfrak{S}(t)$ and thereby change nothing. We now have

$$X_F(t) = \frac{k_\sigma}{r} \sum_{x \in \mathbf{J}^1_K \backslash \mathbf{J}^1_K \mathfrak{S}(s,\sigma)/\mathbf{J}^1_K \mathfrak{N}} \Big(\sum_{y \in \mathbf{J}^1_K x \mathbf{J}^1_K \mathfrak{N}/\mathbf{J}^1_K \mathfrak{N}} \mathrm{tr}\, \bar{\eta}^\omega_K(y^{-1}s\sigma y)\Big).$$

(14.8) *Lemma. — An element $g \in G(K)$ satisfies $g^{-1}s\sigma g \in \mathbf{K}_K$ if and only if $g = ah$, where $a \in K^\times$ and $h \in \mathfrak{S}(s, \sigma)$. Moreover, if $h \in \mathfrak{S}(s, \sigma)$ and $a \in K^\times$, then $ah \in \mathfrak{S}(s, \sigma)$ if and only if $a \in F^\times\, \mathbf{U}^1(\mathfrak{o}_K)$.*

Proof. — If $h \in \mathfrak{S}(s, \sigma)$ and $a \in K^\times$, then $(ah)^{-1}s\sigma ah = a^{-1}\sigma(a)\,h^{-1}s\sigma h$. This proves the last assertion and one implication of the first. Now let $g \in G(K)$ and assume that $g^{-1}s\sigma g = as'\sigma$, with $a \in K^\times$ and $s' \in \mathbf{J}^1_K$. Appealing to (12.3), we can adjust g on the right by an element of $\mathbf{J}^1_K$ to ensure that $\mathcal{N}_\sigma s' \in \mathbf{J}^1_F$. We then get $\mathcal{N}_\sigma(as') = N_{K/F}(a)\,\mathcal{N}_\sigma s'$, which must be $G(F)$-conjugate to $t = \mathcal{N}_\sigma s \in \mathbf{J}^1_F$. This is only possible if $N_{K/F}(a) \in \mathbf{U}^1(\mathfrak{o}_F)$. Since K/F is tamely ramified, we have $N_{K/F}(\mathbf{U}^1(\mathfrak{o}_K)) = \mathbf{U}^1(\mathfrak{o}_F)$. In other words, we can adjust a by a 1-unit to achieve $N_{K/F}(a) = 1$. By Hilbert 90, a is of the form $b^{-1}\sigma(b)$ for some $b \in K^\times$. This gives $(b^{-1}g)^{-1}s\sigma b^{-1}g \in \mathbf{J}^1_K$, as required for the Lemma. □

Now we return to the main argument. Lemma (14.8) says that

$$\mathfrak{S}(s, \sigma)\, \mathbf{K}_K = \dot{\bigcup_{a \in \mathbf{K}_K/F^\times \mathbf{J}^1_K}} \mathfrak{S}(s, \sigma)\, a.$$

Therefore

$$X_F(t) = \frac{k_\sigma}{r(\mathbf{K}_K : F^\times \mathbf{J}_K^1)} \sum_{x \in \mathbf{J}_K^1 \backslash \mathfrak{S}(s,\sigma)\mathbf{K}_K/\mathbf{J}_K^1 \mathfrak{N}} \Big(\sum_{y \in \mathbf{J}_K^1 x \mathbf{J}_K^1 \mathfrak{N}/\mathbf{J}_K^1 \mathfrak{N}} \operatorname{tr} \bar{\boldsymbol{\eta}}_K^\omega(y^{-1} s\sigma y)\Big).$$

According to (14.8), we can replace $\mathfrak{S}(s, \sigma)\, \mathbf{K}_K$ by G(K) here without changing anything. By construction (and (14.4)), the outer series here has only finitely many non-zero terms. A coset $\mathbf{J}_K^1 x \mathbf{K}_K$ is the disjoint union of $(\mathbf{K}_K : \mathbf{J}_K^1 \mathfrak{N})$ distinct $(\mathbf{J}_K^1, \mathbf{J}_K^1 \mathfrak{N})$-double cosets, each of which contributes equally to the sum. The coset spaces $\mathbf{J}_K^1 x \mathbf{J}_K^1 \mathfrak{N}/\mathbf{J}_K^1 \mathfrak{N}$ and $\mathbf{J}_K^1 x \mathbf{K}_K/\mathbf{K}_K$ are effectively identical, so we now have

$$X_F(t) = \frac{k_\sigma}{r}\, \frac{(\mathbf{K}_K : \mathbf{J}_K^1 \mathfrak{N})}{(\mathbf{K}_K : F^\times \mathbf{J}_K^1)} \sum_{x \in \mathbf{J}_K^1 \backslash G(K)/\mathbf{K}_K}\ \sum_{y \in \mathbf{J}_K^1 x \mathbf{K}_K/\mathbf{K}_K} \operatorname{tr} \bar{\boldsymbol{\eta}}_K^\omega(y^{-1} s\sigma y).$$

Since $d = (F^\times : \mathfrak{N}) = (F^\times \mathbf{J}_K^1 : \mathfrak{N}\mathbf{J}_K^1)$, we have proved the theorem. □

Remark. — The identity (14.5) is distinctly intriguing. The left hand side can be interpreted, via (14.4), as a finite sum of characters of irreducible supercuspidal representations, using the Mackey formula derived from the description of these representations as induced representations. The right hand side has very much the same appearance, and, as we shall see, there is a special case in which it admits an analogous description. However, no such interpretation is available in general.

We now specialize to the case in which *the algebra* $K \otimes_F E$ *is a field.* We tend to abbreviate $K \otimes E = KE$. The constructions of (11.2) then yield a simple stratum $[\mathfrak{C}, n_M, 0, \beta]$ in $\mathrm{End}_K(K \otimes E)$, and the groups $\mathbf{H}_K^1$, $\mathbf{J}_K^1$ are the more familiar $H^1(\beta, \mathfrak{C})$, $J^1(\beta, \mathfrak{C})$. We therefore just denote them by H_K^1, J_K^1. Likewise, the character $\boldsymbol{\theta}_K$ is a simple character in the ordinary sense, and an element of $\mathscr{C}(\mathfrak{C}, 0, \beta)$. We thus denote it θ_K. Its K-endo-class is the unique K/F lift of the endo-class of the original character θ_F.

For convenience of reference, we summarize our hypotheses and notations.

(14.9) *Hypotheses.* — (i) *Let* $[\mathfrak{A}, n, 0, \beta]$ *be a simple stratum in* $A_F = \mathrm{End}_F(E)$, *where* E *denotes the field* $F[\beta]$. *(Thus* $\mathfrak{A} = \mathfrak{A}(E)$.*) Let* $\theta_F \in \mathscr{C}(\mathfrak{A}, 0, \beta)$ *be a simple character, and let* η_F *be the unique irreducible representation of the group* $J^1(\beta, \mathfrak{A})$ *which contains* θ_F.

(ii) *Let* K/F *be a finite cyclic field extension, which is either unramified or totally tamely ramified. Fix a generator* σ *of* $\Gamma = \mathrm{Gal}(K/F)$. *Write* $\mathfrak{N} = N_{K/F}(K^\times)$, *and suppose that the algebra* $E \otimes_F K$ *is a field.*

(iii) *Let* $\mathfrak{C}$ *be the unique hereditary* $\mathfrak{o}_K$*-order in* $A_K = \mathrm{End}_K(E \otimes K)$ *which is normalized by* $(E \otimes K)^\times$, *so that* $[\mathfrak{C}, n_K, 0, \beta]$ *is a simple stratum, for some* n_K. *Let* $\theta_K \in \mathscr{C}(\mathfrak{C}, 0, \beta)$ *be the simple character defined by* θ_F *as in* § 11. *Let* η_K *be the unique irreducible representation of* $J^1(\beta, \mathfrak{C})$ *which contains* θ_K. *Fix an extension* $\bar{\eta}_K$ *of* η_K *to a representation of* $J^1(\beta, \mathfrak{C}) \rtimes \Gamma$.

(iv) *Let* ω_F *be a quasicharacter of* $\mathfrak{N}$ *which agrees with* θ_F *on* $\mathfrak{N} \cap H^1(\beta, \mathfrak{A}) = \mathbf{U}^1(\mathfrak{o}_F)$, *and put* $\omega_K = \omega_F \circ N_{K/F}$. *Let* η_F^ω *denote the representation of* $\mathfrak{N}J^1(\beta, \mathfrak{A})$ *which extends* η_F *and is a multiple of* ω_F *on* $\mathfrak{N}$. *Likewise define a representation* $\bar{\eta}_K^\omega$ *of* $K^\times J^1(\beta, \mathfrak{C})$.

(v) *Let* $\mathscr{S}_F$ *denote the set of equivalence classes of irreducible representations of* $G(F) = \mathrm{Aut}_F(E)$ *which contain* η_F^ω.

(vi) *Let* $\mathscr{S}_K$ *denote the set of equivalence classes of irreducible representations* ρ *of the group* $G(K) = \mathrm{Aut}_K(KE) \cong GL(N, K)$ *which contain* η_K^ω *and satisfy* $\rho^\sigma \cong \rho$.

We recall from (14.3) that the set $\mathscr{S}_F$ consists of classes of *supercuspidal* representations. We now show that a similar property holds for $\mathscr{S}_K$.

(14.10) *Proposition. — In the notation of* (14.9), *we have:*

(i) *The set* $\mathscr{S}_K$ *consists of classes of supercuspidal representations of* $G(K)$, *and* $\sharp\mathscr{S}_K = d^{-1}\,\sharp\mathscr{S}_F$.

(ii) *If* $\rho \in \mathscr{S}_K$, *there exists a unique extension* $\bar\rho$ *of* ρ *to a representation of* $G(K) \rtimes \Gamma$ *such that* $\bar\rho$ *contains* $\bar\eta_K^\omega$. *Indeed, there is a uniquely determined representation* $\bar\Lambda$ *of the group* $(KE)^\times J_K^1$ *which extends* $\bar\eta_K^\omega$ *such that*

$$\bar\rho \cong c\text{-Ind}(\bar\Lambda : (KE)^\times J_K^1 \rtimes \Gamma, G(K) \rtimes \Gamma).$$

Proof. — We can argue exactly as in (14.3): an irreducible representation ρ containing η_K^ω is supercuspidal, and is induced from a uniquely determined representation Λ of $(KE)^\times J_K^1$ which extends η_K^ω. We deduce that $\rho^\sigma \cong \rho$ if and only if $\Lambda^\sigma \cong \Lambda$.

We first show there exists a σ-invariant extension Λ. We start with extensions of η_K to the group $J_K = J(\beta, \mathfrak{C})$. We have $J_K = \mu(KE) \ltimes J_K^1$, where $\mu(KE)$ is the group of p-prime roots of unity in the field KE. An extension κ of η_K to J_K is thus determined by the character $\det\kappa \mid \mu(KE)$, and any character of $\mu(KE)$ can arise in this context. The representation κ thus agrees with ω_K on $\mu(KE)$ if and only if $\det\kappa \mid \mu(K) = \omega_K^{\dim \eta_K}$. However, ω_K can be extended to a σ-invariant character of $\mu(KE)$: we simply choose a character χ of $\mu(E)$ which agrees with ω_F on $\mathfrak{N} \cap \mu(F)$, and $\chi \circ N_{KE/E}$ is a σ-stable extension of ω_K to $\mu(KE)$. Thus if κ is an extension of η_K^ω with $\det\kappa \mid \mu(KE) = (\chi \circ N_{KE/E})^{\dim \eta_K}$, we have $\kappa^\sigma \cong \kappa$, as desired. The step to $(KE)^\times J_K^1$ is trivial if K/F is unramified, since this group of generated by a prime of E and J_K. If K/F is totally ramified, the group $(KE)^\times J_K^1$ is generated by J_K and any prime ϖ of KE. We can choose ϖ so that $\varpi^\sigma = \varpi\zeta$, where ζ is a d-th root of unity lying in F. Our extension Λ satisfies $\Lambda^\sigma(\varpi) = \Lambda(\varpi)\,\omega_K(\zeta)$, while $\omega_K(\zeta) = \omega_F(N_{K/F}(\zeta)) = \omega_F(\zeta^d) = 1$. Thus, in all cases, an extension Λ of η_K^ω is σ-invariant if and only if $\Lambda \mid J_K$ is σ-invariant. We conclude that σ-invariant extensions Λ exist.

We can identify $(KE)^\times J_K^1/J_K^1$ with $(KE)^\times/\mathbf{U}^1(\mathfrak{o}_{KE})$ as Γ-module. The σ-invariant extensions of η_K^ω are thus classified by the σ-invariant characters of $(KE)^\times$ which are trivial on $K^\times\,\mathbf{U}^1(\mathfrak{o}_{KE})$. The extension KE/E is cyclic and tamely ramified, so the σ-invariant characters of $(KE)^\times$ are precisely those which factor through $N_{KE/E}$. Our desired set of characters is thus in bijection with the group $N_{KE/E}((KE)^\times)/\mathfrak{N}\mathbf{U}^1(\mathfrak{o}_E)$, which has precisely $e(E \mid F)\,(k_E^\times : k_F^\times)$ elements. Comparing with (14.3), we see that we have proved (i).

Now take $\rho \in \mathscr{S}_K$. Since Γ is cyclic, we can certainly extend ρ to a representation ρ'

of $G(K) \rtimes \Gamma$. This contains an irreducible representation ξ of the group $(KE)^\times J_K^1 \rtimes \Gamma$ which itself contains η_K^ω. Thus ξ is an extension of η_K^ω. Since $\boldsymbol{\rho}$ contains η_K^ω with multiplicity one, the representations $\boldsymbol{\rho}'$, ξ determine each other. We can now choose the extension $\boldsymbol{\rho}'$ so that $\xi = \bar{\eta}_K^\omega$ on $\mathbf{K}_K \Gamma$. The final relation is now immediate. □

(14.11) *Theorem. — Use the notation and hypotheses of* (14.9), (14.10). *For* $\boldsymbol{\pi} \in \mathscr{S}_F$, *let* $\boldsymbol{\Theta}_\pi$ *denote the locally constant function on the set of quasi-regular elements of* $G(F)$ *which represents the character of* $\boldsymbol{\pi}$ *there. We use a similar notation for the character of* $\bar{\boldsymbol{\rho}}$, $\boldsymbol{\rho} \in \mathscr{S}_K$. *We have*

$$\sum_{\rho \in \mathscr{S}_K} \boldsymbol{\Theta}_{\bar{\rho}}(s\sigma) = d^{-1} \frac{\operatorname{tr} \bar{\eta}_K^\omega(\sigma)}{\dim \eta_F} \sum_{\pi \in \mathscr{S}_F} \boldsymbol{\Theta}_\pi(t),$$

for every pair (s, t), $s \in G(K)$, $t \in G(F)_{qr}$, *such that* t *is* $G(K)$*-conjugate to* $\mathcal{N}_\sigma s$.

Proof. — It will be convenient to abbreviate

$$\mathbf{J}_K = (KE)^\times J_K^1.$$

We shall also need the group

$$\mathfrak{K} = \mathbf{K}_K . \operatorname{Ker}(N_{KE/E}).$$

(14.12) *Lemma.* — (i) *The group* $\mathfrak{K}\Gamma$ *is a normal subgroup of* $\mathbf{J}_K \Gamma$. *There is a unique irreducible representation* λ *of* $\mathfrak{K}\Gamma$ *such that* $\lambda \mid \mathbf{K}_K \Gamma = \bar{\eta}_K^\omega$.

(ii) *For any* $g \in G(K)$, *we have*

$$\operatorname{tr} \lambda(g\sigma) = (\mathbf{J}_K : \mathfrak{K})^{-1} \sum_{x \in \mathbf{J}_K/\mathbf{K}_K} \operatorname{tr} \bar{\eta}_K^\omega(x^{-1} g\sigma x).$$

Proof. — Part (i) is essentially contained in the proof of (14.10). In part (ii), both sides vanish unless $g \in \mathfrak{K}$. However, any element of $\mathfrak{K}\sigma$ is $\mathbf{J}_K$-conjugate to an element of $\mathbf{K}_K\, \sigma$. For, the typical element of $\mathfrak{K}$ is $\sigma^{-1}(\alpha^{-1})\, \alpha j$, for some $\alpha \in KE$ and $j \in \mathbf{K}_K$. But

$$\sigma^{-1}(\alpha)\, \sigma^{-1}(\alpha^{-1})\, \alpha j \sigma \sigma^{-1}(\alpha^{-1}) = \alpha j \alpha^{-1}\, \sigma.$$

So, to prove (ii), we might just as well take $g \in \mathbf{K}_K$. The sum

$$\sum_{x \in \mathbf{J}_K/\mathbf{K}_K} \operatorname{tr} \bar{\eta}_K^\omega(x^{-1} g\sigma x)$$

is the value at $g\sigma$ of the character of the representation, call it κ, of $\mathbf{J}_K\, \Gamma$ induced by $\bar{\eta}_K^\omega$. We can divide κ into two parts. First, every extension of $\bar{\eta}_K^\omega$ to $\mathbf{J}_K\, \Gamma$ occurs in κ with multiplicity one. Write κ_0 for the sum of such subrepresentations of κ. We observe that κ_0 is the representation of $\mathbf{J}_K\, \Gamma$ induced by λ. Let κ_1 be the complement of κ_0 in κ, and let τ be an irreducible component of κ_1. Take an irreducible component τ_1 of $\tau \mid \mathbf{J}_K$, and let Δ denote the group of $\gamma \in \Gamma$ such that $\tau_1^\gamma \cong \tau_1$. By definition, $\Delta \neq \Gamma$, and τ is induced from some extension of τ_1 to $\mathbf{J}_K\,\Delta$ (which is a normal subgroup of $\mathbf{J}_K\,\Gamma$). It follows that $\operatorname{tr}\tau(h\sigma) = 0$ for every $h \in \mathbf{J}_K$. We deduce

$$\sum_{x \in \mathbf{J}_K/\mathbf{K}_K} \operatorname{tr} \bar{\eta}_K^{\omega}(x^{-1} g\sigma x) = \operatorname{tr} \kappa_0(g\sigma),$$

for every $g \in \mathbf{J}_K$. However, since κ_0 is induced by λ, we have

$$\begin{aligned}\operatorname{tr} \kappa_0(g\sigma) &= \sum_{x \in \mathbf{J}_K/\mathfrak{K}} \operatorname{tr} \lambda(x^{-1} g\sigma x) \\ &= (\mathfrak{K} : \mathbf{K}_K)^{-1} \sum_{x \in \mathbf{J}_K/\mathbf{K}_K} \operatorname{tr} \lambda(x^{-1} g\sigma x) \\ &= (\mathbf{J}_K : \mathfrak{K}) \operatorname{tr} \lambda(g\sigma),\end{aligned}$$

as required. □

Now we rearrange our definition of X_K. Abbreviate

$$k_\sigma = \frac{\dim \eta_F}{\operatorname{tr} \eta_K(\sigma)}.$$

For the moment, we let $g \in \mathbf{K}_K$ be such that $\mathcal{N}_\sigma\, g$ is quasi-regular in $G(F)$. Theorem (14.5) allows us to group terms

$$X_K(g\sigma) = k_\sigma \sum_{x \in \mathbf{J}_K^1 \backslash G(K)/\mathbf{J}_K} \; \sum_{y \in \mathbf{J}_K^1 x \mathbf{J}_K/\mathbf{K}_K} \operatorname{tr} \bar{\eta}_K^{\omega}(y^{-1} g\sigma y).$$

We can rewrite the inner sum here as

$$\sum_{y \in \mathbf{J}_K^1 x \mathbf{K}_K/\mathbf{K}_K} \; \sum_{t \in \mathbf{J}_K/\mathbf{K}_K} \operatorname{tr} \bar{\eta}_K^{\omega}(t^{-1} y^{-1} g\sigma y t) = (\mathbf{J}_K : \mathfrak{K}) \sum_{y \in \mathbf{J}_K^1 x \mathbf{K}_K/\mathbf{K}_K} \operatorname{tr} \lambda(y^{-1} g\sigma y).$$

However, λ is a representation of $\mathfrak{K}\Gamma$, which is a normal subgroup of $\mathbf{J}_K\,\Gamma$, so the summand here is invariant under $\mathbf{J}_K$. This gives us

$$X_K(g\sigma) = k_\sigma (\mathbf{J}_K : \mathfrak{K}) \sum_{x \in \mathbf{J}_K^1 \backslash G(K)/\mathbf{J}_K} \; \sum_{y \in \mathbf{J}_K^1 x \mathbf{J}_K/\mathbf{J}_K} \operatorname{tr} \lambda(y^{-1} g\sigma y). \tag{14.13}$$

The sum in the right hand side of (14.13) is susceptible of another interpretation. Let $\boldsymbol{\rho}_i$, $1 \leqslant i \leqslant s = (\mathbf{J}_K : \mathfrak{K})$ be the elements of $\mathscr{S}_K$, and set

$$\boldsymbol{\Pi} = \bar{\boldsymbol{\rho}}_1 \oplus \bar{\boldsymbol{\rho}}_2 \oplus \dots \oplus \bar{\boldsymbol{\rho}}_s.$$

We have

$$\operatorname{Ind}(\lambda : \mathfrak{K}\Gamma, \mathbf{J}_K\,\Gamma) = \Lambda_1 \oplus \dots \oplus \Lambda_s,$$

and we can assume that Λ_i induces $\boldsymbol{\rho}_i$. Rearranging the character expressions (A.14), as we may by the same result, we get

$$\begin{aligned}\Theta_{\Pi}(g\sigma) &= \sum_{i=1}^{s} \; \sum_{x \in \mathbf{J}_K^1 \backslash G(K)/\mathbf{J}_K} \; \sum_{y \in \mathbf{J}_K^1 x \mathbf{J}_K/\mathbf{J}_K} \operatorname{tr} \Lambda_i(y^{-1} g\sigma y) \\ &= \sum_{x \in \mathbf{J}_K^1 \backslash G(K)/\mathbf{J}_K} \; \sum_{y \in \mathbf{J}_K^1 x \mathbf{J}_K/\mathbf{J}_K} \; \sum_{i=1}^{s} \operatorname{tr} \Lambda_i(y^{-1} g\sigma y).\end{aligned}$$

However, since $\mathfrak{K}\Gamma$ is normal in $\mathbf{J}_K\,\Gamma$, we have (as functions on $G(K)\Gamma$)

$$\sum_{i=1}^{s} \operatorname{tr}\Lambda_i = (\mathbf{J}_K : \mathfrak{K}) \operatorname{tr}\lambda.$$

Thus $\boldsymbol{\Theta}_\Pi(g\sigma)$ is given by

$$\textbf{(14.14)}\qquad \boldsymbol{\Theta}_\Pi(g\sigma) = (\mathbf{J}_K : \mathfrak{K}) \sum_{x \in J_K^1\backslash G(K)/\mathbf{J}_K} \Big(\sum_{y \in J_K^1 x \mathbf{J}_K/\mathbf{J}_K} \operatorname{tr}\lambda(y^{-1} g\sigma y)\Big),$$

for all $g \in G(K)$ such that $\mathcal{N}_\sigma\, g$ is quasi-regular. In particular, if $s \in \mathbf{K}_K$ has quasi-regular norm, combining (14.13) and (14.14) we obtain

$$\boldsymbol{\Theta}_\Pi(s\sigma) = k_\sigma^{-1}\, X_K(s\sigma).$$

If we further assume that $t = \mathcal{N}_\sigma\, s$ lies in $J_F^1\,\mathfrak{N}$, we can use (14.5), (14.4) to get

$$\textbf{(14.15)}\qquad \boldsymbol{\Theta}_\Pi(s\sigma) = k_\sigma^{-1}\, d^{-1} \sum_{\pi \in \mathcal{S}_F} \boldsymbol{\Theta}_\pi(t),$$

provided $s \in \mathbf{K}_K$ and $t \in J_F^1\,\mathfrak{N}$ is $\mathbf{J}_K^1$-conjugate to $\mathcal{N}_\sigma\, s$. However, $\boldsymbol{\Theta}_\Pi$, $\boldsymbol{\Theta}_\pi$ are class functions, so (14.15) holds for all pairs (s, t) as in (14.11) such that $s\sigma$ is $G(K)$-conjugate to an element of $\mathbf{K}_K\,\sigma$. We therefore assume that $s\sigma$ is not conjugate to an element of $\mathbf{K}_K\,\sigma$. We have $\boldsymbol{\Theta}_\Pi(s\sigma) = 0$ by (14.12), (14.14). If t is not conjugate to an element of $J_F^1\,\mathfrak{N}$, we also have $\Sigma_\pi\, \boldsymbol{\Theta}_\pi(t) = 0$ by (14.4). This leaves only the case where t is conjugate to an element of $J_F^1\,\mathfrak{N}$. We may as well, therefore, take $t \in J_F^1\,\mathfrak{N}$. By (12.3), there exists $s_1 \in \mathbf{K}_K$ such that $\mathcal{N}_\sigma\, s_1 = t$. By (12.1), the element $s_1\,\sigma$ is $G(K)$-conjugate to $s\sigma$, which contradicts the hypothesis on s.

Thus (14.15) holds for all pairs (s, t) as in (14.11), and completes the proof of the theorem. □

Now let $\Delta_{K/F}$ denote the group of characters of $F^\times$ which vanish on $\mathfrak{N}$. If $\boldsymbol{\pi}$ is an irreducible representation of $G(F)$ and $\chi \in \Delta_{K/F}$, we write $\boldsymbol{\pi}\chi$ rather than $\boldsymbol{\pi} \otimes \chi \circ \det$.

(14.16) *Lemma.* — (i) *We have* $\boldsymbol{\pi}\chi \in \mathcal{S}_F$, *for every* $\boldsymbol{\pi} \in \mathcal{S}_F$, $\chi \in \Delta_{K/F}$.
(ii) *For* $\boldsymbol{\pi} \in \mathcal{S}_F$, $\chi \in \Delta_{K/F}$, *we have* $\boldsymbol{\pi}\chi \cong \boldsymbol{\pi}$ *if and only if* $\chi = 1$.

Proof. — Part (i) follows from the definitions, and (ii) from [BK1] (5.2.2), (6.1.2). □

Now let us assume that F *has characteristic zero*. We invoke [AC] Ch. 1 Th. 6.2. Let $\boldsymbol{\rho} \in \mathcal{S}_K$. Since $\boldsymbol{\rho}^\sigma \cong \boldsymbol{\rho}$, there exists an irreducible smooth representation $\boldsymbol{\pi}(\boldsymbol{\rho})$ of $G(F)$ with the property

$$\textbf{(14.17)}\qquad c(\bar{\boldsymbol{\rho}})\, \boldsymbol{\Theta}_{\pi(\rho)}(\mathcal{N}_\sigma\, g) = \boldsymbol{\Theta}_{\bar\rho}(g\sigma)$$

for all elements $g \in G(K)$ such that $\mathcal{N}_\sigma\, g$ is (conjugate to a) *regular* element of $G(F)$. (We note that regular elements are automatically quasi-regular. We write $G(F)_{\mathrm{reg}}$ for the set of regular elements of $G(F)$.) The constant $c(\bar{\boldsymbol{\rho}})$ depends only on the choice of extension of ρ to $\mathrm{G(K)}\,\Gamma$ (and we have normalized such an extension in (14.10)). The representation ρ is then the *base-change lift* of $\pi(\rho)$. The next point to observe is:

(14.18) *Lemma. — The representation* $\pi(\rho)$ *is supercuspidal.*

Proof. — We observe first that $\pi(\rho)$ is a discrete series representation: otherwise, the argument on [AC] p. 53 shows that ρ cannot be σ-discrete (in the sense of [AC]) hence not supercuspidal. Next, if $\pi(\rho)$ is discrete series but not supercuspidal, then [AC] Ch. 1 Lemma 6.12 shows that the same applies to ρ. □

We return to the relation (14.17). We have $\det \mathcal{N}_\sigma g = \mathrm{N}_{\mathrm{K/F}}(\det g)$, so the same relation holds with $\pi(\rho)\,\chi$ in place of $\pi(\rho)$, $\chi \in \Delta$. Summing over χ and ρ, we get

$$\Theta_\Pi(g\sigma) = \sum_{\rho \in \mathcal{S}_\mathrm{K}} \Theta_{\tilde\rho}(g\sigma) = d^{-1} \sum_{\rho \in \mathcal{S}_\mathrm{K}} c(\bar\rho)^{-1} \sum_{\chi \in \Delta} \Theta_{\pi(\rho)\,\chi}(\mathcal{N}_\sigma g),$$

for $g \in \mathrm{G(K)}$ with regular norm. Combining with (14.18), we get

$$d^{-1} \sum_{\rho \in \mathcal{S}_\mathrm{K}} c(\bar\rho)^{-1} \sum_{\chi \in \Delta} \Theta_{\pi(\rho)\,\chi}(h) = k_\sigma^{-1}\, d^{-1} \sum_{\pi \in \mathcal{S}_\mathrm{F}} \Theta_\pi(h) \tag{14.19}$$

for all $h \in \mathrm{G(F)}_{\mathrm{reg}}$ which are conjugate to $\mathcal{N}_\sigma g$, for some $g \in \mathrm{G(K)}$. Now we appeal to [AC] Ch. 1 Lemma 1.4: an *elliptic* regular element $h \in \mathrm{G(F)}$ is conjugate to $\mathcal{N}_\sigma g$, for some $g \in \mathrm{G(K)}$, if and only if $\det h \in \mathfrak{N}$. We deduce that (14.19) holds for all $h \in \mathrm{G(F)}_{\mathrm{reg}}^{\mathrm{ell}}$ with $\det h \in \mathfrak{N}$. However, viewed as a function of $h \in \mathrm{G(F)}_{\mathrm{reg}}$, the left hand side of (14.19) vanishes if $\det h \notin \mathfrak{N}$. The same applies to the right hand side by (14.16). We deduce that (14.19) holds for all $h \in \mathrm{G(F)}_{\mathrm{reg}}^{\mathrm{ell}}$. Since the representations π, $\pi(\rho)$ are all supercuspidal, we can appeal to a consequence of the Jacquet-Langlands correspondence: characters of inequivalent irreducible supercuspidal representations of $\mathrm{G(F)}$ are linearly independent on $\mathrm{G(F)}_{\mathrm{reg}}^{\mathrm{ell}}$. (This follows from [Ro] Th. 5.8 or Th. A.4.1 of [DKV].) We deduce that

$$\mathcal{S}_\mathrm{F} = \{\, \pi(\rho)\,\chi : \rho \in \mathcal{S}_\mathrm{K},\, \chi \in \Delta \,\},$$

and also that

$$c(\bar\rho) = k_\sigma = \frac{\dim \eta_\mathrm{F}}{\operatorname{tr} \bar\eta_\mathrm{K}(\sigma)} \tag{14.20}$$

for all ρ. We have proved:

(14.21) *Corollary. — Suppose that* F *has characteristic zero, and use the notation* (14.9). *Let* $\mathcal{S}_\mathrm{F}$ *be the set of equivalence classes of irreducible smooth representations of* $\mathrm{Aut_F(E)} \cong \mathrm{GL(N, F)}$ *which contain the representation* η_F^ω. *Let* $\mathcal{S}_\mathrm{K}$ *be the set of equivalence classes of irreducible representations* ρ *of* $\mathrm{Aut_K(KE)} \cong \mathrm{GL(N, K)}$ *which contain the representation* η_K^ω *and satisfy* $\rho^\sigma \cong \rho$. *Then:*

(i) $\mathcal{S}_\mathrm{K}$ *is precisely the set of* K/F *base-change lifts of the elements of* $\mathcal{S}_\mathrm{F}$.

(ii) *Let* π *be an irreducible smooth representation of* GL(N, F). *The base-change lift of* π *lies in* $\mathcal{S}_\mathrm{K}$ *if and only if* $\pi \in \mathcal{S}_\mathrm{F}$.

Remark. — (ii) The relation (14.20) is worthy of a second look. Given $\rho \in \mathcal{S}_{\mathrm{K}}$, one can choose an extension $\tilde{\rho}$ to G(K) Γ such that $c(\tilde{\rho}) = 1$. This is proved in [AC] p. 10 by reference to Whittaker models. In our situation, we can therefore choose our extension $\bar{\eta}_{\mathrm{K}}$ to give $\operatorname{tr} \bar{\eta}_{\mathrm{K}}(\sigma) = \dim \eta_{\mathrm{F}}$. (This elementary fact is, of course, easy to obtain directly.) Using this particular $\bar{\eta}_{\mathrm{K}}$, (14.20) says that the normalization of extensions to G(K) Γ given in (14.10) is identical to the Whittaker model normalization.

To conclude, we have to give:

Proof of (14.7). — We first have to show that

(14.22) $$\mathbf{J}^1_{\mathrm{K}}\, x \mathbf{J}^1_{\mathrm{K}} \cap \mathrm{G}(\mathrm{F}) = \mathrm{J}^1_{\mathrm{F}}\, x \mathrm{J}^1_{\mathrm{F}},$$

whenever $x \in \mathrm{G}(\mathrm{F})$ and K/F is a finite cyclic extension which is either unramified or totally tamely ramified. As before, we put $\Gamma = \mathrm{Gal}(\mathrm{K}/\mathrm{F})$. Nothing is changed if we replace x by ax, for some $a \in \mathrm{F}^\times$. We can therefore assume that $x \in \mathfrak{J}^t_{\mathrm{K}}$, for some integer $t \geqslant 1$. The key step in the proof is

(14.23) *Lemma.* — *Let m be a non-negative integer. For an integer q, write* $q' = \left[\dfrac{q-1}{e}\right] + 1$, $e = e(\mathrm{K} \mid \mathrm{F})$. *Then*

$$((\mathfrak{J}^1_{\mathrm{K}}\, x + x\mathfrak{J}^1_{\mathrm{K}}) \cap \mathfrak{J}^{m+t+1}_{\mathrm{K}} + \mathfrak{J}^{m+t+2}_{\mathrm{K}})^{\Gamma}$$
$$= (\mathfrak{J}^1_{\mathrm{F}}\, x + x\mathfrak{J}^1_{\mathrm{F}}) \cap \mathfrak{J}^{(m+t+1)'}_{\mathrm{F}} + \mathfrak{J}^{(m+t+2)'}_{\mathrm{F}}.$$

If we accept this for the moment, we can prove (14.22) by imitating exactly the proof of [BK1] (1.6.1): there seems little point in repeating the obnoxious details.

Now we deduce the second assertion of (14.7) from (14.22). When K/F is totally ramified, this is immediate from the construction of the map φ. We therefore assume that K/F is unramified. Again this follows from the construction of φ once we know that

(14.24) $$\mathbf{J}^1_{\tilde{\mathrm{F}}}\, x \mathbf{J}^1_{\tilde{\mathrm{F}}} \cap \mathrm{G}(\mathrm{F}) = \mathrm{J}^1_{\mathrm{F}}\, x \mathrm{J}^1_{\mathrm{F}}, \quad x \in \mathrm{G}(\mathrm{F}),$$

where, as in § 12, $\tilde{\mathrm{F}}$ denotes the completion of the maximal unramified extension of F. To prove this, suppose we have elements $x \in \mathrm{G}(\mathrm{F})$, $u, v \in \mathbf{J}^1_{\tilde{\mathrm{F}}}$ such that $y = uxv \in \mathrm{G}(\mathrm{F})$. By continuity, we can find an integer $m \geqslant 1$ such that $x\mathbf{J}^m_{\tilde{\mathrm{F}}}\, x^{-1} \subset \mathbf{J}^1_{\tilde{\mathrm{F}}}$. Moreover, for any m, (11.14) gives

$$\mathbf{J}^1_{\tilde{\mathrm{F}}}/\mathbf{J}^m_{\tilde{\mathrm{F}}} = \bigcup_{\mathrm{L}/\mathrm{F}} \mathbf{J}^1_{\mathrm{L}}/\mathbf{J}^m_{\mathrm{L}},$$

where L/F ranges over all finite extensions in $\tilde{\mathrm{F}}/\mathrm{F}$. We can therefore write $y = u'\, xv'$, where $u' \in \mathbf{J}^1_{\tilde{\mathrm{F}}}$ and $v' \in \mathbf{J}^1_{\mathrm{L}}$, for some finite unramified extension L/F. Thus, in particular, $y \in \mathrm{G}(\mathrm{L})$ and $u' \in \mathbf{J}^1_{\mathrm{L}}$. Therefore $y \in \mathbf{J}^1_{\mathrm{L}}\, x\mathbf{J}^1_{\mathrm{L}} \cap \mathrm{G}(\mathrm{F}) = \mathrm{J}^1_{\mathrm{F}}\, x\mathrm{J}^1_{\mathrm{F}}$ by (14.22). This proves (14.24), and we are left with the task of proving (14.23).

Suppose first that K/F is totally ramified. We start by showing that

$$(\mathfrak{J}_K^q)^\Gamma = \mathfrak{J}_F^{q'}.$$

The case $q = 1$ is given by (11.6) while, in general,

$$\begin{aligned}(\mathfrak{J}_K^q)^\Gamma &= (\mathfrak{J}_K^1)^\Gamma \cap \mathfrak{J}_K^q \\ &= \mathfrak{J}_F^1 \cap \mathfrak{J}_K^q \\ &= \mathfrak{J}_F^1 \cap \mathfrak{J}^q(\beta, \mathfrak{A}_M),\end{aligned}$$

in the notation of § 11. The assertion now follows from (11.6), (10.20) and [BK1] (7.1.12).

Now let L_1, L_2 be Γ-stable $\mathfrak{o}_K$-lattices in $\mathrm{End}_K(E \otimes K)$. The cohomology group $H^1(\Gamma, L_1 \cap L_2)$ is trivial, so

$$(L_1 + L_2)^\Gamma = L_1^\Gamma + L_2^\Gamma.$$

Taking fixed points certainly commutes with intersection, so (14.23) follows in this case.

We now prove (14.23) when K/F is unramified. For any $\mathfrak{o}_F$-lattices L_1, L_2 in an F-vector space W, we have

$$(L_1 + L_2) \otimes_{\mathfrak{o}_F} \mathfrak{o}_K = L_1 \otimes_{\mathfrak{o}_F} \mathfrak{o}_K + L_2 \otimes_{\mathfrak{o}_F} \mathfrak{o}_K,$$

$$(L_1 \cap L_2) \otimes_{\mathfrak{o}_F} \mathfrak{o}_K = L_1 \otimes_{\mathfrak{o}_F} \mathfrak{o}_K \cap L_2 \otimes_{\mathfrak{o}_F} \mathfrak{o}_K,$$

as lattices in $W \otimes_F K$, since $\mathfrak{o}_K$ is a flat $\mathfrak{o}_F$-module. This gives us

$$\begin{aligned}&(\mathfrak{J}_K^1 x + x\mathfrak{J}_K^1) \cap \mathfrak{J}_K^{m+t+1} + \mathfrak{J}_K^{m+t+2} \\ &\qquad = ((\mathfrak{J}_F^1 x + x\mathfrak{J}_F^1) \cap \mathfrak{J}_F^{m+t+1} + \mathfrak{J}_F^{m+t+2}) \otimes_{\mathfrak{o}_F} \mathfrak{o}_K,\end{aligned}$$

as a consequence of (11.14). The assertion of (14.23) is now immediate. □

15. Comparison with automorphic induction

Again, let $[\mathfrak{A}(E), n, 0, \beta]$ be a simple stratum in $\mathrm{End}_F(E)$, where $n > 0$ and E denotes the field $F[\beta]$. Let $\theta_F \in \mathscr{C}(\mathfrak{A}(E), 0, \beta)$. We continue our comparison of the set of irreducible representations of $G(F) = \mathrm{Aut}_F(E)$ containing θ_F with the set of Galois-stable irreducible representations of $G(K) = \mathrm{Aut}_K(K \otimes_F E)$ containing $\boldsymbol{\theta}_K$ (in the notation of § 11), for a tame cyclic extension K/F. Here we consider the opposite extreme to § 14, and assume that K *embeds in* E *over* F. In this situation, it is not so easy to use base change directly since we do not know how to interpret the formula (14.5). We therefore approach the problem indirectly, using the " dual " process of *automorphic induction*, as in [HH].

Let us summarize our hypotheses.

(15.1) *Hypotheses.* — *Let* $[\mathfrak{A}(E), n, 0, \beta]$ *be a simple stratum, where* $n > 0$ *and* $E = F[\beta]$. *Let* $\theta_F \in \mathscr{C}(\mathfrak{A}(E), 0, \beta)$. *Let* K/F *be a cyclic extension of degree* d, *with* $\Gamma = \mathrm{Gal}(K/F)$, *such that there exists an* F*-embedding* $K \to E$. *Suppose that either*

(i) K/F *is unramified, or else*
(ii) K/F *is totally tamely ramified and* E/F *is totally ramified.*

The extra restriction in (15.1) (ii) simplifies various matters considerably and will be easily circumvented in the applications in § 16.

For the time being, we impose no restriction on the characteristic of F.

In the situation of (15.1), the algebra $K \otimes_F E$ is a direct sum of $d = [K : F]$ fields, and these are permuted transitively by Γ. We fix an F-embedding $K \to E$. This is equivalent to choosing a field factor of $K \otimes E$ or to fixing a K/F-lift of β. We write

$$A = \mathrm{End}_F(E), \qquad G = \mathrm{Aut}_F(E),$$
$$B = \mathrm{End}_K(E), \qquad H = \mathrm{Aut}_K(E).$$

The field $K \subset E$ surely normalizes the order $\mathfrak{A}(E)$, so we can form the principal $\mathfrak{o}_K$-order

$$\mathfrak{C} = \mathfrak{A}(E) \cap B.$$

We are in the interior lifting situation of § 7, so we have

$$H^1(\beta, \mathfrak{A}(E)) \cap H = H^1(\beta, \mathfrak{C}),$$

and likewise for J-groups, by (7.1). We put

$$\theta_K = \theta_F \mid H^1(\beta, \mathfrak{C}),$$

and this lies in $\mathscr{C}(\mathfrak{C}, 0, \beta)$, by (7.7). Indeed, up to endo-equivalence, θ_K is the K/F-lift of θ_F corresponding to choice of K/F-lift of β implicit in our choice of embedding $K \to E$. It will be convenient to have the abbreviations

$$H^1_F = H^1(\beta, \mathfrak{A}(E)), \qquad J^1_F = J^1(\beta, \mathfrak{A}(E)),$$
$$H^1_K = H^1(\beta, \mathfrak{C}), \qquad J^1_K = J^1(\beta, \mathfrak{C}),$$

and the obvious variations on these. As before, if π is an irreducible representation of G and χ is a quasicharacter of $F^\times$, we abbreviate $\pi\chi = \pi \otimes \chi \circ \det$.

(15.2) *Proposition.* — (i) *Let* π *be an irreducible smooth representation of* G *which contains* θ_F. *Then* π *is supercuspidal.*

(ii) *Let* κ *be a character of* $F^\times$ *which generates the dual of the finite cyclic group* $F^\times/N_{K/F}(K^\times)$. *Then* $\pi\kappa \cong \pi$, *for any irreducible smooth representation* π *of* G *containing* θ_F.

(iii) *Let* σ *be an irreducible smooth representation of* H *which contains* θ_K. *Then* σ *is supercuspidal.*

Proof. — Parts (i) and (iii) are immediate. In part (ii), we note that π is induced from some irreducible representation Λ of the group $E^\times J^0(\beta, \mathfrak{A}(E)) = E^\times J^1(\beta, \mathfrak{A}(E))$ which contains θ_F. The image of this group under the determinant on G is contained in $\det(E^\times)\, \mathbf{U}^1(\mathfrak{o}_F)$. The character κ is null on $\mathbf{U}^1(\mathfrak{o}_F)$. The group $\det(E^\times)$ is just $N_{E/F}(E^\times)$. Since K/F is an abelian subextension of E/F, we have $N_{E/F}(E^\times) \subset N_{K/F}(K^\times)$ and κ is null on this last group. Altogether, $\Lambda \otimes \kappa \circ \det \cong \Lambda$, and $\Lambda \otimes \kappa \circ \det$ induces the representation $\pi\kappa$. □

Now we need to take account of the action of Γ. The choice of a K-basis of E induces an isomorphism $E \cong K^M$ of K-vector spaces, where $M = [E : K] = [E : F]/d$. It is convenient to choose an $\mathfrak{o}_K$-basis of the lattice chain $\mathscr{L} = \{ \mathfrak{p}_E^j : j \in \mathbf{Z} \}$ for this purpose. We transport the obvious action of Γ on K^M back to an action on E, and use this to define an action of Γ on H and G. We write this as $(\gamma, x) \mapsto \gamma(x)$, $x \in G$. Thus, if σ is a smooth representation of H and $\gamma \in \Gamma$, we can define a representation σ^γ of H by

$$\sigma^\gamma : h \mapsto \sigma(\gamma(h)), \quad h \in H.$$

At the level of equivalence classes, this action does not depend on the choice of basis.

This also reflects the natural action of Γ on simple characters. With our above choice of basis, the orders $\mathfrak{A}(E)$, $\mathfrak{C}$ are Γ-stable, and the elements $\gamma(\beta)$, viewed as defining simple pairs over K, are the K/F-lifts of the original β. We also get $\gamma^{-1}(H^1(\beta, \mathfrak{A}(E)) = H^1(\gamma^{-1}(\beta), \mathfrak{A}(E))$, and the character $\theta_K^\gamma = \theta_K \circ \gamma$ of this group is the lift of θ_F corresponding to the lift $\gamma^{-1}(\beta)$ of β.

(15.3) *Proposition. — Let σ be an irreducible smooth representation of* H *which contains* θ_K. *Then $\sigma^\gamma \not\cong \sigma$, for any $\gamma \in \Gamma$, $\gamma \neq 1$.*

Proof. — The representation σ^γ contains the character θ_K^γ, which is a simple character lying in $\mathscr{C}(\mathfrak{C}, 0, \gamma^{-1}(\beta))$. The endo-class of θ_K^γ is the K/F-lift of θ_F corresponding to the K/F-lift $\gamma^{-1}(\beta)$ of β. Hence, if $\gamma \neq 1$, the endo-classes of θ_K, θ_K^γ are distinct, by (9.6). In particular, these realizations of these endo-classes cannot intertwine, so $\sigma^\gamma \not\cong \sigma$, as required. □

We now need some definitions. We write μ_F for the group of p-prime roots of unity in F, where p is the residual characteristic of F. We use the analogous notation for finite extension fields of F.

(15.4A) *Suppose that* K/F *is unramified. Let $\zeta \in \mu_K$ be such that* $K = F[\zeta]$. *Fix a prime element π_F of* F. *Choose a representation λ_F of the group $J_F = J(\beta, \mathfrak{A}(E)) = \mathfrak{o}_E^\times J_F^1$ with the following properties*:

(i) *the restriction of λ_F to J_F^1 is equivalent to η_F, the unique irreducible representation of J_F^1 which contains θ_F*;

(ii) *λ_F is a β-extension of η_F*;

(iii) λ_F *is trivial on* μ_F;
(iv) *the character* $\det(\lambda_F) \mid \mu_E$ *is trivial.*

Let Λ *be a representation of the group* $\tilde{J}_F = E^\times J_F$ *such that* $\Lambda \mid J_F = \lambda_F$.

For the phrase " β-extension ", see [BK1] (5.2). The existence of a representation λ_F with the required properties is guaranteed by [BK1] (5.2.2). The existence of the extension Λ is given by [BK1] (6.1.2). Since $\dim \lambda_F = \dim \eta_F$ is a power of p, condition (iii) is a consequence of (iv).

In the other case of (15.1), we use the notation:

(15.4B) *Suppose that* K/F *is totally ramified. Let* π_K *be a prime element of* K *such that* $\pi_F = \pi_K^d$ *is a prime element of* F. *Write* $J_F = J(\beta, \mathfrak{A}(E)) = \mu_F J_F^1$ *and* $\tilde{J}_F = E^\times J_F^1$. *Let* Λ *denote an irreducible representation of* $\tilde{J}_F$ *such that*

(i) $\Lambda \mid J_F^1 = \eta_F$;
(ii) Λ *is trivial on* μ_F;
(iii) $\det \Lambda(\pi_K) = 1$.

We also write $\lambda_F = \Lambda \mid J_F$.

The existence of Λ is easily established as before. Note that λ_F is again a β-extension of η_F, while (iii) and the choices of primes imply that Λ is trivial on π_F.

When convenient, we unify the two cases (15.4A) and (15.4B) by setting

$$\xi = \begin{cases} \zeta & \text{if K/F is unramified,} \\ \pi_K & \text{if K/F is totally ramified.} \end{cases}$$

Now let χ range over the characters of the group

$$\Xi = \tilde{J}_F / \langle \pi_F \rangle J_F^1 = E^\times / \langle \pi_F \rangle \mathbf{U}^1(\mathfrak{o}_E).$$

The representations $\Lambda\chi$, $\chi \in \Xi$, are then distinct, and indeed do not intertwine in G by [BK1] (5.2.2), (6.1.2). They are precisely the representations of $\tilde{J}_F$ which agree with Λ on the group generated by π_F and J_F^1. Write

$$\boldsymbol{\pi}_\chi = c\text{-Ind}(\Lambda\chi).$$

This is an irreducible supercuspidal representation of G, and the $\boldsymbol{\pi}_\chi$, $\chi \in \Xi$, are distinct. Indeed:

(15.5) *An irreducible smooth representation* $\boldsymbol{\pi}$ *of* G *is of the form* $\boldsymbol{\pi}_\chi$, *for some* $\chi \in \Xi$, *if and only if it contains* θ_F *and its central character agrees with that of* Λ *at the prime element* π_F *of* F.

By (15.2), we have

$$\boldsymbol{\pi}_\chi \kappa \cong \boldsymbol{\pi}_\chi, \quad \chi \in \Xi,$$

where κ is as in (15.2) (ii).

Fix χ, and let $\mathscr{V}$ denote the representation space of π_χ. We view $\pi_\chi\,\kappa$ as also acting on $\mathscr{V}$. The equivalence $\pi_\chi\,\kappa \cong \pi_\chi$ says that there is a linear automorphism A_χ of $\mathscr{V}$, uniquely determined up to a scalar factor, with the property

$$A_\chi\, \pi_\chi(g) = \kappa(\det g)\, \pi_\chi(g)\, A_\chi, \quad g \in G.$$

We have $\Lambda\chi \otimes \kappa \circ \det \cong \Lambda\chi$. It follows that A_χ stabilizes the isotypic space $\mathscr{V}^{\Lambda\chi}$. Since the representation $\Lambda\chi$ occurs in π_χ with multiplicity one, A_χ must act as a scalar on $\mathscr{V}^{\Lambda\chi}$. We normalize A_χ by the condition

$$A_\chi \mid \mathscr{V}^{\Lambda\chi} = 1.$$

With this normalization, let Θ^κ_χ denote the *κ-character of* π_χ. This (cf. [HH] (3.7)) is the distribution

$$\varphi \mapsto \Theta^\kappa_\chi(\varphi) = \mathrm{tr}(\pi_\chi(\varphi) \circ A_\chi),$$

for locally constant, compactly supported, functions φ on G. We form the distribution

$$\Theta = \frac{1}{r} \sum_{\chi \in \Xi} \bar{\chi}(\xi)\, \Theta^\kappa_\chi, \tag{15.6}$$

where $r = \#\Xi$.

Next we choose a compact open subgroup K_1 of G such that

$$K_1 \subset G_0 = \mathrm{Ker}(\kappa \circ \det).$$

As before, we write G_{reg} for the set of regular elements of G.

(15.7) *Proposition.* — *Let $h \in H \cap G_{\mathrm{reg}}$. Then*

$$\Theta(h) = \sum_{x \in K_1\backslash G/\tilde{J}_F} \kappa \circ \det(x) \sum_{y \in K_1 x \tilde{J}_F/\tilde{J}_F} \vartheta(y^{-1} h y),$$

where ϑ is the function supported on $\tilde{J}_F = E^\times J^1_F$ and such that

$$\vartheta(x) = \frac{1}{r} \sum_{\chi \in \Xi} \bar{\chi}(\xi)\, \mathrm{tr}\, \Lambda\chi(x), \quad x \in \tilde{J}_F.$$

The support of the function θ is actually contained in $\xi \langle \pi_F \rangle J^1_F$, and ϑ agrees with the function $\mathrm{tr}\,\Lambda$ *on that coset.*

Proof. — As we observed in the proof of (15.2), we have $\tilde{J}_F \subset G_0$, so we can form the representation

$$\rho_\chi = c\text{-}\mathrm{Ind}^{G_0}_{\tilde{J}_F}(\Lambda\chi)$$

of G_0, and then π_χ is induced by ρ_χ. Write $\mathscr{V}$ for the representation space of π_χ and $\mathscr{V}_0 \subset \mathscr{V}$ for that of ρ_χ. As G_0-space, we have

$$\mathscr{V} = \coprod_{g \in G/G_0} \pi_\chi(g)\, \mathscr{V}_0.$$

The action of G_0 on the space $\pi_\chi(g)\,\mathscr{V}_0$ is given by the conjugate representation $\rho_\chi^{g^{-1}}$, and the conjugates ρ_χ^g, $g \in G/G_0$, are mutually inequivalent. It follows that the intertwining operator A_χ above preserves each space $\pi_\chi(g)\,\mathscr{V}_0$ and acts there as the scalar $\kappa(\det g)$. It follows (cf. [HH] (3.9)) that the distribution Θ_χ^κ is null outside G_0, while on G_0 it is given by

$$\Theta_\chi^\kappa = \sum_{g \in G/G_0} \kappa(\det g^{-1})\ \mathrm{tr}\ \rho_\chi^g,$$

where $\mathrm{tr}\ \rho_\chi^g$ is the character of ρ_χ^g. The representation ρ_χ^g is induced by the conjugate $(\Lambda\chi)^g$. We apply (A.14) to ρ_χ^g to get

$$\mathrm{tr}\ \rho_\chi^g(h) = \sum_{x \in K_1\backslash G_0/\widetilde{J}_F} \Big(\sum_{y \in K_1 x \widetilde{J}_F/\widetilde{J}_F} \mathrm{tr}(\Lambda\chi)\ (y^{-1} ghg^{-1} y)\Big),$$

for $h \in G_0 \cap G_{qr}$. We can replace K_1 by $g^{-1} K_1 g$ in (A.14) without changing anything; this allows us to replace y by gy to get

$$\mathrm{tr}\ \rho_\chi^g(h) = \sum_{x \in K_1\backslash gG_0/\widetilde{J}_F} \Big(\sum_{y \in K_1 x \widetilde{J}_F/\widetilde{J}_F} \mathrm{tr}(\Lambda\chi)\ (y^{-1} hy)\Big).$$

This series is absolutely convergent; in fact, the outer sum has only finitely many non-zero terms by (A.14). We can now substitute this expression into the definition of Θ and rearrange to get the result. $\square$

Now let us consider the corresponding situation in H. Let η_K denote the unique irreducible representation of J_K^1 which contains θ_K.

(15.8) *Proposition. — There is a unique* $\varepsilon \in \{\pm 1\}$ *such that*

$$\mathrm{tr}\ \Lambda(\xi x) = \varepsilon\ \mathrm{tr}\ \eta_K(x), \quad x \in J_K^1.$$

Proof. — Suppose first that K/F is unramified. The natural conjugation action of μ_E on J_F^1 defines an injection of $\mu_E/\mu_F \to \mathrm{Aut}(J_F^1)$. By construction, λ_F is the inflation to J_F of the representation $\bar\lambda_F$ of the semi-direct product $\mu_E/\mu_F \ltimes J_F^1 \cong J_F/\mu_F$ defined by the conditions:

$$\bar\lambda_F \mid J_F^1 = \eta_F,$$
$$\det \bar\lambda_F \mid \mu_E/\mu_F = 1.$$

The set of fixed points of ζ in J_F^1 is precisely J_K^1, since, by definition, $K = F[\zeta]$. We now appeal to [G] Th. 3. This gives a unique irreducible representation ρ of J_K^1 and a unique sign ε such that

$$\mathrm{tr}\ \lambda_F(\zeta x) = \varepsilon\ \mathrm{tr}\ \rho(x), \quad x \in J_K^1.$$

Let us apply this identity first to $x = 1$ and then to an element $y \in H_K^1$. We have $\theta_F(y) = \theta_K(y)$, so we get

$$\mathrm{tr}\ \lambda_F(\zeta)\ \theta_K(y) = \varepsilon \dim \eta_K \cdot \theta_K(y) = \varepsilon\ \mathrm{tr}\ \rho(y).$$

We deduce that $\rho \mid H^1_K$ is a multiple of θ_K, whence $\rho = \eta_K$, and this proves the Proposition when K/F is unramified.

The proof in the totally ramified case is parallel. □

We now need to choose an irreducible representation Λ_K of the group $\tilde{J}_K = E^\times J^1_K$. Let ω_F denote the central character of our chosen representation Λ as in (15.4).

(15.9A) *Suppose that* K/F *is unramified. Let* Λ_K *be a representation of* $\tilde{J}_K$ *such that*

(i) $\Lambda_K \mid J^1_K = \eta_K$;

(ii) $\Lambda_K(\zeta) = 1$;

(iii) *the restriction of* Λ_K *to* $K^\times$ *is a multiple of a character* ω_K *which satisfies*

$$\omega_K \mid F^\times = \omega_F\, \kappa^{Md(d-1)/2},$$

where, as before, $d = [K:F]$, $M = [E:K]$.

(15.9B) *Suppose that* K/F *is totally ramified. Let* Λ_K *be a representation of* $\tilde{J}_K$ *such that*

(i) $\Lambda_K \mid J^1_K = \eta_K$;

(ii) *the restriction of* Λ_K *to* $K^\times$ *is a multiple of a character* ω_K *which satisfies*

$$\omega_K \mid F^\times = \omega_F\, \kappa^{Md(d-1)/2},$$

where, as before, $d = [K:F]$, $M = [E:K]$.

Of course, in (15.9B), the character ω_F is trivial. In both cases of (15.9), the character $\kappa^{d(d-1)/2}$ is independent of the choice of κ among generators of the dual of $F^\times/N_{K/F}(K^\times)$: it is the unique character in this group of order 2 when d is even and is trivial when d is odd.

We observe that the conditions in (15.9) are consistent, by the definitions of Λ and θ_K.

Once Λ_K is chosen, we can form the representations $\Lambda_K\chi$, $\chi \in \Xi$, of $\tilde{J}_K$. Just as before, these do not intertwine in H. Let τ_χ denote the (irreducible supercuspidal) representation of H induced by $\Lambda_K\chi$. An irreducible representation of H is of the form τ_χ, for some $\chi \in \Xi$, if and only if it contains η_K and its central character agrees with that of Λ_K at π_F. We now form the distribution

$$\textbf{(15.10)} \qquad \boldsymbol{\Theta}_K = \frac{1}{r} \sum_{\chi \in \Xi} \bar{\chi}(\xi)\, \boldsymbol{\Theta}_{\tau_\chi},$$

where $\boldsymbol{\Theta}_{\tau_\chi}$ denotes the character of the representation τ_χ. We regard this as a locally constant function on the set of regular elements of H. For $h \in H_{\mathrm{reg}}$, (A.14) then gives us

$$\textbf{(15.11)} \qquad \boldsymbol{\Theta}_K(h) = \sum_{K_2\backslash H/\tilde{J}_K} \; \sum_{y \in K_2 x \tilde{J}_K/\tilde{J}_K} \vartheta_K(y^{-1}hy),$$

where K_2 is some chosen compact open subgroup of H and ϑ_K is the function supported on $\tilde{J}_K$ and defined there by

$$\vartheta_K(x) = \frac{1}{r} \sum_{\chi \in \Xi} \bar{\chi}(\xi) \operatorname{tr} \Lambda_K \chi(x), \quad x \in \tilde{J}_K.$$

Again we see that ϑ_K is actually supported on the coset $\xi \langle \pi_F \rangle J_K^1$ and is equal to $\operatorname{tr} \Lambda_K$ on ξJ_K^1. In particular, we have (recalling (15.8))

$$\textbf{(15.12A)} \qquad \vartheta_K(\pi_F^a \zeta j) = \varepsilon(\kappa^{d(d-1)/2}(\pi_F))^{aM} \vartheta(\pi_F^a \zeta j), \quad a \in \mathbf{Z}, \ j \in J_K^1,$$

in the case where K/F is unramified. When K/F is totally ramified, the corresponding relation is

$$\textbf{(15.12B)} \qquad \vartheta_K(\pi_F^a \pi_K j) = \varepsilon(\kappa^{d(d-1)/2}(\pi_F))^{aM} \omega_K(\pi_K) \vartheta(\pi_K j), \quad a \in \mathbf{Z}, \ j \in J_F^1.$$

For $g \in G$, let us write $\Gamma_G(g)$ for the conjugacy class $\{ygy^{-1} : y \in G\}$, and use a similar notation in H. For $h \in H \cap G_{reg}$, there is a finite set $X(h)$ such that $1 \in X(h)$ and

$$\textbf{(15.13)} \qquad \Gamma_G(h) \cap H = \dot{\bigcup_{x \in X(h)}} \Gamma_H(xhx^{-1}).$$

For $h \in H \cap G_{reg}$, let $\Delta(h) = \Delta_G^H(h)$ be the transfer factor of [HH] (3.3). We recall the definition in more detail below, in the proof of (15.21). Note that the definition of Δ does depend on certain auxiliary choices, but these only affect a constant factor which we do not specify anyway. It will also be convenient to have the quantity

$$\boldsymbol{\delta}(h) = \Delta(h)/|\Delta(h)|^2, \quad h \in H \cap G_{reg}.$$

The absolute value here is the ordinary complex one. In the notation of [HH] recalled below, we have $\Delta(h) = \Delta^1(h)\,\Delta^2(h)$, while $\boldsymbol{\delta}(h) = \Delta^2(h)/\Delta^1(h)$.

Finally, we can state our main result:

(15.14) *Theorem. — Define* $\boldsymbol{\Theta}$ *by* (15.6) *and* $\boldsymbol{\Theta}_K$ *by* (15.10). *There is then a nonzero constant* ε' *such that*

$$\boldsymbol{\Theta}(h) = \varepsilon' \sum_{x \in X(h)} \kappa(x)^{-1} \boldsymbol{\delta}(x^{-1}hx)\, \boldsymbol{\Theta}_K(x^{-1}hx),$$

for all $h \in H \cap G_{reg}$.

We shall prove this below. Before doing so, we derive some consequences. We now assume that F *has characteristic zero*. This enables us to use the main results of [HH], but it is worth emphasizing that, up to and including the proof of (15.14), this restriction has been unnecessary. Here, and for the rest of the section, we write $\kappa(g)$ to mean $\kappa(\det g)$, $g \in G$. For $\chi \in \Xi$, let $\boldsymbol{\pi}_\chi$ be the representation of G induced by $\Lambda\chi$, as above.

The property (15.2) (ii) says exactly that the representation π_χ is obtained from an irreducible representation σ_χ of H by *automorphic induction*, as in [HH] (5.4). We have the defining relation (*loc. cit.* (3.11))

$$\Theta_\chi^\kappa(h) = c_\chi \sum_{x \in X(h)} \kappa(x)^{-1}\, \delta(x^{-1}hx)\, \Theta_{\sigma_\chi}(x^{-1}hx), \quad h \in H \cap G_{reg},$$

where Θ_{σ_χ} is the character of σ_χ and c_χ is some non-zero complex constant. Now let us restrict to the case $h \in H \cap G_{reg}^{ell}$, where G_{reg}^{ell} is the set of elliptic regular elements in G. We can then choose $X(h)$ so that its image in Aut(H) coincides with that of Γ (*loc. cit.* (3.10)). This done, we get the identity

$$\Delta(h) = \kappa(x^{-1})\, \Delta(x^{-1}hx)$$

(see [HH] (4.3) Corollary), and hence

$$\delta(h) = \kappa(x^{-1})\, \delta(x^{-1}hx).$$

The relation above then simplifies to

$$\Theta_\chi^\kappa(h) = c_\chi\, \delta(h) \sum_{\gamma \in \Gamma} \Theta_{\sigma_\chi}(\gamma(h)), \quad h \in H \cap G_{reg}^{ell}$$

(*loc. cit.* (3.11) Remark (2)). We can now substitute this into the definition (15.6) of Θ to get the expression

$$\textbf{(15.15)} \qquad \Theta(h) = \frac{1}{r}\, \delta(h) \sum_{\chi \in \Xi} c_\chi\, \bar{\chi}(\xi) \sum_{\gamma \in \Gamma} \Theta_{\sigma_\chi}(\gamma(h)), \quad h \in H \cap G_{reg}^{ell}.$$

On the other hand, we can incorporate the same simplifications into the expression for Θ given by (15.14). Thus, using the definition (15.10) of Θ_K, we get

$$\begin{aligned} \Theta(h) &= \varepsilon' \sum_{x \in X(h)} \kappa(x^{-1})\, \delta(x^{-1}hx)\, \Theta_K(x^{-1}hx) \\ &= \varepsilon'\, \delta(h) \sum_{\gamma \in \Gamma} \Theta_K(\gamma(h)) \\ &= \frac{1}{r}\, \varepsilon'\, \delta(h) \sum_{\gamma \in \Gamma} \sum_{\chi \in \Xi} \bar{\chi}(\xi)\, \Theta_{\tau_\chi}(\gamma(h)), \end{aligned}$$

valid for $h \in H \cap G_{reg}^{ell}$. Here, as before, τ_χ denotes the (irreducible supercuspidal) representation of H induced by $\Lambda_K \chi$. We now combine this last expression with (15.15) to get the relation

$$\textbf{(15.16)} \qquad \varepsilon' \sum_{\chi \in \Xi} \bar{\chi}(\xi) \sum_{\gamma \in \Gamma} \Theta_{\tau_\chi}(\gamma(h)) = \sum_{\chi \in \Xi} c_\chi\, \bar{\chi}(\xi) \sum_{\gamma \in \Gamma} \Theta_{\sigma_\chi}(\gamma(h)),$$

again valid for all $h \in H \cap G_{reg}^{ell}$. The next point to note is:

(15.17) *Lemma. — For $\chi_j \in \Xi$, $\gamma_k \in \Gamma$, we have $\tau_{\chi_1}^{\gamma_1} \cong \tau_{\chi_2}^{\gamma_2}$ if and only if $\gamma_1 = \gamma_2$ and $\chi_1 = \chi_2$.*

Proof. — The representation $\boldsymbol{\tau}_{\chi_j}^{\gamma_j}$ contains the simple character $\theta_K^{\gamma_j}$. These two simple characters intertwine in H if and only if $\gamma_1 = \gamma_2$: we have already noted this in the proof of (15.3). On the other hand, the representations $\Lambda\chi_j$ intertwine in H if and only if $\chi_1 = \chi_2$, and the Lemma follows. □

In all, the representations $\boldsymbol{\tau}_\chi^\gamma$ appearing in the left hand side of (15.16) are distinct. On the other hand, since $\boldsymbol{\pi}_\chi$ is supercuspidal, the representation $\boldsymbol{\sigma}_\chi$ is supercuspidal by [HH] (4.8), (5.5). However, by *loc. cit.* (5.2), characters of inequivalent discrete series representations of H remain linearly independent on restriction to $H \cap G_{reg}^{ell}$. This shows first that $\boldsymbol{\pi}_\chi$ determines the Γ-orbit of $\boldsymbol{\sigma}_\chi$ via the relation (15.15). Therefore a relation $\boldsymbol{\sigma}_{\chi_1}^{\gamma_1} \cong \boldsymbol{\sigma}_{\chi_2}^{\gamma_2}$ with $\chi_j \in \Xi$, $\gamma_j \in \Gamma$, implies $\chi_1 = \chi_2$. It now follows from (15.16) and the corresponding property (15.17) for the $\boldsymbol{\tau}_\chi$ that the Γ-conjugates of a given $\boldsymbol{\sigma}_\chi$ are mutually inequivalent, and then that the sets of equivalence classes of representations of H,

$$\{ \boldsymbol{\tau}_\chi^\gamma : \chi \in \Xi,\ \gamma \in \Gamma \},$$
$$\{ \boldsymbol{\sigma}_\chi^\gamma : \chi \in \Xi,\ \gamma \in \Gamma \},$$

are identical. We have proved:

(15.18) *Corollary. — In the situation of* (15.1), *suppose that* F *has characteristic zero. Fix a prime element* π_F *of* F *and an element* $\alpha \in \mathbf{C}^\times$. *In the case where* K/F *is totally ramified, choose* π_F *of the form* π_K^d, *for some prime element* π_K *of* K. *Let* κ *be a character which generates the dual of* $F^\times/N_{K/F}(K^\times)$.

Let $\mathscr{S}_F$ *denote the set of irreducible smooth representations* $\boldsymbol{\pi}$ *of* G *which contain* θ_F *and whose central quasicharacter satisfies* $\omega_\pi(\pi_F) = \alpha$. *Let* $\mathscr{S}_K$ *denote the set of irreducible smooth representations* $\boldsymbol{\sigma}$ *of* H *which contain* θ_K *and whose central quasicharacter satisfies*

$$\omega_\sigma(\pi_F) = \alpha\kappa(\pi_F)^{Md(d-1)/2},$$

where $Md = [E : F]$.

Let $\boldsymbol{\pi}$ *be an irreducible smooth representation of* G. *Then* $\boldsymbol{\pi} \in \mathscr{S}_F$ *if and only if* $\boldsymbol{\pi}$ *is automorphically induced by* $\boldsymbol{\sigma}$, *for some* $\boldsymbol{\sigma} \in \mathscr{S}_K$.

The only point to be made here is that, in the constructions above in the ramified case, we imposed the condition $\alpha = 1$ for technical convenience. This is easily removed by tensoring with an unramified character of $F^\times$.

Now we have the task of proving (15.14). We revert to the case where F has arbitrary characteristic. The relation to be established is invariant under G-conjugation so, by (15.7) and (15.11), we may as well take $h \in \langle \pi_F \rangle \xi J_F^1$. The relations (15.12) and formal properties [HH] (4.2) of Δ allow us to eliminate the contribution from $\langle \pi_F \rangle$, so we take $h \in \xi J_F^1$. We need a list of Lemmas.

(15.19) *Lemma. — Let* $h \in \xi J_F^1$. *There exists* $y \in J_F^1$ *such that* $yhy^{-1} \in \xi J_K^1$.

(15.20) *Lemma.* — *Let* $h, h' \in \xi J_K^1$. *Suppose* $yhy^{-1} = h'$ *for some* $y \in G$. *Then* $y \in H$.

(15.21) *Lemma.* — *The functions* Δ, $\boldsymbol{\delta}$ *are constant on the coset* ξJ_K^1. *Moreover, for* $h \in \xi J_K^1$, *we have*

(i) $\boldsymbol{\Theta}_K(x^{-1}hx) = 0$ *for all* $x \in X(h)$, $x \neq 1$;
(ii) $\boldsymbol{\Theta}(h) = \varepsilon' \boldsymbol{\Theta}_K(h)$, *for some constant* $\varepsilon' \neq 0$.

If we assume (15.19), it is enough to prove Theorem (15.14) for $h \in \xi J_K^1$, but, for such h, it follows from (15.21).

Now let us prove (15.19). We write

$$h = \xi j(1+x), \quad j \in J_K^1, \quad x \in \mathfrak{J}_F^r, \quad r \geqslant 1,$$

and proceed by induction on r. Let $y \in \mathfrak{J}_F^r$, and consider

$$(1+y)\, h(1+y)^{-1} \equiv \xi j(1 + x + \xi^{-1} y \xi - y) \ (\text{mod}\ \xi J_F^{r+1}).$$

Conjugation by ξ induces a semisimple automorphism Z of the k_F-vector space $\mathsf{V}^r = \mathfrak{J}_F^r / \mathfrak{J}_F^{r+1}$, so that

$$\mathsf{V}^r = \operatorname{Im}(\mathsf{Z} - 1) \oplus \operatorname{Ker}(\mathsf{Z} - 1).$$

Moreover, by a standard cohomology-vanishing argument, $\operatorname{Ker}(\mathsf{Z}-1)$ is the image of $\mathfrak{J}_K^r$ in V^r. We can now choose y so that $x + \xi^{-1} y \xi - y + \mathfrak{J}_F^{r+1} \in \operatorname{Ker}(\mathsf{Z}-1)$, and we then have $(1+y)\, h(1+y)^{-1} \in \xi J_K^1 J_F^{r+1}$. The lemma now follows.

Now we turn to (15.20). Suppose first that K/F is unramified. Put $q = \# k_K$, and define

$$\ell = \lim_{n \to \infty} h^{q^n}.$$

Defining ℓ' similarly, we have $\ell' = y \ell y^{-1}$. We will show that $\ell = \ell' = \zeta$, and the Lemma will then follow. However, we can write $h = \zeta(1+x)$, with $x \in \mathfrak{J}_K^1$, giving us

$$h^{q^n} \equiv \zeta^{q^n} \equiv \zeta \ (\text{mod}\ \mathfrak{J}_K^n), \quad n \geqslant 1.$$

The same applies to h', so we have the result in this case. When K/F is totally ramified, consider the sequence $\ell_n = \pi_F^{-(q^n-1)/d} h^{q^n} \in \pi_K J_K^1$, and define $\{\ell'_n\}$ similarly. (Note here that, by hypothesis on K, the field $k_F = k_K$ contains a primitive d-th root of unity, so $q \equiv 1 \ (\text{mod}\ d)$.) We have a strictly increasing sequence $\{n_i\}$ of positive integers such that $\{\ell_{n_i}\}$, $\{\ell'_{n_i}\}$ are both convergent, with limits ℓ, ℓ' respectively. We again have $\ell' = y \ell y^{-1}$. Now we finish the proof as before.

We now prove (15.21). To prove the first assertion, we have to recall the definition of Δ from [HH] (3.2-3). Fix a generator σ of $\Gamma = \operatorname{Gal}(K/F)$ and choose an element

$e_0 \in K^\times$ such that $\sigma(e_0) = (-1)^{m(d-1)} e_0$. Let $h, h' \in H$, and let $\alpha_1, \ldots, \alpha_m, \alpha'_1, \ldots, \alpha'_m$ be the eigenvalues of h, h' respectively in some finite extension of K. We set

$$\mathrm{r}(h, h') = \prod_{i,j=1}^{m} (\alpha_i - \alpha'_j),$$

and, for $h \in H$, we put

$$\widetilde{\Delta}(h) = \prod_{0 \leqslant i < j \leqslant d-1} \mathrm{r}(\sigma^i(h), \sigma^j(h)).$$

We have $e_0 \widetilde{\Delta}(h) \in F$ and $\widetilde{\Delta}(h)^2 \in F$. Let $|| \ ||_F$ be the normalized absolute value on F and let $\det h$ be the determinant of h viewed as an element of G. We put

$$\Delta^1(h) = || \widetilde{\Delta}(h)^2 ||_F^{1/2} \, || \det h ||_F^{(m-n)/2} \quad \text{(cf. [HH] Lemma [4.1])},$$
$$\Delta^2(h) = \kappa(e_0 \widetilde{\Delta}(h)),$$
$$\Delta(h) = \Delta^1(h)\, \Delta^2(h).$$

Observe that Δ^1 is real and positive, while Δ^2 is a root of unity. Thus, as we said earlier, $\boldsymbol{\delta} = \Delta/|\Delta|^2 = \Delta^2/\Delta^1$.

Now let $h \in \xi J^1_K$; we want to show that $\Delta(h) = \Delta(\xi)$. Let L be a finite Galois extension of F containing K and all the eigenvalues $\alpha_1, \ldots, \alpha_m$ of H. Since $h \in \xi J^1_K$, there are elements $u_k \in \mathbf{U}^1(\mathfrak{o}_L)$ such that $\alpha_k = \xi u_k$, $1 \leqslant k \leqslant m$. The definition of ξ shows that the element $\sigma^i(\xi) - \sigma^j(\xi)$ has the same valuation as ξ whenever $0 \leqslant i < j \leqslant d-1$. It follows that $\widetilde{\Delta}(h)$, $\widetilde{\Delta}(\xi)$ differ by an element of $\mathbf{U}^1(\mathfrak{o}_L)$ and, in particular, $\Delta^2(h) = \Delta^2(\xi)$ since κ is tamely ramified. On the other hand, $\det h$ and $\det \xi$ differ by an element of $\mathbf{U}^1(\mathfrak{o}_F)$ and hence $\Delta^1(h) = \Delta^1(\xi)$. This proves the first assertion of (15.21).

Part (i) of (15.21) is a consequence of (15.20). This leaves us with (15.21) (ii). We recall the formula (15.7):

$$\boldsymbol{\Theta}(h) = \sum_{x \in K_1\backslash G/\widetilde{J}_F} \kappa \circ \det(x) \sum_{y \in K_1 x \widetilde{J}_F/\widetilde{J}_F} \vartheta(y^{-1} hy).$$

A conjugate $y^{-1} hy$ contributes to this sum if and only if $y^{-1} hy \in \xi J^1_F$. By (15.19), we can choose y in its $\widetilde{J}_F$-coset so that $y^{-1} hy \in \xi J^1_K$, and then $y \in H$ by (15.20). We have $\widetilde{J}_F \cap H = \widetilde{J}_K$ and $\kappa \circ \det$ is null on H, so we get

$$\boldsymbol{\Theta}(h) = \sum_{x \in K_1\backslash G/\widetilde{J}_F} \ \sum_{y \in K_1 x \widetilde{J}_F \cap H/\widetilde{J}_K} \vartheta(y^{-1} hy).$$

At this point, we recall that our open subgroups $K_1 \subset G$, $K_2 \subset H$ were chosen rather arbitrarily. It will now be convenient to set $K_2 = K_1 \cap H$. For $x \in H$, the set $K_1 x \widetilde{J}_F \cap H$ is a finite union of $(K_2, \widetilde{J}_K)$-double cosets. We define an equivalence relation $\sim$ on H by $h_1 \sim h_2$ if $K_1 h_1 \widetilde{J}_F = K_1 h_2 \widetilde{J}_F$, and choose a set X of representatives for this relation. Rewriting the last formula for $\boldsymbol{\Theta}(h)$ and incorporating (15.12), we have

$$\boldsymbol{\Theta}(h) = \varepsilon' \sum_{x \in X} \ \sum_{y \in K_1 x \widetilde{J}_F \cap H/\widetilde{J}_K} \vartheta_K(y^{-1} hy)$$

for some nonzero constant ε'. On the other hand,

$$\boldsymbol{\Theta}_{\mathrm{K}}(h) = \sum_{x \in \mathrm{K}_2 \backslash \mathrm{H}/\tilde{\mathrm{J}}_{\mathrm{K}}} \sum_{y \in \mathrm{K}_2 x \tilde{\mathrm{J}}_{\mathrm{K}}/\tilde{\mathrm{J}}_{\mathrm{K}}} \vartheta_{\mathrm{K}}(y^{-1} hy),$$

by (15.11). However, absolute convergence (A.14) allows us to group terms here, and this gives us the relation $\boldsymbol{\Theta}(h) = \varepsilon'' \boldsymbol{\Theta}_{\mathrm{K}}(h)$ for some constant $\varepsilon'' \neq 0$, as required for the Lemma. □

We have now completed the proof of (15.14) and hence that of (15.18).

16. Comparison with base change II

We now compare our local lifting process $\theta_{\mathrm{F}} \mapsto \boldsymbol{\theta}_{\mathrm{K}}$ (notation of § 11) with base change for an arbitrary tame Galois extension K/F.

Let us briefly recall a variation of the Zelevinsky product notation, of which we make frequent use in this section. If we have an irreducible smooth representation π_i of $\mathrm{GL}(\mathrm{N}_i, \mathrm{F})$, $1 \leqslant i \leqslant r$, we form a representation

$$\pi = \pi_1 \times \pi_2 \times \ldots \times \pi_r$$

of GL(N, F), $\mathrm{N} = \Sigma_i \mathrm{N}_i$, as follows. We form the representation $\pi_1 \otimes \pi_2 \otimes \ldots \otimes \pi_r$ of $\mathrm{M} = \prod \mathrm{GL}(\mathrm{N}_i, \mathrm{F})$, and identify M with a Levi subgroup of GL(N, F). We choose a parabolic subgroup P of GL(N, F) with Levi component M, extend $\bigotimes \pi_i$ to P by triviality, and induce to GL(N, F). This is the representation π. It depends on the choice of P, but its set of composition factors does not. We shall only use this notation when either π is irreducible (and hence independent of P) or when we are only concerned with composition factors. We use a similar notation in the global situation.

We have to start by establishing a few formal properties of local base change and automorphic induction. This is mainly a matter of rearranging some arguments from [AC] and comparing them with [HH]. First, let k denote an algebraic number field and $\mathbf{A}_k$ the adèle ring of k. Let π be an automorphic representation of $\mathrm{GL}(\mathrm{N}, \mathbf{A}_k)$. Thus π decomposes as a restricted tensor product [Fl]

$$\pi = \bigotimes_v \pi_v,$$

where v runs over the places of k and π_v is an irreducible admissible representation of $\mathrm{GL}(\mathrm{N}, k_v)$. There is a finite set S of places of k, containing all the Archimedean ones, such that π_v is *spherical* for all $v \notin \mathrm{S}$. That is to say, for such v, the representation π_v contains the trivial character of $\mathrm{GL}(\mathrm{N}, \mathfrak{o}_v)$, where $\mathfrak{o}_v$ is the discrete valuation ring in the completion k_v. Thus, for $v \notin \mathrm{S}$, the representation π_v is parametrized by an element $t_{\pi, v}$ of $(\mathbf{C}^\times)^{\mathrm{N}}$ modulo permutation. We now have ([JS] or [AC] Ch. 3 (2.4) and (4.1)):

(16.1) *Let* π, π' *be automorphic representations of* GL(N, $\mathbf{A}_k$), *and suppose they are both induced from cuspidal. Then* $\pi \cong \pi'$ *if and only if* $t_{\pi, v} = t_{\pi', v}$ *for almost all non-Archimedean places* v *of* $\boldsymbol{k}$.

(For the term " induced from cuspidal ", see [AC] Ch. 3 § 4.)

Now let l/k be a finite field extension, and π (resp. Π) an automorphic representation of GL(N, $\mathbf{A}_k$) (resp. GL(N, $\mathbf{A}_l$)). We say (cf. [AC] Ch. 3 § 1) that Π is a *weak lift of* π if

$$t_{\Pi, w} = t_{\pi, v}^{f_w}$$

for almost all non-Archimedean places v of k and all places w of l lying over v, where $f_w = f(l_w \mid k_v)$ is the local residue class degree.

(16.2) *Proposition. — Suppose that the extension* l/k *is soluble Galois. Let* π *be an automorphic representation of* GL(N, $\mathbf{A}_k$) *which is induced from cuspidal. There then exists a unique automorphic representation* $\ell_{l/k}\,\pi$ *of* GL(N, $\mathbf{A}_l$) *which is a weak lift of* π. *This representation is induced from cuspidal. Moreover, if* $k \subset l' \subset l$ *and* l'/k *is Galois, then*

$$\ell_{l/k}(\pi) = \ell_{l/l'}(\ell_{l'/k}(\pi)).$$

The Proposition follows immediately from [AC] Ch. 3 Th. 4.2 and the obvious transitivity property of weak lifting.

Now we return to our non-Archimedean local field F, *which henceforward has characteristic zero*. If K/F is a finite cyclic extension and π is an irreducible smooth representation of GL(N, F), we write $b_{K/F}\,\pi$ for the K/F-base change lift of π: if π is tempered, it is defined by the Shintani relation [AC] Ch. 1, 6.1 (which we used for supercuspidal representations in § 14). For general representations π, it is defined indirectly via the Langlands classification; see [AC] remarks on p. 59 following Prop. 6.8.

(16.3) *Proposition. — Let* K/F *be a finite cyclic extension and let* K'/F *be a subextension of* K/F. *We then have*

$$b_{K/F}(\pi) = b_{K/K'}(b_{K'/F}(\pi)),$$

for every irreducible smooth representation π *of* GL(N, F).

Proof. — The remarks cited above reduce us to the case where π is tempered. When π is tempered, it is irreducibly (parabolically) induced from a discrete series representation, and $b_{K'/F}\,\pi$ is irreducibly induced from a Gal(K'/F)-discrete representation. Local base change commutes with irreducible parabolic induction, so we are reduced to the case where π is *discrete series*.

We next recall the construction of $b_{K/F}\,\pi$, for π discrete series, from [AC] Ch. 1 § 6.3. We find a number field k and a place v of k such that $k_v \cong$ F. We then find a cyclic extension l/k in which v does not split such that the local extension l_w/k_v is isomorphic to K/F, where w is the unique place of $\boldsymbol{l}$ lying over v. There is then a cuspidal automorphic representation $\mathbf{\Pi}$ of $GL(N, \mathbf{A}_k)$ such that $\mathbf{\Pi}_v \cong \boldsymbol{\pi}$. (The subsidiary conditions imposed on $\mathbf{\Pi}$ in *loc. cit.* Lemma 6.5 are irrelevant here.) We form the weak lift $\ell_{l/k}\,\mathbf{\Pi}$, and the desired representation $\boldsymbol{b}_{K/F}\,\boldsymbol{\pi}$ is the component of $\ell_{l/k}\,\mathbf{\Pi}$ at w. Observe that this condition is independent of all choices (because $\boldsymbol{\pi}$ and $\boldsymbol{b}_{K/F}\,\boldsymbol{\pi}$ satisfy the Shintani relation). Indeed (16.1) shows that, for the purposes of this construction, we need only have taken $\mathbf{\Pi}$ to be induced from cuspidal.

Now let us prove the Proposition. We can use induction on the degree $[K' : F]$, so we may as well assume that this is *prime*. We choose a cuspidal automorphic representation $\mathbf{\Pi}$ of $GL(N, \mathbf{A}_k)$ whose v-component is $\boldsymbol{\pi}$. Let $\boldsymbol{l}'/\boldsymbol{k}$ be the subextension of $\boldsymbol{l}/\boldsymbol{k}$ corresponding to K'/F, and let w' be the unique place of $\boldsymbol{l}'$ above v. Consider the representation $\boldsymbol{b}_{K'/F}\,\boldsymbol{\pi}$. This representation is Gal(K'/F)-discrete (by the argument on [AC] p. 56) and hence, by [AC] Ch. 1 Lemma 2.8, is of the form

$$\boldsymbol{b}_{K'/F}\,\boldsymbol{\pi} = \boldsymbol{\pi}' \times \boldsymbol{\pi}'^{\sigma} \times \ldots \times \boldsymbol{\pi}'^{\sigma^{g-1}},$$

where $\boldsymbol{\pi}'$ is discrete series, σ generates the Galois group of K'/F, and $g \geqslant 1$ is the least integer for which $\boldsymbol{\pi}'^{\sigma^g} \cong \boldsymbol{\pi}'$. By hypothesis on the degree $[K' : F]$, we have only the cases $g = 1$ or $g = [K' : F]$.

The case $g = 1$ is easy: the weak K'/F-lift of $\mathbf{\Pi}$ is induced from cuspidal and $\boldsymbol{\pi}'$ is discrete so, by the general construction, $\boldsymbol{b}_{K/K'}(\boldsymbol{b}_{K'/F}(\boldsymbol{\pi}))$ is the w-component of $\ell_{K/K'}(\ell_{K'/F}(\mathbf{\Pi})) = \ell_{K/F}(\mathbf{\Pi})$, as required.

We therefore assume that $g = [K' : F]$. In this case, we need to choose the global representation $\mathbf{\Pi}$ with more care. We use the fact ([AC] Ch. I Prop. 6.6, 6.7) that, in these circumstances, any irreducible representation $\boldsymbol{\pi}_1$ of $GL(N, F)$ with $\boldsymbol{b}_{K'/F}\,\boldsymbol{\pi}_1 \cong \boldsymbol{b}_{K'/F}\,\boldsymbol{\pi}$ must be equivalent to $\boldsymbol{\pi}$. We first find a cuspidal representation $\mathbf{\Pi}'$ of $GL(N/g, \mathbf{A}_{l'})$ whose w'-component is $\boldsymbol{\pi}'$. In particular, $\mathbf{\Pi}'$ is not equivalent to any of its Gal(K'/F)-conjugates. We let $\mathbf{\Pi}$ be the representation of $GL(N, \mathbf{A}_k)$ automorphically induced by $\mathbf{\Pi}'$, in the sense of [AC] Ch. 3 § 6. This representation is induced from cuspidal. Invoking [AC] Ch. 3 Cor. 6.5 and Lemma 6.6, the representation $\mathbf{\Pi}$ has weak lift $\mathbf{\Pi}' \times \mathbf{\Pi}'^{\sigma} \times \ldots \times \mathbf{\Pi}'^{\sigma^{g-1}}$. The w'-component of this induced representation is induced from the w'-component of $\bigotimes \mathbf{\Pi}'^{\sigma^j}$, by [L2] Lemma 1, and hence is equivalent to $\boldsymbol{\pi}' \times \boldsymbol{\pi}'^{\sigma} \times \ldots \times \boldsymbol{\pi}'^{\sigma^{g-1}}$. On the other hand, since $[K' : F] = [\boldsymbol{l}' : \boldsymbol{k}]$ is prime, we can use [AC] Ch. 3 Th. 5.1 to show $\boldsymbol{b}_{l'_{w'}/k_v}\,\mathbf{\Pi}_v \cong \boldsymbol{b}_{K'/F}\,\boldsymbol{\pi}$. Thus, by the facts recalled above, we have $\mathbf{\Pi}_v \cong \boldsymbol{\pi}$. Therefore, we might as well have assumed at the beginning that $\boldsymbol{\pi} = \mathbf{\Pi}_v$, for a representation $\mathbf{\Pi}$ of $GL(N, \mathbf{A}_k)$ which is induced from cuspidal and such that $\ell_{K'/F}\,\mathbf{\Pi} = \mathbf{\Pi}_1 \times \mathbf{\Pi}_1^{\sigma} \times \ldots \times \mathbf{\Pi}_1^{\sigma^{g-1}}$ for some cuspidal representation $\mathbf{\Pi}_1$ of $GL(N/g, \mathbf{A}_{l'})$ with w'-component $\boldsymbol{\pi}'$. The representation $\boldsymbol{b}_{K/K'}(\boldsymbol{b}_{K'/F}(\boldsymbol{\pi}))$ is now of the form $\boldsymbol{b}_{K/K'}(\boldsymbol{\pi}') \times \ldots \times \boldsymbol{b}_{K/K'}(\boldsymbol{\pi}')^{\sigma^{g-1}}$, which is the w-component of the weak K/K'-lift of $\mathbf{\Pi}_1 \times \ldots \times \mathbf{\Pi}_1^{\sigma^{g-1}}$. The result again follows from transitivity of weak lifting. □

Remark. — This argument also shows that weak lifting is the same as " strong lifting " (terminology of [AC] Ch. 3 § 1) for arbitrary cyclic extensions, thus removing the prime degree hypothesis imposed in [AC] Ch. 3 Th. 5.1.

We need a complementary result of this kind.

(16.4) *Proposition. — Let* K_1/F, K_2/F *be finite cyclic extensions which are linearly disjoint over* F. *We then have*

$$\boldsymbol{b}_{K_1 K_2/K_1} \circ \boldsymbol{b}_{K_1/F} = \boldsymbol{b}_{K_1 K_2/K_2} \circ \boldsymbol{b}_{K_2/F}.$$

Proof. — Using (16.3), a simple inductive argument reduces us to the case where the field degrees $[K_i : F]$ are both prime. We globalize as before: we choose a number field $\boldsymbol{k}$ and a place v of $\boldsymbol{k}$ such that $\boldsymbol{k}_v \cong F$. For $i = 1, 2$, we can find a cyclic extension $\boldsymbol{l}_i/\boldsymbol{k}$ in which v is inert such that $\boldsymbol{l}_{i, w_i} \cong K$ over $\boldsymbol{k}_v = F$, where w_i is the unique place of $\boldsymbol{l}_i$, over v. We only need verify the identity for discrete series representations π of GL(N, F). We identify π with the v-component of a cuspidal representation $\boldsymbol{\Pi}$ of $GL(N, \mathbf{A}_k)$. An argument identical to the one in (16.3) shows that $\boldsymbol{b}_{K_1 K_2/K_i}(\boldsymbol{b}_{K_i/F}(\pi))$ is the w_{12}-component of the weak $K_1 K_2/F$-lift of $\boldsymbol{\Pi}$, where w_{12} is the unique place of $\boldsymbol{l}_1 \boldsymbol{l}_2$ above v. (Indeed, the hypothesis that K/F is cyclic is not used in the final paragraphs of the proof of (16.3).) □

We now assume that our local extension K/F is *Galois and tamely ramified.* Let K_0/F be the maximal unramified subextension of K/F. The extensions K/K_0, K_0/F are then cyclic, so we can unambiguously define

$$\boldsymbol{b}_{K/F}(\pi) = \boldsymbol{b}_{K/K_0}(\boldsymbol{b}_{K_0/F}(\pi)), \tag{16.5}$$

for any irreducible smooth representation π of GL(N, F). (Both this definition and the next result are susceptible of further generalization, but we do not pursue the matter here.)

(16.6) *Proposition. — Let* L/F *be a tamely ramified Galois extension, and let* K/F *be a Galois subextension of* L/F. *We then have*

$$\boldsymbol{b}_{L/F} = \boldsymbol{b}_{L/K} \circ \boldsymbol{b}_{K/F}.$$

Proof. — Let L_0/K, K_0/F be the maximal unramified subextensions of L/K, K/F respectively. Let E/K_0 be the unramified extension of K_0 of degree $[L_0 : K]$. Since K/K_0 is totally ramified, the fields E, K are linearly disjoint over K_0 and we have $L_0 = EK$. Moreover, E/F is the maximal unramified subextension of L/F. Thus

$$\begin{aligned}\boldsymbol{b}_{L/F} &= \boldsymbol{b}_{L/E} \circ \boldsymbol{b}_{E/F} \text{ (definition)}\\ &= \boldsymbol{b}_{L/L_0} \circ \boldsymbol{b}_{L_0/E} \circ \boldsymbol{b}_{E/K_0} \circ \boldsymbol{b}_{K_0/F} \text{ (16.3)}\\ &= \boldsymbol{b}_{L/L_0} \circ \boldsymbol{b}_{L_0/K} \circ \boldsymbol{b}_{K/K_0} \circ \boldsymbol{b}_{K_0/F} \text{ (16.4)}\\ &= \boldsymbol{b}_{L/K} \circ \boldsymbol{b}_{K/F} \text{ (definition)},\end{aligned}$$

which proves the result. □

We now need a special case of the relation between base change and automorphic induction. (We use the latter term in the sense of [HH] at this stage.)

(16.7) *Proposition. — Let* K/F *be finite cyclic, and let* $\boldsymbol{\sigma}$ *be an irreducible supercuspidal representation of* GL(N, K) *such that* $\boldsymbol{\sigma}^\gamma \not\cong \boldsymbol{\sigma}$ *for any* $\gamma \in \mathrm{Gal}(K/F)$, $\gamma \neq 1$. *Let* $\boldsymbol{\pi}$ *denote the representation of* GL(Nd, F), $d = [K : F]$, *automorphically induced by* $\boldsymbol{\sigma}$. *Then* $\boldsymbol{b}_{K/F}\,\boldsymbol{\pi}$ *is the Zelevinsky product of the representations* $\boldsymbol{\sigma}^\gamma$, $\gamma \in \mathrm{Gal}(K/F)$.

Proof. — We follow the arguments of [HH] (8.8) which construct $\boldsymbol{\pi}$ from $\boldsymbol{\sigma}$. We choose a number field $\boldsymbol{k}$ and a place v of $\boldsymbol{k}$ such that $\boldsymbol{k}_v \cong F$, and a cyclic extension $\boldsymbol{l}/\boldsymbol{k}$ in which v is inert such that the extension $\boldsymbol{l}_w/\boldsymbol{k}_v$ is isomorphic to K/F. As usual, w here denotes the unique place of $\boldsymbol{l}$ above v. We also observe all the subsidiary conditions of *loc. cit.* Those constructions first give a cuspidal automorphic representation $\boldsymbol{\Sigma}$ of $\mathrm{GL}(N, \mathbf{A}_l)$ whose w-component is $\boldsymbol{\sigma}$, and then a cuspidal automorphic representation $\boldsymbol{\Pi}$ of $\mathrm{GL}(Nd, \mathbf{A}_k)$ whose v-component is $\boldsymbol{\pi}$. Let $\boldsymbol{\Pi}'$ denote the automorphic representation of $\mathrm{GL}(Nd, \mathbf{A}_k)$ automorphically induced by $\boldsymbol{\Sigma}$, in the sense of [AC] Ch. 3 § 6. By construction, the v'-components of $\boldsymbol{\Pi}$, $\boldsymbol{\Pi}'$ agree at almost all finite places v' of $\boldsymbol{k}$. Since $\boldsymbol{\Pi}$ is cuspidal and $\boldsymbol{\Pi}'$ is induced by cuspidal ([AC] Ch. 3 Th. 6.2), we have $\boldsymbol{\Pi} \cong \boldsymbol{\Pi}'$ and, in particular, $\boldsymbol{\pi} \cong \boldsymbol{\Pi}'_v$. The assertion follows from combining [AC] Ch. 3 (6.5) and (6.6). □

We now come to our main result.

(16.8) *Theorem. — Suppose that* F *has characteristic zero, and let* K/F *be a finite, tamely ramified Galois extension. Let* $[\mathfrak{A}, n, 0, \beta]$ *be a simple stratum in* $\mathbf{M}(N, F)$ *with* $n > 0$ *and* $[F[\beta] : F] = N$. *Let* $\theta_F \in \mathscr{C}(\mathfrak{A}, 0, \beta)$. *Define a subgroup* $\mathbf{H}^1_K$ *of* GL(N, K) *and a character* $\boldsymbol{\theta}_K$ *of* $\mathbf{H}^1_K$ *as in* (11.1).

Let $\boldsymbol{\pi}$ *be an irreducible smooth representation of* GL(N, F). *Then* $\boldsymbol{\pi}$ *contains the simple character* θ_F *if and only if the* K/F-*base change* $\boldsymbol{b}_{K/F}\,\boldsymbol{\pi}$ *of* $\boldsymbol{\pi}$ *contains the character* $\boldsymbol{\theta}_K$ *of* $\mathbf{H}^1_K$.

Proof. — We identify

$$\mathrm{GL}(N, F) = \mathrm{Aut}_F(F[\beta]),$$
$$\mathrm{GL}(N, K) = \mathrm{Aut}_F(K \otimes F[\beta]).$$

We write $K \otimes F[\beta]$ as the product of the fields $K[\beta_i]$, where β_i, $1 \leqslant i \leqslant r$, are the K/F-lifts of β. We let M_K be the Levi subgroup $\prod_i \mathrm{Aut}_K(K[\beta_i])$ of GL(N, K), and we choose a parabolic subgroup P_K with Levi factor M_K. As in (11.2), $\mathbf{H}^1_K$ has Iwahori decomposition with respect to (M_K, P_K). In particular, there is a unique hereditary $\mathfrak{o}_K$-order $\mathfrak{C}_i$ in $\mathrm{End}_K(K[\beta_i])$ normalized by $K[\beta_i]$, and we have

$$\mathbf{H}^1_K \cap M_K = \prod_i H^1(\beta_i, \mathfrak{C}_i).$$

Moreover, the character $\boldsymbol{\theta}_K \mid \mathbf{H}^1_K \cap M_K$ is the tensor product of characters $\boldsymbol{\theta}^i_K \in \mathscr{C}(\mathfrak{C}_i, 0, \beta_i)$.

(16.9) *Lemma.* — *For* $1 \leqslant i \leqslant r$, *let* ρ_i *be an irreducible representation of* $\mathrm{Aut}_{\mathrm{K}}(\mathrm{K}[\beta_i])$ *containing* θ^i_{K}. *Then* ρ_i *is supercuspidal. Moreover, the representation* $\rho_1 \times \rho_2 \times \ldots \times \rho_r$ *of* $\mathrm{GL}(\mathrm{N}, \mathrm{K})$ *is irreducible.*

Proof. — The first assertion is immediate. By (9.6), the K-endo-classes defined by the θ^i_{K} are distinct, so in the terminology of [BK1] (8.4), the representations ρ_i are inertially inequivalent. The second assertion now follows from [BK1] (8.5.1). □

Note that (16.9) does not require the hypothesis that K/F be Galois. However, if K/F is Galois and $\gamma \in \mathrm{Gal}(\mathrm{K}/\mathrm{F})$, we can form

$$(\rho_1 \times \ldots \times \rho_r)^\gamma \cong \rho_1^\gamma \times \ldots \times \rho_r^\gamma,$$

since we can take the ρ_i in any order [BK1] (8.5.1). In particular, this induced representation is Galois-stable if and only if the ρ_i form a single Galois orbit. Of course, the simple characters θ^i_{K} do form a single Galois orbit, by construction.

Now we need a fundamental property of the character θ_{K}:

(16.10) *Let* σ *be an irreducible smooth representation of* $\mathrm{GL}(\mathrm{N}, \mathrm{K})$. *The representation* σ *then contains the character* θ_{K} *if and only if*

$$\sigma \cong \rho_1 \times \ldots \times \rho_r,$$

where ρ_i *is an irreducible smooth representation of* $\mathrm{Aut}_{\mathrm{K}}(\mathrm{K}[\beta_i])$ *containing the simple character* θ^i_{K}.

The proof of this will appear elsewhere.

(16.11) *Lemma.* — *Let* L/F *be a tamely ramified Galois extension, and let* K/F *be a Galois subextension of* L/F. *Suppose that* (16.8) *holds for the extensions* L/K *and* K/F. *It then holds for* L/F.

Proof. — Let β_i be the set of K/F-lifts of β and β_{ij} the set of L/K-lifts of β_i. Use the notation θ^i_{K}, θ^{ij}_{L} for the associated simple characters.

Let π be an irreducible representation of $\mathrm{GL}(\mathrm{N}, \mathrm{F})$ containing θ_{F}. By hypothesis, the representation $b_{\mathrm{K/F}}\,\pi$ is of the form $\rho_1 \times \ldots \times \rho_r$, where ρ_i is supercuspidal and contains θ^i_{K}. We then have

$$b_{\mathrm{L/F}}(\pi) = b_{\mathrm{L/K}}(\rho_1) \times \ldots \times b_{\mathrm{L/K}}(\rho_r),$$

and, by hypothesis, $b_{\mathrm{L/K}}(\rho_i)$ is a product of supercuspidal representations σ_{ij}, with σ_{ij} containing θ^{ij}_{L}. By (16.10) therefore, $b_{\mathrm{L/F}}(\pi)$ contains the character θ_{L}.

The argument reverses to give the converse. □

We can now prove the theorem. The transitivity lemma (16.11) first reduces us to the case where K/F is cyclic and either unramified or totally tamely ramified. A second application further reduces us to the case where either $\mathrm{K} \otimes_{\mathrm{F}} \mathrm{F}[\beta]$ is a field or K/F embeds in $\mathrm{F}[\beta]/\mathrm{F}$. The first of these cases is dealt with by (14.21). Let us now assume that K/F embeds in F[β]/F and, when K/F is totally ramified, that F[β]/F is totally ramified. Choose some index i, that is, some K/F-lift β_i of β. By (15.18), an irreducible smooth representation π of GL(N, F) contains θ_F if and only if it is automorphically induced from a smooth representation σ of some GL(M, K) containing θ_K^i. We have $\sigma^\gamma \not\cong \sigma$, for any $\gamma \in \mathrm{Gal}(K/F)$, $\gamma \neq 1$, by (15.3). In this case therefore, (16.7) shows that $b_{K/F}\,\pi$ is the Zelevinsky product of the σ^γ, and the result follows from (16.10). In particular, we see that (16.8) holds unconditionally when K/F is unramified.

We therefore assume that K/F is totally tamely ramified and that F[β]/F is not totally ramified. We also assume that K/F embeds in F[β]/F. Let L/F denote the maximal unramified subextension of F[β]/F. If π is an irreducible representation of GL(N, F) containing θ_F, we then know from (15.18) that π is automorphically induced from a supercuspidal representation σ of GL(N/d, L) containing a simple character θ_L^1 corresponding to an L/F-lift β_1 of β. Here, $d = [L : F]$, and we have implicitly identified L with the unique unramified extension of F of degree d in a given algebraic closure (which also contains K). By (16.7), the base change $b_{L/F}\,\pi$ is the Zelevinsky product of the σ^γ, with γ ranging over Gal(L/F). However, the " Iwahori components " of the character θ_L are just the Galois conjugates of θ_L^1. Thus $b_{L/K}\,\pi$ contains the character θ_L, by (16.10). For each L/F-lift β' of β, the field extension L[β']/L is totally ramified. Thus, when we apply KL/L-base change, we can use (15.18) in each of the blocks σ^γ and deduce that $b_{KL/F}\,\pi$ contains the character θ_{KL}. Since the extension KL/K is unramified, we deduce that $b_{K/F}\,\pi$ contains θ_K, as required. The converse argument is similar. □

(16.12) *Remarks.* — (i) It needs to be emphasized that the main theorem (16.8) is conditional on the statement (16.10). This is best treated elsewhere as a special case of the rather different arguments of [BK4].

(ii) It is worthwhile to consider the places where the characteristic zero hypothesis enters the arguments above. First, we use it in § 14 and § 15 to get the character formulae which define local base change and local automorphic induction. The Jacquet-Langlands correspondence likewise enters, but only via its consequence that discrete series characters are linearly independent on the elliptic regular set. In the present section, we only use the *statements* (16.3), (16.4) and (16.7). All other arguments are characteristic-independent.

Appendix: Some character formulas

We gather together the explicit character formulas we need in §§ 14, 15. These are scattered in the literature, and usually proved under inconvenient and unnecessary restrictions. We are thus forced to give this fairly detailed survey in the degree of generality we require. Apart from some comments on the supercuspidal case (see (A.14)), there is nothing new here: we follow the relevant parts of [HC1], [HC2], [RS] quite closely. We deal only with (variations on) the group GL(N). However, many of the arguments can be carried through much more generally; we comment briefly on this at the end.

As before, F is a non-Archimedean local field *of arbitrary characteristic*. We shall be concerned here with three different groups G:

(A.1) (i) $G = GL(N, F)$, *for some* $N \geqslant 1$;

(ii) $G = GL(N, F) \rtimes \Gamma$, *where* F/F_0 *is a finite cyclic field extension and* $\Gamma = Gal(F/F_0)$;

(iii) G *is an open subgroup of* GL(N, F) *containing* SL(N, F).

We refer to (A.1) (i) as *the standard case*, to (A.1) (ii) as *the twisted case*, and to the remaining one as *the subgroup case*. In all cases, we put

$$G^0 = GL(N, F),$$
$$\mathfrak{g} = \mathbf{M}(N, F).$$

An element $g \in G^0$ is called *regular*, in the terminology of, e.g., [HC1], if its characteristic polynomial in F[X] has N distinct roots in an algebraic closure of F. This is equivalent to saying that the centralizer of g in $\mathbf{M}(N, F)$ is a product of *separable* field extensions of F. We write G^0_{reg} for the set of regular element of G^0. It will be convenient for us to use a slightly more general notion. We say that $g \in G^0$ is *quasi-regular over* F if its centralizer in $\mathbf{M}(N, F)$ is just a product of field extensions of F. Equivalently, g is quasi-regular if its characteristic polynomial has no repeated *irreducible* factor in F[X]. We write G^0_{qr} for the set of F-quasi-regular elements of G^0.

One can describe this property of quasi-regularity in various ways. We view $\mathfrak{g}$ as the Lie algebra of G^0; both groups G, G^0 act on $\mathfrak{g}$ by conjugation, for which we use the notation

$$g : x \mapsto \mathrm{Ad}\, g(x) = gxg^{-1}, \quad x \in \mathfrak{g},$$

where g is in G or G^0.

(A.2) *Proposition. — Let* $G^0 = GL(N, F)$ *as above, and let* $g \in G^0$. *The following are equivalent*:

(i) g *is quasi-regular over* F;

(ii) *the vector space* $\mathfrak{k}_g = \mathrm{Ker}(\mathrm{Ad}\, g - 1)$ *contains no nonzero nilpotent element of* $\mathfrak{g}$;

(iii) *if* U *is the unipotent radical of a parabolic subgroup of* G^0 *and* $\mathfrak{u} \subset \mathfrak{g}$ *is the Lie algebra of* U, *then* $\mathfrak{k}_g \cap \mathfrak{u} = \{0\}$.

Proof. — The equivalence of (ii) and (iii) is immediate. The equivalence of (i) and (ii) is a consequence of the rational Jordan canonical form. □

Of course, (A.2) holds without change in the subgroup case when $g \in G$, and we say that $g \in G$ is quasi-regular if $g \in G \cap G^0_{qr}$. In the twisted case, we say that $g \in G$ is quasi-regular if it satisfies the conditions (A.2) (ii), (iii) (which are again equivalent in this context). Indeed, in all cases, we could equally well define $g \in G$ to be quasi-regular if it satisfies (A.2) (iii). We write G_{qr} for the set of quasi-regular elements of G.

It is well known that the set G^0_{reg} is open dense in G^0 with respect to the Zariski topology, hence also the F-topology. Indeed, G^0_{reg} is defined by the non-vanishing of the principal coefficient in the characteristic polynomial of the endomorphism $\mathrm{Ad}\, g - 1$ of $\mathfrak{g}$. On the other hand, our set G_{qr} is not geometrically defined, but we now show that it inherits the density property.

(A.3) *Proposition. — The set* G_{qr} *is open and dense in* G *in the* F*-topology.*

Proof. — To save a little notation, we first assume we are in the standard case, so that $G^0 = G$. For a matrix $x = (x_{ij}) \in \mathfrak{g}$, we define

$$| x | = \max_{i,j} \| x_{ij} \|,$$

where $\| \ \|$ denotes the absolute value on F. The set $\mathscr{B}$ of elements $x \in \mathfrak{g}$ with $| x | = 1$ is then compact in $\mathfrak{g}$. The set $\mathscr{B}_0$ of nilpotent elements of $\mathscr{B}$ is closed, being the set of zeros in $\mathscr{B}$ of the function $x \mapsto x^N$. Thus $\mathfrak{B}_0$ is compact.

We show that the set $G - G_{qr}$ is closed. To that end, let $\{ g_n \}_{n \geqslant 1}$ be a convergent sequence of elements of $G - G_{qr}$ with limit $g \in G$. By (A.2), we can find $x_n \in \mathscr{B}_0$ such that $g_n^{-1} x_n g_n = x_n$, for each n. Since $\mathscr{B}_0$ is compact, we can pass to a subsequence and assume that $x_n \to x \in \mathscr{B}_0$ as $n \to \infty$. It follows that $g_n^{-1} x_n g_n \to g^{-1} x g = x$, whence $g \notin G_{qr}$. Thus $G - G_{qr}$ is closed and G_{qr} is open. Since $G_{qr} \supset G_{reg}$, is it certainly dense, and we have proved the Proposition in the standard case.

The subgroup case is now immediate. In the twisted case, the identical argument shows that G_{qr} is open. Take an element $g\gamma \in G$, with $g \in G^0$, $\gamma \in \Gamma$. Write F_1 for the fixed field of γ and $d = [F : F_1]$. Nothing will be changed if we replace $g\gamma$ by a G^0-conjugate, so we can assume that the element $h = (g\gamma)^d$ actually lies in $GL(N, F_1)$. (Of course, h is just the γ-norm of g as in [AC], and § 12 above.) Suppose that h is a regular element of $GL(N, F_1)$. Using the characteristic polynomial characterization, it follows that h is a regular element of G^0 and so the kernel of $\mathrm{Ad}\, h - 1$ on $\mathfrak{g}$ contains no non-trivial nilpotent. The same therefore applies to $g\gamma$, whence $g\gamma \in G_{qr}$. However, the condition on h amounts to saying that g is γ-regular: see [AC], proof of Prop. 2.2 in Ch. I. Such elements can be equally defined by the non-vanishing of a certain polynomial function, and they are therefore dense in G^0. Thus $G_{qr} \cap G^0 \gamma$ is dense in $G^0 \gamma$, and we have proved the result. □

It is perhaps worth pointing out that, in the twisted case, we have proved more than necessary. We have shown that, if we fix γ, the set of $g\gamma \in G^0 \gamma$ such that $\mathscr{N}_\gamma g$ is regular in $GL(N, F^{\langle\gamma\rangle})$ is dense in $G^0 \gamma$, and this set is contained in $G_{qr} \cap G^0 \gamma$.

(A.4) *Theorem. — Let* $g \in G_{qr}$ *and let* P^0 *be an* F*-parabolic subgroup of* G^0*. In the subgroup case, put* $P = P^0 \cap G$*, and* $P = P^0$ *in the other cases. The maps*

$$\begin{aligned} \varphi_g &: G \to G/P, \\ x &\mapsto x^{-1} g x P, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \psi_g &: G \times P \to G, \\ (x, p) &\mapsto x^{-1} g x p \end{aligned}$$

are submersive.

Proof. — For $g \in G^0_{\mathrm{reg}}$, this is proved in [HC2] Th. 1, which we essentially copy. We have the relations

$$\varphi_g(yx) = \varphi_{y^{-1}gy}(x),$$
$$\psi_g(yx, pq) = \psi_{y^{-1}gy}(x, p)\, q$$

for $x, y \in G$, $p, q \in P$. Thus we need only check that φ_g is submersive at $x = 1$ and that ψ_g is submersive at $(x, p) = (1, 1)$. The subgroup case will therefore follow from the standard one, so we can now exclude it. Computing derivatives at these points, we see that both submersivity statements are equivalent to

(A.5) $$(\mathrm{Ad}\, g^{-1} - 1)\,(\mathfrak{g}) + \mathfrak{p} = \mathfrak{g},$$

where $\mathfrak{g} = \mathbf{M}(N, F)$ is the Lie algebra of G and $\mathfrak{p} \subset \mathfrak{g}$ that of P. However, $\mathfrak{g}$ carries the Ad G-invariant symmetric bilinear form $(x, y) \mapsto \mathrm{Tr}_{F/F_0}(\mathrm{tr}(xy))$, where F_0 is the fixed field of Γ in the twisted case, $F_0 = F$ in the standard one. Relative to this form, the orthogonal complement of $\mathfrak{p}$ is the Lie algebra $\mathfrak{u}$ of the unipotent radical of P. The orthogonal complement of $(\mathrm{Ad}\, g^{-1} - 1)\,(\mathfrak{g})$ is, by an elementary computation, just $\mathrm{Ker}(\mathrm{Ad}\, g - 1)$. Since $g \in G_{\mathrm{qr}}$, we have

$$\mathrm{Ker}(\mathrm{Ad}\, g - 1) \cap \mathfrak{u} = \{0\}$$

by (A.2). Taking orthogonal complements, we get the desired relation (A.5). □

We now take $g \in G_{\mathrm{qr}}$ and a subgroup P of G defined by a parabolic subgroup P^0 of G^0 as in (A.4). We fix a Haar measure dx on G and a left Haar measure $d_l\, x$ on P. We use the symbol C_c^∞ to denote " smooth functions of compact support " (with complex values). Appealing to [HC1] Th. 11 p. 49 (which, we note, involves no hypothesis on the characteristic or connectivity), we get:

(A.6) *For each* $\alpha \in C_c^\infty(G \times P)$, *there is a unique function* $f_{\alpha, g} \in C_c^\infty(G)$ *with the property*

$$\int_{G \times P} \alpha(x, p)\, \Phi(x^{-1} gxp)\, dx\, d_l\, p = \int_G f_{\alpha, g}(x)\, \Phi(x)\, dx$$

for all functions $\Phi \in C_c^\infty(G)$.

(A.7) *Lemma.* — *Fix* $\alpha \in C_c^\infty(G \times P)$. *The mapping*

$$G_{\mathrm{qr}} \to C_c^\infty(G),$$
$$g \mapsto f_{\alpha, g}$$

is locally constant.

This is proved exactly as in [HC3] Lemma 1, noting that G_{qr} is open.

We now need the standard Cartan decomposition in G^0. We let P^0 denote the usual Borel subgroup of upper triangular matrices in G^0, and T^0 the maximal torus of diagonal matrices. Further, we let K^0 be the maximal compact subgroup $GL(N, \mathfrak{o}_F)$, so that $G^0 = K^0 P^0$. We write T^0_+ for the set of matrices $\mathrm{diag}(t_1, t_2, \ldots, t_N) \in T^0$ such that $\| t_1 \| \geqslant \| t_2 \| \geqslant \ldots \geqslant \| t_N \|$. This gives us $G^0 = K^0 T^0_+ K^0$. In the standard case, we put $P = P^0$, $K = K^0$, $T_+ = T^0_+$. In the twisted case, we put $P = P^0$, $T_+ = T^0_+$, $K = K^0 \Gamma$. In the subgroup case, we put $P = P^0 \cap G$, $T_+ = T^0_+ \cap G$, $K = K^0 \cap G$. In all cases then, we have

$$G = KT_+ K.$$

(This is immediate in the standard and twisted cases. In the subgroup case, if we have $k_1 tk_2 \in G$, with $k_i \in K^0$, $t \in T^0_+$, then $t \in T_+$ and there are diagonal matrices u_i with unit entries such that $k_1 u_1$, $u_2 k_2$ have determinants in $\det G$ and hence lie in K. It follows that $u_1^{-1} tu_2^{-1} \in T_+$.)

We fix a compact open subgroup K_1 of K, and let dk_1 be the Haar measure on K_1 normalized so that $\int_{K_1} dk_1 = 1$.

Let $(\pi, \mathscr{V})$ be an admissible representation of G. We write $\mathrm{End}_0(\mathscr{V})$ for the canonical image of $\check{\mathscr{V}} \otimes \mathscr{V}$ in $\mathrm{End}_{\mathbf{C}}(\mathscr{V})$, where $(\check{\pi}, \check{\mathscr{V}})$ is the contragredient of $\mathscr{V}$. Equivalently, $\mathrm{End}_0(\mathscr{V})$ is the space of linear maps $T : \mathscr{V} \to \mathscr{V}$ with the property that the maps $G \to \mathrm{End}_{\mathbf{C}}(\mathscr{V})$ given by $g \mapsto \pi(g)\, T$, $g \mapsto T\pi(g)$ are both locally constant: see [Ca] p. 122.

(A.8) *Theorem. — Let* $(\pi, \mathscr{V})$ *be a smooth representation of* G *of finite composition length. For* $g \in G_{qr}$, *define*

$$T_g = \int_{K_1} \pi(k_1^{-1} gk_1)\, dk_1.$$

Then $T_g \in \mathrm{End}_0(\mathscr{V})$, *and the mapping* $G_{qr} \to \mathrm{End}_0(\mathscr{V})$, $g \mapsto T_g$, *is locally constant.*

Proof. — In the case $g \in G_{reg}$, $K = K_1$, this is [HC2] Theorem 2. The hypothesis $K = K_1$ is removed in [RS], whose proof we follow (assuming only $g \in G_{qr}$).

We can assume without loss of generality that K_1 is a normal subgroup of K. Let $K_1, K_2, \ldots, K_n$ be the distinct cosets of K_1 in K. For $g \in G_{qr}$ and $1 \leqslant i \leqslant n$, put

$$T_{g,i} = \int_{K_i} \pi(k^{-1} gk)\, dk,$$

where dk is the Haar measure on K which extends dk_1. Thus $T_g = T_{g,1}$. Moreover:

(A.9) *If* $k \in K$, *there exists* $i = i(k)$ *such that*

$$T_g \circ \pi(k) = \pi(k) \circ T_{g,i}.$$

Since $\mathscr{V}$ has finite composition length, we can find an open normal subgroup K_0 of K such that $\mathscr{V}$ is generated over G by its subspace $\mathscr{V}^{K_0}$ of K_0-fixed vectors. Moreover, $\mathscr{V}$ is admissible, so $\mathscr{V}^{K_0}$ is finite-dimensional. We further choose a compact open subgroup P_0 of P with the property $t^{-1} P_0 t \subset K_0$ for all $t \in T_+$: in all cases, we can choose an integer $n \geqslant 1$ such that K contains the group $C_n = 1 + \mathfrak{p}_F^n \mathbf{M}(n, \mathfrak{o}_F)$ and then set $P_0 = P \cap C_n$. Let $\alpha_i \in C_c^\infty(G \times P)$ denote the characteristic function of $K_i \times P_0$, for $1 \leqslant i \leqslant n$. By (A.6), there is a unique function $f_{\alpha_i, g} \in C_c^\infty(G)$ such that

$$\int_{K_i \times P_0} \Phi(k^{-1} gkp)\, dk\, d_l p = \int_G f_{\alpha_i, g}(y)\, \Phi(y)\, dy$$

for all $\Phi \in C_c^\infty(G)$ and $1 \leqslant i \leqslant n$. This same formula is true when Φ is only locally integrable on G: see [HC1] Corollary p. 49. Applying this to the coefficients of π (which are smooth, therefore locally integrable, functions on G) we get

$$\textbf{(A.10)} \qquad \int_{P_0} T_{g,i}\, \pi(p)\, d_l p = \int_{K_i \times P_0} \pi(k^{-1} gkp)\, dk\, d_l p = \pi(f_{\alpha_i, g}),$$

for $1 \leqslant i \leqslant n$. Each function $g \mapsto f_{\alpha_i, g}$, $g \in G_{qr}$, is locally constant by (A.7). Therefore, given $g_0 \in G_{qr}$, we can choose a neighbourhood W of g_0 in G_{qr} and an open normal subgroup $\widetilde{K}$ of K, with $\widetilde{K} \subset K_1$, such that $f_{\alpha_i, g}$ is $\widetilde{K}$-bi-invariant for all i and all $g \in W$.

Our definitions say that $\mathscr{V}$ is spanned over $\mathbf{C}$ by the elements $\pi(x)\, v$, $x \in G$, $v \in \mathscr{V}^{K_0}$. However, we have $G = KT_+ K$, and $\mathscr{V}^{K_0}$ is invariant under K. Thus $\mathscr{V}$ is spanned by the elements $\pi(kt)\, v$, for $k \in K$, $t \in T_+$ and $v \in \mathscr{V}^{K_0}$. For such k, t, v and $g \in W$, we have

$$T_g\, \pi(kt)\, v = \pi(k)\, T_{g,i}\, \pi(t)\, v,$$

by (A.9), where $i = i(k)$. But, substituting in the relation (A.10), we get

$$\begin{aligned} \pi(f_{\alpha_i, g})\, \pi(t)\, v &= \int_{P_0} T_{g,i}\, \pi(p)\, \pi(t)\, v\, d_l p \\ &= T_{g,i} \int_{P_0} \pi(pt)\, v\, d_l p \\ &= T_{g,i}\, \pi(t) \int_{P_0} \pi(t^{-1} pt)\, v\, d_l p \\ &= c_i(t)\, T_{g,i}\, \pi(t)\, v, \end{aligned}$$

for some $c_i(t) > 0$, since $t^{-1} P_0 t \subset K_0$. Hence

$$\begin{aligned} T_g\, \pi(kt)\, v &= \pi(k)\, T_{g,i}\, \pi(t)\, v \\ &= c_i(t)^{-1}\, \pi(k)\, \pi(f_{\alpha_i, g})\, \pi(t)\, v. \end{aligned}$$

Now let $\chi_{\widetilde{K}}$ denote $\operatorname{meas}(\widetilde{K})^{-1}$ times the characteristic function of $\widetilde{K}$. We then have

$$\pi(\chi_{\widetilde{K}})\, T_g\, \pi(kt)\, v = \pi(\chi_{\widetilde{K}})\, c_i(t)^{-1}\, \pi(k)\, \pi(f_{\alpha_i, g})\, \pi(t)\, v.$$

The operators $\pi(k)$, $\pi(\chi_{\widetilde{K}})$ commute, since K normalizes $\widetilde{K}$. This last expression is therefore

$$\begin{aligned} c_i(t)^{-1}\, \pi(k)\, \pi(\chi_{\widetilde{K}} * f_{\alpha_i, g})\, \pi(t)\, v &= c_i(t)^{-1}\, \pi(k)\, \pi(f_{\alpha_i, g})\, \pi(t)\, v \\ &= T_g\, \pi(kt)\, v, \end{aligned}$$

since, by the choice of $\widetilde{K}$, we have $\chi_{\widetilde{K}} * f_{\alpha_i, g} = f_{\alpha_i, g}$ when $g \in W$. Since the vectors $\pi(kt)\, v$ span $\mathscr{V}$, we conclude that

$$\pi(\chi_{\widetilde{K}})\, T_g = T_g,$$

for all $g \in W$. Returning to the original definition of T_g, we see that it commutes with $\pi(\chi_{\widetilde{K}})$, so $T_g \in \mathrm{End}_0(\mathscr{V})$. Moreover, the equation $\pi(\chi_{\widetilde{K}})\, T_g = T_g\, \pi(\chi_{\widetilde{K}}) = T_g$ says that

$$T_{xgy} = T_g, \quad x, y \in \widetilde{K}, \; g \in W,$$

so $g \mapsto T_g$ is a locally constant mapping $G_{qr} \to \mathrm{End}_0(\mathscr{V})$, as required. $\square$

The elements of $\mathrm{End}_0(\mathscr{V})$ are finite-rank linear operators on $\mathscr{V}$. In particular, the trace $\mathrm{tr}(T_g)$ is defined, for $g \in G_{qr}$.

(A.11) *Corollary. — The character of* π *is represented on* G_{qr} *by the locally constant function*

$$\Theta_\pi(g) = \mathrm{tr}(T_g), \quad g \in G_{qr}.$$

That is, if $\varphi \in C_c^\infty(G)$ *has support contained in* G_{qr}, *then*

$$\mathrm{tr}(\pi(\varphi)) = \int_G \Theta_\pi(g)\, \varphi(g)\, dg,$$

where dg *is the Haar measure on* G *used to define* $\pi(\varphi) = \int_G \varphi(g)\, \pi(g)\, dg$.

Proof. — For $\varphi \in C_c^\infty(G)$, put

$$\varphi^0(x) = \int_{K_1} \varphi(k_1 x k_1^{-1})\, dk_1, \quad x \in G.$$

Then $\varphi^0 \in C_c^\infty(G)$ and $\mathrm{tr}\, \pi(\varphi) = \mathrm{tr}\, \pi(\varphi^0)$. If $\varphi \in C_c^\infty(G_{qr})$, then $\varphi^0 \in C_c^\infty(G_{qr})$ and

$$\int_G \varphi(x)\, \mathrm{tr}(T_x)\, dx = \mathrm{tr}\left(\int_G \varphi(x)\, T_x\, dx\right)$$

since $g \mapsto T_g$ is locally constant and $T_g \in \mathrm{End}_0(\mathscr{V})$. Hence

$$\begin{aligned}
\int_G \varphi(x)\, \mathrm{tr}(T_x)\, dx &= \mathrm{tr} \int_G \int_{K_1} \varphi(x)\, \pi(k_1^{-1} x k_1)\, dk_1\, dx \\
&= \mathrm{tr} \int_{G \times K_1} \varphi(k_1 x k_1^{-1})\, \pi(x)\, dk_1\, dx \\
&= \mathrm{tr} \int_G \varphi^0(x)\, \pi(x)\, dx \\
&= \mathrm{tr}\, \pi(\varphi^0),
\end{aligned}$$

as required. $\square$

Let $(\pi, \mathscr{V})$ be an irreducible discrete series representation of G, i.e. such that the matrix coefficients of π are square-integrable modulo the centre of G. Write Z for the centre of G. Let $\langle\, ,\, \rangle$ be some G-invariant Hermitian inner product on $\mathscr{V}$. We write $\mathscr{A}(\pi)$ for the vector space of coefficients of π. We fix a Haar measure $d\dot{x}$ on G/Z and let $d(\pi)$ be the corresponding formal degree. Thus we have the Schur orthogonality relation [HC1]

$$\langle u_1, u_2 \rangle \overline{\langle v_1, v_2 \rangle} = d(\pi) \int_{G/Z} \langle \pi(x)\, u_1, v_1 \rangle \overline{\langle \pi(x)\, u_2, v_2 \rangle}\, d\dot{x}$$

for $u_i, v_i \in \mathscr{V}$.

(A.12) *Theorem.* — *Let* $(\pi, \mathscr{V})$ *be an irreducible discrete series representation of* G. *Let* Θ_π *be as in* (A.11). *Then*

$$\varphi(1)\, \Theta_\pi(g) = d(\pi) \int_{G/Z} \int_{K_1} \varphi(x^{-1} k^{-1} gkx)\, dk\, d\dot{x},$$

for any $g \in G_{qr}$, $\varphi \in \mathscr{A}(\pi)$. *The outer integral here converges absolutely and uniformly on compact subsets of* G_{qr}.

Suppose that π *is supercuspidal, and let* C *be a compact subset of* G_{qr}. *There exists a compact mod centre subset* $S = S_C$ *of* G *such that*

$$\int_{K_1} \varphi(x^{-1} k^{-1} gkx)\, dk = 0$$

whenever $g \in C$ *and* $x \notin S$.

Proof. — This is copied from [RS] Th. 2. It is enough to treat the case where π is unitarizable and φ is of the form

$$\varphi(x) = \langle \pi(x)\, u, v \rangle, \quad x \in G,$$

for some $u, v \in \mathscr{V}$.

We suppose given a compact set C in G_{qr}. For $g_0 \in C$, we saw in the proof of (A.8) that there is a neighbourhood $W(g_0)$ of g_0 in G_{qr} and a compact open subgroup $\widetilde{K}(g_0)$ of G such that T_g is constant on the double coset $\widetilde{K}(g_0)\, g\widetilde{K}(g_0)$ for all $g \in W(g_0)$. We can cover C by a finite collection of these W's, so there exists a compact open subgroup $\widetilde{K}$ of G such that T_g is constant on double cosets $\widetilde{K}g\widetilde{K}$ for all $g \in C$. In particular, we have

$$\pi(k)\, T_g = T_g\, \pi(k) = T_g, \quad k \in \widetilde{K},\ g \in C.$$

This means that T_g lies in the subspace $\check{\mathscr{V}}^{\widetilde{K}} \otimes \mathscr{V}^{\widetilde{K}}$ of $\mathrm{End}_0(\mathscr{V})$. We abbreviate $\widetilde{\mathscr{V}} = \mathscr{V}^{\widetilde{K}}$ and set $M = \dim_{\mathbf{C}} \widetilde{\mathscr{V}}$. We choose an orthonormal basis $\{e_1, e_2, \ldots, e_M\}$ of $\widetilde{\mathscr{V}}$. In our present terms, T_g is then the operator

$$T_g : w \mapsto \sum_{i, j=1}^{M} \langle T_g\, e_i, e_j \rangle \langle w, e_i \rangle\, e_j, \quad w \in \widetilde{\mathscr{V}}.$$

We thus have

$$\begin{aligned}\int_{K_1} \varphi(x^{-1} k^{-1} gkx)\, dk &= \int_{K_1} \langle \pi(k^{-1} gk)\, \pi(x)\, u, \pi(x)\, v \rangle\, dk \\ &= \langle T_g\, \pi(x)\, u, \pi(x)\, v \rangle \\ &= \sum_{i,j=1}^{M} \langle T_g\, e_i, e_j \rangle \langle \pi(x)\, u, e_i \rangle \overline{\langle \pi(x)\, v, e_j \rangle} \\ &= \sum_{i,j=1}^{M} \langle T_g\, e_i, e_j \rangle \langle \pi(x^{-1})\, e_j, v \rangle \overline{\langle \pi(x^{-1})\, e_i, u \rangle}.\end{aligned}$$

If π is supercuspidal, this last expression vanishes identically outside of some compact mod centre set, which proves the last assertion of the Theorem. Returning to the general case, the local constancy of T_g now implies

(A.13)
$$\left| \int_{K_1} \varphi(x^{-1} k^{-1} gkx)\, dk \right| \leqslant b_C \sum_{i,j} | \langle \pi(x^{-1})\, e_j, v \rangle |\, | \langle \pi(x^{-1})\, e_i, u \rangle |, \quad g \in C,$$

for some $b_C > 0$.

Returning to the original equality, we now have

$$\begin{aligned} d(\pi) \int_{G/Z} \int_{K_1} & \varphi(x^{-1} k^{-1} gkx)\, dk\, d\dot{x} \\ &= \sum_{i,j=1}^{M} \langle T_g\, e_i, e_j \rangle\, d(\pi) \int_{G/Z} \langle \pi(x^{-1})\, e_j, v \rangle \overline{\langle \pi(x^{-1})\, e_i, u \rangle}\, d\dot{x} \\ &= \sum_{i,j=1}^{M} \langle T_g\, e_i, e_j \rangle \langle e_j, e_i \rangle \langle u, v \rangle \\ &= \sum_{i=1}^{M} \langle T_g\, e_i, e_i \rangle\, \varphi(1) \\ &= \mathrm{tr}(T_g)\, \varphi(1) = \Theta_\pi(g)\, \varphi(1), \end{aligned}$$

as required for the Theorem. Moreover, the bound (A.13) and the Schwarz inequality give

$$\int_{G/Z} \left| \int_{K_1} \varphi(x^{-1} k^{-1} gkx)\, dk \right| d\dot{x} \leqslant b_C \sum_{i,j} \| \varphi_{j,v} \|_2 \, \| \varphi_{i,u} \|_2, \quad g \in C,$$

where $\| \ \|_2$ is the L^2-norm on G/Z and $\varphi_{k,w} \in \mathscr{A}(\pi)$ denotes the function $x \mapsto \langle \pi(x)\, e_k, w \rangle$, $1 \leqslant k \leqslant M$, $w \in \mathscr{V}$. Uniformity of convergence then follows from the " Weierstrass M-test ". □

Of course, (A.12) applies equally to essentially square-integrable representations of G. In particular, it holds whenever $\boldsymbol{\pi}$ is supercuspidal. We apply this theorem to the case where $\boldsymbol{\pi}$ is induced from an irreducible smooth representation ρ of an open compact mod centre subgroup H of G (and therefore supercuspidal). Moreover, if we put

$$\boldsymbol{\Theta}_\rho(x) = \begin{cases} \mathrm{tr}(\rho(x)), & x \in \mathrm{H}, \\ 0, & x \in \mathrm{G}, \quad x \notin \mathrm{H}, \end{cases}$$

then $\boldsymbol{\Theta}_\rho$ lies in $\mathscr{A}(\boldsymbol{\pi})$, the space of coefficients of $\boldsymbol{\pi}$.

(A.14) *Theorem.* — *Let* $\boldsymbol{\pi} = c\text{-}\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\,\rho$, as *above. We have*:

(i) $d(\boldsymbol{\pi}) = \dim \rho/\mathrm{meas}(\mathrm{H}/\mathrm{Z}, d\dot{x})$;

(ii) *for* $g \in \mathrm{G}_{\mathrm{qr}}$,

$$\boldsymbol{\Theta}_\pi(g) = \sum_{x \in \mathrm{G}/\mathrm{H}} \int_{\mathrm{K}_1} \boldsymbol{\Theta}_\rho(x^{-1}k^{-1}gkx)\, dk.$$

Moreover, the series in (ii) *has only finitely many terms*: *if* C *is a compact subset of* G_{qr}, *there is a finite subset* S_{C} *of* G/H *such that the inner integral vanishes when* $g \in \mathrm{C}$ *and* $x\mathrm{H} \notin \mathrm{S}_{\mathrm{C}}$.

Further,

$$\boldsymbol{\Theta}_\pi(g) = \sum_{x \in \mathrm{K}_1\backslash \mathrm{G}/\mathrm{H}} \Big(\sum_{y \in \mathrm{K}_1 x\mathrm{H}/\mathrm{H}} \boldsymbol{\Theta}_\rho(y^{-1}gy)\Big), \quad g \in \mathrm{G}_{\mathrm{qr}}.$$

For $g \in \mathrm{C}$, *all but finitely many of the terms*

$$\sum_{y \in \mathrm{K}_1 x\mathrm{H}/\mathrm{H}} \boldsymbol{\Theta}_\rho(y^{-1}gy)$$

are identically zero.

Proof. — The definition of $d(\boldsymbol{\pi})$ gives

$$d(\boldsymbol{\pi}) \int_{\mathrm{G}/\mathrm{Z}} \boldsymbol{\Theta}_\rho(x)\, \overline{\boldsymbol{\Theta}_\rho(x)}\, d\dot{x} = \dim \rho,$$

while

$$\int_{\mathrm{G}/\mathrm{Z}} \boldsymbol{\Theta}_\rho(x)\, \overline{\boldsymbol{\Theta}_\rho(x)}\, d\dot{x} = \int_{\mathrm{H}/\mathrm{Z}} \boldsymbol{\Theta}_\rho(x)\, \overline{\boldsymbol{\Theta}_\rho(x)}\, d\dot{x} = \mathrm{meas}(\mathrm{H}/\mathrm{Z}, d\dot{x})$$

since ρ is irreducible. This proves (i).

Applying (A.12) with $\varphi = \boldsymbol{\Theta}_\rho$ and $g \in \mathrm{G}_{\mathrm{qr}}$, we get

$$\dim(\rho)\, \boldsymbol{\Theta}_\pi(g) = d(\boldsymbol{\pi}) \int_{\mathrm{G}/\mathrm{Z}} \int_{\mathrm{K}_1} \boldsymbol{\Theta}_\rho(x^{-1}k^{-1}gkx)\, dk\, d\dot{x}.$$

As a function of x, the integrand here is constant on the coset $x\mathrm{H}$, so we can use (i) to get

$$\boldsymbol{\Theta}_\pi(g) = \sum_{x \in \mathrm{G}/\mathrm{H}} \int_{\mathrm{K}_1} \boldsymbol{\Theta}_\rho(x^{-1}k^{-1}gkx)\, dk,$$

which proves (ii). The next assertion follows from (A.12), and the others are immediate. □

The second formula in (A.14) (ii) generalizes that of [Sy] (1.9) (which does not have the finiteness observation).

Remark. — The ideas above can be generalized quite considerably. Indeed, they apply virtually without change when G is a connected reductive group over F. They thereby extend the arguments of, in particuler, [HC1], [RS], to quasi-regular elements (as defined by (A.2) (iii)) and arbitrary characteristic. However, we defer discussion of this matter to another occasion.

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C. B., Department of Mathematics,
King's College,
Strand,
London WC2R 2LS.
c.bushnell©kcl.ac.uk.

G. H., Mathématiques,
Bâtiment 425,
Université de Paris XI
et URA 752 du CNRS,
91405 Orsay.
henniart©dmi.ens.fr.

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