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PROJECTIVE VARIETIES WITH NON-RESIDUALLY FINITE FUNDAMENTAL GROUP (1)

by Domingo TOLEDO

1. Introduction

A well known folklore problem, often attributed to J.-P. Serre, asks whether there is a complex algebraic variety whose fundamental group is not residually finite. The purpose of this paper is to construct examples that give an affirmative answer to this question. These examples are actually smooth projective varieties.

Recall that a group is said to be residually finite if the intersection of all its subgroups of finite index consists of the identity element alone. A well known theorem, going back to Malcev (cf. Theorems VII and VIII of [13]), asserts that a finitely generated group of matrices (with coefficients in a field, and in fact any commutative ring) is residually finite. Our examples therefore provide fundamental groups of smooth projective varieties with no faithful linear representations. No examples with this property were known before.

Our groups do have plenty of non-faithful linear representations. In fact each example admits a surjective homomorphism to a lattice in the Lie group $SO(2, n)$ whose kernel is a free group of infinite rank. From the construction it is clear that this kernel in turn contains a subgroup of infinite rank that must be mapped trivially by any homomorphism to a linear group or a finite group.

There is another well known problem, also attributed to J.-P. Serre, that asks whether a complex algebraic variety exists whose fundamental group is non-trivial but has no non-trivial finite quotients. A more specific form of this question, for the Higman 4-group, is stated in Chapter I, § 1.4 of [19]. Since our examples have plenty of homomorphisms to finite groups, they shed no light on this more subtle question. But they show that residual-finitness problems can indeed arise in algebraic geometry. W. Fulton pointed out to us that, in relation to the fundamental group of the complement of a plane algebraic curve, Zariski had already posed the question of existence of non-residually finite groups, cf. § 1 (and Appendix 1) to Chapter VIII of [21] (2). More recently, residual finiteness questions have appeared in relation to the problem of understanding

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(2) (Added in Proof.) See also S. Abhyankar, Tame coverings and fundamental groups of algebraic varieties, Part I, *American Jour. Math.*, **81** (1959), 46-94, Remark 15 on p. 93.

restrictions on fundamental groups of algebraic varieties. Most of the known restrictions seem to involve linear representations of such groups, and at present Gromov's techniques on L_2-cohomology [8] (and more recent techniques of Gromov and Schoen) seem to be the only ones that completely avoid linear representations.

The groups we consider are the fundamental groups of the complements of smooth totally geodesic divisors in locally symmetric varieties for the group $SO(2, n)$ where $n \geqslant 4$. To avoid unnecessary technicalities, we will also assume that n is even. One has to show that these groups have the desired properties. The fact that they are not residually finite follows from a theorem of Raghunathan ([18], Main Theorem) in the following way. From the geometry of the situation it is clear that the fundamental group of the boundary of a tubular neighborhood of the divisor injects into the fundamental group of the complement of the divisor. But the fundamental group of the boundary of the tubular neighborhood is easily seen to be a co-compact lattice in a certain covering group of $SO(2, n-1)$. Raghunathan's theorem asserts that any lattice in a related covering group must contain a certain fixed infinite cyclic subgroup of its center, and this easily implies the same assertion for our group. From this it follows that the fundamental group of the complement of the divisor contains a non-residually finite subgroup, hence it is not residually finite.

A point to be observed here is that the covering groups in question are not linear groups, hence Malcev's theorem does not apply to their finitely generated subgroups, and there is no reason to believe that lattices in these covering groups are residually finite. To produce non-residually finite lattices in specific non-linear Lie groups is more subtle. Examples have been given by Millson [16] and general criteria, together with a wider class of examples, have been given by Deligne [5], but Raghunathan's theorem seems to be the only one that applies to co-compact lattices, which is the situation that we need.

The fact that these groups are fundamental groups of smooth projective varieties is proved by first showing the projectivity of identification spaces M/D, where M is a compact locally symmetric variety and D is a smooth totally geodesic divisor in M with negative normal bundle. Namely, we prove that M/D has the structure of a projective variety V with a single singular point p such that $M - D$ is biholomorphic with $V - \{ p \}$. We originally proved this fact by first applying Grauert's criterion [7] to blow down D in the analytic category, and then applying an ampleness criterion to a suitable line bundle on M/D to show projectivity. The more direct proof presented here was later shown to us by P. Deligne. Note that the hypotheses on M and D imply that the universal cover of M is the symmetric space for either $SU(1, n)$ or $SO(2, n)$, $n \geqslant 2$ and that the embedding of the universal cover of D corresponds to the standard embedding of $SU(1, n-1)$ or $SO(2, n-1)$ respectively (otherwise there are no totally geodesic complex hypersurfaces with negative normal bundle).

The point of proving that $V = M/D$ is a projective variety is that it has hyperplane sections. If Z denotes a generic hyperplane section of V (in particular not containing the singular point p), then the appropriate version of the Lefschetz theorem on hyperplane sections implies that the inclusion of Z in $V - \{p\} = M - D$ induces an isomorphism of fundamental groups if $n \geqslant 3$, hence there is a smooth projective variety with the same fundamental group as $M - D$. This construction works when the universal cover of M is the symmetric space for either $SU(1, n)$ or $SO(2, n)$. It seems possible that in both situations one could obtain non-residually finite groups, provided n is large. But it is only for $SO(2, n)$ that Raghunathan's theorem is available, hence we give the details only for this group.

Motivated by the results of this paper, M. Nori, and independently F. Catanese and J. Kollár, found another way of using Raghunathan's theorem to produce projective varieties with non-residually finite fundamental group (as branched covers of locally symmetric varieties, branched over ample divisors). A description of these examples appears in [1]. The fundamental groups they obtain are quite different in nature from ours, in that the intersection of all subgroups of finite index is a finite cyclic group, while in our examples this intersection is a free group of infinite rank.

We would like to point out that the main technique used here, already used in [20], of taking non-singular hyperplane sections of singular projective varieties, appears to be a powerful way of constructing smooth projective varieties with interesting fundamental groups. More precisely, it gives a way of showing that the fundamental groups of certain smooth quasi-projective varieties are also fundamental groups of smooth projective varieties.

This technique also gives a way of constructing new smooth varieties with more standard fundamental groups. We came to this construction from a different consideration, namely showing the projectivity of the analytic space obtained by blowing down a codimension two complex geodesic subvariety in quotients of the three ball. This was motivated by a question raised by M. Gromov, whether a smooth projective surface with fundamental group a co-compact lattice in $SU(1, 3)$ must be birationally equivalent to a hyperplane section of the corresponding quotient of the three-ball. Non-singular hyperplane sections of these blown-down spaces show that there are more possibilities. This is discussed in the last section of this paper.

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2. Projectivity of identification spaces

Let X_n denote the symmetric space for the group $SO(2, n)$. As a homogeneous space, $X_n = SO(2, n)/S(O(2) \times O(n)) = SO^+(2, n)/SO(2) \times SO(n)$, where $SO^+(2, n)$ denotes the component of the identity of (the two-component group) $SO(2, n)$. To obtain concrete realizations of X_n corresponding to these homogeneous space descriptions, choose a quadratic form q of signature $(2, n)$ on $\mathbf{R}^{n+2}$, for instance

$x_1^2 + x_2^2 - x_3^2 - \ldots - x_{n+2}^2$. Then X_n can be identified with the open subset of the Grassmannian of (unoriented) two planes in $\mathbf{R}^{n+2}$ on which the restriction of q is positive definite. Its pre-image in the Grassmannian of oriented two planes splits into two components, $SO(2, n)$ acts transitively on this pre-image, and $SO^+(2, n)$ preserves these components and acts transitively on each. We fix one component and use it as our model of X_n.

The center of $SO^+(2, n)$ is trivial if n is odd and consists of $\pm$ id if n is even. Let $PSO^+(2, n)$ denote the quotient of $SO^+(2, n)$ by its center. This group then acts effectively on X_n.

The symmetric space X_n is actually hermitian symmetric, and its invariant complex structure can be described as follows. Let b denote the real bilinear form associated to the quadratic form q. Given a point in X_n, let x, y be a positively oriented basis for the corresponding two-plane, where x and y are b-orthogonal and of the same length, and let $z = x + iy \in \mathbf{C}^{n+2}$. Denote also by b the complex-bilinear extension of b to $\mathbf{C}^{n+2}$. Then it is easy to see that the complex line determined by z depends only on the oriented two-plane determined by x and y, and thus there is a well defined map of X_n into the subset of $P(\mathbf{C}^{n+2})$ defined by the conditions

$$(2.1) \qquad b(z, z) = 0, \quad b(z, \bar{z}) > 0.$$

The latter is the open subset of the quadric hypersurface $Q_n \subset \mathbf{P}^{n+1}$ with equation $b(z, z) = 0$ defined by the inequality $b(z, \bar{z}) > 0$. Again one sees that this open set splits into two connected components (which are interchanged by complex conjugation), and that we obtain an $SO^+(2, n)$-equivariant identification of X_n with one of these components. This identification gives the invariant complex structure on X_n.

The goal of this section is to prove the projectivity of certain spaces obtained from embeddings of compact quotients of X_{n-1} into compact quotients of X_n. To this end we need to recall certain simple identities for equivariant line bundles on X_n. Let S denote the tautological line sub-bundle of the trivial bundle $X_n \times \mathbf{C}^{n+2}$, and let K denote the canonical bundle of X_n.

Lemma **(2.2)**. — *1.* $K \approx S^{\otimes n}$ *as* $SO^+(2, n)$*-bundles on* X_n.
2. With its $SO^+(2, n)$*-invariant hermitian metric,* S *is a positive line bundle on* X_n.

Proof. — The first part follows from the fact that S and K are homogeneous line bundles which extend to Q_n, and the corresponding fact over Q_n. An alternative argument would be the following.

Let $S^\perp$ denote the b-orthogonal complement of S in $X_n \times \mathbf{C}^{n+2}$, and let T denote the holomorphic tangent bundle of X_n. Then T is the holomorphic sub-bundle of the tangent bundle of $\mathbf{P}^{n+1}$ on which the first equation (2.1) is infinitesimally preserved, therefore there is an isomorphism of holomorphic $SO(2, n)^+$-bundles

$$T \approx \mathrm{Hom}(S, S^\perp/S).$$

From the exact sequences of holomorphic $SO^+(2, n)$-bundles

$$0 \to S^\perp \to X_n \times \mathbf{C}^{n+2} \to S^* \to 0, \qquad 0 \to S \to S^\perp \to S^\perp/S \to 0,$$

one obtains easily (tensoring with S^* and computing top exterior powers) the isomorphism asserted in the first part of the lemma.

The second assertion is well-known, cf. [12]; alternatively, the line bundle S over Q_n is the negative line bundle $\mathcal{O}_{Q_n}(-1)$, thus, by the usual change of sign in Hirzebruch proportionality [10], S is a positive line bundle over X_n. The proof is complete.

If $\Gamma \subset SO^+(2, n)$ is a discrete, torsion-free subgroup, and $M = \Gamma \backslash X_n$, then S and K descend to line bundles on M which will be denoted by S_M and K_M, and the assertions of Lemma (2.2) hold for these line bundles.

Let $v \in \mathbf{R}^{n+2}$ be a negative vector for q, i.e. $q(v) < 0$. For example, if q is as above, then $v = (0, \ldots, 0, 1)$ is negative. If $v^\perp$ denotes the b-orthogonal complement of v, then the restriction of q to $v^\perp$ has signature $(2, n-1)$. The subgroup of $SO^+(2, n)$ that fixes v is isomorphic to $SO^+(2, n-1)$, leaves $v^\perp$ invariant, and its action there is equivalent to the standard action of $SO^+(2, n-1)$ on $\mathbf{R}^{n+1}$. We will denote this subgroup by $SO^+(2, n-1)_v$.

The choice of a negative vector v gives an embedding of X_{n-1} in X_n as the set of positive and oriented two-planes contained in $v^\perp$. In the above embedding of X_n in the quadric $Q_n \subset \mathbf{P}^{n+1}$, X_{n-1} corresponds to the points lying in the hyperplane with equation $b(v, \ ) = 0$. Thus X_{n-1} is a complex analytic hypersurface in X_n. The following lemma is easily verified:

Lemma **(2.3)**. — *If n is even,* $SO^+(2, n-1)_v$ *injects into* $PSO^+(2, n)$ *under the natural projection* $\pi : SO^+(2, n) \to PSO^+(2, n)$ *and its image coincides with the subgroup of* $PSO^+(2, n)$ *that leaves the image of the corresponding embedding of* X_{n-1} *invariant.*

Let $v \in \mathbf{R}^{n+2}$ be a negative vector, and $X_{n-1} \subset X_n$ the corresponding embedding. Then v defines a section s_v of S^* by the formula

$$s_v(w) = b(v, w).$$

It is clear that s_v is an $SO^+(2, n-1)_v$-invariant section of S^* which vanishes precisely on X_{n-1} and (being a linear function in affine coordinates) it is clear that it vanishes to first order.

This observation can be applied to the following situation. Suppose that n is even and that $\Gamma \subset SO^+(2, n)$ is a discrete, torsion-free, co-compact subgroup. Since the kernel of the natural projection $\pi : SO^+(2, n) \to PSO^+(2, n)$ is a torsion group, the restriction of π is an isomorphism of Γ with $\pi(\Gamma)$. Thus Γ acts effectively on X_n, and if we let $M = \Gamma \backslash X_n$, then M is a compact manifold with universal cover X_n and fundamental group isomorphic to Γ. Let $\Gamma' = \pi^{-1}(\pi(\Gamma)) \subset SO^+(2, n)$.

Suppose M contains a compact, non-singular, totally geodesic complex hypersurface D obtained in the following way. Take a negative vector v, let $X_{n-1} \subset X_n$ be the corresponding embedding, and let $\Gamma_0 = \Gamma' \cap SO^+(2, n-1)_v$. From Lemma (2.3) we see that $\pi(\Gamma_0)$ is the subgroup of $\pi(\Gamma)$ that leaves X_{n-1} invariant, hence the map $\Gamma_0\backslash X_{n-1} \to M$ induced by the inclusion $X_{n-1} \subset X_n$ is a totally geodesic immersion which is generically one to one. Our assumptions will be that this immersion is actually an embedding, and that Γ_0 is co-compact in $SO^+(2, n-1)_v$. Let D denote the image of this embedding. Then D will be a smooth totally geodesic divisor in M such that $\pi_1(D) \approx \Gamma_0$. Examples of such embeddings will be given at the end of this section. Observe that from the construction of Γ_0 the natural injection $\Gamma_0 \to \Gamma$ is not necessarily that given as subgroups of $SO^+(2, n)$, but rather the injection of their isomorphic projections to $PSO^+(2, n)$.

Lemma **(2.4).** — *Let* $D \subset M$ *be as in the previous paragraph. Then there is a neighborhood* U *of* D *in* M *so that* $\mathcal{O}(D) \mid U$ *is isomorphic to* $S^*_M \mid U$.

Proof. — There is a neighborhood V of X_{n-1} in X_n so that $\Gamma_0\backslash V$ embeds in M. Let U denote the image of thie embedding. Then $S^* \mid V$ has the $SO^+(2, n-1)_v$-invariant hence Γ_0-invariant section s_v which vanishes precisely on X_{n-1} and to first order. This section descends to a section s_D of $S^*_M \mid U$ that vanishes precisely on D and to first order. It follows that $\mathcal{O}(D) \mid U$ is isomorphic to $S^*_M \mid U$. In fact, $\mathcal{O}(D)$ is the line bundle obtained from $S^*_M \mid U$ and the trivial line bundle over $M - D$ by identifying these two bundles over $U - D$ by the isomorphism that takes s_D into the constant section 1.

The main technical point of this section is contained in the following proposition, which is an immediate consequence of the last two lemmas:

Proposition **(2.5).** — *Let* M *and* D *be as in Lemma (2.4). Then the line bundle* $K_M(nD)$ *is trivial in a neighborhood of* D *in* M.

Observe that the normal bundle of D in M is negative, hence by Grauert's criterion (Satz 8, § 3 of [7]) D can be blown down analytically, i.e. the identification space M/D is an analytic space. The proposition means that $K(nD)$ descends to a line bundle L on M/D (i.e., the direct image sheaf is locally trivial). Our original proof of the projectivity of M/D was to start from its analytic structure and use Grauert's criterion for projectivity (Satz 2, § 3 of [7]) to show that L is an ample line bundle on M/D. P. Deligne then showed us a more direct approach that avoids analytic arguments. The following proposition, and its proof, are due to Deligne.

Proposition **(2.6).** — *Let* M *be a smooth projective variety, let* D *be a smooth irreducible divisor on* M, *let* H *be a positive line bundle on* M, *and suppose that there exists a positive integer* ℓ *so that the line bundle* $H(\ell D)$ *is trivial in a neighborhood of* D.

Then there is a positive integer k *so that* $H(\ell D)^{\otimes k}$ *is base-point free and so that the corresponding map from* M *to a projective space is a bijection of the identification space* M/D *to the image* *of* M, *and is an embedding on* M — D. *Thus, if* V *denotes the image of* M *and* $p \in$ V *the image of* D, *then* M/D *can be given the structure of a projective variety* V *such that* M — D *is bihilomorphic to* V — $\{p\}$.

Proof. — Let s_D denote the canonical section of $\mathcal{O}(D)$. Then the map $t \mapsto t \otimes s_D^{k\ell}$ defines an embedding $H^{\otimes k} \to H(\ell D)^{\otimes k}$, and under the assumption that $H(\ell D)$ is trivial in a neighborhood of D we see that the quotient sheaf is $\mathcal{O}/\mathcal{O}(-k\ell D)$. Thus we have an exact sequence of sheaves

$$0 \to H^{\otimes k} \to H(\ell D)^{\otimes k} \to \mathcal{O}/\mathcal{O}(-k\ell D) \to 0. \tag{2.7}$$

Since M is projective and H is positive, we can choose k large enough so that for all P, Q $\in$ M,

$$H^1(M, H^{\otimes k} \otimes \mathcal{I}_{P,Q}) = 0, \qquad H^1(M, H^{\otimes k} \otimes \mathcal{I}_P^2) = 0,$$

where $\mathcal{I}_P$, $\mathcal{I}_{P,Q}$ denote the ideal of functions vanishing at P, respectively P and Q. These conditions in turn imply that

$$H^1(M, H^{\otimes k}) = 0$$

and that $H^{\otimes k}$ is base-point free and embeds M in a projective space.

From this we first see that $H(\ell D)^{\otimes k}$ is base-point free: using the image of $H^0(M, H^{\otimes k})$ in $H^0(M, H(\ell D)^{\otimes k})$ it is clear that given any $P \in M - D$ there is a section of $H(\ell D)^{\otimes k}$ not vanishing at P, i.e., $H(\ell D)^{\otimes k}$ is base point free in M — D. From the exact cohomology sequence of (2.7) and the vanishing of $H^1(M, H^{\otimes k})$ it follows that

$$H^0(M, H(\ell D)^{\otimes k}) \to H^0(M, \mathcal{O}/\mathcal{O}(-k\ell D))$$

is surjective. In particular, given any $P \in D$ there is a section of $H(\ell D)^{\otimes k}$ not vanishing at P, and therefore $H(\ell D)^{\otimes k}$ is base-point free.

To prove the bijectivity mod D of the corresponding map to projective space, observe that if P, Q $\in$ M — D, there is an exact sequence

$$0 \to H^{\otimes k} \otimes \mathcal{I}_{P,Q} \to H(\ell D)^{\otimes k} \to \mathcal{O}/\mathcal{O}(-k\ell D) \oplus H(\ell D)^{\otimes k}/(H(\ell D)^{\otimes k} \otimes \mathcal{I}_{P,Q}) \to 0. \tag{2.8}$$

From the corresponding cohomology sequence and the vanishing of $H^1(M, H^{\otimes k} \otimes \mathcal{I}_{P,Q})$ we obtain the surjectivity of the evaluation map

$$H^0(M, H(\ell D)^{\otimes k}) \to H^0(M, \mathcal{O}/\mathcal{O}(-k\ell D)) \oplus H(\ell D)_P^{\otimes k} \oplus H(\ell D)_Q^{\otimes k},$$

where the subscripts P, Q denote the fibers at P, Q of the indicated line bundle. This in turn implies the bijectivity of the map from M/D to the image of M. Repeating this argument with $\mathcal{I}_P^2$ instead of $\mathcal{I}_{P,Q}$ we see that the map in an embedding on M — D (i.e., its differential is injective), and the proof is complete.

Remark. — If instead of assuming that D is irreducible we allow it to have several components (necessarily disjoint), then the above argument embeds the space obtained by identifying each component separately to a point in a projective space, and a suitable subspace of the space of all sections embeds M/D in a projective space.

From this proposition we obtain the main theorem of this section:

Theorem **(2.9)**. — *Let* M *and* D *be as in Proposition* (2.5). *Let* V = M/D *be the space obtained from* M *by identifying* D *to a point. Then* V *is a projective variety.*

Proof. — Let $H = K_M$ and $\ell = n$. Then by Proposition (2.5) the assumption of Proposition (2.6) is satisfied, and V is a projective variety.

Corollary **(2.10)**. — *Let* M *and* D *be as in Proposition* (2.5), *and let* $n \geqslant 4$. *Then* $\pi_1(M - D)$ *is the fundamental group of a smooth projective variety.*

Proof. — Fix a projective embedding of V = M/D and let Z be a generic hyperplane section of V, disjoint from the unique singular point p of V. Then Z is a smooth projective variety and the standard Morse-theoretic proof of the Lefschetz theorem on hyperplane sections gives that $V - \{p\}$ is homotopy equivalent to a space obtained from Z by attaching cells of dimension at least n. This is very easy to check in the present situation of an open variety $V - \{p\}$, where V is a projective variety with a unique singular point p, cf. the Proposition in § 2 of [20]. It also follows from the Lefschetz theorem for open, non-singular varieties proved by Goresky and MacPherson in [6] (stated on p. 24).

Thus we get an isomorphism on fundamental groups as soon as $n \geqslant 3$. Since we are assuming n even, we need $n \geqslant 4$. Since $V - \{p\}$ is biholomorphic to M — D, the corollary follows.

Remark. — It is important to realize that in the present situation the Lefschetz theorem gives an isomorphism $\pi_1(Z) \approx \pi_1(V - \{p\})$ rather than $\pi_1(Z) \approx \pi_1(V)$. If V were non-singular then both isomorphisms are of course equivalent and well-known. But if V has a unique singular point p, then the first is always true but the second is false in general. In fact, for V as in Theorem (2.9), it is easy to see that $\pi_1(V)$ is a finite group. Namely by Van Kampen's theorem, $\pi_1(V)$ is isomorphic to the quotient of Γ by the normal closure of Γ_0. Since Γ is a lattice in a higher-rank simple Lie group, by a well-known theorem of Margulis (Theorem 2.3.2 of [14]), any infinite normal subgroup of Γ has finite index, hence $\pi_1(V)$ is finite. If one examines what the Morse-theoretic proof of the Lefschetz theorem would say when applied to V, it is easy to see where the difference lies. The " last " attaching map (meaning the one corresponding to the maximum value of the distance function used in § 2 of [20]), which would be that of a cell in the non-singular case, becomes that of the cone on a non-simply connected space in our situation, thus $\pi_1(Z)$ and $\pi_1(V)$ need not be isomorphic. But if the same argument is applied to $V - \{p\}$, this last attaching map does not come up, and one obtains the desired isomorphism of fundamental groups.

Finally we need to give examples of manifolds M, D to which Theorem (2.9) applies. We review briefly the standard examples.

Let n be even, and let q be the quadratic form $\sqrt{2}(x_1^2 + x_2^2) - x_3^2 - \ldots x_{n+2}^2$. Let $O(q)$, $SO^+(q)$, $PSO^+(q)$ denote the corresponding orthogonal, connected component of special orthogonal and projective special orthogonal group of q. These are isomorphic respectively to $O(2, n)$, $SO^+(2, n)$, $PSO^+(2, n)$. Let Δ denote the subgroup of $O(q)$ consisting of matrices with entries in the ring of integers in the field $\mathbf{Q}(\sqrt{2})$. It is well known that Δ is a co-compact subgroup of $O(q)$ and that Δ contains a normal, torsion-free subgroup Δ' of finite index, cf. § 4.3 of [2]. Let $\Gamma = \Delta' \cap SO^+(q)$. Then $\Gamma\backslash X_n$ is a compact manifold with universal cover X_n and fundamental group Γ.

Let $\sigma : \mathbf{R}^{n+2} \to \mathbf{R}^{n+2}$ be the linear transformation defined by

$$(x_1, \ldots, x_{n+1}, x_{n+2}) \to (x_1, \ldots, x_{n+1}, -x_{n+2}).$$

Then $\sigma \in \Delta$. Since Δ' is a normal subgroup of Δ, σ normalizes Γ, hence induces a holomorphic involution, still denoted by σ, of the quotient manifold $M = \Gamma\backslash X_n$. It is clear that $\sigma \neq \mathrm{id}_M$.

Now let $v = (0, \ldots, 0, 1)$, which is a negative vector for q, let $X_{n-1} \subset X_n$ be the corresponding embedding, and let Γ_0 be the subgroup of Γ' that fixes v, i.e., $\Gamma_0 = \Gamma' \cap SO^+(q)_v$, where $\Gamma' = \pi^{-1}(\pi(\Gamma))$ is as the paragraph preceeding Lemma (2.4). Then (using again § 4.3 of [2]) Γ_0 is co-compact in $SO^+(q)_v$, and the image of the natural map $\Gamma_0\backslash X_{n-1} \to M$ is an immersed submanifold of complex dimension $n - 1$ contained in the fixed point of set of σ. Since the fixed point set of an involution is always a smooth submanifold (possibly disconnected), it is clear that each component of the fixed point set of σ is a smooth submanifold of complex dimension at most $n - 1$. It follows easily that the image of this map is a smooth submanifold of M, denoted by D, which must in fact be a component of the fixed point set of σ. Finally, from Lemma (2.3) we see that the image of Γ_0 in $\pi(\Gamma)$ is the subgroup of $\pi(\Gamma)$ that leaves X_{n-1} invariant. From this it is clear that the natural map $\Gamma_0\backslash X_{n-1} \to D$ is bijective, therefore all the hypotheses of Theorem (2.9) are satisfied.

3. Study of the fundamental group

Let $n \geqslant 4$ be an even integer, let Γ, Γ_0, $M = \Gamma\backslash X_n$, $D = \Gamma_0\backslash X_{n-1}$ be as in the previous section, and let Φ denote the fundamental group of $M - D$. In the last section we saw that Φ is the fundamental group of a smooth projective variety. The purpose of this section is to show that Φ is not residually finite.

Let K denote the kernel of the homomorphism $\Phi \to \Gamma \approx \pi_1(M)$ induced by the inclusion $M - D \subset M$, and let $\pi : X_n \to M = \Gamma\backslash X_n$ denote the projection.

Lemma **(3.1)**. — *The sequence* $1 \to K \to \Phi \to \Gamma \to 1$ *is exact, and* K *is isomorphic to* $\pi_1(X_n - \pi^{-1}(D))$.

Proof. — Since D has real codimension two in M, the natural homomorphism $\pi_1(M - D) \to \pi_1(M)$ is surjective, thus the above sequence is exact. Also from codimension two it follows that $X_n - \pi^{-1}(D)$ is connected, hence (by the restriction of π) is a covering space of $M - D$, and therefore $\pi_1(X_n - \pi^{-1}(D))$ maps injectively onto a subgroup of Φ (which will be denoted by the same symbol). Since X_n is simply connected, it is clear that $\pi_1(X_n - \pi^{-1}(D)) \subset K$.

To show the equality of these two groups, let $X = X_n - \pi^{-1}(D)$ and let Y denote the covering space of $M - D$ corresponding to K. Since $\pi_1(X) \subset K$, there is a covering projection $X \to Y$. From general covering space considerations, the natural map $Y \to X_n$ is injective. Since $X \to X_n$ is also injective, it follows that the covering $X \to Y$ is one-sheeted, hence the two groups in question are equal, as asserted in the lemma.

Now let N denote a tubular neighborhood of D in M, let ∂N denote its boundary, and let $\Phi_0 = \pi_1(\partial N)$.

Lemma **(3.2).** — *The homorphism $\Phi_0 \to \Phi$ corresponding to the inclusion $\partial N \to M - D$ is injective.*

Proof. — First observe that Φ_0 is a central extension of Γ_0:

$$1 \to Z \to \Phi_0 \to \Gamma_0 \to 1$$

where Z is an infinite cyclic group that can be described as follows. Choose a component of $\pi^{-1}(D)$, let $\hat{N}$ denote a tubular neighborhood of this component which projects to N, choose $x \in \partial\hat{N}$, and let γ denote the loop in $\partial\hat{N}$ based at x that runs once along the fibre of $\partial\hat{N}$ through x in a chosen orientation. Then Z is the infinite cyclic subgroup of $\pi_1(\partial N, \pi(x))$ generated by the loop $\pi\gamma$. Compute all fundamental groups using the basepoints x or $\pi(x)$ as dictated by the context. Then the inclusions $\partial N \subset M - D \subset M$ and the projection $\partial N \to D$ induce a map of the exact sequence

$$1 \to Z \to \Phi_0 \to \Gamma_0 \to 1$$

to the exact sequence

$$1 \to K \to \Phi \to \Gamma \to 1.$$

Now it is clear (from Mayer-Vietoris and excision) that $H_1(X_n - \pi^{-1}(D))$, the abelianization of K, is a free abelian group on infinitely many generators, one generator for each component of $\pi^{-1}(D)$, and that the image of γ in $H_1(X_n - \pi^{-1}(D))$ is one of these generators, hence the homomorphism $Z \to K$ is injective. Since by construction the homomorphism $\Gamma_0 \to \Gamma$ is injective, it follows that $\Phi_0 \to \Phi$ is injective, as asserted in the lemma.

Remark. — The group K itself is free on infinitely many generators, one for each component of $\pi^{-1}(D)$. This follows from the appropriate Morse theory applied to $f: X_n \to \mathbf{R}$, where f is the square of the distance from a generic point in $X_n - \pi^{-1}(D)$. It is clear that f has a unique critical point on X_n, and the restriction of f to each component of $\pi^{-1}(D)$ has a unique critical point. An immediate application of Theorem 10.8 (p. 122) of [6] gives that $X_n - \pi^{-1}(D)$ has the homotopy type of a bouquet of circles, one for each component of $\pi^{-1}(D)$. This last statement also follows from § 4 of [15], where the Torelli group in genus 2 is studied in the same way (in fact it corresponds to the inclusions $SO(2, 2) \subset SO(2, 3)$).

For simplicity of notation, let $m = n - 1$, which by assumption is odd and at least three. The group $SO^+(2, m)$ is centerless and acts effectively on X_m. Also $SO^+(2, m)$ is homotopy equivalent to its maximal compact subgroup $SO(2) \times SO(m)$, hence its fundamental group is isomorphic to $\mathbf{Z} \times \mathbf{Z}/2$, where the splitting into factors corresponds to the splitting into factors of the maximal compact subgroup. We will denote by $\widehat{SO}(2, m)$ the covering group of $SO^+(2, m)$ corresponding to the infinite cyclic factor in this decomposition of the fundamental group. Observe that $\widehat{SO}(2, m)$ is not a linear group.

Lemma **(3.3).** — *The group Φ_0 is a co-compact lattice in $\widehat{SO}(2, m)$.*

Proof. — We have seen in § 2 that the normal bundle of D in M is S^*, but it is more convenient to state this in terms of real bundles, namely the normal bundle of X_m in X_n is isomorphic, as a homogeneous $SO^+(2, m)$-bundle, to the tautological 2-plane sub-bundle E of $X_m \times \mathbf{R}^{m+2}$. For any fixed positive constant c, the set of vectors in E of length c is isomorphic, as a homogeneous $SO^+(2, m)$-bundle, to the bundle $SO^+(2, m)/SO(m) \to SO^+(2, m)/SO(2) \times SO(m) = X_m$. Dividing by the discrete group Γ_0 we see that $\partial N = \Gamma_0 \backslash SO^+(2, m)/SO(m)$, and the projection $\partial N \to D$ corresponds to the natural projection of this double coset space to the space $\Gamma_0 \backslash SO^+(2, m)/SO(2) \times SO(m)$.

Let Ψ denote the pre-image of Γ_0 in $\widehat{SO}(2, m)$, which is a discrete subgroup of $\widehat{SO}(2, m)$. The natural map $\Psi \backslash \widehat{SO}(2, m) \to \Gamma_0 \backslash SO^+(2, m)$ is bijective, thus Ψ is co-compact in $\widehat{SO}(2, m)$. From the definition of $\widehat{SO}(2, m)$ it is clear that there is a unique homomorphism $SO(m) \to \widehat{SO}(2, m)$ covering the inclusion $SO(m) \subset SO^+(2, m)$, and we regard $SO(m)$ as a subgroup of $\widehat{SO}(2, m)$ via this homomorphism. Therefore we obtain a natural surjective map of double coset spaces

$$\Psi \backslash \widehat{SO}(2, m)/SO(m) \to \Gamma_0 \backslash SO^+(2, m)/SO(m).$$

Using the fact that the kernel of the projection $\widehat{SO}(2, m) \to SO^+(2, m)$ is central, one checks that this map is also injective.

Finally, from the definition of $\widehat{SO}(2, m)$ it also follows that $\widehat{SO}(2, m)/SO(m)$ is simply connected, hence it is the universal cover of $\partial N \approx \Gamma_0 \backslash SO^+(2, m)/SO(m)$, and $\Psi \approx \pi_1(\partial N) \approx \Phi_0$. Hence Φ_0 is a co-compact lattice in $\widehat{SO}(2, m)$ as asserted in the lemma.

Remark. — It follows from the proof of Lemma (3.3) that the kernel Z of the central extension $Z \to \Phi_0 \to \Gamma_0$ is isomorphic to the kernel, also denoted by Z, of the central extension $Z \to \widehat{SO}(2, m) \to SO^+(2, m)$, and that the first extension is isomorphic to the restriction to Γ_0 of the second extension.

Lemma **(3.4).** — *Let* Z *denote the kernel of the natural homomorphism* $\Phi_0 \to \Gamma_0$. *Let* Δ *be a subgroup of finite index in* Φ_0. *Then* Δ *contains the subgroup* 8Z.

Proof. — Let Spin$(2, m)$ denote the Spin group of the quadratic form used to define SO$(2, m)$. It is the double-cover of $SO^+(2, m)$ corresponding to the homomorphism $\mathbf{Z} \times \mathbf{Z}/2 \to \mathbf{Z}/2$ which is non-zero on the generators of each factor. This can be checked, say, as follows. The covering Spin$(2, m) \to SO^+(2, m)$ is the restriction to $SO^+(2, m)$ of the corresponding covering Spin$(2 + m, \mathbf{C}) \to SO(2 + m, \mathbf{C})$ of the complexifications of these groups. The homomorphism of fundamental groups can be computed by passing to the corresponding homotopy-equivalent maximal compact subgroups, and the homomorphism of fundamental groups corresponding to the inclusion $SO(2) \times SO(m) \to SO(2 + m)$ is the asserted homomorphism $\mathbf{Z} \times \mathbf{Z}/2 \to \mathbf{Z}/2$.

Let Spin$(2, m)^\sim$ denote the universal cover of Spin$(2, m)$, which is also the universal cover of $SO^+(2, m)$ and $\widehat{SO}(2, m)$. Since the center of $SO^+(2, m)$ is trivial, the center of Spin$(2, m)^\sim$ is naturally isomorphic to $\pi_1(SO^+(2, m))$, which we continue to identify with $\mathbf{Z} \times \mathbf{Z}/2$ as before. Then the kernel Z_1 of the projection Spin$(2, m)^\sim \to$ Spin$(2, m)$ is the subgroup $\{(x, \varepsilon) \in \mathbf{Z} \times \mathbf{Z}/2 : \bar{x} + \varepsilon = 0\}$, where $\bar{x}$ denotes the mod 2 reduction of x. Also $\widehat{SO}(2, m) =$ Spin$(2, m)^\sim/(\mathbf{Z}/2)$, and the homomorphism $Z_1 \to Z = \ker(\widehat{SO}(2, m) \to SO^+(2, m)) \approx (\mathbf{Z} \times \mathbf{Z}/2)/(\mathbf{Z}/2)$ corresponds to the composition $Z_1 \subset \mathbf{Z} \times \mathbf{Z}/2 \to Z$. Therefore Z_1 maps isomorphically to Z under the projection Spin$(2, m)^\sim \to \widehat{SO}(2, m)$.

Finally, let Δ be as in the lemma. Then the pre-image of Δ in Spin$(2, m)^\sim$ is a co-compact lattice, and the Main Theorem in [18] asserts that it must contain the subgroup $8Z_1$. Since Z_1 projects isomorphically to Z, it follows that Δ contains 8Z, and the proof is complete.

Theorem **(3.5).** — *The group* Φ *is not residually finite, and it is the fundamental group of a smooth projective variety.*

Proof. — Since by Lemma (3.3) Φ contains Φ_0 as a subgroup, and by Lemma (3.4) Φ_0 is not residually finite, we see that Φ is not residually finite. We proved in the previous section that Φ is the fundamental group of a smooth projective variety, hence the proof of the theorem is complete.

Remark. — We have used the assumption that *n* is even to simplify various technical considerations of components, center and fundamental group of the various Lie groups involved. The arguments of § 2 remain valid, with more care, as long as $n \geqslant 3$, regardless of parity. The Main Theorem of [18] is also true for lattices that project to an arithmetic group in $SO(2, m)$ defined by a quadratic form, cf. Theorem 3.4 of [18]. Thus the arguments of this section can be made to work as long as $n \geqslant 4$, regardless of parity.

4. An application to ball quotients

We consider another application of the technique of proving projectivity of blown-down spaces and taking hyperplane sections. The motivation comes from the following problem. Suppose that Γ is a torsion-free co-compact lattice in $SU(1, 3)$, and that X is a smooth projective surface with fundamental group Γ. The problem is to classify all (minimal models of) such surfaces.

Let $M = \Gamma\backslash B^3$, where B^3 denotes the unit ball in $\mathbf{C}^3$, which, with its Bergmann metric, is the symmetric space for $SU(1, 3)$. Then M is a smooth projective variety which is a classifying space for the discrete group Γ. Let X be as above. It is not hard to see that there is a (essentially unique) holomorphic mapping $f: X \to M$ inducing an isomorphism on fundamental groups and whose image is a divisor on M. This follows from the theory of harmonic mappings (existence and Siu's rigidity), and can be proved, say, along the lines of Theorems (7.2*b*) and (8.1) of [4]. Now the obvious way to construct such X and the associated map $X \to M$ is to take smooth hyperplane sections of projective embeddings of M, and M. Gromov raised the question of whether this it the only way. We proceed to construct some examples that show that the class of such surfaces is at least somewhat larger.

We start with Γ and M constructed in analogy with the standard examples of § 2. Let h be the Hermitian form $\sqrt{2} \mid z_0 \mid^2 - \mid z_1 \mid^2 - \mid z_2 \mid^2 - \mid z_3 \mid^2$ on $\mathbf{C}^4$, and let $U(h)$, $SU(h)$ denote the corresponding unitary and special unitary groups, which act naturally on B^3 when the latter is identifyied with the set of lines in $\mathbf{C}^4$ on which h is positive. Let R denote the subring of $\mathbf{C}$ obtained by adjoining $\sqrt{-1}$ to the ring of integers in $\mathbf{Q}(\sqrt{2})$, and let Δ denote the subgroup of $U(h)$ consisting of matrices with entries in R. It is known that Δ is discrete and co-compact in $U(h)$ and that it contains a normal torsion-free subgroup Δ' of finite index, cf. § 4.3 of [2]. Let $\Gamma = \Delta' \cap SU(h)$. Then Γ operates freely on B^3 and $M = \Gamma\backslash B^3$ is a compact manifold.

Let σ_1 and σ_2 denote the involutions of $\mathbf{C}^4$ defined respectively by

$$(z_1, z_2, z_3, z_4) \to (z_1, z_2, -z_3, z_4)$$

and $(z_1, z_2, z_3, z_4) \to (z_1, z_2, z_3, -z_4)$. Then $\sigma_1, \sigma_2 \in \Delta$ and therefore induce involutions of M which will be denoted by the same symbol. Let $v_1 = (0, 0, 1, 0)$ and $v_2 = (0, 0, 0, 1)$, and let Γ_i, $i = 1, 2$ denote the subgroup of Γ that leaves invariant the line determined by v_i. Then Γ_i is the subgroup of Γ that leaves invariant the lines in $\mathbf{C}^4$ contained in the h-orthogonal complement of v_i. Each such set of lines of isomorphic to the two-dimensional ball B^2, thus we obtain corresponding immersions of $\Gamma_i \backslash B^2$ in M.

Arguing as at the end of § 2 with the involutions σ_i, we see that these immersions are embeddings. Let D_i, $i = 1, 2$ denote the images of these embeddings. Then D_i is a smooth totally geodesic divisor on M which is isomorphic to the complex surface $\Gamma_i \backslash B^2$.

The elements of Γ_i need only fix v_i up to a multiple. Each such multiple must be an element a of the ring R defined above satisfying $|a|^2 = 1$. One checks that a must be a fourth root of unity, thus an eigenvalue of an element of Γ which is a root of unity. Thus, passing to a neat subgroup of finite index in Γ (cf. § 17.1, 17.4 and 17.7 of [3]), we may (and do) assume that Γ_i actually fixes v_i.

Let S denote the restriction to B^3 of the tautological line sub-bundle of $\mathbf{P}^3 \times \mathbf{C}^4$. Then S is a homogeneous SU(1, 3) line bundle, hence it descends to a line bundle, still denoted by S, one any quotient of B^3 (or of an invariant subset of B^3) by a subgroup of Γ. The formula

$$s_i(w) = h(w, v_i)$$

defines a section s_i of S^* over $\Gamma_i \backslash B^3$ which vanishes precisely on $\Gamma_i \backslash B^2$ and to first order. Since $\Gamma_i \backslash B^2$ has a neighborhood in $\Gamma_i \backslash B^3$ that projects bijectively to a neighborhood U_i of D_i in M, the following lemma is clear:

Lemma **(4.1).** — *Let* M *and* D_i *be as above. Then* D_i *has a neighborhood* U_i *such that* $\mathcal{O}(D_i) \mid U_i \approx S^* \mid U_i$.

The proportionality statements for SU(1, 3) analogous to Lemma (2.2) are easily proved and are as follows:

Lemma **(4.2).** — *The line bundle* S *over* M *is positive. If* K *denotes the canonical bundle of* M, *then* $K \approx S^{\otimes 4}$.

Now let $Y = D_1 \cap D_2 \subset M$. Then $Y \neq \emptyset$ since it contains the projection to M of the lines contained in the orthogonal complement of $\{v_1, v_2\}$. It may be disconnected, but this will not affect the subsequent arguments. Since D_1 and D_2, being distinct, connected, totally geodesic submanifolds of M, intersect transversally, we see that each component of Y is a totally geodesic smooth complex curve on M. From the last two Lemmas we see that the following is true:

Lemma **(4.3).** — *The line bundle* $K(2D_1 + 2D_2)$ *is trivial in a neighborhood of* Y *in* M.

To prove that M/Y is a projective variety we will also need the following Lemma:

Lemma **(4.4).** — *For* $i = 1, 2$, $K(2D_i)$ *is an ample line bundle on* M.

Proof. — We use Kleiman's criterion (Theorem 2, § IV.1 of [11]), as formulated in Chapter I, § 8 of [9]: A line bundle L on M is ample if and only if there is a positive constant ε such that for all curves C in M, $L.C \geqslant \varepsilon \| C \|$. Here L.C, respectively $\| C \|$, denotes the evaluation on the fundamental class of C of the first Chern class of L, respectively some fixed norm on $H_2(M, \mathbf{R})$.

Since K is an ample line bundle on M, there exists such a constant, say 2ε, for K. We give a lower bound for

$$(\mathbf{4.5}) \qquad (K(2D_i)).C = K.C + 2D_i.C$$

by considering two cases. First, if C is not contained in D_i, then the second term in (4.5) is $\geqslant 0$, hence the sum is at least $2\varepsilon \| C \|$. If $C \subset D_i$, then the second term in (4.5) is $2(\mathcal{O}(D_i) \mid D_i).C$, which, by Lemmas (4.1) and (4.2) is the same as $2S^*.C = -(K.C)/2$, so (4.5) becomes K.C/2 which by assumption is at least $\varepsilon \| C \|$. Therefore the constant ε works for all C and the proof is complete.

Proposition (**4.6**). — *Let* M *and* $Y = D_1 \cap D_2$ *be as above. Then there is a positive integer k so that the line bundle* $K(2D_1 + 2D_2)$ *is base-point free and so that the corresponding map from* M *to a projective space is an embedding on* M — Y *and is a bijection of the identification space* M/Y *to the image of* M. *Thus* M/Y *has the structure of a projective variety* V *with a unique singular point p so that* M — Y *is biholomorphic to* $V - \{p\}$.

Proof. — We sketch the modification that has to be made to the proof of Proposition (2.6) to yield the result. Let $\mathscr{I}_i$ denote the ideal sheaf of D_i, and, for $k > 0$, let $(\mathscr{I}_1^{2k}, \mathscr{I}_2^{2k})$ denote the sub-sheaf of $\mathcal{O}$ generated by $\mathscr{I}_1^{2k}$ and $\mathscr{I}_2^{2k}$. Then we replace the sequence (2.7) by the exact sequence (Koszul resolution of the sheaf $\mathcal{O}/(\mathscr{I}_1^{2k}, \mathscr{I}_2^{2k})$)

$$(*) \qquad 0 \to K^{\otimes k} \to K(2D_1)^{\otimes k} \oplus K(2D_2)^{\otimes k} \to K(2D_1 + 2D_2)^{\otimes k} \to \mathcal{O}/(\mathscr{I}_1^{2k}, \mathscr{I}_2^{2k}) \to 0,$$

where the maps are $t \to (-t \otimes s_1^{2k}, t \otimes s_2^{2k})$ and $(t_1, t_2) \to t_1 \otimes s_2^{2k} + t_2 \otimes s_1^{2k}$, where s_i denotes the canonical section of $\mathcal{O}(D_i)$, and the identification of the last term uses Lemma (4.3). By the ampleness of K and of $K(2D_i)$ (Lemma (4.4)), for large enough k one has the vanishing of $H^j(M, L^{\otimes k} \otimes \mathscr{I}_{P,Q})$ for $j = 1, 2$, all $P, Q \in M$ and L any of the line bundles K, $K(2D_i)$. Choose k also large enough for the vanishing of the corresponding cohomology groups with $\mathscr{I}_P^2$. Then the resulting vanishing theorems for hypercohomology of $(*)$ and of the exact sequence analogous to (2.8) (obtained from $(*)$ by tensoring the first two terms with $\mathscr{I}_{P,Q}$ and modifying the last term accordingly) can be used to prove Proposition (4.6) along the same lines as the proof of Proposition (2.6). If Y is disconnected we proceed as in the remark following the proof of Proposition (2.6) to complete the proof.

Theorem (**4.7**). — *Let* Γ, M *and* V *be as above. Choose any projective embedding of* V, *and let* X *denote the intersection of* V *with a hyperplane transversal to* V *(in particular not containing* *the singular point p). Then* X *is a smooth projective surface with fundamental group isomorphic to* Γ *but is not homotopy equivalent to any hyperplane section of* M.

Proof. — From Morse theory as in the proof of Corollary (2.10) it is clear that X is a smooth projective surface with the same fundamental group as $V - \{p\} \approx M - Y$. Since Y has real codimension 4 in M, $\pi_1(M - Y) \approx \pi_1(M)$, hence $\pi_1(X) \approx \Gamma$, thus proving the first assertion.

To prove the second assertion, observe that the composition of

$$X \subset V - \{p\} \approx M - Y$$

gives a holomorphic embedding $X \subset M$ not as a hyperplane section, since it is disjoint from the complex curve Y. If X were homotopy equivalent to a hyperplane section Z of M, then the composition of this homotopy equivalence with the inclusion $Z \subset M$ would give a second map $X \to M$ inducing an isomorphism on fundamental groups. Since $M = K(\Gamma, 1)$, this two maps $X \to M$ would differ (up to homotopy) by a self-homotopy equivalence φ of M. By Mostow's rigidity theorem [17], φ is homotopic to an isometry ψ of M. But then $\pm\,\psi(Y)$ is a complex curve in M and the intersection number $Z.\psi(Y) = 0$, which contradicts the assumption that Z is a hyperplane section.

Remarks. — 1. The divisor $X \subset M$ constructed in Theorem (4.7) is not ample, but lies in the boundary of the ample cone in M. Thus the most stringent possible characterization of the surfaces X as in the first paragraph would be that they are divisors in the closure of the ample cone in M, and that the only complex curves with which they have zero intersection number are totally geodesic.

2. It is clear that many similar constructions can be carried out on suitable locally symmetric varieties for $SU(1, n)$ and $SO(2, n)$ for $n \geqslant 3$.

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