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FINITENESS THEOREMS FOR DISCRETE SUBGROUPS OF BOUNDED COVOLUME IN SEMI-SIMPLE GROUPS

by Armand BOREL and Gopal PRASAD*

Introduction

1. It is well-known that a real non-compact simple Lie group not locally isomorphic to $SL_2(\mathbf{R})$ or $SL_2(\mathbf{C})$ has only finitely many conjugacy classes of discrete subgroups of covolumes bounded by a given constant [44]. Motivated by the results of [17], J. Tits asked whether the same would be true for p-adic groups, not only for a single ambient group, but also when the ground field and the group are allowed to vary, with a specific universal normalization of Haar measures. This problem was our starting point. We were naturally led to consider also an analogue for groups over $\mathbf{R}$ or $\mathbf{C}$ and then, as a common generalization, irreducible discrete subgroups of products of simple groups over local fields.

2. In this introduction we shall outline some of the main results obtained so far, referring to §§7, 8 for the most precise assumptions and general statements. We let k be a global field; V, V_∞, V_f respectively be the set of places, of archimedean places, and of nonarchimedean places of k and k_v the completion of k at $v \in V$. Let G be an absolutely almost simple simply connected k-group and G' be a k-group centrally k-isogenous to G. If $v \in V_f$, we let μ'_v be the Haar measure on $G'(k_v)$ which assigns the volume one to the stabilizer of a chamber in the Bruhat-Tits building of $G(k_v)$. This is (essentially) the normalization proposed by J. Tits, so μ'_v will be called here the Tits measure. If v is archimedean, and k_v is identified with $\mathbf{R}$ or $\mathbf{C}$, then μ'_v is the Haar measure which gives the volume one to a maximal compact subgroup of $R_{k_v/\mathbf{R}}(G')(\mathbf{C})$. (Originally, we had considered the measure associated to the Killing form. This μ'_v was suggested to us by P. Deligne.) For a finite set of places $S \subset V$, we let μ'_S be the Haar measure on $G'_S = \prod_{v \in S} G'(k_v)$ which is the product of the μ'_v's. When $G' = G$, we set $\mu'_S = \mu_S$. Then we have (7.8):

Theorem A. — Let $c > 0$ be given. Assume k runs through the number fields. Then there are only finitely many choices of k, of G'/k of absolute rank $\geqslant 2$ up to k-isomorphism, of a finite set S of places of k containing all the archimedean places, of arithmetic $\Gamma' \subset G'_S$ up to conjugacy, such that $\mu'_S(G'_S/\Gamma') \leqslant c$.

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[The proof will also yield the finiteness of the number of natural equivalence classes of (k, G', S, Γ') in the function field case, under some mild restrictions. We note also that, in view of the arithmeticity results of [23] and [43], irreducible discrete subgroups of finite covolume of simple groups over local fields are of the type considered here under rather general assumptions. This leads to an apparently different formulation of this finiteness theorem. See Remark 7.9.]

3. The starting point of the proof is a formula of [31] for $\mu_S(G_S/\Lambda_0)$, where Λ_0 is a " principal " S-arithmetic subgroup contained in $G(k)$. (The volume formula in [31] involves the Tamagawa number $\tau_k(G)$ of G, which has recently been proved to be equal to one if k is a number field.) To deal with a subgroup of G'_S commensurable with the image of Λ_0, we need an estimate for the index of the latter in its normalizer. This is done by consideration of the first Galois cohomology set with coefficients in the center C of G (or flat cohomology if C is not reduced) via a slight generalization of an exact sequence due to Rohlfs [32] (see §§2, 5). The proof uses number theoretical estimates, in particular some involving discriminants given in §6. These arguments yield first the finiteness of the set of triples (k, G, S) in 7.3. The finiteness of the number of conjugacy classes of Γ' in a given G'_S, which follows from [3] in characteristic zero, is proved in 7.7 with respect to conjugacy under (Ad G) (k). For the proof of the finiteness theorems in §7, we have to know that given a finite subset $\mathscr{R}$ of V, the set of inner forms of G which are k_v-isomorphic to G for all $v \notin \mathscr{R}$ is finite. This is well-known in characteristic zero [5]. A proof in the function field case is supplied in Appendix B.

4. Another possible natural normalization of Haar measures in the nonarchimedean case is the absolute value of the Euler-Poincaré measure introduced by J.-P. Serre in [33]. If $v \in V_\infty$, we may use on $G'(k_v)$ a similar measure, provided $G'(k_v)$ has a compact Cartan subgroup. If this condition is fulfilled for every $v \in V_\infty \cap S$, then the corresponding product measure on G'_S is also a Haar measure. It may be smaller then μ'_S, but by a controllable factor (see 4.4) and the estimates are good enough to ensure that Theorem A also holds for this choice of the Haar measure in these cases except maybe if G is of type A_2. With that *proviso*, it yields therefore the finiteness of the number of (k, G', S, Γ') such that $0 \neq |\chi(\Gamma')| \leqslant c$ where χ is the Euler-Poincaré characteristic in the sense of C. T. C. Wall (see 7.3, 7.8).

5. Earlier results pertaining to p-adic groups in characteristic zero, announced in [4], were proved in a completely different way, by comparing the index of an Iwahori subgroup in a maximal parahoric subgroup with an estimate for the order of finite subgroups of G(K), where K is a nonarchimedean local field of characteristic zero and G is a semi-simple group defined over K. This method does not depend on any information on Tamagawa numbers and allows us to vary G, and also K among local fields having a bounded absolute ramification index. This is the subject matter of §8.

6. The main result of [17] gives an explicit list of triples (F, G, Γ) where F is a nonarchimedean local field, G an absolutely almost simple F-group of F-rank $\geqslant 2$ and Γ a discrete subgroup of G(F) which acts transitively on the chambers of the Bruhat-Tits building of G(F). In particular this set is finite. It is clear from the definition of the Tits measure μ_T of G(F) that, in that case, $\mu_T(G(F)/\Gamma) \leqslant 1$. Therefore this finiteness assertion follows from Theorem A. More generally, we show the finiteness of the number of triples (F, G, Γ) consisting of a nonarchimedean local field F of characteristic zero, an absolutely almost simple F-group G of absolute rank $\geqslant 2$, and a discrete subgroup Γ of G(F) which is transitive on the set of the facets of a given type of the Bruhat-Tits building of G(F). In fact, we shall establish more general results in the S-arithmetic framework (see 7.10, 7.11).

7. Let G be as in 2. Let S be a finite subset of V containing V_∞. A collection $P = (P_v)_{v \in V_f - S}$, where P_v is a parahoric subgroup of $G(k_v)$, is said to be *coherent* if the product of the P_v's by $G_S = \prod_{v \in S} G(k_v)$ is an open subgroup of the adèle group G(A).

It is known that if either k is a number field or G is anisotropic over k, and $(P_v)_{v \in V_f}$ is a coherent collection of parahoric subgroups, then the "class number"

$$\mathfrak{c}(P) = \# ((G_\infty \times \prod_{v \in V_f} P_v)\backslash G(A)/G(k))$$

is finite (and, by strong approximation, equal to one if G_∞ is non-compact), where $G_\infty = \prod_{v \in V_\infty} G_v$. Arguments similar, in fact in part common, to those of 7.3 and 7.7 yield (see 7.2, 7.6):

Theorem B. — Let $c \in \mathbf{N}$ be given. Then there are, up to natural equivalence, only finitely many number fields k, absolutely almost simple simply connected k-groups G *with* G_∞ *compact, and coherent collections* P *of parahoric subgroups such that* $\mathfrak{c}(P) \leqslant c$.

We thank Moshe Jarden and A. M. Odlyzko for conversations and correspondence on discriminant and class numbers of global fields. We are indebted to J. Tits for having kindly provided more conceptual proofs of two properties of volumes of parahoric subgroups stated in 3.1 and proved in Appendix A, and for his careful reading of the manuscript and his helpful suggestions.

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0. Notation, conventions and preliminaries

In this section, we recall or fix some notation and conventions, often to be used without reference. In addition we prove some facts about global fields (mostly function fields), for which we could not give references.

0.0. As usual $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ will denote respectively the fields of rational, real and complex numbers; $\mathbf{Z}$ the ring of rational integers.

The number of elements of a finite set S will be denoted by $\# S$.

0.1. Throughout this paper k is a global field i.e. a number field or the function field of a curve over a finite field, and A is the k-algebra of adèles of k endowed with the usual locally compact topology. Let V be the set of places of k, and V_∞ (resp. V_f) the subset of archimedean (resp. nonarchimedean) places. For a set S of places of k, let $S_f = S \cap V_f$, and $S_\infty = S \cap V_\infty$.

For $v \in V$, k_v denotes the completion of k at v and $| \ |_v$ the normalized absolute value on k_v. For $v \in V_f$, let $\hat{k}_v$ be the maximal unramified extension of k_v; let $\mathfrak{o}_v$ and $\hat{\mathfrak{o}}_v$ be the ring of integers of k_v and $\hat{k}_v$ respectively; let q_v be the cardinality of the residue field of k_v and $v(x)$ the normalized additive valuation of $x \in k_v^\times$. Recall that, for $x \in k_v^\times$,

$$| x |_v = [\mathfrak{o}_v : x\mathfrak{o}_v]^{-1} = q_v^{-v(x)} \quad \text{if } x \in \mathfrak{o}_v,$$
$$| x |_v = [x\mathfrak{o}_v : \mathfrak{o}_v] = q_v^{-v(x)} \quad \text{if } x \notin \mathfrak{o}_v.$$

0.2. Except in §8, G will be an absolutely almost simple, simply connected algebraic group defined over k, $\overline{G}$ its adjoint group (i.e. the group of its inner automorphisms), $\varphi : G \to \overline{G}$ the natural central isogeny and G′ a k-group centrally k-isogeneous to G. We fix a central k-isogeny $\iota : G \to G'$ and let $\varphi' : G' \to \overline{G}$ be the unique central isogeny such that $\varphi = \varphi' . \iota$; it is defined over k.

Let C be the center of G and C′ that of G′. Let r be the absolute rank of G and for $v \in V_f$, let r_v be its rank over $\hat{k}_v$.

0.3. For a subset $\mathscr{X}$ of V, let $G_{\mathscr{X}}$ (resp. $G'_{\mathscr{X}}$) denote the direct product of the $G(k_v)$ (resp. $G'(k_v)$), $v \in \mathscr{X}$, if $\mathscr{X}$ is finite, and their restricted direct product if $\mathscr{X}$ is infinite. The group $G(k)$ (resp. $G'(k)$) will always be viewed as a subgroup of $G_{\mathscr{X}}$ (resp. $G'_{\mathscr{X}}$) in terms of its diagonal embedding.

For $v \in V$ and $\mathscr{X} \subset V$, the homomorphisms $G(k) \to G'(k)$, $G(k_v) \to G'(k_v)$, $G_{\mathscr{X}} \to G'_{\mathscr{X}}$, induced by ι will also be denoted by ι.

0.4. Let S be a finite set of places of k containing all the archimedean ones. We assume that for every nonarchimedean $v \in S$, G is isotropic over k_v, or, equivalently, $G(k_v)$ is noncompact. Let $\mathscr{S} = \mathscr{S}(G)$ be the subset of S consisting of the places v such that G is isotropic over k_v. We assume further that $\mathscr{S}$ is nonempty.

0.5. We shall assume familiarity with the Bruhat-Tits theory of reductive groups over nonarchimedean local fields. All we need is stated in [41], and the proofs of most of the results can be found in [8].

For $v \in V_f$, we shall let X_v denote the Bruhat-Tits building of $G(k_v)$. We recall that $G(k_v)$ acts on X_v by *special* simplicial automorphisms; in particular any simplex stable under an element $g \in G(k_v)$ is pointwise fixed by g.

0.6. Let $\mathscr{K}$ be a compact open subgroup of G_{V-S}. Let $\Lambda = G(k) \cap \mathscr{K}$. Any subgroup of G_S (resp. G'_S) commensurable with Λ (resp. $\iota(\Lambda)$) is called an S-arithmetic subgroup.

Let $G_S \to G_{\mathscr{S}}$ and $G'_S \to G'_{\mathscr{S}}$ be the natural projections. Then any subgroup of $G_{\mathscr{S}}$ (resp. $G'_{\mathscr{S}}$) commensurable with the projection of an S-arithmetic subgroup of G_S (resp. G'_S) will be called an arithmetic subgroup. Arithmetic subgroups are discrete and of finite covolume.

0.7. If K is a number field, $a(K)$ will denote the number of its archimedean places, D_K the absolute value of its discriminant over $\mathbf{Q}$, and h_K its class number.

0.8. If K is a global function field, let $a(K) = 1$, g_K be its genus, q_K be the cardinality of its field of constants, and h_K be its "class number" i.e. the order of the quotient of the group of its divisors of degree zero by the subgroup of principal divisors. Let $D_K = q_K^{2g_K-2}$ in this case. The following bounds for the class number h_K are known.

$$(1) \qquad (q_K^{1/2} - 1)^{2g_K} \leqslant h_K \leqslant (q_K^{1/2} + 1)^{2g_K}.$$

For the sake of expository completeness we sketch a proof pointed out to us by Manohar Madan: The zeta-function $\zeta_K(s)$ of K can be written in the form

$$\zeta_K(s) = \frac{P(q_K^{-s})}{(1 - q_K^{-s})(1 - q_K^{1-s})},$$

where P is a polynomial of degree $2g_K$ with integral coefficients, $P(0) = 1$ and $P(1) = h_K$ ([45: Chapter VII, §6, Theorem 4]). According to the "Riemann hypothesis" for curves over finite fields proved by A. Weil (see [1] for an elementary proof), the roots of P have absolute value $q_K^{-1/2}$. This at once implies the above bounds.

It is a well known result of Hermite and Minkowski that (up to isomorphism) there are only finitely many number fields with a given discriminant (see [20: Chapter V, Theorem 5]). For global function fields the following finiteness assertion holds. Its proof was supplied to us by Moshe Jarden and Dinesh Thakur.

0.9. *Proposition. — For given g and q, there are only finitely many global function fields of genus g and field of constants of cardinality q.*

For its proof we need the following lemma.

0.10. *Lemma. — Let K be a global function field of genus g and field of constants K_0. Suppose that K/K_0 has a prime divisor P of degree 1. Then $K = K_0(x,y)$, where (x,y) satisfy an equation $f(x,y) = 0$ with coefficients in K_0, of degree at most $4g$.*

Proof. — To each divisor D of K we associate the K_0-vector space

$$\mathscr{L}(D) = \{ x \in K \mid (x) + D \geqslant 0 \}, \quad \text{and set} \quad \dim(D) = \dim(\mathscr{L}(D)).$$

If $g = 0$, then $K = K_0(x)$ with x transcendental over K_0 ([11: §18, Theorem]). So, assume that $g > 0$. By the Riemann-Roch theorem, $\dim(nP) = n + 1 - g$ if $n > 2g - 2$. Hence, $\mathscr{L}((2g-1)P) \subset \mathscr{L}(2gP) \subset \mathscr{L}((2g+1)P)$. Choose

$$x \in \mathscr{L}(2gP) - \mathscr{L}((2g-1)P) \quad \text{and} \quad y \in \mathscr{L}((2g+1)P) - \mathscr{L}(2gP).$$

Then $v_P(x) = -2g$, $v_P(y) = -(2g+1)$ and $(x)_\infty = 2gP$ i.e., $\deg(x)_\infty = 2g$, where v_P is the additive valuation associated with P.

If i and j are integers between 0 and $4g$, then

$$v_p(x^i y^j) = -2gi - (2g+1)j \geqslant -16g^2 - 4g,$$

and therefore, $x^i y^j \in \mathscr{L}((16g^2 + 4g)P)$. As

$$\dim \mathscr{L}((16g^2 + 4g)P) = 16g^2 + 3g + 1 < (4g+1)^2,$$

there exist $a_{ij} \in K_0$, $0 \leqslant i, j \leqslant 4g$, not all zero, such that $\sum_{i,j} a_{ij} x^i y^j = 0$. We prove that $K = K_0(x,y)$.

Note that $[K : K_0(x)] = \deg(x)_\infty$ by the theorem on [11: p. 25]. Therefore, by the above, $[K : K_0(x)] = 2g$. Hence, in order to prove that $K = K_0(x,y)$, it suffices to show that $[K_0(x,y) : K_0(x)] \geqslant 2g$. If we had $[K_0(x,y) : K_0(x)] < 2g$, there would exist $b_{ij} \in K_0$ with $0 \leqslant j \leqslant 2g - 1$, not all 0, such that $\Sigma\, b_{ij} x^i y^j = 0$. Hence there would exist distinct pairs (i,j) and (r,s) with $0 \leqslant j, s \leqslant 2g - 1$ such that $v_P(x^i y^j) = v_P(x^r y^s)$. Thus $2gi + (2g+1)j = 2gr + (2g+1)s$. As $2g$ and $2g+1$ are relatively prime, this would imply that $2g$ divides $s - j$. It would then follow that $s = j$ and $r = i$. This contradiction concludes the proof.

Proof of Proposition 0.9. If $g = 0$, then K is either a rational function field over K_0 or $K = K_0(x,y)$ where (x,y) satisfy a quadratic equation with coefficients in K_0 ([12]). If $g = 1$, then, by a theorem of F. K. Schmidt, K has a prime divisor of degree 1 [9]. So in view of the above lemma we may (and we shall) assume that $g \geqslant 2$.

Denote the unique extension of degree $(2g-2)!$ of K_0 by K'_0. As K has a prime divisor of degree $\leqslant 2g-2$ ([11: p. 52]), $K' = K'_0 K$ has a prime divisor of degree 1. By the preceding lemma, $K' = K'_0(x,y)$, where (x,y) satisfy an equation of degree $\leqslant 4g$ with coefficients in K'_0. There are therefore only finitely many possibilities for K'. For each of these possibilities K is an intermediate field between $K_0(x)$ and K'. Let p be the characteristic of K. Since $[K_0(x) : K_0(x)^p] = p$, the field K' is generated over K_0 by one element [12: Lemma 24.31]. Hence there are only finitely many possibilities for K[21].

0.11. *Lemma.* — *A global function field* L *contains only finitely many subfields* K *such that* L/K *is a Galois extension.*

Any subfield K of L such that L/K is a Galois extension is the fixed field of a suitable subgroup of the automorphism group of L. Now the lemma follows from the well-known result that the automorphism group of any global function field is finite.

Let K be a global field and n be a positive integer. Let K_n be the subgroup of $K^\times$ consisting of all $x \in K^\times$ such that for every normalized nonarchimedean valuation v of $K^\times$, $v(x) \in n\mathbf{Z}$. Clearly, $K_n \supset K^{\times n}$.

The proof of the following proposition was suggested by Moshe Jarden and Dipendra Prasad.

0.12. *Proposition.* — $\#(K_n/K^{\times n}) \leqslant h_K\, n^{a(K)}$.

Proof. — If K is a number field (resp. global function field), let $\mathscr{P}$ be the group of all fractional principal ideals (resp. principal divisors) of K and $\mathscr{I}$ be the group of all fractional ideals (resp. divisors of degree zero). We shall use multiplicative notation for the group operation in both $\mathscr{I}$ and $\mathscr{P}$. The kernel of the natural map $x \mapsto (x)$ of $K^\times$ onto $\mathscr{P}$ is precisely the group U of units. This gives us our first short exact sequence

(1) $$1 \to U \to K^\times \to \mathscr{P} \to 1.$$

Let $\mathscr{C} = \mathscr{I}/\mathscr{P}$; then the class number h_K equals $\#\mathscr{C}$ and we have a second short exact sequence

(2) $$1 \to \mathscr{P} \to \mathscr{I} \to \mathscr{C} \to 1.$$

Now note first that since $U \cap K^{\times n} = U^n$, (1) gives rise to the following short exact sequence,

(3) $$1 \to U/U^n \to K^\times/K^{\times n} \to \mathscr{P}/\mathscr{P}^n \to 1.$$

As $U \subset K_n$, (3) yields another short exact sequence:

(4) $$1 \to U/U^n \to K_n/K^{\times n} \to (\mathscr{P} \cap \mathscr{I}^n)/\mathscr{P}^n \to 1.$$

On the other hand, let $\mathscr{C}_n$ be the subgroup of all elements of $\mathscr{C}$ whose order is a divisor of n. If for $x \in K^\times$, there exists $I \in \mathscr{I}$ such that $(x) = I^n$, then I is unique. Therefore the map $(x) \mapsto I\mathscr{P}$ induces an isomorphism

(5) $$(\mathscr{P} \cap \mathscr{I}^n)/\mathscr{P}^n \cong \mathscr{C}_n.$$

Combining (4) and (5) we get

$$[K_n : K^{\times n}] = [U : U^n] \, \sharp\mathscr{C}_n.$$

Obviously, $\sharp\mathscr{C}_n \leqslant h_K$. So it suffices to prove that $[U : U^n] \leqslant n^{a(K)}$.

If K is a number field, then by Dirichlet's unit theorem, $U \cong \mu(K) \times \mathbf{Z}^{a(K)-1}$ where $\mu(K)$ is the cyclic group of roots of unity in K ([45:IV, Theorem 9]). Thus U is the direct product of $a(K)$ cyclic groups. From this we conclude that $[U : U^n] \leqslant n^{a(K)}$.

If K is a global function field, then U is the group of non-zero elements of the field of constants (*loc. cit.*). As the latter field is finite, U is cyclic. Therefore, $[U : U^n] \leqslant n$.

1. Remarks on arithmetic subgroups

In this section, for the sake of completeness, we prove in our framework some properties of arithmetic subgroups which are well-known in characteristic zero.

1.1. Let $v \in V_f$. We observe first that the fixed point set F of a compact open subgroup H of $G(k_v)$ on the Bruhat-Tits building X_v of $G(k_v)$ is compact. In fact, H acts continously on the compactification $\overline{X}_v$ of X_v constructed in [6]. If F were not compact, then H would have a fixed point in $\overline{X}_v - X_v$. But there, by construction, the isotropy subgroups are of the form $P(k_v)$, where P is a proper parabolic k_v-subgroup of G, and those subgroups do not contain any open subgroup of $G(k_v)$.

1.2. *Proposition. — Let* Γ' *be an arithmetic subgroup of* $G'_{\mathscr{S}}$. *Then* $\varphi'(\Gamma')$ *is contained in* $\overline{G}(k)$ *and is Zariski-dense. The subgroups* $\Gamma' \cap G'(k)$ *and* $\Gamma' \cap \iota(G(k))$ *are normal subgroups of* Γ'.

The subgroup $\Gamma' \cap \iota(G(k))$ is of finite index in Γ', hence contains a subgroup Γ'_0 which is normal, of finite index, in Γ'. Since $G'(k)$ is contained in the commensurability group of Γ'_0, the latter is Zariski-dense in G', and hence $\varphi'(\Gamma'_0)$ is a Zariski-dense subgroup of $\overline{G}$. For $\gamma' \in \Gamma'$, the element $\varphi'(\gamma')$ normalizes $\varphi'(\Gamma'_0)$, so it is a k-automorphism of G. This implies that $\varphi'(\Gamma') \subset \overline{G}(k)$ and that Γ' normalizes $G'(k)$ and $\iota(G(k))$, hence also $\Gamma' \cap G'(k)$ and $\Gamma' \cap \iota(G(k))$.

1.3. For $v \in V_f$, $\mathrm{Aut}(G(k_v))$, and so in particular $\overline{G}(k_v)$, acts on the building X_v by simplicial automorphisms. In view of 1.2, this allows one to define an action of any arithmetic subgroup of $G'_{\mathscr{S}}$ on X_v $(v \in V_f)$. This will be used in the sequel without further reference.

A compact open subgroup $\mathscr{K}$ of G_{V-S} contains, as a subgroup of finite index, a direct product $\prod_v \mathscr{K}_v$, where, for $v \notin S$, $\mathscr{K}_v$ is a compact open subgroup of $G(k_v)$ which is hyperspecial for all the v's outside some finite subset T of V containing S; see [41: 3.9]. If $\mathscr{K}$ is such a group, then $\Lambda_{\mathscr{K}} = G(k) \cap \mathscr{K}$ is an S-arithmetic subgroup of G_S, and in its natural embedding in G_{V-S}, its closure is $\mathscr{K}$ by strong approximation ([30], [22]).

1.4. *Proposition.* — *Let* Γ' *be an arithmetic subgroup of* $G'_{\mathscr{S}}$ *and* Λ *be the inverse image in* $G(k)$ *of* $\Gamma' \cap \iota(G(k))$ *under* ι.

(i) *The fixed point set of* Γ' *in* X_v $(v \notin S)$ *is compact, not empty.*

(ii) *For any field extension* K *of* k, *the normalizer of* $\varphi'(\Gamma')$ *in* $\overline{G}(K)^{\mathscr{S}}$ *is contained in* $\overline{G}(k)$ *(*$\overline{G}(k)$ *embedded in* $\overline{G}(K)^{\mathscr{S}}$ *diagonally),* $\varphi'(\Gamma')$ *is of finite index in its normalizer, and the normalizer* $N(\Gamma')$ *of* Γ' *in* $G'_{\mathscr{S}}$ *is arithmetic.*

(iii) Γ' *is contained in only finitely many arithmetic subgroups.*

(iv) *If* Γ' *is maximal, then for* $v \notin S$, *the closure* P_v *of* Λ *in* $G(k_v)$ *is a parahoric subgroup of* $G(k_v)$, $\Lambda = G(k) \cap \Pi_v P_v$, *and* Γ' *is the normalizer of* $\iota(\Lambda)$ *in* $G'_{\mathscr{S}}$.

Proof. — By strong approximation, the projection of Λ in G_{V-S} is dense in a compact open subgroup. Therefore, its fixed point set $\mathscr{F}_v$ in X_v is compact (1.1), non-empty (by the fixed-point theorem of Bruhat-Tits [8: I, 3.2.4]), and reduced to the unique fixed point of a hyperspecial parahoric subgroup P_v for $v \in V - T$, where T is a suitable finite subset of V containing S (1.3). Since $\iota(\Lambda)$ is of finite index in Γ', the group of automorphisms of X_v (for $v \notin S$) determined by Γ' is relatively compact, therefore its fixed point set F_v is not empty; F_v is obviously contained in $\mathscr{F}_v$ and so in particular it is compact and (i) is proved.

By 1.2, $\varphi'(\Gamma')$ is contained and Zariski-dense in $\overline{G}(k)$. Therefore its normalizer in $\overline{G}(K)^{\mathscr{S}}$ is contained in $\overline{G}(k)$ and so it coincides with the normalizer $N(\varphi'(\Gamma'))$ of $\varphi'(\Gamma')$ in $\overline{G}(k)$. Obviously, F_v is stable under the natural action of $N(\varphi'(\Gamma'))$ on X_v. Hence, for all $v \notin S$, $N(\varphi'(\Gamma'))$ is a relatively compact subgroup of $\overline{G}(k_v)$. From this we conclude that $N(\varphi'(\Gamma'))$ is a discrete subgroup of $\overline{G}_{\mathscr{S}} := \Pi_{v \in \mathscr{S}} \overline{G}(k_v)$, and as it contains $\varphi'(\Gamma')$, which is a discrete subgroup of $\overline{G}_{\mathscr{S}}$ of finite covolume, the index of $\varphi'(\Gamma')$ in it is finite*. This implies in particular that the normalizer $N(\Gamma')$ of Γ' in $G'_{\mathscr{S}}$ is arithmetic, which proves (ii).

For $v \in T - S$, let $\mathscr{P}_v$ be the (finite) set of parahoric subgroups of $G(k_v)$ which fix some facet contained in $\mathscr{F}_v$. For $P = \Pi_{v \notin S} P_v$, where $P_v \in \mathscr{P}_v$ if $v \in T - S$, and P_v is the hyperspecial parahoric subgroup as above if $v \in V - T$, let $\Lambda_P = G(k) \cap P$, $\Lambda'_P = \iota(\Lambda_P)$ and $N(\Lambda'_P)$ be the normalizer of Λ'_P in $G'_{\mathscr{S}}$. As (by (i)) any arithmetic subgroup containing Γ' has a fixed point in $\mathscr{F}_v$, $v \notin S$, it is contained in the normalizer of Λ'_P for a suitable P. Since according to (ii), $N(\Lambda'_P)$ itself is an arithmetic subgroup, it follows that $\Gamma' = N(\Lambda'_P)$ for some P if Γ' is maximal. This proves (iv). Also, since there are only finitely many P's and, for each P, $[N(\Lambda'_P) : \Gamma']$ is finite, we conclude that the arithmetic subgroups of $G'_{\mathscr{S}}$ containing Γ' are finite in number, which proves (iii).

1.5. The group Λ defined in 1.4 (iv) will be called the *principal* S-*arithmetic subgroup* determined by the coherent collection $P = (P_v)_{v \in V_f - S}$ of parahoric subgroups. We shall also say that Λ and $\Gamma' = N(\iota(\Lambda))$ are *associated* to P.

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* For a different proof, see §1.5 in Lattices in semi-simple groups over local fields by G. Prasad, *Advances in Math. Studies in Algebra and Number Theory*, Academic Press (1979).

2. The action of the first Galois cohomology group of the center of G on Δ_v

2.1. For v nonarchimedean, let T_v be a maximal $\hat{k}_v$-split torus of G which is defined over k_v and contains a maximal k_v-split torus of G; according to the Bruhat-Tits theory such a torus exists. Let $\hat{I}_v$ be an Iwahori subgroup of $G(\hat{k}_v)$ defined over k_v (i.e., stable under the Galois group of $\hat{k}_v/k_v$) such that the chamber in the Bruhat-Tits building of $G/\hat{k}_v$ fixed by $\hat{I}_v$ lies in the apartment corresponding to T_v, and let $I_v = \hat{I}_v \cap G(k_v)$. Let $\hat{\Delta}_v$ be the basis of the affine root system of $G/\hat{k}_v$ relative to T_v, determined by the Iwahori subgroup $\hat{I}_v$, and Δ_v be the basis of the affine root system of G/k_v relative to the maximal k_v-split torus contained in T_v, determined by the Iwahori subgroup I_v of $G(k_v)$.

2.2. Any subset $\Theta_v \subset \Delta_v$ determines a parahoric subgroup P_{Θ_v}, of $G(k_v)$, containing I_v (which is assigned to the empty set); moreover any parahoric subgroup of $G(k_v)$ is conjugate to a unique subgroup of the form P_{Θ_v}. A parahoric subgroup of $G(k_v)$ which is conjugate to P_{Θ_v} is said to be of *type* Θ_v.

$\mathrm{Aut}(G(k_v))$, and so in particular $\overline{G}(k_v)$, acts on the set of parahoric subgroups of $G(k_v)$, and there is a homomorphism

$$\xi_v : \overline{G}(k_v) \to \mathrm{Aut}(\Delta_v)$$

such that for $g \in \overline{G}(k_v)$, the conjugate of P_{Θ_v} $(\Theta_v \subset \Delta_v)$ under g is a parahoric subgroup of type $\xi_v(g)$ (Θ_v).

There is a similar homomorphism

$$\hat{\xi}_v : \overline{G}(\hat{k}_v) \to \mathrm{Aut}(\hat{\Delta}_v).$$

Furthermore, ξ_v (resp. $\hat{\xi}_v$) is trivial on $\varphi(G(k_v))$ (resp. $\varphi(G(\hat{k}_v))$). Let Ξ_v (resp. $\hat{\Xi}_v$) be its image.

2.3. *Lemma.* — *Let* $g \in \overline{G}(k_v)$.

(i) *If* $\hat{\xi}_v(g)$ *is trivial, then so is* $\xi_v(g)$. *In particular,* Ξ_v *is a subquotient of* $\hat{\Xi}_v$.

(ii) *Assume* G *to be quasi-split over* k_v. *If* $\xi_v(g) = 1$, *then* $\hat{\xi}_v(g) = 1$.

Proof. — (i) The first assertion follows immediately from the fact that two parahoric subgroups of $G(\hat{k}_v)$ which are defined over k_v are conjugate in $G(\hat{k}_v)$ if, and only if, their intersections with $G(k_v)$ are conjugate in $G(k_v)$ [8: II, Proposition 5.2.10 (ii)].

(ii) Assume G to be quasi-split over k_v. If it does not split over $\hat{k}_v$, then it is *residually split* over k_v; $\hat{\Delta}_v$ then has a natural identification with Δ_v and the second assertion of the lemma is obvious. We assume therefore that G splits over $\hat{k}_v$. Then $G(k_v)$ has a hyperspecial parahoric subgroup ([41:1.10.2]) to which corresponds a hyperspecial vertex of $\hat{\Delta}_v$. If $\xi_v(g) = 1$, then this vertex is fixed under $\hat{\xi}_v(g)$. But, by [16: 1.8], the group $\hat{\Xi}_v$ operates freely on the set of hyperspecial vertices of $\hat{\Delta}_v$. Therefore, $\hat{\xi}_v(g) = 1$, whence (ii).

Remark. — The conclusion of (ii) may fail if G is not quasi-split over k_v; it fails, for example, if G is anisotropic over k_v or if it is an inner form of type D_r whose k_v-rank is $r-2$.

2.4. Let K be a field and H be an affine algebraic group-scheme over K. If K is of characteristic zero, then $H^1(K, H)$ denotes as usual the first Galois cohomology set with coefficients in H. If K is of positive characteristic, then we let it stand for the set denoted $\check{H}^1(\mathrm{Spec}(K)_{fl}, H)$ in [25: III, §§3, 4], or $H^1_f(K, H)$, $H^1(K, H)$, in [34]. If H is commutative, similar groups are defined in all positive degrees. The usual exact sequence in Galois cohomology associated to a short exact sequence of group schemes is also available [25: III, Prop. 4.5] as well as the long exact cohomology sequence associated to a short exact sequence of commutative group schemes [34]. Moreover, if H is smooth, then these two cohomology sets are canonically isomorphic [25: III, Theorem 3.9]. (It is assumed there that the group-scheme is commutative, and the assertion is proved for cohomology groups in any degree $i \geqslant 0$, but this assumption is not used for $i = 1$, as is tacitly understood later in 4.8.) From this it follows that we need not distinguish between the two cases in our discussion below of cohomology with coefficients in C.

2.5. Let C be the center of G. It is k-isomorphic to the center of the unique simply connected, quasi-split *inner* k-form $\mathscr{G}$ of G.

The natural central k-isogeny $\varphi : G \to \overline{G}$ gives rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & C(k) & \longrightarrow & G(k) & \xrightarrow{\varphi} & \overline{G}(k) & \xrightarrow{\delta} & H^1(k, C) & \longrightarrow & H^1(k, G) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & C(k_v) & \longrightarrow & G(k_v) & \xrightarrow{\varphi} & \overline{G}(k_v) & \xrightarrow{\delta_v} & H^1(k_v, C) & \longrightarrow & H^1(k_v, G), \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & C(\hat{k}_v) & \longrightarrow & G(\hat{k}_v) & \xrightarrow{\varphi} & \overline{G}(\hat{k}_v) & \xrightarrow{\hat{\delta}_v} & H^1(\hat{k}_v, C) & \longrightarrow & H^1(\hat{k}_v, G).
\end{array}$$

Since for any nonarchimedean v, $H^1(k_v, G)$ and $H^1(\hat{k}_v, G)$ vanish, ([19], [8: III]; [38]) δ_v and $\hat{\delta}_v$ are both surjective, therefore we have a commutative diagram

$$(1)\qquad \begin{array}{ccc}
\overline{G}(k_v)/\varphi(G(k_v)) & \xrightarrow[\cong]{\delta_v} & H^1(k_v; C) \\
\downarrow & & \downarrow \\
\overline{G}(\hat{k}_v)/\varphi(G(\hat{k}_v)) & \xrightarrow[\cong]{\hat{\delta}_v} & H^1(\hat{k}_v; C)
\end{array}$$

As ξ_v and $\hat{\xi}_v$ are trivial on $\varphi(G(k_v))$ and $\varphi(G(\hat{k}_v))$ respectively, they induce homomorphisms

$$(2)\qquad H^1(k_v, C) \to \mathrm{Aut}(\Delta_v), \qquad H^1(\hat{k}_v, C) \to \mathrm{Aut}(\hat{\Delta}_v),$$

also to be denoted ξ_v and $\hat{\xi}_v$ respectively.

2.6. Let $\varepsilon = \varepsilon(G) = 2$ if G is of type D_r with r even, and let it be 1 otherwise. Let $n = n(G) = r + 1$ if G is of type A_r; $n = 2$ if G is of type B_r, C_r (r arbitrary), or D_r (with r even), or E_7; $n = 3$ if G is of type E_6; $n = 4$ if G is of type D_r with r odd; $n = 1$ if G is of type E_8, F_4 or G_2.

Let μ_n^ε be the kernel of the endomorphism $m_n : x \mapsto x^n$ of $(GL_1)^\varepsilon$. If G splits over some field K, then C is isomorphic to μ_n^ε over K. For any field K, $H^1(K, \mu_n^\varepsilon)$ is canonically isomorphic to $(K^\times/K^{\times n})^\varepsilon$.

Let now $v \in V_f$ be such that G splits over $\hat{k}_v$. Then C is isomorphic to μ_n^ε over $\hat{k}_v$. Moreover, it is known [16: 1.8] that $\hat{\Xi}_v$ is isomorphic to $(\mathbf{Z}/n\mathbf{Z})^\varepsilon$. The second assertion of 2.3 (i) then shows that the order of Ξ_v is a divisor of n^ε. We identify C with μ_n^ε in terms of a fixed $\hat{k}_v$-isomorphism $\theta : C \to \mu_n^\varepsilon$. This then provides an identification of $H^1(\hat{k}_v, C)$ with $(\hat{k}_v^\times/\hat{k}_v^{\times n})^\varepsilon$, with respect to which we have:

2.7. *Proposition. — Let v be a nonarchimedean place of k such that* G *splits over $\hat{k}_v$. Then the kernel of ξ_v is the subgroup $(\hat{\mathfrak{o}}_v^\times \hat{k}_v^{\times n}/\hat{k}_v^{\times n})^\varepsilon$ of $(\hat{k}_v^\times/\hat{k}_v^{\times n})^\varepsilon$.*

Proof. — Let us write $\mathscr{C}$ for $(GL_1)^\varepsilon$. Let $H = (\mathscr{C} \times G)/C_\theta$, where

$$C_\theta = \{ (\theta(x), x^{-1}) \mid x \in C \},$$

and Z be a maximal $\hat{k}_v$-split torus of H; it is a maximal torus of H and $T := G \cap Z$ is a maximal torus of G. Since in a split torus, every subtorus is split and a direct factor, there exists a $\hat{k}_v$-subtorus D of Z, of dimension ε, such that $Z = D \times T$, hence such that $H = D \ltimes G$ is a semi-direct product of D and the normal subgroup G. Let p be the projection of H onto D. Then we have a sequence of isomorphisms:

$$(1) \qquad H^1(\hat{k}_v, C) \xrightarrow{\hat{\delta}_v^{-1}} \overline{G}(\hat{k}_v)/\varphi(G(\hat{k}_v)) \xrightarrow{\cong} H(\hat{k}_v)/\mathscr{C}(\hat{k}_v)\, G(\hat{k}_v) \xrightarrow{\cong} D(\hat{k}_v)/D(\hat{k}_v)^n.$$

We extend φ to a homomorphism of H onto $\overline{G}$, also denoted φ. Its kernel is precisely $\mathscr{C}$. Since the latter is $\hat{k}_v$-split, the homomorphism $H(\hat{k}_v) \to \overline{G}(\hat{k}_v)$ is surjective (Hilbert's Theorem 90). The inverse image in $H(\hat{k}_v)$ of $\varphi(G(\hat{k}_v))$ is $\mathscr{C}(\hat{k}_v)\, G(\hat{k}_v)$, whence the second isomorphism in (1). The kernel of $p : H(\hat{k}_v) \to D(\hat{k}_v)$ is $G(\hat{k}_v)$. By restriction to $\mathscr{C}$, p defines a $\hat{k}_v$-morphism

$$\mathscr{C} = (GL_1)^\varepsilon \to D$$

whose kernel is C, and with a suitable identification of D with $(GL_1)^\varepsilon$, this homomorphism is the homomorphism m_n defined above. Therefore, the image of $\mathscr{C}(\hat{k}_v)$ under p is $D(\hat{k}_v)^n$. This yields the third isomorphism in (1). The aforementioned identification of D with $(GL_1)^\varepsilon$ gives an identification of $D(\hat{k}_v)/D(\hat{k}_v)^n$ with $(\hat{k}_v^\times/\hat{k}_v^{\times n})^\varepsilon$, and with this identification, the isomorphism $H^1(\hat{k}_v, C) \to (\hat{k}_v^\times/\hat{k}_v^{\times n})^\varepsilon$, induced by θ, is the composite of the three isomorphisms in (1).

The composite $\xi_v . \varphi$ defines a homomorphism $H(\hat{k}_v) \to \mathrm{Aut}(\hat{\Delta}_v)$, which is trivial on $\mathscr{C}(\hat{k}_v)\, G(\hat{k}_v)$ and also on the maximal bounded subgroup Z_c of $Z(\hat{k}_v)$ (see [41: 2.5]).

But it is obvious that in the identification of $D(\hat{k}_v)/D(\hat{k}_v)^n$ with $(\hat{k}_v^\times/\hat{k}_v^{\times n})^e$, the image of the maximal bounded subgroup of $D(\hat{k}_v)$ in $D(\hat{k}_v)/D(\hat{k}_v)^n$ is $(\hat{\mathfrak{o}}_v^\times \hat{k}_v^{\times n}/\hat{k}_v^{\times n})^e$. This shows that in the identification of $H^1(\hat{k}_v, C)$ with $(\hat{k}_v^\times/\hat{k}_v^{\times n})^e$, the kernel of $\hat{\xi}_v$ contains $(\hat{\mathfrak{o}}_v^\times \hat{k}_v^{\times n}/\hat{k}_v^{\times n})^e$. As the groups $\hat{\Xi}_v$ and $(\hat{k}_v^\times/\hat{\mathfrak{o}}_v^\times \hat{k}_v^{\times n})^e$ have equal order ($= n^e$, see 2.6), the kernel of $\hat{\xi}_v$ cannot be bigger. This proves the proposition.

2.8. For $v \notin S$, let P_v be a parahoric subgroup of $G(k_v)$ such that $G_S . \prod_{v \notin S} P_v$ is an open subgroup of $G(A)$. Let $\Theta_v(\subset \Delta_v)$ be the type of P_v. Let $\Lambda = G(k) \cap \prod_{v \notin S} P_v$ and $\Lambda' = \iota(\Lambda)$. In the sequel, we shall view Λ and Λ' as arithmetic subgroups of $G_{\mathscr{S}}$ and $G'_{\mathscr{S}}$ respectively. Let Γ' be the normalizer of Λ' in $G'_{\mathscr{S}}$; it is an arithmetic subgroup of $G'_{\mathscr{S}}$ (see 1.4 (ii)). We recall that $\varphi'(\Gamma')$ is contained in $\overline{G}(k)$; see 1.2. Hence the natural homomorphism $\delta : \overline{G}(k) \to H^1(k, C)$, whose kernel is $\varphi(G(k))$, induces a homomorphism

$$\partial : \Gamma'/\Lambda' \to H^1(k, C).$$

Let Ξ_v be as in 2.2, and let Ξ be the direct sum of the Ξ_v, $v \notin S$. Then Ξ acts on $\Delta := \prod_{v \notin S} \Delta_v$. Let $\Theta = \prod_{v \notin S} \Theta_v(\subset \Delta)$; let Ξ_Θ be the stabilizer of Θ in Ξ, and Ξ_{Θ_v} that of Θ_v in Ξ_v.

For $c \in H^1(k, C)$, let c_v denote the cohomology class in $H^1(k_v, C)$ determined by c. The maps ξ_v's induce a map $\xi : H^1(k, C) \to \Xi$ given by $\xi(c) = (\xi_v(c_v))_{v \in V-S}$ ($c \in H^1(k, C)$). Let

$$H^1(k, C)_\Theta = \{ c \in H^1(k, C) \mid \xi(c) \in \Xi_\Theta \},$$

$$H^1(k, C)'_\Theta = \{ c \in H^1(k, C)_\Theta \mid c_v \in \delta_v\, \varphi'(G'(k_v)) \quad \text{for all } v \in \mathscr{S} \}$$

and

$$\delta(\overline{G}(k))'_\Theta = \delta(\overline{G}(k)) \cap H^1(k, C)'_\Theta.$$

Let $\gamma' \in \Gamma'$. Then $\varphi'(\gamma')$ belongs to $\overline{G}(k)$ (1.2), and it stabilizes Λ, hence also P_v for all $v \notin S$. Therefore, $\delta_v\, \varphi'(\gamma') \in \Xi_v$. This shows that ∂ maps Γ'/Λ' into $\delta(\overline{G}(k))'_\Theta$.

In the notation introduced above we have:

2.9. *Proposition. — The following sequence is exact*

$$1 \to (\prod_{v \in \mathscr{S}} C'(k_v))/(C'(k) \cap \Lambda') \to \Gamma'/\Lambda' \xrightarrow{\partial} \delta(\overline{G}(k))'_\Theta \to 1.$$

Apart from minor modifications, the above proposition is due to J. Rohlfs when G is k-split [32]. It was already remarked in [24] that the proof of [32] goes over without change to the more general case if k is a number field. Since our context is slightly more general (for example, we allow k to be of positive characteristic), we repeat the proof.

We begin by showing that ∂ is surjective. Let $c \in \delta(\overline{G}(k))'_\Theta$, and $g \in \overline{G}(k)$ be such that $\delta(g) = c$. Then the parahoric subgroup $g(P_v)$ is of the same type as P_v ($v \notin S$). There exist therefore $h_v \in G(k_v)$ such that $g(P_v) = h_v P_v h_v^{-1}$. Moreover, for v outside a finite set T of places containing S, we have $g(P_v) = P_v$ and so $h_v \in P_v$. By strong approximation ([30], [22]), we can find an $h \in G(k)$ such that for all $v \notin T$, $h \in P_v$, and which is so close to h_v, for $v \in T - S$, that $h_v P_v h_v^{-1} = hP_v h^{-1}$. This last equality is then true for all $v \notin S$. Therefore, $\varphi(h)^{-1} g$ stabilizes P_v for all $v \notin S$. Also,

$$\delta(\varphi(h)^{-1} g) = \delta(g) = c.$$

As $c \in \delta(\overline{G}(k))'_\Theta$, there is, for $v \in \mathscr{S}$, a $\gamma'_v \in G'(k_v)$ such that $\varphi'(\gamma'_v) = \varphi(h)^{-1} g$. Then since $\varphi(h)^{-1} g$ stabilizes P_v for all $v \notin S$, the element $\gamma' = (\gamma'_v)_{v \in \mathscr{S}}$ belongs to Γ'. Therefore ∂ is surjective. If now $\delta\varphi'(\gamma') = 1$, then $\varphi'(\gamma') = \varphi(g)$ for some $g \in G(k)$ and, consequently, $\gamma' \in \iota(g).\Pi_{\mathscr{S}} C'(k_v)$. From this the exactness on the left follows.

2.10. As $\delta(\overline{G}(k))'_\Theta \subset H^1(k, C)'_\Theta$, Proposition 2.9 gives the following exact sequence:

$$1 \to \big(\prod_{v \in \mathscr{S}} C'(k_v)\big)/(C'(k) \cap \Lambda') \to \Gamma'/\Lambda' \xrightarrow{\partial} H^1(k, C)'_\Theta.$$

Now let $\quad H^1(k, C)_\xi = \{c \in H^1(k, C) \mid \xi(c) = 1\},$

and $\quad H^1(k, C)'_\xi = \{c \in H^1(k, C)_\xi \mid c_v \in \delta_v \varphi'(G'(k_v))\}.$

It is obvious that

$$(*) \qquad \# H^1(k, C)'_\Theta \leqslant \# H^1(k, C)'_\xi . \prod_{v \in V - S} \# \Xi_{\Theta_v},$$

and since $\# C'(k_v) \leqslant n^\varepsilon$ for all v, we conclude from the above exact sequence that

$$\begin{aligned}[\Gamma' : \Lambda'] &\leqslant \# \prod_{v \in \mathscr{S}} C'(k_v) . \# H^1(k, C)'_\xi . \prod_{v \in V - S} \# \Xi_{\Theta_v} \\ &\leqslant n^{\varepsilon \# \mathscr{S}} . \# H^1(k, C)_\xi . \prod_{v \in V - S} \# \Xi_{\Theta_v}.\end{aligned}$$

3. Lower bound for the covolumes of arithmetic subgroups

We shall continue to use the notation introduced in §§0 and 2.

3.1. In the sequel, we shall use the fact that for $v \in V_f$ a parahoric subgroup P_v^m of $G(k_v)$ of maximal volume is necessarily *special*. We also need to know that if P_v is a parahoric subgroup of $G(k_v)$, of type Θ_v, such that $P_v^m \cap P_v$ contains an Iwahori subgroup I_v, then

$$(*) \qquad [P_v^m : I_v] \geqslant [P_v : I_v] \ (\# \Xi_{\Theta_v}).$$

This could be rather laboriously checked case by case, using the " reduction mod $\mathfrak{p}$ " of the parahoric subgroup P_v (see 3.5, 3.7 in [41]) to compute the index of an Iwahori subgroup it contains. More conceptual proofs are given in Appendix A.

3.2. Let Γ' be a maximal arithmetic subgroup of $G'_{\mathscr{S}}$, $\Lambda' = \Gamma' \cap \iota(G(k))$ and Λ be its inverse image in $G(k)$ under ι. Then according to Proposition 1.4 (iv), for $v \notin S$, the closure P_v of Λ in $G(k_v)$ is a parahoric subgroup of $G(k_v)$, and $\Lambda = G(k) \cap \prod_{v \notin S} P_v$. Let $\Theta_v (\subset \Delta_v)$ be the type of P_v and $\Theta = \prod_{v \notin S} \Theta_v$.

For all but finitely many v, P_v is a hyperspecial parahoric subgroup of $G(k_v)$ and so is of maximum volume ([41: 3.8.2]). Let T be the smallest set of places containing S such that for $v \notin T$, the parahoric subgroup P_v is of maximum volume. Then for all $v \notin T$, as P_v is special (3.1), $\Xi_{\Theta_v} = \{1\}$.

For every $v \in T - S$, we fix a parahoric subgroup P_v^m of $G(k_v)$ of maximum volume such that $P_v^m \cap P_v$ contains an Iwahori subgroup I_v. Let

$$\Lambda^m = G(k) \cap \Big(\prod_{v \in T - S} P_v^m . \prod_{v \notin T} P_v \Big).$$

Then Λ^m is an arithmetic subgroup.

3.3. Using strong approximation, we see at once that

$$\frac{[\Lambda^m : \Lambda^m \cap \Lambda]}{[\Lambda : \Lambda^m \cap \Lambda]} = \prod_{v \in T - S} \frac{[P_v^m : I_v]}{[P_v : I_v]}.$$

Also, for $v \in T - S$,

$$\frac{[P_v^m : I_v]}{[P_v : I_v]} \geqslant \# \Xi_{\Theta_v} \quad \text{(see 3.1)}.$$

Hence,

$$\frac{[\Lambda^m : \Lambda^m \cap \Lambda]}{[\Lambda : \Lambda^m \cap \Lambda]} \geqslant \prod_{v \in V - S} \# \Xi_{\Theta_v}.$$

3.4. For $v \in V_f$, let μ_v (resp. μ'_v) be the Haar measure on $G(k_v)$ (resp. $G'(k_v)$) with respect to which the volume of any Iwahori subgroup of $G(k_v)$ (resp. the volume of the stabilizer in $G'(k_v)$ of any chamber in X_v) is 1.

It is known that $\iota(G(k_v))$ is a closed normal subgroup of $G'(k_v)$ and that $G'(k_v)/\iota(G(k_v))$ is a compact abelian group ([7: 3.19 (i)]). Let I_v be an Iwahori subgroup of $G(k_v)$ and I'_v be the stabilizer in $G'(k_v)$ of the chamber pointwise fixed by I_v. Then $I'_v . \iota(G(k_v)) = G'(k_v)$ and $\iota(I_v) = I'_v \cap \iota(G(k_v))$. Using these facts it is easy to see that μ'_v is the measure determined by the Haar measure on the closed normal subgroup $\iota(G(k_v))$ with respect to which $\iota(I_v)$ has volume 1, and the normalized Haar measure on the compact group $G'(k_v)/\iota(G(k_v))$.

(We note that I'_v is not always an Iwahori subgroup, as defined in Tits [41: 3.7], but it contains a unique such subgroup, necessarily of finite index.)

3.5. For v archimedean, let μ_v (resp. μ'_v) be the Haar measure on $G(k_v)$ (resp. $G'(k_v)$) such that in the induced measure, any maximal compact subgroup of $R_{k_v/\mathbf{R}}(G)$ $(\mathbf{C})$

(resp. $R_{k_v/\mathbf{R}}(G')$ $(\mathbf{C})$) has volume 1. In particular, if $k_v = \mathbf{R}$ and G is anisotropic over k_v, then $\mu_v(G(k_v)) = 1 = \mu'_v(G'(k_v))$.

3.6. Let $\mu_{\mathscr{S}}$ (resp. $\mu'_{\mathscr{S}}$) denote the product measure $\Pi_{v\in\mathscr{S}}\,\mu_v$ (resp. $\Pi_{v\in\mathscr{S}}\,\mu'_v$) on $G_{\mathscr{S}}$ (resp. $G'_{\mathscr{S}}$) as well as the induced measure on their quotients by discrete subgroups. Then

$$\mu'_{\mathscr{S}}(G'_{\mathscr{S}}/\Gamma') = [\Gamma' : \Lambda']^{-1}.\,\mu'_{\mathscr{S}}(G'_{\mathscr{S}}/\Lambda'),$$

and it follows, using the alternate description of the Haar measure μ'_v given in 3.4, that

$$\mu'_{\mathscr{S}}(G'_{\mathscr{S}}/\Lambda') \geqslant \mu_{\mathscr{S}}(G_{\mathscr{S}}/\Lambda) = \frac{[\Lambda^m : \Lambda^m \cap \Lambda]}{[\Lambda : \Lambda^m \cap \Lambda]}\,\mu_{\mathscr{S}}(G_{\mathscr{S}}/\Lambda^m).$$

Hence (see 3.3)

$$(1)\qquad \mu'_{\mathscr{S}}(G'_{\mathscr{S}}/\Gamma') \geqslant \frac{\mu_{\mathscr{S}}(G_{\mathscr{S}}/\Lambda)}{[\Gamma' : \Lambda']} \geqslant \frac{\Pi_{v\in V-S}\,\#\Xi_{\Theta_v}}{[\Gamma' : \Lambda']}\,\mu_{\mathscr{S}}(G_{\mathscr{S}}/\Lambda^m).$$

Now as

$$(2)\qquad [\Gamma' : \Lambda'] \leqslant n^{\varepsilon\#\mathscr{S}}.\,\#H^1(k, C)_{\xi}.\prod_{v\in V-S}\#\Xi_{\Theta_v}$$

(cf. 2.10), we conclude that

$$(*)\qquad \mu'_{\mathscr{S}}(G'_{\mathscr{S}}/\Gamma') \geqslant n^{-\varepsilon\#\mathscr{S}}(\#H^1(k, C)_{\xi})^{-1}\,\mu_{\mathscr{S}}(G_{\mathscr{S}}/\Lambda^m).$$

3.7. In [31] the volumes of arithmetic quotients of semi-simple groups have been computed. We shall now describe the result. We begin by observing that since for $v \in S - \mathscr{S}$, G is anisotropic over k_v, $G(k_v)$ is compact and $\mu_v(G(k_v)) = 1$, and hence for any S-arithmetic subgroup Λ of $G(k)$, $\mu_S(G_S/\Lambda) = \mu_{\mathscr{S}}(G_{\mathscr{S}}/\Lambda)$; where μ_S is the measure on G_S/Λ induced by the product measure $\Pi_{v\in S}\,\mu_v$ on G_S.

We recall that r is the absolute rank of G, and, for $v \in V_f$, r_v the rank of G over the maximal unramified extension $\widehat{k}_v$ of k_v. Let $\mathscr{G}$ be the unique quasi-split, simply connected inner k-form of G. If G is not a k-form of type ${}^6\mathsf{D}_4$, let ℓ be the smallest Galois extension of k over which $\mathscr{G}$ splits. If G is a k-form of type ${}^6\mathsf{D}_4$, let ℓ be a fixed cubic extension of k contained in the Galois extension, of degree 6, over which $\mathscr{G}$ splits.

Let $\mathfrak{s} = \mathfrak{s}(\mathscr{G}) = 0$ if $\mathscr{G}$ splits over k; if $\mathscr{G}$ does not split over k (i.e. if G is an *outer* form of a split group), then let $\mathfrak{s} = \frac{1}{2}(r-1)(r+2)$ if G is an outer form of type A_r with r odd, $\mathfrak{s} = \frac{1}{2}r(r+3)$ if G is an outer form of type A_r with r even, $\mathfrak{s} = 2r - 1$ if G is an outer form of type D_r (r arbitrary), and $\mathfrak{s} = 26$ if G is an outer form of type E_6; see [31: 0.4]. In particular, we have

$$\mathfrak{s}(\mathscr{G}) \geqslant \begin{cases} 5 & \text{if } \mathscr{G} \text{ does not split over } k \\ 7 & \text{if } \mathscr{G} \text{ is an outer form of type } \mathsf{D}_r \ (r \geqslant 4). \end{cases}$$

Let m_i $(1 \leqslant i \leqslant r)$ be the exponents of the compact simply connected real-analytic Lie group of the same type as G; see [31: 1.5]. Note that $r + 2\sum_1^r m_i = \dim G$.

Let $\tau_k(G)$ be the *Tamagawa number* of G/k (see, for example, [31: 3.3]).

With these notations we have ([31: Theorem 3.7]): Let $P = (P_v)_{v \in V_f - S}$ be a coherent collection of parahoric subgroups and Λ the principal S-arithmetic subgroup determined by P (1.5). Then

$$\mu_{\mathscr{S}}(G_{\mathscr{S}}/\Lambda) = \mu_S(G_S/\Lambda)$$
$$= D_k^{\frac{1}{2}\dim G}(D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}s}\left(\prod_{v \in V_\infty}\left|\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\right|_v\right)\tau_k(G)\ \mathscr{E}(P),$$

where $\mathscr{E}(P) = \prod_{v \in S_f} e(I_v) . \prod_{v \in V - S} e(P_v)$; the $e(I_v)$ and $e(P_v)$ are positive real numbers computable in terms of P, the structure of G/k and the Bruhat-Tits theory. For $v \in S_f$ (resp. $v \in V - S$), $e(I_v)$ (resp. $e(P_v)$) is the inverse of the volume of any Iwahori subgroup of $G(k_v)$ (resp. of P_v) with respect to the Haar measure $\gamma_v\, \omega_v^*$; where γ_v is defined in §1.3 and ω_v^* in §2.1 of [31]. In this paper we need the following information, see [31: 3.10, 2.10, 2.11] (the unexplained notation is as in [31]):

(1) for all $v \in S_f$, $e(I_v) > 1$ and for all $v \in V - S$, $e(P_v) > 1$;

(2) $e(I_v) = (\#\overline{T}_v(\mathfrak{f}_v))^{-1} . q_v^{(r_v + \dim \mathscr{M}_v)/2} \geqslant (q_v + 1)^{-r_v} . q_v^{r_v(r_v+3)/2}$;

(3) $e(I_v) = (q_v - 1)\,(q_v^{d_v} - 1)^{-(r+1)/d_v}\, q_v^{r(r+3)/2} > (q_v - 1)\, q_v^{r(r+1)/2 - 1}$

if $G(k_v) = SL_{(r+1)/d_v}(\mathfrak{D}_v)$, where $\mathfrak{D}_v$ is a central division algebra of degree $d_v < (r+1)$ over k_v, and $v \in S_f$.

(4) $e(P_v) = q_v^{(\dim \overline{M}_v + \dim \mathscr{M}_v)/2} . (\#\overline{M}_v(\mathfrak{f}_v))^{-1}$ $(v \in V_f - S)$.

Moreover:

(5) $e(P_v) \geqslant (q_v + 1)^{-1} . q_v^{r_v + 1}$

if $v \in V_f - S$ and either G is not quasi-split over k_v, or P_v is not special, or G splits over $\hat{k}_v$ but P_v is not hyperspecial. Also,

(6) $e(P_v) \geqslant (q_v - 1)\, q_v^{(r^2 + 2r - (r+1)^2 d_v^{-1} - 1)/2}$

if $G(k_v) = SL_{(r+1)/d_v}(\mathfrak{D}_v)$, where $\mathfrak{D}_v$ is a central division algebra of degree $d_v \leqslant r + 1$ over k_v, and

(7) $e(P_v) \geqslant q_v^{(r+1)/2}$

if G is an outer form of type A_r, r odd, of k_v-rank $(r-1)/2$, which does not split over $\hat{k}_v$.

In the sequel, we write e_v *for* $e(I_v)$ *and* e_v^m *for* $e(P_v)$, *where* P_v *is a parahoric subgroup of* $G(k_v)$ *of maximal volume.*

3.8. As every arithmetic subgroup of $G'_{\mathscr{S}}$ is contained in a maximal one (1.4 (iii)), combining the bound $(*)$ of 3.6 and the formula for the volume of $G_{\mathscr{S}}/\Lambda$ given above, we obtain the following:

$$\mu'_{\mathscr{S}}(G'_{\mathscr{S}}/\Gamma') \geqslant n^{-\varepsilon\#\mathscr{S}}(\#H^1(k, C)_{\xi})^{-1} D_k^{\frac{1}{2}\dim G}(D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}s} \left(\prod_{v \in V_\infty} \left| \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v \right) \tau_k(G)\, \mathscr{E},$$

where, in the notation of 3.7,

$$\mathscr{E} = \prod_{v \in S_f} e_v \cdot \prod_{v \in V_f - S} e_v^{\mathfrak{m}}.$$

This shows that *the volumes* $\mu'_{\mathscr{S}}(G'_{\mathscr{S}}/\Gamma')$ *have a strictly positive lower bound, as* Γ' *runs through the arithmetic subgroups of* $G'_{\mathscr{S}}$. This is then, of course, true with respect to any Haar measure on $G'_{\mathscr{S}}$.

In §5 we shall give an upper bound for the order of $H^1(k, C)_{\xi}$.

3.9. *Proposition. — Let* K' *be a compact open subgroup of the restricted product* G'_{V-S} *of the groups* $G'(k_v)$ $(v \in V - S)$. *Then the number of double cosets* $G'(k)\backslash G'(A)/(G'_S K')$ *is finite.*

This is the finiteness of the class number of G' (at any rate for G'_S non-compact, which is a standing assumption in this paper). It is well-known in the number field case [2], but we do not know of a reference in the function field case (except when G' is anisotropic over k, where S may be taken empty [14: 2.2.7 (iii)]).

We fix a Haar measure ν on $G'(A) = G'_S \times G'_{V-S}$. It is a product of Haar measures ν_S and ν_{V-S} on G'_S and G'_{V-S} respectively. The double cosets mod $G'_S K'$ and $G'(k)$ correspond bijectively to the orbits of $G'_S K'$ on $G'(k)\backslash G'(A)$, which are all open. Since $G'(k)\backslash G'(A)$ has finite Haar measure, it is enough to show that the volumes of these orbits have a strictly positive lower bound. The double cosets are represented by elements of G'_{V-S}; it suffices therefore to consider the orbit of the image of an element $x \in G'_{V-S}$. It is isomorphic to $\Gamma_x\backslash G'_S\, xK'x^{-1}$, where $\Gamma_x = G'(k) \cap G'_S\, xK'x^{-1}$. Let Γ'_x be the projection of Γ_x into G'_S, with respect to the decomposition $G'(A) = G'_S \times G'_{V-S}$. Then $\nu(G'(k)\backslash G'(k)\, xG'_S K') = \nu_S(\Gamma'_x\backslash G'_S) \cdot \nu_{V-S}(K')$. As $xK'x^{-1}$ is a compact open subgroup of G'_{V-S}, Γ'_x is an S-arithmetic subgroup. Then the next to last assertion in 3.8 yields our claim.

3.10. *Proposition. — Let* $\mathscr{R}$ *be a finite subset of* V *containing* S, *such that* G *is quasi-split over* k_v *and splits over* $\hat{k}_v$ *for all* $v \notin \mathscr{R}$. *Then the set of arithmetic subgroups* Γ' *of* G'_S *associated to coherent collections* $(P_v)_{v \notin S}$ *of parahoric subgroups (see* 1.5*) which are hyperspecial for* $v \notin \mathscr{R}$ *form finitely many classes with respect to* $\overline{G}(k)$*-conjugacy.*

In view of the construction of the Γ' (see 1.5), it is equivalent to show that the P's in which P_v is hyperspecial for all $v \notin \mathscr{R}$ form finitely many classes under $\overline{G}(k)$-conjugacy. For any $v \in V_f$, the parahoric subgroups of $G(k_v)$ form finitely many conjugacy classes under $G(k_v)$, hence a fortiori under $\overline{G}(k_v)$. It suffices therefore to consider the P's in which P_v belongs to a given conjugacy class of parahoric subgroups in $G(k_v)$ for $v \in \mathscr{R} - S$. Let P and P′ be two such coherent collections. Of course, $P_v = P'_v$ for almost all v's. For $v \notin \mathscr{R}$, any two hyperspecial subgroups of $G(k_v)$ are conjugate under $\overline{G}(k_v)$ [41: 2.5]. There exists then $g \in \overline{G}(A)$ such that ${}^gP = P'$. Let $\overline{P}_v$ be the stabilizer of P_v in $\overline{G}(k_v)$ $(v \notin S)$. Then $\overline{P} = \Pi_{v \notin S} \overline{P}_v$ is a compact open subgroup of $\overline{G}_{V-S}$ and $\overline{G}_S \overline{P}$ is the stabilizer of P in $\overline{G}(A)$. The $\overline{G}(k)$-conjugacy classes of the P's satisfying our conditions correspond therefore to the double cosets of $\overline{G}(A)$ mod $\overline{G}(k)$ and $\overline{G}_S \overline{P}$. They are finite in number by 3.9 and the proposition follows.

4. Euler-Poincaré characteristic of arithmetic groups

We assume in this section that if k is of positive characteristic, then the k-rank of G is zero. Then any arithmetic subgroup of $G'_{\mathscr{S}}$ has a torsion free subgroup of finite index ([33: Theorem 4]) and there exists a $G_{\mathscr{S}}$-invariant measure $\mu^{EP}_{\mathscr{S}}$ on $G_{\mathscr{S}}$ such that, for any arithmetic subgroup Γ of $G_{\mathscr{S}}$,

$$|\chi(\Gamma)| = \mu^{EP}_{\mathscr{S}}(G_{\mathscr{S}}/\Gamma),$$

where $\chi(\Gamma)$ is the Euler-Poincaré characteristic of Γ in the sense of C. T. C. Wall (see [33: §§1.8, 3]).

4.1. It follows from [33: Proposition 25] that, up to sign, $\mu^{EP}_{\mathscr{S}}$ is the product of the Euler-Poincaré measures on the groups $G(k_v)$ $(v \in \mathscr{S})$ introduced in [33: §3], and to be denoted here by μ^{EP}_v. Also, for any nonarchimedean v, μ^{EP}_v is a non-zero multiple $a_v \mu_v$ of the Tits measure μ_v defined in 3.4; here

$$a_v = \mu^{EP}_v(I_v) = (-1)^{s_v}(W_v(\mathbf{q}^{-1}))^{-1},$$

where I_v is an Iwahori subgroup of $G(k_v)$, s_v is the k_v-rank of G and $W_v(\mathbf{q})$ is the Poincaré series associated with the Tits system on $G(k_v)$ whose " B " is an Iwahori subgroup (of $G(k_v)$) and " N " is the group of k_v-rational elements of the normalizer of a suitable maximal k_v-split torus of G ([33: Theorem 6]).

If $v \in \mathscr{S}_\infty$, μ^{EP}_v is non-zero if and only if $G(k_v)$ contains a compact Cartan subgroup ([33: Proposition 23]). Thus if k is a global function field, then $\mu^{EP}_{\mathscr{S}}$ is non-zero; if k is a number field, and $\mu^{EP}_{\mathscr{S}} \neq 0$, then k is necessarily totally real.

4.2. For $v \in \mathscr{S}_\infty$, the Hirzebruch proportionality principle ([33: §3.2]) at once implies that if $G(k_v)$ contains a compact Cartan subgroup, then, up to sign, μ^{EP}_v equals $c_v \mu_v$, where μ_v is the Haar measure on $G(k_v)$ defined in 3.5 and c_v is the Euler-Poincaré characteristic of the compact dual of the symmetric space associated with $G(k_v)$ (i.e., the quotient of a suitable maximal compact subgroup of $G(\mathbf{C})$ by a maximal compact subgroup of $G(k_v)$). The constant c_v is therefore a non-zero integer.

4.3. Assume that $G_{\mathscr{S}_\infty} = \Pi_{v \in \mathscr{S}_\infty} G(k_v)$ has a compact Cartan subgroup. Then, combining the above observations, we conclude that for any arithmetic subgroup Γ of $G_{\mathscr{S}}$,

$$|\chi(\Gamma)| \geqslant \prod_{v \in \mathscr{S}_f} |W_v(\mathbf{q}^{-1})|^{-1} \mu_{\mathscr{S}}(G_{\mathscr{S}}/\Gamma).$$

4.4. *A lower bound for* $|W_v(\mathbf{q}^{-1})|^{-1}$. As before, for $v \in V_f$, let $\hat{k}_v$ be the maximal unramified extension of k_v. Let σ_v denote the Frobenius automorphism of $\hat{k}_v/k_v$. Then there is a natural action of σ_v on the affine Weyl group of $G/\hat{k}_v$ and the subgroup of the fixed points is the affine Weyl group of G/k_v. Now the results contained in 1.10.1, 1.11 and 3.3.1 of Tits [41] together with those in 1.30, 1.32, 1.33 and 3.10 of Steinberg [39] imply that

$$(W_v(\mathbf{q}^{-1}))^{-1} = \prod_{j=1}^{r_v} \frac{(1-\varepsilon_j^v q_v^{1-d_v(j)})(1-\varepsilon_{oj}^v q_v^{-1})}{(1-\varepsilon_j^v q_v^{-d_v(j)})},$$

where the $d_v(j)$'s are certain positive integers $\geqslant 2$, and ε_j^v, ε_{oj}^v are certain roots of unity (see Steinberg [39: Theorem 3.10]).

From the above expression for $(W_v(\mathbf{q}^{-1}))^{-1}$, it is obvious that as the $d_v(j)$'s and the q_v's are $\geqslant 2$,

$$(1) \qquad |W_v(\mathbf{q}^{-1})|^{-1} \geqslant \left(\frac{(1-q_v^{-1})^2}{1+q_v^{-2}}\right)^{r_v} = \left(\frac{(q_v-1)^2}{q_v^2+1}\right)^{r_v} \quad (\geqslant 5^{-r_v}).$$

As a consequence, we have in particular

$$(2) \qquad |\chi(\Gamma)| = \mu^{\mathrm{EP}}(G_{\mathscr{S}}/\Gamma) \geqslant 5^{-c(S,G)} \mu_{\mathscr{S}}(G_{\mathscr{S}}/\Gamma), \qquad (c(S,G) = \sum_{v \in \mathscr{S}_f} r_v).$$

4.5. For the proof of Theorem 7.3, we need to know $|W_v(\mathbf{q}^{-1})|^{-1}$ explicitly for certain G and v. Using Proposition 24 and Theorem 6 of [33] and the Bruhat-Tits theory, $|W_v(\mathbf{q}^{-1})|^{-1}$ can be easily computed; the values are given in Appendix C, as they are needed.

4.6. Now let Γ', Λ' and Λ be as in 3.2. Then

$$|\chi(\Gamma')| = [\Gamma' : \Lambda']^{-1} |\chi(\Lambda')|$$

and it is obvious that $|\chi(\Lambda')| \geqslant |\chi(\Lambda)|$. Therefore, under the hypothesis of 4.3 we have

$$\begin{aligned} |\chi(\Gamma')| &\geqslant [\Gamma' : \Lambda']^{-1} |\chi(\Lambda)| \\ &\geqslant [\Gamma' : \Lambda']^{-1} \prod_{v \in \mathscr{S}_f} |W_v(\mathbf{q}^{-1})|^{-1} \mu_{\mathscr{S}}(G_{\mathscr{S}}/\Lambda) \\ &\geqslant n^{-\varepsilon \# \mathscr{S}} (\# H^1(k, C)_\xi)^{-1} \prod_{v \in \mathscr{S}_f} |W_v(\mathbf{q}^{-1})|^{-1} \mu_{\mathscr{S}}(G_{\mathscr{S}}/\Lambda^{\mathrm{m}}) \end{aligned}$$

(cf. 3.6), where Λ^{m} is as in 3.2. By 3.7,

$$\mu_{\mathscr{S}}(G_{\mathscr{S}}/\Lambda^{\mathrm{m}}) = D_k^{\frac{1}{2}\dim G}(D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}s} \left(\prod_{v \in V_\infty} \left| \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v\right) \tau_k(G)\, \mathscr{E},$$

where $\mathscr{E}$ is as in 3.8; therefore we get the following bound:

$$|\chi(\Gamma')| \geqslant n^{-\varepsilon\#\mathscr{S}}(\#\mathrm{H}^1(k, \mathrm{C})_\xi)^{-1} \prod_{v\in\mathscr{S}_f} |\mathrm{W}_v(\mathbf{q}^{-1})|^{-1} \mathrm{D}_k^{\frac{1}{2}\dim \mathrm{G}}(\mathrm{D}_\ell/\mathrm{D}_k^{[\ell:k]})^{\frac{1}{2}s} \cdot \left(\prod_{v\in \mathrm{V}_\infty}\left|\prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}}\right|_v\right)\tau_k(\mathrm{G})\,\mathscr{E}$$

for every arithmetic subgroup Γ'.

5. An upper bound for the order of $\mathrm{H}^1(k, \mathrm{C})_\xi$

In this section we shall give a " good " upper bound for the order of the group $\mathrm{H}^1(k, \mathrm{C})_\xi$, introduced in 2.10.

As in 3.7, let $\mathscr{G}$ be the unique simply connected, quasi-split inner k-form of G. Then the center C of G is k-isomorphic to that of $\mathscr{G}$. We shall begin by considering the case where $\mathscr{G}$ is k-split, i.e. G/k is an inner k-form (of a k-split group). As recalled in 2.6, C is k-isomorphic, in this case, to μ_n^ε, where ε, n and μ_n^ε are as in 2.6. We identify C with μ_n^ε in terms of a fixed k-isomorphism. This then provides an identification of $\mathrm{H}^1(\mathrm{K}, \mathrm{C})$ with $(\mathrm{K}^\times/\mathrm{K}^{\times n})^\varepsilon$ for any field extension K of k. For $x \in (\mathrm{K}^\times)^\varepsilon$, we denote by $\bar{x}$ the element of $\mathrm{H}^1(\mathrm{K}, \mathrm{C})$ which it determines. For $v \in \mathrm{V}$, $\bar{x}_v$ will denote the cohomology class in $\mathrm{H}^1(k_v, \mathrm{C})$ determined by $x \in (k^\times)^\varepsilon$.

Each $v \in \mathrm{V}_f$ gives a homomorphism $(k^\times)^\varepsilon \to \mathbf{Z}^\varepsilon$, which will be denoted again by v. Let now T be the (finite) set of places $v \notin \mathrm{S}$ such that G does not split over k_v. Then in view of Lemma 2.3, it follows from Proposition 2.7 that for $v \notin \mathrm{S} \cup \mathrm{T}$ and $x \in (k^\times)^\varepsilon$, $\xi_v(\bar{x}_v)$ is trivial if and only if $v(x) \in (n\mathbf{Z})^\varepsilon$. From this we conclude that $\mathrm{H}^1(k, \mathrm{C})_\xi \cap (k_n/k^{\times n})^\varepsilon$ is a subgroup of $\mathrm{H}^1(k, \mathrm{C})_\xi$ of index $\leqslant n^{\varepsilon\#(\mathrm{S}_f\cup\mathrm{T})}$, where k_n is the subgroup of $k^\times$ consisting of the elements x such that $v(x) \in n\mathbf{Z}$ for all nonarchimedean v. As $\#(k_n/k^{\times n}) \leqslant h_k\, n^{a(k)}$ (Proposition 0.12), this implies the following:

5.1. *Proposition. — If* G/k *is an inner form of a split group,*

$$\#\mathrm{H}^1(k, \mathrm{C})_\xi \leqslant h_k^\varepsilon\, n^{\varepsilon a(k)+\varepsilon\#(\mathrm{S}_f\cup\mathrm{T})}.$$

5.2. In the rest of this section we treat the case where G/k is an outer form. Then $\mathscr{G}$ is a non-split, quasi-split group of type A_r, or D_r, or E_6. Let ℓ be as in 3.7. Note that ℓ is a separable quadratic extension of k except when G is a triality form of type D_4 in which case it is a separable (but not necessarily Galois) extension of k of degree 3. For v nonarchimedean, let $\ell_v = \ell \otimes_k k_v$. If ℓ_v is a field, let $\tilde{v}$ denote its normalized additive valuation (i.e. the additive valuation whose set of values is $\mathbf{Z}$). Its restriction to k_v is a multiple of v. If v splits over ℓ, let $\tilde{v}_i$ ($i = 1, 2$ and possibly 3) be the normalized additive valuations of ℓ " lying " over v (i.e. whose restriction to $k^\times$ is a multiple of v); in this case ℓ_v is a direct sum of 2 or 3 local fields.

5.3. Let n be as in 2.6 and μ_n be the kernel of the endomorphism $x \mapsto x^n$ of GL_1. Then, except in the case where G/k is a form of type ${}^2\mathrm{D}_r$ with r even, the center C of G is k-isomorphic to the kernel of the norm map

$$\mathrm{N}_{\ell/k} : \mathrm{R}_{\ell/k}(\mu_n) \to \mu_n.$$

If G/k is of type ${}^2\mathrm{D}_r$ with r even, then C is k-isomorphic to $\mathrm{R}_{\ell/k}(\mu_2)$.

Assume first that G/k is not of type ${}^2\mathrm{D}_r$ with r even. Using the above description of the center C, we get the following commutative diagram:

$$\begin{array}{ccccccc}
\mu_n(k)/\mathrm{N}_{\ell/k}(\mu_n(\ell)) & \longrightarrow & \mathrm{H}^1(k, \mathrm{C}) & \longrightarrow & \ell^\times/\ell^{\times n} & \xrightarrow{\mathrm{N}_{\ell/k}} & k^\times/k^{\times n} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mu_n(k_v)/\mathrm{N}_{\ell/k}(\mathrm{R}_{\ell/k}(\mu_n)\,(k_v)) & \longrightarrow & \mathrm{H}^1(k_v, \mathrm{C}) & \longrightarrow & (\ell\otimes_k k_v)^\times/(\ell\otimes_k k_v)^{\times n} & \xrightarrow{\mathrm{N}_{\ell/k}} & k_v^\times/k_v^{\times n} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mu_n(\hat{k}_v)/\mathrm{N}_{\ell/k}(\mathrm{R}_{\ell/k}(\mu_n)\,(\hat{k}_v)) & \longrightarrow & \mathrm{H}^1(\hat{k}_v, \mathrm{C}) & \longrightarrow & (\ell\otimes_k \hat{k}_v)^\times/(\ell\otimes_k \hat{k}_v)^{\times n} & \xrightarrow{\mathrm{N}_{\ell/k}} & \hat{k}_v^\times/\hat{k}_v^{\times n}
\end{array}$$

in which the rows are exact. Note that the order of $\mu_n(k)/\mathrm{N}_{\ell/k}(\mu_n(\ell))$ is at most 2, and this group is trivial if either n is odd or $[\ell : k] = 3$. This is evident from the fact that $\mathrm{N}_{\ell/k}(\mu_n(\ell))$ contains $\mu_n(k)^{[\ell:k]}$, and if $[\ell : k] = 3$, then (G/k is a triality form of type D_4 and) $n = 2$. Next we assert that if v is a nonarchimedean place such that G splits over $\hat{k}_v$, then the image of $\mu_n(k_v)/\mathrm{N}_{\ell/k}(\mathrm{R}_{\ell/k}(\mu_n)\,(k_v))$ in $\mathrm{H}^1(k_v, \mathrm{C})$ acts trivially on Δ_v. To prove this, in view of Lemma 2.3 (i), it suffices to note that if G splits over $\hat{k}_v$, then $\ell\otimes_k \hat{k}_v$ is a direct sum of $[\ell : k]$ ($\geqslant 2$) copies of $\hat{k}_v$, hence $\mathrm{N}_{\ell/k}(\mathrm{R}_{\ell/k}(\mu_n)\,(\hat{k}_v)) = \mu_n(\hat{k}_v)$ and so the image of $\mu_n(k_v)/\mathrm{N}_{\ell/k}(\mathrm{R}_{\ell/k}(\mu_n)\,(k_v))$ in $\mathrm{H}^1(\hat{k}_v, \mathrm{C})$ is trivial.

Assume now that G/k is of type ${}^2\mathrm{D}_r$ with r even. Then $n = [\ell : k] = 2$. In this case C is k-isomorphic to $\mathrm{R}_{\ell/k}(\mu_2)$ and will be identified with it in terms of a fixed k-isomorphism. For any field extension K of k, the group $\mathrm{H}^1(\mathrm{K}, \mathrm{C})$ is canonically isomorphic to $(\ell\otimes_k \mathrm{K})^\times/(\ell\otimes_k \mathrm{K})^{\times 2}$. In particular, $\mathrm{H}^1(k, \mathrm{C}) = \ell^\times/\ell^{\times 2}$, and $\ell^\times$ acts on Δ_v and $\hat{\Delta}_v$ through the quotient $\ell^\times/\ell^{\times 2}$; we shall denote the induced homomorphism $\ell^\times \to \Xi_v(\subset \mathrm{Aut}\,\Delta_v)$ by ξ_v in the sequel.

5.4. *Lemma.* — *Let* $v \in \mathrm{V}_f$.

(i) *Assume* G/k *is not of type* ${}^2\mathrm{D}_r$ *with* r *even. Let* $\mathsf{L} = \{ x \in \ell^\times \mid \mathrm{N}_{\ell/k}(x) \in k^{\times n} \}$ *and* $x \in \mathsf{L}$. *If* v *does not split over* ℓ, *then* $\tilde{v}(x) \in n\mathbf{Z}$ *if* v *is ramified in* ℓ, *or if one of* n, $[\ell : k]$ *is odd. In particular, if* v *does not split over* ℓ, $\tilde{v}(x) \in 2\mathbf{Z}$ *if* G *is a triality form of type* D_4.

(ii) *Assume* G *to be split over* $\hat{k}_v$ *and quasi-split over* k_v. *Then* $\tilde{v}(x) \in n\mathbf{Z}$ *if* v *does not split over* ℓ, *and* $\tilde{v}_i(x) \in n\mathbf{Z}$ *for all* i *if* v *splits over* ℓ, *where* $x \in \ell^\times$ *if* G *is of type* ${}^2\mathrm{D}_r$ *with* r *even, and* $x \in \mathsf{L}$ *otherwise and* $\xi_v(x) = 1$.

(If G is not of type ${}^2\mathrm{D}_r$ with r even, then the image of $\mathrm{H}^1(k, \mathrm{C})$ in $\ell^\times/\ell^{\times n}$ is $\mathsf{L}/\ell^{\times n}$, see 5.3. At any nonarchimedean place v such that G splits over $\hat{k}_v$, ξ_v induces a homomorphism of L into Ξ_v ($\subset \mathrm{Aut}\,\Delta_v$) which we have also denoted by ξ_v.)

Proof. — If v does not split over ℓ, and ℓ_v is a ramified extension of k_v, then for $x \in \ell^\times, \tilde{v}(x) = v(\mathrm{N}_{\ell/k}(x))$; if ℓ_v is an unramified extension of k_v, then $\tilde{v}(x) = v(\mathrm{N}_{\ell/k}(x))/[\ell : k]$, so for $x \in \mathsf{L}$, it is an integral multiple of $n/[\ell : k]$. Now assertion (i) of the lemma is obvious. Note that if $[\ell : k]$ is odd, then G is a triality form of type D_4 and $n = 2$.

The second assertion of the lemma follows from 2.3 and 2.7.

5.5. Let $\mathscr{R}$ (resp. T) be the set of places $v \notin \mathrm{S}$ such that G does not split over $\hat{k}_v$ (resp. splits over $\hat{k}_v$ but is not quasi-split over k_v). Both $\mathscr{R}$ and T are finite. Let S_f^0 be the subset of S_f consisting of all places v such that either v splits over ℓ or ℓ_v is an unramified extension of k_v.

Let ℓ_n be the subgroup of $\ell^\times$ consisting of the elements x such that $\tilde{v}(x) \in n\mathbf{Z}$ for every normalized nonarchimedean valuation $\tilde{v}$ of ℓ. Then 5.4 implies that if G/k is not of type ${}^2\mathsf{D}_r$ with r even, then the subgroup $\mathrm{H}^1(k, \mathrm{C})_\xi$ mapping into $\ell_n/\ell^{\times n}$ has index $\leqslant n^{\#(\mathrm{S}_f^0 \cup \mathrm{T})}$ if G is not a triality form, and index $\leqslant 2^{\#\mathscr{R} + 2\#(\mathrm{S}_f^0 \cup \mathrm{T})}$ if G is a triality form. It also implies that if G is of type ${}^2\mathsf{D}_r$ with r even, then $\mathrm{H}^1(k, \mathrm{C})_\xi \cap (\ell_2/\ell^{\times 2})$ is a subgroup of $\mathrm{H}^1(k, \mathrm{C})_\xi$ of index at most $2^{\#\mathscr{R} + 2\#(\mathrm{S}_f^0 \cup \mathrm{T})}$.

By 0.12, the order of $\ell_n/\ell^{\times n}$ is $\leqslant h_\ell\, n^{a(\ell)}$. Moreover, $2^{\#\mathscr{R}} \leqslant \mathrm{D}_\ell/\mathrm{D}_k^{[\ell:k]}$, see [31: Appendix], and as we saw in 5.3, if G is not of type ${}^2\mathsf{D}_r$ with r even, the kernel $\mu_n(k)/\mathrm{N}_{\ell/k}(\mu_n(\ell))$ of the homomorphism $\mathrm{H}^1(k, \mathrm{C}) \to \ell^\times/\ell^{\times n}$ is trivial if G is a triality form and is of order at most 2 in all other cases. Combining all this information, we get:

5.6. *Proposition.* — *Assume that* G *is an outer form (of a split group). Then*

(i) *If* G *is of type* D_r *with r even (including the triality forms of type* D_4*),*

$$\#\mathrm{H}^1(k, \mathrm{C})_\xi \leqslant h_\ell\, 2^{a(\ell) + 2\#(\mathrm{S}_f^0 \cup \mathrm{T})}\, \mathrm{D}_\ell/\mathrm{D}_k^{[\ell:k]}.$$

(ii) *In all the other cases,*

$$\#\mathrm{H}^1(k, \mathrm{C})_\xi \leqslant 2h_\ell\, n^{a(\ell) + \#(\mathrm{S}_f^0 \cup \mathrm{T})}.$$

6. A number theoretic result

In this section we shall assume that k is a number field and prove the following proposition, which is needed for the proof of the finiteness theorems in §7.

Let ε, n be as in 2.6 and $m_1 \leqslant \ldots \leqslant m_r$ be the exponents of G (3.7). Recall that $n^\varepsilon \leqslant r + 1$ and $\varepsilon \leqslant 2$. As before, $a(k)$ will denote the number of archimedean places of k.

6.1. *Proposition.* — *Given a positive real number c and a nonnegative integer a, there exist effectively computable positive integers* $\mathfrak{m}_c$, $\mathfrak{m}_{c,a}$ *and* $\mathfrak{n}_c$, $\mathfrak{n}_{c,a}$ *such that*

(i) if either $r > \mathfrak{m}_c$ *or* $\mathrm{D}_k > \mathfrak{n}_c$, *then*

$$\text{(i)} \qquad \mathrm{D}_k^{\frac{1}{2}\dim \mathrm{G}} \left(\prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right)^{[k:\mathbf{Q}]} > c;$$

(ii) if G *is an inner k-form (of a split group) of type other than* A_1 *and* A_2 *and either* $r > \mathfrak{m}_{c,a}$ *or* $D_k > \mathfrak{n}_{c,a}$*, then*

$$\text{(ii)} \qquad n^{-2\varepsilon a(k)}\, h_k^{-\varepsilon}\, D_k^{\frac{1}{2}\dim G} \left(\prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}}\right)^{[k:\mathbf{Q}]} > c n^{\varepsilon a};$$

(iii) if G/k *is an outer form of type other than* A_2 *and either* $r > \mathfrak{m}_{c,a}$ *or* $D_k > \mathfrak{n}_{c,a}$*, then*

$$\text{(iii)} \qquad n^{-([\ell:k]+\varepsilon)a(k)}\, h_\ell^{-1}\, D_k^{\frac{1}{2}\dim G}(D_\ell/D_k^{[\ell:k]}) \left(\prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}}\right)^{[k:\mathbf{Q}]} > c n^{\varepsilon a};$$

(iv) if $D_k > \mathfrak{n}_{c,a}$*, then*

$$\text{(iv)} \qquad (2^4\pi^5)^{-[k:\mathbf{Q}]}\, 3^{-2a(k)}\, h_k^{-1}\, D_k^{4} > 3^a c.$$

(v) if k is totally real and $D_k > \mathfrak{n}_{c,a}$*, then*

$$\text{(v)} \qquad (2^3\pi^2)^{-[k:\mathbf{Q}]}\, h_k^{-1}\, D_k^{3/2} > 2^a c.$$

(vi) There is a positive integer $\mathfrak{n}'_{c,a}$ *such that if* $D_\ell > \mathfrak{n}'_{c,a}$*, then*

$$\text{(vi)} \qquad (2^4\pi^5)^{-[k:\mathbf{Q}]}\, 3^{-a(k)-a(\ell)}\, h_\ell^{-1}\, D_k^{4}(D_\ell/D_k^2)^2 > 3^a c.$$

Proof. — In the proof of assertions *(v)* and *(vi)* of this proposition we shall use some ideas of [10].

For a number field K, let D_K, h_K, R_K be respectively the absolute value of its discriminant, its class number and regulator. Let $\zeta_K(s)$ $(= \Pi_\mathfrak{p}(1 - (N\mathfrak{p})^{-s})^{-1})$ be its Dedekind zeta-function. Recall that $\zeta_K(s)$ has a simple pole at $s = 1$ and the residue is $2^{r_1(K)}(2\pi)^{r_2(K)}\, h_K\, R_K/w_K\, D_K^{1/2}$, where $r_1(K)$ (resp. $r_2(K)$) is the number of real (resp. complex) places of K and w_K is the order of the finite group of roots of unity in K. Let

$$Z_K(s) = -\zeta'_K(s)/\zeta_K(s) = \sum_{\mathfrak{p}} \log(N\mathfrak{p})/((N\mathfrak{p})^s - 1)$$

be the negative of the logarithmic derivative of $\zeta_K(s)$.

According to the Brauer-Siegel theorem ([35: Hilfssatz 1]), for all real $s > 1$,

$$\text{(1)} \qquad h_K\, R_K \leqslant w_K\, s(s-1)\, 2^{-r_1(K)}\, \Gamma\left(\frac{s}{2}\right)^{r_1(K)} \Gamma(s)^{r_2(K)}\, (2^{-2r_2(K)}\, \pi^{-[K:\mathbf{Q}]}\, D_K)^{s/2}\, \zeta_K(s).$$

R. Zimmert [47] has given the following lower bound for the regulator:

$$\text{(2)} \qquad \begin{aligned} R_K &\geqslant .02 w_K \exp(.46 r_1(K) + .1 r_2(K)) \\ &\geqslant .02 w_K \exp(.1 a(K)), \end{aligned}$$

where $a(K) = r_1(K) + r_2(K)$ is the number of archimedean places of K.

A. Odlyzko ([27: Theorem 1]; see also [29]) has provided the following lower bound for D_K:

$$\text{(3)} \qquad \text{If } [K:\mathbf{Q}] > 10^5, \text{ then } D_K \geqslant (55)^{r_1(K)}\,(21)^{2r_2(K)}.$$

Moreover, it follows from his results that there exist absolute positive constants (i.e. constants not depending on K) c_1, c_2 ($\leqslant 2$) such that for all $s \in (1, 1 + c_2)$

$$\text{(4)} \qquad D_K \geqslant (55)^{r_1(K)} (21)^{2r_2(K)} \exp(2Z_K(s) - 2(s-1)^{-1} - c_1).$$

Since the absolute value of the logarithmic derivative of the Gamma-function is bounded above in the interval $\left[\frac{1}{2}, 2\right]$, there exists a constant c_3 such that, for $1 \leqslant s \leqslant 2$,

$$\text{(5)} \qquad \begin{cases} \Gamma\left(\dfrac{s}{2}\right) \leqslant \pi^{1/2} \exp(c_3(s-1)) \\ \Gamma(s) \quad \leqslant \exp(c_3(s-1)). \end{cases}$$

Also, it follows at once from [26: Lemma 2] that there is an absolute constant c_4 such that for all $s > 1$,

$$\text{(6)} \qquad \zeta_K(s) \leqslant \exp(Z_K(s) + c_4(s-1)\, a(K)).$$

Taking $s = 2$ in (1) and using (2) and (3) we obtain

$$\text{(7)} \qquad h_K \leqslant 10^2 \left(\frac{\pi}{12}\right)^{[K:Q]} D_K$$

$$\left(\text{as } \zeta_K(2) \leqslant (\zeta_Q(2))^{[K:Q]} = \left(\frac{\pi^2}{6}\right)^{[K:Q]}\right).$$

We shall now prove the assertions *(i)*, *(ii)* and *(iii)* of the proposition. We begin by recalling that for at most one i, $m_i = m_{i+1}$ (and if $m_i = m_{i+1}$ for some i, then G is of type D_r with r even) and $m_r \to \infty$ with $r \to \infty$; see [31: 1.5]. From this it is clear that, as $m! \gg (2\pi)^{m+1}$ for all $m \gg 0$, there exist positive integers $\mathfrak{m}_c \leqslant \mathfrak{m}_{c,a}$ such that if $r \geqslant \mathfrak{m}_c$ (resp. $r \geqslant \mathfrak{m}_{c,a}$), then

$$\prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} > c + 1 \qquad \left(\text{resp. } (r+1)^{-(5+2a)} \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} > 10^2(c+1)\right).$$

Now as D_k is a positive integer, inequality (i) holds if $r \geqslant \mathfrak{m}_c$. Also as

$$\dim G > 2 \max(\varepsilon, [\ell : k]),$$

$n^{\varepsilon} \leqslant r + 1$ and both $\varepsilon a + 2\varepsilon a(k)$ and $\varepsilon a + ([\ell : k] + \varepsilon)\, a(k)$ are $\leqslant (5 + 2a)\, [k : \mathbf{Q}]$, the inequalities (ii) and (iii) evidently hold for $r \geqslant \mathfrak{m}_{c,a}$ in view of the bound for the class number given by (7). Let us now assume that $2 \leqslant r < \mathfrak{m}_{c,a}$. We observe that, if $r \geqslant 2$ and G is not an inner or outer form of type A_2, then

$$n^{-2\varepsilon} \prod_{i=1}^{r} m_i! \geqslant \frac{3}{4}, \qquad n^{-[\ell:k]-\varepsilon} \prod_{i=1}^{r} m_i! \geqslant \frac{3}{16},$$

and

$$n^{-\varepsilon} \prod_{i=1}^{r} m_i! > 1, \qquad n^{-([\ell:k]+\varepsilon)/2} \prod_{i=1}^{r} m_i! > 1.$$

Now recall that $\dim G = r + 2\sum_{i=1}^{r} m_i$. Using this it is easy to see that if G is not of type A_1 or A_2, then

$$r + \sum_{i=1}^{r} m_i \leqslant \frac{18}{11}\left(\frac{1}{2}\dim G - \max(\varepsilon, [\ell : k])\right).$$

Also $r + \Sigma m_i \geqslant 6$. Let $t_{c,a}$ be the smallest positive integer such that

$$((2\pi)^{-18/11}.21)^{t_{c,a}} > (10^4\, \mathfrak{m}_{c,a}^a\, c)^{8/11},$$

then using (3) and (7), for $K = k$ and ℓ, we conclude that if $[k : \mathbf{Q}] \geqslant \max(t_{c,a}, 10^5)$, then (ii) (hence also (i)) and (iii) hold. On the other hand, using (7) it is seen that there is a positive integer $u_{c,a}$ such that if $[k : \mathbf{Q}] \leqslant \max(t_{c,a}, 10^5)$ and $r \leqslant \mathfrak{m}_{c,a}$, then for $D_k > u_{c,a}$, *(i)*, *(ii)* and *(iii)* hold. Let $\mathfrak{n}_c = u_{c,0}$. We shall later choose an integer $\mathfrak{n}_{c,a} \geqslant u_{c,a}$.

We shall now prove that, for all sufficiently large D_k, the inequality (iv) holds. For this we note that using (7) (for $K = k$), we have

$$(2^4\,\pi^5)^{-[k:\mathbf{Q}]}\,3^{-2a(k)}\,h_k^{-1}\,D_k^4 \geqslant 10^{-2}(((2^2.3)^{1/3}\,\pi^2)^{-r_1(k)}(2^{2/3}\,\pi^2)^{-2r_2(k)}\,D_k)^3,$$

so if $[k : \mathbf{Q}] > 10^5$, in view of (3),

$$\begin{aligned}(2^4\,\pi^5)^{-[k:\mathbf{Q}]}\,3^{-2a(k)}\,h_k^{-1}\,D_k^4 &\geqslant 10^{-2}\left(\left(\frac{55}{(2^2.3)^{1/3}\,\pi^2}\right)^{r_1(k)}\left(\frac{21}{2^{2/3}\,\pi^2}\right)^{2r_2(k)}\right)^3\\ &> 10^{-2}((2.4)^{r_1(k)}\,(1.3)^{2r_2})^3\\ &\geqslant 10^{-2}(1.3)^{3[k:\mathbf{Q}]},\end{aligned}$$

which implies that there is a positive integer $n_{c,a} > 10^5$ such that for $[k : \mathbf{Q}] > n_{c,a}$,

$$(2^4\,\pi^5)^{-[k:\mathbf{Q}]}\,3^{-2a(k)}\,h_k^{-1}\,D_k^4 > 3^a\,c.$$

It is obvious that we can find a positive integer $u'_{c,a}$ such that the inequality

$$(((2^2.3)^{1/3}\,\pi^2)^{-r_1(k)}\,(2^{2/3}\,\pi^2)^{-2r_2(k)}\,D_k)^3 > 10^2.3^a\,c$$

holds for all k with $[k : \mathbf{Q}] \leqslant n_{c,a}$ and $D_k > u'_{c,a}$. Hence for all k with $D_k > u'_{c,a}$, the inequality (iv) holds.

We shall now prove that there is a positive integer $u''_{c,a}$ such that if $D_k > u''_{c,a}$, (v) holds.

(1) and (2) for $K = k$ give us the following (recall that in *(v)*, k is assumed to be totally real):

$$h_k \leqslant 50s(s-1)\,2^{-r_1(k)}\,\Gamma\left(\frac{s}{2}\right)^{r_1(k)}\,(\pi^{-[k:\mathbf{Q}]}\,D_k)^{s/2}\,\zeta_k(s)\,\exp(-\,.1a(k)).$$

This, along with (4), (5) and (6) imply that if $[k : \mathbf{Q}] > 10^5$,

$$\begin{aligned}(2^3\,\pi^2)^{-r_1(k)}\,h_k^{-1}\,D_k^{3/2} &\geqslant .02(s(s-1))^{-1}\left(\frac{(55)^{(3-s)/2}}{2^2\,\pi^{(5-s)/2}}\right)^{r_1(k)}\\ &\quad.\exp\left((2-s)\,Z_k(s) - \frac{1}{2}\,c_1(3-s) - (3-s)\,(s-1)^{-1}\right.\\ &\qquad\left.+ (.1 - (c_3 + c_4)\,(s-1))\,r_1(k)\right).\end{aligned}$$

Now observe that $55/2^2\pi^2 > 1.3$, and $\exp((2-s)\,Z_k(s)) \geqslant 1$ if $s < 2$. So by choosing $s(>1)$ sufficiently close to 1, the above gives the following bound:

There is an absolute constant c_5, such that

$$(8)\qquad (2^3\pi^2)^{-r_1(k)}\, h_k^{-1}\, D_k^{3/2} \geqslant c_5(1.3)^{r_1(k)} \quad \text{for all } k \text{ with } [k:\mathbf{Q}] > 10^5.$$

On the other hand, using (7) we find that

$$(2^3\pi^2)^{-r_1(k)}\, h_k^{-1}\, D_k^{3/2} \geqslant 10^{-2}(2\pi^3/3)^{-r_1(k)}\, D_k^{1/2}.$$

From this and (8) it is obvious that there is a positive integer $u''_{c,a}$ such that for all k with $D_k > u''_{c,a}$, the inequality (v) holds.

Take $\mathfrak{n}_{c,a} = \max(u_{c,a}, u'_{c,a}, u''_{c,a})$.

We now finally prove that there exists a positive integer $\mathfrak{n}'_{c,a}$ such that if $D_\ell > \mathfrak{n}'_{c,a}$, then (vi) holds. Since

$$a(k) \leqslant [k:\mathbf{Q}] = \frac{1}{2}(r_1(\ell) + 2r_2(\ell)),$$

it suffices to prove that there is a positive integer $\mathfrak{n}'_{c,a}$ such that if $D_\ell > \mathfrak{n}'_{c,a}$, then

$$\begin{aligned}(2^4\pi^5)^{-[\ell:\mathbf{Q}]/2}\, 3^{-3r_1(\ell)/2-2r_2(\ell)}\, h_\ell^{-1}\, D_k^4(D_\ell/D_k^2)^2 \\ = (2^2.3^{3/2}.\pi^{5/2})^{-r_1(\ell)}\,(2^4.3^2.\pi^5)^{-r_2(\ell)}\, h_\ell^{-1}\, D_\ell^2 > 3^a c.\end{aligned}$$

Now using (1), (2), (4), (5) and (6) for $K=\ell$, we conclude that if $[\ell:\mathbf{Q}) > 10^5$, then

$$\begin{aligned}&(2^2.3^{3/2}.\pi^{5/2})^{-r_1(\ell)}\,(2^4.3^2.\pi^5)^{-r_2(\ell)}\, h_\ell^{-1}\, D_\ell^2 \\ &\geqslant .02(s(s-1))^{-1}\left(\frac{(55)^{(4-s)/2}}{2.3^{3/2}.\pi^{(6-s)/2}}\right)^{r_1(\ell)}\left(\frac{(21)^{(4-s)}}{2^{(4-s)}.3^2.\pi^{(5-s)}}\right)^{r_2(\ell)} \\ &\quad .\exp\Big((3-s)\,Z_\ell(s) - \frac{1}{2}(4-s)\,c_1 - (4-s)\,(s-1)^{-1} \\ &\qquad + (.1 - (c_3+c_4)\,(s-1))\,a(\ell)\Big)\end{aligned}$$

Now as

$$(55)^{3/2}/2.3^{3/2}.\pi^{5/2} > 2.2, \qquad (21)^3/2^3.3^2.\pi^4 > 1.3,$$

and $\exp((3-s)\,Z_\ell(s)) \geqslant 1$ if $s < 2$,

by choosing $s(>1)$ sufficiently close to 1, we infer that there is an absolute constant c_6 such that

$$(9)\qquad (2^2.3^{3/2}.\pi^{5/2})^{-r_1(\ell)}\,(2^4.3^2.\pi^5)^{-r_2(\ell)}\, h_\ell^{-1}\, D_\ell^2 \geqslant (2.2)^{r_1(\ell)}\,(1.3)^{r_2(\ell)}\, c_6,$$

for all ℓ with $[\ell:\mathbf{Q}] > 10^5$. Also, using (7) for $K=\ell$, we find that

$$(10)\qquad (2^2.3^{3/2}.\pi^{5/2})^{-r_1(\ell)}\,(2^4.3^2.\pi^5)^{-r_2(\ell)}\, h_\ell^{-1}\, D_\ell^2 \geqslant 10^{-2}(3^{1/2}.\pi^{7/2})^{-r_1(\ell)}\,\pi^{-7r_2(\ell)}\, D_\ell.$$

From (9) and (10) it is clear that there exists a positive integer $\mathfrak{n}'_{c,a}$ such that for all k and ℓ with $D_\ell > \mathfrak{n}'_c$, the inequality (vi) holds.

7. The finiteness theorems

This section is devoted to the proof of the main results of this paper (Theorems 7.2, 7.3, 7.8 and 7.11).

7.1. Let $\mathscr{C}$ be a set of pairs (k, G) consisting of a global field k and an absolutely almost simple, simply connected algebraic group G defined over k such that (i) there is a non-zero lower bound τ for the Tamagawa numbers $\tau_k(G)$ for $(k, G) \in \mathscr{C}$ and (ii) if k is a global function field of genus zero, then G is anisotropic over k i.e. k-rank $G = 0$. We recall here that over a global function field, any absolutely almost simple anisotropic group is necessarily an inner or outer form of type A ([15: §3, Korollar 1]).

It was conjectured by A. Weil that the Tamagawa number of any simply connected semi-simple group, defined over an arbitrary global field, is 1. This conjecture has recently been proved over number fields ([18]; see also [31: 3.3]). The Tamagawa number of any simply connected group of inner type A over an arbitrary global function field is known to be 1 (see [46]). However, whether this is the case in general over a global function field is not yet known.

In view of the above results, we may assume $\mathscr{C}$ to contain all pairs (k, G) such that either k is a number field and G is arbitrary, or k is a global function field and G is of inner type A.

7.2. *Theorem.* — *Let c be a positive integer and let $\mathscr{C}_c$ be the subset of $\mathscr{C}$ consisting of the pairs (k, G) such that* (i) *if k is a global function field, its genus is* > 0; (ii) G *is anisotropic over k and* $G_\infty := \prod_{v \in V_\infty} G(k)$ *is compact*; (iii) *the class number*

$$\mathfrak{c}(P) := \#(G_\infty \textstyle\prod_{v \in V_f} P_v \backslash G(A)/G(k))$$

of G/k with respect to some coherent collection of parahoric subgroups $(P_v)_{v \in V_f}$ is $\leqslant c$. Then (up to natural equivalence) $\mathscr{C}_c$ is finite.

We recall that a collection $P = (P_v)_{v \in V_f}$ of parahoric subgroups P_v of $G(k_v)$ is said to be coherent if $\prod_{v \in V_\infty} G(k_v) . \prod_{v \in V_f} P_v$ is an open subgroup of the adèle group $G(A)$.

7.3. *Theorem.* — *Let c be a positive real number and $\mathscr{V}_c$ be the set of triples (k, G, S) such that* (i) $(k, G) \in \mathscr{C}$ *and the absolute rank of* G *is at least* 2 *(i.e.* G *is not a form of* SL_2*),* (ii) S *is a finite set of places of k containing all the archimedean ones so that for all nonarchimedean $v \in S$,* G *is isotropic at v and the subset* S(G) *of* S *consisting of the places where* G *is isotropic is nonempty,* (iii) *there is a k-group G' which is centrally k-isogenous to* G *and an arithmetic subgroup Γ' of $G'_{S(G)}$ such that either $\mu'_{S(G)}(G'_{S(G)}/\Gamma') < c$, or Γ' is virtually free*, $0 \neq |\chi(\Gamma')| < c$ and* G *is not of type A_2, where $\mu'_{S(G)}$ is as in* 3.6 *and $\chi(\Gamma')$ is the Euler-Poincaré characteristic of Γ' in the sense of C. T. C. Wall. Then (up to natural equivalence) $\mathscr{V}_c$ is finite.*

We shall prove these theorems together.

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* Equivalently, G is anisotropic over k if the latter is a number field.

7.4. Let $(k, G) \in \mathscr{C}$. As before, let r denote the absolute rank of G and r_v its rank over the maximal unramified extension $\hat{k}_v$ of k_v if $v \in V_f$. Let ℓ, $\mathscr{G}$, and $\mathfrak{s}(\mathscr{G})$ be as in 3.7 and let $\overline{\mathscr{G}}$ be the adjoint group of $\mathscr{G}$.

a) Let $(k, G) \in \mathscr{C}_c$. Then since G_∞ is assumed to be compact, if k is a number field, it is totally real. Let $P = (P_v)_{v \in V_f}$ be a coherent collection of parahoric subgroups such that the class number $\mathfrak{c}(P)$ of G with respect to P is $\leqslant c$. It follows from [31: 4.3, 2.10 and 2.11] that

(1) $$c \geqslant \mathfrak{c}(P) \geqslant C(\mathscr{G}/k)\tau\zeta(P),$$

where

(2) $$C(\mathscr{G}/k) = D_k^{\frac{1}{2}\dim \mathscr{G}}(D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}\mathfrak{s}(\mathscr{G})} \prod_{v \in V_\infty} \left| \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v$$

and $\zeta(P) = \prod_{v \in V_f} e(P_v)$, with

(3) $$e(P_v) = q_v^{(\dim \overline{M}_v + \dim \overline{\mathscr{M}}_v)/2}.(\# \overline{M}_v(\mathfrak{f}_v))^{-1} > 1 \quad (v \in V_f).$$

(The unexplained notation is as in [31].) We have

(4) $$e(P_v) \geqslant (q_v + 1)^{-1} q_v^{r_v+1}$$

if either G is not quasi-split over k_v, or P_v is not special, or G splits over $\hat{k}_v$ but P_v is not hyperspecial (3.7 (5)), and

(5) $$e(P_v) \geqslant (q_v - 1)\, q_v^{(r^2+2r-(r+1)^2 d_v^{-1} - 1)/2}$$

if $G(k_v) = SL_{(r+1)/d_v}(\mathfrak{D}_v)$, where $\mathfrak{D}_v$ is a central division algebra of degree d_v over k_v (3.7 (6)).

(1) and (3) yield

(6) $$C(\mathscr{G}/k) \leqslant c/\tau$$

or, more generally,

(6)′ $$C(\mathscr{G}/k) \prod_{v \in \mathscr{R}} e(P_v) \leqslant c/\tau \quad (\mathscr{R} \subset V_f).$$

b) If $(k, G, S) \in \mathscr{V}_c$, then from the result stated in 3.7, 3.8 and the bounds obtained in §§4, 5, we get that

(7) $$\text{either } c \geqslant B(\mathscr{G}/k)\tau\mathscr{F} \quad \text{or} \quad c \geqslant B(\mathscr{G}/k)\tau\mathscr{F}^{EP},$$

where

(8) $$B(\mathscr{G}/k) = 2^{-1} n^{-\varepsilon a(k) - \varepsilon' a(\ell)} h_\ell^{-\varepsilon'} D_k^{\frac{1}{2}\dim \mathscr{G}}(D_\ell/D_k^{[\ell:k]})^{\mathfrak{s}'(\mathscr{G})} \prod_{v \in V_\infty} \left| \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v.$$

The constants n, ε are as in 2.6, $\varepsilon' = \varepsilon$ if $\mathscr{G}$ is k-split, $\varepsilon' = 1$ otherwise, and

$$\mathfrak{s}'(\mathscr{G}) = \begin{cases} s(\mathscr{G})/2 - 1 & \text{if } \mathscr{G}/k \text{ is an outer form of type } \mathsf{D}_r,\ r \text{ even}, \\ s(\mathscr{G})/2 & \text{otherwise}, \end{cases}$$

(9) $$\mathscr{F} = \prod_{v \in V_f} f_v, \qquad \mathscr{F}^{EP} = \prod_{v \in V_f} f_v^{EP},$$

with

(10) $$f_v = e_v n^{-\varepsilon}, \qquad f_v^{\text{EP}} = e_v n^{-\varepsilon} \mid W_v(\mathbf{q}^{-1}) \mid^{-1} = f_v \mid W_v(\mathbf{q}^{-1}) \mid^{-1}$$

if $v \in S_f$ and $\ell_v = \ell \otimes_k k_v$ is a ramified field extension of k_v,

(11) $$f_v = e_v n^{-2\varepsilon}, \qquad f_v^{\text{EP}} = e_v n^{-2\varepsilon} \mid W_v(\mathbf{q}^{-1}) \mid^{-1} = f_v \mid W_v(\mathbf{q}^{-1}) \mid^{-1}$$

otherwise ($v \in S_f$),

(12) $$f_v = f_v^{\text{EP}} = e_v^{\mathfrak{m}} n^{-\varepsilon}$$

if $v \in T(G)$, where $T(G)$ is the set of places $v \notin S$ such that G splits over $\hat{k}_v$ but is not quasi-split over k_v, and finally

(13) $$f_v = f_v^{\text{EP}} = e_v^{\mathfrak{m}} \quad \text{if} \quad v \notin S \cup T(G).$$

(The e_v and $e_v^{\mathfrak{m}}$'s are as in 3.7). Also recall from 4.4 that

(14) $$\mid W_v(\mathbf{q}^{-1}) \mid^{-1} > ((q_v - 1)^2 (q_v^2 + 1)^{-1})^{r_v} \, (\geqslant 5^{-r_v}) \quad (v \in V_f).$$

Now we claim that, for $v \in V_f$,

(15) $$f_v > 1, \qquad f_v^{\text{EP}} > 1, \quad \text{unless G is of type } \mathsf{A}_2 \text{ and } q_v \leqslant 3.$$

If $r_v > 4$, this already follows from the previous inequalities. It will be checked in all cases in Appendix C.

We get then from (7) and (15)

(16) $$B(\mathscr{G}/k) < c/\tau$$

or again, more generally, for any subset $\mathscr{R}$ of V_f

(16)′ $$B(\mathscr{G}/k) \prod_{v \in \mathscr{R}} f_v < c/\tau, \qquad B(\mathscr{G}/k) \prod_{v \in \mathscr{R}} f_v^{\text{EP}} < c/\tau.$$

c) Next we remark that $D_\ell / D_k^{[\ell:k]} \geqslant 1$. This follows, e.g., from Theorem A in the Appendix of [31].

d) Let now k be a number field. Then we deduce from 6.1 (for $a = 0$) the existence of integers $\mathfrak{m}_c$, $\mathfrak{n}_c$ and $\mathfrak{n}'_c$ such that

(17) $$C(\mathscr{G}/k) \geqslant c \quad (k \text{ is assumed to be totally real when } r = 1), \qquad B(\mathscr{G}/k) \geqslant c \quad (r \geqslant 2),$$

if either $r > \mathfrak{m}_c$ or $D_k > \mathfrak{n}_c$ or $D_\ell > \mathfrak{n}'_c$.

e) Assume now k to be a function field. We want to prove a similar assertion. If G is of type A, which is necessarily the case if G is anisotropic over k by [15: §3, Kor. 1], we let A(G) denote the set of places v of k where G is a non-split inner form of type A. We now claim that there exist positive integers g_c, g'_c, m_c and q_c such that:

(i) *If k is a global function field of genus > 1 and either $g_k > \mathsf{g}_c$ or $g_\ell > \mathsf{g}'_c$, or the absolute rank of $\mathscr{G}$ is greater than m_c, or $q_\ell > \mathsf{q}_c$, then* $C(\mathscr{G}/k) \geqslant B(\mathscr{G}/k) > c/\tau$.

(ii) *If k is a global function field of genus 1 and* G *is anisotropic over k, then* $C(\mathscr{G}/k) \,.\, \Pi_{v \in A(G)}\, e(P_v) > c/\tau$ *if either* $g_\ell > \mathfrak{g}'_c$, *or the absolute rank of* G *is greater than* $\mathfrak{m}_c$, *or* $q_\ell > \mathfrak{q}_c$.

(iii) *If k is a global function field of genus 1, then both* $B(\mathscr{G}/k) \,.\, \Pi_{v \in S}\, f_v$ *and* $B(\mathscr{G}/k) \,.\, \Pi_{v \in S}\, f_v^{EP}$ *are greater than c/τ if either* $g_\ell > \mathfrak{g}'_c$, *or the absolute rank of* G *is greater than* $\mathfrak{m}_c$, *or* $q_\ell > \mathfrak{q}_c$.

(iv) *If k is a global function field of genus zero, and* G *is anisotropic over k, then both* $B(\mathscr{G}/k) \,.\, \Pi_{v \in S \cup A(G)}\, f_v$ *and* $B(\mathscr{G}/k) \,.\, \Pi_{v \in S \cup A(G)}\, f_v^{EP}$ *are greater than c/τ if either* $g_\ell > \mathfrak{g}'_c$, *or the absolute rank of* G *is greater than* $\mathfrak{m}_c$, *or* $q_\ell > \mathfrak{q}_c$.

(If the genus of k is $\leqslant 1$, then $D_k \leqslant 1$ and (6), (16) do not allow one to limit dim G and therefore r. But the point of (ii), (iii) and (iv) is to show that we can compensate for that by multiplying $B(\mathscr{G}/k)$ or $C(\mathscr{G}/k)$ by some of the factors $e(P_v)$ or f_v, f_v^{EP}, which is allowed in view of (6)', (16)'.)

(i) and (iii) follow easily, we only need to use the upper bound for the class number given in 0.8 (1) and the estimate for e_v provided by 3.7 (2).

We already pointed out that in (ii), (iv), G is a form of type A_r. If it is an inner one, then there is a central division algebra $\mathfrak{D}$ of degree $r+1$ over k such that $G = SL_1(\mathfrak{D})$. It is well-known from class field theory that if d_v is the order of $\mathfrak{D} \otimes_k k_v$ in the Brauer group, then $d_v = 1$ for all but finitely many v's, $r+1$ is the least common multiple of the d_v's and the local invariants m_v/d_v of $\mathfrak{D}$, where m_v is an integer prime to d_v, add up to zero mod 1. This implies that one of the following three conditions is fulfilled:

($\cdot$) The number of places where $d_v = r+1$, i.e. where $\mathfrak{D}_v = \mathfrak{D} \otimes_k k_v$ is a division algebra, or, equivalently, where G is anisotropic, is at least two.

($\cdot\cdot$) $r \geqslant 5$. There is exactly one place where $d_v = r+1$, at least another one where $d_v \geqslant 2$ and a third one where $d_v \geqslant 3$.

($\cdot\cdot\cdot$) $r \geqslant 5$. There are as least one place where $d_v \geqslant 2$ and two other places where $d_v \geqslant 3$.

If G is an outer form (of type A_r), then there exists a central division algebra $\mathscr{D}$ over a separable quadratic extension ℓ of k and an involution σ of $\mathscr{D}$, of the second kind, such that $G(k) = \{ d \in \mathscr{D}^\times \mid d\sigma(d) = 1 \text{ and } \mathrm{Nrd}(d) = 1 \}$. The local invariant of $\mathscr{D}$ at any place of ℓ which is fixed under the Galois conjugation of ℓ/k is zero. On the other hand, the sum of the local invariants of $\mathscr{D}$ at any two conjugate places of ℓ is zero. This implies that one of the following conditions is fulfilled:

($\cdot$) There is a place v of k where G/k_v is an anisotropic inner form of type A_r, i.e. $G(k_v) = SL_1(\mathfrak{D}_v)$, where $\mathfrak{D}_v$ is a central division algebra of degree $r+1$ over k_v.

($\cdot\cdot$) There are two places v_1, v_2 of k which split over ℓ, such that $G(k_{v_i}) = SL_{(r+1)/d_i}(\mathfrak{D}_i)$, where $\mathfrak{D}_i$ is a central division algebra of degree d_i over k_{v_i} and $d_1 \geqslant 2$, $d_2 \geqslant 3$.

The assertions (ii) and (iv) can now be proved using the estimates for e_v given by 3.7 (2), (3), (6), and the upper bound for the class number given in 0.8 (1). Note that for $v \in S_f$ if $G(k_v) = SL_{(r+1)/d_v}(\mathfrak{D}_v)$, where $\mathfrak{D}_v$ is a central division algebra of degree d_v over k_v, then

$$f_v^{EP} = (r+1)^{-2} q_v^{r(r+3)/2}(q_v - 1)(q_v^{r+1} - 1)^{-1}.$$

f) It was proved by Hermite and Minkowski (see [20: Chapter V, Theorem 5]) that there are only finitely many number fields k and ℓ such that $D_k \leqslant \mathfrak{n}_c$ and $D_\ell \leqslant \mathfrak{n}'_c$. Also it follows from Proposition 0.9 and Lemma 0.11 that there are only finitely many global function fields k, each of them having only finitely many separable extensions ℓ of degree $\leqslant 3$, such that $g_k \leqslant \mathfrak{g}_c$, $g_\ell \leqslant \mathfrak{g}'_c$ and $q_\ell \leqslant \mathfrak{q}_c$. Since $\mathscr{G}$, being quasi-split, is uniquely determined by its absolute type and the fields k, ℓ, we now conclude that there is a finite set $\mathscr{Q}_c$ of pairs $(k, \mathscr{G})$ consisting of a global field k and an absolutely almost-simple, simply connected, quasi-split k-group $\mathscr{G}$ such that if either $(k, G) \in \mathscr{C}_c$ or $(k, G, S) \in \mathscr{V}_c$, then G is an inner k-form of $\mathscr{G}$ for some $(k, \mathscr{G})$ in this finite set. Over the finite set $\mathscr{Q}_c$ both $B(\mathscr{G}/k)$ and $C(\mathscr{G}/k)$ have a strictly positive lower bound.

Fix $(k, \mathscr{G}) \in \mathscr{Q}_c$. Then n and r are fixed and $r_v \leqslant r$. It is then clear from (4) and (14) that $e(P_v)$, f_v and f_v^{EP} tend to infinity with q_v if G is not quasi-split over k_v. Therefore we conclude that if there is an inner k-form G of $\mathscr{G}$ such that $(k, G) \in \mathscr{C}_c$ or $(k, G, S) \in \mathscr{V}_c$ for some S, then the cardinality of the residue fields at all non-archimedean places where G fails to be quasi-split is bounded by a constant depending only on k, $\mathscr{G}$, c; moreover, in the latter case, the cardinality of the residue fields at places contained in S_f is also bounded in view of 3.7 (2). Since the set of places of a global field where the cardinality of the residue field is less than a given integer is finite, we see now that there are only a finite number of possibilities for S and that there exists a finite subset $\mathscr{R}$ of V such that G is quasi-split outside $\mathscr{R}$, hence such that the element of $H^1(k, \overline{\mathscr{G}})$ which defines the inner k-form G of $\mathscr{G}$ belongs to the kernel of the natural map

$$\lambda_{\mathscr{R}} : H^1(k, \overline{\mathscr{G}}) \to \prod_{v \in V - \mathscr{R}} H^1(k_v, \overline{\mathscr{G}}).$$

But this kernel is known to be finite, see Appendix B. This shows that there are only finitely many possibilities for G and concludes the proof of 7.2 and 7.3.

7.5. In order to complete the proofs of Theorem A and B of the introduction, there still remains to prove a finiteness assertion for the P's in 7.2 and the Γ''s in 7.3. In view of these theorems, it suffices to show this for one group. Note that, as long as we deal with one group, some of the restrictions made in 7.2 and 7.3 are not necessary.

The group $(\mathrm{Aut}\, G')(A)$ operates canonically on $G'(A)$ and similarly $(\mathrm{Aut}\, G')_S$ operates on G'_S. In particular $\overline{G}(A)$, $\overline{G}_S$ and $\overline{G}(k_v)$ act on $G'(A)$, G'_S, $G'(k_v)$ respectively. This will be referred to as $\overline{G}(A)$ or $\overline{G}_S$ or $\overline{G}(k_v)$-conjugacy.

7.6. *Theorem. — Assume* G *is anisotropic over* k *and* G_∞ *is compact. Let* $c > 0$. *Then, up to* $\overline{G}(A)$-*conjugacy, there are only finitely many coherent collections* $P = (P_v)_{v \in V_f}$ *of parahoric subgroups such that* $\mathfrak{c}(P) \leqslant c$.

Let $c_0 = \mathrm{C}(\mathscr{G}/k)\,\tau$. Then $\mathfrak{c}(\mathrm{P}) > c_0$ for any P (see 7.4 (1), (3)), therefore we may assume $c > c_0$.

There is a finite subset $\mathscr{R}$ of V with the following properties: (i) $\mathscr{R} \supset \mathrm{V}_\infty$; (ii) for $v \notin \mathscr{R}$, G is quasi-split over k_v and splits over $\hat{k}_v$; (iii) $q_v > 3c/2c_0$.

Let P be a coherent collection of parahoric subgroups. Assume that, for some $v \notin \mathscr{R}$, the group P_v is not hyperspecial. Then, since $e(\mathrm{P}_v) \geqslant (q_v + 1)^{-1} q_v^{r_v+1}$ (see 7.4 (4)) and $q_v \geqslant 2$, we have $e(\mathrm{P}_v) \geqslant c/c_0$, whence

$$\mathfrak{c}(\mathrm{P}) > c.$$

As a consequence, if $\mathfrak{c}(\mathrm{P}) \leqslant c$, then P_v is hyperspecial for $v \notin \mathscr{R}$. Since any two hyperspecial subgroups of $\mathrm{G}(k_v)$ are conjugate under $\overline{\mathrm{G}}(k_v)$, $(v \notin \mathscr{R})$, [41: 2.5], it follows that $(\mathrm{P}_v)_{v \in \mathrm{V} - \mathscr{R}}$ is determined uniquely up to $\overline{\mathrm{G}}(\mathrm{A})$-conjugacy. But for a given $v \in \mathscr{R}$ there are only finitely many possibilities for P_v up to conjugacy in $\mathrm{G}(k_v)$, whence the theorem.

7.7. *Theorem. — Fix* $(k, \mathrm{G}, \mathrm{S})$, *a central isogeny* $\iota : \mathrm{G} \to \mathrm{G}'$ *and* $c > 0$. *We assume* $\mathrm{S} \supset \mathrm{V}_\infty$ *and* G_S *is not compact. Let* $\mathscr{S}$ *be the subset of* S *consisting of all places where* G *is isotropic. Then, up to* $\overline{\mathrm{G}}(k)$*-conjugacy,* $\mathrm{G}'_{\mathscr{S}}$ *contains only finitely many finitely generated arithmetic subgroups (for the k-structure defined by* G/k) *such that either* $\mu'_{\mathscr{S}}(\mathrm{G}'_{\mathscr{S}}/\Gamma') \leqslant c$, *or* Γ' *is virtually torsion-free and* $0 \neq |\chi(\Gamma')| \leqslant c$.

As there is a constant $\mathfrak{e}$ such that we have, for every Γ', $|\chi(\Gamma')| = \mathfrak{e}\mu'_{\mathscr{S}}(\mathrm{G}'_{\mathscr{S}}/\Gamma')$, it suffices, in order to prove the theorem, to show that there are only finitely many finitely generated arithmetic subgroups Γ' with $\mu'_{\mathscr{S}}(\mathrm{G}'_{\mathscr{S}}/\Gamma') < c$.

Since a finitely generated group contains only finitely many subgroups of a given finite index, it suffices, in view of 1.4 (iii), to prove that $\mathrm{G}'_{\mathscr{S}}$ has only finitely many maximal arithmetic subgroups Γ' such that $\mu'_{\mathscr{S}}(\mathrm{G}'_{\mathscr{S}}/\Gamma') \leqslant c$. Let then Γ' be maximal. According to 1.4 (iv), there exists a coherent collection $\mathrm{P} = (\mathrm{P}_v)_{v \in \mathrm{V}-\mathrm{S}}$ of parahoric subgroups such that Γ' is the normalizer of $\iota(\Lambda)$ where $\Lambda = \mathrm{G}(k) \cap \prod_v \mathrm{P}_v$. For $v \in \mathrm{V} - \mathrm{S}$, let Θ_v be the type of P_v and Ξ_{Θ_v} be as in 2.8.

It follows from the first inequality of 3.6 (1), 3.6 (2) and the formula for the volume $\mu_{\mathscr{S}}(\mathrm{G}_{\mathscr{S}}/\Lambda)$ given in 3.7 that there is a constant C depending only on G, k and S such that

$$(*) \qquad \mu'_{\mathscr{S}}(\mathrm{G}'_{\mathscr{S}}/\Gamma') \geqslant \mathrm{C} \prod_{v \in \mathrm{V}-\mathrm{S}} (\#\Xi_{\Theta_v})^{-1}.e(\mathrm{P}_v),$$

where, for $v \in \mathrm{V} - \mathrm{S}$, $e(\mathrm{P}_v)$ is as in 3.7. Now let $e_v^{\mathfrak{m}}$ be as in 3.7. Then the inequalities 3.1 (*) and 3.7 (1) imply at once:

$$(1) \qquad (\#\Xi_{\Theta_v})^{-1}.e(\mathrm{P}_v) \geqslant e_v^{\mathfrak{m}} > 1.$$

Let T be the smallest subset of V containing S such that, for all $v \notin \mathrm{T}$, the group G is quasi-split over k_v and splits over $\hat{k}_v$. If for a $v \notin \mathrm{T}$, P_v is not hyperspecial, then

$$(2) \qquad e(\mathrm{P}_v) \geqslant (q_v + 1)^{-1} q_v^{r_v+1}.$$

Let $\mathscr{R}$ be a finite subset of V, containing T, such that

(3) $$q_v > 3c(r+1)/2C \quad \text{for } v \notin \mathscr{R}.$$

If for some $v \notin \mathscr{R}$, P_v is not hyperspecial, then $e(P_v) > c(r+1)/C$, as easily follows from (2) and (3); now since $\# \Xi_{\Theta_v} \leqslant r+1$ for every v, we conclude from (*) that $\mu'_{\mathscr{S}}(G'_{\mathscr{S}}/\Gamma') > c$. Thus if $\mu'_{\mathscr{S}}(G'_{\mathscr{S}}/\Gamma') \leqslant c$, then P_v is hyperspecial for $v \notin \mathscr{R}$. The finiteness of the number of $\overline{G}(k)$-conjugacy classes of the Γ''s now follows from 3.10.

7.8. *Theorem.* — a) *Under the assumptions of* 7.2, *the set of* (k, G, P) *such that* $\mathfrak{c}(P) \leqslant c$ *is finite under natural equivalence.*

b) *Under the assumptions of* 7.3, *the set of* (k, G, S, G', Γ') *such that* $\mu_{S(G)}(G_{S(G)}/\Gamma') \leqslant c$ *(resp.* Γ' *is virtually torsion-free,* $0 < |\chi(\Gamma')| \leqslant c$ *and* G *is not of type* A_2*) is finite under natural equivalence.*

Theorem 7.2 reduces the proof of a) to the consideration of the possible P's for a given (k, G), in which case it follows from 7.6. Similarly, 7.3 reduces the proof of b) to the case of one system (k, G, S) and, since G has only finitely many centrally isogeneous groups, of the arithmetic subgroups of one G′, which is settled by 7.7.

7.9. *Remark.* — In characteristic zero, the arithmeticity results of Margulis [23] allow us to express the previous finiteness results in a different way:

We consider the 4-tuples (S, k_S, H_S, Γ), where $S = S_\infty \cup S_f$ is a finite set, k_S stands for a collection k_s of local fields of characteristic zero which are archimedean for $s \in S_\infty$ and non-archimedean for $s \in S_f$, H_S is a product of groups $H_s(k_s)$, where H_s is an absolutely almost simple k_s-group and Γ an irreducible discrete subgroup of finite covolume of H_S. Assume moreover that the groups H_s are isotropic for $s \in S_f$ and that the sum of the k_s-ranks of the H_s $(s \in S)$ is at least two. If Γ is not cocompact, then [23] shows that Γ is S-arithmetic for a suitable choice of k having the completions k_s and of a k-group G′ isomorphic to H_s over k_s for $s \in S$. If Γ is cocompact, then we may have possibly to enlarge S_∞ and use a k-group G′ which is anisotropic at the new archimedean places. It follows that 7.8 implies the finiteness of the 4-tuples (S, k_S, H_S, Γ) under natural equivalence.

In positive characteristic, we deduce from [43] a similar result if we assume moreover Γ to be finitely generated.

7.10. *Corollary.* — *We keep the assumptions of* 7.3 *and assume moreover that* G *is anisotropic over* k, *isotropic over* k_v *for* $v \in S_f$ *and, in case* k *is a number field, that* $G(k_v)$ *is compact for* v *archimedean. Fix an integer* $c > 0$. *Let* X_S *be the product of the Bruhat-Tits buildings* X_v *of* G *over* k_v $(v \in S_f)$. *Then, up to natural equivalence, there exist only finitely many* 5*-tuples* (k, G, G', S_f, Γ') *such that* Γ' *has at most* c *orbits on the set of chambers of* X_S.

Let I'_S be the stabilizer of a chamber in X_S. The number of orbits of Γ' in the set of chambers is also the number of orbits of Γ' on G'_S/I'_S, which, in turn, is equal to the number of orbits of I'_S on G'_S/Γ'. By definition $\mu'_S(I'_S) = 1$, therefore each orbit of I'_S in G'_S/Γ' has volume $\leqslant 1$. Moreover, if k is a number field and v is archimedean, then $G'(k_v)$ is compact by assumption, hence $\mu'_v(G'(k_v)) = 1$ by definition (cf. 3.5). Consequently $\mu'_S(G'_S/\Gamma') \leqslant c$, and the corollary now follows from 7.8.

Remarks. — (1) If the discrete subgroup Γ' of G'_S has finitely many orbits on the set of chambers of X_S, then, as pointed out above, the stability group of a chamber has finitely many orbits on G'_S/Γ' and the latter quotient is necessarily compact. Therefore the supplementary assumptions made here on G are necessary. In the function field case, they imply that G is of type **A**.

(2) Assume now that Γ' consists of special automorphisms of X_S and has finitely many orbits on the set of facets of some given type. Then, as before, we see that the stability group of one such facet has finitely many orbits on G'_S/Γ', hence the latter quotient is compact. However its volume is not bounded by a universal constant, and tends to infinity with the relative ranks at S_f (if the facet is not a chamber). But if the number of elements of S_f is bounded, then the growth of the volume is sufficiently slow so that a minor modification of the previous arguments will again yield a finiteness theorem:

7.11. *Theorem. — Let a, c be two positive integers. Then up to natural equivalence, there exist only finitely many pairs (k, G) consisting of a number field k and an absolutely almost simple, simply connected k-group G such that* (i) *G is anisotropic at all the archimedean places of k,* (ii) *there is a k-group G' k-isogenous to G, a finite set $\mathscr{S}$ of nonarchimedean places of k of cardinality a and an arithmetic subgroup Γ' of $G'_{\mathscr{S}}$ which acts by special automorphisms on the product $X_{\mathscr{S}} = \prod_{v\in\mathscr{S}} X_v$ of the Bruhat-Tits buildings X_v of G/k_v, $v \in \mathscr{S}$, with at most c orbits in the set of facets conjugate to some facet $F = \prod_{\mathscr{S}} F_v$. Moreover, up to natural equivalence, there are only finitely many 5-tuples (k, G, S, G', Γ') such that $\# \mathscr{S} = a$ and Γ' has at most c orbits in the set of facets conjugate to some facet $F = \prod_{\mathscr{S}} F_v$, where none of the F_v's is a vertex.*

Proof. — Let k, G, G', $\mathscr{S}$ be such that $\# \mathscr{S} = a$ and $G'_{\mathscr{S}}$ contains an arithmetic subgroup Γ' which acts by special automorphisms on the product $X_{\mathscr{S}}$ of the Bruhat-Tits buildings X_v of G/k_v, $v \in \mathscr{S}$, with at most c orbits in the set of facets conjugate to some facet $F = \prod_{\mathscr{S}} F_v$. Let C be a chamber of $X_{\mathscr{S}}$ containing F. Then C is a product $\prod_v C_v$, where C_v is a chamber of X_v and F_v a facet of C_v, $v \in \mathscr{S}$. Let I_v (resp. P_v) be the stabilizer of C_v (resp. F_v) in $G(k_v)$. Let $\mu'_{\mathscr{S}}$ be the product of the Tits measures on $G'(k_v)$, $v \in \mathscr{S}$. We want to show first

$$(1) \qquad \mu'_{\mathscr{S}}(G'_{\mathscr{S}}/\Gamma') \leqslant c \,.\, \prod_{v\in\mathscr{S}} [P_v : I_v].$$

Let I'_v (resp. P'_v) be the stabilizer of C_v (resp. F_v) in $G'(k_v)$ and $G'_{v,0}$ be the subgroup of $G'(k_v)$ operating on X_v by special automorphisms. The latter is open of finite index in $G'(k_v)$ and contains $\iota(G(k_v))$. Let $P'_{v,0} = G'_{v,0} \cap P'_v$ and $I'_{v,0} = G'_{v,0} \cap I'_v$; we have $G'(k_v) = \iota(G(k_v)) . I'_v$, $P'_{v,0} = \iota(P_v) . I'_{v,0}$, hence

$$(2) \qquad [P'_{v,0} : I'_{v,0}] = [P_v : I_v].$$

The group $G'_{\mathscr{S},0} := \prod_{v\in\mathscr{S}} G'_{v,0}$ is transitive on the facets of any given type and $\Gamma' \subset G'_{\mathscr{S},0}$. We have therefore, by assumption,

(3) $$\#(\Gamma'\backslash G'_{\mathscr{S},0}/P'_{\mathscr{S},0}) \leqslant c, \quad \text{where } P'_{\mathscr{S},0} = \prod_{v\in\mathscr{S}} P'_{v,0},$$

which implies

(4) $$\mu'_{\mathscr{S}}(G'_{\mathscr{S},0}/\Gamma') \leqslant c.\mu'_{\mathscr{S}}(P'_{\mathscr{S},0}) = c.\prod_{v\in\mathscr{S}} [P'_{v,0} : I'_{v,0}]\,[I'_v : I'_{v,0}]^{-1},$$

and as

$$\mu'_{\mathscr{S}}(G'_{\mathscr{S}}/\Gamma') = [G'_{\mathscr{S}} : G'_{\mathscr{S},0}]\,\mu'_{\mathscr{S}}(G'_{\mathscr{S},0}/\Gamma') = \prod_{v\in\mathscr{S}} [I'_v : I'_{v,0}].\mu'_{\mathscr{S}}(G'_{\mathscr{S},0}/\Gamma')$$

we conclude that

(5) $$\mu'_{\mathscr{S}}(G'_{\mathscr{S}}/\Gamma') \leqslant c.\prod_{v\in\mathscr{S}} [P'_{v,0} : I'_{v,0}],$$

so that (1) follows from (2) and (5). Proceeding as in 7.4 and taking (1) into account we get first (recall that $\tau_k(G) = 1$ as k is a number field)

(6) $$c \geqslant B'(\mathscr{G}/k).(\prod_{v\in\mathscr{S}} e_v[P_v : I_v]^{-1}).\prod_{v\in V_f-\mathscr{S}} f_v,$$

where $$B'(\mathscr{G}/k) = B(\mathscr{G}/k)\,n^{-2\mathfrak{s}a}$$

and e_v, f_v, $B(\mathscr{G}/k)$ are as in 7.4. In the notation of [31: 2.2],

(7) $$e_v[P_v : I_v]^{-1} = q_v^{(\dim \overline{M}_v + \dim \overline{\mathscr{M}}_v)/2}(\#\overline{M}_v(\mathfrak{f}_v))^{-1}.$$

By definition $e_v[P_v : I_v]^{-1}$ is the $e(P_v)$ of 7.4, therefore it satisfies

(8) $$e_v[P_v : I_v]^{-1} > 1 \quad (v \in \mathscr{S})$$

and

(9) $$e_v[P_v : I_v]^{-1} \geqslant (q_v + 1)^{-1}\,q_v^{r_v+1}$$

if either G is not quasi-split over k_v, or P_v is not maximal (i.e. if F_v is not a vertex). Since $f_v > 1$ (Appendix C), we deduce first from 6.1, (6) and (8), as in 7.4, that there are only finitely many possibilities for k, $\mathscr{G}$. Then from (9) and 3.7 (5), we see that, for given $(k, \mathscr{G})$, there are only finitely many $v \in V_f$ where G may not be quasi-split, whence the finiteness of the G's (hence also of G''s). Moreover, as P_v is maximal if and only if F_v is a vertex, if for no $v \in \mathscr{S}$, F_v is a vertex, then we conclude from (9) that the cardinality of the residue fields at all v in $\mathscr{S}$ is bounded by a constant depending only on k, G and c, which implies the finiteness of the possible $\mathscr{S}$'s; the finiteness of the possible Γ' now follows from 7.8 *b)*.

7.12. The following example shows the necessity of the restriction imposed by (ii) in 7.1, namely that if k is a global function field of genus zero, then G is anisotropic.

Let $n \geqslant 2$ be an integer, q be a power of a prime and F_q be the finite field with q elements. Let k_q be the global function field $F_q(t)$. It is of genus zero and its zeta-function is

$$\zeta_q(s) = (1 - q^{-s})^{-1}.(1 - q^{1-s})^{-1}.$$

Let $\Gamma_{n,q} = SL_n(F_q[t^{-1}])$. Then $\Gamma_{n,q}$ is an arithmetic subgroup of $SL_n(F_q((t)))$, and with respect to the Tits measure on the latter, its covolume is

$$q^{-(n^2-1)}(\# SL_n(F_q))\,(q-1)^{-(n-1)}\,q^{-\frac{1}{2}n(n-1)}\prod_{m=1}^{n-1}\zeta_q(m+1).$$

(See, for example, [31: Theorem 3.7].) Since $q^{(n^2-1)} > \# SL_n(F_q)$, we find that the covolume of $\Gamma_{n,q}$ in $SL_n(F_q((t)))$ tends to zero if either n or $q \to \infty$.

Similarly, 7.11 is not valid in general over a function field. In fact, [42] provides infinitely many examples of arithmetic subgroups of anisotropic forms of $\mathbf{A}_2$ which are transitive on the edges of a given type on the building of SL_3 over $k((y))$, where k runs through finite fields.

8. Upper bound for the order of finite subgroups and a lower bound for the covolumes of discrete subgroups

In this section we shall sketch an alternative approach to get a lower bound for the covolume of discrete subgroups of the group of rational points of a connected semi-simple isotropic group defined over a nonarchimedean local field of characteristic zero and finite products of groups of this form. This approach was announced in [4]. For arithmetic subgroups, it does not give bounds as sharp as those obtained earlier, and it does not allow one to vary the ground field arbitrarily. However it applies to arbitrary (i.e., not necessarily arithmetic) discrete subgroups and it does not require any information on Tamagawa numbers. It depends on the following two results (Propositions 8.1, 8.2) on upper bounds for the order of finite subgroups, which may be of some independent interest.

Let K be a finite extension of the field $\mathbf{Q}_p$ of p-adic numbers. Let e be its ramification index over $\mathbf{Q}_p$ and q be the cardinality of its residue field.

8.1. *Proposition.* — (i) *The order of any finite abelian subgroup of* $SL_n(K)$ *is less than* $(2e+1)^n\,(q+1)^{n-1}$.

(ii) *There exists an absolute constant c, not depending on* K *or n, such that the order of any finite subgroup of* $SL_n(K)$ *is less than* $2^{cn^2/\log n}(2e+1)^n\,(q+1)^{n-1}$.

8.2. *Proposition.* — *Let* G *be a simply connected semi-simple* K*-subgroup of* SL_n. *Let r be the rank of* G *over the maximal unramified extension of* K *and w be the order of its absolute Weyl group. Then*

(i) *The order of any finite abelian subgroup of* G(K) *is less than* $w(2e+1)^n\,(q+1)^r$.

(ii) *There exists a constant d, depending on n but not on* K, *such that the order of any finite subgroup of* G(K) *is less than* $dw(2e+1)^n\,(q+1)^r$.

We will prove these two propositions together.

Let A be a finite abelian subgroup of either G(K) or $SL_n(K)$. Let A_p be the

p-primary component of A and A$'$ be the sum of prime-to-p-primary components of A. Then $A = A_p \oplus A'$.

Let $\mathfrak{o}$ be the ring of integers of K, $\mathfrak{p}$ the unique maximal ideal of $\mathfrak{o}$ and $F = \mathfrak{o}/\mathfrak{p}$ be the residue field. After replacing A by a conjugate under an element of $GL_n(K)$, we may (and do) assume that A is contained in the maximal compact subgroup $SL_n(\mathfrak{o})$ of $SL_n(K)$. Now since the kernel of the " reduction mod $\mathfrak{p}$ " $SL_n(\mathfrak{o}) \to SL_n(F)$ is a pro-p group, this map is injective on A$'$. Let $\overline{A}'$ denote the image of A$'$, let V be the natural n-dimensional representation of $SL_n(F)$ and let $V = \bigoplus_{1 \leqslant i \leqslant a} V_i$ be the decomposition of V as a direct sum of irreducible $F[\overline{A}']$-submodules. Set $\dim V_i = m_i$. Then $\sum_{i=1}^{a} m_i = n$. It is clear that

$$(1) \qquad \# A' = \# \overline{A}' \leqslant (q-1)^{-1} . \big(\prod_{i=1}^{a} (q^{m_i} - 1)\big) \leqslant (q+1)^{n-1}.$$

Now assume that A is a finite abelian subgroup of G(K) and let P be a maximal parahoric subgroup of G(K) containing A$'$. Since G is simply connected, the "reduction mod $\mathfrak{p}$" of P is a connected linear algebraic group defined over the residue field F, see [41: §§3.4, 3.5]. Let M be the quotient of this linear algebraic group by its unipotent radical. Then M is a reductive F-group of absolute rank $\leqslant r$, and the order of its absolute Weyl group is as most w. As A$'$ is a finite abelian group of order prime to p, the natural homomorphism of P into M(F), maps it isomorphically onto an abelian subgroup $\overline{A}'$ of M(F). Now according to a result of Springer and Steinberg [37: Chapter II, Theorem 5.16], $\overline{A}'$ normalizes a maximal F-torus T of M(F) and hence (see [31: Lemma 2.8])

$$(2) \qquad \# A' = \# \overline{A}' \leqslant w \# T(F) \leqslant w(q+1)^r.$$

We shall now estimate the order of A_p. For this purpose we consider a maximal commutative semi-simple K-subalgebra $\mathscr{A}$ of the matrix algebra $M_n(K)$ containing A_p. Being semi-simple, $\mathscr{A}$ is a direct sum of certain field extensions K_i of K; $1 \leqslant i \leqslant b$. Let $[K_i : K] = n_i$. Then $\sum_{i=1}^{b} n_i = n$, and so, in particular, $1 \leqslant b \leqslant n$. Now recall that any finite subgroup of the multiplicative group of a field is cyclic, and let c_i be the largest positive integer such that K_i contains a primitive p^{c_i}-th root of unity. Then it is obvious that $\# A_p \leqslant \prod_{i=1}^{b} p^{c_i}$. On the other hand, the field extension obtained by adjoining a primitive p^{c_i}-th root of unity to $\mathbf{Q}_p$ has ramification index $p^{c_i - 1}(p-1)$ over $\mathbf{Q}_p$, and the ramification index of K_i over $\mathbf{Q}_p$ is at most en_i; hence, $p^{c_i-1}(p-1) \leqslant en_i$, which implies that $p^{c_i} \leqslant ep(p-1)^{-1} n_i$. Therefore,

$$\# A_p \leqslant \prod_{i=1}^{b} p^{c_i} \leqslant (ep(p-1)^{-1})^b \prod_{i=1}^{b} n_i \leqslant (ep(p-1)^{-1})^b \, (n/b)^b$$

(since $\sum_{i=1}^{b} n_i = n$). Now it is easily seen, by computing the maxima of the function $f(x) = (ep(p-1)^{-1})^x \, (n/x)^x$ in the range $[1, n]$, that

$$(3) \qquad \# A_p < (2e+1)^n.$$

The assertions 8.1 (i) and 8.2 (i) now follow from (1), (2) and (3).

Let now $\mathscr{F}$ be a (not necessarily abelian) finite subgroup of $SL_n(K)$. Then, as K is embeddable as a subfield of the field of complex numbers, the quantitative version of a theorem of Jordan proved by Frobenius (see [36: §70, Satz 200]), implies that $\mathscr{F}$ contains an abelian normal subgroup A whose index is $\leqslant n!\, 12^{n(\pi(n+1)+1)}$, where $\pi(n+1)$ is the number of positive primes $\leqslant (n+1)$. Using now the bound for the order of finite abelian subgroups obtained above, we conclude that

$$\sharp\mathscr{F} \leqslant n!\, 12^{n(\pi(n+1)+1)}(2e+1)^n\,(q+1)^{n-1}, \tag{4}$$

and if $\mathscr{F}$ is a finite subgroup of G(K), that

$$\sharp\mathscr{F} \leqslant n!\, 12^{n(\pi(n+1)+1)}\, w(2e+1)^n\,(q+1)^r. \tag{5}$$

Thus 8.2 (ii) is satisfied with $d = n!\, 12^{n(\pi(n+1)+1)}$.

To prove the second assertion of Proposition 8.1, we note that according to the prime number theorem, $\pi(n+1)\log(n+1)/(n+1) \to 1$ as $n \to \infty$. Moreover, $n! < n^n$ and, for every i, $(\log n)^i/n \to 0$ as $n \to \infty$. There exists therefore an absolute constant c such that $n!\, 12^{n(\pi(n+1)+1)} < 2^{cn^2/\log n}$. Together with (4), this proves 8.1 (ii).

8.3. Let us now assume that G is a simply connected semi-simple K-subgroup of SL_n. Let μ be the Tits measure on G(K) i.e. the Haar measure with respect to which every Iwahori subgroup of G(K) has volume 1. Let Γ be a discrete subgroup of G(K) and P be a parahoric subgroup of maximum volume. The G(K)-invariant measure on $G(K)/\Gamma$ induced by μ will also be denoted by μ. The group $P \cap \Gamma$, being compact and discrete, is finite. Also, $\mu(P) = [P : I]$. As the natural inclusion of P in G induces an injective map $P/P \cap \Gamma \to G/\Gamma$, we conclude that

$$\mu(G/\Gamma) \geqslant \mu(P)\,.\,(\sharp(P \cap \Gamma))^{-1} = [P : I]\,(\sharp(P \cap \Gamma))^{-1}.$$

Using the "reduction mod $\mathfrak{p}$" and the Bruhat-Tits theory (see [41: §§3.5, 3.7]) it is easy to give a good lower bound for [P : I] and Propositions 8.1, 8.2 provide an upper bound for the order of finite subgroups of G(K). Combining these we get a lower bound for the volume of G/Γ. For example, if G is an absolutely simple group of type E_8, then G is K-split, P is hyperspecial and $[P : I] > q^{120}$ (recall that the root system of type E_8 has 240 roots), and considering the embedding of G in SL_{248} given by the adjoint representation, we find from Proposition 8.2 that there is a constant c, which does not depend on K, such that the order of any finite subgroup of G(K), and so in particular of $P \cap \Gamma$, is less than $c(2e+1)^{248}\,(q+1)^8$. Hence

$$\mu(G/\Gamma) > c^{-1}(2e+1)^{-248}\,(q+1)^{-8}\,q^{120}.$$

Note that for a fixed e, the above lower bound goes to infinity with q.

Appendix A: Volumes of parahoric subgroups

This section provides in particular the proofs of two assertions made in 3.1. The arguments are minor modifications of those communicated to us by J. Tits.

A.1. We let K be a non-archimedean local field, q the order of its residue field, H an isotropic absolutely almost simple simply connected K-group, and X the Bruhat-Tits building of H(K). We fix an apartment **A** of X, a chamber C in **A** and let Δ be the set of vertices of C. As usual, the elements of Δ represent either a basis of the affine root system Φ_{af} of H/K or the vertices of the local Dynkin diagram $\mathscr{D}$. Let I be the stability group of C in H(K). It is an Iwahori subgroup. Let **W** be the affine Weyl group of H/K. The isotropy group of an element $c \in \overline{\mathrm{C}}$ in H(K) (resp. **W**) is denoted P_c, (resp. $\mathbf{W}_c$). We let μ_{T} be the Tits measure on H(K). Therefore if the parahoric subgroup P contains I, then $\mu_{\mathrm{T}}(\mathrm{P}) = [\mathrm{P} : \mathrm{I}]$.

A.2. In the classification tables of [41], each vertex β of $\mathscr{D}$ is equipped with a positive integer $d(\beta)$ (written explicitely only if it differs from 1). If r_β is the fundamental reflection associated to $\beta \in \Delta$, i.e., to the wall of $\overline{\mathrm{C}}$ opposite β, then $q^{d(\beta)} = \#(\mathrm{I}r_\beta\, \mathrm{I}/\mathrm{I})$ [41: 3.3.1]. Moreover, if w is in the affine Weyl group **W** and $w = r_1 \ldots r_t$ is a reduced decomposition of w, where the r_i are fundamental reflections, then

$$(1) \qquad \#(\mathrm{I}w\mathrm{I}/\mathrm{I}) = q_w = \prod_i q^{d(\beta_i)},$$

where $\beta_i \in \Delta$ is the vertex representing r_i (loc. cit.). This also shows that $q_{w.w'} = q_w \cdot q_{w'}$ if $\ell(w.w') = \ell(w) + \ell(w')$.

A.3. We have to refine and reformulate this. Let T be the maximal K-split torus in H such that T(K) stabilizes **A**. We let Φ^{nd} be the system of non-divisible roots in the relative root system $\Phi = \Phi(\mathrm{H}, \mathrm{T})$. We view it as a subset of $\mathrm{X}^*(\mathrm{T}) \otimes \mathbf{R}$, which, in turn, is identified with the dual ${}^v\mathbf{A}^*$ of the space of translations ${}^v\mathbf{A}$ of **A**. Given an affine root α, there is a unique element $\overline{\alpha} \in \Phi^{nd}$ such that $\overline{\alpha}$ is a positive rational multiple of the vector part of α, and any $a \in \Phi^{nd}$ occurs in this way. For $a \in \Phi^{nd}$, let $\Gamma_a = \cup\, \alpha^{-1}(0)$, where the union is over all the affine roots with vector part proportional to a. It is a union of parallel hyperplanes in **A**. If $c \in \overline{\mathrm{C}} \cap \Gamma_a$, we let $r_{a,c}$ be the reflection in the hyperplane of Γ_a containing c. It belongs to **W**. Our previous r_β is then $r_{\overline{\beta},b}$ for any b in the interior of the closed facet of codimension one of $\overline{\mathrm{C}}$ not containing β. If now $\alpha \in \Delta$ is such that $c \in \Gamma_{\overline{\alpha}}$, then

$$(1) \qquad \#(\mathrm{I}r_{\overline{\alpha},c}\, \mathrm{I}/\mathrm{I}) = q^{d(\alpha,c)}, \quad \text{where } 1 \leqslant d(\alpha, c) \leqslant d(\alpha).$$

In fact, $d(\alpha, c)$ can take at most two values as c varies (besides zero when $c \notin \Gamma_{\overline{\alpha}}$). The group $\mathbf{W}_c$ is generated by the $r_{\overline{\beta},c}$, where β runs through the set $\Delta_{(c)}$ of vertices of Δ defining the type of the facet of $\overline{\mathrm{C}}$ containing c. Then $\overline{\Delta}_c = \{\overline{\beta} \mid \beta \in \Delta_{(c)}\}$ is a basis of the sub-root system Φ_c of Φ^{nd} given by

$$(2) \qquad \Phi_c = \{\overline{\alpha} \mid c \in \Gamma_{\overline{\alpha}},\ (\alpha \in \Phi_{\mathrm{af}})\}.$$

By the Bruhat decomposition we have $\mathrm{P}_c = \coprod_{w \in \mathbf{W}_c} \mathrm{I}w\mathrm{I}$, whence

$$(3) \qquad [\mathrm{P}_c : \mathrm{I}] = \sum_{w \in \mathbf{W}_c} q(w, c), \quad \text{where } q(w, c) := \#(\mathrm{I}w\mathrm{I}/\mathrm{I}).$$

As above, if $w = r_1 r_2 \dots r_t$ is a reduced decomposition of w in $\mathbf{W}_c$, where r_i is one of the $r_{\bar\beta, c}(\beta \in \Delta_{(c)})$, then

(4) $$q(w, c) = \prod_i q(r_i, c).$$

A.4. For the purpose of this discussion, we shall say that a special vertex c of $\overline{\mathbf{C}}$ is *very special* if $d(c)$ has the smallest possible value among the $d(b)$'s for b special. (There are in fact at most two possible values.) We have $d(b) = 1$ if b is hyperspecial, as is shown by inspection of the tables in [41], or could be deduced from 3.8.1 there, hence any hyperspecial point is very special. The parahoric subgroup P_c is said to be very special if c is so.

In the sequel we fix a very special vertex c_0. We have

(1) $$d(\beta, c_0) = d(\beta) \quad (\beta \in \Delta - \{ c_0 \}),$$

hence

(2) $$d(\beta, c_0) \geqslant d(\beta, c) \quad (\beta \in \Delta - \{ c_0 \},\ c \in \overline{\mathbf{C}}).$$

The $\bar\beta$'s for $\beta \in \Delta - \{ c_0 \}$ form a basis Δ_0 of Φ and $-\bar c_0$ is the dominant root (with respect to Δ_0). We identify $\mathbf{W}_{c_0}$ in this way with the Weyl group W of Φ. For $c \in \overline{\mathbf{C}}$, we now identify $\mathbf{W}_c$ with the subgroup W_c of W generated by the reflections $r_{\bar\beta}$ $(\beta \in \Delta_{(c)})$. Then Φ_c is the subroot system of Φ^{nd} generated by the corresponding roots and W_c is the Weyl group of Φ_c. If $\alpha, \beta \in \Delta_{(c)}$ are transformed into one another by an element of W_c, then $d(\alpha, c)$ and $d(\beta, c)$ are equal. We may therefore extend the definition of $d(\alpha, c)$ to all α such that $\bar\alpha \in \Phi_c$ and $c \in \Gamma_{\bar\alpha}$ by requiring that it be W_c-invariant. We fix the ordering on Φ^{nd} defined by the basis Δ_0 and, for $c \in \overline{\mathbf{C}}$, let Φ_c^+ (resp. Φ_c^-) be the set of roots in Φ_c which are positive (resp. negative) under this ordering. In view of the relation between reduced decompositions and positive roots transformed into negative ones, we can also write A.3 (4) as

(3) $$q(w, c) = \prod_{\bar\alpha \in \Phi_c^+,\ w\bar\alpha < 0} q^{d(\alpha, c)}.$$

A.5. *Proposition. — Let* μ *be a Haar measure on* H(K).

(i) $\mu(\mathrm{P}_c)$ $(c \in \overline{\mathbf{C}})$ *is maximal among the volumes of parahoric subgroups of* H(K) *if and only if* c *is very special.*

(ii) *Assume* $c \in \overline{\mathbf{C}}$ *is not special. Then*

(1) $$\mu(\mathrm{P}_{c_0}) \geqslant \mu(\mathrm{P}_c) . (1 + q([\mathrm{W} : \mathrm{W}_c] - 1)).$$

In the proof we may assume that $\mu = \mu_{\mathrm{T}}$. Let first c be special. In view of A.3 (3) and A.4 (3), we have $q(w, c_0) \geqslant q(w, c)$ for all $w \in \mathrm{W}$. But, if c is not very special, we have a strict inequality for at least one w, therefore, by A.3 (3), $\mu(\mathrm{P}_{c_0}) > \mu(\mathrm{P}_c)$. This shows (i) for c special. On the other hand, the second factor on the right hand side of (1) is $\geqslant 2$. Therefore (ii) implies (i) for nonspecial c's. There remains to prove (ii), which we now proceed to do.

Assume c to be nonspecial. Let

$$W^c = \{ w \in W \mid w(\Phi_c^+) \subset \Phi^+ \}.$$

This is a set of representatives for the left cosets W/W_c. Let $u \in W^c$, $w \in W_c$ and $a \in \Phi_c^+$. If $w.a < 0$, then $w.a \in \Phi_c^-$ and therefore $uw.a < 0$, hence

$$w^{-1}\Phi^- \cap \Phi_c^+ = (uw)^{-1}\Phi^- \cap \Phi_c^+.$$

In view of A.4 (3), this shows that

(2) $$q(uw, c_0) \geqslant q(w, c)$$

and

(3) $$q(uw, c_0) \geqslant q.q(w, c) \quad \text{if } (uw)^{-1}\Phi^- \cap \Phi^+ - w^{-1}\Phi^- \cap \Phi^+ \neq \emptyset.$$

If $(uw)^{-1}\Phi^- \cap \Phi^+ = w^{-1}\Phi^- \cap \Phi_c^-$, then

$$\{ a \in \Phi_c^+, w.a < 0 \} = \{ a \in \Phi^+, uw.a < 0 \}.$$

Given w, this determines uw, hence can happen for at most one $u \in W^c$; therefore

(4) $$\sum_{u \in W^c} q(uw, c_0) \geqslant q(w, c)\ (1 + q([W : W_c] - 1)).$$

Then, in view of A.3 (3), the assertion (ii) follows from (4) by summing over $w \in W_c$.

A.6. As in 2.4, we let Ξ be the group of automorphisms of Δ defined by (Ad H) (K). For $c \in \overline{C}$, let Θ_c be the type of the face of C containing c, i.e., the subdiagram of Δ whose vertices correspond to the faces of codimension one of $\overline{C}$ containing c, and Ξ_c be the subgroup of Ξ leaving Θ_c stable. Then we have the following corollary, which is 3.1 (*) in a different notation.

A.7. *Corollary.* — $\mu(P_{c_0}) \geqslant \mu(P_c)\ (\#\Xi_c)$.

If c is special, then $\Xi_c = 1$ and the assertion follows from A.5 (i). Let now c be non-special. In view of A.5 (ii), it suffices to show that

(1) $$1 + q([W : W_c] - 1) \geqslant \#\Xi_c.$$

The left-hand side being $\geqslant 3$, we have only to consider the cases where $\#\Xi_c \geqslant 4$. Then H is either an inner form of type $\mathbf{A}$, of K-rank r ($r \geqslant 2$), or a K-split form of type $\mathbf{D}_r$ ($r \geqslant 4$). In the former case, Ξ is a cyclic group of order $r + 1$. Since $\Xi_c \neq 1$, it is a cyclic group of some order m dividing $r + 1$ and Φ_c is isomorphic to the direct product of m copies of the Weyl group of $\mathbf{A}_s$ for some $s \leqslant d$, where $d + 1 = (r + 1)/m$. Then W_c has order $\leqslant ((d + 1)!)^m$, therefore

$$[W : W_c] \geqslant r + 1,$$

and hence the left-hand side of (1) is at least $2r + 1$.

In the second case, Ξ_c is of order 4. It is easily verified that no subgroup of index 2 of W is the Weyl group of a subroot system. Hence the left-hand side of (1) is > 4.

Appendix B: A theorem in Galois cohomology

At the end of the proof of 7.2 and 7.3, we have used a finiteness theorem in Galois cohomology which is well-known in the number field case, but for which we do not know of a reference in the function field case. The purpose of this appendix is to supply a proof. The groups G and G′ are as before.

B.1. *Theorem. — The fibres of the canonical map*

$$\lambda^1_{G'} : H^1(k, G') \to \prod_{v \in V} H^1(k_v, G') \tag{1}$$

are finite.

[In other words, $\lambda^1_{G'}$ is proper with respect to the discrete topology.]

If k is a number field, this follows from Theorem 7.1 in [5]. From now on k is a function field. Let N be the (scheme theoretic) kernel of the central isogeny $\iota : G \to G'$. It is a finite group scheme of multiplicative type, contained in any maximal torus of G. By definition, we have an exact sequence

$$1 \to N \to G \to G' \to 1 \tag{2}$$

and, similarly, if T is a maximal k-torus of G and $T' = \iota(T)$, an exact sequence

$$1 \to N \to T \to T' \to 1. \tag{3}$$

By [15] and [8: III],

$$H^1(k, G) = 0 = H^1(k_v, G) \quad (v \in V). \tag{4}$$

From this and the exact sequence associated to (2),

$$\ldots \to H^1(k, G) \to H^1(k, G') \xrightarrow{\delta} H^2(k, N), \tag{5}$$

it follows that δ is injective. At first, it shows only that $\delta^{-1}(0)$ is the zero element. But the case of an arbitrary fibre of δ is reduced to the previous one by the familiar trick of twisting by a cocycle c representing a given element of $H^1(k, G')$ and replacing the original exact sequence (2) by

$$1 \to N \to G_c \to G'_c \to 1,$$

noting that G_c is also semisimple and simply connected. See e.g. [5: 1.10], in the Galois cohomology case, i.e., if N is reduced. But all this formalism is also available in the flat cohomology case, as is shown in much greater generality in [13: IV, 4.3.4].

Similarly, $\delta_v : H^1(k_v, G') \to H^2(k_v, N)$ is injective. Since $H^2(k_v, N)$ is finite (see Proposition 78 in [34]), this shows that $H^1(k_v, G')$ is finite. [This had already been pointed out by J.-C. Douai, *C. R. Acad. Sci. Paris*, **280** (1975), 321-323, who has showed moreover that δ_v is bijective, but we shall not need this result.]

As a consequence, we are reduced to showing that the fibres of the analogous map

(6) $$\lambda^2_{\mathrm{N}} : \mathrm{H}^2(k, \mathrm{N}) \to \prod_{v \in \mathrm{V}} \mathrm{H}^2(k_v, \mathrm{N})$$

are finite. But we now deal with commutative groups, so this amounts to proving that $\ker \lambda^2_{\mathrm{N}}$ is finite. We consider the following commutative diagram with exact rows associated to the exact sequence (3):

$$\begin{array}{ccccccc} \mathrm{H}^1(k,\mathrm{T}) & \xrightarrow{\alpha} & \mathrm{H}^1(k,\mathrm{T}') & \xrightarrow{\beta} & \mathrm{H}^2(k,\mathrm{N}) & \xrightarrow{\gamma} & \mathrm{H}^2(k,\mathrm{T}) \\ \downarrow{\scriptstyle \lambda^1_{\mathrm{T}}} & & \downarrow{\scriptstyle \lambda^1_{\mathrm{T}'}} & & \downarrow{\scriptstyle \lambda^2_{\mathrm{N}}} & & \downarrow{\scriptstyle \lambda^2_{\mathrm{T}}} \\ \prod_v \mathrm{H}^1(k_v,\mathrm{T}) & \xrightarrow{\tilde{\alpha}} & \prod_v \mathrm{H}^1(k_v,\mathrm{T}') & \xrightarrow{\tilde{\beta}} & \prod_v \mathrm{H}^2(k_v,\mathrm{N}) & \xrightarrow{\tilde{\gamma}} & \prod_v \mathrm{H}^2(k_v,\mathrm{T}). \end{array}$$

By [28: IV, 2.7], the kernel of λ^2_{T} is finite. This reduces our task to proving that $\mathrm{M} = \ker \gamma \cap \ker \lambda^2_{\mathrm{N}}$ is finite. An element $x \in \mathrm{M}$ is the image of some element $y \in \mathrm{H}^1(k, \mathrm{T}')$ such that $\lambda^1_{\mathrm{T}'}(y)$ belongs to the kernel of $\tilde{\beta}$, hence to the image of $\tilde{\alpha}$. Recall that for a connected smooth group scheme, the image of the localization map λ^1 belongs to the subset of elements all but finitely many components of which are zero; following [28] we denote it by $\coprod_v$. By §2.6 in [28: IV], the kernels and cokernels of

$$\lambda^1_{\mathrm{T}} : \mathrm{H}^1(k, \mathrm{T}) \to \coprod_v \mathrm{H}^1(k_v, \mathrm{T}) \quad \text{and} \quad \lambda^1_{\mathrm{T}'} : \mathrm{H}^1(k, \mathrm{T}') \to \coprod_v \mathrm{H}^1(k_v, \mathrm{T}')$$

are finite. By diagram chasing, we see that the set of possible y's is finite modulo the image of α and the (finite) kernel of $\lambda^1_{\mathrm{T}'}$. Its image under β is therefore finite, as was to be proved.

B.2. *Corollary. — Let* $\mathscr{R}$ *be a finite subset of* V. *Then the kernel of the map*

$$\lambda^1_{\mathrm{G}', \mathscr{R}} : \mathrm{H}^1(k, \mathrm{G}') \to \prod_{v \notin \mathscr{R}} \mathrm{H}^1(k_v, \mathrm{G}')$$

is finite.

This follows from B.1 and the fact that $\mathrm{H}^1(k_v, \mathrm{G}')$ is finite (see [5] in characteristic zero, and the previous proof otherwise).

Appendix C: Verification of the inequalities $f_v > 1$ and $f_v^{\mathrm{EP}} > 1$

C.1. In this appendix, we use the notation of §7 freely. Our goal is to check the assertion 7.4 (15), namely

(1) $$f_v > 1 \quad (v \in \mathrm{V}_f),$$

(2) $$f_v^{\mathrm{EP}} > 1 \quad \text{unless G is of type } \mathsf{A}_2 \quad \text{and} \quad q_v \leqslant 3 \quad (v \in \mathrm{V}_f).$$

If $v \notin \mathrm{S}_f \cup \mathrm{T(G)}$, then (see 7.4 (13)) f_v and f_v^{EP} are both equal to e_v^{m}, which is > 1 by 3.7 (1). If $v \in \mathrm{T(G)}$, then $f_v = f_v^{\mathrm{EP}}$ by 7.4 (12). We have therefore to consider f_v for $v \in \mathrm{S}_f \cup \mathrm{T(G)}$ and f_v^{EP} for $v \in \mathrm{S}_f$.

C.2. *Proof of* (1) *for* $v \in \mathrm{T(G)}$. In that case

$$f_v = f_v^{\mathrm{EP}} = e_v^{\mathfrak{m}}\, n^{-\varepsilon}.$$

Now recall from 3.7 (5) that

$$e_v^{\mathfrak{m}} \geqslant (q_v + 1)^{-1}\, q_v^{r_v+1}.$$

(i) G *is of type* B, C, or E_7: Then $r_v = r$, $n^\varepsilon = 2$ and

$$f_v \geqslant (q_v + 1)^{-1}\, q_v^{r_v+1}\, 2^{-1} > 1.$$

(ii) G *is of type* E_6: Then $n^\varepsilon = 3$ and $r_v = 6$, hence

$$f_v \geqslant (q_v + 1)^{-1}\, q_v^{7}\, 3^{-1} > 1.$$

(iii) G *is of type* D_r: Then $r_v = r \geqslant 4$ and $n^\varepsilon = 4$, so

$$f_v \geqslant (q_v + 1)^{-1}\, q_v^{r_v+1}\, 4^{-1} \geqslant (q_v + 1)^{-1}\, q_v^{r_v-1} > 1.$$

(iv) G *is a form of type* A_r *(which splits over* $\hat{k}_v$ *since* $v \in \mathrm{T(G)}$*)*: Then $r_v = r$, $n^\varepsilon = r + 1$ and

$$f_v \geqslant (q_v + 1)^{-1}\, q_v^{r+1}(r + 1)^{-1} > q_v^{r-1}(r + 1)^{-1} \geqslant 1 \quad \text{if } r \geqslant 3.$$

If $r = 2$, then, as $v \in \mathrm{T(G)}$, G/k_v is anisotropic and $\mathrm{G}(k_v) \cong \mathrm{SL}_1(\mathfrak{D}_v)$, where $\mathfrak{D}_v$ is a division algebra of degree 3. By the inequality in 3.7 (6),

$$e_v^{\mathfrak{m}} \geqslant (q_v - 1)\, q_v^2,$$

so

$$f_v \geqslant (q_v - 1)\, q_v^2\, 3^{-1} > 1.$$

C.3. Now let us assume that $v \in \mathrm{S}_f$. Then $f_v = e_v\, n^{-\varepsilon}$ if $\ell_v = \ell \otimes_k k_v$ is a ramified field extension of k_v, and $f_v = e_v\, n^{-2\varepsilon}$ otherwise.

Recall from 3.7 (2) that

$$e_v \geqslant (q_v + 1)^{-r_v}\, q_v^{r_v(r_v+3)/2}.$$

(i) G *is of type* B, C *or* E_7: Then $n^\varepsilon = 2$, and $r_v = r \geqslant 2$. So

$$e_v\, n^{-2\varepsilon} \geqslant (q_v + 1)^{-r}\, q_v^{r(r+3)/2}\, 2^{-2}$$
$$> q_v^{r(r-1)/2}\, 2^{-2} > 1 \quad \text{if } r \geqslant 3.$$

If $r = 2$, then G is of type B_2 and we need to use the exact value of e_v:

$$e_v = (q_v - 1)^{-2}\, q_v^6 \quad \text{if G splits over } k_v,$$
$$e_v = (q_v^2 - 1)^{-1}\, q_v^6 \quad \text{if G is of rank 1 over } k_v.$$

In both cases $e_v > q_v^4$ and

$$f_v = e_v\, 2^{-2} > q_v^4\, 2^{-2} > 1.$$

(ii) G *is of type* E_6: Then $n^\varepsilon = 3$, r_v equals 4 or 6 and

$$f_v \geqslant (q_v + 1)^{r_v}\, q_v^{r_v(r_v+3)/2}\, 3^{-2} > 1.$$

(iii) G *is of type* D_r: Then $r_v \geqslant 2$.

a) If $r_v = 2$, $G/\hat{k}_v$ is a triality form of type D_4. In this case,

$$e_v = (q_v - 1)^{-2}\, q_v^8$$

and
$$f_v \geqslant e_v\, n^{-2\varepsilon} = (q_v - 1)^{-2}\, q_v^8\, 4^{-2} > 1.$$

b) If $G/\hat{k}_v$ is not a triality form, then $r_v \geqslant 3$ and, using 3.7 (2), we get

$$f_v \geqslant e_v\, n^{-2\varepsilon} > (q_v + 1)^{-r_v}\, q_v^{r_v(r_v+3)/2}\, 4^{-2},$$

which is > 1 if $r_v > 3$. On the other hand, if $r_v = 3$, then G is of type D_4 and

$$e_v = (\# \overline{T}_v(\mathfrak{f}_v))^{-1}\, q_v^{(r_v + \dim \overline{\mathscr{M}}_v)/2} \geqslant (q_v + 1)^{-3}\, q_v^{12}$$

(note that $\overline{\mathscr{M}}_v$ is a group of type B_3; therefore, its dimension is 21) and

$$f_v \geqslant e_v\, n^{-2\varepsilon} > (q_v + 1)^{-3}\, q_v^{12}\, 4^{-2} > 1.$$

(iv) *a)* G *is of type* A_r *and splits over* $\hat{k}_v$: Then $r_v = r$, $n^{\varepsilon} = r + 1$ and, by 3.7 (2),

$$e_v \geqslant (q_v + 1)^{-r}\, q_v^{r(r+3)/2},$$

so
$$f_v \geqslant e_v\, n^{-2\varepsilon} \geqslant (q_v + 1)^{-r}\, q_v^{r(r+3)/2}(r + 1)^{-2} > 1 \quad \text{if } r > 2.$$

Let now $r = 2$, G/k_v must be isotropic since $v \in S_f$. Then

$$e_v = (q_v - 1)^{-2}\, q_v^5 \quad \text{if } G/k_v \text{ is of inner type } A_2,$$
$$e_v = (q_v^2 - 1)^{-1}\, q_v^5 \quad \text{if } G/k_v \text{ is of outer type } A_2,$$

and in both cases,

$$f_v = e_v\, n^{-2\varepsilon} = e_v\, 3^{-2} > 1 \quad \text{for all } q_v.$$

(iv) *b)* G *is of type* A_r, *not splitting over* $\hat{k}_v$: $f_v = e_v(r + 1)^{-1}$ in this case. $r_v = r/2$ if r is even, and equals $(r + 1)/2$ if r is odd. By the inequality in 3.7 (2),

$$f_v \geqslant (q_v + 1)^{-r_v}\, q_v^{r_v(r_v+3)/2}(r + 1)^{-1},$$

and it can be easily checked that the number on the right-hand side is greater than 1 if $r_v \geqslant 3$. Let now $r_v = 2$. Then $G/\hat{k}_v$ is an outer form of type A_2, A_3 or A_4. Making use of the fact that $\# \overline{T}_v(\mathfrak{f}_v) \leqslant (q_v + 1)^{r_v}$ ([31: 2.8]), we see from the equality in 3.7 (2) that, if $G/\hat{k}_v$ is of type ${}^2A_3$ or ${}^2A_4$, then

$$e_v \geqslant (q_v + 1)^{-2}\, q_v^6$$

so
$$f_v \geqslant (q_v + 1)^{-2}\, q_v^6\, 5^{-1} > 1 \quad \text{for all } q_v.$$

If $G/\hat{k}_v$ is of type ${}^2A_2$, then

$$e_v = (q_v - 1)^{-1}\, q_v^2$$

and
$$f_v = (q_v - 1)^{-1}\, q_v^2\, 3^{-1} > 1 \quad \text{for all } q_v.$$

C.4. We now take up the verification of $f_v^{\text{EP}} > 1$ for $v \in S_f$ and G not of type A_2.

(i) G *is of type* B, C, or E_7: Then $r_v \geqslant 2$, $n^{\mathfrak{e}} = 2$, and

$$f_v^{\text{EP}} \geqslant e_v\, 2^{-2}\, |\, W_v(\mathbf{q}^{-1})\,|^{-1},$$

where

$$e_v = (\#\overline{T}_v(\mathfrak{f}_v))^{-1}\, q_v^{(r_v + \dim \overline{\mathscr{M}}_v)/2}.$$

Now recall that $\#\overline{T}_v(\mathfrak{f}_v) \leqslant (q_v + 1)^{r_v}$ ([31: 2.8]), $|\, W_v(\mathbf{q}^{-1})\,|^{-1} \geqslant 5^{-r_v}$ (4.4) and $\overline{\mathscr{M}}_v$ is an absolutely almost simple group of the same type as G. As G is of type B, C, or E, we conclude that $\dim \overline{\mathscr{M}}_v \geqslant r_v(2r_v + 1)$, and so

$$f_v^{\text{EP}} \geqslant (q_v + 1)^{-r_v}\, q_v^{r_v(r_v+1)}\, 2^{-2}\, 5^{-r_v} > 1 \quad \text{if } r_v \geqslant 4.$$

Let $r_v = 3$. If G/k_v is split, then

$$e_v = (q_v - 1)^{-3}\, q_v^{12}$$

and, as

$$|\, W_v(\mathbf{q}^{-1})\,|^{-1} \geqslant \left(\frac{(q_v - 1)^2}{q_v^2 + 1}\right)^3 \quad (4.4),$$

we get

$$f_v^{\text{EP}} \geqslant (q_v - 1)^{-3}\, q_v^{12}\, 2^{-2} \left(\frac{(q_v - 1)^2}{q_v^2 + 1}\right)^3 = \frac{2^{-2}\, q_v^{12}(q_v - 1)^3}{(q_v^2 + 1)^3} > 1.$$

If G/k_v is a form of B_3 of relative rank 2, then

$$\#\overline{T}_v(\mathfrak{f}_v) = (q_v - 1)^2\, (q_v + 1),$$

and so

$$f_v^{\text{EP}} \geqslant (q_v - 1)^{-2}\, (q_v + 1)^{-1}\, q_v^{12}\, 2^{-2} \left(\frac{(q_v - 1)^2}{q_v^2 + 1}\right)^3 = \frac{2^{-2}\, q_v^{12}(q_v - 1)^4}{(q_v^2 + 1)^3\, (q_v + 1)} > 1$$

for all q_v.

If G/k_v is a form of type C_3 of relative rank 1, then

$$\#\overline{T}_v(\mathfrak{f}_v) = (q_v + 1)^2\, (q_v - 1),$$

and

$$|\, W_v(\mathbf{q}^{-1})\,|^{-1} = (q_v^5 - 1)\, (q_v^3 + 1)^{-1}\, (q_v^2 + 1)^{-1}$$

so

$$f_v^{\text{EP}} = \frac{q_v^{12}(q_v^5 - 1)}{4(q_v^3 + 1)\, (q_v^2 + 1)\, (q_v + 1)^2\, (q_v - 1)}$$

$$= \frac{q_v^{12}(q_v^4 + q_v^3 + q_v^2 + q_v + 1)}{4(q_v^3 + 1)\, (q_v^2 + 1)\, (q_v + 1)^2} > 1.$$

Let G now be of type B_2. If it is split over k_v, then

$$\#\overline{T}_v(\mathfrak{f}_v) = (q_v - 1)^2,$$

$$|\, W_v(\mathbf{q}^{-1})\,|^{-1} = (q_v^3 - 1)\, (q_v - 1)\, (q_v^2 + 1)^{-1}\, (q_v + 1)^{-2},$$

and if it is a form of type B_2 of k_v-rank 1,

$$\#\overline{T}_v(\mathfrak{f}_v) = (q_v^2 - 1)$$

$$|\, W_v(\mathbf{q}^{-1})\,|^{-1} = (q_v^3 - 1)\, (q_v^2 + 1)^{-1}\, (q_v + 1)^{-1}$$

and so

$$f_v^{\text{EP}} = 2^{-2}\, q_v^6(q_v^2 + q_v + 1)\, (q_v^2 + 1)^{-1}\, (q_v + 1)^{-2} > 1,$$

in both cases.

(ii) G *is of type* D_r:

a) $G/\hat{k}_v$ is a triality form. Then

$$e_v = (q_v - 1)^{-2}\, q_v^8,$$
$$|\, W_v(\mathbf{q}^{-1})\,|^{-1} = (q_v^5 - 1)\,(q_v - 1)^2\,(q_v^6 - 1)^{-1}\,(q_v + 1)^{-1},$$

so
$$f_v^{\text{EP}} \geqslant 4^{-2}\, q_v^8(q_v^5 - 1)\,(q_v^6 - 1)^{-1}\,(q_v + 1)^{-1} > 1.$$

b) If $G/\hat{k}_v$ is not a triality form, then $r_v \geqslant 3$, $\overline{T}_v(\mathfrak{f}_v) \leqslant (q_v + 1)^{r_v}$ ([31: 2.8]); $\overline{\mathscr{M}}_v$ is of type D_r if G splits over $\hat{k}_v$ and in this case the dimension of $\overline{\mathscr{M}}_v$ is $r(2r - 1)$. If G is not split over $\hat{k}_v$, $r_v = r - 1$ and $\overline{\mathscr{M}}_v$ is of type B_{r-1}, its dimension is $(2r - 1)\,(r - 1)$.

We take up first the case where G (is of type D_r and) splits over $\hat{k}_v$. Then

$$e_v \geqslant (q_v + 1)^{-r}\, q_v^{r^2}$$

and so
$$f_v^{\text{EP}} \geqslant q_v^{r^2}(q_v + 1)^{-r}\, 4^{-2}\, 5^{-r} > 1 \quad \text{if } r \geqslant 5.$$

Let us assume now that $r = 4$. Then there are the following possibilities for G/k_v.

(1) G splits over k_v. In this case $\#\overline{T}_v(\mathfrak{f}_v) = (q_v - 1)^4$ and

$$e_v = (q_v - 1)^{-4}\, q_v^{16},$$

therefore,

$$f_v^{\text{EP}} \geqslant q_v^{16}(q_v - 1)^{-4}\, 4^{-2}\, 5^{-4} > 1$$

for all q_v.

(2) G/k_v is of type ${}^2\mathsf{D}^{(1)}_{4,3}$ (and it splits over $\hat{k}_v$). In this case

$$\#\overline{T}_v(\mathfrak{f}_v) = (q_v^2 - 1)\,(q_v - 1)^2,$$

so
$$e_v = (q_v^2 - 1)^{-1}\,(q_v - 1)^{-2}\, q_v^{16},$$

and hence, for all q_v,

$$f_v^{\text{EP}} \geqslant q_v^{16}(q_v^2 - 1)^{-1}\,(q_v - 1)^{-2}\, 4^{-2}\, 5^{-4} > 1.$$

(3) G/k_v is of type ${}^1\mathsf{D}^{(1)}_{4,2}$ (and it splits over $\hat{k}_v$). Then $\#\overline{T}_v(\mathfrak{f}_v) = (q_v^2 - 1)^2$, so $e_v = (q_v^2 - 1)^{-2}\, q_v^{16}$. In this case we need to know the precise value of $|\, W_v(\mathbf{q}^{-1})\,|^{-1}$, which is

$$(q_v^5 - 1)\,(q_v^3 - 1)\,(q_v^3 + 1)^{-1}\,(q_v^2 + 1)^{-2}\,(q_v + 1)^{-1}.$$

Hence

$$f_v^{\text{EP}} = 4^{-2}\, q_v^{16}(q_v^5 - 1)\,(q_v^3 - 1)\,(q_v^4 - 1)^{-2}\,(q_v^3 + 1)^{-1}\,(q_v + 1)^{-1} > 1$$

for all q_v.

(4) G/k_v is of type ${}^3\mathsf{D}_{4,2}$ (and it splits over $\hat{k}_v$). Then $\#\overline{T}_v(\mathfrak{f}_v) = (q_v^3 - 1)\,(q_v - 1)$, hence

$$e_v = (q_v^3 - 1)^{-1}\,(q_v - 1)^{-1}\, q_v^{16}.$$

In this case

$$|\, W_v(\mathbf{q}^{-1})\,|^{-1} = (q_v^9 - 1)\,(q_v^5 - 1)\,(q_v^2 + 1)\,(q_v - 1)^2\,(q_v^6 + 1)^{-1}\,(q_v^6 - 1)^{-2}.$$

Hence, for all q_v,

$$f_v^{\text{EP}} = \frac{q_v^{16}(q_v^9 - 1)(q_v^5 - 1)(q_v^2 + 1)(q_v - 1)}{4^2(q_v^6 + 1)(q_v^6 - 1)^2(q_v^3 - 1)} > 1.$$

(5) G/k_v is of type ${}^2\mathsf{D}_{4,1}^{(2)}$ (and it splits over $\hat{k}_v$). In this case $\#\overline{T}_v(\mathfrak{f}_v) = (q_v^4 - 1)$

$$e_v = (q_v^4 - 1)^{-1} q_v^{16}$$

$$|W_v(\mathbf{q}^{-1})|^{-1} = (q_v^5 - 1)(q_v^4 + 1)^{-1}(q_v + 1)^{-1}$$

and so, for all q_v,

$$f_v^{\text{EP}} = 4^{-2} q_v^{16}(q_v^5 - 1)(q_v^8 - 1)^{-1}(q_v + 1)^{-1} > 1.$$

c) Let us assume that G is a form of type D_r which does not split over $\hat{k}_v$. Then $r_v = r - 1$ and $\overline{\mathscr{M}}_v$ is of type B_{r-1}. Therefore

$$\dim \overline{\mathscr{M}}_v = (2r - 1)(r - 1) \quad \text{and} \quad e_v \geqslant (q_v + 1)^{-(r-1)} q_v^{r(r-1)},$$

so

$$f_v^{\text{EP}} \geqslant 4^{-1} q_v^{r(r-1)}(q_v + 1)^{-(r-1)} 5^{-(r-1)}.$$

From this it is easily seen that if $r \geqslant 5$, then $f_v^{\text{EP}} > 1$. Let $r = 4$. Then

$$\#\overline{T}_v(\mathfrak{f}_v) = (q_v - 1)^3 \quad \text{if G is of rank 3 over } k_v$$

and

$$\#\overline{T}_v(\mathfrak{f}_v) = (q_v^2 - 1)(q_v + 1) \quad \text{if G is of rank 1 over } k_v.$$

In the first case

$$f_v^{\text{EP}} \geqslant 4^{-1}(q_v - 1)^{-3} q_v^{12} 5^{-3} > 1 \quad \text{for all } q_v.$$

In the second case, we need to know the value of $|W_v(\mathbf{q}^{-1})|^{-1}$ which is

$$(q_v^5 - 1)(q_v^3 + 1)^{-1}(q_v^2 + 1)^{-1}.$$

We get

$$f_v^{\text{EP}} \geqslant 4^{-1} q_v^{12}(q_v^5 - 1)(q_v^4 - 1)^{-1}(q_v^3 + 1)^{-1}(q_v + 1)^{-1} > 1 \quad \text{for all } q_v.$$

(iii) *Let G/k_v be an inner form of type* A_r: Let $\mathfrak{D}_v$ be the central division algebra such that $G(k_v) \cong SL_{n_v}(\mathfrak{D}_v)$ and d_v be its degree. Then $d_v n_v = r + 1$,

$$\#\overline{T}_v(\mathfrak{f}_v) = (q_v^{d_v} - 1)^{n_v}(q_v - 1)^{-1}$$

so

$$e_v = (q_v^{d_v} - 1)^{-n_v}(q_v - 1) q_v^{r(r+3)/2}.$$

As

$$|W_v(\mathbf{q}^{-1})|^{-1} = (q_v^{d_v} - 1)^{n_v}(q_v^{r+1} - 1)^{-1},$$

we have

$$f_v^{\text{EP}} = (r + 1)^{-2} q_v^{r(r+3)/2}(q_v - 1)(q_v^{r+1} - 1)^{-1},$$

which is easily seen to be greater than 1 if either $r > 2$ or $q_v \geqslant 3$, and less than 1 if $r = 2$ and $q_v = 2$.

(iv) *Let G/k_v be an outer form of type A_r which splits over $\hat{k}_v$*: In this case $r_v = r$ and

$$e_v \geqslant (q_v + 1)^{-r} q_v^{r(r+3)/2}.$$

So

$$f_v^{\text{EP}} \geqslant (r + 1)^{-2} 5^{-r}(q_v + 1)^{-r} q_v^{r(r+3)/2}.$$

This implies that $f_v^{\mathrm{EP}} > 1$ if $r > 6$. If $r = 6$, then $\# \overline{\mathrm{T}}_v(\mathfrak{f}_v) = (q_v^2 - 1)^3$,

$$e_v = (q_v^2 - 1)^{-3} q_v^{27}$$

and

$$f_v^{\mathrm{EP}} \geqslant 7^{-2}\, 5^{-6} (q_v^2 - 1)^{-3} q_v^{27} > 1 \quad \text{for all } q_v.$$

If $r = 5$, $\# \overline{\mathrm{T}}_v(\mathfrak{f}_v) = (q_v^2 - 1)^2 (q_v - 1)$ or $(q_v^2 - 1)^2 (q_v + 1)$ depending on whether the k_v-rank of G is 3 or 2;

$$e_v = (q_v^2 - 1)^{-2} (q_v - 1)^{-1} q_v^{20}$$

in the first case, and

$$e_v = (q_v^2 - 1)^{-2} (q_v + 1)^{-1} q_v^{20}$$

in the second case. As

$$f_v^{\mathrm{EP}} \geqslant 6^{-2}\, 5^{-5}\, e_v,$$

we conclude that in the first case $f_v^{\mathrm{EP}} > 1$. In the second case, the value of $|\, \mathrm{W}_v(\mathbf{q}^{-1}) \,|^{-1}$ is

$$(q_v^8 - 1)(q_v^5 - 1)(q_v^5 + 1)^{-1}(q_v^3 + 1)^{-2}(q_v^2 + 1)^{-1}.$$

It is now simple to see that, for all q_v,

$$f_v^{\mathrm{EP}} = 6^{-2}\, e_v \,|\, \mathrm{W}_v(\mathbf{q}^{-1}) \,|^{-1} > 1.$$

Let now $r = 4$. Then

$$\# \overline{\mathrm{T}}_v(\mathfrak{f}_v) = (q_v^2 - 1)^2, \qquad |\, \mathrm{W}_v(\mathbf{q}^{-1}) \,|^{-1} = \frac{(q_v^4 + 1)(q_v^3 - 1)(q_v^2 - 1)}{(q_v^5 + 1)(q_v^3 + 1)(q_v + 1)}$$

and

$$e_v = (q_v^2 - 1)^{-2} q_v^{14}.$$

So

$$f_v^{\mathrm{EP}} = \frac{5^{-2}\, q_v^{14} (q_v^4 + 1)(q_v^3 - 1)}{(q_v^5 + 1)(q_v^3 + 1)(q_v^2 - 1)(q_v + 1)} > 1.$$

We assume now that $r = 3$. Then $\# \overline{\mathrm{T}}_v(\mathfrak{f}_v) = (q_v^2 - 1)(q_v - 1)$ if k_v-rank $\mathrm{G} = 2$ and $\# \overline{\mathrm{T}}_v(\mathfrak{f}_v) = (q_v^2 - 1)(q_v + 1)$ if k_v-rank $\mathrm{G} = 1$. In the first case

$$|\, \mathrm{W}_v(\mathbf{q}^{-1}) \,|^{-1} = (q_v^3 - 1)(q_v - 1)(q_v^3 + 1)^{-1}(q_v + 1)^{-1}$$

and in the second case it is equal to $(q_v^3 - 1)(q_v^3 + 1)^{-1}$. So, in both cases,

$$f_v^{\mathrm{EP}} = 4^{-2}\, q_v^9 (q_v^3 - 1)(q_v^2 - 1)^{-1}(q_v^3 + 1)^{-1}(q_v + 1)^{-1} > 1.$$

Let $r = 2$. Then $\# \overline{\mathrm{T}}_v(\mathfrak{f}_v) = (q_v^2 - 1)$, $|\, \mathrm{W}_v(\mathbf{q}^{-1}) \,|^{-1} = (q_v^2 + 1)(q_v - 1)(q_v^3 + 1)^{-1}$ and

$$f_v^{\mathrm{EP}} = 3^{-2}\, q_v^5 (q_v^2 + 1)(q_v^3 + 1)^{-1}(q_v + 1)^{-1}$$

which is > 1 if $q_v \geqslant 3$ and < 1 if $q_v = 2$.

(v) *Let* G/k_v *be an outer form of type* A_r *which does not split over* $\hat{k}_v$ *and* $r > 2$:

If $r = 2n$, $\overline{\mathscr{M}}_v$ is an absolutely almost simple group of type B_n, its dimension is $n(2n+1)$,

$$e_v = (q_v - 1)^{-n}\, q_v^{n(n+1)}$$

and

$$f_v^{\mathrm{EP}} \geqslant (2n+1)^{-1}\,(q_v-1)^{-n}\, q_v^{n(n+1)}\, 5^{-n},$$

so $f_v^{\mathrm{EP}} > 1$ if $n \geqslant 3$. If $n = 2$,

$$|\,W_v(\mathbf{q}^{-1})\,|^{-1} = (q_v^3 - 1)\,(q_v - 1)\,(q_v^2+1)^{-1}\,(q_v+1)^{-2},$$

so, in this case, for all q_v,

$$f_v^{\mathrm{EP}} = 5^{-1}\, q_v^6 (q_v^2 + q_v + 1)\,(q_v^2+1)^{-1}\,(q_v+1)^{-2} > 1.$$

Now let $r = 2n+1$ $(n \geqslant 1)$. In this case the group $\overline{\mathscr{M}}_v$ is of type C_{n+1}, its dimension is $(n+1)(2n+3)$. Moreover $\#\overline{T}_v(\mathfrak{f}_v) = (q_v-1)^{n+1}$ if k_v-rank $(G) = n+1$ and $\#\overline{T}_v(\mathfrak{f}_v) = (q_v-1)^{n}\,(q_v+1)$ if k_v-rank $(G) = n$. So

$$f_v^{\mathrm{EP}} \geqslant (2n+2)^{-1}\,(q_v-1)^{-n-1}\, q_v^{(n+1)(n+2)}\, 5^{-n-1}$$

in the first case; in the second case

$$f_v^{\mathrm{EP}} \geqslant (2n+2)^{-1}\,(q_v-1)^{-n}\,(q_v+1)^{-1}\, q_v^{(n+1)(n+2)}\, 5^{-n-1}.$$

In both cases, $f_v^{\mathrm{EP}} > 1$ if $n \geqslant 2$. So let us assume that $n = 1$. Then, G/k_v is an outer form of type A_3 which does not split over $\hat{k}_v$, and we have

$$|\,W_v(\mathbf{q}^{-1})\,|^{-1} = (q_v^3 - 1)\,(q_v - 1)\,(q_v^2+1)^{-1}\,(q_v+1)^{-2}$$

if k_v-rank $(G) = 2$, and

$$|\,W_v(\mathbf{q}^{-1})\,|^{-1} = (q_v^3 - 1)\,(q_v^2+1)^{-1}\,(q_v+1)^{-1}$$

if k_v-rank $G = 1$. So

$$f_v^{\mathrm{EP}} = 4^{-1}\, q_v^6 (q_v^2 + q_v + 1)\,(q_v^2+1)^{-1}\,(q_v+1)^{-2} > 1$$

in both cases.

(vi) If G is of type E_6, E_8, F_4 or G_2, the verification of $f_v^{\mathrm{EP}} > 1$ is easy.

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