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# $p$-adic etale cohomology

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# $p$-ADIC ETALE COHOMOLOGY

*by* SPENCER BLOCH* *and* KAZUYA KATO

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## 0. Global introduction

The purpose of this paper is to apply the results of [12], to the global study of $p$-adic etale cohomology and the associated $p$-adic Galois representations. We fix a field K of characteristic 0 which is complete with respect to a discrete valuation, with residue field $k$ of characteristic $p > 0$ and valuation ring $\Lambda$. The generic (resp. special) point of $S = \operatorname{Sp} \Lambda$ is denoted $\eta$ (resp. $s$). We consider a diagram of schemes

$$\begin{array}{ccccc} V = X_\eta & \xrightarrow{j} & X & \xleftarrow{i} & X_s = Y \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Sp} K = \eta & \longrightarrow & S = \operatorname{Sp} \Lambda & \longleftarrow & s = \operatorname{Sp} k \end{array} \tag{0.1}$$

with all vertical arrows smooth and proper. A bar will either indicate algebraic or integral closure (viz. $\overline{K}$, $\overline{\Lambda}$) or base extension ($\overline{X} = X_{\bar{s}} = X \times_S \overline{S}$, $\overline{V} = V_{\overline{K}}, \ldots$). Finally, $G = \operatorname{Gal}(\overline{K}/K)$ and $C_p \underset{\text{def}}{=} \hat{\overline{K}}$, the completion of $\overline{K}$.

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The basic global objects are the etale cohomology groups $H^*_{et}(\overline{V}, \mathbf{Q}_p)$, which we study using the spectral sequence

**(0.2)** $$E_2^{s,t} = H^s(\overline{Y}, \bar{i}^* R^t \bar{j}_*(\mathbf{Z}/p^\cdot \mathbf{Z})) \Rightarrow H^{s+t}_{et}(\overline{V}, \mathbf{Z}/p^\cdot \mathbf{Z}).$$

This spectral sequence induces a G-stable filtration

**(0.3)** $$F^\cdot(H^*_{et}(\overline{V}, \mathbf{Z}_p))$$

such that $F^0 H^q = H^q$ and $F^n H^q = (0)$ for $n \geq q + 1$. We write

**(0.4)** $$\mathrm{gr}^n H^q = F^n H^q / F^{n+1} H^q.$$

Recall that one also has the de Rham-Witt cohomology [3], [10]

$$H^s(\overline{Y}, W\Omega^t_{\overline{Y}})$$

and the crystalline cohomology [2]

$$H^*_{crys}(\overline{Y}/W(\bar{k})) \quad (W(\bar{k}) = \text{Witt vectors over } \bar{k}),$$

which depend only on the special fibre $\overline{Y}$ and are linked via the slope spectral sequence

**(0.5)** $$E_1^{s,t} = H^t(\overline{Y}, W\Omega^s) \Rightarrow H^{s+1}_{crys}(\overline{Y}/W(\bar{k})).$$

$H^*_{crys}$ has a canonical endomorphism F (Frobenius) and we write

**(0.6)** $$H^q_{crys}(\overline{Y}/W(\bar{k}))^{(i)}_{\mathbf{Q}}$$

for the $p^i$-eigenspace of F on $H^q_{crys} \otimes \mathbf{Q}$. Roughly speaking we will say $\overline{Y}$ is *ordinary* if the rank of (0.6) equals the rank of the $\bar{k}$-vector space (Hodge group)

$$H^{q-i}(\overline{Y}, \Omega^i_{\overline{Y}})$$

for all $i$ and $q$. (This definition is not quite correct in the presence of torsion in $H^*_{crys}$. For a more detailed discussion see § 7 below.) An abelian variety of dimension $d$ is ordinary if and only if it has $p^d$ geometric points of order $p$.

By Deligne (unpublished but cf. [20], p. 143), ordinary hypersurfaces of any given degree make up an open dense set in the moduli space.

*Theorem* **(0.7)**. — *Let notation be as above and assume* $\overline{Y}$ *is ordinary. Then there exist functorial* G-*module isomorphisms*

(i) $\mathrm{gr}^{q-i} H^q_{et}(\overline{V}, \mathbf{Q}_p) \cong H^q_{crys}(\overline{Y}/W(\bar{k}))^{(i)}_{\mathbf{Q}}(-i)$
(ii) $\mathrm{gr}^{q-i} H^q_{et}(\overline{V}, \mathbf{Q}_p) \otimes_{\mathbf{Z}_p} W(\bar{k}) \cong H^{q-i}(\overline{Y}, W\Omega^i)(-i)_{\mathbf{Q}}$
(iii) $\mathrm{gr}^{q-i} H^q_{et}(\overline{V}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_p \cong H^{q-i}(V, \Omega^i_V) \otimes_K \mathbf{C}_p(-i)$.

(The notation $(-i)$ means twist $i$ times by the dual of the $p$-adic cyclotomic character on G. Also G acts in the natural way on $\mathbf{C}_p$.)

Recall that a $\mathbf{Q}_p[G]$-module M is said to admit a Hodge-Tate decomposition if the module $M \otimes_{\mathbf{Q}_p} \mathbf{C}_p$ with semi-linear G-action is isomorphic to a direct sum

$$\bigoplus_n M_n(n)$$

with $M_n \cong \mathbf{C}_p^{r_n}$ as a G-module. Assume now $k$ is perfect. Tate has shown [18] that the Tate module of a $p$-divisible group admits a Hodge-Tate decomposition. By using this fact, Tate and Raynaud proved that $H^1_{et}(\overline{V}, \mathbf{Q}_p)$ has the Hodge-Tate decomposition

$$H^1_{et}(\overline{V}, \mathbf{Q}_p) \otimes \mathbf{C}_p \cong [H^1(V, \mathcal{O}_V) \otimes_K \mathbf{C}_p] \oplus [H^0(V, \Omega^1_V) \otimes_K \mathbf{C}_p(-1)]$$

for any smooth proper variety V over K.

*Corollary* **(0.8)**. — *Assume* $\overline{Y}$ *ordinary and* $k$ *perfect. Then for all* $q$,

$$H^q_{et}(\overline{V}, \mathbf{Q}_p) \underset{\mathbf{C}_p}{\cong} \bigoplus_i (H^{q-i}(V, \Omega^i_V) \otimes_K \mathbf{C}_p(-i)),$$

*so* $H^q_{et}(\overline{V})$ *has a Hodge-Tate decomposition.*

The proof is straightforward from (0.7) (iii) together with the result of Tate:

**(0.9)** If $n \neq 0$, $H^0(G, \mathbf{C}_p)(n) = 0$. If $k$ is perfect and $n \neq 0$, then $H^1(G, \mathbf{C}_p(n)) = 0$.

We continue to assume now that $\overline{Y}$ is ordinary, and we suppose in addition that the residue field $k$ is separably closed (not necessary perfect). There is some geometric interest in considering the extensions

**(0.10)** $$0 \to gr^{i+1} H^q \to F^i H^q / F^{i+2} H^q \to gr^i H^q \to 0.$$

If $H^*_{crys}(\overline{Y})$ is torsion free, the isomorphisms of (0.7) exist before being tensored by $\mathbf{Q}$ (see (9.6)). Thus the extension class lies in

**(0.11)** $$\mathrm{Hom}_{\mathbf{Z}_p}(H^q_{crys}(\overline{Y})^{(q-i)}, H^q_{crys}(\overline{Y})^{(q-i-1)}) \otimes_{\mathbf{Z}_p} H^1(G, \mathbf{Z}_p(1)).$$

One has (cf. [19], prop. (2.2))

$$H^1(G, \mathbf{Z}_p(1)) \cong \varprojlim K^*/K^{*p^n} = \hat{K}^*,$$

the $p$-adic completion of the multiplicative group of K. If a basis $f_\beta$ for the Hom in (0.11) is fixed, one gets (dual) functions

$$f^*_\beta : \begin{Bmatrix} \text{liftings of Y} \\ \text{over } \Lambda \end{Bmatrix} \to \hat{K}^*.$$

This situation is understood in the case of abelian varieties (Katz [13]) and also K-3 surfaces (Deligne-Illusie [6]). It should be the case that the image of $f^*_\beta$ lands in the group of principal units $U^{(1)} \subset \hat{K}^*$. The $f^*_\beta$ could then reasonably be thought of as $p$-adic modular functions, i.e. as $p$-adic functions on the moduli space (in fact, on the " period space " !?).

Briefly, the content of the various sections of the paper are as follows. Sections 1-6 are local. § 1 describe the local setup and states the main local results, (0.4) (0.5). In section 2 we identify the mod $p$ Milnor K-theory of a field in characteristic $p$ with the group of logarithmic Kahler differentials. § 3 contains a lemma about Galois

cohomology which enables us to prove in § 5 that the Galois cohomology of a henselian field is expressed by Milnor K-theory. § 4 is preparatory, giving elementary properties of symbols which will be used in later sections. § 6 " sheafifies " these results. Sections 7, 8 and 9 are global. In § 7 we discuss *ordinary* varieties in characteristic $p$. We characterize these by the vanishing of all cohomology groups of sheaves of locally exact differentials. Finally, § 8 and § 9 are devoted to the proof of the main global result (0.7).

A summary of this work is published in [5].

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## 1. Local results

**(1.1)** Recall the situation of (0.1)

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d] & X \arrow[d] & Y \arrow[l, hook', "i"'] \arrow[d] \\ \operatorname{Sp} K \arrow[r] & \operatorname{Sp} \Lambda & \operatorname{Sp} k \arrow[l] \end{tikzcd}$$

but do not assume vertical arrows proper. In the local study § 1-§ 6, we are principally interested in the structures of the étale sheaves on Y;

$$M_n^q = i^* R^q j_*(\mathbf{Z}/p^n \mathbf{Z}(q)) \quad (n, q \geq 0).$$

These are localizations on Y of the $p$-adic étale cohomology of V in suitably twisted coefficients. For $y \in Y$, the stalk $M_{n,\bar{y}}^q$ is isomorphic to the étale cohomology group $H^q\left(\operatorname{Sp}\left(\mathcal{O}_{X,\bar{y}}\left[\frac{1}{p}\right]\right), \mathbf{Z}/p^n \mathbf{Z}(q)\right)$, where $\mathcal{O}_{X,\bar{y}}$ denotes the strict henselization of $\mathcal{O}_{X,y}$. In the case X is proper over $\Lambda$, the spectral sequence (0.2) relates the limit $\overline{M}_n^q = i^* R^q \bar{j}_*(\mathbf{Z}/p^n \mathbf{Z}(q))$ of $M_n^q$ to the $p$-adic étale cohomology $H^*(\overline{V}, \mathbf{Z}_p)$ of V.

We study $M_n^q$ by using symbols and a natural filtration. We shall see that $M_n^q$ is related to differential modules on X and Y, and to the De Rham-Witt complex on Y.

**(1.2)** First, we define the symbols. The exact sequence of Kummer on V

$$0 \longrightarrow \mathbf{Z}/p^n \mathbf{Z}(1) \longrightarrow \mathcal{O}_V^* \xrightarrow{p^n} \mathcal{O}_V^* \longrightarrow 0$$

induces an exact sequence on Y

$$i^* j_* \mathcal{O}_V^* \xrightarrow{p^n} i^* j_* \mathcal{O}_V^* \longrightarrow M_n^1 \longrightarrow 0.$$

For local sections $x_1, \ldots, x_q$ of $i^* j_* \mathcal{O}_V^*$, let $\{x_1, \ldots, x_q\}$ be the local section of $M_n^q$ defined as the cup product of the images of $x_i$ $(1 \leq i \leq q)$ in $M_n^1$. Then,

$$\{x, -x\} = 0, \quad \{x, y\} + \{y, x\} = 0, \quad \{z, 1 - z\} = 0$$

for any local sections $x, y, z$ such that $1 - z$ is invertible. (The proofs of these identities are essentially the same as Tate's proof of the existence of the cohomological symbol $K_2 \to H^2_{\mathrm{Gal}}( \ , \mathbf{Z}/p^n \mathbf{Z}(2))$ for fields. They also follow from Soulé's Chern class homomorphism $K_2 \to H^2( \ , \mathbf{Z}/p^n \mathbf{Z}(2))$ for rings and the corresponding identities in $K_2$ (cf. [17], [19]).)

Next we define the filtration of $M_n^q$. For $m \geq 1$, let $U^m M_n^q$ be the subsheaf of $M_n^q$ generated locally by local sections of the form $\{x_1, \ldots, x_q\}$ such that $x_1 - 1 \in \pi^m i^* \mathcal{O}_X$, where $\pi$ is a prime element of K. It is possible to compute the subquotients

$$\mathrm{gr}^m(M_n^q) = \begin{cases} M_n^q / U^1 M_n^q & (m = 0) \\ U^m M_n^q / U^{m+1} M_n^q & (m \geqslant 1) \end{cases}$$

for those $m$ such that $0 \leq m < e' = \dfrac{ep}{p-1}$, where $e$ denotes the absolute ramification index of K. If $n = 1$, $U^m M_1^q = 0$ for $m \geq e'$ and thus we obtain a precise picture of $M_1^q$. The result is very similar to the structure theorems of the K-theoretic sheaf $S\hat{C}K_q(\mathcal{O}_Y)$ and of the De Rham-Witt complex of Y (cf. Bloch [3], Illusie [10]). Indeed, if $0 < m < e'$,

$$\mathrm{gr}^m(M_n^q) \cong \mathrm{gr}^m(SC_\infty K_q(\mathcal{O}_Y)/p^n SC_\infty K_q(\mathcal{O}_Y))$$

for the filtration $\{U^m SC_\infty K_q\}_{m \geq 1}$ on $SC_\infty K_q$ which is defined by modifying the filtration fil* of [3] II, § 4, as

$$U^m SC_\infty K_q = \mathrm{fil}^{m-1} SC_\infty K_q + \{\mathrm{fil}^{m-1} SC_\infty K_{q-1}, T\}.$$

(Cf. also [11] § 2.)

But this precise analogy holds only in this range of $m$, and the structure of $\mathrm{gr}^m(M_n^q)$ for $m \geq e'$ has rather different aspects which are not yet well understood.

**(1.3)** Let $\Omega_Y^\bullet = \Omega_{Y/\mathbf{Z}}^\bullet$ be the exterior algebra over $\mathcal{O}_Y$ of the sheaf $\Omega_{Y/\mathbf{Z}}^1$ of absolute differentials on Y. If $k$ is perfect, this coincides with the usual $\Omega_{Y/k}^\bullet$, but is bigger than the latter in general. As in [3], [10], define subsheaves $B_i^q$ and $Z_i^q$ $(i \geq 0)$ of $\Omega_Y^q$ such that

$$0 = B_0^q \subset B_1^q \subset \ldots \subset Z_1^q \subset Z_0^q = \Omega_Y^q$$

by the relations

$$B_1^q = \mathrm{Image}(d : \Omega_Y^{q-1} \to \Omega_Y^q)$$
$$Z_1^q = \mathrm{Ker}(d : \Omega_Y^q \to \Omega_Y^{q+1})$$
$$B_i^q \xrightarrow[\cong]{C^{-1}} B_{i+1}^q / B_1^q, \quad Z_i^q \xrightarrow[\cong]{C^{-1}} Z_{i+1}^q / B_1^q$$

where $C^{-1}$ is the inverse Cartier operator:

$$\Omega^q_Y \underset{\cong}{\to} Z^q_1/B^q_1; \quad x\frac{dy_1}{y_1}\wedge\ldots\wedge\frac{dy_q}{y_q}\mapsto x^p\frac{dy_1}{y_1}\wedge\ldots\wedge\frac{dy_q}{y_q}$$

($y_1,\ldots,y_q$ invertible). Define

$$\Omega^q_{Y,\log} = \mathrm{Ker}(1 - C^{-1}: \Omega^q_Y \to \Omega^q_Y/B^q_1).$$

This is in fact the part of $\Omega^q_Y$ generated étale locally by local sections of the forms $\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_q}{x_q}$ ([10] Th. 02.4.2). Let $W_n\Omega^\bullet_Y$ be the De Rham-Witt complex of $Y_{et}$ and let $W_n\Omega^q_{Y,\log}$ be the part of $W_n\Omega^q_Y$ generated étale locally by local sections of the form $d\log(x_1)\ldots d\log(x_q)$. Note that, since all local rings of Y are inductive limits of smooth algebras over $F_p$, the theory of the De Rham-Witt complex over a perfect base ([10]) applies to $W_n\Omega^\bullet_Y$.

Our results are the following:

*Theorem* **(1.4).** — *The sheaves* $M^q_n$ *are generated locally by symbols, and*

(i) $\mathrm{gr}^0(M^q_n) \cong W_n\Omega^q_{Y,\log} \oplus W_n\Omega^{q-1}_{Y,\log}$.

(ii) *For* $m \geq 1$, *there is a surjective homomorphism*

$$\rho_m: \Omega^{q-1}_Y \oplus \Omega^{q-2}_Y \to \mathrm{gr}^m(M^q_n).$$

(iii) *Let* $1 \leqslant m < e' = \frac{ep}{p-1}$ *and let* $m = m_1 p^s$, $s \geq 0$, $p \nmid m_1$. *Then, for* $0 \leq n \leq s$ (*resp.* $n > s$), *the above homomorphism* $\rho_m$ *induces an isomorphism*

$$\Omega^{q-1}_Y/Z^{q-1}_n \oplus \Omega^{q-2}_Y/Z^{q-2}_n \underset{\cong}{\to} \mathrm{gr}^m(M^q_n)$$

(*resp. an exact sequence*

$$0 \to \Omega^{q-2}_Y \xrightarrow{\theta} \Omega^{q-1}_Y/B^{q-1}_s \oplus \Omega^{q-2}_Y/B^{q-2}_s \to \mathrm{gr}^m(M^q_n) \to 0,$$

*where* $\theta(\omega) = (C^{-s}(d\omega),\ (-1)^q m_1 C^{-s}(\omega))$, $C^{-s} = C^{-1}\circ\ldots\circ C^{-1}$ ($s$ times)).

*Corollary* **(1.4.1).** — *The sheaf* $M^q_1$ *has the following structure.*

(i) $\mathrm{gr}^0(M^q_1) \cong \Omega^q_{Y,\log} \oplus \Omega^{q-1}_{Y,\log}$.

(ii) *If* $1 \leq m < e'$ *and* $m$ *is prime to* $p$,

$$\mathrm{gr}^m(M^q_1) \cong \Omega^{q-1}_Y.$$

(iii) *If* $1 \leq m < e'$ *and* $p \mid m$,

$$\mathrm{gr}^m(M^q_1) \cong \Omega^{q-1}_Y/Z^{q-1}_1 \oplus \Omega^{q-2}_Y/Z^{q-2}_1.$$

(iv) *For* $m \geq e'$, $U^m M^q_1 = 0$.

The surjective homomorphism

$$M^q_n \to W_n\Omega^q_{Y,\log} \oplus W_n\Omega^{q-1}_{Y,\log}$$

given by (1.4) (i) is a homomorphism such that

$$\{\tilde{x}_1, \ldots, \tilde{x}_q\} \mapsto (d\log(x_1) \ldots d\log(x_q), 0)$$
$$\{\tilde{x}_1, \ldots, \tilde{x}_{q-1}, \pi\} \mapsto (0, d\log(x_1) \ldots d\log(x_{q-1})),$$

where $\pi$ is a fixed prime element of K, $x_1, \ldots, x_q$ are any local sections of $\mathcal{O}_Y^*$, and $\tilde{x}_i$ are any liftings of $x_i$ $(1 \leq i \leq q)$ to $i^* \mathcal{O}_X^*$. An analogous homomorphism is given by

*Theorem* **(1.5)**. — *There exists a unique homomorphism* $M_n^q \to \Omega^q_{X/S}/p^n \Omega^q_{X/S}$ *which satisfies*

$$\{f_1, \ldots, f_q\} \mapsto \frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_q}{f_q}$$
$$\{f_1, \ldots, f_{q-1}, c\} \mapsto 0$$

*for any local sections* $f_1, \ldots, f_q$ *of* $i^* \mathcal{O}_X^*$ *and for any* $c \in K^*$. *Here we regard* $\Omega^q_{X/S}/p^n \Omega^q_{X/S}$ *as a sheaf on* $Y_{et}$ *in the natural way.*

In conclusion, one might say that the $p$-adic étale cohomology $M_n^q$, the De Rham-Witt complex $W_n \Omega_Y^\bullet$, and the De Rham complex $\Omega^\bullet_{X/S}$, live in completely different worlds, and there is no unified cohomology theory at present which combine them in an intrinsic manner. We must therefore use some presentation of them by symbols in the study of their relations. It becomes clear that the symbols play important roles in the algebraic geometry of mixed characteristic, though we do not know from what world the symbols come.

## 2. The differential symbol

Let $K_*^M$ be the Milnor K-theory of fields [15].

For a field F of characteristic $p > 0$, we write

$$k_q(F) = K_q^M(F)/pK_q^M(F),$$
$$\nu_F^q = \mathrm{Ker}(\Omega_F^q \xrightarrow{1-C^{-1}} \Omega_F^q/d\Omega_F^{q-1}),$$
$$\psi = \psi^q : k_q(F) \to \nu_F^q; \quad \psi(\{x_1, \ldots, x_q\}) = \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_q}{x_q}.$$

The following result was proved independently by O. Gabber.

*Theorem* **(2.1)**. — $\psi$ *is an isomorphism.*

We give here the proof of the injectivity of $\psi$. The proof of the surjectivity is similar to the proof of Proposition (2.4) below and is given in [12], § 1.

We fix $q$ so that Theorem (2.1) holds for all $q' < q$. We use the method in [4].

*Lemma* **(2.2)**. — *If* $\psi^q$ *is injective for* F, *it is injective for any purely transcendental extension of* F.

This follows from the commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & k_q(\mathrm{F}) \arrow[r] \arrow[d, "\text{by } \psi"] & k_q(\mathrm{F}(t)) \arrow[r, "(\partial_m)_m"] \arrow[d, "\text{by } \psi"] & \coprod_m k_{q-1}(\mathrm{F}[t]/m) \arrow[r] \arrow[d, "(i_m)_m"] & 0 \\
0 \arrow[r] & \Omega^q_{\mathrm{F}[t]} \arrow[r] & \Omega^q_{\mathrm{F}(t)} \arrow[r] & \coprod_m \Omega^q_{\mathrm{F}(t)}/\Omega^q_{\mathrm{F}[t]_m} &
\end{tikzcd}$$

where $m$ ranges over all maximal ideals of $\mathrm{F}[t]$ and $\partial_m$ denotes the tame symbol for each $m$ ([1], Ch. I, §§ 4 and 5). The homomorphism $i_m$ is the composition of $k_{q-1}(\mathrm{F}[t]/m) \to \Omega^{q-1}_{(\mathrm{F}[t]/m)}$ with the canonical injective homomorphism $\Omega^{q-1}_{(\mathrm{F}[t]/m)} \to \Omega^q_{\mathrm{F}(\mathrm{T})}/\Omega^q_{\mathrm{F}[\mathrm{T}]_m}$, which is defined by

$$x_0\, dx_1 \wedge \ldots \wedge dx_{q-1} \mapsto \tilde{x}_0\, d\tilde{x}_1 \wedge \ldots \wedge d\tilde{x}_{q-1} \wedge \pi_m^{-1}\, d\pi_m$$

for any $x_0, \ldots, x_{q-1} \in \mathrm{F}$, any prime element $\pi_m$ at $m$, and for any lifting $\tilde{x}_i$ of $x_i$ $(1 \leq i \leq q-1)$.

*Corollary* **(2.2.1)**. — *$\psi^q$ is injective for* K *if* K *is purely transcendental over a perfect field.*

**(2.3)** For a semi-local Dedekind domain R with field of fractions K such that $\mathrm{char}(\mathrm{K}) = p > 0$, let

$$k_q(\mathrm{R}) = \mathrm{Ker}(k_q(\mathrm{K}) \xrightarrow{(\partial_m)_m} = \coprod_m k_{q-1}(\mathrm{R}/m)),$$

where $m$ ranges over all maximal ideals of R. Let I be the radical of R, let

$$k_q(\mathrm{R}) \to k_q(\mathrm{R}/\mathrm{I}) \underset{\mathrm{dfn}}{=} \coprod_m k_q(\mathrm{R}/m)$$

be the specialization map induced by the homomorphism in Lemma (2.3.2) below, and let $k_q(\mathrm{R}, \mathrm{I})$ be its kernel. Assume R has a $p$-base so that the Cartier and the inverse Cartier operators are defined, and let

$$\nu^q_{\mathrm{R}} = \mathrm{Ker}(1 - \mathrm{C}^{-1} : \Omega^q_{\mathrm{R}} \to \Omega^q_{\mathrm{R}}/d(\Omega^{q-1}_{\mathrm{R}})),$$
$$\nu^q_{\mathrm{R},\mathrm{I}} = \mathrm{Ker}(\nu^q_{\mathrm{R}} \to \nu^q_{\mathrm{R}/\mathrm{I}}).$$

By Lemma (2.3.2) below, we obtain a diagram (commutative with exact rows)

**(2.3.1)**
$$\begin{tikzcd}
0 \arrow[r] & k_q(\mathrm{R}, \mathrm{I}) \arrow[r] \arrow[d, "\psi"] & k_q(\mathrm{R}) \arrow[r] \arrow[d, "\psi"] & k_q(\mathrm{R}/\mathrm{I}) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \nu^q_{\mathrm{R},\mathrm{I}} \arrow[r] & \nu^q_{\mathrm{R}} \arrow[r] & \nu^q_{\mathrm{R}/\mathrm{I}} &
\end{tikzcd}$$

*Lemma* **(2.3.2)**. — *Let* R *be a discrete valuation ring with quotient field* K *and with residue field* F *such that* $\mathrm{char}(\mathrm{K}) = p > 0$.

(i) *$k_q(\mathrm{R})$ is generated by symbols $\{x_1, \ldots, x_q\}$ $(x_1, \ldots, x_q \in \mathrm{R}^x)$.*

(ii) *There is a unique homomorphism $k_q(\mathrm{R}) \to k_q(\mathrm{F})$ such that*

$$\{a_1, \ldots, a_q\} \mapsto \{\bar{a}_1, \ldots, \bar{a}_q\}.$$

(iii) *If R has a $p$-base, there is a unique homomorphism $\psi : k_q(\mathrm{R}) \to \nu^q_{\mathrm{R}}$ such that*

$$\{x_1, \ldots, x_q\} \mapsto \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_q}{x_q}.$$

*Proof.* — (i) follows from [1], I (4.5) *b)* and (ii) is the $\partial^0_\pi$ of (loc. cit.) (4.4). The homomorphism in (iii) is induced by $\psi : k_q(\mathrm{R}) \to \nu^q_{\mathrm{K}}$ by virtue of (i).

For a finitely generated field F over $\mathbf{F}_q$, we can find a discrete valuation ring R which is a local ring of a finitely generated algebra over $\mathbf{F}_p$, such that $\mathrm{R}/m \cong \mathrm{F}$ and such that the field of fractions K of R is purely transcendental over $\mathbf{F}_p$. Since $k_q(\mathrm{R}) \to \nu^q_{\mathrm{R}}$ is injective by Corollary (2.2.1), the diagram (2.3.1) shows that to prove (2.1) it suffices to prove

*Proposition* **(2.4)**. — *Let $k$ be a perfect field of characteristic $p > 0$, let R be a semi-local Dedekind domain which is obtained as a localization of a finitely generated $k$-algebra. Then,*

$$\psi : k_q(\mathrm{R}, \mathrm{I}) \to \nu^q_{\mathrm{R},\mathrm{I}}$$

*is surjective.*

*Proof of* (2.4). — To begin with, $k_q$ has a norm compatible with the trace on $\nu^q$ and carrying $k_q(\mathrm{R}', \sqrt{\mathrm{IR}'})$ to $k_q(\mathrm{R}, \mathrm{I})$ for R′ the normalization of R in a finite extension K′ of K (cf. for example, [11], § (3.3), Lemma 13). The diagram

$$\begin{tikzcd}
k_q(\mathrm{R}', \sqrt{\mathrm{IR}'}) \arrow[r, "\psi"] \arrow[d, shift left, "\mathrm{Norm}"] & \nu^q_{\mathrm{R}', \sqrt{\mathrm{IR}'}} \arrow[d, shift left, "\mathrm{tr}"] \\
k_q(\mathrm{R}, \mathrm{I}) \arrow[r, "\psi"] \arrow[u, shift left] & \nu^q_{\mathrm{R},\mathrm{I}} \arrow[u, shift left, "f^*"]
\end{tikzcd}$$

and the formula $\mathrm{tr}.f^* =$ multiplication by $[\mathrm{K}' : \mathrm{K}]$ reduce us to showing that for a given $\mathrm{A} \in \nu^q_{\mathrm{R},\mathrm{I}}$ there exists K′ with $[\mathrm{K}' : \mathrm{K}]$ prime to $p$ such that $f^* \mathrm{A} \in \mathrm{Im}\, \psi$.

We now follow closely the arguments of [12]. Choose a $p$-basis $b_1, \ldots, b_n$ of K such that $b_1, \ldots, b_{n-1} \in \mathrm{R}^*$ and these elements mod I form a $p$-basis for R/I and such that the valuation of $b_n$ at each maximal ideal is prime to $p$. Strictly increasing functions $s : \{1, \ldots, q\} \to \{1, \ldots, n\}$ are ordered lexicographically so $s < t$ if for some $i \in \{1, \ldots, q\}$ we have $s(i') = t(i')$ $i' < i$ and $s(i) < t(i)$. Write

$$\omega_s = \frac{db_{s(1)}}{b_{s(1)}} \wedge \ldots \wedge \frac{db_{s(q)}}{b_{s(q)}}.$$

An element $\Sigma a_s \omega_s$ lies in $\nu^q_K$ if and only if

$$\Sigma(a_s^p - a_s)\, \omega_s \in d\Omega_K^{q-1}.$$

It lies in $\nu^q_{R,I}$ if and only if, in addition, $a_s \in I$ for all $s$. The notation $\Omega^r_{K,s}$ (resp. $\Omega^r_{K,<s}$) for $s:\{1, \ldots, r\} \to \{1, \ldots, n\}$ will mean the sub-K-vector space of $\Omega^r_K$ spanned by $\omega_t$ for $t \leq s$ (resp. $t < s$).

*Lemma* **(2.5)**. — *Let* $a \in I$ *and let* $s:\{1, \ldots, q\} \to \{1, \ldots, n\}$ *be strictly increasing. Assume*

$$(a^p - a)\, \omega_s \in \Omega^q_{K,<s} + d\Omega_K^{q-1}$$

*Then replacing* K *by some finite prime to* $p$ *extension* K′ *which is a succession of Galois extensions and replacing* R *and* I *by* R′ *and* $\sqrt{IR'}$ *as above, there exist* $y_1, \ldots, y_q \in K$ *such that* $\{y_1, \ldots, y_q\} \in k_q(R, I)$, $\alpha = \dfrac{dy_1}{y_1} \wedge \ldots \wedge \dfrac{dy_q}{y_q} \in \Omega^q_{K,s}$, *and* $a\omega_s - \alpha \in \Omega^q_{R,I} \cap \Omega^q_{K,<s}$, *where* $\Omega^q_{R,I} = \mathrm{Ker}(\Omega^q_R \to \Omega^q_{R/I})$.

Note that this lemma suffices to prove (2.4) and (2.1). In fact, given $\Sigma a_s \omega_s \in \nu^q_{R,I}$ we can by the lemma subtract $\alpha \in \mathrm{Im}(k_q(R, I) \to \nu^q_{R,I})$ and decrease the "size" of the maximal $s$ with $a_s \neq 0$.

*Proof of* (2.5). — Adjoining the $(p-1)$-st root of some element in R we obtain as in [12]

$$\textbf{(2.6)} \qquad a\omega_s = a'\, \omega_{s'} \wedge \frac{dc}{c} + \tau$$

where $s' : \{1, \ldots, q-1\} \to \{1, \ldots, n\}$, $s'(i) = s(i+1)$, $a' \in K$, $\tau \in \Omega^q_{K,<s}$, $c \in K^p(b_1, \ldots, b_{s(1)})$, and

$$(a'^p - a')\, \omega_{s'} \in \Omega^{q-1}_{K,<s} + d\Omega_K^{q-2}.$$

We have

$$\frac{dc}{c} = \sum_{i=1}^{i=s(1)} \gamma_i \frac{db_i}{b_i} \quad (\gamma_i \in K), \quad \pm a = a'\, \gamma_{s(1)}.$$

Define

$$J = \bigcap_{\substack{\mathfrak{M} \supset I \\ \gamma_{s(1)} \in \mathfrak{M}}} \mathfrak{M}, \qquad L = \bigcap_{\substack{\mathfrak{M} \supset I \\ \mathfrak{M} \not\supset J}} \mathfrak{M}$$

and let $R_J$, $R_L$ be localizations, so that $JR_J$ and $LR_L$ are the Jacobson radicals. Note that $a' \in LR_L$, so, by induction on $q$, we may assume

$$\textbf{(2.7)} \qquad \begin{aligned} &a'\, \omega_{s'} = \beta + v, \quad \beta = \psi\{y_1, \ldots, y_{q-1}\}, \quad \{y_1, \ldots, y_{q-1}\} \in k_{q-1}(R, L) \\ &\beta \in \Omega^{q-1}_{K,s'}, \quad v \in \Omega^{q-1}_{K,<s'}. \end{aligned}$$

Write $T = R_J \cap K^p(b_1, \ldots, b_{s(1)-1})$. Let $H = R_J/JR_J$ and $P = T/J \cap T$. The image of $\dfrac{dc}{c}$ in $\Omega^1_{R_J/T}$ dies in $\Omega^1_{H/P}$ and is fixed under the Cartier operator. The diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & (1 + JR_J)/(T^* \cap (1 + JR_J)^*) & & \\
 & & \downarrow & & \\
0 & \longrightarrow & R_J^*/T^* & \xrightarrow{d\log} & \Omega^1_{R_J/T} \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^*/P^* & \xrightarrow{d\log} & \Omega^1_{H/P}
\end{array}$$

shows that there exists $\delta \in 1 + JR_J$ such that

$$\frac{dc}{c} = \frac{d\delta}{\delta} + \eta \text{ in } \Omega^1_K$$

with
$$\eta \in R_J.\mathrm{Im}(\Omega^1_T \to \Omega^1_{R_J}) \subseteq \Omega^1_{K,<s(1)}$$
$$\frac{d\delta}{\delta} \in \Omega^1_{K,s(1)}.$$

By (2.6) and (2.7) we get

$$a\omega_s = (\beta + v) \wedge \left(\frac{d\delta}{\delta} + \eta\right) + \tau$$
$$\equiv \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_{q-1}}{y_{q-1}} \wedge \frac{d\delta}{\delta} \pmod{\Omega^q_{K,<s}}.$$

Note, quite generally, that if $B_1 \in k_\ell(R_J, JR_J)$ and $B_2 \in k_m(R_L, LR_L)$ the product $B_1.B_2$ belongs to $k_{\ell+m}(R, I)$. This is a simple consequence of the fact that

$$k_\ell(R_m, mR_m).k_1(K) \subseteq k_{\ell+1}(R_m, mR_m).$$

In particular, $\{y_1, \ldots, y_{q-1}, \delta\} \in k_q(R, I)$, q.e.d.

*Corollary* **(2.8)**. — *Let* F *be a field of characteristic* $p > 0$. *Then the p-primary torsion subgroup of* $K^M_q(F)$ *is infinitely divisible, and*

$$K^M_q(F)/p^n K^M_q(F) \underset{\simeq}{\to} W_n \Omega^q_{F,\log}.$$

*Here* $W_n \Omega^q_{F,\log}$ *is the group of global sections of* $W_n \Omega^q_{\log}$ *on* $(S_p F)_{et}$.

*Proof*. — For a discussion of $W_n \Omega^q_{F,\log}$ see [10]. In particular we have

$$\begin{array}{ccccccc}
 & & K^M_q(F)/p & \longrightarrow & K^M_q(F)/p^n & \longrightarrow & K^M_q(F)/p^{n-1} \longrightarrow 0 \\
 & & \wr\| & & \downarrow & & \wr\| \\
0 & \longrightarrow & \Omega^q_{F,\log} & \longrightarrow & W_n \Omega^q_{F,\log} & \longrightarrow & W_{n-1} \Omega^q_{F,\log}
\end{array}$$

where the bottom sequence is exact by *op. cit.* (5.7.5). The left hand vertical arrow is an isomorphism by (2.1) and the right hand arrow is an isomorphism by induction. This establishes the isomorphism. The first assertion follows from the exact sequence of $\mathrm{Tor}(K_q^M, \cdot)$ applied to $0 \to \mathbf{Z}/p \to \mathbf{Z}/p^n \to \mathbf{Z}/p^{n-1} \to 0$.

## 3. A basic cohomological lemma

Let K be a field, $p$ a prime number prime to char(K). The cohomological symbol defined by Tate gives a map [19]

$$K_r^M(K)/p^n K_r^M(K) \to H^r_{et}(\mathrm{Sp}\, K, \mathbf{Z}/p^n \mathbf{Z}(r)),$$

which one conjectures to be an isomorphism quite generally. It is useful to formulate a relative conjecture. Let $(\mathbf{Q}/\mathbf{Z})'$ denote the prime to char(K) torsion in $\mathbf{Q}/\mathbf{Z}$, let $\chi \in H^1(\mathrm{Sp}\, K, (\mathbf{Q}/\mathbf{Z})')$ and let K′ be the cyclic extension of K corresponding to $\chi$.

*Conjecture* **(3.1).** — The sequence

$$K_{r-1}^M(K') \xrightarrow{N} K_{r-1}^M(K) \xrightarrow{\chi\cup} H^r(\mathrm{Sp}\, K, (\mathbf{Q}/\mathbf{Z})'(r-1)) \longrightarrow H^r(\mathrm{Sp}\, K', (\mathbf{Q}/\mathbf{Z})'(r-1))$$

is exact. Here N is the norm map in Milnor-K-theory [11], § (1.7), and "$\chi \cup$" is the map $\chi \to \chi \cup h(\chi)$ with

$$h: K_{r-1}^M(K) \to H^{r-1}(\mathrm{Sp}\, K, \mathbf{Z}_p(r-1)), \text{ the cohomological symbol.}$$

See [14] for definitive results on these conjectures in the $K_2$ case.

The following lemma is taken from [12]. It is the essential tool we will use in studying these questions.

*Lemma* **(3.2).** — *Let notation be as above, but take* $[K' : K] = p$. *Regard* $\chi$ *as an element of* $H^1(\mathrm{Sp}\, K, \mathbf{Z}/p\mathbf{Z})$, *and let* $G = \mathrm{Gal}(K'/K) \cong \mathbf{Z}/p\mathbf{Z}$. *Then*

(i) *The sequence*

$$\textbf{(3.2.1)} \qquad H^{q-1}(\mathrm{Sp}\, K, \mathbf{Z}/p\mathbf{Z}) \xrightarrow{\chi\cup} H^q(\mathrm{Sp}\, K, \mathbf{Z}/p\mathbf{Z}) \longrightarrow H^q(\mathrm{Sp}\, K', \mathbf{Z}/p)$$

*is exact if and only if the sequence*

$$\textbf{(3.2.2)} \qquad H^{q-1}(\mathrm{Sp}\, K, \mathbf{Z}/p\mathbf{Z}) \xrightarrow{\mathrm{res}} H^{q-1}(\mathrm{Sp}\, K', \mathbf{Z}/p\mathbf{Z})_G \xrightarrow{\mathrm{cor}} H^{q-1}(\mathrm{Sp}\, K, \mathbf{Z}/p\mathbf{Z})$$

*is exact.*

(ii) *The sequence*

$$\textbf{(3.2.3)} \qquad H^{q-1}(\mathrm{Sp}\, K', \mathbf{Z}/p\mathbf{Z}) \xrightarrow{\mathrm{cor}} H^{q-1}(\mathrm{Sp}\, K, \mathbf{Z}/p\mathbf{Z}) \xrightarrow{\chi\cup} H^q(\mathrm{Sp}\, K, \mathbf{Z}/p\mathbf{Z})$$

*is exact if and only if the sequence*

$$\textbf{(3.2.4)} \qquad H^q(\mathrm{Sp}\, K, \mathbf{Z}/p\mathbf{Z}) \longrightarrow H^q(\mathrm{Sp}\, K', \mathbf{Z}/p\mathbf{Z})^G \xrightarrow{\mathrm{cor}} H^q(\mathrm{Sp}\, K, \mathbf{Z}/p\mathbf{Z})$$

*is exact.*

(For a G-module M, $M^G$ = invariants of G acting on M and $M_G$ = co-invariants $= M/(\sum_{g \in G} (1 - g) M)$.)

*Proof.* — We will only prove (i). The proof of (ii) is similar, and it will not be used in the sequel. Adjoining a $p$-th root $\zeta$ of 1 involves an extension of degree prime to $p$, and hence induces injections on the homology of the complexes (3.2.1) and (3.2.2). Thus we may assume $\zeta \in K$.

*Sublemma* **(3.3).** — *Assume* $\zeta \in K$, *and identify* $\mathbf{Z}/p\mathbf{Z} \xrightarrow{\sim} \mu_p$ *via* $1 \mapsto \zeta$. *Thus* $H^1(\mathrm{Sp}\, K, \mathbf{Z}/p\mathbf{Z}) \cong K^*/K^{*p}$ *and* $\zeta \in K^*$ *gives a class* $[\zeta] \in H^1(\mathrm{Sp}\, K, \mathbf{Z}/p\mathbf{Z})$. *Let*

$$\beta : H^1(\mathrm{Sp}\, K, \mathbf{Z}/p\mathbf{Z}) \to H^2(\mathrm{Sp}\, K, \mathbf{Z}/p\mathbf{Z})$$

*be the Bockstein associated to the exact sequence*

$$\textbf{(3.3.1)} \qquad 0 \to \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p^2\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0.$$

*Then* $\beta(\chi) = \chi \cup [\zeta]$.

*Proof.* — An element $t \in K^*$ maps to the class $a(\cdot) : \mathrm{Gal}(K^{sep}/K) \to \mathbf{Z}/p\mathbf{Z}$ where

$$\zeta^{a(\sigma)} = (t^{1/p})^\sigma / t^{1/p}.$$

Let $\rho^p = \zeta$, $\theta^{p^2} = t$. The cocycle $w(\sigma, \tau)$ associated to $\beta(t)$ is given by

$$\zeta^{w(\sigma,\tau)} = \theta^{\sigma\tau}\, \theta / \theta^\sigma\, \theta^\tau.$$

Note that

$$\theta^\sigma = \rho^{A(\sigma)}\, \theta, \qquad A(\sigma) \equiv a(\sigma) \pmod p.$$

From this one gets easily

$$\zeta^{w(\sigma,\tau)} = (\rho^\tau/\rho)^{A(\sigma)} = (\rho^\tau/\rho)^{a(\sigma)}.$$

The cohomology class represented by the right side is $t \cup [\zeta]$, q.e.d.

*Sublemma* **(3.4)**, — *Let* S *be a profinite group,* $p$ *a prime number,* $\chi$ *a non-zero element of* $H^1(S, \mathbf{Z}/p\mathbf{Z})$, *and* $T = \mathrm{Ker}(\chi : S \to \mathbf{Z}/p\mathbf{Z})$. *Let* $\beta : H^*(S, \mathbf{Z}/p\mathbf{Z}) \to H^{*+1}(S, \mathbf{Z}/p\mathbf{Z})$ *be the Bockstein. For* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *a complex, call* $\mathrm{Ker}(g)/\mathrm{Im}(f)$ *the homology.*

(i) *Let* $q \geqq 2$. *Then, the following two complexes have isomorphic homology groups.*

$$\textbf{(3.4.1)} \qquad H^{q-1}(S, \mathbf{Z}/p) \oplus H^{q-2}(S, \mathbf{Z}/p) \xrightarrow{(\chi\cup,\, \beta(\chi)\cup)} H^q(S, \mathbf{Z}/p) \xrightarrow{\mathrm{res}} H^q(T, \mathbf{Z}/p).$$

$$\textbf{(3.4.2)} \qquad H^{q-1}(S, \mathbf{Z}/p) \xrightarrow{\mathrm{res}} H^{q-1}(T, \mathbf{Z}/p)_{S/T} \xrightarrow{\mathrm{cor}} H^{q-1}(S, \mathbf{Z}/p).$$

(ii) *For* $q \geqq 1$, *the following two complexes have isomorphic homology groups.*

$$\textbf{(3.4.3)} \qquad H^{q-1}(T, \mathbf{Z}/p) \xrightarrow{\mathrm{cor}} H^{q-1}(S, \mathbf{Z}/p) \xrightarrow{(\chi\cup,\, \beta(\chi)\cup)} H^q(S, \mathbf{Z}/p) \oplus H^{q+1}(S, \mathbf{Z}/p)$$

$$\textbf{(3.4.4)} \qquad H^q(S, \mathbf{Z}/p) \xrightarrow{\mathrm{res}} H^q(T, \mathbf{Z}/p)^{S/T} \xrightarrow{\mathrm{cor}} H^q(S, \mathbf{Z}/p).$$

*Remark.* — These sequences are exact if $p = 2$, but need not be exact in the case $p \neq 2$. For example, let $p$ be an odd prime number, and let S be the semi-direct product $\mathbf{Z}_p[\zeta] \times_\tau \mathbf{Z}_p$, where $\zeta_p$ denotes a primitive $p$-th root of 1 and $\tau$ is the homomorphism

$$\mathbf{Z}_p \to \mathrm{Aut}(\mathbf{Z}_p[\zeta_p]); \qquad a \mapsto (x \mapsto \zeta_p^a x).$$

Let $\chi : \mathrm{S} \to \mathbf{Z}_p$ be the homomorphism induced by the second projection $\mathrm{S} \to \mathbf{Z}_p$. Then, the sequence (3.4.2) is not exact in the case $q = 2$. Thus, though S is torsion free, S can not be isomorphic to $\mathrm{Gal}(k_s/k)$ for any field $k$.

*Proof of* (3.4). — Since the proofs of (i) and (ii) are rather similar, we present here only the proof of (i). Let X be the S-module of all functions $\mathrm{S/T} \to \mathbf{Z}/p$, $s$ an element of S such that $\chi(s) = 1$, and Y the image of $s - 1 : \mathrm{X} \to \mathrm{X}$. Let $g : \mathrm{X} \to \mathrm{Y}$ (resp. $h : \mathrm{Y} \to \mathrm{X}$, resp. $i : \mathbf{Z}/p \to \mathrm{Y}$) be the map induced by $s - 1$ (resp. the inclusion map, resp. the embedding as constant functions). Since there is a canonical isomorphism $\mathrm{H}^q(\mathrm{S}, \mathrm{X}) \cong \mathrm{H}^q(\mathrm{T}, \mathbf{Z}/p)$ for any $q$, the exact sequences of S-modules

$$0 \longrightarrow \mathbf{Z}/p \xrightarrow{h.i} \mathrm{X} \xrightarrow{g} \mathrm{Y} \longrightarrow 0, \qquad 0 \longrightarrow \mathrm{Y} \xrightarrow{h} \mathrm{X} \xrightarrow{j} \mathbf{Z}/p \longrightarrow 0$$

($j$ is defined by $j(f) = \sum_{x \in \mathrm{S/T}} f(x)$ for all $f \in \mathrm{X}$) induce a commutative diagram

$$\begin{tikzcd}
 & \mathrm{H}^{q-2}(\mathrm{S}, \mathbf{Z}/p) \arrow[d, "\partial"] \arrow[dr, "\beta(\chi)\cup"] & & \\
\mathrm{H}^{q-1}(\mathrm{T}, \mathbf{Z}/p) \arrow[r, "g"] \arrow[dr, "s-1"'] & \mathrm{H}^{q-1}(\mathrm{S}, \mathrm{Y}) \arrow[r, "\partial"] \arrow[d, "h"] & \mathrm{H}^q(\mathrm{S}, \mathbf{Z}/p) \arrow[r, "\mathrm{res}"] & \mathrm{H}^q(\mathrm{T}, \mathbf{Z}/p\mathbf{Z}) \\
 & \mathrm{H}^{q-1}(\mathrm{T}, \mathbf{Z}/p) \arrow[d, "\mathrm{cor}"'] & \mathrm{H}^{q-1}(\mathrm{S}, \mathbf{Z}/p) \arrow[l, "\mathrm{res}"] \arrow[ul, "i"'] \arrow[u, "\chi\cup"'] & \\
 & \mathrm{H}^{q-1}(\mathrm{S}, \mathbf{Z}/p) & &
\end{tikzcd}$$

with two long exact sequences. Here $\partial$ denote the connecting homomorphisms. (Note that the restriction maps and the corestriction maps are induced by $h.i$ and $j$, respectively. The commutativity of the diagram follows from (3.5) below.) The assertion (i) follows from this diagram. This proves (3.2) and (3.4).

*Lemma* **(3.5)**. — (i) *The image of* $1 \in \mathrm{H}^0(\mathrm{S}, \mathbf{Z}/p)$ *under the composite map* $\mathrm{H}^0(\mathrm{S}, \mathbf{Z}/p) \xrightarrow{\partial} \mathrm{H}^1(\mathrm{S}, \mathrm{Y}) \xrightarrow{\partial} \mathrm{H}^2(\mathrm{S}, \mathbf{Z}/p)$ *coincides with* $\beta(\chi)$.

(ii) *The image of* $1 \in \mathrm{H}^0(\mathrm{S}, \mathbf{Z}/p)$ *under the composite map*

$$\mathrm{H}^0(\mathrm{S}, \mathbf{Z}/p) \xrightarrow{i} \mathrm{H}^0(\mathrm{S}, \mathrm{Y}) \xrightarrow{\partial} \mathrm{H}^1(\mathrm{S}, \mathbf{Z}/p)$$

*coincides with* $\chi$.

*Proof.* — (ii) is easy and so we give here the proof of (i). By functoriality, we may assume $T = \{1\}$. Let $f \in X$ be the function defined by $f(1) = 1$ and $f(\sigma) = 0$ for $\sigma \neq 1$. Then, $j(f) = 1$. So, $\partial(1) \in H^1(S, Y)$ is represented by the cocycle $S \to Y$, $\sigma \mapsto f'_\sigma$, where

$$f'_\sigma(\tau) = \begin{cases} 1 & \text{if } \sigma \neq 1 \text{ and } \tau = \sigma^{-1} \\ -1 & \text{if } \sigma \neq 1 \text{ and } \tau = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For $\sigma \in S$, define $f''_\sigma \in X$ by

$$f'_{s^m}(s^n) = \begin{cases} 1 & \text{if } m + n > p \text{ or if } m \geq 1 \text{ and } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$(0 \leq m < p,\ 0 \leq n < p)$. Then, $g(f''_\sigma) = f'_\sigma$. So, $\partial\,\partial(1) \in H^2(S, \mathbf{Z}/p)$ is represented by the cocycle $G \times G \to \mathbf{Z}/p \subset X$,

$$(s^m, s^n) \mapsto f''_{s^m} \circ s^n - f''_{s^{m+n}} + f''_{s^n} = \begin{cases} 1 & \text{if } m + n \geq p \\ 0 & \text{if } m + n < p \end{cases}$$

$(0 \leq m < p,\ 0 \leq n < p)$. But this cocycle also represents $\beta(\chi)$ as is easily seen.

## 4. Filtration on Symbols

In this section, A denotes a ring additively generated by $A^*$ (e.g. A local), and $\pi$ denotes a non-zero divisor of A contained in the Jacobson radical of A.

Let $K_q^M$ $(q \geq 0)$ be the group

$$\Big(\underbrace{A\left[\frac{1}{\pi}\right]^* \otimes \ldots \otimes A\left[\frac{1}{\pi}\right]^*}_{q \text{ times}}\Big)\Big/J$$

where J denotes the subgroup of the tensor product generated by elements of the form $x_1 \otimes \ldots \otimes x_q$ such that $x_i + x_j = 1$ or $0$ for some $0 \leq i < j \leq q$. An element $x_1 \otimes \ldots \otimes x_q \bmod J$ of $K_q^M$ will be denoted by $\{x_1, \ldots, x_q\}$. One has of course $\{x, 1 - x\} = 0$ $\left(x, 1 - x \in A\left[\frac{1}{\pi}\right]^*\right)$, $\{x, -x\} = 0$ and also $\{x, y\} = -\{y, x\}$. In this section, we give some elementary lemmas concerning the structure of $K_q^M$, which will be useful in later sections. The arguments are essentially the same as in [3], Ch. II, § 3, where Quillen's K-functor is studied for $A = R[[T]]$ and $\pi = T$.

For $m \geq 1$, let $U^m K_q^M$ be the subgroup of $K_q^M$ generated by symbols of the form

$$\{1 + x\pi^m, y_1, \ldots, y_{q-1}\}$$

such that $x \in A$ and $y_1, \ldots, y_{q-1} \in A\left[\frac{1}{\pi}\right]^*$.

*Lemma* (4.1). — $\{U^i K_q^M, U^j K_r^M\} \subset U^{i+j} K_{q+r}^M$.

For $a, b \in A^*$, we have

$$\{1 + a\pi^i, 1 + b\pi^j\} \equiv \{1 + a\pi^i(1 + b\pi^j), 1 + b\pi^j\} \bmod U^{i+j} \tag{4.1.1}$$
$$= -\{1 + a\pi^i(1 + b\pi^j), -a\pi^i\}$$
$$= -\left\{1 + \frac{ab\pi^{\ j}}{1 + a\pi^i}, -a\pi^i\right\}.$$

The lemma follows easily.

For a ring R, $\Omega_R^1$ will denote the module of (absolute) Kahler differentials of R. We write $\Omega_R^r = \Lambda_R^r \Omega_R^1$. Define the homomorphism

$$\delta_r : R \otimes (R^*)^{\otimes r} \to \Omega_R^r,$$

by

$$\delta_r(x \otimes y_1 \otimes \ldots \otimes y_r) = x \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_r}{y_r}.$$

*Lemma* (4.2). — *Assume* R *is additively generated by* $R^*$ (*e.g.* R *local*). *Then* $\delta_r$ *is surjective, and* Ker $\delta_r$ *is generated by elements of the following types*:

(4.2.1) $x \otimes y_1 \otimes \ldots \otimes y_r$ *with* $y_i = y_j$ *for some* $1 \leq i < j \leq r$.

(4.2.2) $$\sum_{i=1}^{m} x_i \otimes x_i \otimes y_1 \otimes \ldots \otimes y_{r-1} - \sum_{i=1}^{l} x_i' \otimes x_i' \otimes y_1 \otimes \ldots \otimes y_{r-1}$$
$$x_i, x_i' \in R^*, \quad \sum_{i=1}^{m} x_i = \sum_{i=1}^{l} x_i'.$$

*Proof.* — Straightforward and left to the reader.

Let $R = A/\pi A$, and for any $m \geq 1$, define

(4.3) $$\rho_m : \Omega_R^{q-1} \oplus \Omega_R^{q-2} \to \mathrm{gr}^m K_q^M = U^m K_q^M / U^{m+1} K_q^M$$

by

$$\rho_m\left(x \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_{q-1}}{y_{q-1}}, 0\right) = \{1 + \tilde{x}\pi^m, \tilde{y}_1, \ldots, \tilde{y}_{q-1}\}$$
$$\rho_m\left(0, x \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_{q-2}}{y_{q-2}}\right) = \{1 + \tilde{x}\pi^m, \tilde{y}_1, \ldots, \tilde{y}_{q-2}, \pi\},$$

$\tilde{x} \in A$, $\tilde{y}_i \in A^*$ lifting $x \in R$, $y_i \in R^*$.

The fact that $\rho_m$ is well defined is an easy consequence of (4.1) and (4.2).

From now on, let $p$ be a prime number and assume that $R = A/\pi A$ is essentially smooth over a field of characteristic $p$. Note that

(4.4) $$(1 + \pi^m x)^p \equiv 1 + \pi^{mp} x^p \bmod \pi^{mp+1} A$$

if $p \in \pi^{m(p-1)+1} A$.

*Lemma* **(4.5)**. — *Let* $m = m_1 p^s$, $s \geq 0$, $p \nmid m_1$ *and assume that* $p^p \in \pi^{m(p-1)+1} A$. *Then*

(i) $\rho_m(B_s^{q-1} \oplus B_s^{q-2}) = (0)$

(ii) *Define* $\theta : \Omega_R^{q-2} \to (\Omega_R^{q-1}/B_s^{q-1}) \oplus (\Omega_R^{q-2}/B_s^{q-2})$

*by* $$\theta(\omega) = (C^{-s}(d\omega), (-1)^q m_1 C^{-s}(\omega)).$$

*Then* $\rho_m \circ \theta(\Omega_R^{q-2}) = (0)$.

(See (1.3) for the notation $B_s^*$ and $C^{-1}$.)

*Proof*. — Let $0 \leq t < s$. Part (i) follows from

$$\{1 + x^{p^t} \pi^{m_1 p^s}, x\} \equiv p^t \{1 + x\pi^{m_1 p^{s-t}}, x\} \bmod U^{m+1},$$

$$p^t\{1 + x\pi^{m_1 p^{s-t}}, x\} = -\{1 + x\pi^{m_1 p^{s-t}}, (-1)^{p^t} \pi^m\}$$
$$= -\{1 + x^{p^t} \pi^m, -1\} - p^{s-t} m_i \{1 + x^{p^t} \pi^m, \pi\} \in U^{m+1}.$$

(Use (4.4) with $mp$ in place of $m$.)

Part (ii) amounts to the assertion

$$\{1 + x^{p^s} \pi^m, x, y_1, \ldots, y_{q-2}\} \equiv (-1)^{q-1} m_1 \{1 + x^{p^s} \pi^m, y_1, \ldots, y_{q-2}, \pi\},$$

i.e. $$\{1 + x^{p^s} \pi^m, x\pi^{m_1}\} \in U^{m+1}.$$

This is again straightforward.

*Lemma* **(4.6)**. — *Let* $m$, $m_1$ *and* $s$ *be as in* (4.5) *and let* $0 \leq n \leq s$. *Then* $\rho_m(Z_n^{q-1} \oplus Z_n^{q-2}) = (0)$ *in* $\mathrm{gr}^m(K_q^M/p^n K_q^M) = (U^m K_q^M + p^n K_q^M)/(U^{m+1} K_q^M + p^n K_q^M)$.

*Proof*. — Let $m' = mp^{-n}$. Note that

$$p^n\{1 + x\pi^{m'}, y\} \equiv \{1 + x^{p^n} \pi^m, y\} \bmod U^{m+1}.$$

Since $Z_n$ is generated by $B_n$ together with differentials $x^{p^n} \dfrac{dy_1}{y_1} \wedge \ldots$, the lemma follows.

Let $m$, $m_1$ and $s$ be as in (4.5) and let $n \geq 0$. Define the group ${}^mG_n^q$ to be

**(4.7)** $$(\Omega_R^{q-1}/Z_n^{q-1}) \oplus (\Omega_R^{q-2}/Z_n^{q-2}) \quad \text{if } n \leq s,$$
$$\mathrm{Coker}(\Omega_R^{q-2} \xrightarrow[(4.5)]{\theta} (\Omega_R^{q-1}/B_s) \oplus (\Omega_R^{q-2}/B_s)) \quad \text{if } n > s.$$

We have established surjections ${}^mG_n^q \to \mathrm{gr}^m(K_q^M/p^n K_q^M)$.

*Remark* **(4.8)**. — These surjective homomorphisms are in fact bijective. Indeed, by localization, the question of injectivity is reduced to the case where R is a field. If $\mathrm{char}\left(A\left[\frac{1}{\pi}\right]\right) = 0$, injectivity will be proved in § 5 and § 6 by using the cohomological symbol. If $\mathrm{char}\left(A\left[\frac{1}{\pi}\right]\right) = p$, injectivity follows from [3], Ch. II, § 4 (cf. also [11], § 2).

Note that in the mixed characteristic case, the condition on $m$ in (4.5) is actually restrictive. The structure of $\mathrm{gr}^m(\mathrm{K}_q^{\mathrm{M}})$ for large $m$ such that $p^p \notin \pi^{m(p-1)}\mathrm{A}$ (this is equivalent to $m > e' = \dfrac{ep}{p-1}$ with the notation of § 5, § 6) is not yet known.

## 5. Galois cohomology

In this section, K denotes a henselian discrete valuation field with residue field F such that $\mathrm{char}(\mathrm{K}) = 0$ and $\mathrm{char}(\mathrm{F}) = p > 0$. In the next section, we shall apply the results of this section to the quotient field of the strict henselian discrete valuation ring $\mathcal{O}_{\mathrm{X},\bar{\nu}}$ where $\nu$ is the generic point of Y (not to the base field K of § 0).

Let

$$k_q(\mathrm{K}) = \mathrm{K}_q^{\mathrm{M}}(\mathrm{K})/p\mathrm{K}_q^{\mathrm{M}}(\mathrm{K}),$$
$$h^q(\mathrm{K}) = \mathrm{H}^q_{\mathrm{et}}(\mathrm{Sp}\,\mathrm{K}, \mathbf{Z}/p\mathbf{Z}(q)).$$

The aim of this section is to determine the structures of these groups and to prove that the cohomological symbol gives an isomorphism

$$\mathrm{K}_q^{\mathrm{M}}(\mathrm{K})/p^n\,\mathrm{K}_q^{\mathrm{M}}(\mathrm{K}) \cong \mathrm{H}^q(\mathrm{Sp}\,\mathrm{K}, \mathbf{Z}/p^n\,\mathbf{Z}(q))$$

for all $q$ and $n$.

We define the filtration $\mathrm{U}^m\,\mathrm{K}_q^{\mathrm{M}}(\mathrm{K})$ $(m \geq 1)$ as in § 4. Here we take the valuation ring $\mathcal{O}_{\mathrm{K}}$ of K as A and a prime element of K as $\pi$. Note that the homomorphism $\rho_m : \Omega_{\mathrm{F}}^{q-1} \oplus \Omega_{\mathrm{F}}^{q-2} \to \mathrm{gr}^m\,\mathrm{K}_q^{\mathrm{M}}(\mathrm{K})$ depends upon a choice of a prime element $\pi$ of K, which, we will assume, has been fixed.

Let $\mathrm{U}^0\,\mathrm{K}_q^{\mathrm{M}}(\mathrm{K}) = \mathrm{K}_q^{\mathrm{M}}$. Let $\mathrm{U}^m k_q(\mathrm{K}) \subset k_q(\mathrm{K})$ $(m \geq 0)$ be the image of $\mathrm{U}^m\,\mathrm{K}_q^{\mathrm{M}}(\mathrm{K})$, and let $\mathrm{U}^m h^q(\mathrm{K}) \subset h^q(\mathrm{K})$ be its image under the cohomological symbol map $k_q(\mathrm{K}) \to h^q(\mathrm{K})$.

Let $\mathrm{ord}_{\mathrm{K}}$ be the normalized additive discrete valuation of K, let

$$\mathrm{U}_{\mathrm{K}}^{(m)} = \{x \in \mathrm{K},\ \mathrm{ord}_{\mathrm{K}}(x-1) \geq m\} \quad \text{for } m \geq 1,$$

let $e = \mathrm{ord}_{\mathrm{K}}(p)$ the absolute ramification index of K, and let $e' = \dfrac{ep}{p-1}$.

*Lemma* **(5.1).** — (i) $\mathrm{U}^m k_q(\mathrm{K}) = 0$ *for* $m > e'$.

(ii) *Assume that $e'$ is an integer and let $a$ be the residue class of $p\pi^{-e}$. Then, the surjective homomorphism* (4.3)

$$\rho_{e'} : \Omega_{\mathrm{F}}^{q-1} \oplus \Omega_{\mathrm{F}}^{q-2} \to \mathrm{U}^{e'} k_q(\mathrm{K})$$

*annihilates* $(1 + a\mathrm{C})\,\mathrm{Z}_1^{q-1} \oplus (1 + a\mathrm{C})\,\mathrm{Z}_1^{q-2}$, *where* C *is the Cartier operator. If* F *is separably closed, then* $\mathrm{U}^{e'} k_q(\mathrm{K}) = 0$.

*Proof.* — (i) follows from

$$\mathrm{U}_{\mathrm{K}}^{(m)} \subset (\mathrm{K}^*)^p \quad \text{if } m > e'.$$

The proof of (ii) is similar to the proof of (4.6) using

$$(1 + x\pi^{e'/p})^p \equiv 1 + (x^p + xp\pi^{-e})\,\pi^{e'} \mod \pi^{e'+1}.$$

*Lemma* **(5.2)**. — *Let* $1 \leq m < e'$ *and let the group* ${}^m\mathrm{G}_1^q$ *be as in* (4.7) *with* $\mathrm{R} = \mathrm{F}$. *Then,*

$${}^m\mathrm{G}_1^q \cong \mathrm{gr}^m k_q(\mathrm{K}) \cong \mathrm{gr}^m h^q(\mathrm{K}).$$

*Proof.* — By a limit argument we may assume that F is finitely generated over $\mathbf{F}_p$ of transcendence degree $d$. We may also suppose that K contains the $p$-th roots of 1 (a straightforward reduction using norms, which we leave for the reader). Then the group $\mathrm{U}^{e'} h^{d+2}(\mathrm{K})$ is non-zero by [11], § 1, Th. 2 (cf. also [12], page 227). Note

$$\textbf{(5.2.1)} \qquad {}^m\mathrm{G}_1^q \cong \begin{cases} \Omega_{\mathrm{F}}^{q-1} & 0 < m < e',\ p \nmid m \\ \mathrm{B}_1^q \oplus \mathrm{B}_1^{q-1} & 0 < m < e',\ p \mid m. \end{cases}$$

We now consider a diagram of pairings

$$\begin{tikzcd}
{}^m\mathrm{G}_1^q \times {}^{e'-m}\mathrm{G}_1^{d+2-q} \arrow[r, "\bar{\rho}_m \times \bar{\rho}_{e'-m}"] \arrow[d, "(2)"] & \mathrm{gr}^m h^q \times \mathrm{gr}^{e'-m} h^{d+2} \arrow[d, "\text{cup product}"] \\
\Omega_{\mathrm{F}}^d/\mathrm{B}_1^d \arrow[r, "(1)"] & {}^{e'}\mathrm{G}_1^{d+2} \arrow[r, "\bar{\rho}_{e'}"] & \mathrm{U}^{e'} h^{d+2}
\end{tikzcd}$$

where arrow (1) is the natural surjection which exists because

$$\mathrm{B}_1^d = (1 + a\mathrm{C})\,\mathrm{B}_1^d \subset (1 + a\mathrm{C})\,\Omega_{\mathrm{F}}^d,$$

and arrow (2) corresponds under the isomorphism (5.2.1) to wedge product of forms if $p \nmid m$ (resp. to

$$(d\omega_1, d\omega_2) \times (dy_1, dy_2) \mapsto \omega_1\, dy_2 + \omega_2\, dy_1$$

if $p \mid m$). It is a simple exercise with symbols (calculated as in (4.1)) to show that this diagram commutes upto an $(\mathbf{F}_p)^\times$-multiple. Also $\Omega_{\mathrm{F}}^d/\mathrm{B}_1^d$ is a 1-dimensional $\mathbf{F}_p$ vector space and (2) is a perfect pairing of $\mathbf{F}_p$ vector spaces. Injectivity of $\bar{\rho}_m$ follows. Since the arrows from left to right in the statement of (5.2) are already known to be surjective, we are done.

*Lemma* **(5.3)**. — $\nu_{\mathrm{F}}^q \oplus \nu_{\mathrm{F}}^{q-1} \cong \mathrm{gr}^0 k_q(\mathrm{K}) \cong \mathrm{gr}^0 h^q(\mathrm{K})$.

*Proof.* — Results in [1] give an isomorphism

$$\mathrm{gr}^0 k_q(\mathrm{K}) \cong k_q(\mathrm{F}) \oplus k_{q-1}(\mathrm{F})$$

so, from (2.1), we get a map $\bar{\rho}_0$ defined as the composition

$$\bar{\rho}_0 : \nu_F^q \oplus \nu_F^{q-1} \cong \mathrm{gr}^0 k_q(K) \to \mathrm{gr}^0 h^q(K).$$

Let $K' \supset K$ be the quotient field of a henselian discrete valuation ring $\mathcal{O}_{K'} \supset \mathcal{O}_K$ with the property that $K'$ is unramified over $K$, with residue field

$$F' = \mathcal{O}_{K'}/\pi\mathcal{O}_{K'} \cong F(z),$$

where $z$ is transcendental over $F$. Let $\tilde{z} \in \mathcal{O}_{K'}^*$ lift $z$. Multiplication by $1 + \tilde{z}\pi$ gives

$$\nu_F^q \oplus \nu_F^{q-1} \xrightarrow{\bar{\rho}_0} \mathrm{gr}^0 h^q(K) \xrightarrow{(1+\tilde{z}\pi)} \mathrm{gr}^1 h^{q+1}(K') \cong \Omega^q_{F(z)}.$$

The composition is easily seen to be

$$\frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_q}{f_q} \mapsto z\,\frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_q}{f_q}$$

$$\frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_{q-1}}{f_{q-1}} \mapsto \pm\, dz \wedge \frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_{q-1}}{f_{q-1}}.$$

Injectivity of $\bar{\rho}_0$ is now immediate.

Our next objective is to prove that $k_q(K) \cong h^q(K)$. Let $Sh^q(K) = U^0 h^q(K)$ be the image of $k_q(K)$ in $h^q(K)$. We first prove $Sh^q(K) = h^q(K)$ in the case $F$ is separably closed and $K$ contains a primitive $p$-th root $\zeta_p$ of 1. To apply the basic lemma of § 3, we devote ourselves in (5.4)-(5.11) to proving

*Proposition* **(5.4)**. — *Assume that* $F$ *is separably closed and* $\zeta_p \in K$. *Let* $b \in O_K^*$ *be such that the image* $\beta$ *of* $b$ *in* $F$ *is not a p-th power. Let* $a = b^{1/p}$ *be a p-th root of* $b$, $L = K(a)$, $\alpha = \beta^{1/p}$, $E = F(\alpha)$ *with* $G = \mathrm{Gal}(L/K)$. *Then, the sequences*

**(5.4.1)** $$Sh^q(K) \xrightarrow{\mathrm{res}} Sh^q(L)^G \xrightarrow{\mathrm{cor}} Sh^q(K)$$

**(5.4.2)** $$Sh^q(K) \xrightarrow{\mathrm{res}} Sh^q(L)_G \xrightarrow{\mathrm{cor}} Sh^q(K)$$

*are exact for all* $q$.

Note that we already know the precise structure of $Sh^q(K)$ and $Sh^q(L)$, for $\mathrm{gr}^{e'} h^q = 0$ by (5.1) (ii).

We begin with some lemmas concerning differentials. Let $i : \Omega_F^q \to \Omega_E^q$ be the canonical homomorphism, and let $\mathrm{Tr} : \Omega_E^q \to \Omega_F^q$ be the trace map characterized by

(i) $$\mathrm{Tr}(E.i(\Omega_F^q)) = \mathrm{Tr}(dE \wedge i(\Omega_F^{q-1})) = 0$$

(ii) $$\text{For } \omega \in \Omega_F^{q-1} \text{ and } f \in E^*, \quad \mathrm{Tr}\left(i(\omega) \wedge \frac{df}{f}\right) = \omega \wedge \frac{df^p}{f^p}.$$

A proof of the existence of Tr is that the norm on $S\hat{C}K_{q+1}$ ([13]) induces this homomorphism Tr on its subquotient $\Omega^q$. (The assumptions $p \neq 2$ and $p > q$ in [3], II, § 4, Th. (4.1) are unnecessary by [11], § 2, Prop. 2.) In (5.5)-(5.9), we need not assume $F$ separably closed.

*Lemma* **(5.5)**. — (i) *For* $\omega \in \Omega_F^q$, *the three conditions* a) $\omega \wedge d\beta = 0$, b) $\omega \in \Omega_F^{q-1} \wedge d\beta$, c) $i(\omega) = 0$, *are equivalent.*

(ii) *Let* $I^q = i(\Omega_F^q) \subset \Omega_E^q$. *Then the map*

$$(E \otimes_F I^q) \oplus (E \otimes_F I^{q-1}) \to \Omega_E^q$$

*defined by*

$$(x \otimes \omega, 0) \mapsto x\omega$$

$$(0, x \otimes \omega) \mapsto x\omega \wedge \frac{d\alpha}{\alpha}$$

*is an isomorphism.*

The proof is left to the reader.

*Lemma* **(5.6)**. — *The sequence*

$$\nu_E^q \xrightarrow{\mathrm{Tr}} \nu_F^q \xrightarrow{i} \nu_E^q \xrightarrow{\mathrm{Tr}} \nu_F^q$$

*is exact.*

*Proof.* — By (2.1), the assertion is equivalent to the exactness of

$$k_q(E) \xrightarrow{N_{E/F}} k_q(F) \longrightarrow k_q(E) \xrightarrow{N_{E/F}} k_q(F).$$

We use the fact that the composite

$$K_q^M(E) \xrightarrow{N} K_q^M(F) \xrightarrow{i} K_q^M(E)$$

is multiplication by $p$. This fact is reduced to the case where any finite extension of F is of degree a power of $p$. In this case, $K_q^M(E)$ is generated by elements $\{x, y_1, \ldots, y_{q-1}\}$ such that $x \in E^*$, $y_1, \ldots, y_{q-1} \in F^*$ ([1], Ch. I (5.3)).

Now assume $x \in K_q^M(E)$ and $N(x) = py$, $y \in K_q^M(F)$. Then, $px = i \circ N(x) = pi(y)$. Since the $p$-primary torsion part of $K_q^M(E)$ is divisible by (2.8), we have $x - i(y) \in pK_q^M(E)$. This shows the exactness of $k_q(F) \to k_q(E) \to k_q(F)$. The exactness of $k_q(E) \to k_q(F) \to k_q(E)$ is proved similarly.

Now, we analyze the sequences (5.4.1) and (5.4.2) using the filtration on $Sh^q$.

*Lemma* **(5.7)**. — (i) $\mathrm{cor}(U^m h^q(L)) \subset U^m h^q(K)$ *for any* $m$.

(ii) *The following diagrams commute.*

$$\begin{array}{ccc} \nu_E^q \oplus \nu_E^{q-1} & \cong & \mathrm{gr}^0 h^q(L) \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{cor}} \\ \nu_F^q \oplus \nu_F^{q-1} & \cong & \mathrm{gr}^0 h^q(K) \end{array} \qquad \begin{array}{ccc} \Omega_E^{q-1} \oplus \Omega_E^{q-2} & \longrightarrow & \mathrm{gr}^m h^q(L) \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{cor}} \\ \Omega_F^{q-1} \oplus \Omega_F^{q-2} & \longrightarrow & \mathrm{gr}^m h^q(K) \end{array}$$

($m \geq 1$). *Here in the diagram on the right, the horizontal arrows are induced by* $\rho_m$ *defined using the same prime element* $\pi$.

*Proof.* — For $m \geq 1$, let $T_m$ be the image of

$$U^m h^1(L) \otimes S h^{q-1}(K) \to S h^q(L)$$
$$x \otimes y \mapsto x \cup \mathrm{res}(y).$$

By using (5.5) (ii), we can prove easily that

**(5.7.1)** For any $m \geq 1$, $U^m h^q(L)$ is generated by $T_m$ and $\mathrm{res}(U^m h^{q-1}(K)) \cup \{a\}$, where $\{a\}$ denotes the class of $a$ in $h^1(L)$.

By using

$$N_{L/K}(1 - xa^i) = 1 - x^p b^i \quad \text{for } 0 < i < p \text{ and } x \in K,$$

we have

**(5.7.2)**
$$N_{L/K}(U_L^{(m)}) \subset U_K^{(mp)} \quad \text{for } 1 \leq m \leq \frac{e}{p-1}$$
$$N_{L/K}(U_L^{(m)}) \subset U_K^{(m+e)} \quad \text{for } m \geq \frac{e}{p-1}.$$

Note that (5.7.1) and (5.7.2) prove (i). The commutativity of the diagrams in (ii) follows easily.

Now, for $m \geq 0$, let $S_m$ be the homology group of the complex

$$\mathrm{gr}^m h^q(K) \xrightarrow{\mathrm{res}} \mathrm{gr}^m h^q(L) \xrightarrow{\mathrm{cor}} \mathrm{gr}^m h^q(K).$$

By (5.6) and (5.7), we have

*Corollary* **(5.8)**. — $S_0 = (0)$.

*Lemma* **(5.9)**. — *For* $1 \leq m < e'$, *we have an isomorphism*

$$(EI^{q-1}/I^{q-1}) \oplus (EI^{q-2}/I^{q-2}) \cong S_m$$

*characterized by*

$$x\left(\frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_{q-1}}{y_{q-1}}, 0\right) \mapsto \{1 + \tilde{x}\pi^m, \tilde{y}_1, \ldots, \tilde{y}_{q-1}\}$$
$$\left(0, x\frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_{q-2}}{y_{q-2}}\right) \mapsto \{1 + \tilde{x}\pi^m, \tilde{y}_1, \ldots, \tilde{y}_{q-2}, \pi\}$$

*for* $x \in E$ *and* $y_1, \ldots, y_{q-1} \in F^*$, *where tildas indicate liftings.*

*Proof.* — Assume $p \mid m$. By (5.5) (ii), we have a commutative diagram for any $q$

$$\begin{array}{ccc}
EI^q/(I^q \cap Z^q_{1,E}) \oplus \sum_{i=1}^{p-1} \alpha^i I^{q-1} \oplus I^{q-1}/(I^{q-1} \cap Z^{q-1}_{1,E}) & \cong & \Omega^q_E/Z^q_{1,E} \\
\Big\downarrow \mathrm{pr}_2 & & \Big\downarrow \mathrm{Tr} \\
I^{q-1}/(I^{q-1} \cap Z^{q-1}_{1,E}) & \hookrightarrow & \Omega^q_F/Z^q_{1,F}
\end{array}$$

where the upper horizontal isomorphism is

$$(\omega, \omega', \omega'') \mapsto \omega + (\omega' + \omega'') \wedge \frac{d\alpha}{\alpha}$$

and the lower horizontal injection is $i(\omega) \mapsto \omega \wedge \frac{d\beta}{\beta}$. Now, (5.9) follows from (5.7) and (5.2) in this case. The proof for the case $p \nmid m$ is similar and is left to the reader.

To proceed further, we need

*Lemma* **(5.10)**. — *Assume* F *separably closed. Let* $\sigma$ *be a generator of* Gal(L/K) *and let* $e'' = \frac{e}{p-1}$. *Then,*

(i) $U_L^{(e'')} \subset (L^*)^{\sigma-1}.K^*$.
(ii) $U_L^{(e)} \subset (L^*)^p.K^*$.

*Proof.* — Note that $p : U_K^{(e'')} \to U_K^{(e')}$ is surjective (cf. [22], § (1.7)). Let $x \in U_L^{(e'')}$. Then by (5.7.2), we have $N(x) \in U_K^{(e')}$. Hence there is an element $y$ of $K^*$ such that $N(x) = y^p$. Since $N(xy^{-1}) = 1$, we have $x \in (L^*)^{\sigma-1}.K^*$ by Hilbert's theorem 90. Next, let $x \in U_L^{(e)}$. Then, $x^{\sigma-1} \in U_L^{(e')}$ since $\sigma$ acts on $O_L/\pi^{e''} O_L$ trivially. Hence there is an element $y$ of $U_L^{(e'')}$ such that $x^{\sigma-1} = y^p$. By (5.7.2), $N(y)$ is a $p$-th root of 1 contained in $U_K^{(e')}$. Since $\mathrm{ord}_K(\zeta_p - 1) = e''$, we have $N(y) = 1$. By Hilbert's theorem 90, $x^{\sigma-1} = (z^{\sigma-1})^p$ for some $z \in L^*$, and thus we obtain $xz^{-p} \in K^*$.

**(5.11)** Now we can prove (5.4). First we consider the sequence (5.4.1). Let $\sigma$ and $e''$ be as in (5.10). We have

$$(\sigma - 1)\,(U^m h^q(L)) \subset U^{m+e''} h^q(L) \quad \text{for } m \geq 0$$

and this induces $\sigma - 1 : S_m \to S_{m+e''}$, $m \geq 0$. We claim

**(5.11.1)** $\qquad \sigma - 1 : S_m \to S_{m+e''}$

is an isomorphism for $0 < m < e$.

Indeed, for $x \in O_K$ and letting $\zeta_p = \sigma(a)/a$, we have

$$\begin{aligned}(1 + a^i x\pi^m)^{\sigma-1} &= (1 + a^i \zeta_p^i x\pi^m)/(1 + a^i x\pi^m) \\ &\equiv 1 + ia^i x\pi^m(\zeta_p - 1) \mod \pi^{m+e''+1}.\end{aligned}$$

Our claim follows from this and from (5.9).

Now assume $x \in Sh^q(L)^G$ and $\mathrm{cor}(x) = 0$. We prove $x \in \mathrm{res}(Sh^q(K))$. We are reduced to the case $x \in U^1 h^q(L)$ by (5.8), and then to the case $x \in U^e h^q(L)$ by (5.11.1). Then we have $x \in T_e$ by (5.9) ($T_m$ is as in (5.7)). But $T_e \subset \mathrm{res}(Sh^q(K))$ by (5.10) (ii).

Next we consider the sequence (5.4.2). Assume $x \in Sh^q(L)$ and $\mathrm{cor}(x) = 0$.

We shall prove $x \in \mathrm{res}(\mathrm{S}h^q(\mathrm{K})) + (\sigma - 1)\,\mathrm{S}h^q(\mathrm{L})$. We are reduced to the case $x \in \mathrm{U}^1\,\mathrm{S}h^q(\mathrm{L})$ by (5.8). We prove

**(5.11.2)** If $x \in \mathrm{T}_m + \mathrm{U}^i h^q(\mathrm{L})$ with $1 \leq m \leq e''$ and $(m-1)p < i < mp$, and if $\mathrm{cor}(x) = 0$, then $x \in \mathrm{T}_m + \mathrm{U}^{i+1} h^q(\mathrm{L})$.

**(5.11.3)** If $x \in \mathrm{T}_m + \mathrm{U}^{mp} h^q(\mathrm{L})$ with $1 \leq m < e''$ and if $\mathrm{cor}(x) = 0$, then $x \in \mathrm{res}(\mathrm{U}^m h^q(\mathrm{K})) + \mathrm{T}_{m+1} + \mathrm{U}^{mp+1} h^q(\mathrm{L})$.

Once we have these assertions, we are reduced to the case

$$x \in \mathrm{T}_{e''} + \mathrm{U}^{e''p} h^q(\mathrm{L}) = \mathrm{T}_{e''}$$

and then we have $x \in \mathrm{res}(\mathrm{S}h^q(\mathrm{K})) + (\sigma - 1)\,\mathrm{S}h^q(\mathrm{L})$ by (5.10) (i).

The assertion (5.11.2) follows from (5.7.2) and (5.9) easily. To prove (5.11.3), let $\Phi$ be a subset of F such that $\Phi \amalg \{\beta\}$ form a $p$-base of F. For each $\varphi \in \Phi$, we fix a representative $\tilde{\varphi}$ of $\varphi$ in $\mathrm{O}_\mathrm{K}$. We endow $\Phi$ with a structure of totally order set. For $q \geq 0$, let $\Phi_q$ be the set of all strictly increasing functions $\{1, \ldots, q\} \to \Phi$, and let $\mathrm{E}\Phi_q$ be the free E-module with base $\Phi_q$. We obtain a surjective homomorphism

$$\mathrm{E}\Phi_{q-1} \oplus \mathrm{E}\Phi_{q-2} \oplus \mathrm{I}^{q-2}/(\mathrm{I}^{q-2} \cap \mathrm{Z}^{q-2}_{1,\mathrm{E}}) \oplus \mathrm{I}^{q-3}/(\mathrm{I}^{q-3} \cap \mathrm{Z}^{q-3}_{1,\mathrm{E}})$$
$$\to (\mathrm{T}_m + \mathrm{U}^{mp} h^q(\mathrm{L}))/(\mathrm{T}_{m+1} + \mathrm{U}^{mp+1} h^q(\mathrm{L}))$$

$$((\sum_{r=0}^{p-1} x_{r,\varphi}\,\alpha^r)_{\varphi \in \Phi_{q-1}}, 0, 0, 0) \mapsto \sum_\varphi \{1 + \sum_{r=0}^{p-1} \tilde{x}_{r,\varphi}\, a^r \pi^m, \widetilde{\varphi(1)}, \ldots, \widetilde{\varphi(q-1)}\},$$

$$(0, (\sum_{r=0}^{p-1} x_{r,\varphi}\,\alpha^r)_{\varphi \in \Phi_{q-2}}, 0, 0) \mapsto \sum_\varphi \{1 + \sum_{r=0}^{p-1} \tilde{x}_{r,\varphi}\, a^r \pi^m, \widetilde{\varphi(1)}, \ldots, \widetilde{\varphi(q-2)}, \pi\},$$

$$(0, 0, \omega, \omega') \mapsto \bar{\rho}_{mp}\left(\omega \wedge \frac{d\alpha}{\alpha}, \omega' \wedge \frac{d\alpha}{\alpha}\right),$$

where $x_{r,\varphi} \in \mathrm{F}$. The composite of this homomorphism with

$$(\mathrm{T}_m + \mathrm{U}^{mp} h^q(\mathrm{L}))/(\mathrm{T}_{m+1} + \mathrm{U}^{mp+1} h^q(\mathrm{L})) \xrightarrow{\mathrm{cor}} \mathrm{gr}^{mp} h^q(\mathrm{K}) \cong \mathrm{B}^q_{1,\mathrm{F}} \oplus \mathrm{B}^{q-1}_{1,\mathrm{F}}$$

is given by $\theta_{q-1} \oplus \theta_{q-2}$, where

$$\theta_q : \mathrm{E}\Phi_q \oplus \mathrm{I}^{q-1}/(\mathrm{I}^{q-1} \cap \mathrm{Z}^{q-1}_{1,\mathrm{E}}) \mapsto \mathrm{B}^{q+1}_{1,\mathrm{F}}$$

$$((\sum_{r=0}^{p-1} x_{r,\varphi}\,\alpha^r)_\varphi, i(\omega) \mod \mathrm{Z}^{q-1}_{1,\mathrm{E}})$$
$$\mapsto \left(\left(\sum_\varphi \sum_{r=1}^{p-1} x^p_{r,\varphi}\, r\beta^r \frac{d\varphi(1)}{\varphi(1)} \wedge \ldots \wedge \frac{d\varphi(q)}{\varphi(q)}\right) \pm d\omega\right) \wedge \frac{d\beta}{\beta}.$$

If $((\sum_{r=0}^{p-1} x_{r,\varphi}\,\alpha^r)_\varphi, i(\omega) \mod \mathrm{Z}^{q-1}_{1,\mathrm{E}})$ is contained in $\mathrm{Ker}(\theta_q)$, since $\Phi$ is a part of a $p$-base $\Phi \amalg \{\alpha\}$ of E, we have $x_{r,\varphi} = 0$ for $0 < r < p$ and for all $\varphi \in \Phi_q$, and we also have $i(\omega) \mod \mathrm{Z}^{q-1}_{1,\mathrm{E}} = 0$. This proves (5.11.3).

*Theorem* **(5.12).** — *Let* K *be a henselian discrete valuation field with residue field* F *such that* $\mathrm{char}(K) = 0$ *and* $\mathrm{char}(F) = p > 0$. *Then, the cohomological symbol*

$$h^q_{p^n,K} : K^M_q(K)/p^n K^M_q(K) \to H^q(\mathrm{Sp}\, K, \mathbf{Z}/p^n \mathbf{Z}(q))$$

*is bijective for any* $q$ *and any* $n$.

We are reduced to the case $n = 1$ and $\zeta_p \in K$ by the following general lemma.

*Lemma* **(5.13).** — *Let* $k$ *be a field and* $p$ *a prime number which is invertible in* $k$. *Let* $E = k(\zeta_p)$ *where* $\zeta_p$ *is a primitive* $p$*-th root of* 1. *Fix* $q \geq 0$.

(i) *If the cohomological symbol* $h^q_{p,E}$ *is surjective,* $h^q_{p^n,K}$ *is surjective for any* $n$.

(ii) *If* $h^q_{p,E}$ *is injective and* $h^{q-1}_{p,E}$ *is surjective, then* $h^q_{p^n,k}$ *is injective for any* $n$.

The proof is identical with the case $q = 2$ treated in [19].

**(5.14)** We prove the surjectivity of $h^q_{p,K}$ in the case F is separably closed and $\zeta_p \in K$. Let $C(K)$ be the quotient $h^q(K)/Sh^q(K)$. For the proof that $C(K) = 0$, it is sufficient to show the injectivity of $C(K) \to C(L)$ for any extension $L/K$ of the type of (5.4). Indeed, as an inductive limit of successions of such extensions, one obtains a henselian discrete valuation field $\widetilde{K}$ with algebraically closed residue field. The cohomological dimension of $\widetilde{K}$ is one ([16], Ch. II, (3.3)), whence $h^q(\widetilde{K}) = 0$ for $q \geq 2$. Hence $C(\widetilde{K}) = 0$ and this will imply $C(K) = 0$ if we prove the injectivity of $C(K) \to C(L)$. Let $G = \mathrm{Gal}(L/K)$ and consider the diagram with exact rows

**(5.14.1)**
$$\begin{tikzcd}
0 \arrow[r] & Sh^q(K) \arrow[r] \arrow[d] & h^q(K) \arrow[r] \arrow[d, "\mathrm{res}"] & C(K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Sh^q(L)^G \arrow[r] & h^q(L)^G \arrow[r] \arrow[d, "\mathrm{cor}"] & C(L)^G & \\
 & & h^q(K) & &
\end{tikzcd}$$

(note that $\mathrm{cor} \circ \mathrm{res} = 0$). By induction on $q$, we may assume $h^{q-1}(K) = Sh^{q-1}(K)$ and $h^{q-1}(L) = Sh^{q-1}(L)$. Then, by (5.4), the sequence

$$h^{q-1}(K) \xrightarrow{\mathrm{res}} h^{q-1}(L)_G \xrightarrow{\mathrm{cor}} h^{q-1}(K)$$

is exact. Hence, by (3.2), the sequence

**(5.14.2)** $$Sh^{q-1}(K) \xrightarrow{\cup\{b\}} h^q(K) \xrightarrow{\mathrm{res}} h^q(L)$$

is exact. By the diagram (5.14.1), the injectivity of $C(K) \to C(L)$ follows from the exactness of (5.14.2) and that of (5.4.1).

**(5.15)** Now we prove the bijectivity of $h^q_{p,K}$ assuming $\zeta_p \in K$. Note that we have already

$$k_q(K)/U^{e'} k_q(K) \xrightarrow{\simeq} Sh^q(K)/U^{e'} h^q(K).$$

Let $K_{nr}$ denote the maximal unramified extension of K, $F_s$ the separable closure of F, and let $G_F = \mathrm{Gal}(F_s/F) \cong \mathrm{Gal}(K_{nr}/K)$. One has $\Omega^*_{F_s} = \Omega^*_F \otimes_F F_s$, whence also $B^*_{F_s} \cong B^*_F \otimes_{F^p} F^p_s$. In particular by (5.2), $\mathrm{gr}^m h^q(K_{nr}) \cong \mathrm{gr}^m h^q(K) \otimes_{F^p} F^p_s$ for $1 \leq m < e'$, so

$$H^0(G_F, U^1 h^q(K_{nr})) \cong U^1 h^q(K)/U^{e'} h^q(K)$$
$$H^r(G_F, U^1 h^q(K_{nr})) = 0 \quad r > 0.$$

The exact sequences (cf. (5.3))

$$0 \longrightarrow U^1 h^q(K_{nr}) \longrightarrow h^q(K_{nr}) \longrightarrow \nu^q_{F_s} \oplus \nu^{q-1}_{F_s} \longrightarrow 0$$
$$0 \longrightarrow \nu^q_{F_s} \longrightarrow Z^q_{1,F_s} \xrightarrow{1-C} \Omega^q_{F_s} \longrightarrow 0$$

give

$$H^0(G_F, h^q(K_{nr})) = Sh^q(K)/U^{e'} h^q(K) \cong k_q(K)/U^{e'} k_q(K)$$
$$H^1(G_F, h^q(K_{nr})) = (\Omega^q_F/(1 - C) Z^q_{1,F}) \oplus (\Omega^{q-1}_F/(1 - C) Z^{q-1}_{1,F})$$
$$H^r(G_F, h^q(K_{nr})) = 0 \quad r \geq 2.$$

The spectral sequence with $\mathbf{Z}/p\mathbf{Z}$ coefficients

$$H^r(G_F, h^q(K_{nr})) \Rightarrow h^{q+r}(K)$$

yields exact sequences

**(5.15.1)**
$$0 \to (\Omega^{q-1}_F/(1 - C) Z^{q-1}_1) \oplus (\Omega^{q-2}_F/(1 - C) Z^{q-2}_1)$$
$$\to h^q(K) \to k_q(K)/U^{e'} k_q(K) \to 0.$$

As in (5.1), let $a$ be the residue class of $p\pi^{-e}$. The congruence

$$(1 - \zeta_p)^{p-1} \equiv -p \bmod \pi^{e+1}$$

shows that multiplication by the residue class of $(1 - \zeta_p)^{p\pi^{-e'}}$ gives a morphism

$$\Omega^*_F/(1 - C) Z^*_1 \to \Omega^*_F/(1 + aC) Z^*_1.$$

So by (5.1), the exact sequence (5.15.1) shows that $U^{e'} k_q(K) \cong U^{e'} h^q(K)$ and also $k_q(K)/U^{e'} k_q(K) \cong h^q(K)/U^{e'} h^q(K)$.

## 6. The sheaf $M^q_n$

Our objective in this section is to prove theorems (1.4) and (1.5) describing the sheaf $M^q_n$ on $Y_{et}$.

We first determine the structure of $M^q_1$. Let $U^m M^q_1 \subset M^q_1$ $(m \geq 1)$ be as in (1.2) and let $U^0 M^q_1 = M^q_1$. Without loss of generality, we may assume that Y is connected. Let $\nu$ be the generic point of Y, and let $\tau : \nu \to Y$ be the canonical map. Note that the structure of $\tau^* M^q_1$ is known by the preceeding section, for the stalk $M^q_{1,\bar{\nu}}$ is isomorphic to the Galois cohomology group of the quotient field of the strictly henselian discrete valuation ring $\mathcal{O}_{x,\bar{\nu}}$.

*Proposition* **(6.1).** — *Let the notation be as above. Then,*

(i) $M_1^q \to \tau_* \tau^* M_1^q$ *is injective.*

(ii) *For any* $m \geq 0$, *the inverse image of* $\tau_* \tau^*(U^m M_1^q)$ *in* $M_1^q$ *coincides with* $U^m M_1^q$.

(iii) *The graded sheaves* $\mathrm{gr}^m(M_1^q)$ *are described as in* (1.4.1).

In the first version of this paper, we proved this structure of $M_1^q$ using an injectivity theorem of O. Gabber. For $y \in Y$, let $\mathcal{O}_{x,\bar{y}}$ be the strict henselization, and let $\widetilde{K}$ be the quotient field of the henselization of $\mathcal{O}_{X,y}$ at the generic point of the special fiber of $\mathrm{Sp}\, \mathcal{O}_{X,\bar{y}}$. Gabber proved that

$$M_{n,\bar{y}}^q = H^q\left(\mathrm{Sp}\left(\mathcal{O}_{X,\bar{y}}\left[\frac{1}{p}\right]\right), \mathbf{Z}/p^n\,\mathbf{Z}(q)\right) \to H^q(\mathrm{Sp}\,\widetilde{K}, \mathbf{Z}/p^n\,\mathbf{Z}(q))$$

is injective, as a consequence of his general results [7] [8]. In the case $n = 1$, this is nearly (6.1) (i). It is possible to prove (6.1) using this injectivity, but in this paper we adopt another simpler method found later.

*Proof of* (6.1). — We first prove the injectivity of $M_1^q \to \tau_* \tau^* M_1^q$. Let T be its kernel. Since the problem is etale local, we may assume that $X = \mathbf{P}_\Lambda^n$ and $k$ is separably closed. Furthermore, by induction on $n$, we may assume $n \geq 1$ and that the stalk of T at any non-closed point of $\mathbf{P}_k^n$ is zero. We may assume also that $\zeta_p \in K$, by a trace argument. Let $G = \mathrm{Aut}(\mathbf{P}_\Lambda^n)$ be the projective general linear group, $\mathbf{Z}[G]$ the group ring, and let $I \subset \mathbf{Z}[G]$ be the augmentation ideal. The ring $\mathbf{Z}[G]$ acts on the cohomology groups $H^t(\mathbf{P}_k^n, M_1^r)$. Since T is a skyscraper sheaf,

$$(0) \neq T \Rightarrow I^N . \Gamma(\mathbf{P}_k^n, M_1^q) \neq (0), \quad \text{any } N \geq 1.$$

On the other hand, by induction on $q$ we may assume that (6.1) holds for $M_1^{q-t}$ for any $t \geq 1$. In particular, $M_1^{q-t}$ will have a filtration stable under G whose graded pieces are direct sums of sheaves like

$$\Omega^*_{\mathbf{P}_k^n, \log}, \quad \Omega^*_{\mathbf{P}_k^n}, \quad d\Omega^*_{\mathbf{P}_k^n}.$$

These are absolute differentials and not relative to $k$, but $\Omega^q_{\mathbf{P}_k^n}$ has a filtration stable under G whose graded pieces are isomorphic to $\Omega^i_{\mathbf{P}_k^n/k} \otimes \Omega_k^{q-i}$, and there are exact sequences

$$0 \longrightarrow \Omega^q \xrightarrow{C^{-1}} \Omega^q/d\Omega^{q-1} \longrightarrow d\Omega^q \longrightarrow 0$$

$$0 \longrightarrow \Omega^q_{\log} \longrightarrow \Omega^q \xrightarrow{1-C^{-1}} \Omega^q/d\Omega^{q-1} \longrightarrow 0.$$

Thus we will have (since I kills $H^*(\mathbf{P}_k^n, \Omega^q_{\mathbf{P}^n/k})$)

$$I^N H^*(\mathbf{P}_k^n, M_1^{q-t}) = (0), \quad t \geq 1, \ N \gg 0.$$

This implies

$$I^N . E_\infty^{0,q} \neq (0)$$

in the spectral sequence

$$E_2^{s,t} = H^s(\mathbf{P}_k^n, M_1^t) \Rightarrow H^{s+t}(\mathbf{P}_k^n, \mathbf{Z}/p\mathbf{Z}).$$

But $H^q(\mathbf{P}^n_K, \mathbf{Z}/p\mathbf{Z})$ maps surjectively on $E^{0,q}_\infty$ and

$$H^q(\mathbf{P}^n_K) \cong \bigoplus_\ell H^\ell(\mathbf{P}^n_{\bar{K}}) \otimes H^{q-\ell}(\mathrm{Sp}\, K)$$

so
$$I^N . H^q(\mathbf{P}^n_K) = (0) \quad \text{for } N \geq 1.$$

This contradiction implies $T = (0)$.

Let $V^m M^q_1 \subset M^q_1$ $(m \geq 0)$ be the inverse image of $\tau_* \tau^* U^m M^q_1$, and let $\mathrm{gr}^m_V(M^q_1) = V^m M^q_1 / V^{m+1} M^q_1$. By the injectivity of $M^q_1 \to \tau_* \tau^* M^q_1$, we have using (5.1) (ii) $V^m M^q_1 = 0$ for $m \geq e' = \dfrac{ep}{p-1}$ with $e = e_K$. Furthermore, we have

$$\left.\begin{array}{ll} \Omega^q_{Y,\log} \oplus \Omega^{q-1}_{Y,\log} & m = 0 \\ \Omega^{q-1}_Y & 0 < m < e' \\ & p \nmid m \\ B^q_1 \oplus B^{q-1}_1 & 0 < m < e' \\ & p \mid m \end{array}\right\} \underset{(*)}{\hookrightarrow} \mathrm{gr}^m_V(M^q_1)$$

$$\mathrm{gr}^m_V(M^q_1) \hookrightarrow \tau_* \tau^* \mathrm{gr}^m M^q_1 = \begin{cases} \tau_* \tau^* \Omega^q_{Y,\log} \oplus \tau_* \tau^* \Omega^{q-1}_{Y,\log} \\ \tau_* \tau^* \Omega^{q-1}_Y \\ \tau_* \tau^* B^q_1 \oplus \tau_* \tau^* B^{q-1}_1, \end{cases}$$

and we must show that the inclusion $(*)$ is an equality for $0 \leq m < e'$. Indeed, for $0 < m < e'$, the sheaves on the left map onto $\mathrm{gr}^m_U M^q_1$ as in § 4. Since $\Omega^q_Y \to \tau_* \tau^* \Omega^q_Y$ is injective, we see that $(*)$ is an injection. For $m = 0$, the inclusion

$$\Omega^q_{Y,\log} \oplus \Omega^{q-1}_{Y,\log} \hookrightarrow \mathrm{gr}^m_V(M^q_1)$$

follows from the fact ([10], Th. (02.4.2)) that $\Omega^q_{Y,\log}$ is generated etale locally by logarithmic forms.

Now, let T be the cokernel of the map $(*)$. We may assume again that T is a skyscraper sheaf. We proceed just as above. By downward induction on $m$, $V^{m+1} M^q_1 = U^{m+1} M^q_1$ and it has the structure described in (6.1). Hence

$$I^N . H^*(\mathbf{P}^n_k, V^{m+1} M^q_1) = (0) \quad \text{for some } N \geq 0.$$

From this, we have again

$$(0) \neq T \Rightarrow I^N . \Gamma(\mathbf{P}^n_k, M^q_1) \neq (0) \quad \text{any } N \geq 1.$$

Now the same argument as above proves that $T = (0)$.

Next we study $M^q_n$ for $n \geq 1$.

*Corollary* **(6.1.1)**. — *For* $n \geq 1$, $M^q_n$ *is generated locally by symbols.*

This is reduced to the case $n = 1$ by induction on $n$ using the exact sequence $M^q_{n-1} \to M^q_n \to M^q_1$, and the case $n = 1$ follows from (6.1).

For $0 < m < e'$ and $n, q \geq 0$, let $G_n^q$ be the sheaf on Y defined to be the sheafification of $U \mapsto {}^m Gr_n^q$ for $R = \mathcal{O}(U)$ defined in (4.7).

*Proposition* **(6.2)**. — ${}^m G_n^q \xrightarrow{\cong} \mathrm{gr}^m M_n^q$ *for* $0 < m < e'$.

Since the problem is to prove the injectivity, we are reduced to the case $\zeta_p \in K$. The exact sequence on V

$$0 \longrightarrow \mathbf{Z}/p^{n-1}\mathbf{Z}(q) \xrightarrow{\text{"}p\text{"}} \mathbf{Z}/p^n\mathbf{Z}(q) \longrightarrow \mathbf{Z}/p\mathbf{Z}(q) \longrightarrow 0$$

and the isomorphism $M_1^{q-1} \cong i^* R^{q-1} j_*(\mathbf{Z}/p\mathbf{Z}(q))$ induced by $\mathbf{Z}/p\mathbf{Z} \xrightarrow{\cong} \mathbf{Z}/p\mathbf{Z}(1)$; $1 \to \zeta_p$ yield an exact sequence

$$M_1^{q-1} \xrightarrow{\delta} M_{n-1}^q \xrightarrow{\text{"}p\text{"}} M_n^q \longrightarrow M_1^q \longrightarrow 0.$$

Here the surjectivity of $M_n^q \to M_1^q$ follows from the fact that $M_1^q$ is generated locally by symbols.

*Lemma* **(6.3)**. — *The boundary map* $\delta$ *satisfies*

$$\delta(\{x_1, \ldots, x_{r-1}\}) = \{\zeta_p, x_1, \ldots, x_{r-1}\}.$$

*In particular,* $\delta(M_1^{q-1}) \subset U^{e''} M_{n-1}^q$ *where* $e'' = \dfrac{e}{p-1}$. (*Note that* $\mathrm{ord}_K(\zeta_p - 1) = e''$.)

The proof is straightforward and left to the reader.

*Lemma* **(6.4)**. — *Let* $1 \leq m < e'$ *and let* $m_0$ *be the smallest integer such that* $m \leq pm_0$. *Then the sequence*

$$U^{m_0} M_{n-1}^q \xrightarrow{\text{"}p\text{"}} U^m M_n^q \longrightarrow U^m M_1^q \longrightarrow 0$$

*is exact.*

*Proof.* — If $\ell \geq e'$ is an integer, then as sheaves in the etale topology

$$1 + \pi^\ell i^* \mathcal{O}_X \subset (1 + \pi^{\ell - e} i^* \mathcal{O}_X)^p$$

whence $\qquad U^\ell M_n^q \subset \text{"}p\text{"}\, U^{\ell - e} M_{n-1}^q.$

Taking ${}^m G_n^q = (0)$ if $m$ is not an integer, the lemma will follow from the exactness of the sequences

**(6.4.1)** $\qquad {}^{m/p}G_{n-1}^q \xrightarrow{\text{"}p\text{"}} {}^m G_n^q \longrightarrow {}^m G_1^q \longrightarrow 0, \quad 1 \leqslant m < e',$

and from ${}^m G_1^q \cong \mathrm{gr}^m M_1^q$ (6.1) (iii).

We show the exactness of (6.4.1). If $p \nmid m$, then

$${}^m G_n^q \cong \Omega_Y^{q-1} \cong {}^m G_1^q.$$

Also ${}^m G_{n'}^q \twoheadrightarrow {}^m G_n^q$ for $n' > n$ so it suffices to consider the case $n > s$, $p \mid m$. In this case, the map "$p$" is induced by the inverse Cartier operator $C^{-1}$ and one finds

$${}^m G_n^q / (\text{"}p\text{"}\, {}^{m/p}G_{n-1}^q) \cong (\Omega_Y^{q-1}/Z_1^{q-1}) \oplus (\Omega_Y^{q-1}/Z_1^{q-2}) \cong {}^m G_1^q.$$

*Lemma* **(6.5)**. — *For* $1 \leq m < e'$, *the sequences*

$$0 \longrightarrow \mathrm{gr}^{m/p}\, \mathrm{M}^q_{n-1} \xrightarrow{\text{“}p\text{”}} \mathrm{gr}^m\, \mathrm{M}^q_n \longrightarrow \mathrm{gr}^m\, \mathrm{M}^q_1 \longrightarrow 0$$

*are exact.* (*By convention* $\mathrm{gr}^x = 0$ *if* $x \notin \mathbf{Z}$.)

If $p \nmid m$, this follows from

$$\begin{array}{ccc} {}^m\mathrm{G}^q_n & \cong & {}^m\mathrm{G}^q_1 \\ \downarrow & & \wr\| \\ \mathrm{gr}^m\, \mathrm{M}^q_n & \longrightarrow & \mathrm{gr}^m\, \mathrm{M}^q_1. \end{array}$$

If $p \mid m$, by (6.3) and (6.4), we have exact sequences

$$\begin{array}{ccccccccc} \mathrm{M}^{q-1}_1 & \xrightarrow{\delta} & \mathrm{U}^{m/p+1}\, \mathrm{M}^q_{n-1} & \xrightarrow{\text{“}p\text{”}} & \mathrm{U}^{m+1}\, \mathrm{M}^q_n & \longrightarrow & \mathrm{U}^{m+1}\, \mathrm{M}^q_1 & \longrightarrow & 0 \\ \| \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{M}^{q-1}_1 & \xrightarrow{\delta} & \mathrm{U}^{m/p}\, \mathrm{M}^q_{n-1} & \xrightarrow{\text{“}p\text{”}} & \mathrm{U}^m\, \mathrm{M}^q_n & \longrightarrow & \mathrm{U}^m\, \mathrm{M}^q_1 & \longrightarrow & 0 \end{array}$$

which prove the lemma.

Now, the proof of (6.2) follows using (6.5), (6.4.1) and induction on $n$.

**(6.6)** Fix a prime element $\pi$ of K. We prove that there is an isomorphism

**(6.6.1)** $$\mathrm{gr}^0\, \mathrm{M}^q_n \to \mathrm{W}_n\, \Omega^q_{\mathrm{Y},\log} \oplus \mathrm{W}_n\, \Omega^{q-1}_{\mathrm{Y},\log}$$

such that

$$\{f_1, \ldots, f_q\} \mapsto (d \log \bar{f}_1 \ldots d \log \bar{f}_q, 0)$$
$$\{f_1, \ldots, f_{q-1}, \pi\} \mapsto (0, d \log \bar{f}_1 \ldots d \log \bar{f}_{q-1})$$

for any local sections $f_1, \ldots, f_q$ of $i^*(\mathcal{O}^*_{\mathrm{X}})$. Here $\bar{f}_i$ is the image of $f_i$ in $\mathcal{O}^*_{\mathrm{Y}}$ and $d \log : \mathcal{O}^*_{\mathrm{Y}} \to \mathrm{W}_n\, \Omega^1_{\mathrm{Y},\log}$ is the homomorphism of [10], (3.23.1). We prove also the existence of the canonical homomorphism

**(6.6.2)** $$\mathrm{M}^q_n \to \Omega^q_{\mathrm{X}/\mathrm{S}}/p^n\, \Omega^q_{\mathrm{X}/\mathrm{S}}$$

stated in (1.5).

By (5.12), $\mathrm{M}^q_{n,\bar{\mathrm{v}}}$ is isomorphic to the mod $p^n$-Milnor K-group of the quotient field of $\mathcal{O}^q_{\mathrm{X},\bar{\mathrm{v}}}$. This proves the existence of the vertical arrows having the desired properties in the following diagrams:

$$\begin{tikzcd}
 & M_n^q \arrow[r] & \tau_* \tau^* M_n^q \arrow[d] \\
W_n \Omega^q_{Y,\log} \oplus W_n \Omega^{q-1}_{Y,\log} \arrow[rr, hook] & & \tau_* \tau^*(W_n \Omega^q_{Y,\log} \oplus W_n \Omega^{q-1}_{Y,\log})
\end{tikzcd}$$

$$\begin{tikzcd}
 & M_n^q \arrow[r] & \tau_* \tau^* M_n^q \arrow[d] \\
\Omega^q_{X/S}/p^n \Omega^q_{X/S} \arrow[rr, hook] & & \tau_* \tau^*(\Omega^q_{X/S}/p^n \Omega^q_{X/S})
\end{tikzcd}$$

The existence of the homomorphisms (6.6.1) and (6.6.2) follows from the injectivity of the lower horizontal arrows and from the fact that $M_n^q$ is generated locally by symbols.

The bijectivity of the homomorphism (6.6.1) is reduced to the case $n = 1$ by the diagram

$$\begin{tikzcd}
 & M_{n-1}^q/U^1 M_{n-1}^q \arrow[r, "\text{“}p\text{”}"] \arrow[dl] & M_n^q/U^1 M_n^q \arrow[r] \arrow[d] & M_1^q/U^1 M_1^q \arrow[r] \arrow[dl] & 0 \\
0 \arrow[r] & W_{n-1} \Omega^q_{Y,\log} \oplus W_{n-1} \Omega^{q-1}_{Y,\log} \arrow[r] & W_n \Omega^q_{Y,\log} \oplus W_n \Omega^{q-1}_{Y,\log} \arrow[r] & \Omega^q_{Y,\log} \oplus \Omega^{q-1}_{Y,\log} \arrow[r] & 0
\end{tikzcd}$$

Lastly, we give a description of a sheaf $L_n$ on $Y_{zar}$ closely related to $M_n^q$. Let $\varepsilon : Y_{et} \to Y_{zar}$ be the canonical morphism of sites. Let $L_n^q$ be the Zariski sheaf associated to the presheaf $U \mapsto H^q_{et}(U, i^* \mathbf{R}j_*(\mathbf{Z}/p^n \mathbf{Z}(q)))$, so,

$$L_n^q = R^q \varepsilon_* i^* \mathbf{R}j_*(\mathbf{Z}/p^n \mathbf{Z}(q)),$$

where the notations $i^*$ and $\mathbf{R}j_*$ are used in the sense of etale topology as before. Then, the etale sheafification of $L_n^r$ is $M_n^r$. By Gabber [9], the stalk $L_{n,y}^q$ of $L_n^q$ at $y \in Y$ coincides with the etale cohomology group $H^q_{et}\left(\mathrm{Sp}\left(A\left[\frac{1}{p}\right]\right), \mathbf{Z}/p^n \mathbf{Z}(q)\right)$ where $A = (\varepsilon_* i^* \mathcal{O}_X)_y$ is the "henselization along Y" of $\mathcal{O}_{X,y}$. Thus the study of $L_n^q$ is a natural generalization of the study of the Galois cohomology of henselian discrete valuation fields in § 5. Define the filtration of $L_n^q$ in the same way as in the case of $M_n^q$. As in (5.15), by using the spectral sequence

$$E_2^{st} = R^s \varepsilon_* M_1^t \Rightarrow L_1^{s+t} \quad \text{assuming } \zeta_p \in K,$$

we can deduce from (1.4.1) a structure theorem of $L_1^q$. The structures of $\mathrm{gr}^m(L_n^q)$ with $n \geq 2$ and $0 \leq m \leq e'$ are obtained by the methods of (6.2) and (6.3).

*Theorem* **(6.7)**. — (i) $\mathrm{gr}^0(\mathrm{L}_n^q) \cong \mathrm{W}_n\,\Omega^q_{\mathrm{Y},\log} \oplus \mathrm{W}_n\,\Omega^{q-1}_{\mathrm{Y},\log}$ *and* $\mathrm{gr}^m(\mathrm{L}_n^q) \cong {}^m\mathrm{G}_n^q$ *for* $0 < m < e'$. *Here* $\mathrm{W}_n\,\Omega^*_{\mathrm{Y},\log}$ *and* ${}^m\mathrm{G}_n^q$ *denote the restrictions to the Zariski site of their etale versions.*

(ii) *Assume that* $e'$ *is an integer and let* $e' = p^s r$, $s \geq 0$, $p \nmid r$. *Take a prime element* $\pi$ *of* K *and let* $a \in k$ *be the residue class of* $p\pi^{-e}$. *Then, for* $n \leq s$, $\mathrm{gr}^{e'}(\mathrm{L}_n^q)$ *is isomorphic to*

$$(\Omega_{\mathrm{Y}}^{q-1}/(1 + a\mathrm{C})\,\mathrm{Z}_n) \oplus (\Omega_{\mathrm{Y}}^{q-2}/(1 + a\mathrm{C})\,\mathrm{Z}_n)$$

*where the quotients are taken in the sense of Zariski topology. For* $n > s$, $\mathrm{gr}^{e'}(\mathrm{L}_n^q)$ *is isomorphic to the cokernel of*

$$\Omega_{\mathrm{Y}}^{q-2} \to (\Omega_{\mathrm{Y}}^{q-1}/(1 + a\mathrm{C})\,\mathrm{B}_s) \oplus (\Omega_{\mathrm{Y}}^{q-2}/(1 + a\mathrm{C})\,\mathrm{B}_s)$$
$$\omega \mapsto ((1 + a\mathrm{C})\,\mathrm{C}^{-s}(d\omega), (-1)^q r(1 + a\mathrm{C})\,\mathrm{C}^{-s}(\omega)).$$

*Remark* **(6.8)**. — Contrary to the case $0 \leq m < e'$, $\mathrm{gr}^{e'}(\mathrm{L}_n^q)$ and $\mathrm{gr}^{e'}(\mathrm{M}_n^q)$ are not determined by only $n$, $q$ and $e$. Their structures depend in a subtle way upon the nature of K. The structures of $\mathrm{gr}^m(\mathrm{L}_n^q)$ and $\mathrm{gr}^m(\mathrm{M}_n^q)$ with $m > e'$ seem to be closely related to the number of roots of 1 of $p$-primary orders contained in K.

## 7. Ordinary Varieties

Throughout this paragraph, Y will denote a complete, smooth variety over a perfect field $k$ of characteristic $p > 0$. For simplicity we write

$$\mathrm{Z}^q = \mathrm{Z}_1^q = \mathrm{Ker}(d : \Omega_{\mathrm{Y}}^q \to \Omega_{\mathrm{Y}}^{q-1}),$$
$$\mathrm{B}^q = \mathrm{B}_1^q = \mathrm{Im}(d : \Omega_{\mathrm{Y}}^{q-1} \to \Omega_{\mathrm{Y}}^q).$$

Sheaves and cohomology will be taken with respect to the etale topology unless otherwise indicated. The results of this section overlap with results of L. Illusie [21, IV (4.12), (4.13)].

*Lemma* **(7.1)**. — *Assume the field k above is algebraically closed. Fix an integer* $r \geq 0$, *and assume*

$$\mathrm{H}^q(\mathrm{Y}, \mathrm{B}^r) = (0) \textit{ for all } q.$$

*Then*:

(i) *The natural maps*

$$k \otimes_{\mathbf{Z}/p\mathbf{Z}} \mathrm{H}^q(\mathrm{Y}, \Omega^r_{\mathrm{Y},\log}) \to \mathrm{H}^q(\mathrm{Y}, \Omega^r)$$

*are injective for all q.*

(ii) *The homomorphisms in* (i) *are bijective for all q if and only if* $\mathrm{H}^q(\mathrm{B}^{r+1}) = (0)$ *for all q.*

(iii) *If the equivalent conditions in* (ii) *are satisfied, the map*

$$W_n(k) \otimes_{\mathbf{Z}/p^n\mathbf{Z}} H^q(W_n \Omega^r_{Y,\log}) \to H^q(W_n \Omega^r_Y)$$

*is an isomorphism for all $q$ and $n$.*

(iv) *Assume*

$$(*) \qquad (\textit{multiplicity of slope } r \textit{ in } H^{q+r}_{\mathrm{crys}}(Y/W)) = \dim_k H^q(Y, \Omega^r_Y) \textit{ for all } q.$$

*Then the groups* $H^q(Y, W\Omega^r_{Y,\log})$ *are torsion free for all $q$, and the equivalent conditions* (ii) *hold. In the absence of torsion in* $H^*(W\Omega^r_{Y,\log})$, *condition* $(*)$ *is equivalent to* (ii).

*Proof.* — For a later application, we prove that $H^q(B^r) = (0)$ implies $H^q(\Omega^r_{Y,\log}) \otimes k \hookrightarrow H^q(\Omega^r_Y)$ fixing $q$ and $r$. Consider the sequence

$$0 \longrightarrow \Omega^r_{Y,\log} \longrightarrow \Omega^r_Y \xrightarrow{1-C^{-1}} \Omega^r_Y/B^r \longrightarrow 0.$$

By $H^q(B^r) = 0$, the homomorphism $H^{q-1}(\Omega^r_Y) \to H^{q-1}(\Omega^r_Y/B^r)$ induced by the projection $\Omega^r_Y \to \Omega^r_Y/B^r$ is surjective, and hence $1 - C^{-1} : H^{q-1}(\Omega^r_Y) \to H^{q-1}(\Omega^r_Y/B^r)$. This shows that $H^q(\Omega^r_{Y,\log}) \hookrightarrow H^q(\Omega^r_Y)$. Let $i : H^q(\Omega^r_Y) \to H^q(\Omega^r_Y/B^r)$ be the injection induced by the projection $\Omega^r_Y \to \Omega^r_Y/B^r$, and take a $k$-linear map $t : H^q(\Omega^r_Y/B^r) \to H^q(\Omega^r_Y)$ such that $t \circ i$ is the identity map. Then, the $p$-linear map $t \circ C^{-1}$ acts on $H^q(\Omega^r_Y)$. Since $H^q(\Omega^r_{Y,\log}) \hookrightarrow \mathrm{Ker}(1 - t \circ C^{-1})$, we have $H^q(\Omega^r_{Y,\log}) \otimes k \hookrightarrow H^q(\Omega^r_Y)$ by $p$-linear algebra.

Now fix $r$ and assume $H^q(B^r) = 0$ for all $q$. Then, the above homomorphism $t$ is bijective and

$$\mathrm{Ker}(1 - t \circ C^{-1}) = \mathrm{Ker}(1 - C^{-1}) = H^q(\Omega^r_{Y,\log}).$$

By $p$-linear algebra, $H^q(\Omega^r_{Y,\log}) \otimes k \cong H^q(\Omega^r_Y)$ if and only if $t \circ C^{-1}$ is bijective. This proves (ii).

To prove (iii), the exact sequence ([10], I (3.9.1))

$$0 \to \Omega^r_Y/B^r_n \to \mathrm{Ker}(W_n \Omega^r_Y \to W_{n-1} \Omega^r_Y) \to \Omega^{r-1}_Y/Z^{r-1}_n \to 0$$

together with the isomorphisms (op. cit. (0.2.2))

$$Z^{r-1}_n/Z^{r-1}_{n+1} \cong B^r_{n+1}/B^r_n, \qquad B^r_n \cong B^r_{n+1}/B^r_1$$

implies (under the hypotheses of (ii))

$$H^q(\Omega^r_Y) \underset{\cong}{\to} H^q(\mathrm{Ker}(W_n \Omega^r_Y \to W_{n-1} \Omega^r_Y)).$$

The result now follows by induction on $n$, using the exact sequence

$$0 \to \Omega^r_{Y,\log} \to W_n \Omega^r_{Y,\log} \to W_{n+1} \Omega^r_{Y,\log} \to 0$$

and the five lemma.

To prove (iv), note the string of inequalities

$$\begin{aligned}(\text{multiplicity of slope } r \text{ in } H^{q+r}_{\mathrm{crys}}(Y/W)) &= \mathrm{rank}_{\mathbf{Q}_p} H^q(W\Omega^r_{\log}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \\ &\leq \mathrm{rank}_{\mathbf{F}_p}(H^q(W\Omega^r_{\log})/p) \leq \mathrm{rank}_{\mathbf{F}_p} H^q(\Omega^r_{Y,\log}) \leq \mathrm{rank}_k H^q(\Omega^r_Y).\end{aligned}$$

Equality of the extremes forces (ii) to hold and the torsion subgroups of $H^q(W\Omega^r_{\log})$ and $H^{q+1}(W\Omega^r_{\log})$ to vanish. In the absence of torsion, (ii) is equivalent to this equality.

*Definition* **(7.2)**. — Let Y be a smooth, proper variety over a perfect field $k$ of characteristic $p > 0$. We say Y is *ordinary* if $H^m(Y, B^r) = (0)$ for all $m$ and $r$.

Let $\bar{k}$ be the algebraic closure of $k$, $\overline{Y} = Y_{\bar{k}}$.

*Proposition* **(7.3)**. — *The following conditions* (1)-(5) *are equivalent.*

(1) Y *is ordinary.*

(2) $H^q(\overline{Y}, \Omega^r_{\overline{Y}/\bar{k},\log}) \otimes_{F_p} \bar{k} \xrightarrow{\sim} H^q(\overline{Y}, \Omega^r_{\overline{Y}/\bar{k}})$ *for any* $q$, $r$.

(3) $H^q(\overline{Y}, W_n\,\Omega^r_{\overline{Y},\log}) \otimes_{Z/p^nZ} W_n(\bar{k}) \xrightarrow{\sim} H^q(\overline{Y}, W_n\,\Omega^r_{\overline{Y}})$ *for any* $q$, $r$, $n$.

(4) $H^q(\overline{Y}, W\Omega^r_{\overline{Y},\log}) \otimes_{Z_p} W(\bar{k}) \xrightarrow{\sim} H^q(\overline{Y}, W\Omega^r_{\overline{Y}})$ *for any* $q$, $r$.

(5) $F : H^q(Y, W\Omega^r_Y) \to H^q(Y, W\Omega^r_Y)$ *is bijective for any* $q$ *and* $r$.

*Moreover,* Y *is ordinary and* $H^q_{\mathrm{crys}}(Y/W)$ *is torsion free for all* $q$ *if and only if the following condition holds*:

(6) *For any* $q$, *the Newton polygon defined by the slopes of the action of frobenius on* $H^q_{\mathrm{crys}}(Y/W)$ *coincides with the Hodge polygon defined by the numbers* $\dim_k H^{q-i}(Y, \Omega^i_{Y/k})$.

*If* $H^q(Y, W\Omega^r_Y)$ *is torsion free for any* $q$ *and* $r$, *these conditions* (1)-(6) *are also equivalent to*

(7) *For any* $q$, *the slopes of frobenius on* $H^q_{\mathrm{crys}}(Y/W)$ *are all integers.*

*Proof.* — The implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are clear from (7.1). To prove (5) $\Rightarrow$ (2) we may assume that the ground field is algebraically closed. Bijectivity of F implies vanishing of cohomology for the pro-sheaves $W\Omega^*/F$, and hence the exact sequences

$$0 \to W\Omega^r/F \xrightarrow{V} W\Omega^r/p \to W\Omega^r/V \to 0$$

$$0 \to W\Omega^{r-1}/F \xrightarrow{dV} W\Omega^r/V \to \Omega^r \to 0$$

([10], I (3.15), (3.19)) yield

**(7.3.1)** $$H^*(W\Omega^r/p) \cong H^*(\Omega^r_Y).$$

In particular, $H^*(W\Omega^r/p)$ are finite dimensional over $k$. Hence the exact sequences

$$0 \longrightarrow \Omega^r_{\log} \longrightarrow W\Omega^r/p \xrightarrow{1-F} W\Omega^r/p \longrightarrow 0$$

(op. cit., I (3.5), (5.7.2)) induce

$$0 \longrightarrow H^q(\Omega^r_{\log}) \longrightarrow H^q(W\Omega^r/p) \xrightarrow{1-F} H^q(W\Omega^r/p) \longrightarrow 0 \text{ (exact)}$$

for all $q$ and $r$. By the bijectivity of F, $p$-linear algebra gives

$$k \otimes H^q(\Omega^r_{\log}) \cong H^q(W\Omega^r/p).$$

By (7.3.1), this proves (2).

Assume now that Y is ordinary, $k = \bar{k}$, and $H^*_{\mathrm{crys}}(Y/W)$ is torsion free. Consider

the complex of pro-sheaves $W\Omega^{\bullet}_{\log} \otimes W(k)$ with 0 differentials. The map on hypercohomology

$$\textbf{(7.3.2)} \qquad H^*(Y, W\Omega^{\bullet}_{\log}) \otimes W(k) \to H^*(Y, W\Omega^{\bullet}) = H^*_{\mathrm{crys}}(Y/W)$$

gives rise by (4) to an isomorphism on the $E_1$ terms of the corresponding spectral sequences, and hence is itself an isomorphism. Thus $H^q(W\Omega^r_{\log})$ is torsion free for all $q$ and $r$. Condition (6) now follows from (7.1) (iv).

Assume that (6) holds. We apply (7.1) (iv) inductively starting with $r = 0$ to deduce that Y is ordinary and the $H^q(W\Omega^r_{\log})$ are torsion free. Using (7.3.2), we see that $H^*_{\mathrm{crys}}(Y/W)$ is torsion free also.

Finally, if $H^q(Y, W\Omega^r)$ is torsion free for all $q$ and $r$ then the slope spectral sequence degenerates at $E_1$ [10] and the slopes $s$ of $H^{q+r}_{\mathrm{crys}}(Y/W)$ with $r \leq s < r + 1$ are given by the slopes of $p^r F$ on $H^q(Y, W\Omega^r)$. Conditions (5) and (7) are then seen to be equivalent. The proof is complete.

*Example* **(7.4)**. — Let Y be an abelian variety over $k$. Then, Y is ordinary in the sense (7.2) if and only if it is ordinary in the classical sense (i.e. ${}_pY(\bar{k}) \cong (\mathbf{Z}/p\mathbf{Z})^{\dim(Y)}$). Indeed, for an abelian variety Y, there are isomorphisms

$$H^0(\overline{Y}, \Omega^1_{Y,\log}) \cong {}_p\mathrm{Pic}(\overline{Y}), \qquad H^1(\overline{Y}, \mathbf{Z}/p\mathbf{Z}) \cong \mathrm{Hom}({}_pY(\bar{k}), \mathbf{Z}/p\mathbf{Z}).$$

The orders of these groups are $p^{\dim(Y)}$ if and only if the equivalent conditions

$$\bar{k} \otimes H^0(\overline{Y}, \Omega^1_{Y,\log}) \cong H^0(\overline{Y}, \Omega^1_Y), \qquad \bar{k} \otimes H^1(\overline{Y}, \mathbf{Z}/p\mathbf{Z}) \cong H^1(\overline{Y}, \mathcal{O}_{\overline{Y}})$$

are satisfied. Assume these conditions are satisfied. Then,

$$H^q(Y, \Omega^r_Y) \cong \overset{q}{\wedge} H^1(Y, \mathcal{O}) \otimes_k \overset{r}{\wedge} H^0(Y, \Omega^1_Y)$$

shows that

$$\bar{k} \otimes H^q(\overline{Y}, \Omega^r_{Y,\log}) \to H^q(\overline{Y}, \Omega^r_Y).$$

By induction on $r$ using (7.1) (i) (ii), it follows that Y is ordinary in our sense.

## 8. A vanishing theorem

Let the notation be as in (0.1). In particular, X is smooth proper over $S = \mathrm{Sp}\,\Lambda$, $V = X_\eta$ is the generic fiber and $Y = X_s$ is the closed fiber.

By base change, we obtain diagrams

$$\begin{tikzcd}
V \otimes_K K' \arrow[r, "j'"] \arrow[d] & X \otimes_\Lambda \Lambda' \arrow[d] & Y \otimes_k k' \arrow[l, "i'"'] \arrow[d] \\
\mathrm{Sp}\,K' \arrow[r] & S' & \mathrm{Sp}\,k' \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{V} \arrow[r, "\bar{j}"] \arrow[d] & \overline{X} \arrow[d] & \overline{Y} \arrow[l, "\bar{i}"'] \arrow[d] \\
\mathrm{Sp}\,\overline{K} \arrow[r] & \overline{S} & \mathrm{Sp}\,\bar{k} \arrow[l]
\end{tikzcd}$$

where $K'$ is a finite extension of $K$, $S' = \operatorname{Sp} \Lambda'$ is the integral closure of $S = \operatorname{Sp} \Lambda$ in $K'$ and $k'$ is the residue field of $K'$. Let

$$\begin{aligned} M^q_{n,K'} &= i'^* R^q j'_*(\mathbf{Z}/p^n \mathbf{Z}(q)), \\ \overline{M}^q_n &= \bar{i}^* R^q \bar{j}_*(\mathbf{Z}/p^n \mathbf{Z}(q)) = \varinjlim_{K'} M^q_{n,K'}, \\ U\overline{M}^q_n &= \varinjlim_{K'} U^1 M^q_{n,K'} \subset \overline{M}^q_n. \end{aligned}$$

By (6.6), we get an exact sequence

$$\textbf{(8.0.1)} \qquad 0 \to U\overline{M}^q_n \to \overline{M}^q_n \to W_n \Omega^q_{Y,\log} \to 0,$$

for the symbols $\{x_1, \ldots, x_{q-1}, \pi\}$ die in the limit $\overline{M}^q_n$. Since $\overline{M}^q_n$ is generated locally by symbols (1.4), the long exact sequence

$$\ldots \to \overline{M}^{q-1}_n(1) \to \overline{M}^q_n \to \overline{M}^q_{m+n} \to \overline{M}^q_n \to \overline{M}^{q+1}_m(-1) \to \ldots$$

breaks up. So in the diagram

$$\textbf{(8.0.2)} \qquad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & U\overline{M}^q_n \arrow[r] \arrow[d] & U\overline{M}^q_{n+1} \arrow[r] \arrow[d] & U\overline{M}^q_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \overline{M}^q_n \arrow[r] \arrow[d] & \overline{M}^q_{n+1} \arrow[r] \arrow[d] & \overline{M}^q_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & W_n \Omega^q_{Y,\log} \arrow[r] \arrow[d] & W_{n+1} \Omega^q_{Y,\log} \arrow[r] \arrow[d] & \Omega^q_{Y,\log} \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

the bottom two rows are exact as are all three columns. It follows that the top row is also exact.

The aim of this section is to prove

*Theorem* **(8.1)**. — *Fix $q$ and $r$, and assume* $H^q(\overline{Y}, B^r_Y) = (0)$. *Then,* $H^q(\overline{Y}, U\overline{M}^r_n) = (0)$ *for any $n$.*

*Corollary* **(8.2)**. — *If $\overline{Y}$ is ordinary, we have* $H^q(\overline{Y}, U\overline{M}^r_n) = (0)$ *for any $q$, $r$, $n$.*

The proof of (8.1) is rather long and complicated. There is a shorter proof of (8.2), and we give it first.

*Proof of* (8.2). — We may assume that the residue field $k$ of $\Lambda$ is algebraically closed. By (8.0.2), it suffices to show that $H^q(Y, U\overline{M}^r_1) = (0)$. For this it is enough

to show that the map $H^q(Y, U^1 M_1^r) \to H^q(Y, U\overline{M}_1^r)$ is zero. We proceed by induction on $r$. When $r = 0$, $M_1^0 = (\mathbf{Z}/p\mathbf{Z})_Y$ and $U^1 M_1^0 = (0)$. Assume the result for all $t < r$. In particular, after ramifying, we may assume that the maps

$$H^*(Y, M_1^t) \to H^*(Y, \Omega^t_{Y,\log})$$

are onto for all $t < r$. For any integer $m \geq 1$ and any $u \in W(k)$ consider the diagram

$$\begin{tikzcd}
H^*(Y, U^{m+1} M_1^r) \arrow[r] & H^*(Y, U^m M_1^r) \arrow[r] & H^*\left(Y, \begin{Bmatrix} 0 \\ \Omega_Y^{r-1} \\ B_Y^{r-1} \oplus B_Y^r \end{Bmatrix}\right) \\
 & H^*(Y, M_1^{r-1}) \arrow[u] \arrow[r] & H^*(Y, \Omega^{r-1}_{Y,\log}) \arrow[u]
\end{tikzcd}$$

where the left hand vertical arrow is $\alpha \mapsto (1 + u\pi^m).\alpha$ and the right one is induced by the natural inclusion $\Omega^{r-1}_{Y,\log} \subset \Omega_Y^{r-1}$ followed by multiplication by the residue class $\bar{u}$ of $u$ in $k$. Note that $U^m M_1^r = (0)$ for $m \gg 0$. Proceeding by downward induction on $m$ and using the hypothesis that Y is ordinary, we see that any class in $H^*(Y, U^1 M_1^r)$ can be written as a finite product

$$\prod_i (1 + u_i \pi^{m_i}).\alpha_i$$

with $u_i \in W(k)$ and $\alpha_i \in H^*(M_1^{r-1})$. Since $(1 + u_i \pi^{m_i})^{1/p} \in \bar{\Lambda}$, we conclude that the map $H^*(Y, U^1 M_1^r) \to H^*(Y, U\overline{M}_1^r)$ is zero as claimed.

Now we give the proof of (8.1). In (8.3)-(8.6) below, we do not assume $H^q(B^r) = (0)$. However, we shall always assume $k = \bar{k}$ and $\zeta_p \in K$ without loss of generality.

*Definition* **(8.3)**. — Fix $r \geq 0$. For an element $b$ of $K^*$, let $U_b^m$ be the kernel of $U^m M_1^r \xrightarrow{\cup\{b\}} M_1^{r+1}$. Let $\mathrm{gr}_b^m = U_b^m / U_b^{m+1}$.

*Lemma* **(8.4)**. — *Assume that* $\mathrm{ord}_K(b)$ *is prime to* $p$.

(i) *For* $m \geq 1$, $U_b^m$ *is generated locally by local sections of the forms*

**(8.4.1)** $\{x, -b\}$ *with* $x \in U^m M_1^{r-1}$,

**(8.4.2)** $\{1 - f^p b^s c^p, g_1, \ldots, g_{r-1}\}$

*where* $f, g_1, \ldots, g_{r-1} \in i^* \mathcal{O}_X^*$, $c \in K^*$, *and* $s$ *is an integer such that* $p \nmid s$ *and* $\mathrm{ord}_K(b^s c^p) \geq m$.

(ii) *For* $0 < m < e'$, *we have an exact sequence*

$$0 \longrightarrow \mathrm{gr}_b^m \longrightarrow \mathrm{gr}^m(M_1^r) \xrightarrow{\rho'_m} B_Y^r \longrightarrow 0$$

*where* $\rho'_m$ *is the homomorphism induced by* (1.4.1). (*Irrespective of whether or not* $p \mid m$, (1.4.1) *gives a surjection* $\mathrm{gr}^m(M_1^r) \to \Omega^{r-1}/\mathbf{Z}_1^{r-1} \cong B^r$.)

*Proof.* — Let $0 < m < e'$. It is clear that the local sections (8.4.1) and (8.4.2) are annihilated by $\cup\{b\}$. Furthermore, the class of these local sections fill out $\mathrm{Ker}(p'_m)$. So we obtain diagrams

$$\begin{tikzcd} \mathrm{B}^r_{\mathrm{Y}} \arrow[r, "\pm\,\mathrm{ord}_{\mathrm{K}}(b).\mathrm{incl}"] \arrow[d] & \Omega^r_{\mathrm{Y}} \arrow[d, phantom, "\wr"] \\ \mathrm{gr}^m(\mathrm{M}^r_1)/\mathrm{gr}^m_b \arrow[r, "\cup\{b\}"] & \mathrm{gr}^m(\mathrm{M}^{r+1}_1) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{B}^r_{\mathrm{Y}} \arrow[r, "\pm\,\mathrm{ord}_{\mathrm{K}}(b)"] \arrow[d] & \mathrm{B}^r_{\mathrm{Y}} \arrow[d, phantom, "\cap"] \\ \mathrm{gr}^m(\mathrm{M}^r_1)/\mathrm{gr}^m_b \arrow[r, "\cup\{b\}"] & \mathrm{gr}^m(\mathrm{M}^{r+1}_1) \end{tikzcd}$$

$$(p \nmid m) \qquad\qquad\qquad\qquad (p \mid m)$$

which prove $\mathrm{gr}^m_b = \mathrm{Ker}(\rho'_m)$.

*Definition* **(8.5)**. — Let L be a finite extension of K. For $0 < m < \dfrac{e_{\mathrm{L}}\, p}{p-1}$ $(e_{\mathrm{L}} = \mathrm{ord}_{\mathrm{L}}(p))$, let $n = \dfrac{e_{\mathrm{L}}\, p}{p-1} - m$, and define

$$\mathrm{U}_m\, \mathrm{M}^r_{1,\mathrm{L}} = \mathrm{U}^n\, \mathrm{M}^r_{1,\mathrm{L}}, \qquad \mathrm{gr}_m(\mathrm{M}^r_{1,\mathrm{L}}) = \mathrm{gr}^n(\mathrm{M}^r_{1,\mathrm{L}}).$$

We have thus an increasing filtration

$$(0) = \mathrm{U}_0\, \mathrm{M}^r_{1,\mathrm{L}} \subset \mathrm{U}_1\, \mathrm{M}^r_{1,\mathrm{L}} \subset \ldots$$

Since $e_{\mathrm{L}}$ varies with L, it is not true that

$$\mathrm{res}_{\mathrm{L}/\mathrm{K}}(\mathrm{U}_m\, \mathrm{M}^r_{1,\mathrm{K}}) \subset \mathrm{U}_m\, \mathrm{M}^r_{1,\mathrm{L}}.$$

However, we have the following result.

*Lemma* **(8.6)**. — *Let $b$ be an element of* $\mathrm{K}^*$ *such that* $\mathrm{ord}_{\mathrm{K}}(b)$ *is prime to $p$. Let* $0 < m < e' = \dfrac{e_{\mathrm{K}}\, p}{p-1}$, *and let* $\mathrm{U}_{b,m} = \mathrm{U}^n_b$, $\mathrm{gr}_{b,m} = \mathrm{gr}^n_b$ *where* $n = e' - m$. *Consider the following two cases.*

*Case* 1. — *Let* $\mathrm{L} = \mathrm{K}(a)$ *where* $(-a)^p = -b$, *and* $t = e'$.

*Case* 2. — *Assume* $0 < \mathrm{ord}_{\mathrm{K}}(b) < e'$. *Let* $\mathrm{L} = \mathrm{K}(a)$ *where* $(1-a)^p = 1 - b$ *and let* $t = e' - \mathrm{ord}_{\mathrm{K}}(b)$. *Then, in both cases, we have the following* (i) (ii) (iii).

(i) *For* $m \leq t$,

$$\mathrm{res}_{\mathrm{L}/\mathrm{K}}(\mathrm{U}_{b,m}) \subset \mathrm{U}_m\, \mathrm{M}^r_{1,\mathrm{L}}.$$

(ii) *If* $m < t$ *and* $p \mid m$,

$$\mathrm{res}_{\mathrm{L}/\mathrm{K}}(\mathrm{U}_{b,m}) \subset \mathrm{U}_{m-1}\, \mathrm{M}^r_{1,\mathrm{L}}.$$

(iii) *If* $m < t$ *and* $p \nmid m$, *we have the following commutative diagram:*

$$\begin{tikzcd} Z_Y^{r-1} \arrow[r, "C"] \arrow[d, "\varphi_m"', "\wr"] & \Omega_Y^{r-1} \arrow[d, "\psi_m", "\wr"'] \\ \mathrm{gr}_{b,m} \arrow[r, "\mathrm{res}"] & \mathrm{gr}_m(M_{1,L}^r) \end{tikzcd}$$

*where* C *is the Cartier operator, and* $\varphi_m$ *and* $\psi_m$ *are isomorphisms defined below.*

(iv) *In Case* 2, *we have the following commutative diagram:*

$$\begin{tikzcd} Z_Y^{r-1} \arrow[r, "C-1"] \arrow[d, "\varphi_t"', "\wr"] & \Omega_Y^{r-1} \arrow[d, "\psi_t", "\wr"'] \\ \mathrm{gr}_{b,t} \arrow[r, "\mathrm{res}"] & \mathrm{gr}_t(M_{1,L}^r) \end{tikzcd}$$

*where* $\varphi_t$ *and* $\psi_t$ *are isomorphisms defined below.*

In (iii), the definitions of $\varphi_m$ and $\psi_m$ are as follows. Fix an integer $s$ and an element $c$ of $K^*$ such that $\mathrm{ord}_K(b^s c^p) = e' - m$. Then, $\varphi_m$ is the homomorphism induced by

$$\Omega_Y^{r-1} \xrightarrow{\cong} \mathrm{gr}_m(M_{1,K}^r)$$

$$x \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_{r-1}}{y_{r-1}} \mapsto \{1 - b^s c^p \tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{r-1}\},$$

and $\psi_m$ is the homomorphism

$$\Omega_Y^{r-1} \xrightarrow{\cong} \mathrm{gr}_m(M_{1,L}^r)$$

$$x \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_{r-1}}{y_{r-1}} \mapsto \{1 + pa^s c\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{r-1}\}.$$

The homomorphisms $\varphi_t$ and $\psi_t$ in (iv) are defined in the same way for the particular choices $m = t$, $s = 1$ and $c = 1$.

*Proof.* — Let $0 < m < e' = \dfrac{e_K p}{p-1}$, $b, c \in K^*$, and $a \in L^*$ be as above. Note that L is a totally ramified extension of K of degree $p$ and $\mathrm{ord}_L(a) = \mathrm{ord}_K(b)$ both in Case 1 and Case 2.

In Case 1 (resp. Case 2), (8.6) will be the consequence of the following (8.6.1)

and (8.6.5) (resp. (8.6.3), (8.6.4), (8.6.7), (8.6.8)) and of the fact that $Z^{r-1}$ is generated by $B^{r-1}$ and elements of the form $x^p \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_{r-1}}{y_{r-1}}$.

Let $f \in i^* \mathcal{O}_X$ and let $m_L$ be the maximal ideal of L. By the binomial theorem, we have

$$(1 - a^s cf)^p \equiv 1 - pa^s cf - a^{sp} c^p f^p \bmod m_L^{e'p-m+1} i^* \mathcal{O}_X$$

with
$$\operatorname{ord}_L(pa^s c) = e'p - m$$
$$\operatorname{ord}_L(a^{sp} c^p) = e'p - mp.$$

In Case 1, this shows that

**(8.6.1)** $$\{1 - b^s c^p f^p\} = \{1 + pa^s cf\} \bmod U^{e'p-m+1} M^1_{1,L}$$

in $M^1_{1,L}$. In Case 2, the equation $(1 - a)^p = 1 - b$ shows that

**(8.6.2)** $$\frac{a^p}{b} \equiv 1 - p\frac{a}{b} \bmod m_L^{(p-1)t+1}$$

with
$$\operatorname{ord}_L\left(p\frac{a}{b}\right) = (p-1)t.$$

Hence we have

$$(1 - a^s cf)^p \equiv 1 - pa^s cf - \left(1 - p\frac{a}{b}\right)^s b^s c^p f^p$$
$$\equiv 1 - pa^s cf - b^s c^p f^p + spab^{s-1} c^p f^p \bmod m_L^{e'p-m+1} i^* \mathcal{O}_X$$

with
$$\operatorname{ord}_L(b^s c^p) = e'p - mp$$
$$\operatorname{ord}_L(pab^{s-1} c^p) = e'p - m + (p-1)(t-m).$$

This shows that, in $M^1_{1,L}$,

**(8.6.3)** $$\{1 - b^s c^p f^p\} \equiv \{1 + pa^s cf\} \bmod U^{e'p-m+1} M^1_{1,L}$$

for $0 < m < t$, and (put $m = t$, $s = 1$ and $c = 1$)

**(8.6.4)** $$\{1 - bf^p\} \equiv \{1 + pa(f - f^p)\} \bmod U^{e'p-t+1} M^1_{1,L}.$$

On the other hand, in Case 1,

**(8.6.5)** $$\operatorname{res}(\{M^1_{1,K}, -b\}) = 0 \text{ in } M^1_{1,L}.$$

In Case 2, by (8.6.2)

**(8.6.6)** $$\{b\} \equiv \{-b\} \equiv \left\{1 + p\frac{a}{b}\right\} \bmod U^{(p-1)t+1} M^1_{1,L}.$$

Hence we have in this case, for $0 < m < t$,

**(8.6.7)** $$\begin{aligned}\operatorname{res}(\{U^{e'-m} M^1_{1,K}, -b\}) &\subset \{U^{e'p-mp} M^1_{1,L}, U^{(p-1)t} M^1_{1,L}\} \\ &\subset U^{e'p-mp+(p-1)t} M^2_{1,L} \text{ by } (4.1) \\ &\subset U^{e'p-m+1} M^2_{1,L}.\end{aligned}$$

Also (8.6.6) shows in Case 2

**(8.6.8)**
$$\{1 - bf, -b\} \equiv \left\{1 - bf, 1 + p\frac{a}{b}\right\}$$
$$\equiv \left\{1 + paf, p\frac{a}{b}\right\} \bmod U^{e'p-t+1} M^2_{1,L}$$

by the proof of (4.1).

*Lemma* **(8.7)**. — *Assume* $H^q(Y, B^r) = (0)$. *Let* $b$ *be an element of* $K^*$ *such that* $\mathrm{ord}_K(b)$ *is prime to* $p$, *and let* $0 < m < e'$, $p \nmid m$. *Then, in the commutative diagram*

$$\begin{array}{ccc} H^q(Y, U^m_b) & \xrightarrow{i} & H^q(Y, U^m M^r_1) \\ \Big\downarrow{\scriptstyle (8.4)\,(ii)} & & \Big\downarrow \\ H^q(Y, Z^{r-1}) & \xrightarrow{j} & H^q(Y, \Omega^{r-1}_Y) \end{array}$$

*the homomorphisms* $i$ *and* $j$ *are surjective and* $\mathrm{Ker}(i) \to \mathrm{Ker}(j)$ *is also surjective.*

*Proof.* — By (8.4) (ii), $U^m M^r_1/U^m_b$ has a filtration whose successive quotients are all isomorphic to $B^r$. Hence $H^q(Y, U^m M^r_1/U^m_b) = (0)$ and

$$H^{q-1}(Y, U^m M^r_1/U^m_b) \to H^{q-1}(Y, B^r).$$

The lemma follows from the diagram

$$\begin{array}{ccccccc} H^{q-1}(U^m M^r_1/U^m_b) & \longrightarrow & H^q(U^m_b) & \longrightarrow & H^q(U^m M^r_1) & \longrightarrow & H^q(U^m M^r_1/U^m_b) = (0) \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ H^{q-1}(B^r) & \longrightarrow & H^q(Z^{r-1}) & \longrightarrow & H^q(\Omega^{r-1}_Y) & \longrightarrow & H^q(B^r) = (0) \end{array}$$

**(8.8)** Now we are ready to prove Theorem (8.1). We prove the following fact by induction on $m \geq 0$.

**(8.8.1)** *For any finite extension* $K'$ *of* $K$ *such that* $m < \dfrac{e_{K'} p}{p-1}$, *the map*

$$H^q(Y, U_m M^r_{1,K'}) \to H^q(Y, U\overline{M}^r_1)$$

*is zero.*

First assume $p \mid m$ and $m \geq 1$. We replace $K'$ by $K$. Let

$$x \in H^q(Y, U_m M^r_{1,K}) \quad 1 \leqslant m < \frac{e_K p}{p-1}, \quad p \mid m.$$

Let $b$ be an element of $K^*$ such that $\mathrm{ord}_K(b)$ is prime to $p$, and let $L = K((-b)^{1/p})$. By (8.7), $x$ comes from $H^q(Y, U_{b,m})$. By (8.6), we have $\mathrm{res}_{L/K}(U_{b,m}) \subset U_{m-1} M^r_{1,L}$. Thus, the image of $x$ in $H^q(Y, U\overline{M}^r_1)$ is contained in the image of $H^q(Y, U_{m-1} M^r_{1,L})$, and this completes the induction in this case.

Next, we consider the case $p \nmid m$. Fix a $k$-linear section $s$ of the surjection $H^q(Y, Z^{r-1}) \to H^q(Y, \Omega^{r-1}_Y)$. Then, $C \circ s$ acts on $H^q(Y, \Omega^{r-1}_Y)$. We say that an element $\omega$ of $H^q(Y, \Omega^{r-1}_Y)$ is of order $\leq i$, if there are elements $\beta_1, \ldots, \beta_i$ of $k$ such that

$$(C \circ s - \beta_1) \circ \ldots \circ (C \circ s - \beta_i)(\omega) = 0.$$

($C \circ s - \beta$ means $\omega \mapsto C \circ s(\omega) - \beta\omega$.) Then, any element of $H^q(Y, \Omega^{r-1}_Y)$ is of finite order. For $b \in K^*$ such that $\mathrm{ord}_K(b) = e' - m$, the isomorphism

$$\rho_b : \Omega^{r-1}_Y \xrightarrow{\cong} \mathrm{gr}_m(M^r_1);$$

$$x \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_{r-1}}{y_{r-1}} \mapsto \{1 - \tilde{x}b, \tilde{y}_1, \ldots, \tilde{y}_{r-1}\}$$

induces a homomorphism

$$\rho'_b : H^q(Y, U_m M^r_1) \to H^q(Y, \Omega^{r-1}_Y).$$

It is easily seen that for an element $x$ of $H^q(Y, U_m M^r_1)$, the order of $\rho'_b(x)$ is independent of the choice of $b$. We call this independent order of $\rho'_b(x)$ the order of $x$. We prove the following assertion by induction on $i$.

**(8.8.2)** *For any* $K'$ *such that* $m < \dfrac{e_{K'} p}{p-1}$ *and for any* $x \in H^q(Y, U_m M^r_{1,K'})$ *of order* $\leq i$, *the image of* $x$ *in* $H^q(Y, U\overline{M}^r_1)$ *is zero.*

We replace $K'$ by $K$ and let $x$ be an element of $H^q(Y, U_m M^r_{1,K})$ of order $i \geq 1$. By easy $p$-linear algebra, we can find an element $b$ of $K^*$ such that $\mathrm{ord}_K(b) = e' - m$ and such that one of $C \circ s(\rho'_b(x))$ and $(C \circ s - 1)(\rho'_b(x))$ is of order $\leq i - 1$. By (8.7), there is an element $y$ of $H^q(Y, U_{b,m})$ whose image in $H^q(Y, U_m M^r_1)$ is $x$ and whose image in $H^q(Y, Z^{r-1})$ under

$$U_{b,m} \to \mathrm{gr}_{b,m} \overset{\text{by } \rho_b}{\cong} Z^{r-1}$$

is $s(\rho'_b(x))$. In the case $C \circ s(\rho'_b(x))$ (resp. $(C \circ s - 1)(\rho'_b(x))$) is of order $\leq i - 1$, let $L = K((-b)^{1/p})$ (resp. $L = K((1-b)^{1/p})$). The image of $y$ in $H^q(Y, U_m M^r_{1,L})$ is of order $\leq i - 1$ by (8.6), and it has the same image as $x$ in $H^q(Y, U\overline{M}^r_1)$. This completes the induction on $i$ and hence proves (8.1).

## 9. p-adic cohomology

Keep the notations of § 8. We consider the spectral sequences

$$E_2^{s,t} = H^s(\overline{Y}, \overline{M}^t_\bullet)(-t) \Rightarrow H^{s+t}_{et}(\overline{V}, \mathbf{Z}/p^\bullet \mathbf{Z})$$

and the associated filtration. Let

$$\mathrm{H}^q_{\mathrm{crys}}(\overline{\mathrm{Y}}/\mathrm{W}(\bar{k}))^{(i)} = \mathrm{Ker}(\mathrm{F} - p^i : \mathrm{H}^q_{\mathrm{crys}} \to \mathrm{H}^q_{\mathrm{crys}}).$$

*Theorem* **(9.1).** — *Fix* $q \geq 0$ *and assume* $\mathrm{H}^{q-i}(\overline{\mathrm{Y}}, \mathrm{B}^i_{\overline{\mathrm{Y}}}) = 0$ *for all* $i$. *Then, we have canonical isomorphisms of* $\mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K})$*-modules*

**(9.1.1)** $$\mathrm{gr}^{q-i}\,\mathrm{H}^q_{\mathrm{et}}(\overline{\mathrm{V}}, \mathbf{Q}_p) \cong \mathrm{H}^q_{\mathrm{crys}}(\overline{\mathrm{Y}}/\mathrm{W}(\bar{k}))^{(i)}_{\mathbf{Q}}\,(-i)$$

**(9.1.2)** $$\mathrm{gr}^{q-i}\,\mathrm{H}^q_{\mathrm{et}}(\overline{\mathrm{V}}, \mathbf{Q}_p) \otimes_{\mathbf{Z}_p} \mathrm{W}(\bar{k}) \cong \mathrm{H}^{q-i}(\overline{\mathrm{Y}}, \mathrm{W}\Omega^i)_{\mathbf{Q}}\,(-i)$$

*for all* $i$.

*Proof.* — By (8.1), the exact sequences

$$0 \to \mathrm{U}\overline{\mathrm{M}}^i_n \to \overline{\mathrm{M}}^i_n \to \mathrm{W}_n\,\Omega^i_{\mathrm{Y},\log} \to 0$$

give injections

$$\mathrm{H}^{q-i}(\overline{\mathrm{Y}}, \overline{\mathrm{M}}^i_n) \to \mathrm{H}^{q-i}(\overline{\mathrm{Y}}, \mathrm{W}_n\,\Omega^i_{\mathrm{Y},\log}).$$

Furthermore, $\mathrm{H}^{q-i}(\overline{\mathrm{Y}}, \Omega^i_{\overline{\mathrm{Y}},\log})$ is finite by the proof of (7.1) (i) and hence $\mathrm{H}^{q-i}(\overline{\mathrm{Y}}, \mathrm{W}_n\,\Omega^i_{\overline{\mathrm{Y}},\log})$ are finite. Since passing to the inverse limit preserves the exactness for systems of finite groups,

$$\mathrm{gr}^{q-i}\,\mathrm{H}^q(\overline{\mathrm{V}}, \mathbf{Z}_p) \underset{\mathrm{def}}{=} \varprojlim_n \mathrm{gr}^{q-i}\,\mathrm{H}^q(\overline{\mathrm{V}}, \mathbf{Z}/p^n\,\mathbf{Z})$$

is canonically isomorphic to a subquotient of $\varprojlim_n \mathrm{H}^{q-i}(\overline{\mathrm{Y}}, \mathrm{W}_n\,\Omega^i_{\overline{\mathrm{Y}},\log})$. Thus we have

$$\begin{aligned}\dim_{\mathbf{Q}_p} \mathrm{H}^q(\overline{\mathrm{V}}, \mathbf{Q}_p) &\leq \sum_i \dim_{\mathbf{Q}_p} \mathrm{H}^{q-i}(\overline{\mathrm{Y}}, \mathrm{W}\Omega^i_{\overline{\mathrm{Y}},\log})_{\mathbf{Q}} \\ &\leq \sum_i \dim_{\mathrm{W}(\bar{k})_{\mathbf{Q}}} \mathrm{H}^{q-i}(\overline{\mathrm{Y}}, \mathrm{W}\Omega^i_{\overline{\mathrm{Y}},})_{\mathbf{Q}} \\ &= \dim_{\mathrm{W}(\bar{k})_{\mathbf{Q}}} \mathrm{H}^q_{\mathrm{crys}}(\overline{\mathrm{Y}}/\mathrm{W}(\bar{k}))_{\mathbf{Q}} = \dim_{\mathbf{Q}_p} \mathrm{H}^q(\overline{\mathrm{V}}, \mathbf{Q}_p).\end{aligned}$$

Hence all the inequalities are in fact equalities. This proves our theorem.

*Corollary* **(9.1.1).** — *If* $k$ *is perfect and* $\mathrm{H}^{q-i}(\mathrm{Y}, \mathrm{B}^i_{\mathrm{Y}}) = 0$ *for all* $i$, $\mathrm{H}^q_{\mathrm{et}}(\overline{\mathrm{V}}, \mathbf{Q}_p)$ *admits a Hodge-Tate decomposition* (cf. § 0).

This follows from (9.1) and the result of Tate (0.9).

We prove some integral statements assuming Y ordinary.

*Theorem* **(9.2).** — *Assume* $\overline{\mathrm{Y}}$ *ordinary. Let* $\overline{\mathrm{S}}_n = \mathrm{Spec}(\overline{\Lambda}/p^n\,\overline{\Lambda})$ *and* $\overline{\mathrm{X}}_n = \mathrm{X} \times_{\mathrm{S}} \overline{\mathrm{S}}_n$. *Then, we have, for all* $r$ *and* $n$,

**(9.2.1)** $$\mathrm{H}^q(\overline{\mathrm{Y}}, \overline{\mathrm{M}}^r_n) \cong \mathrm{H}^q(\overline{\mathrm{Y}}, \mathrm{W}_n\,\Omega^r_{\overline{\mathrm{Y}},\log})$$

**(9.2.2)** $$\mathrm{H}^q(\overline{\mathrm{Y}}, \overline{\mathrm{M}}^r_n) \otimes_{\mathbf{Z}/p^n\mathbf{Z}} \mathrm{W}_n(\bar{k}) \cong \mathrm{H}^q(\overline{\mathrm{Y}}, \mathrm{W}_n\,\Omega^r_{\overline{\mathrm{Y}}})$$

**(9.2.3)** $$\mathrm{H}^q(\overline{\mathrm{Y}}, \overline{\mathrm{M}}^r_n) \otimes_{\mathbf{Z}/p^n\mathbf{Z}} \overline{\Lambda}/p^n\,\overline{\Lambda} \cong \mathrm{H}^q(\overline{\mathrm{X}}_n, \Omega^r_{\overline{\mathrm{X}}_n/\overline{\mathrm{S}}_n}).$$

*Proof.* — The first two isomorphisms are clear from (8.1) and (7.3). To give (9.2.3), we use the map $\overline{\mathrm{M}}^q_n \to \Omega^q_{\overline{\mathrm{X}}_n/\overline{\mathrm{S}}_n}$ of (1.5).

For $n = 1$ we consider the diagram $(\overline{\Lambda}_n = \overline{\Lambda}/p^n\overline{\Lambda})$

$$\begin{array}{ccc} H^r(\overline{M}_1^q) \otimes \overline{\Lambda}_1 \cong H^r(\overline{Y}, \Omega^q_{\overline{Y},\log}) \otimes \overline{\Lambda}_1 & \xrightarrow{(*)} & H^r(\overline{X}_1, \Omega^q) \\ \downarrow & & \downarrow \\ H^r(\Omega^q_{\overline{Y},\log}) \otimes k & \xrightarrow{\cong} & H^r(\overline{Y}, \Omega^q) \end{array}$$

By passing to the limit over discrete valuation rings contained in $\overline{\Lambda}$, one sees that $(*)$ is an isomorphism, proving (9.2.3) for $n = 1$. We now consider the diagram on cohomology associated to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{M}_n^q \otimes \overline{\Lambda} & \longrightarrow & \overline{M}_{n+1}^q \otimes \overline{\Lambda} & \longrightarrow & \overline{M}_1^q \otimes \overline{\Lambda} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^q_{\overline{X}_n} & \longrightarrow & \Omega^q_{\overline{X}_{n+1}} & \longrightarrow & \Omega^q_{\overline{X}_1} & \longrightarrow & 0 \end{array}$$

and apply the 5-lemma. Q.E.D.

*Corollary* **(9.3)**. — *Assume* $\overline{Y}$ *ordinary and let* $D = \varprojlim_n \overline{\Lambda}_n$ *be the ring of integers in* $\mathbf{C}_p$. *Then,*

**(9.3.1)** $$(\varprojlim_n H^q(\overline{Y}, \overline{M}_n^r)) \otimes_{\mathbf{Z}_p} D \cong H^q(X_D, \Omega^r_{X_D/D})$$

*for all* $q$ *and* $r$. *If* $k$ *is perfect, we have for all* $q$,

**(9.3.2)** $$H^q_{\text{et}}(\overline{V}, \mathbf{Q}_p) \otimes \mathbf{C}_p = \bigoplus_{i \in \mathbf{Z}} H^{q-i}(V, \Omega^i_{V/K}) \otimes_K \mathbf{C}_p(-i).$$

*Corollary* **(9.4)**. — *Assume that* $\overline{Y}$ *is ordinary. Then the spectral sequences*

$$E_2^{s,t} = \varprojlim_n H^s(\overline{Y}, \overline{M}_n^t)(-t) \Rightarrow H^{s+t}_{\text{et}}(\overline{V}, \mathbf{Z}_p)$$

*degenerate modulo torsion at* $E_2$. *If* $\dim(V) < \dfrac{p-1}{(e_K, p-1)}$, *the spectral sequences*

$$E_2^{s,t} = H^s(\overline{Y}, \overline{M}_n^t)(-t) \Rightarrow H^{s+t}_{\text{et}}(\overline{V}, \mathbf{Z}/p^n\mathbf{Z})$$

*degenerate at* $E_2$ *for all* $n$.

Indeed, the last assertion in (9.4) follows from the facts that

$$\mathrm{Hom}_{\mathrm{Gal}(\overline{K}/K_{nr})}(\mathbf{Z}/p(i)(\overline{K}), \mathbf{Z}/p(j)(\overline{K})) = (0)$$

if $0 < |i-j| < \dfrac{p-1}{(e_K, p-1)}$ and that $\overline{M}_n^q = 0$ if $q > \dim(V)$.

*Lemma* **(9.5)**. — *Assume $k$ perfect and* Y *ordinary, and fix* $q \geq 0$. *Then, the following four conditions are equivalent*:

(i) $H^q_{crys}(Y/W(k))$ *is torsion free*;
(ii) $H^{q-i}(Y, W\Omega^i_Y)$ *are torsion free for all* $i$;
(iii) $H^q_{DR}(X/S) = H^q(X, \Omega^\bullet_{X/S})$ *is torsion free*;
(iv) $H^{q-i}(X, \Omega^i_{X/S})$ *are torsion free for all* $i$.

*Proof*. — The equivalence of (i) and (ii) are proved in the proof of (7.3) by showing that

$$H^q_{crys}(\overline{Y}/W(\bar{k})) \cong \bigoplus_i H^{q-i}(\overline{Y}, W\Omega^i_{\overline{Y}}).$$

Similarly, consider the complex of sheaves on $\overline{Y}$

$$\overline{M}^\bullet_n \otimes D$$

with zero differentials. The map of complexes $\overline{M}^\bullet_n \otimes D \to \Omega^\bullet_{X_n/S_n}$, which induces an isomorphism of $E_1$-terms, shows that

$$H^q_{DR}(\overline{X}_n/\overline{S}_n) \cong \bigoplus_i H^{q-i}(\overline{X}_n, \Omega^i_{\overline{X}_n/\overline{S}_n}).$$

Hence, (iii) and (iv) are equivalent. The equivalence of (ii) and (iv) follows from (9.2).

The following result is now deduced from (9.2).

*Theorem* **(9.6)**. — *Assume that* $\overline{Y}$ *is ordinary and that* $H^q_{crys}(\overline{Y}/W(\bar{k}))$ *and* $H^{q+1}_{crys}(\overline{Y}/W(\bar{k}))$ *are torsion free. Then, for all* $i$, *we have*

**(9.6.1)** $$gr^{q-i} H^q_{et}(\overline{V}, \mathbf{Z}_p) \cong H^q_{crys}(\overline{Y}/W(\bar{k}))^{(i)}(-i),$$

**(9.6.2)** $$gr^{q-i} H^q_{et}(\overline{V}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} W(\bar{k}) \cong H^q_{crys}(\overline{Y}, W\Omega^i)(-i),$$

**(9.6.3)** $$gr^{q-i} H^q_{et}(\overline{V}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} D \cong H^{q-i}(X_D, \Omega^i_{X_D/D})(-i).$$

## REFERENCES


[1] Bass, H., and Tate, J., The Milnor ring of a global field, in *Algebraic* K-*theory*, II, *Springer Lecture Notes in Math.*, **342** (1972), 349-442.

[2] Berthelot, P., *Cohomologie cristalline des schémas de charactéristique* $p > 0$, *Springer Lecture Notes in Math.*, **407** (1974).

[3] Bloch, S., Algebraic K-theory and crystalline cohomology, *Publ. Math. I.H.E.S.*, **47** (1974), 187-268.

[4] Bloch, S., Torsion algebraic cycles, $K_2$, and Brauer groups of function fields, in *Groupe de Brauer, Springer Lecture Notes in Math.*, **844** (1981), 75-102.

[5] Bloch, S., $p$-adic étale cohomology, in *Arithmetic and Geometry*, vol. I, **13-27**, Birkhäuser (1983).

[6] Deligne, P. (with Illusie, L.), Cristaux ordinaires et coordonnées canoniques, in *Surfaces algébriques, Springer Lecture Notes in Math.*, **868** (1981), 80-137.

[7] Gabber, O., *An injectivity property for étale cohomology*, preprint, 1981.

[8] Gabber, O., *Gersten's conjecture for some complexes of vanishing cycles*, preprint, 1981.

[9] Gabber, O., *Affine analog of proper base change theorem*, preprint, 1981.

[10] Illusie, L., Complexe de De Rham-Witt et cohomologie cristalline, *Ann. Sci. Ec. Norm. Sup.*, 4e sér., **12** (1979), 501-661.

[11] KATO, K., A generalization of local class field theory by using K-groups, II, *J. Fac. Sci. Univ. Tokyo*, **27** (1980), 603-683.
[12] KATO, K., Galois cohomology of complete discrete valuation fields, in *Algebraic K-theory, Springer Lecture Notes in Math.*, **967** (1982), 215-238.
[13] KATZ, N., Serre-Tate local moduli, in *Surfaces algébriques, Springer Lecture Notes in Math.*, **868** (1981), 138-202.
[14] MERKURJEV, A. S., and SUSLIN, A. A., K-cohomology of Severi-Brauer varieties and norm residue homomorphism, *Math. U.S.S.R. Izvestiya*, **21** (1983), 307-340.
[15] MILNOR, J., Algebraic K-theory and quadratic forms, *Inv. Math.*, **9** (1970), 318-344.
[16] SERRE, J.-P., *Cohomologie Galoisienne, Springer Lecture Notes in Math.*, **5** (1965).
[17] SOULÉ, C., K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale, *Inv. Math.*, **55** (1979), 251-295.
[18] TATE, J., $p$-divisible groups, in *Proceedings of a Conference on local fields*, Springer-Verlag, 1967, 158-183.
[19] TATE, J., Relation between $K_2$ Galois cohomology, *Inv. Math.*, **36** (1976), 257-274.
[20] DELIGNE, P., La conjecture de Weil, II, *Publ. Math. I.H.E.S.*, **52** (1980), 313-428.
[21] ILLUSIE, L., and RAYNAUD, M., Les suites spectrales associées au complexe de De Rham-Witt, *Publ. Math. I.H.E.S.*, **57** (1983), 73-212.
[22] SERRE, J.-P., Sur les corps locaux à corps résiduel algébriquement clos, *Bull. Soc. Math. France*, **8** (1961), 105-154.

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*Added in proof.* Since this paper was written, there has been considerable progress. Faltings has shown that the cohomology of any smooth proper variety over K has a Hodge-Tate decomposition, and work of Fontaine and Messing has thrown much light on the structure of these representations in the non-ordinary case.