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METAPLECTIC FORMS

by D. A. KAZHDAN *and* S. J. PATTERSON

INTRODUCTION

The purpose of this work is to study automorphic forms on a class of groups known as metaplectic groups. To give some idea of what these are, let k be an **A**-field (algebraic number field or field of functions over a finite field) containing the full group of n-th roots of 1, denoted by $\mu_n(k)$, where n is coprime to the characteristic of k. Let $G = GL_r$ and G_k, $G_{\mathbf{A}}$ be the group of rational points and the associated adelic group respectively. Then there exists a non-trivial extension of groups

$$(1) \longrightarrow \mu_n(k) \xrightarrow{i} \widetilde{G}_{\mathbf{A}} \xrightarrow{p} G_{\mathbf{A}} \longrightarrow (1)$$

which is constructed from analogous local extensions

$$(1) \longrightarrow \mu_n(k_v) \xrightarrow{i_v} \widetilde{G}_v \xrightarrow{p_v} G_v \longrightarrow (1).$$

In the case $n = 2$ these have been known implicitly for a long time but were first made explicit in Weil's memoir [49], although there it is symplectic groups which appear. Theta functions are most naturally regarded in this context. The general case was discovered independently and simultaneously as the result of two different lines of thought. In [25] and [26] T. Kubota constructed quite explicitly the n-fold extension in the case $r = 2$. He had been led to this by the derivation of the law of quadratic reciprocity from the theory of theta functions, or, ultimately from the fact that the 2-fold extension exists and splits over $G_k \subset G_{\mathbf{A}}$. He showed instead how the reciprocity law led to the existence of such an extension, and that this splits over G_k (this is generally true, not just in the case $n = 2$).

On the other hand R. Steinberg [47] and C. C. Moore [35] investigated the general algebraic problem of determining the central extensions of simply-connected Chevalley groups over arbitrary fields, and they also found metaplectic groups. The relation between the two lines of thought was recognized quite quickly, c.f. the remarks in [2] where the relation to the " congruence subgroup problem " is also discussed. This direction of investigation was completed by Matsumoto [32], who gave the general construction of the metaplectic extension of a Chevalley group; recently a yet more general and intrinsic construction has been given by P. Deligne [5].

Once these groups have been constructed and one has the splitting referred to above, viz,

$$\begin{array}{ccccccccc} (1) & \longrightarrow & \mu_n(k) & \xrightarrow{i} & \widetilde{G}_{\mathbf{A}} & \xrightarrow{p} & G_{\mathbf{A}} & \longrightarrow & (1), \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & G_k & & \end{array}$$

one is naturally led to look at automorphic forms (in the sense of [23] § 10, although some further explanation is needed). These we call *metaplectic forms*. In this respect there are two features of the " classical " theory which can guide us. One is the theory of theta functions, which form a very exceptional class of automorphic forms. We can, and do, seek analogues of these; the construction of such analogues is one of the principal objectives of this work. This goal was first proposed by Kubota (cf. [26]). In a sense it was achieved completely in one very special case ($n = 3, r = 2, k = \mathbf{Q}(\sqrt{-3})$) in [37]. A new and more conceptual treatment of the more general case ($n = 3$, $r = 2$) was given by Deligne [6], who followed Gelbart, Piatetski-Shapiro et al. [9], [10], [11] in taking the property of a representation being *distinguished* (i.e. having unique Whittaker model) as the fundamental one which relates local and global representations. They had used this concept to great effect in studying the classical theta-functions from a representation-theoretic point of view. Deligne showed that in the special case ($n = 3$, $r = 2$) there is one factor of a reducible principal series representation (locally) which has a unique Whittaker model. He then constructed a global representation by means of residues of Eisenstein series (as was originally proposed by Kubota [26]) and he showed that the local factors of these were just the representations which he had considered locally. The uniqueness of local Whittaker models then implied that the " Fourier coefficients " of a global form have a product structure; indeed they turn out to be cubic Gauss sums. This type of argument first appears in [23].

This is the problem which we shall study in general, and we shall extend the above argument to as general a case as we can. It turns out that one can construct certain local representations which have several interesting properties, as quotients of reducible principal series representations. One can show that they have a unique Whittaker model only when $r = n - 1$ or n; it is for this reason that we find ourselves compelled to look at groups of higher rank.

In the case $n = 3$, $r = 2$, $k = \mathbf{Q}(\sqrt{-3})$ considered in [37] the " L-series " of the metaplectic form was equal to the " Fourier coefficient " of an Eisenstein series. The construction in [37] was carried out by making this identification and applying a " converse theorem ". The same kind of identity still holds; that is, the " L-series " of the form with $r = n - 1$ is the " Fourier-coefficient " of an Eisenstein series for $r = 2$; unfortunately our formalism does not allow us to state this so directly, but it is contained in Theorems II.2.3 and I.4.2. In the case $r = n$ analogous considerations suggest that the metaplectic form is associated with a ζ-function in the same way that $\sum_{n\in\mathbf{Z}} e^{2\pi i n^2 z}$ is associated with the Riemann ζ-function.

The other " classical " aspect of the theory of forms of half-integral weight is the Shimura correspondence. This is a correspondence (both local and global) between representations of $\widetilde{G}_F$ (F local) or automorphic representations of $\widetilde{G}_{\mathbf{A}}$ and those of G_F or $G_{\mathbf{A}}$ whose characters satisfy certain symmetry conditions. This has been much discussed in the case $n = 2$, $r = 2$, and it has been constructed for arbitrary n and $r = 2$ by Flicker [7]. One hopes that in view of the progress on the Selberg trace formula made recently by J. Arthur and others it will be possible in the future to prove this in general. In any case the formulation in [7] makes it quite clear what one should expect and we hope to discuss this in detail in a separate publication. Such a result would be significant for the theory of representations of metaplectic groups on three grounds. Firstly, as described in § I.5, one can use a knowledge of the characters of the local representations considered here to understand their Whittaker models. Secondly, similar considerations lead to a general and " universal " formula for the " number " of Whittaker models of a general local cuspidal representation of a metaplectic group. Finally, it suggests the existence of certain very special cuspidal representations, both locally and globally, when $r = n$. In the special case $r = n = 2$ these would be the cuspidal r_χ of [11]; compare also the remarks in [7] §§ 2.2, 5.2.

This work is organized into three chapters. Chapter 0 contains essentially known material—namely a discussion of the construction of the metaplectic group, both locally and globally, and of the representation theory of Heisenberg groups. Chapter I is devoted to the local theory, mainly the construction of the representations alluded to above and the investigation of their properties. The techniques are by now fairly standard (cf. [3], [46]) but some of the results are unexpected. In the case $r = 2$ similar investigations have been carried out by H. Aritürk ([1], $n = 3$), C. Moen ([34], n general)—we have not seen this work—and P. Deligne ([6]). We do not presume to undertake a full study of the representation theory of metaplectic groups.

In Chapter II we develop the global theory. The local representations of Chapter I are shown to be the local factors of certain automorphic representations. This is achieved by an application of the theory of Eisenstein series. The final results are to be found in § II.2. The global considerations also allow us to complete the local theory in some essential points. In § II.3 we make our results more explicit in order to bring out their arithmetical content.

In [39] the second author promised a work with the same title as this. The original manuscript has now become obsolete. Most of the material referred to in [39] can be found here, especially in § II.3.

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0. — PRELIMINARIES

1. The metaplectic group

For any field F let

$$\mu_n(F) = \{x \in F^\times : x^n = 1\}.$$

In this paragraph we shall consider a local field F such that

$$\mathrm{Card}(\mu_n(F)) = n.$$

We shall summarize here the construction of the metaplectic cover of $GL_r(F)$.

Let $(.,.)$ be the n-th order Hilbert symbol on F (see [2] or [50] XIII-5); it is a map $(.,.) : F^\times \times F^\times \to \mu_n(F)$ satisfying

$$(a, b).(a', b) = (aa', b),$$

$$(a, b)\,(b, a) = 1,$$

$$(a, 1-a) = 1$$

and $\{a : (a, x) = 1 \quad \text{for all } x \in F^\times\} = F^{\times n}$

where $F^{\times n} = \{a^n : a \in F^\times\}$.

The group which we would like to construct is a central extension

$$(1) \longrightarrow \mu_n(F) \xrightarrow{i_F} \widetilde{GL}_r(F) \xrightarrow{p_F} GL_r(F) \longrightarrow (1)$$

but there are several related but non-isomorphic such extensions which we shall later have to examine. We shall first construct one such extension along with a section $GL_r(F) \xrightarrow{s_F} \widetilde{GL}_r(F)$, so that the extension will be described by a 2-cocycle σ. At this point we should remark that the corresponding extension of $SL_r(F)$ is essentially unique (cf. [33], [35] or [47]) but that this not true of $GL_r(F)$. Once we have described the construction of $GL_r(F)$ we will construct various " twisted " forms of it.

To construct $\widetilde{GL}_r(F)$ we shall adapt the construction of $\widetilde{SL}_r(F)$ given by Milnor in [33] Ch. 12. Note that $GL_r(F)$ can be regarded as a subgroup of $SL_{r+1}(F)$ by $g \mapsto \begin{pmatrix} \det(g)^{-1} & 0 \\ 0 & g \end{pmatrix}$, which shows that, by restriction, one can construct $\widetilde{GL}_r(F)$ once one has constructed $\widetilde{SL}_{r+1}(F)$. We first set up some notations. Let $G = GL_r(F)$, and let H be the subgroup of diagonal matrices. If $h_i \in F^\times (1 \leqslant i \leqslant r)$ let $\mathrm{diag}(h_i)$

be the diagonal matrix $(\delta_{ij} h_i)$. Let W be the group of permutation matrices; i.e. those matrices such that each row or column has just one non-zero entry, which is 1. Thus $W \cong S_r$, the symmetric group on r symbols. We let Φ be the set of roots of G, which we shall often consider as the set of pairs (i,j), $1 \leqslant i, j \leqslant r$, $i \neq j$. If $h = \operatorname{diag}(h_i)$, $\alpha = (i,j)$ we write

$$h^\alpha = h_i/h_j;$$

and this definition can be extended to the lattice spanned by Φ (root lattice). Let now N_+ be the group of unipotent upper-triangular matrices, N_- the group of unipotent lower-triangular matrices, and Φ^+ the set of positive roots with respect to N_+. We shall write $\alpha > 0$ if $\alpha \in \Phi^+$ and $\alpha < 0$ if $-\alpha \in \Phi^+$. We shall write $\langle\ ,\ \rangle$ for the Killing form on Φ.

If $\alpha \in \Phi$ let N_α be the corresponding subgroup of unipotent matrices. So, if $n \in N_\alpha$, $n = I + \xi \mathbf{e}_\alpha$ where $\xi \in F$ and $\mathbf{e}_\alpha$ is the elementary matrix with 1 at the α-th position and 0 elsewhere. Thus

$$hnh^{-1} = I + h^\alpha \xi \mathbf{e}_\alpha.$$

The group W acts on H by $h \mapsto h^w = w^{-1} h w$, and on Φ by $\alpha \mapsto w\alpha$ where $(h^w)^\alpha = h^{(w\alpha)}$. Clearly $\mathbf{e}_{w\alpha} = w \mathbf{e}_\alpha w^{-1}$.

We begin by constructing a central extension with a preferred section

$$0 \longrightarrow \mu_n(F) \xrightarrow{\ i\ } \widetilde{H} \underset{s_F}{\overset{p}{\rightleftarrows}} H \longrightarrow (1)$$

by the 2-cocycle

$$\sigma(h, h') = \prod_{i<j} (h_i, h'_j) \qquad h = \operatorname{diag}(h_i), \qquad h' = \operatorname{diag}(h'_i).$$

Note that, if $h, h' \in H$ and $\widetilde{h}, \widetilde{h}' \in \widetilde{H}$ are such that $p(\widetilde{h}) = h$, $p(\widetilde{h}') = h'$, then

$$\widetilde{h}\widetilde{h}'\widetilde{h}^{-1}\widetilde{h}'^{-1} = i(\prod_k (h_k, h'_k)^{-1} . (\det(h), \det(h'))).$$

One can verify that the cohomology class of σ is invariant under W (although the cocycle σ itself is not). We extend σ now to the group $M = HW$ by defining

$$\sigma(w, w') = 1 \qquad (w, w' \in W)$$

$$\sigma(h, w) = 1 \qquad (w \in W,\ h \in H)$$

$$\sigma(w, h) = \prod_{(i,j) \in \Phi(w)} (h_i, h_j)^{-1} . (-1, h_i/h_j)\,(\det(w), \det(h)) \qquad (w \in W,\ h \in H)$$

where $\Phi(w) = \{\alpha \in \Phi^+ : w(\alpha) < 0\}$. Then

$$\sigma(h_1 w_1, h_2 w_2) = \sigma(h_1, h_2^{w_1^{-1}})\, \sigma(w_1, h_2)$$

defines a 2-cocycle on M, and a corresponding extension $\widetilde{M}$ and a section $\mathbf{s}_F : M \to \widetilde{M}$.

The construction of $\widetilde{G}$ is now achieved by means of the beautiful technique of Matsumoto. Let $R : G \to M$ be such that $R(m) = m$ $(m \in M)$ and $R(ngn') = R(g)$ $(n, n' \in N_+)$; this exists and is unique, as one sees from the Bruhat decomposition. Let

$$X = \{(g, \widetilde{m}) : g \in G, \widetilde{m} \in \widetilde{M}, R(g) = p(m)\},$$

where $p : \widetilde{M} \to M$ is the natural projection.

We shall construct two groups of automorphisms of X, LAut(X) and RAut(X), which will be generated by certain elements which we now describe. First LAut(X) will be generated by

$$\lambda(n)\ (n \in N_+) \qquad \text{where } \lambda(n)\,(g, \widetilde{m}) = (ng, \widetilde{m})$$

$$\lambda(h)\ (h \in \widetilde{H}) \qquad \text{where } \lambda(h)\,(g, \widetilde{m}) = (p(h)\,g, h\widetilde{m})$$

and $\lambda(s)$ (s a simple reflection in W), which we now specify. (Recall that s is simple if there is just one $\alpha \in \Phi^+$ such that $s\alpha < 0$.) If $m \in M$ let $\widetilde{m} = \mathbf{s}_F(m) \in \widetilde{M}$. Then $\lambda(s)$ is defined to be

$$\lambda(s)\,.\,(g, \widetilde{m}) = (sg, (R(sg)\,R(g)^{-1})^{\sim}\,\widetilde{m}).$$

It is clear that LAut(X) acts transitively on X. Now define RAut(X) which is generated by

$$\lambda^*(n)\ (n \in N_+) \qquad \text{where } (g, \widetilde{m})\,\lambda^*(n) = (gn, \widetilde{m})$$

$$\lambda^*(h)\ (h \in \widetilde{H}) \qquad \text{where } (g, \widetilde{m})\,\lambda^*(h) = (gp(h), \widetilde{m}h)$$

$\lambda^*(s)$ (s a simple reflection) where $(g, \widetilde{m})\,\lambda^*(s) = (gs, \widetilde{m}(R(gs)^{-1}\,R(g))^{\sim\,-1})$; RAut(X) is also a transitive group of transformations of X. One must next verify that if $g \in \mathrm{LAut}(X)$, $g^* \in \mathrm{RAut}(X)$, $x \in X$ then

$$(gx)\,g^* = g(xg^*).$$

Granted that this true, one sees that LAut(X), and RAut(X) act simply transitively on X; for if $g_1\,x_0 = g_2\,x_0$ for some x_0, g_1, g_2 then $g_1(x_0\,g^*) = g_2(x_0\,g^*)$, and as RAut(X) acts transitively, this means that $g_1 = g_2$. Thus X is a principal homogenous space of both LAut(X) and RAut(X), which are now seen to be isomorphic. Now as $X \to G$; $(g, \widetilde{m}) \mapsto g$ has fibre isomorphic to $\mu_n(F)$ and as LAut(X) acts transitively on X, it follows that LAut(X) is an extension of G by $\mu_n(F)$. Moreover, by construction the subgroup of LAut(X) preserving $\{(p(m), \widetilde{m}) : \widetilde{m} \in \widetilde{M}\}$ contains $\widetilde{M}$. So LAut(X) is the group we are seeking; and we write $\widetilde{G} = \mathrm{LAut}(X)$, and

$$(1) \longrightarrow \mu_n(F) \xrightarrow{\ i\ } \widetilde{G} \xrightarrow{\ p\ } G \longrightarrow (1).$$

Before going further we shall say a little about the structure of the proof that $g(xg^*) = (gx)\,g^*$. It suffices to check it for g (resp. g^*) a generator of LAut(X) (resp. RAut(X)). It is easy to check this in all cases except when $g = s_\alpha$, $g^* = s_\beta$ and s_α (resp. s_β) is the simple reflection associated with the simple root α (resp. β). This one treats by considering $x = (n_\alpha\, m n_\beta, \widetilde{m})$ $(n_\alpha \in N_\alpha, n_\beta \in N_\beta)$ and computing; only the cases $\alpha = \pm m\beta$ are difficult, and one merely has to have the patience to check these (cf. [33]). One should note that when $\alpha = -m\beta$ one requires that $(x, 1-x) = 1$, and only in this case.

When $r = 2$ one can supplement this construction by describing the cocycle σ defining $\widetilde{GL}_2(F)$ explicitly; this was given by Kubota [26], and the following formula is a simpler version of his. It is

$$\sigma(g_1, g_2) = \left(\frac{x(g_1 g_2)}{x(g_1)}, \frac{x(g_1 g_2)}{x(g_2)}\right)\left(\det(g_1), \frac{x(g_1 g_2)}{x(g_1)}\right)$$

where

$$x\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\begin{cases} = c & \text{if } c \neq 0 \\ = d & \text{if } c = 0. \end{cases}$$

Now we have described an extension

$$(1) \longrightarrow \mu_n(F) \xrightarrow{i} \widetilde{GL}_r(F) \underset{\mathbf{s}}{\overset{p}{\rightleftarrows}} GL_r(F) \longrightarrow (1)$$

where the section $\mathbf{s}$ is defined by

$$\mathbf{s}(nmn') = \lambda(n)\, \widetilde{m} \lambda(n') \qquad (n, n' \in N_+, m \in M)$$

and $\widetilde{M}$ is considered as above as a subgroup of LAut(X). With respect to $\mathbf{s}$ this extension is described by a cocycle σ, which extends the σ previously defined on M, and which satisfies $\sigma(ng, g'n') = \sigma(g, g')$ $(n, n' \in N_+)$.

The " twisted " extensions, for $c \in \mathbf{Z}/n\mathbf{Z}$,

$$(1) \longrightarrow \mu_n(F) \xrightarrow{i} \widetilde{GL}_r^{(c)}(F) \underset{\mathbf{s}}{\overset{p^{(c)}}{\rightleftarrows}} GL_r(F) \longrightarrow (1)$$

are defined by a cocycle $\sigma^{(c)}$ where

$$\sigma^{(c)}(g_1, g_2) = \sigma(g_1, g_2)\,(\det(g_1), \det(g_2))^c.$$

We shall also write $\widetilde{G}^{(c)}$ for $\widetilde{GL}_r^{(c)}(F)$.

Remarks. — It is apposite here to make a simple remark concerning a subtle distinction. Let $\varepsilon : \mu_n(f) \to \mathbf{C}^\times$ be an embedding. In our applications we can distinguish between two metaplectic coverings

$$(1) \to \mu_n(F) \to \widetilde{G}_j \to G \to (1) \qquad (j = 1, 2)$$

only if the induced extensions

$$(1) \to \mathbf{C}^\times \to \widetilde{G}_{j,\varepsilon} \to G \to (1)$$

are inequivalent. This does not depend on the choice of ε. In our case $\widetilde{G}^{(c)}$ and $\widetilde{G}^{(c')}$ are indistinguishable in this sense if and only if

$$2(c - c') \equiv 0 \pmod{n}.$$

It may or may not happen that the original extensions are equivalent.

The reason for this is the following. Let $\varepsilon_1 : \mu_2(F) \to \mathbf{C}^\times$. Then there exists a function (cf. [49], p. 176)

$$\gamma : F^\times \to \mu_4(\mathbf{C})$$

which factors through $F^\times/F^{\times 2}$, so that

$$\varepsilon(x, y)_{2,F} = \gamma(xy)/\gamma(x)\,\gamma(y).$$

Thus $\varepsilon \circ (\ ,\ )_{2,F}$, as a 2-cocycle on $F^\times$ is trivial, whereas $(\ ,\ )_{2,F}$, with values in $\mu_2(F)$ need not be.

In this connection it is worth remarking that J. Klose has shown that $(\ ,\ )_{2,F}$ is trivial in $H^2(F^\times, \mu_2(F))$ if and only if

$$(-1, -1)_{2,F} = 1.$$

Now let $B = HN_+$ be the standard Borel subgroup of G. $\widetilde{B}^{(c)} = (p^{(c)})^{-1}(B)$ the group covering B in $\widetilde{G}^{(c)}$. Let also

$$Z^{(c)} = \{\lambda I : \lambda^{r-1+2rc} \in F^{\times n}\}$$

and

$$\widetilde{Z}^{(c)} = (p^{(c)})^{-1}(Z^{(c)}).$$

Proposition **0.1.1.** — *$\widetilde{Z}^{(c)}$ is the centre of $\widetilde{G}^{(c)}$ and of $\widetilde{B}^{(c)}$.*

Proof. — Consider first the centre of $\widetilde{B}^{(c)}$; this consists of elements of the form $z = \mathbf{s}(\lambda I)\, i(\zeta)$. Then if $h \in H$, $h = \operatorname{diag}(h_i)$ and if $\widetilde{h}$ is such that $p^{(c)}(\widetilde{h}) = h$

$$\widetilde{h}z\widetilde{h}^{-1} z^{-1} = i(\prod (h_i, \lambda)^{-1} . (\det(h), \lambda^r)^{1+2c}).$$

By considering the case when all h_i but one are 1 one sees that the righthand side is identically 1 if and only if $\lambda^{r(1+2c)-1} \in F^{\times n}$. This shows that $\widetilde{Z}^{(c)}$ is the centre of $\widetilde{B}^{(c)}$. If now $z \in \widetilde{Z}^{(c)}$ and if $g \in G$, $\widetilde{g} \in \widetilde{G}^{(c)}$ with $p^{(c)}(\widetilde{g}) = g$ then we define a homomorphism $\theta : G \to \mu_n(F)$ by $\widetilde{g}z\widetilde{g}^{-1} z^{-1} = i(\theta(g))$. As it is trivial on B, θ is trivial and hence z is in the centre of $\widetilde{G}^{(c)}$.

Now we begin the discussion of the topology on $\widetilde{G}^{(c)}$. If F is archimedean then there are two cases. If $F = \mathbf{C}$, $(\cdot, \cdot)$ is trivial and the cover is trivial. We shall not discuss this case further. If $F = \mathbf{R}$ then $n = 2$ and we have constructed the double (spin) cover of $GL_r(\mathbf{R})$; this shall presume known (cf. [33]).

Thus henceforth we shall take F to be non-archimedean. Let R_F be the ring of integers in F, P_F the maximal ideal of R_F. Let $|\cdot|_F$ be the normalized valuation (as in [50] Ch. 1) on F. Then one has

Proposition **0.1.2.** — *There exists an open subgroup $K \subset GL_r(R_F)$ on which $\sigma^{(c)}$ splits. If $|n|_F = 1$ we can take $K = GL_r(R_F)$.*

Proof. — See [35] pp. 54-56.

Let us write $\sigma(g_1, g_2) = \kappa(g_1 g_2)/\kappa(g_1)\,\kappa(g_2)$ for a splitting with $g_1, g_2 \in K$ (which we take to be $GL_r(R_F)$ if $|n|_F = 1$). Note that we can find K as above so that $(\det(k_1), \det(k_2)) = 1$ $(k_1, k_2 \in K)$. Hence κ also splits $\sigma^{(c)}$.

In the case $r = 2$ Kubota ([26] p. 19) has shown that

$$\kappa(g) = (c, d/\det(g)) \quad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, 0 < |c|_F < 1\right)$$
$$= 1 \qquad\qquad (|c|_F = 0, 1).$$

Let now κ, K be as above and $\kappa^* : K \to \widetilde{G}^{(c)}$, $g \mapsto i(\kappa(g))\, \mathbf{s}(g)$. Although κ^* is not unique, if κ_1^* is another such map then κ^* and κ_1^* are equal on an open subgroup $K' \subset K$, and $\kappa_1^* = (i.\theta).\kappa^*$, where θ is a locally constant homomorphism $K \to \mu_n(F)$. Thus we can define a topology on $\kappa^*(K) \subset \widetilde{G}^{(c)}$ by giving it the topology of K. Next, if $g \in \widetilde{G}^{(c)}$, the uniqueness of the germ of κ^* shows that there exists a neighbourhood U of I in K so that $g^{-1}\kappa^*(U)\, g \subset \kappa^*(K)$, and hence we can give $\widetilde{G}^{(c)}$ the structure of a topological group which agrees with the structure just described on $\kappa^*(K)$. With this topology one sees in

$$(1) \longrightarrow \mu_n(F) \xrightarrow{i} \widetilde{G}^{(c)} \xrightarrow{p^{(c)}} G \longrightarrow (1)$$

that $\widetilde{G}^{(c)}$ is a Hausdorff group and $p^{(c)}$ is a local homeomorphism.

The metaplectic extensions $\widetilde{G}^{(c)}$ can be defined as follows. Let $r' > r$ and let $t_1, \ldots, t_{r'-r}$ be integers such that $\sum_j t_j = -1$. Then we have an embedding of GL_r into $SL_{r'}$ by

$$g \longmapsto \begin{pmatrix} g & 0 & \ldots & 0 \\ 0 & (\det g)^{t_1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & (\det g)^{t_{r'-r}} \end{pmatrix}.$$

By examining the restriction of σ (on $SL_{r'}$) to the image of H_r we see that σ pulls back to $\sigma^{(c)}$ with

$$c = -1 + \sum_{i<j} t_i t_j$$

and it is easy to realize every value of c by a suitable choice of r', (t_j).

We can therefore construct κ^* on $SL_{r'}$ and pull it back to GL_r. One can verify that, if $|n|_F = 1$, there is no homomorphism

$$SL_{r'}(R_F) \to \mu_n(F)$$

and hence κ^* on $SL_{r'}(R_F)$ is uniquely defined by the condition that it should be a lifting. Such a lift we call " canonical ".

Before we proceed we observe that

$$N_+ \to \widetilde{G}^{(c)}; \quad n \mapsto \mathbf{s}(n)$$

is a homomorphism and a continuous section of $p^{(c)}$. As such, it is unique. We denote the image of this map in $\widetilde{G}^{(c)}$ by N_+^*.

Proposition **0.1.3** — *Suppose that* $|n|_F = 1$, *and let*

$$\kappa^* : GL_r(R_F) \to \widetilde{G}^{(c)}$$

be a canonical lift. Then

i) $\kappa^* \mid (H \cap K) = \mathbf{s} \mid (H \cap K)$

ii) $\kappa^* \mid W = \mathbf{s} \mid W$

iii) $\kappa^* \mid (N_+ \cap K) = \mathbf{s} \mid (N_+ \cap K)$.

Remarks. — 1) i) and ii) (resp. iii)) could have been combined to

$$\kappa^* \mid M = \mathbf{s} \mid M \quad (\text{resp. } \kappa^* \mid B = \mathbf{s} \mid B).$$

2) Since the three groups generate $GL_r(R_F)$ one sees that κ^* is determined by Proposition 0.1.3.

3) It is a consequence of this that the " canonical " lift does not depend on the choice of $r', t_1, \ldots, t_{r'-r}$.

Proof. — Consider the restriction of κ^* on SL_r, to a copy of $SL_2(F)$. Since κ^* is uniquely determined on $SL_2(F)$, this restriction is determined. Moreover if $SL_2(F)$ is that copy associated with a positive root α then one sees from an examination of the construction of $\widetilde{SL}_{r'}(F)$ that *a)* the restriction of the covering to $SL_2(F)$ is $\widetilde{SL}_2(F)$ and *b)* the restriction of $\mathbf{s}$ to $SL_2(F)$ is the corresponding section for $SL_2(F)$.

Thus if $h \in H \cap K$ one can write h as a product in $SL_{r'}(F)$, $\Pi\, h$, where each h_j lies in such a copy of $SL_2(F)$. Thus as

$$\kappa^*(\prod_j h_j) = \prod_j \kappa^*(h_j)$$

the assertion follows. The same argument is valid on $N_+ \cap K$. This proves (i) and (iii).

The same argument shows that it is only necessary to prove that $\kappa^*(w) = \mathbf{s}(w)$ for w a simple reflection as $\mathbf{s} \mid W$ is a homomorphism. We write w as δw_1 where $\delta \in H$, $\delta^2 = 1$ and w_1 belongs to the corresponding copy of $SL_2(F)$. As $\kappa^*(w) = \kappa^*(\delta) . \kappa^*(w_1)$ and as

$$\kappa^*(\delta) = \mathbf{s}(\delta) \quad (\text{by (i)})$$
$$\kappa^*(w_1) = \mathbf{s}(w_1) \quad (\text{by Kubota's formula})$$

one has only to verify that

$$\mathbf{s}(w) = \mathbf{s}(\delta) . \mathbf{s}(w_1)$$

and this follows from the construction of σ on M which we have described above. This proves the proposition.

Observe also that, as G is totally disconnected, there is a continuous section $G \to \widetilde{G}^{(c)}$, but this is not unique. In particular we can choose a continuous 2-cocycle, $\tau^{(c)}$, cohomologous to $\sigma^{(c)}$, so that $\tau^{(c)} \mid K \times K = \{1\}$.

We shall conclude our investigation of the structure of $\widetilde{G}^{(c)}$ by considering its conjugacy classes. Let then $C \subset G$ be a conjugacy class. Then $(p^{(c)})^{-1}(C)$ is a union of conjugacy classes in $\widetilde{G}^{(c)}$. If $\widetilde{C}_1$ is one of these let $\mu(C) = \{\zeta \in \mu_n(F) : i(\zeta)\,\widetilde{C}_1 = \widetilde{C}_1\}$; this is a subgroup of $\mu_n(F)$ and

$$(p^{(c)})^{-1}(C) = \bigcup_{\zeta \in \mu_n(F)/\mu(C)} i(\zeta)\,\widetilde{C}_1.$$

This can be characterized in another way. Let $x \in C$ and let $Z_G(x)$ be the centralizer of x in G. Let $\widetilde{x} \in (p^{(c)})^{-1}(x)$. Define

$$\theta_x : Z_G(x) \to \mu_n(F); \qquad z \mapsto i^{-1}(\widetilde{x}^z\, \widetilde{x}^{-1})$$

where we use the fact that G acts on $\widetilde{G}^{(c)}$ by conjugation. Then θ_x is a homomorphism and its image is $\mu(C)$.

We shall be interested in those classes C such that $(p^{(c)})^{-1}(C)$ consists of n conjugacy classes in $\widetilde{G}^{(c)}$, since in any application of the Selberg trace formula these are the only ones that would contribute.

Proposition **0.1.4.** — *Suppose* $C \subset G$ *is a conjugacy class of regular elements. Then* $\mu(C) = \{1\}$ *if and only if for* $g \in C$, $g \in Z^{(c)}.\{\gamma^n : \gamma \in T_g\}$ *where* T_g *is the maximal torus to which* g *belongs.*

Remarks. — If γ is regular and $g = \gamma^n$ then it does not necessarily follow that $\mu(g^G) = \{I\}$. As an example let n be even, $r = 2$, $\gamma = \begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix}$ where $\theta \in F^\times$, $\theta^{n/2} \notin F^{\times n}$. Then $g = \gamma^n = \begin{pmatrix} \theta^{n/2} & 0 \\ 0 & \theta^{n/2} \end{pmatrix}$ and as $(\theta^{n/2})^{2-1+2.c.2} \notin F^{\times n}$ it follows that $\mu(g^G) \neq \{I\}$.

Proof. — The proof of this proposition depends on another proposition, which we shall now formulate.

Let $A \subset M_r(F)$ be a commutative F-algebra of dimension r. Then we can represent A as $\bigoplus E_j$ where the E_j are fields and

$$\Sigma[E_j : F] = r.$$

If $u, v \in A^\times$ then choose $u', v' \in \widetilde{G}^{(c)}$ so that

$$p^{(c)}(u') = u, \; p^{(c)}(v') = v.$$

Then
$$u'.v'.u'^{-1}.v'^{-1}$$

lies in $i(\mu_n(F))$, and does not depend on the choice of u', v'. We can therefore define

$$[u, v]_A = i^{-1}(u'\,v'.u'^{-1}.v'^{-1}).$$

Let $\alpha : \bigoplus E_j \to A$ be the isomorphism between $\bigoplus E_j$ and A.

Let $(\ ,\ )_E$ be the n-th order Hilbert symbol in E_j.

Proposition **0.1.5.** — *With the notations above, let* $u = (u_j)$, $v = (v_j) \in \bigoplus E_j$. *Then*

$$[\alpha(u), \alpha(v)]_A = \prod_j (u_j, v_j)_{E_j}^{-1} (\det(\alpha(u)), \det(\alpha(v))_F^{1+2c}.$$

Before we proceed to the proof of this proposition let us show how it leads to the proof of Proposition 0.1.4. Let C be a regular conjugacy class as in the statement of the proposition. Let $g \in C$. Then the centralizer of g, T_g say, is a maximal torus in G. As such there exists a sub-F-algebra A of $M_r(F)$ such that $T_g = A^\times$. The algebra A itself is isomorphic to $\bigoplus E_j$ for some fields E_j with $\Sigma[E_j : F] = r$, as above. Suppose $g = \alpha(u)$. Then

$$\mu(C) = \{[u, v]_A : v \in \bigoplus E_j^\times\}$$

by the second definition. Thus if $\mu(C) = \{1\}$ we must have

$$[u, v]_A = 1$$

for all $v \in \bigoplus E_j^\times$. Let i be one of the indices and let $v = (v_j)$ with $v_j = 1 (j \neq i)$. Then we demand that

$$(u_i, v_i)_{E_i}^{-1} (\det(u), N_{E_i/F}(v_i)_F^{1+2c}) = 1.$$

Equivalently, one has, by a functorial property of the Hilbert symbol ([33], p. 177)

$$(u_i^{-1}.\det(u)^{1+2c}, v_i)_{E_i} = 1,$$

or, if $\lambda = \det(u)^{1+2c}$

we demand that

$$u_i \in \lambda . E_i^{\times n}.$$

But since $\det(u) = \prod_j N_{E_j/F}(u_j)$

we have $\lambda^{r(1+2c)-1} \in F^{\times n}$.

This means that u is of the form $u_1^n . \lambda$ with $\lambda^{r(1+2c)-1} \in F^{\times n}$. Conversely, if u is of this form then $[u, v]_A = 1$ for all $v \in \bigoplus E_j^\times$. These statements are precisely the assertion of the proposition.

Proof of Proposition **0.1.5.** — The proof of this proposition is based on the use of the transfer map in K-theory. We shall reduce the proof to the case that A is a field. To do this we observe that from the definition of $[\cdot, \cdot]_A$ one has that

$$[\gamma u \gamma^{-1}, \gamma v \gamma^{-1}]_{\gamma A \gamma^{-1}} = [u, v]_A$$

for any $\gamma \in G$. We shall then choose γ so that $\gamma A \gamma^{-1} \subset M_{(r_1, \ldots, r_t)}(F)$ where

$$r_j = [E_j : F],$$

$M_{(r_1,\dots,r_t)}(F)$ is the subalgebra of $M_r(F)$ consisting of blocks of size $r_1 \times r_1, r_2 \times r_2, \dots$ astride the diagonal and $\alpha(E_j)$ lies in the j-th block. Note that

$$M_{(r_1,\dots,r_t)}(F)^\times = GL_{r_1}(F) \times GL_{r_2}(F) \times \dots \times GL_{r_t}(F).$$

Denote by ρ the partition $(r_1, r_2, \dots, r_t)$ and by G_ρ

$$M_{(r_1,\dots,r_t)}(F)^\times.$$

Fix i, i^*; $1 \leq i,\ i^* \leq t,\ i \neq i^*$. Suppose $g, g^* \in G_\rho$ so that $g = (g_j)$, $g^* = (g_j^*)$ with $g_j = I$ if $j \neq i$, and $g_j^* = I$ if $j \neq i^*$. Let $\tilde{g}, \tilde{g}^* \in \tilde{G}^{(c)}$ with

$$p^{(c)}(\tilde{g}) = g, \quad p^{(c)}(\tilde{g}) = g^*.$$

Define $[g, g^*]$ by

$$i([g, g^*]) = \tilde{g}\tilde{g}^*\tilde{g}^{-1}\tilde{g}^{*-1}.$$

Again this does not depend on the choice of g, g^*. The map

$$GL_{r_i}(F) \times GL_{r_{i^*}}(F) \to \mu_n(F); \quad (g, g^*) \mapsto [g, g^*]$$

(where we have identified $GL_{r_j}(F)$ as its realization as a subgroup of G_ρ), is a bilinear map. As such it is a function only of $\det(g)$ and $\det(g^*)$. Examining this function on the diagonal subgroup we see that

$$[g, g^*] = (\det(g), \det(g^*))_F^{1+2c}.$$

This permits us to compute the commutant of $\alpha(E_i)$ and $\alpha(E_{i^*})$ $(i \neq i^*)$, where we have made use of an evident abuse of language. It therefore remains to compute the commutant of two elements of $\alpha(E_i^\times)$, and this is essentially the same problem as when A is a field. We shall, however, proceed slightly differently, and take A to be of the form $E \oplus E \oplus E$ where E is a field. We consider the subgroup

$$S = \{(x, x', x'') \in E^\times \times E^\times \times E^\times \mid xx'x'' = 1\}$$

of $A^\times$. This maps into $SL_{3[E:F]}(F)$. By a simple observation made in [33] p. 95, to know the commutator on $\alpha(S)$ determines the commutator we need (Hint: consider the commutator of $(x, x^{-1}, 1)$ and $(y, 1, y^{-1})$). We shall therefore explain how the commutator can be determined. In doing this we shall use the concept of the functor K_2, and we shall draw freely on the theory of this functor as developed in [33].

Recall that the Hilbert symbol induces

$$h_{n,F} : K_2(F) \to \mu_n(F)$$

such that, if

$$\{\ ,\ \}_F : F^\times \otimes_{\mathbf{Z}} F^\times \to K_2(F)$$

is the natural symbol map, then

$$h_{n,F}(\{x, y\}_F) = (x, y)_{n,F}.$$

Let E/F be a finite extension of fields. Then in [33] the transfer map

$$\mathrm{tr}: \ \mathrm{K}_2(\mathrm{E}) \to \mathrm{K}_2(\mathrm{F})$$

is defined. It has the property *(loc. cit.)*

$$\mathrm{tr}\{x,y\}_{\mathrm{E}} = \{x, \mathrm{N}_{\mathrm{E/F}}(y)\}_{\mathrm{F}} \qquad (x \in \mathrm{F}^\times, y \in \mathrm{E}^\times)$$

where $\mathrm{N}_{\mathrm{E/F}}$ denotes the usual norm. Since

$$(x, \mathrm{N}_{\mathrm{E/F}}(y))_{n,\mathrm{F}} = (x,y)_{n,\mathrm{E}} \qquad (x \in \mathrm{F}^\times, y \in \mathrm{E}^\times)$$

it follows that the following diagram commutes

$$\begin{array}{ccc} \mathrm{K}_2(\mathrm{E}) & \xrightarrow{\mathrm{tr}} & \mathrm{K}_2(\mathrm{F}) \\ \downarrow{\scriptstyle h_{n,\mathrm{E}}} & & \downarrow{\scriptstyle h_{n,\mathrm{F}}} \\ \mu_n(\mathrm{E}) & = & \mu_n(\mathrm{F}) \end{array}$$

To see this, note that $h_{n,\mathrm{F}} \circ \mathrm{tr}$ must be a power of $h_{n,\mathrm{E}}$ by Moore's theorem ([33] p. 165, [35] Theorem (3.1)). Moreover we need only to prove the assertion for E/F totally ramified or unramified. But in these cases there exist $x \in \mathrm{F}^\times$, $y \in \mathrm{E}^\times$ so that $(x, \mathrm{N}_{\mathrm{E/F}}(y))_{n,\mathrm{F}}$ is a primitive n-th root of unity, as one can easily verify. From these facts the assertion follows.

Now, as earlier in this section, we can construct metaplectic covers

(I) $$(1) \to \mathrm{K}_2(\mathrm{F}) \to \widetilde{\mathrm{SL}}_{r_1}(\mathrm{F}) \to \mathrm{SL}_{r_1}(\mathrm{F}) \to (1)$$

(II) $$(1) \to \mathrm{K}_2(\mathrm{E}) \to \widetilde{\mathrm{SL}}_{r_2}(\mathrm{E}) \to \mathrm{SL}_{r_2}(\mathrm{E}) \to (1)$$

which are the universal covers if $r_1, r_2 \geq 2$. Starting from an identification $\mathrm{F}^{r_1} \otimes_{\mathrm{F}} \mathrm{E} \cong \mathrm{F}^{r_1[\mathrm{E}:\mathrm{F}]}$ we embed

$$\alpha: \ \mathrm{SL}_{r_2}(\mathrm{E}) \to \mathrm{SL}_{r_2[\mathrm{E}:\mathrm{F}]}(\mathrm{F})$$

from which we obtain the map $\mathrm{tr}: \mathrm{K}_2(\mathrm{E}) \to \mathrm{K}_2(\mathrm{F})$ as the map of fundamental groups.

On the other hand, if h_1, h_2 lie in the diagonal torus of $\mathrm{SL}_{r_2}(\mathrm{E})$ then their commutator in (II) is

$$\prod_i \{h_{1,i}, h_{2,i}\}_{\mathrm{E}}^{-1} \quad (\in \mathrm{K}_2(\mathrm{E}))$$

and thus the commutator of $\alpha(h_1)$ and $\alpha(h_2)$ in (I) with

$$r_1 = r_2[\mathrm{E}:\mathrm{F}]$$

is, by definition of tr, $\prod_i \mathrm{tr}\{h_{1,i}\, h_{2,i}\}_{\mathrm{E}}^{-1}$.

If we now apply $h_{n,\mathrm{F}}$ we see that the commutator of $\alpha(h_1)$ and $\alpha(h_2)$ in

$$(1) \to \mu_n(\mathrm{F}) \to \widetilde{\mathrm{SL}}_{r_2[\mathrm{E}:\mathrm{F}]}(\mathrm{F}) \to \mathrm{SL}_{r_2[\mathrm{E}:\mathrm{F}]}(\mathrm{F}) \to (1)$$

is
$$\prod_{i<j} (h_{1,i}, h_{2,j})_{\mathrm{E}}^{-1}.$$

This is (in the special case that $r_2 = 3$) the result that we need. This completes the proof of the proposition.

Remarks. — 1) If g is not regular then the situation is more complicated. Suppose that $n \in \mathrm{C}$ where $n \in \mathrm{N}_+$. Then $\mu(\mathrm{C}) = \{1\}$. But if $\lambda n \in \mathrm{C}$, $\lambda \in \mathrm{F}^\times$, $n \in \mathrm{N}_+$, $n \neq \mathrm{I}$ then a much more careful analysis is required. Similar questions arise if g is semisimple but not regular.

2) When $r = 2$ one can easily complete the description given above. One then has that if $\lambda n \in \mathrm{C}$, $n \in \mathrm{N}_+ - \{\mathrm{I}\}$ then $\mu(\mathrm{C}) = \{1\}$ if and only if $\lambda^{2(1+4c)} \in \mathrm{F}^{\times n}$, but if $\lambda\mathrm{I} \in \mathrm{C}$ then $\mu(\mathrm{C}) = \{\mathrm{I}\}$ if and only if $\lambda^{1+4c} \in \mathrm{F}^{\times n}$.

2. The global metaplectic group

In this section we describe the construction of the adelic metaplectic groups associated with GL_r over a global field k. This is fairly standard, but it will be useful to set it down here for future reference.

We shall first fix some notation. Let k be a global field and for each place v of k let k_v be the corresponding closure. Let $\mathrm{G} = \mathrm{GL}_r$, and $\mathrm{G}_v = \mathrm{G}(k_v)$, $\mathrm{G}_k = \mathrm{G}(k)$. Let r_v be the ring of integers of k_v, m_v its maximal ideal and $q_v = \mathrm{Card}(r_v/m_v)$. Suppose that

$$\mathrm{Card}(\mu_n(k)) = n.$$

Let now

$$(1) \longrightarrow \mu_n(k_v) \xrightarrow{i_v} \widetilde{\mathrm{G}}_v^{(c)} \xrightarrow{p_v} \mathrm{G}_v \longrightarrow (1)$$

be the extension constructed in § 0.1. Here p_v is a local homeomorphism, a requirement that determines the topology on $\widetilde{\mathrm{G}}_v^{(c)}$. For a non-archimedean place v let $\mathrm{K}_v \subset \mathrm{G}(r_v)$ be a subgroup over which this extension splits, and $\mathrm{K}_v = \mathrm{G}(r_v)$ for almost all places v. We identify $\mu_n(k_v)$ with $\mu_n(k)$. Let us also choose a lift K_v^* of K_v, which for almost all v should be the canonical lift of Proposition 0.1.3. Let S be a finite set of places containing all the archimedean places. We form

$$\widetilde{\mathrm{G}}_{\mathbf{A}}^{(c)}(\mathrm{S}) = (\prod_{v \in \mathrm{S}} \widetilde{\mathrm{G}}_v^{(c)} \times \prod_{w \notin \mathrm{S}} \mathrm{K}_w^*)/\mathrm{M}$$

where
$$\mathrm{M} = \langle \tilde{j}_{v_1} \circ i_{v_1}(\zeta) \cdot \tilde{j}_{v_2} \circ i_{v_2}(\zeta)^{-1} \mid \zeta \in \mu_n(k), v_1, v_2 \in \mathrm{S} \rangle,$$

and, if $v' \in \mathrm{S}$,

$$\tilde{j}_{v'} : \widetilde{\mathrm{G}}_{v'}^{(c)} \to (\prod_{v \in \mathrm{S}} \widetilde{\mathrm{G}}_v^{(c)} \times \prod_{w \notin \mathrm{S}} \mathrm{K}_w^*)$$

is the natural injection. Moreover $\tilde{j}_v$, induces an injection

$$\tilde{G}_{v'}^{(c)} \to \tilde{G}_{\mathbf{A}}^{(c)}(S)$$

which we shall also denote by $\tilde{j}_{v'}$. We give $\tilde{G}_{\mathbf{A}}^{(c)}(S)$ the obvious topology. Let $G_{\mathbf{A}}(S)$ be analogous group defined without the metaplectic coverings. Then one has a topological extension of groups

$$(1) \longrightarrow \mu_n(k) \xrightarrow{i_{\mathbf{A}}} \tilde{G}_{\mathbf{A}}^{(c)}(S) \xrightarrow{p_{\mathbf{A}}(S)} G_{\mathbf{A}}(S) \longrightarrow (1).$$

Let $j_v : G_v \to G_{\mathbf{A}}(S)$ be the canonical injection. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
(1) & \longrightarrow & \mu_n(k) & \xrightarrow{i_{\mathbf{A}}} & \tilde{G}_{\mathbf{A}}^{(c)}(S) & \xrightarrow{p_{\mathbf{A}}(S)} & G_{\mathbf{A}}(S) & \longrightarrow & (1) \\
 & & \Vert & & \uparrow{\scriptstyle \tilde{j}_v} & & \uparrow{\scriptstyle j_v} & & \\
(1) & \longrightarrow & \mu_n(k) & \xrightarrow{i_v} & \tilde{G}_v^{(c)} & \xrightarrow{p_v} & G_v & \longrightarrow & (1)
\end{array}$$

Moreover if $S' \supset S$ then we have an analogous diagram:

$$\begin{array}{ccccccccc}
(1) & \longrightarrow & \mu_n(k) & \xrightarrow{i_{\mathbf{A}}} & \tilde{G}_{\mathbf{A}}^{(c)}(S') & \xrightarrow{p_{\mathbf{A}}(S')} & G_{\mathbf{A}}(S') & \longrightarrow & (1) \\
 & & \Vert & & \uparrow{\scriptstyle \tilde{j}_{S',S}} & & \uparrow{\scriptstyle j_{S',S}} & & \\
(1) & \longrightarrow & \mu_n(k) & \xrightarrow{i_{\mathbf{A}}} & \tilde{G}_{\mathbf{A}}^{(c)}(S) & \xrightarrow{p_{\mathbf{A}}(S)} & G_{\mathbf{A}}(S) & \longrightarrow & (1)
\end{array}$$

where $\tilde{j}_{S',S}$ and $j_{S',S}$ are the natural injections. We can thus form $\tilde{G}_{\mathbf{A}}^{(c)} = \varinjlim \tilde{G}_{\mathbf{A}}^{(c)}(S)$ and $G_{\mathbf{A}} = \varinjlim G_{\mathbf{A}}(S)$ with the appropriate topologies. Thus one has continuous injections $\tilde{j}_S$, j_S and

$$\begin{array}{ccccccccc}
(1) & \longrightarrow & \mu_n(k) & \xrightarrow{i_{\mathbf{A}}} & \tilde{G}_{\mathbf{A}}^{(c)} & \xrightarrow{p_{\mathbf{A}}} & G_{\mathbf{A}} & \longrightarrow & (1) \\
 & & \Vert & & \uparrow{\scriptstyle \tilde{j}_S} & & \uparrow{\scriptstyle j_S} & & \\
(1) & \longrightarrow & \mu_n(k) & \xrightarrow{i_{\mathbf{A}}} & \tilde{G}_{\mathbf{A}}^{(c)}(S) & \xrightarrow{p_{\mathbf{A}}(S)} & G_{\mathbf{A}}(S) & \longrightarrow & (1)
\end{array}$$

Here $G_{\mathbf{A}}$ is the usual adelic group and $\tilde{G}_{\mathbf{A}}^{(c)}$ is endowed with the finest topology with respect to which all the $\tilde{j}_S$ are continuous. Hence $\tilde{G}_{\mathbf{A}}^{(c)} \xrightarrow{p_{\mathbf{A}}} G_{\mathbf{A}}$ is a local homeomorphism, a stipulation which also determines the topology on $\tilde{G}_{\mathbf{A}}^{(c)}$. There is one further fact which will be crucial to us. Let $G_k \to G_{\mathbf{A}}$ be the diagonal embedding. We shall show that this lifts to $G_k \to \tilde{G}_{\mathbf{A}}^{(c)}$; thus

$$\begin{array}{ccccccccc}
(1) & \longrightarrow & \mu_n(k) & \longrightarrow & \tilde{G}_{\mathbf{A}}^{(c)} & \longrightarrow & G_{\mathbf{A}} & \longrightarrow & (1) \\
 & & & & & {\scriptstyle s_0}\nwarrow & \Big| & & \\
 & & & & & & G_k & &
\end{array}$$

Let, for any place v, $\mathbf{s}_v : G_v \to \widetilde{G}_v^{(c)}$ be the original section with respect to which $\sigma^{(c)}$ (over k_v) was defined. We shall define a section $\mathbf{s}_0 : G_k \to \widetilde{G}_{\mathbf{A}}^{(c)}$ such that $\mathbf{s}_0 \mid B(k)$ is an injective homomorphism. From the theory of [33] it will then follow that the restriction of the covering to G_k is trivial, and this will also imply that $\mathbf{s}_0$ is an injective homomorphism.

We shall first define $\mathbf{s}_0$ on $B(k)$ and $W(\subset G_k)$. Let γ belong to either of these two groups. We let S be a finite set of places containing all the archimedean places (if there are any) and such that if $w \notin S$ then K_w^* is the canonical lift, and $\gamma \in K_w$. Then we define $\mathbf{s}_0(\gamma)$ to be the image of

$$\prod_{v\in S} \mathbf{s}_v(\gamma) \times \prod_{w\notin S} \mathbf{s}_w(\gamma) \; (\in \prod_{v\in S} \widetilde{G}_v^{(c)} \times \prod_{w \notin S} K_w^*)$$

in $\widetilde{G}_{\mathbf{A}}^{(c)}(S)$. This is well-defined by Proposition 0.1.3.

By construction

$$\mathbf{s}_0(\gamma_1\,\gamma_2) = \mathbf{s}_0(\gamma_1)\,.\,\mathbf{s}_0(\gamma_2) \qquad (\gamma_1, \gamma_2 \in W).$$

Let us verify this also for $\gamma_1, \gamma_2 \in B(k)$. If we write γ_j as $h_j\,n_j$ with $h_j \in H(k)$, $n_j \in N_+(k)$ then (with $h_j = \operatorname{diag}(h_{j,a})$)

$$\mathbf{s}_v(\gamma_1\,\gamma_2) = \mathbf{s}_v(\gamma_1)\,.\,\mathbf{s}_v(\gamma_2)\,.\,i_v(\prod_{a<b} (h_{1,a}, h_{2,b})_v\,(\det(h_1), \det(h_2))_v^c).$$

Thus if S is choosen to be sufficiently large

$$\mathbf{s}_w(\gamma_1\,\gamma_2) = \mathbf{s}_w(\gamma_1)\,.\,\mathbf{s}_w(\gamma_2) \qquad (w \notin S)$$

whereas $\prod_{v\in S} (\prod_{a<b} (h_{1,a}, h_{2,b})_v\,.\,(\det(h_1), \det(h_2))_v^c) = 1$ by the reciprocity law. Hence in $\widetilde{G}_{\mathbf{A}}^{(c)}(S)$ one has

$$\mathbf{s}_0(\gamma_1\,\gamma_2) = \mathbf{s}_0(\gamma_1)\,.\,\mathbf{s}_0(\gamma_2).$$

Now let $g \in G_k$ be arbitrary. We can write g as $b.w.n$ ($b \in B(k)$, $w \in W$, $n \in N_+(k)$) and can define $\mathbf{s}_0(g)$ by

$$\mathbf{s}_0(g) = \mathbf{s}_0(b)\,.\,\mathbf{s}_0(w)\,.\,\mathbf{s}_0(n).$$

It is easy to see that this does not depend on the choice of b or n. This therefore defines a section. Finally, either by the theory of [33], § 11, or directly, one verifies that if $\tau : G_k \times G_k \to \mu_k$ is defined by

$$\mathbf{s}_0(g_1\,g_2) = i_{\mathbf{A}}(\tau(g_1, g_2))\;\mathbf{s}_0(g_1)\,.\,\mathbf{s}_0(g_2)$$

then $\tau(g_1, g_2) = 1$.

Remarks. — 1) One can also use the method described by Kubota in [26] to prove these results. In this approach one constructs an adelic 2-cocycle on $G_{\mathbf{A}}$. If one does this, then it is again plain that the splitting of the restriction of the cover to G_k is equivalent to the reciprocity law for the Hilbert symbol.

2) We shall denote $\mathbf{s}_0(G_k)$ by G_k^* and $\mathbf{s}_0(\gamma)$ by γ^* $(\gamma \in G_k)$. That $\mathbf{s}_0$ is not unique will cause us no problems.

To complete this section we must discuss the notion of a smooth function on $\widetilde{G}_{\mathbf{A}}^{(c)}$. Let

$$\varepsilon : \mu_n(k) \to \mathbf{C}^\times$$

be an injective homomorphism. Let us assume that we are given, for each v, a smooth function f_v (i.e. smooth at the archimedean places and locally constant at the non-archimedean places) such that

$$f_v(i_v(\zeta)\, g) = \varepsilon(\zeta) f_v(g) \qquad (\zeta \in \mu_n(k)),$$

and such that the following condition holds: *There exists a finite set of places* S *containing all the archimedean places and such that if* $w \notin S$ *then* a) K_w^* *is the canonical lift, and* b)

$$\operatorname{Supp}(f_w) = \widetilde{K}_w,$$

$$f_w(gk) = f_w(g) \qquad (k \in K_w^*),$$

and $$f_w(I) = 1.$$

Then $\otimes f_v$ is defined to be the function on $\widetilde{G}_{\mathbf{A}}^{(c)}(S')$ for any $S' \supset S$, such that on the class of

$$(g_v) \times (g_w) \in \prod_{v \in S'} \widetilde{G}_v^{(c)} \times \prod_{w \notin S'} K_w^*$$

$\otimes f_v$ has the value $\prod_{v \in S'} f_v(g_v)$. This is well-defined. We regard the tensor product as being with respect to $\mathbf{C}[\mu_n(k)]$. The space of smooth functions on $\widetilde{G}_{\mathbf{A}}^{(c)}$ will then be the space of finite linear combinations of such functions.

We note here that we shall also have to consider other spaces of functions on $\widetilde{G}_{\mathbf{A}}^{(c)}$, but we shall postpone their introduction until they are needed.

3. Heisenberg groups

In this section we summarize some of the basic facts about a certain type of group extension, usually known as Heisenberg groups. Most of the material discussed here is known, but again we have collected together those aspects which will be useful to us. An account of much of the basic theory is given by Weil in [49], Ch. I, but as he has a different type of application in mind, it cannot be used here as it stands.

To define a Heisenberg group, we suppose we are given locally compact abelian groups G, A and a bicharacter $(\ ,\ ) : G \times G \to A$, i.e. a continuous function such that $(xx', y) = (x, y)(x', y)$ and $(x, yy') = (x, y)(x, y')$. We shall also assume that A is *cyclic* in the sense that there exists a continuous faithful unitary representation $\varepsilon : A \to \mathbf{C}^\times$, which we consider fixed. In our case A will generally be a finite group, G will be a torus in GL_r and $(\ ,\ )$ will be derived from some Hilbert symbol. In the case considered in [49], G was a vector space (or its adelization), $A = \{z \in \mathbf{C}^\times : |z|^2 = 1\}$ and $(x, y) = e^{2\pi i B(x,y)}$ where B is a bilinear form.

Now, (,) is a 2-cocycle on G and hence there is a locally compact group G and a central extension

$$(1) \longrightarrow A \xrightarrow{i} \widetilde{G} \xrightarrow{p} G \longrightarrow (1)$$

where $\widetilde{G}$ is set-theoretically $G \times A$ and endowed with the multiplication

$$(g, a).(g', a') = (gg', (g, g')\, aa').$$

The group $\widetilde{G}$ takes the product topology, and all the maps are continuous, as are multiplication and inversion in $\widetilde{G}$. Then $\widetilde{G}$ is called a *Heisenberg group.*

Let us note that $\widetilde{G}$ is abelian if and only if $(\cdot, \cdot)$ is symmetric; i.e. if $(g_1, g_2) = (g_2, g_1)$. The extension is trivial if there is a continuous map $\varphi : G \to A$ for which

$$(g_1, g_2) = \varphi(g_1 g_2)\, \varphi(g_1)^{-1}\, \varphi(g_2)^{-1},$$

and which therefore satisfies for $g_1, g_2, g_3 \in G$

$$\varphi(g_1 g_2 g_3)\, \varphi(g_1)\, \varphi(g_2)\, \varphi(g_3) = \varphi(g_1 g_2)\, \varphi(g_2 g_3)\, \varphi(g_3 g_1)\, \varphi(I).$$

Let now, for $g_1, g_2 \in G$,

$$\{g_1, g_2\} = i^{-1}(\tilde{g}_1 \tilde{g}_2 \tilde{g}_1^{-1} \tilde{g}_2^{-1})$$

where $p(\tilde{g}_j) = g_j$. This is well-defined and one has

$$\{g_1, g_2\} = (g_1, g_2)\,(g_2, g_1)^{-1}.$$

This is a skew-symmetric bicharacter. Let $G_0 = \{g \in G : \{g, g'\} = 1$ for all $g' \in G\}$. Now call a subgroup $H \subset G$ *isotropic* if $\{h, h'\} = 1$ for $h, h' \in H$, and *maximal isotropic* if it is maximal with respect to this property. Otherwise expressed, H is such that $\widetilde{H} = p^{-1}(H)$ is a maximal abelian subgroup of G. Observe that H is closed in G, since its closure would have the same properties.

Examples. — 1) An interesting example of a group formed in this way is $\mathbf{Z}/4\mathbf{Z}$ as

$$0 \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0.$$

Here $G = \mathbf{Z}/2\mathbf{Z}$, $A = \mathbf{Z}/2\mathbf{Z}$ and $(x, y) = 0$ if x or y is 0 and $(1, 1) = 1$. In this case $G = \mathbf{Z}/4\mathbf{Z}$ is abelian, but is not isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Naturally { , } is trivial and $H = G$.

2) Let F be a non-archimedean local field in which -1 is not a square. Let (,) be the 2-Hilbert symbol; then (,) is a bicharacter as above ([50], XIII, § 5). In [49], p. 176, Weil has shown that there exists a function c_a $(a \in F^\times)$ with values in the fourth roots of 1 so that

$$c_{ab}\, c_a^{-1}\, c_b^{-1} = (a, b); \quad c_{a^2} = 1.$$

From this, $c_a^2 = (a, -1)$ and so c_a does not only take the values ± 1. In this case $G_0 = G$. If, moreover, $(-1, -1) = -1$ (which happens when $F = \mathbf{Q}_2$), $(c_{-1})^2 = -1$ so that (a, b) cannot be written as $\varphi(ab)/\varphi(a)\,\varphi(b)$ for any $\varphi : F^\times \to \mu_2(F)$. Thus $\widetilde{G}_0$ is not a trivial covering of G_0.

Let us now write, for any locally compact abelian group Γ, $\Gamma^\wedge$ for the group of characters, with its usual topology, and let $\langle\ ,\ \rangle_\Gamma : \Gamma^\wedge \times \Gamma \to \mathbf{C}^\times$; $(\chi, x) \mapsto \chi(x)$ be the natural pairing between $\Gamma^\wedge$ and Γ. Let now H be maximal isotropic in G; then set

$$\alpha : (H/G_0) \to (G/H)^\wedge; \quad h \mapsto \varepsilon(\{h, \cdot\})$$
$$\beta : (G/H) \to (H/G_0)^\wedge; \quad g \mapsto \varepsilon(\{g, \cdot\})$$

and one sees, by the conditions on H, that $\alpha(H/G_0)$ separates points in G/H, and $\beta(G/H)$ separates points in H/G_0. Moreover

$$\langle \alpha(h), g \rangle_{G/H} = \overline{\langle \beta(g), h \rangle_{H/G_0}}$$

so that α, β are the transposes of one another, and are injective. Thus α, β are actually isomorphisms, cf. [48], § 28.

Likewise one can identify G/G_0 and $(G/G_0)^\wedge$.

Now let $\widetilde{G}_0 = p^{-1}(G_0)$ (this need not be isomorphic to $G_0 \times A$, although this is frequently the case). Let $\omega : \widetilde{G}_0 \to \mathbf{C}^\times$ be a quasicharacter, such that $\omega \circ i = \varepsilon$ on A. Then there exists a quasicharacter $\omega' : \widetilde{H} \to \mathbf{C}^\times$ extending ω, (an easy exercise). If ω'' is another such extension of ω then ω''/ω' can be regarded as a quasicharacter on H/G_0 and, in view of our assumption that A be finite, it is a character and hence of the form $\varepsilon(\{g, \cdot\})$. Thus if ω' is as above, every extension can be written in the form $\omega' . \varepsilon(\{g, \cdot\})$. We shall write ε_g for $\varepsilon(\{g, \cdot\})$ henceforth.

Now we turn to the representations of $\widetilde{G}$. We have to bear some topological considerations in mind. We shall assume now that A is a discrete group. Let us first describe a construction of certain representations and then we shall indicate the relations between them. Let H be as above and let ω be a quasicharacter $\widetilde{G}_0$, and ω' an extension of ω to $\widetilde{H}$. Then we consider $\mathrm{ind}_{\widetilde{H}}^{\widetilde{G}}(\omega')$ which we denote by $(\pi_{\omega, \omega', H}, V_{\omega, \omega', H})$—the process of induction we have to describe in rather more detail. To do this we recall that if we consider all pairs of subgroups (L, L') with $L' \subset L \subset G$, and such that L' is compact, L is generated by a compact neighbourhood of I, and L/L' is elementary then G can be "exhausted" by all such pairs (cf. [49], No. 11). We choose L' so that $(L', g) = 1$ and $(g, L') = 1$ for all $g \in L$ and we assume also that ω is trivial on L' considered as a subgroup of $\widetilde{G}_0$. Under our assumptions such L, L' exist exhausting G. We suppose also that $L \cap H$ exhausts H. Then we consider $V_{\omega, \omega', H}(L, L')$ to be the space of the functions $f : \widetilde{L} \to \mathbf{C}$ satisfying

$$f(hg) = \omega'(h) f(g) \qquad (h \in \widetilde{H})$$

and f is a Schwarz function on L/L'. Then this is naturally endowed with the Schwarz topology and the injective limit over the (L, L') yields a space $V_{\omega, \omega', H}$. Compare [49], No. 11. In most cases the construction can be carried out much more easily, but we shall remark on this later. In particular, if G is a p-adic group this coincides with the algebraic induction; cf. [49]. For the sake of brevity, we consider $V_{\omega,\omega',H}$ as a space of functions and ignore the topological considerations from now on. As usual, $\pi_{\omega,\omega',H}$ acts on $V_{\omega,\omega',H}$ by right translation.

By an analogue of Mackey's criterion $(\pi_{\omega,\omega',H}, V_{\omega,\omega',H})$ is irreducible. Its class does not depend on the choice of ω'. Indeed, let ω'' be another choice of extension of ω to $\widetilde{H}$; then $\omega'' = \varepsilon_\gamma \omega'$ and the intertwining map $V_{\omega,\omega',H} \to V_{\omega,\omega'',H}$ is given by

$$f \mapsto (g \mapsto f(\gamma g)).$$

Now we show that in fact it does not depend on the choice of H. To this end let H_1, H_2 be maximal isotropic subgroups and ω_1 (resp. ω_2) an extension of ω to $\widetilde{H}_1$ (resp. $\widetilde{H}_2$). Then let $\widetilde{H}_* = \widetilde{H}_1 \cap \widetilde{H}_2$. Again every character of $\widetilde{H}_*/\widetilde{G}_0$ can be represented as $\varepsilon_\gamma \mid \widetilde{H}_*$; thus for some $\gamma \in G$

$$\omega_1 \mid \widetilde{H}_1 \cap \widetilde{H}_2 = \varepsilon_\gamma . \omega_2 \mid \widetilde{H}_1 \cap \widetilde{H}_2.$$

We shall replace ω_2 by $\varepsilon_\gamma \omega_2$ (making use of the intertwining operator above). Now consider the map

$$F_{21} : V_{\omega,\omega_2,H_2} \to V_{\omega,\omega_1,H_1}; \quad f \mapsto \left(g \mapsto \int_{\widetilde{H}_1/\widetilde{H}_1 \cap \widetilde{H}_2} \omega_1^{-1}(h) f(hg)\, dh\right)$$

where dh is a Haar measure on $\widetilde{H}_1/\widetilde{H}_1 \cap \widetilde{H}_2$. This integral exists—this was the purpose of the topological considerations above. Analogously one defines $F_{12} : V_{\omega,\omega_1,H} \to V_{\omega,\omega_2,H_2}$. These are both $\widetilde{G}$-maps. We need only verify that these two maps are inverses to one another, at least up to scalar multiple. But this is clear as the maps F_{12} and F_{21} are non-trivial and the V_{ω,ω_j,H_j} $(j = 1, 2)$ are irreducible.

We shall close this section by discussing two minor topics which can be most conveniently accomodated here.

For the first of these we shall show how to construct maximal isotropic subgroups in the cases which will be of interest to us. Let Γ be any abelian group and $(\ ,\ )$: $\Gamma \times \Gamma \to \mathbf{C}^\times$ any skew-symmetric bicharacter. Let $G = \Gamma^r$ and, for $c \in \mathbf{Z}$, define $(\ ,\ )$ on G by

$$(g, g') = \prod_{i<j} (g_i, g'_j) . (\prod_k g_k, \prod_l g'_l)^c$$

where $g = (g_i), g' = (g'_i)$. Then

$$\{g, g'\} = \prod_i (g_i, g'_i)^{-1} (\prod g_k, \prod g'_k)^{1+2c}.$$

Proposition **0.3.1.** — *Let notations be as above and suppose there are subgroups* $\Gamma_1 \subset \Gamma_2 \subset \Gamma$ *such that*

$$\Gamma_1 = \{\gamma \in \Gamma : (\gamma, \Gamma_1) = \{1\}\}$$

and

$$\Gamma_2 = \{\gamma \in \Gamma : (\gamma, \Gamma_1)^{r-1+2rc} = \{1\}\} \cap \{\gamma \in \Gamma : (\gamma, \Gamma_2)^{r(r-1+2rc)} = \{1\}\}.$$

Then $$\mathrm{H} = \{(\delta t_j) : t_j \in \Gamma_1, \delta \in \Gamma_2\}$$

is a maximal isotropic subgroup of G.

Proof. — We shall first check that H is isotropic. As $\{\ ,\ \}$ is bimultiplicative we have to check that $\{t, t'\} = 1$, $\{t, \boldsymbol{\delta}\} = 1$ and $\{\boldsymbol{\delta}, \boldsymbol{\delta}'\} = 1$ where $t = (t_j)$, $t' = (t'_j)$, $t_j, t'_j \in \Gamma_1$, $\boldsymbol{\delta} = (\delta, \delta, \ldots)$, $\boldsymbol{\delta}' = (\delta', \delta', \ldots)$, $\delta, \delta' \in \Gamma_2$. That $\{t, t'\} = 1$ is clear as $(\Gamma_1, \Gamma_1) = \{1\}$. Also

$$\{t, \boldsymbol{\delta}\} = (\prod_k t_k, \delta)^{r-1+2rc} = 1$$

as $(\Gamma_1, \Gamma_2)^{r-1+2rc} = 1$. Likewise

$$\{\boldsymbol{\delta}, \boldsymbol{\delta}'\} = (\delta, \delta')^{r(r-1+2rc)} = 1$$

as $(\Gamma_2, \Gamma_2)^{r(r-1+2rc)} = 1$.

Conversely, suppose $g \in \mathrm{G}$, $\{g, h\} = 1$ for all $h \in \mathrm{H}$. Choose h to be $(1, \ldots, t, 1, \ldots, 1)$, $t \in \Gamma_1$ in the j-th entry. Then, if $g = (g_i)$

$$1 = \{g, h\} = (g_j^{-1}(\prod_k g_k)^{1+2c}, t).$$

Hence $g_j^{-1}(\prod_k g_k)^{1+2c} \in \Gamma_1$. Thus we can write g_j as $\mathrm{T}_j \Delta$ where $\mathrm{T}_j \in \Gamma_1$, $\Delta^{r-1+2rc} \in \Gamma_1$. Also, if $h = (\delta, \delta, \ldots)$, $\delta \in \Gamma_2$

$$1 = \{g, h\} = (\Delta, \delta)^{r(r-1+2rc)}.$$

The conditions on Δ show that

$$\Delta \in \{\gamma \in \Gamma : (\gamma, \Gamma_1)^{r-1+2rc} = \{1\}\} \cap \{\gamma \in \Gamma_2 : (\gamma, \Gamma_2)^{r(r-1+2rc)} = \{1\}\} = \Gamma_2.$$

Thus $g \in \mathrm{H}$, as required.

Finally we shall prove the following lemma.

Lemma **0.3.2.** — *Let* F *be a non-archimedean local field and let* $n' = \operatorname{Card} \mu_n(\mathrm{F})$, *where* n *is a natural number. If* $\mathrm{R_F}$ *is the ring of integers of* F *then*

$$[\mathrm{R}_\mathrm{F}^\times : \mathrm{R}_\mathrm{F}^{\times n}] = n' \, |n|_\mathrm{F}^{-1}$$

and $$[\mathrm{F}^\times : \mathrm{F}^{\times n}] = nn' \, |n|_\mathrm{F}^{-1}.$$

Proof. — The second assertion is clearly an immediate consequence of the first. To prove the first let $d^\times x$ be a Haar measure on $\mathrm{R}_\mathrm{F}^\times$. We shall compute the volume of $\mathrm{R}_\mathrm{F}^{\times n}$. Let $a : \mathrm{R}_\mathrm{F}^\times \to \mathrm{R}_\mathrm{F}^\times$ be $a(x) = x^n$. This map is of degree n' and its Jacobian is $|n|_\mathrm{F}$. Hence the measure of the image of a is

$$n'^{-1} \, |n|_\mathrm{F} \operatorname{meas}(\mathrm{R}_\mathrm{F}^\times),$$

which is equivalent to the first assertion. This proves the lemma.

I. — LOCAL THEORY

In this chapter we shall construct principal series representations of $\widetilde{GL}_r^{(c)}(F)$ where F is a non-archimedean local field satisfying $\mathrm{Card}(\mu_n(F)) = n$. These are usually irreducible, but our interest will centre on a certain quotient representation of a reducible principal series representation. This representation we shall call " exceptional ", for it is closely related to the " Ausnahmefall " of Hecke in [20], No. 42. In the case $n = 1$ such representations are 1-dimensional, and when $n = 2$ they are amongst those associated with theta functions. They are amongst those investigated by Serre and Stark in [42] and Gelbart, Piatetski-Shapiro et al. [9], [10], [11]. The nature of this condition is most naturally expressed in terms of coinvariants (as in [6], 5.4), and by the " periodicity theorem ", Theorem 1.2.9 *e*). Note that this does *not* depend on the existence and uniqueness of a Whittaker model of the representation in question.

As was intimated in the introduction, we shall have to understand the Whittaker models of these exceptional representations in order to be able to draw global consequences. They shall be classified in § I.3.

In § I.4 we give some more detailed information about the Whittaker models of these representations and, in particular, we compute the " class-1 Whittaker functions " associated with them.

In § I.5 we establish a connection between the character of a representation and its Whittaker models. In § I.6 we consider briefly the case when F is an archimedean field.

Our notations will be carried over from Chapter 0, except that we shall suppress the dependence of the metaplectic groups on the parameter c. Thus we shall write $\widetilde{G}$ for $\widetilde{G}^{(c)}$ but it should not be forgotten that the dependence on c is important.

In this chapter $\pi \in F$ will be a uniformizer of F and $q = |\pi|_F^{-1}$.

I.1. Principal Series Representations

Let G, H, N_+, $\widetilde{G}$, $\widetilde{H}$, N_+^* be as before, $B = HN_+$, $\widetilde{B} = \widetilde{H}N_+^*$. We also let $H_n = \{h^n : h \in H\}$, $\widetilde{H}_n = p^{-1}(H_n)$. Note that $\widetilde{H}_n \widetilde{Z}$ is the centre of $\widetilde{H}$. We let $\widetilde{H}_* \supset \widetilde{H}_n \widetilde{Z}$ be a maximal abelian subgroup of $\widetilde{H}$. We fix an injective character $\varepsilon : \mu_n(F) \to \mathbf{C}^\times$.

Let Φ be the set of roots of G and let $\Phi(\mathbf{Z})$ be the lattice spanned by Φ in $\mathrm{Hom}_{\mathrm{alg}}(H_r, H_1)$, where H_r is the diagonal sub-algebraic-group of GL_r and $H_1 = GL_1$. For any commutative ring R let $\Phi(R) = \Phi(\mathbf{Z}) \otimes_{\mathbf{Z}} R$. Let $\langle\ ,\ \rangle$ be the Killing form on $\Phi(R)$.

If $u = \sum_{\alpha \in \Phi} u(\alpha)\, \alpha \in \Phi(\mathbf{C})$ and $h \in \mathrm{H}$ then we define $|h^u|_{\mathrm{F}}$ to be $\prod_{\alpha \in \Phi} |h^\alpha|_{\mathrm{F}}^{u(\alpha)}$. We shall also consider the function $h \mapsto |h^u|_{\mathrm{F}}$ as a function on

i) B which factors through $\mathrm{B} \to \mathrm{B}/\mathrm{N}_+ \cong \mathrm{H}$,

ii) $\widetilde{\mathrm{H}}$ which factors through $\widetilde{\mathrm{H}} \xrightarrow{p} \mathrm{H}$, and

iii) $\widetilde{\mathrm{B}}$ which factors through $\widetilde{\mathrm{B}} \xrightarrow{p} \mathrm{B} \to \mathrm{B}/\mathrm{N}_+ \cong \mathrm{H}$.

In particular let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

and set $$\mu(h) = |h^\rho|_{\mathrm{F}},$$

which we also regard as a function on all of the groups listed above.

Let W be the group of permutation matrices and let $\mathbf{s}$ be the section of p constructed in § 0.1. One then has

$$\mathbf{s}(w_1 w_2) = \mathbf{s}(w_1)\, \mathbf{s}(w_2)\, i((\det(w_1), \det(w_2)))^c.$$

Thus $\mathbf{s}$ fails to be a homomorphism only in the case that c is odd and $(-1, -1) = -1$. This last condition implies that $n \equiv 2 \pmod 4$ and so if we let $c' = c + n/2$ then c' is even and $2c \equiv 2c' \pmod n$. We have already remarked in the discussion preceeding Proposition 0.1.1 that we cannot distinguish representation-theoretically between $\widetilde{\mathrm{G}}^{(c)}$ and $\widetilde{\mathrm{G}}^{(c')}$, and so we can *assume that when* $n \equiv 2 \pmod 4$ *the parameter c will be taken to be even.* With this assumption $\mathbf{s}$ is always a homomorphism and we shall henceforth identify $\mathbf{s}(\mathrm{W})$ and W.

Let ω be a quasicharacter of $\widetilde{\mathrm{H}}_n \widetilde{\mathrm{Z}}$ such that $\omega \circ i = \varepsilon$. Extend ω to a quasicharacter ω' of $\widetilde{\mathrm{H}}_*$. Now let $\widetilde{\mathrm{B}}_* = \widetilde{\mathrm{H}}_* \mathrm{N}_+^*$ and extend ω' to $\widetilde{\mathrm{B}}_*$ by

$$\omega'(hn) = \omega'(h) \qquad (h \in \widetilde{\mathrm{H}}_*, n \in \mathrm{N}_+^*).$$

Let $\mathrm{V}(\omega')$ be the space of functions $f : \widetilde{\mathrm{G}} \to \mathbf{C}$ such that

1) $$f(bg) = (\omega' \mu)(b) f(g) \qquad (b \in \widetilde{\mathrm{B}}_*);$$

2) there is an open subgroup $\mathrm{K}_f \subset \widetilde{\mathrm{G}}$ such that $f(gk) = f(g)$ $(k \in \mathrm{K}_f)$, and let $\widetilde{\mathrm{G}}$ act on this space by right translations. This is an admissible representation of $\widetilde{\mathrm{G}}$ and we denote it by $(\pi(\omega'), \mathrm{V}(\omega'))$.

Observe that this can be constructed in several steps: first by inducing ω' to $\widetilde{\mathrm{H}}$, yielding $\pi_{\mathrm{H}}(\omega')$, then by extending this to $\widetilde{\mathrm{B}}$ by pulling back via $\widetilde{\mathrm{B}} \to \widetilde{\mathrm{B}}/\mathrm{N}_+^* \cong \widetilde{\mathrm{H}}$, yielding $\pi_{\mathrm{B}}(\omega')$, and finally inducing $\mu \otimes \pi_{\mathrm{B}}(\omega')$ to $\widetilde{\mathrm{G}}$. This shows that the class $(\pi(\omega'), \mathrm{V}(\omega'))$ depends only on ω, since the class of $\pi_{\mathrm{H}}(\omega')$ depends only on ω, as we saw in § 0.3.

Let us observe that ω^{-1} is a representation on $\widetilde{H}_n$ and that $\omega^{-1} \circ i = \bar{\varepsilon}$. The representation $(\pi(\omega'^{-1}), V(\omega'^{-1}))$ can be identified with the (algebraic) contragredient of $(\pi(\omega'), V(\omega'))$ by means of the pairing

$$V(\omega') \times V(\omega'^{-1}) \to \mathbf{C}$$

$$(f_1, f_2) \mapsto \langle f_1, f_2 \rangle = \sum_{\eta \in \widetilde{H}_n Z \backslash \widetilde{H}} \int_{N_+^*} f_1(\eta w_0 n) f_2(\eta w_0 n)\, dn,$$

where $w_0 = (\delta_{i, r+1-i}) \in W$.
The invariance of this pairing follows in the usual way.

Let now ω be as above; then there exist real numbers $\sigma_{12}, \sigma_{23}, \ldots, \sigma_{r-1,r}, \sigma_0$ so that, if $h \in \widetilde{H}_n \widetilde{Z}$, $p(h) = \operatorname{diag}(h_j)$ then

$$|\omega(h)| = |h_1/h_2|_F^{\sigma_{12}} |h_2/h_3|_F^{\sigma_{23}} \ldots |h_{r-1}/h_r|_F^{\sigma_{r-1,r}} |h_1 \ldots h_r|_F^{\sigma_0}.$$

If we write σ_α for $\sigma_{i,i+1}$ if α is the simple root $(i, i+1)$, then writing

$$\sigma(\omega) = \sum \sigma_\alpha\, \alpha$$

one has in a self-explanatory notation

$$|\omega(h)| = |p(h)^{\sigma(\omega)}|_F\, |\det p(h)|_F^{\sigma_0}.$$

One thinks of $\sigma(\omega)$ as a kind of " real part " of ω.

If ω is a quasicharacter as above then for $w \in W$, ${}^w\omega$, defined by

$$({}^w\omega)(h) = \omega(h^w)$$

is also of the same type. Note that

$$\sigma({}^w\omega) = w\sigma(\omega).$$

We say that ω is *dominant* if $\sigma(\omega)$ lies in the dominant Weyl chamber; that is, if $\alpha > 0$ then $\langle \alpha, \sigma(\omega) \rangle > 0$. This means that if we write $|\omega(h)| = \prod_i |h_i|_F^{t_i}$ then $t_1 > t_2 > t_3 > \ldots > t_r$. We let $\omega^n : H \to \mathbf{C}^\times$ be the quasicharacter defined by

$$\omega^n(h) = \omega(\mathbf{s}(h^n)).$$

We call ω *unramified* if ω^n is trivial on $H \cap GL_r(R_F)$.

We shall end this section with some considerations special to those F with $|n|_F = 1$. We first construct a special H_* which we will take as standard. In this case we apply Proposition 0.3.1 with $\Gamma_1 = R_F^\times F^{\times n}$, $\Gamma_2 = R_F^\times F^{\times n/(n, r-1+2rc)}$. These satisfy the conditions of Proposition 0.3.1 by [50] XIII-5, Prop. 6. Note that the corresponding $\widetilde{H}_*$ is normalized by W. When $|n|_F = 1$ we shall take K^* to be the canonical lift of $GL_r(R_F)$ characterized by Proposition 0.1.3. We now make an observation which simplifies much of our later work.

Lemma **I.1.1.** — *Let* F *be such that* $|n|_F = 1$ *and let* ω *be an unramified quasicharacter of* $\widetilde{H}_n \widetilde{Z}$. *Then there exists a quasicharacter* χ *of* $F^\times$ *such that* $\omega \cdot (\chi \circ \det \circ p)$ *is trivial on* $\widetilde{Z} \cap K^*$.

Proof. — Let for $\lambda \in R_F^\times$, $\lambda^{r-1+2rc} \in F^{\times n}$,

$$\omega_1(\lambda) = \omega(\varkappa^*(\lambda I)).$$

This is a character of $R_F^\times$. As ω is unramified, ω_1 is trivial on $R_F^{\times n}$. Now we can observe (as $\lambda^{r(1+2c)-1} \in F^{\times n}$)

$$\omega_1(\lambda) = \omega_1(\det(\lambda))^{1+2c} \quad (\lambda \in R_F^\times).$$

We now let χ be an extension of $\omega_1^{-(1+2c)}$ to $F^\times$. Then

$$\omega(\chi \circ \det \circ p)$$

is trivial on $\widetilde{Z} \cap K^*$.

Corollary **I.1.2.** — *With the notations above χ can be chosen so that $\omega.(\chi \circ \det \circ p)$ is unitary on $\widetilde{Z}$.*

Proof. — Clear.

When $|n|_F = 1$ we shall call ω *normalized* if ω is trivial on $\widetilde{Z} \cap K^*$. Suppose that ω is an unramified, normalized quasicharacter of $\widetilde{H}_n \widetilde{Z}$. If $\widetilde{H}_*$ is as above then there exists a unique extension ω' of ω to $\widetilde{H}_*$ characterized by

$$\omega' \mid \widetilde{H}_* \cap K^* = 1.$$

These extensions of $\widetilde{H}_n \widetilde{Z}$ and ω we shall also term *canonical.*

Lemma **I.1.3.** — *Suppose that $|n|_F = 1$ and that ω as above is unramified and normalized. Let $(\pi, V) = (\pi(\omega'), V(\omega'))$. Then*

$$\dim\{v \in V : \pi(k)\, v = v \ (k \in K^*)\} = 1.$$

Proof. — We may choose $\widetilde{H}_*$ and ω' to be canonical. Then as $\widetilde{G} = \cup\, \widetilde{B}_*\, \eta K^*$, where η runs through a set of representatives of $\widetilde{H}_*\backslash\widetilde{H}$, a function f in $V(\omega')$ which satisfies $\pi(k) f = f$ is determined by its values $f(\eta)$. If $h \in \widetilde{H} \cap K^*$ then there is a consistency condition, namely that

$$f(\eta h) = f(\eta h \eta^{-1}.\eta)$$

so that $(\omega'\, \mu)(\eta h \eta^{-1}) = 1$ if $f(\eta) \neq 0$. But this condition is verified if $\eta \in \widetilde{H}_*$ (by construction) but not for any $\eta \notin \widetilde{H}_*$ (by the maximality of $\widetilde{H}_*$). This proves the lemma.

Notation. — Under the assumptions of the lemma and supposing that $\widetilde{H}_*$, ω' are canonical we shall let $v_0(\omega') \in V(\omega')$ be that element which is K^*-invariant and for which $v_0(\omega')(I) = 1$. If no confusion should arise we shall write v_0 instead of $v_0(\omega')$.

I.2. Coinvariants and intertwining operators

In this section we shall use the general methods of [3] to investigate the principal series representations introduced in § I.1. In particular, we shall investigate the reducibility of these representations. Similar techniques can also be found in [46] Chapters 1 and 2.

Let $P \subset G$ be a standard parabolic subgroup of G. One has $P = M_P U_P$ where M_P is the standard Levi component and U_P is the unipotent radical of P. Then we let $U_P^* \subset N_+^*$ be the lift of U_P to $\tilde{G}$. Let $\tilde{M}_P = p^{-1}(M_P)$. Then if (π, V) be an admissible representation, we defined

$$\varphi_P(V) = V/\langle \pi(n)\, v - v \mid v \in V,\ n \in U_P^* \rangle.$$

This is an admissible $\tilde{M}_P$-module (Jacquet's theorem, cf. [3] 1.9 (e)), and φ_P is an exact functor from the category of admissible $\tilde{G}$-modules to that of admissible $\tilde{M}_P$-modules (cf. [3] 1.9 (a)). It is often called the *Jacquet functor* (e.g. in [46]), but in [3] it is called the *localization functor*, and in [6] it is called the *functor of coinvariants*.

We shall first make use of this in the case $P = B$, in which case we write φ_0 for φ_B.

Proposition **I.2.1.** — *The* $\tilde{H}$*-module* $\varphi_0((\pi(\omega'), V(\omega')))$ *has a Jordan-Hölder series whose composition factors are*

$$\mathrm{ind}_{w\tilde{H}_* w^{-1}}^{\tilde{H}}({}^w\omega' . \mu) \qquad (w \in W).$$

If ${}^w\omega \neq \omega$ *for all* $w \in W$, $w \neq I$ *then*

$$\varphi_0((\pi(\omega'), V(\omega'))) \cong \bigoplus_{w \in W} \mathrm{ind}_{w\tilde{H}_* w^{-1}}^{\tilde{H}}({}^w\omega'\, \mu).$$

Remark. — The representation $\mathrm{ind}_{\tilde{H}_*}^{\tilde{H}}(\omega' . \mu)$ is the representation (π_ω, V_ω) of § 0.4 (by definition).

Proof. — This is a variant of the Geometrical Lemma 2.12 of [3]. It follows at once from [3] Theorem 5.2, which is proved in sufficient generality to cover our case (cf. [3] 6.4). The last statement is then immediate.

Let us now call ω *regular* if ${}^w\omega \neq \omega$ for all $w \in W$, $w \neq I$.

As an immediate application Proposition I.2.1 one has:

Proposition **I.2.2.** — *Let* ω_1, ω_2 *be two quasicharacters of* $\tilde{H}_n \tilde{Z}$ *and let* ω_1', ω_2' *be extensions to* $\tilde{H}_*$. *Then, if* ω_1 *is regular*

$$\dim \mathrm{Hom}_G(V(\omega_1'), V(\omega_2')) \leqslant 1$$

with equality if and only if $\omega_2 = {}^w\omega_1$ *for some* $w \in W$.

Proof. — This follows at once from Proposition I.2.1 and [3] 1.9 (a), (b).

It is now our intention to construct a homomorphism $V(\omega') \to V({}^{w}\omega')$ explicitly. We do this by writing, for $f \in V(\omega')$

$$(I_w f)(g) = \int_{N_+^*(w)} f(w^{-1} ng)\, dn$$

where $N_+^*(w)$ is the subgroup of N_+^* corresponding to roots $\alpha > 0$ such that $w^{-1}\alpha < 0$. If this converges for all $f \in V(\omega')$ and is non-trivial then it is a generator of $\mathrm{Hom}(V(\omega'), V({}^{w}\omega'))$. Here ${}^{w}\omega'$ is a quasicharacter of $(\widetilde{H}_*)^{w^{-1}}$, which is also a maximal abelian subgroup of $\widetilde{H}$.

Let $s \in \Phi(\mathbf{C})$ (for the notation see § I.1) and let ω_s be the quasicharacter $h \mapsto |h^s|_F$ of H, $\widetilde{H}$, B and $\widetilde{B}$. Write s in the form $\Sigma s_\alpha \alpha$, where the sum is over the positive simple roots. Let, for any positive simple root α,

$$X_\alpha(s) = |\pi|_F^{s_\alpha},$$

where π is a uniformizer of F.

We let $\Omega_s : G \to \mathbf{C}^\times$ be defined as

$$\Omega_s(g) = \omega_s(h)$$

if, in the Iwasawa decomposition $G = N_+ HK$, $g = nhk$. This we also regard as a function on $\widetilde{G}$. Note that if $f \in V(\omega')$ then $\Omega_s f \in V(\omega_s \omega')$; this can be regarded as trivializing the vector bundle whose fibre over the quasicharacter ω of $\widetilde{H}_n \widetilde{Z}$ is $V(\omega')$, when the maps $\Phi(\mathbf{C}) \to \{\text{Quasicharacters on } \widetilde{H}_n \widetilde{Z}\} : s \mapsto \omega_s \omega'$ are used to define the structure of a (disconnected) complex manifold on the set of quasicharacters of $\widetilde{H}_n \widetilde{Z}$. We shall write f_s for $\Omega_s f$.

We now introduce the notion of the length, $l(w)$ of $w \in W$; this is the minimal number of simple reflections $s_1, \ldots, s_l$ needed to write w as $s_1 s_2 \ldots s_l$. There is in W a unique longest element, which we shall denote by w_0; it is characterized by $w_0 \alpha < 0$ for all $\alpha > 0$. It is the same element as was introduced in § I.1.

If ω is a quasicharacter of $\widetilde{H}_n \widetilde{Z}$ and α is a root we let, for $x \in F^\times$,

$$\omega_\alpha^n(x) = \omega^n(\mathrm{diag}(1, \ldots, 1, x, 1, \ldots, x^{-1}, \ldots, 1)),$$

where, if $\alpha = (i, j)$, the x is in the i-th position and x^{-1} in the j-th position. This is a quasicharacter of $F^\times$. Moreover we recall that given any quasicharacter χ of $F^\times$ we can define the associated L-function $L(\chi)$ as

$$\begin{aligned} L(\chi) &= (1 - \chi(\pi))^{-1} \quad &&(\chi \text{ unramified, } \pi \text{ a uniformizer of F}) \\ &= 1 &&(\chi \text{ ramified}). \end{aligned}$$

Proposition **I.2.3.** — a) *Suppose* $w_1, w_2 \in W$, $l(w_1 w_2) = l(w_1) + l(w_2)$. *Suppose also that*

$$I_{w_1} : V({}^{w_2}\omega') \to V({}^{w_1 w_2}\omega'); \quad I_{w_2} : V(\omega') \to V({}^{w_2}\omega')$$

are defined in the sense that the integral above is everywhere absolutely convergent. Then

$$I_{w_1 w_2} : V(\omega') \to V({}^{w_1 w_2}\omega')$$

is defined and

$$I_{w_1 w_2} = I_{w_1} I_{w_2}.$$

b) *If* $w \in W$ *then* I_w *is defined for* ω *satisfying*

$$\langle \alpha, \sigma(\omega) \rangle > 0 \quad \textit{for } \alpha > 0 \quad \textit{such that } w\alpha < 0.$$

Moreover if $t \in \Phi(\mathbf{C})$, $\langle \alpha, \sigma(\omega_t\, \omega) \rangle > 0$ *for* $\alpha > 0$ *such that* $w\alpha < 0$, *then, for* $f \in V(\omega')$,

$$\prod_{\substack{\alpha > 0 \\ w\alpha < 0}} L((\omega_t\, \omega)^n_\alpha)^{-1} . I_w(f_t)$$

is a polynomial in $\{X_\alpha(t), X_\alpha(t)^{-1} : \alpha > 0, w\alpha < 0\}$.

Proposition **I.2.4.** — *Let* F *be such that* $|n|_F = 1$ *and suppose that* ω *is unramified and normalized. Let* $v_0(\omega')$ *be defined as in* § I.1. *Let* $m = \text{meas}(R_F)$ *be the measure of* R_F *with respect to the additive measure. Then if* ω *satisfies the conditions of Proposition* I.2.3

$$I_w\, v_0(\omega') = \{ \prod_{\substack{\alpha > 0 \\ w\alpha < 0}} mL(\omega^n_\alpha)\, L(|\ |_F\, \omega^n_\alpha)^{-1} \}\, v_0({}^w\omega').$$

Proofs. — We have grouped these two propositions together since their proofs involve the same techniques.

First of all, Proposition I.2.3 is a simple application of Fubini's theorem.

Now let us assume we have proved Propositions I.2.3 and I.2.4 when $w = s$, a simple reflection. Then one deduces the general case by induction on $l(w)$. More specifically, assume that for some l that these have been proved for all $w' \in W$ with $l(w') < l$. Suppose that $l(w) = l$. Then we can find a simple reflection s such that $l(w) = l(ws) + 1$. By Proposition I.2.3

$$I_w = I_{ws} . I_s.$$

Moreover, if α is the simple root associated with s then

$$\{\beta : \beta > 0, w\beta < 0\} = \{\alpha\} \cup \{s\beta' > 0 : \beta' > 0, ws(\beta') < 0\},$$

the union being disjoint. From these two facts it is easy to derive the validity of the two assertions for w from the corresponding ones for ws and s.

Thus we have reduced the general case to that when $w = s$, a simple reflection. Recall that, if $f \in V(\omega')$ then

$$I_s f_t(g) = \int_{N^\ddagger_*(s)} f_t(s^{-1}\, ng)\, dn.$$

We have to verify that the integral is absolutely convergent. Consider now a fixed function $f \in V(\omega')$ and let L be an open neighbourhood of I in $\widetilde{G}$ such that $f(gl) = f(g)$ $(l \in L)$. Let η run through a fixed set of representatives R of $\widetilde{H}_* \backslash \widetilde{H}$. Then we can find a compact neighbourhood L_1 of I in $N^*_+(s)$ such that if $n \in N^*_+(s) - L_1$, $s^{-1}ng$ can be written as

$$s^{-1}ng = b(n)\,\eta(n)\,gl(n) \qquad (b(n) \in \widetilde{B}_*,\ \eta(n) \in R,\ l(n) \in L).$$

Thus

$$I_s f_t(g) = \int_{L_1} f_t(s^{-1}ng)\,dn + \sum_{\eta \in R} \left(\int_{\{n \in N^*_+(s) - L_1 : \eta(n) = \eta\}} \omega_t\,\omega'(b(n))\,dn\right) f_t(\eta g) \tag{1}$$

Let α be the root associated with s. Then it is clear that, as L_1 is compact, the first term is a polynomial in $X_\alpha(t)$, $X_\alpha(t)^{-1}$. Thus we have to verify that the integral in the second term is absolutely convergent and we must compute it. The computation involved is very similar to the one involved in computing $I_s v_0(\omega')$, so we shall discuss this and then treat the two computations together. The integral above we shall call the " first case ".

Since v_0 is unique (Lemma I.1.3) $I_s v_0(\omega')$ is a multiple of $v_0({}^s\omega')$. However, as $v_0({}^s\omega')$ can be regarded as a function on $\widetilde{G}$, and its value at I is 1, we see that the multiple is $I_s v_s(\omega')(I)$. This is

$$\int_{N^*_+(s)} v_0(sn)\,dn.$$

This integral is the sum of $m.v_0(s) = m$ and

$$\int_{N^*_+(s) - N^*_+(s) \cap K^*} v_0(sn)\,dn.$$

If $n \in N^*_+(s) - N^*_+(s) \cap K^*$ we can write

$$s^{-1}n = b(n)\,\eta(n)\,k_0(n) \qquad (b(n) \in \widetilde{B}_*,\ \eta(n) \in R,\ k_0(n) \in K^*).$$

Then this second integral is

$$\int_{\{n \in N^*_+(s) - N^*_+(s) \cap K^* : \eta(n) = 1\}} (\omega'\,\mu)(b(n))\,dn$$

which is the same type of integral as above. This we call the " second case ".

If we write $n = \mathbf{s}(I + \xi \mathbf{e}_\alpha)$ then

$$s^{-1}n = h_\alpha(\xi)\,\eta_0\,\mathbf{s}(I - \xi\mathbf{e}_\alpha)\,\mathbf{s}(I + \xi^{-1}\mathbf{e}_{-\alpha})$$

where $h_\alpha(\xi) = \mathbf{s}(\mathrm{diag}(1, \ldots, 1, \xi^{-1}, \xi, 1, \ldots, 1))$ (where, if $\alpha = (i, i+1)$, ξ^{-1} is in the i-th place, ξ in the $(i+1)$-th), and $\eta_0 = \mathbf{s}(\mathrm{diag}(1, 1, \ldots, -1, 1, \ldots, 1))$, where the -1 is in the i-th position. In the first case, if L_1 is large enough $\mathbf{s}(I + \xi^{-1}\mathbf{e}_{-\alpha}) \in gLg^{-1}$, whereas in the second case $\mathbf{s}(I + \xi^{-1}\mathbf{e}_{-\alpha}) \in K^*$. Thus in the first case we can consider the integral split into classes modulo n-th powers, so that it is a sum of integrals of the form

$$n^{-1}\int_{\{\xi : |\xi|_F > c\}} (\omega_t\,\omega'\,\mu)(h_\alpha(\xi^n))\,|n|_F\,|\xi|_F^n\,d^\times\xi$$
$$= n^{-1}\int_{\{\xi : |\xi|_F > c\}} (\omega_t\,\omega)^n_\alpha(\xi)^{-1}\,|n|_F\,d^\times\xi.$$

This is a familiar integral, convergent whenever

$$|(\omega_t\,\omega)^n_\alpha(\pi)|^{-1} > 1,$$

which is precisely the stated condition. Moreover this integral is 0 if ω^n_α is ramified, and is of the form $m\mathrm{X}_\alpha(t)^{\mathrm{K}}/(1 - \mathrm{X}_\alpha(t)^n\,\omega^n_\alpha(\pi))$ if ω_α is unramified. This proves Proposition I.2.3 when $w = s$.

In the second case we write $\xi = \pi^{-1}x$ where $j \equiv 0 \pmod n$, $j > 0$ and x ranges over $\mathrm{R}^\times_\mathrm{F}$. The integral is then

$$\sum_{\substack{j\equiv 0(n)\\ j>0}} \int_{\mathrm{R}^\times_\mathrm{F}} f(h_\alpha(\pi^{-j}x)\,\eta_0)\,dx\,.\,|\pi|_\mathrm{F}^{-j} = \sum_{k=1}^{\infty} \omega^n_\alpha(\pi)^k\, m(1 - |\pi|_\mathrm{F})$$

$$= m(1 - |\pi|)\,\omega^n_\alpha(\pi)/(1 - \omega^n_\alpha(\pi)) \qquad (2)$$

Thus we have

$$\int_{\mathrm{N}^*_\ddagger(s)} v_0(s^{-1}n)\,dn = m(1 + (1 - |\pi|_\mathrm{F})\,\omega^n_\alpha(\pi)/(1 - \omega^n_\alpha(\pi)))$$
$$= m(1 - q^{-1}\,\omega^n_\alpha(\pi))/(1 - \omega^n_\alpha(\pi)),$$

as required. This completes the proofs of the propositions.

As a further application of the computations carried out above we can deduce a rather technical result which will be needed later. Let α be a simple root, s the associated reflection. We shall investigate the case that $\omega^n_\alpha = 1$, or, equivalently, ${}^s\omega = \omega$.

In this case ${}^s\omega'$ is an extension of ω to $\widetilde{\mathrm{H}}^s_*$. Thus $\mathrm{V}(\omega')$ and $\mathrm{V}({}^s\omega')$ are equivalent and we wish to make the relationship between them explicit, following the principles set out in § 0.3.

First of all note that on $\widetilde{\mathrm{H}}_* \cap \widetilde{\mathrm{H}}^s_*$, ω' and ${}^s\omega'$ are extensions of ω. Thus $\omega'.({}^s\omega')^{-1}$ is a character of $\widetilde{\mathrm{H}}_* \cap \widetilde{\mathrm{H}}^s_*$ trivial on $\widetilde{\mathrm{H}}_n\widetilde{\mathrm{Z}}$. Since the quotient group is of exponent n it follows that $\omega'.({}^s\omega')^{-1}$ is of order dividing n. Indeed,

$$\omega'({}^s\omega')^{-1}(h) = \omega'(hs^{-1}h^{-1}s)$$
$$= \varepsilon(-1, h_j \det(h))\, c(h_i/h_j)$$

where $\alpha = (i, j)$ and $c(\xi) = \omega'\,\mathbf{s}(\mathrm{diag}(1, \ldots, 1, \xi, \xi^{-1}, 1, \ldots, 1))$, when this is defined ξ being in the i-th position.

One knows, as $\omega'({}^s\omega')^{-1}$ is a character trivial on $\widetilde{\mathrm{H}}_n\widetilde{\mathrm{Z}}$, that c is a character on a subgroup of $\mathrm{F}^\times$ containing $\mathrm{F}^{\times n}$ and c is trivial on $\mathrm{F}^{\times n}$. As c can be extended to a character of $\mathrm{F}^\times$ we see that $c(\xi) = \varepsilon(x, \xi)$ for some $x \in \mathrm{F}^\times$. Thus, with the notations above,

$$\omega'({}^s\omega')^{-1}(h) = \varepsilon \circ i^{-1}(h_\alpha(x)\,\eta_0)^{-1}\,h(h_\alpha(x)\,\eta_0)\,h^{-1}.$$

Hence we construct a map $\alpha_s : \mathrm{V}({}^s\omega') \to \mathrm{V}(\omega')$ such that

$$\alpha_s(f)(g) = \int_{\widetilde{\mathrm{H}}_*\cap\widetilde{\mathrm{H}}^s_*\backslash\widetilde{\mathrm{H}}_*} (\omega'\,\mu)(y)^{-1}\,|x|_\mathrm{F}^{-1} f((h_\alpha(x)\,\eta_0)^{-1}yg)\,dy.$$

Here the integral is simply a finite sum; let A_0 be the measure of $\widetilde{\mathrm{H}}_* \cap \widetilde{\mathrm{H}}^s_*\backslash\widetilde{\mathrm{H}}_*$ with respect to the chosen measure. Then the lemma referred to above is the following:

Lemma 1.2.5. — *Let notations be as above. Then, for* $f \in V(\omega')$ *one has*

$$\alpha_s(\lim_{t\to 0} L(|\ |_F\,(\omega_t\,\omega)^n_\alpha)\ L((\omega_t\,\omega)^n_\alpha)^{-1}\,I_s f_t) = Af$$

where $$A = mn^{-1}\,|n|_F\,A_0.$$

Proof. — From the formulae (1), (2) encountered in the proof of Proposition I.2.3 one has that

$$\lim_{t\to 0} L(|\ |_F\,(\omega_t)^n_\alpha)\ L((\omega_t)^n_\alpha)^{-1}\,I_s f_t(g)$$
$$= \sum_{\xi\in R_0} |\xi|_F f(h_\alpha(\xi)\,\eta_0 g)\lim_{t\to 0} L(|\ |_F\,(\omega_t)^n_\alpha)^{-1}\int_{\{u\in F^{\times n}:|u|_F>c\}}(\omega_t\,\omega\mu)(h_\alpha(\xi))\,d\xi$$

since $L(|\ |_F) = (1-|\pi|_F)^{-1}$.

The limit appearing here has in essence already been evaluated and the right-hand side of the above equation becomes

$$mn^{-1}\,|n|_F\sum_{\xi\in R_0}|\xi|_F f(h_\alpha(\xi)\,\eta_0\,g)$$

since $[F^\times : F^{\times n}] = n^2\,|n|_F^{-1}$ (Lemma 0.3.2). The result of applying α_s to this is

$$mn^{-1}\,|n|_F\sum_{\xi\in R_0}\int_{\widetilde{H}_*\cap\widetilde{H}_*^s\backslash\widetilde{H}_*}|\xi/x|_F\,(\omega'\,\mu)\,(y)^{-1} f(h_\alpha(\xi)\,\eta_0(h_\alpha(x)\,\eta_0)^{-1}yg)\,dy.$$

After carrying out the integration over y the only non-zero summand is that arising from $\xi = x$. This yields the assertion.

Now we shall discuss the question of the " regularization " of the I_w, that is, the definition of the intertwining operators $V(\omega') \to V({}^w\omega')$ for arbitrary ω. This is done by noting that for t such that $\omega_t\omega$ is dominant $I_w : V(\omega_t\,\omega') \to V({}^w(\omega_t\,\omega'))$ is defined and

$$\prod_{\substack{\alpha>0\\ w\alpha<0}} L((\omega_t\,\omega)^n_\alpha)^{-1}\,I_w(f_t)$$

is a polynomial in $\{X_\alpha(t), X_\alpha(t)^{-1} : \alpha > 0,\ w\alpha < 0\}$. This means that we can define a homomorphism of $\widetilde{G}$-modules

$$\text{“}\prod_{\substack{\alpha>0\\ w\alpha<0}} L(\omega^n_\alpha)^{-1}\,I_w\text{”} : V(\omega') \to V({}^w\omega')$$

by $$\text{“}\prod_{\substack{\alpha>0\\ w\alpha<0}} L(\omega^n_\alpha)^{-1}\,I_w\text{”}f = [\prod_{\substack{\alpha>0\\ w\alpha<0}} L((\omega_t\,\omega)^n_\alpha)^{-1}\,I_w f_t)]_{t=0}.$$

This we shall use as a definition henceforth. Moreover, if $L(\omega^n_\alpha)$ is finite for all $\alpha > 0$ we can speak of I_w as

$$\prod_{\substack{\alpha>0\\ w\alpha<0}} L(\omega^n_\alpha)\ \text{“}\prod_{\substack{\alpha>0\\ w\alpha<0}} L(\omega^n_\alpha)^{-1}\,I_w\text{”}.$$

There is another approach, due to I. N. Bernstein, which can be used to define the I_w in general, and which, since we find it very illuminating, we discuss here. The underlying idea is to remark that if we regard the $X_\alpha(t)$ as indeterminates which we shall now write as $X_{\alpha'}$, the space $V(\omega_t\,\omega')$, which can be regarded as a vector space over $\mathbf{C}(X_{(12)}, X_{(23)}, \ldots, X_{(r-1,r)})$ and which we write as $\mathbf{C}(X_\alpha)$, becomes a representation space of $\widetilde{G}$ over this field.

Let us ask whether there exists a homomorphism of $\widetilde{G}$-modules

$$V((\omega_t\,\omega')) \to V({}^w(\omega_t\,\omega'))$$

where ω_t is regarded as a function of the X_α. Clearly, for such a homomorphism to exist is equivalent to the solubility of a countable set of equations. These equations are polynomial in the X_α ($\alpha > 0$, simple), since only such functions arise. Thus either this set of equations is identically satisfied, or there exists a countable collection of algebraic sets in $(\mathbf{C}^\times)^{r-1}$ such that (X_α) must lie in one of these so that the equations may have a solution. But we have already seen in Proposition I.2.3 that for X_α lying in the open set such that $\omega_t\,\omega$ is dominant there is a solution. Hence this second possibility is untenable, and so the equations have a solution for generic X_α.

This means that there is a map $V((\omega_t\,\omega')) \to V({}^w(\omega_t\,\omega'))$ in this generic sense. However, by Proposition I.2.2, this map is unique up to a scalar multiple. Moreover, it coincides up to a scalar function of the X_α (a priori not necessarily rational) with our I_w. But we have shown in Proposition I.2.3 b) that our I_w do in fact induce a homomorphism $V(\omega_t\,\omega') \to V({}^w(\omega_t\,\omega'))$ of the above type.

Now, for any quasicharacter χ of $F^\times$ let $P_F^{f(\chi)}$ be its conductor. We set

$$I'_w = \prod_{\substack{\alpha>0\\ w\alpha<0}} (|n|_F^{-\frac{1}{2}}\, q^{\frac{1}{2}f(\omega_\alpha^n)}\, m^{-1}\, L(|\ |_F\,\omega_\alpha^n)\, L(\omega_\alpha^n)^{-1}) . I_w$$

which is defined, as a $\widetilde{G}$-homomorphism $V(\omega') \to V({}^w\omega')$, for all ω'.

Theorem **I.2.6.** — $I'_{w_1 w_2} = I'_{w_1}\, I'_{w_2}$.

Proof. — The proof, following [12], Ch. 3, § 7.6, depends on a preliminary reduction to the case $w_1 = w_2 = s$, a simple reflection. To see this, notice that W is generated by the simple reflections subject to the relations

a) $\quad s^2 = 1$

b) $\quad s s^1 s = s^1 s s^1$

where, in b), s and s^1 are " neighbouring " reflections. To verify the relations of the type b) is elementary as

$$l(ss^1 s) = l(ss^1) + 1 = l(s) + 2 = 3$$
$$l(s^1 ss^1) = l(s^1 s) + 1 = l(s^1) + 2 = 3$$

and so b) follows by Proposition I.2.3 (*a*), since

$$I_{ss^1s} = I_s I_{s^1} I_s, \qquad I_{s^1ss^1} = I_s I_{s^1} I_s$$

where these are defined, and this is equivalent to the statement.

Thus we have to prove that $I_s'^2$ is the identity. Let us show that this can be reduced to the corresponding question in $\widetilde{GL}_2$. Let $V_1(\omega)$ be the space of locally constant functions on $\widetilde{G}$ which satisfy

$$f(bg) = (\omega\mu)(b) f(g) \qquad (b \in \widetilde{B}_n).$$

Notice that I_w can be defined on $V_1(\omega)$ just as before. As $\widetilde{H}$ acts on the left on $V_1(\omega)$, we obtain the different $V(\omega')$ by decomposing $V_1(\omega)$ with respect to characters of $\widetilde{H}_*$. Thus all statements about the I_w on $V(\omega')$ are valid for $V_1(\omega)$ and conversely. We shall study the I_s' defined on $V_1(\omega)$. Let us consider a copy of $\widetilde{GL}_2$ in $\widetilde{G}$, containing the roots $\pm\alpha$, where α is associated with s. We can restrict to this (which would not have been valid if we had used $V(\omega')$), and thus it suffices to prove the equality in the case $r = 2$. Note also that, by Proposition I.2.2, $I_s' I_s'$ is a multiple of the identity. Our problem is to determine which multiple. This we do by writing $I_s' I_s'$ as an integral transform and then using techniques from the theory of such transforms.

Let then $r = 2$ and let $f \in V_1(\omega)$. To this we ascribe a family $[f](h, \cdot)$ $(h \in \widetilde{H})$ of functions on F which are defined by

$$[f](h, \xi) = f\left(hs\left(\mathbf{s}\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}\right)\right).$$

Thus $[f](h)$ is locally constant and as $y \to \infty$

$$[f](h, \xi y^n)\, \omega_\alpha^n(y)\, |y|_F^n$$

is ultimately constant (as f is constant near h', for $h' \in \widetilde{H}$). Note also that

$$[f](\eta h, \xi) = \omega\mu(\eta)\, [f](h, \xi) \qquad (\eta \in \widetilde{H}_n).$$

For this reason we can regard $[f](h, \cdot)$, as h runs through a set of representatives of $\widetilde{H}_n \backslash \widetilde{H}$, as a *finite* family of functions satisfying the continuity and asymptotic properties above. Conversely, given such a family we can reconstruct a corresponding f.

After a brief computation one verifies that

$$[I_s f](h, \xi) = \sum_{x \in F^{\times n} \backslash F^\times} n^{-1} |n|_F\, \mu(h)^2 |x|_F^{-1} \int_{F^\times} [f](h^s \Delta(x)\, \eta_0, \xi + xy^n)\, \omega_\alpha^n(y)\, d^\times y$$

where $\Delta(x) = \mathbf{s}(\mathrm{diag}(x, x^{-1}))$ and $\eta_0 = \mathbf{s}(\mathrm{diag}(-1, 1))$. Iterating this yields

$$[I_s I_s f](h, \xi) = \sum_{x_1, x_2} n^{-2} |n|_F^2\, |x_2/x_1|_F\, \varepsilon(x_2, x_1)$$

$$\int_{F^\times} \left\{ \int_{F^\times} [f](h\Delta(-x_1/x_2), \xi + x_1 y_1^n + x_2 y_2^n)\, \omega_\alpha^n(y_1)\, d^\times y_1 \right\} \omega_\alpha^n(y_2)^{-1} d^\times y_2$$

where the integrals are to be understood as their regularizations, and x_1, x_2 run through a set of representatives of $F^{\times n} \backslash F^{\times}$.

Since this is a multiple of $[f](h, \xi)$ it follows that we need only consider those terms with $-x_2/x_1 \in F^{\times n}$. Thus we can assume that $x_2 = -x_1$; the integral is then, on replacing y_2 by $y_1 y$ and writing $F^{\times} = \bigcup x_1 F^{\times n}$,

$$n^{-1} |n|_F \int_{F^\times} \left\{ \int_{F^\times} [f](h, \xi + (y^n - 1)x)\, \omega_\alpha^n(y)\, d^\times y \right\} d^\times x.$$

This can be reformulated as saying that, if φ is a Schwarz function on F, then

$$n^{-1} |n|_F \int_{F^\times} \left\{ \int_{F^\times} \varphi(x(y^n - 1))\, \omega_\alpha^n(y)\, d^\times y \right\} d^\times x$$

is a multiple of $\varphi(0)$. Our problem is to determine which multiple.

By definition of the regularization, the integral above is equal to

$$n^{-1} |n|_F \lim_{\lambda \to 0} \int_{F^\times} \left\{ \int_{F^\times} \varphi(x(y^n - 1))\, \omega_\alpha^n(y)\, d^\times y \right\} |x|_F^\lambda\, d^\times x.$$

The double integral is convergent for $\mathrm{Re}(\lambda)$ large enough, and so we can write it as

$$n^{-1} |n|_F \lim_{\lambda \to 0} \left(\int_{F^\times} \omega_\alpha^n(y)\, |y^n - 1|_F^{-\lambda}\, d^\times y \right) . \int_{F^\times} \varphi(x)\, |x|_F^\lambda\, d^\times x.$$

However

$$\lim_{\lambda \to 0} \lambda \int_{F^\times} \varphi(x)\, |x|_F^\lambda\, d^\times x = m(1 - q^{-1})(\log q)^{-1} . \varphi(0)$$

and thus the multiple which we are trying to find is

$$m n^{-1} |n|_F (1 - q^{-1})(\log q)^{-1} \lim_{\lambda \to 0} \lambda^{-1} \int_{F^\times} \omega_\alpha^n(y)\, |y^n - 1|_F^{-\lambda}\, d^\times y.$$

We shall next express the final integral in more familiar terms. By means of the exact sequence

$$(1) \to \mu_n(F) \to F^\times \to F^{\times n} \to (1)$$

we can regard ω_α^n, which is a quasicharacter of $F^\times$ trivial on $\mu_n(F)$, as a quasicharacter on $F^{\times n}$. Let ω be an extension of ω_α^n to $F^\times$. Let X_n be the group of characters of $F^\times$ of order n, or, equivalently, of characters trivial on $F^{\times n}$. Then as $[F^\times : F^{\times n}] = n^2 |n|_F^{-1}$ (cf. Lemma 0.3.2) we have

$$n^{-1} |n|_F \int_{F^\times} \omega_\alpha^n(y)\, |y^n - 1|_F^{-\lambda}\, d^\times y$$
$$= \sum_{\chi \in X_n} n^{-2} |n|_F \int_{F^\times} (\omega\chi)(y)\, |1 - y|_F^{-\lambda}\, d^\times y.$$

The integrals appearing on the right-hand side of this formula are beta-functions, in the sense of [12] p. 145. By the analogue of Euler's formula *(loc. cit.)* this is

$$n^{-2} |n|_F \sum_{\chi \in X_n} \Gamma(\omega\chi)\, \Gamma(|\ |_F^{1-\lambda}) / \Gamma(\omega\chi\, |\ |_F^{1-\lambda}).$$

Since $\Gamma(|\ |_F^t)$ has a zero at $t = 1$ and, as $t \to 1$,

$$\Gamma(|\ |_F^t) \sim -(\log q)\, m.(1 - q^{-1})^{-1}(t - 1)$$

it follows that the multiple which we are seeking is

$$n^{-2}\, |n|_F . m^2 . \sum_{\chi \in X_n} \Gamma(\omega\chi)/\Gamma(\omega\chi\, |\ |_F).$$

It therefore remains to evaluate this sum.

By [12] p. 150 (18) and (18′)

$$\Gamma(\omega)/\Gamma(\omega|\ |_F) = \frac{(1 - q^{s-1})(1 - q^{-s-1})}{(1 - q^{-s})(1 - q^s)} \quad (\omega \text{ unramified, } \omega(\pi) = q^{-s})$$

$$= q^{-f(\omega)} \quad (\omega \text{ ramified, conductor } P_F^{f(\omega)}).$$

Thus we can compute the multiple above explicitly.

First of all suppose that ω^n is unramified, $\omega_\alpha^n(x) = |x|_F^{nt}$, say. Then we can take $\omega(x) = |x|_F^t$. There exist n characters in X_n which are unramified. Let X_n' be the group of characters on $R_F^\times$, of order n; this group is of order $n\,|n|_F^{-1}$. The multiple is

$$n^{-2}\, |n|_F\, m^2 \sum_{\zeta^n = 1} \frac{(1 - \zeta q^{t-1})(1 - \zeta^{-1} q^{-t-1})}{(1 - \zeta^{-1} q^{-t})(1 - \zeta q^t)} + n^{-1}\, |n|_F\, m^2 \sum_{\chi \in X_n' - \{1\}} q^{-f(\chi)}$$

$$= |n|_F\, m^2 \left\{ -\frac{q^{-nt}}{(1 - q^{-nt})^2}(1 - q^{-1})^2 + n^{-1} q^{-1} + n^{-1} \sum_{\chi \in X_n' - \{1\}} q^{-f(\chi)} \right\}.$$

If, however, ω_α^n is ramified then the multiple is

$$n^{-2}\, |n|_F\, m^2 \sum_{\chi \in X_n} q^{-f(\omega\chi)}.$$

Now let us evaluate these when $|n|_F = 1$. When ω_α^n is unramified then $q^{-f(\chi)} = q^{-1}$ $(\chi \in X_n' - \{1\})$ and so we obtain

$$|n|_F\, m^2 (1 - q^{-nt-1})(1 - q^{nt-1})(1 - q^{-nt})^{-1}(1 - q^{nt})^{-1},$$

as required. If ω_α^n is ramified then the multiplier is, as required, $m^2 . q^{-f(\omega_\alpha^n)}$.

To deal with the cases when $|n|_F < 1$ it seems to be most convenient to resort to global methods. The point is that a global analogue of the functional equations in question can be proved by means of the theory of Eisenstein series (Theorem II.1.4 below). This is independent of any local theorems and therefore can be exploited to complete the proof here. We shall merely sketch the latter; it runs in tandem with the proof of Theorem I.6.3.

Let k be a global field with $\mathrm{Card}(\mu_n(k)) = n$, and which has a place v such that $k_v \cong F$. Then we can make use of the considerations of § II.1. There we have shown that if I_w' is $\bigotimes_u I_{w,u}'$, the tensor product being taken over all places of k, and with respect to $\mu_n(k)$, then

$$(I_s')^2 = \mathrm{Id}.$$

Let now the global ω be such that ω_α^n is ramified at all places u of k with $|n|_u < 1$ except $u = v$. *A priori* we know that at these places one has $(I_{s,u})^2 = (\text{const})\ \text{Id}$. By Theorem I.6.3 we can assume that the archimedean analogue is true. It thus follows from the global functional equation that when $(\omega_\alpha^n)_v$ is unramified, $(I_{s,v})^2$ is of the form

$$(\text{const})\,(1 - q^{-nt-1})\,(1 - q^{nt-1})\,(1 - q^{-nt})^{-1}(1 - q^{nt})^{-1}\ \text{Id}.$$

However from the expression which we have already derived it is clear that the constant here has to be $|n|_F\, m^2$. This proves the theorem in this case.

Finally we observe that $\{(\omega_\alpha^n)_v : \omega \mid H_k^* = 1,\ \omega_\alpha^n$ unramified at u with $|n|_u < 1,\ u \neq v\}$ is dense in the set of all *local* ω_α^n at k_v. Therefore, to prove the local assertion, it suffices to prove it for these $(\omega_\alpha^n)_v$. But since it is true at all other places of k, and is true globally, it follows that it is true at v. This proves the theorem.

Remark. — These methods have shown that

$$\sum_{\chi \in X_n'} q^{-f(\omega\chi)} = 1 + (n-1)\, q^{-1} \qquad (\omega^n \text{ unramified})$$

$$= n q^{-f(\omega^n)} \qquad (\omega^n \text{ ramified}).$$

It seems possible to verify these assertions by purely local considerations; at least Bob Coleman has shown us how to do this in certain cases.

After these rather lengthy preliminaries we can return to our principal objective, which is the understanding of the $V(\omega')$. We base our analysis on the following lemma.

Lemma **I.2.7.** — *Let* (π, W) *be an admissible representation of* $\widetilde{G}$ *and let* φ_0 *be as in Proposition* I.2.1. *Then* $\text{ind}_{H_*}^{\widetilde{H}}(\omega'\,\mu)$ *is a sub-*$\widetilde{H}$*-representation of* $\varphi_0(W)$ *if and only if there exists a non-zero morphism of* $\widetilde{G}$*-representations* $\alpha : W \to V(\omega')$.

Proof. — This follows from [3] 1.9.

Corollary **I.2.8.** — *Suppose that* $\omega_\alpha^n \neq |\ |_F^{\pm 1}$ $(\alpha \in \Phi^+)$. *Then* $V(\omega')$ *is irreducible.*

Proof. — By Theorem I.2.6, under the assumptions of the corollary,

$$\text{“}\prod_{\substack{\alpha > 0 \\ w\alpha < 0}} L(\omega^n)^{-1} I_w\text{''} : V(\omega') \to V({}^w\omega')$$

is an isomorphism. Thus if $W \subset V(\omega')$ is a non-zero subrepresentation W can also be realized as a subrepresentation of $V({}^w\omega')$. Hence, by the lemma, $\varphi_0(W)$ contains $\text{ind}_{\widetilde{H}_*^{w^{-1}}}^{\widetilde{H}}({}^w\omega'\,\mu)$, for each $w \in W$. So, by Proposition I.2.1, $\varphi_0(W) = \varphi_0(V(\omega'))$. As $W \subset V(\omega')$ and as φ_0 is exact, $\varphi_0(V(\omega')/W) = \{0\}$; hence, by [3] 2.4 $V(\omega')/W = \{0\}$. Thus $W = V(\omega')$, and so $V(\omega')$ is irreducible.

We now call ω *exceptional* if $\omega_\alpha^n = |\ |_F$ for all positive simple roots α. Such an ω is clearly dominant. Then we can construct the representations which we shall study.

Theorem **I.2.9.** — *Let* ω *be exceptional. Let*

$$V_0(\omega') = \mathrm{Im}(I_{w_0} : V(\omega') \to V({}^{w_0}\omega')),$$

where w_0 *is the longest element of* W. *Then*

a) $V_0(\omega')$ *is the unique irreducible subrepresentation of* $V({}^{w_0}\omega')$,
b) $V_0(\omega')$ *is the unique irreducible quotient representation of* $V(\omega')$,
c) $V_0(\omega') = \bigcap_s \mathrm{Ker}(I_s : V({}^{w_0}\omega') \to V({}^{sw_0}\omega'))$, *where* s *runs through the simple reflections*,
d) $\mathrm{Ker}(I_{w_0} : V(\omega') \to V({}^{w_0}\omega'))$ *is generated by the set of* $\mathrm{Im}(I_s : V({}^{s}\omega') \to V(\omega'))$ *as* s *runs through the set of simple reflections in* W,
e) $\varphi_0(V_0(\omega')) \cong \mathrm{ind}^{\widetilde{H}}_{w_0\widetilde{H}_* w_0^{-1}}({}^{w_0}\omega'\ \mu)$,
f) *if* $|n|_F = 1$ *and* ω *is normalized then* $V_0(\omega')$ *contains a* K^*-*invariant vector.*

A representation $V_0(\omega')$ will be called *exceptional.*

Proof. — We begin with a general remark concerning the I_w in those cases when $\omega_\alpha^n \neq 1$ for any α. Then $I_w : V(\omega') \to V({}^{w}\omega')$ is defined as the regularized value of

$$I_w f(g) = \int_{N_*^{+}(w)} f(w^{-1} ng)\, dn.$$

Let L be a sufficiently small compact open subgroup of $\widetilde{G}$, and let f be the right L-invariant function in $V(\omega')$ with support $\widetilde{B}_*$ L and such that $f(I) = 1$. Let $g = w$; then the integral is convergent and so can be understood in the usual sense. By choosing L small enough one has $I_w f(w) \neq 0$, and hence $I_w \neq 0$. Notice that ω' is exceptional if and only if $L(|\ |_F\, \omega^n_{-\alpha})^{-1} = 0$ for every positive simple root α.

With ω' fixed as in the theorem let

$$M(w) = \mathrm{ind}^{\widetilde{H}}_{w\widetilde{H}_* w^{-1}}({}^{w}\omega'\ \mu).$$

Then we shall first prove e), *i.e.* that $\varphi_0(V_0(\omega')) \cong M(w_0)$. Clearly by Lemma I.2.7 $M(w_0)$ is a component of $\varphi_0(V_0(\omega'))$. Conversely, if $M(w)$ is another component then, by Lemma I.2.7, there would be a morphism $\alpha : V_0(\omega') \to V({}^{w}\omega')$. Then the composite map $\alpha \circ I_{w_0} : V(\omega') \to V({}^{w}\omega')$ is, by Proposition I.2.2, a multiple of I_w. It cannot be O by construction. Thus for some $c \neq O$ one has

$$\alpha \circ I_{w_0} = c I_w.$$

Suppose now s is a simple reflection such that $l(ws) > l(w)$. Then by Theorem I.2.6 $I_{w_0} I_s = O$ but $I_w I_s \neq O$. This is a contradiction and hence we have proved e).

Since $M(w_0)$ is an irreducible $\widetilde{H}$-module it now follows from [3] 2.4 that $V_0(\omega')$ is irreducible. Part a) of the theorem follows at once from this and Lemma I.2.7.

Next we shall prove b). Let V be another irreducible quotient of $V(\omega')$ and let $M(w)$ be a component of $\varphi_0(V)$. Then V can be realized as a subspace of $V({}^{w}\omega')$, let $\alpha : V(\omega') \to V({}^{w}\omega')$ be the map whose image is V. By Proposition I.2.2 this is a multiple of I_w, and as $I_{w_0} = I_{w_0 w^{-1}} I_w$ it follows that $V_0(\omega')$ is a quotient of V. As V is irreducible $V \cong V_0(\omega')$ (and $w = w_0$). This proves b).

We next prove c). Let V_0 be the space defined by the right-hand side of the statement. Then as $I_s I_{w_0} = O$, $V_0 \supset V_0(\omega')$. Next note that the argument above shows that $M(ww')$ is a component of $\varphi_0(\mathrm{Im}(I_w : V({}^{w'}\omega') \to V({}^{ww'}\omega')))$, and hence $M(ww')$ is not a component of $\varphi_0(\mathrm{Ker}(I_w : V({}^{w'}\omega') \to V({}^{ww'}\omega')))$. Taking $w' = w_0$, we see that $M(ww_0)$ is not a component of $\varphi_0(\mathrm{Ker}(I_w))$. But this shows, as φ_0 is exact, that

$$\varphi_0(\bigcap_{w \neq I} \mathrm{Ker}(I_w\, V({}^{w_0}\omega')))$$

is either O or $M(w_0)$. Since by Theorem I.2.6

$$V_0(\omega') \subset \mathrm{Ker}(I_w\, V({}^{w_0}\omega'))$$

we see that

$$V_0(\omega') = \bigcap_{w \neq I} \mathrm{Ker}(I_w\, V({}^{w_0}\omega')).$$

But now if we choose s such that $l(w) = l(ws) + 1$ then

$$\mathrm{Ker}(I_s\, V({}^{w_0}\omega')) \subset \mathrm{Ker}(I_w\, V({}^{w_0}\omega'))$$

and c) follows at once. Part d) follows by a " dual " argument.

Finally, f) is a corollary of Proposition I.2.4.

Remarks. — The class of the representation $V_0(\omega')$ depends only on ω. Our notation will imply a realization of the representation in the form given by either a) or b) of the theorem.

We shall refer to Theorem I.2.9 e), which plays an especially important role in our investigation, as the " Periodicity Theorem ". The reason for the name will become clearer in § I.4.

I.3. Whittaker models

In this section we shall discuss the Whittaker models of $V(\omega')$ and of $V_0(\omega')$. To explain the relevant concepts, let e_0 be a non-trivial additive character on F and define the character $e : N_+ \to \mathbf{C}^\times$ by

$$e(n) = e_0(\sum_{1 \leqslant i \leqslant r} n_{i,i+1}),$$

which we also regard as a character of N_+^*. Fix a quasicharacter ω_0 of $\widetilde{Z}$, and form the character $\omega_0 \times e$ of $\widetilde{Z}N_+^*$ by

$$(\omega_0 \times e)(zn) = \omega_0(z)\, e(n) \qquad (z \in \widetilde{Z},\ n \in N_+^*).$$

We shall suppose that $\omega_0 \circ i = \varepsilon$ on $\mu_n(F)$. Then we form the representation (ρ_e, L_e) of $\widetilde{G}$ induced from $\omega_0 \times e$; that is, we regard L_e as the space of functions $f: \widetilde{G} \to \mathbf{C}$ such that

a) $f(\gamma g) = (\omega_0 \times e)(\gamma) f(g) \quad (\gamma \in \widetilde{Z}\widetilde{N}_+)$,

b) for each $f \in L_e$ there exists an open subgroup $K_f \subset \widetilde{G}$ such that

$$f(gk) = f(g) \quad (k \in K_f)$$

on which G acts by right translation. The representation (ρ_e, L_e) is algebraic.

We say that a Whittaker model of an admissible representation (π, V) is a injection $(\pi, V) \to (\rho_e, L_e)$. Let us recall how these are in general constructed. Let (π', V') be the dual representation to (π, V), *i.e.* V' is the dual space of V, the action of $v' \in V'$ on $v \in V$ being denoted by $\langle v', v \rangle$. Then π' is defined by

$$\langle \pi'(g) v', v \rangle = \langle v', \pi(g)^{-1} v \rangle.$$

Now suppose that $\lambda \in V'$ and satisfies

$$\pi'(n) \lambda = \bar{e}(n) \lambda$$

and that $\pi \mid \widetilde{Z} = \omega_0$. Then if $v \in V$, $(g \mapsto \langle \lambda, \pi(g) v \rangle) \in L_e$. If (π, V) is irreducible and $\lambda \neq 0$ this is clearly injective. Conversely if $t: (\pi, V) \to (\rho_e, L_e)$ then $\lambda(v) = t(v)(I)$ is an element of V' as above. We denote the vector space of such λ by Wh(V).

Another characterization can be given. Let (π, V) be as above and let

$$\varphi^e(\pi, V) = V/\langle \pi(n) v - e(n) v \mid v \in V, n \in N_+^* \rangle;$$

then $\varphi^e(\pi, V)$ is the e-localization of π in the terminology of [3]. Clearly $\varphi^e(\pi, V)$ is a $\widetilde{Z}N_+^*$-space, and it is dual to Wh(V).

Remark. — Let Z^0 be the centre of G and $\widetilde{Z}^0 = p^{-1}(Z^0)$. Then Wh(V) is naturally a $\widetilde{Z}^0$-module.

Let us now consider the case of $(\pi, V) = (\pi(\omega'), V(\omega'))$ so that $\omega_0 = \omega \mid \widetilde{Z}$. For $\eta \in \widetilde{H}$ we form

$$\langle \lambda_\eta, f \rangle = \int_{N_+^*} \overline{e(n)} f(\eta w_0^{-1} n)\, dn.$$

As in Lemma 1.2.3 *b)* this exists if ω is dominant (as we have shown there that the integral is absolutely convergent). Clearly λ_η satisfies the above condition, so that $\lambda_\eta \in$ Wh(V).

Clearly if $h \in \widetilde{H}_*$ and $\eta \in \widetilde{H}$

$$\lambda_{h\eta} = (\omega' \mu)(h) \lambda_\eta.$$

Lemma **I.3.1.** — *There exists* $f_\eta \in V(\omega')$ *so that*

$$\lambda_{\eta'}(f_\eta) \neq 0 \quad \textit{if } \eta'^{-1}\eta \in \widetilde{H}_*$$
$$= 0 \quad \textit{if } \eta'^{-1}\eta \notin \widetilde{H}_*.$$

Proof. — Choose a fixed set of representatives R of $\widetilde{H}_*\backslash\widetilde{H}$. Then let L be an open compact subgroup of $\widetilde{G}$, which we shall take to be sufficiently small. Let f_η ($\eta \in R$) be defined by

$$f_\eta(b\eta l w_0) = (\omega' \mu)(b) \quad (b \in \widetilde{B}_*, l \in L),$$
$$f_\eta(g) = 0 \qquad (g \notin \widetilde{B}_* \eta L w_0).$$

Then we shall check that this has the required properties. One has

$$\langle \lambda_{\eta'}, f_\eta \rangle = \int_{N_*^\ddagger} f_\eta(\eta' w_0 n) \overline{e(n)} \, dn.$$

But there is only a contribution when

$$\eta' w_0 n \in \widetilde{B}_* \eta L w_0,$$

or

$$(\eta^{-1} \eta')(w_0 n w_0^{-1}) \in \widetilde{B}_* L.$$

This however requires that $w_0 n w_0^{-1} \in L$, $\eta^{-1}\eta' \in \widetilde{H}_*$, if L is sufficiently small. Thus the integral is zero if $\eta^{-1}\eta' \notin \widetilde{H}_*$. If $\eta^{-1}\eta' \in \widetilde{H}_*$ we can assume that $\eta' = \eta$. Then

$$\langle \lambda_\eta, f_\eta \rangle = \int_{N_*^\ddagger \cap w_0^{-1} L w_0} 1 \, dn \neq 0.$$

This proves the lemma.

Lemma **I.3.2.** — *One has*

$$\dim(\mathrm{Wh}(V(\omega'))) = \mathrm{Card}(\widetilde{H}_*\backslash\widetilde{H}).$$

Here Card(S) means the cardinality of S; the notation will only be used for finite sets S.

Proof. — The equivalent statement that

$$\dim(\varphi^e(V(\omega')) = \mathrm{Card}(\widetilde{H}_*\backslash\widetilde{H})$$

is contained in [3], Theorem 5.2.

Thus if ω is dominant the λ_η ($\eta \in \widetilde{H}_*\backslash\widetilde{H}$) span the vector space $\mathrm{Wh}(V(\omega'))$. Since one can easily verify that the $\lambda_\eta(f_t)$ are rational functions of the $X(t)$, the method of Bernstein explained after Lemma I.2.4 applies and shows that the λ_η are defined generically.

The objective of this section is to describe $\varphi^e(V_0(\omega'))$ or equivalently $\mathrm{Wh}(V_0(\omega'))$ from the description of $V_0(\omega')$ given by Theorem I.2.9. We can unfortunately only obtain complete results when $|n|_F = 1$. In § I.5 we shall discuss a different approach which permits us to avoid some of the problems occuring when $|n|_F < 1$.

First of all we shall explain the basic idea of the approach of this section. If $w \in W$ the linear form on $V(\omega')$

$$f \mapsto \langle \lambda_\eta, I_w f \rangle,$$

where $\lambda_\eta \in \mathrm{Wh}(V({}^w\omega'))$, is defined in Bernstein's generic sense. Since it lies in $\mathrm{Wh}(V(\omega'))$ it can be written as

$$\sum_{\eta'} \tau(w, \omega', \eta, \eta') \langle \lambda_{\eta'}, f \rangle,$$

where $\lambda_{\eta'} \in \mathrm{Wh}(\mathrm{V}(\omega'))$. We shall write $\tau_w(\eta, \eta')$ for $\tau(w, \omega', \eta, \eta')$ if no confusion should arise. The sum in η' is taken over $\widetilde{\mathrm{H}}_*\backslash\widetilde{\mathrm{H}}$. These equations define the transpose action of the I_w on $\mathrm{Wh}(\mathrm{V}(\omega'))$.

Now let $\mathbf{c}$ be a function on $\widetilde{\mathrm{H}}$ which satisfies

$$\mathbf{c}(h\eta) = (\omega' \mu)(h)^{-1}\, \mathbf{c}(\eta) \quad (h \in \widetilde{\mathrm{H}}_*, \eta \in \widetilde{\mathrm{H}}).$$

Define $$\lambda(\mathbf{c}) = \sum_{\eta \in \widetilde{\mathrm{H}}_*\backslash\widetilde{\mathrm{H}}} \mathbf{c}(\eta)\, \lambda_\eta.$$

Suppose next that ω' is exceptional. Then $\Lambda \in \mathrm{Wh}(\mathrm{V}_0(\omega'))$ yields an element of $\mathrm{Wh}(\mathrm{V}(\omega'))$ via

$$\mathrm{V}(\omega') \xrightarrow{\mathrm{I}_{w_0}} \mathrm{V}_0(\omega'),$$

and this is then a $\lambda(\mathbf{c})$. Since $\lambda(\mathbf{c})$ is trivial on the kernel of I_{w_0} we must have

$$\langle \lambda(\mathbf{c}), \mathrm{I}_s f \rangle = 0 \quad (f \in \mathrm{V}({}^s\omega'))$$

where s runs through the simple reflections of W. Conversely if $\lambda(\mathbf{c})$ satisfies these conditions it follows from Theorem I.2.9 *d)* that $\lambda(\mathbf{c})$ is trivial on $\mathrm{Ker}(\mathrm{I}_{w_0})$ and hence defines an element of $\mathrm{Wh}(\mathrm{V}_0(\omega'))$.

Moreover $\langle \lambda(\mathbf{c}), \mathrm{I}_s f \rangle = 0$
if and only if

$$\sum \mathbf{c}(\eta)\, \tau({}^s\omega', s, \eta, \eta') = 0.$$

As s runs through the set of simple reflections in W we obtain a system of linear equations in $\mathbf{c}$. If $\mathbf{c}$ is a solution of this system we can define an element $\Lambda(\mathbf{c})$ of $\mathrm{Wh}(\mathrm{V}_0(\omega'))$ by

$$\langle \Lambda(\mathbf{c}), \mathrm{I}_{w_0} f \rangle = \langle \lambda(\mathbf{c}), f \rangle$$

and every element of $\mathrm{Wh}(\mathrm{V}_0(\omega'))$ arises uniquely in this way.

We shall now explain how the $\tau_s(\eta, \eta')$ can be computed. Let

$$\mathrm{K}_m = \{k \in \mathrm{GL}_r(\mathrm{R}_\mathrm{F}) : k \equiv \mathrm{I} \pmod{\mathrm{P}_\mathrm{F}^m}\},$$

and suppose that $m \geq 1$ is large enough so that the covering p splits over K_m. Let $k \mapsto k^*$ be such a splitting, and let K_m^* be the image of K_m in $\widetilde{\mathrm{G}}$. Let $\mathrm{K}_m^+ = \mathrm{N}_+ \cap \mathrm{K}_m$. Then, as in the proof of Lemma I.3.1 we define $f_\eta^{(m)}$ to be that element of $\mathrm{V}(\omega')$ with

a) $\mathrm{Supp}(f_\eta^{(m)}) = \widetilde{\mathrm{B}}_*\, \eta w_0\, \mathrm{K}_m^* = \widetilde{\mathrm{B}}_*\, \eta w_0 (\mathrm{K}_m^+)^*$,
b) $f_\eta^{(m)}$ is right-K_m^*-invariant, and
c) $f_\eta^{(m)}(w_0) = 1$.

If m is sufficiently large, such an $f_\eta^{(m)}$ exists. As we have seen in the proof of Lemma I.3.1:

$$\langle \lambda_\eta, f_{\eta'}^{(m)} \rangle \neq 0 \quad (\eta = \eta')$$
$$= 0 \quad (\eta \neq \eta').$$

Then one has

$$\langle \lambda_\eta, \mathrm{I}_w f_{\eta'}^{(m)} \rangle = \tau_w(\eta, \eta') \langle \lambda_{\eta'}, f_{\eta'}^{(m)} \rangle,$$

so that

$$\tau_w(\eta, \eta') = \langle \lambda_\eta, \mathrm{I}_w f_{\eta'}^{(m)} \rangle / \langle \lambda_{\eta'}, f_{\eta'}^{(m)} \rangle.$$

Moreover in the proof of Lemma I.3.1 we saw that

$$\langle \lambda_{\eta'}, f_{\eta'}^{(m)} \rangle = \mathrm{meas}((\mathrm{K}_m^+)^*)$$

if m is large enough; consequently one has

$$\tau(w, \omega', \eta, \eta') = \lim_{m \to \infty} \langle \lambda_\eta, \mathrm{I}_w f_{\eta'}^{(m)} \rangle / \mathrm{meas}((\mathrm{K}_m^+)^*).$$

Now that we have explained the approach we shall carry it out when $|n|_\mathrm{F} = 1$. We shall take $\widetilde{\mathrm{B}}_*$ to be the standard group constructed at the end of § I.1. Note that $p(\widetilde{\mathrm{H}}_*)$ is then

$$\{\mathrm{diag}(h_j) : \mathrm{ord}_\mathrm{F}(h_j) \equiv (2c+1) \sum_k \mathrm{ord}_\mathrm{F}(h_k) \pmod n\}.$$

Let π be a uniformizer of F, let $\mathbf{f} = (f_1, \ldots, f_r) \in \mathbf{Z}^r$ and let

$$\pi^\mathbf{f} = \mathbf{s}((\mathrm{diag}(\pi^{f_j}))).$$

If we let

$$\Lambda = \{f \in \mathbf{Z}^r : f_j \equiv (2c+1) \sum_k f_k \pmod n\}$$

and construct a set of representatives A for $\mathbf{Z}^r/\Lambda$ then $\{\pi^\mathbf{f} : \mathbf{f} \in \mathrm{A}\}$ is a set of representatives for $\widetilde{\mathrm{H}}_* \backslash \widetilde{\mathrm{H}}$.

Suppose now that the conductor of e_0 is R_F and let dx be the self-dual Haar measure on F. For $\mathrm{J} \in \mathbf{Z}/n\mathbf{Z}$ define the Gauss sum

$$g^{(\mathrm{J})} = q \int_{\mathrm{R}_\mathrm{F}^\times} \varepsilon(\pi, x)^\mathrm{J} \, e_0(x/\pi) \, dx.$$

Suppose next that ω is a quasicharacter of $\widetilde{\mathrm{H}}_n \widetilde{\mathrm{Z}}$ such that ω_α^n is unramified for a given simple positive root α. Since $\widetilde{\mathrm{H}}_*$ is the standard maximal isotropic subgroup we can choose ω' on $\widetilde{\mathrm{H}}_*$ extending ω such that ω' is trivial on

$$\kappa^*\{\mathrm{diag}(1, \ldots, 1, y, y^{-1}, 1, \ldots, 1) : y \in \mathrm{R}_\mathrm{F}^\times\}$$

where κ^* is the canonical lift of K and, if $\alpha = (i, i+1)$, the y (resp. y^{-1}) is in the i-th (resp. $(i+1)$-th) position. Let

$$u_j = \kappa^*(\mathrm{diag}(1, 1, \ldots, 1, -1, 1, \ldots, 1))$$

where the -1 is in the j-th position. If ω, ω' are as above and if $|n|_\mathrm{F} = 1$, let

$$\theta = \omega'(u_j),$$

it is not difficult to see that this is well-defined. Then one has:

Lemma **I.3.3.** — *Suppose that* F *is such that* $|n|_\mathrm{F} = 1$. *Let* $\alpha = (i, i+1)$ *be a simple root and* $s \in \mathrm{W}$ *the associated reflection. Suppose that* ω_α^n *is unramified and that* ω' *is choosen as above. Then*

$$\tau(s, \omega', h, h^*) = \theta(\tau_s^1(h, h^*) + \tau_s^2(h, h^*)),$$

where

a) *for* $j = 1, 2$, $\eta, \eta' \in \widetilde{H}$, $h, h' \in \widetilde{H}_*$ *one has*

$$\tau_s^j(h\eta, h'\eta') = {}^s\omega'\,\mu(h).(\omega'\,\mu)(h')^{-1}\,\tau_s^j(\eta, \eta'),$$

b) *if* $\mathbf{f}, \mathbf{f}^* \in \mathbf{Z}^r$ *then*

$$\tau_s^1(\pi^{\mathbf{f}}, \pi^{\mathbf{f}^*}) = 0 \quad \textit{unless } \mathbf{f}^* \equiv \mathbf{f} \pmod{\Lambda},$$

and, with

$$\mathbf{f}_1 = (f_1, f_2, \ldots, f_{i-1}, f_{i+1} - 1, f_i + 1, f_{i+2}, \ldots, f_r)$$

$$\tau_s^2(\pi^{\mathbf{f}}, \pi^{\mathbf{f}^*}) = 0 \quad \textit{unless } \mathbf{f}^* \equiv \mathbf{f}_1 \pmod{\Lambda},$$

c) *one has*

$$\tau_s^1(\pi^{\mathbf{f}}, \pi^{\mathbf{f}}) = (1 - \omega_\alpha^n(\pi))^{-1}(1 - q^{-1})\, q^{f_i - f_{i+1}}.\,\omega_\alpha^n(\pi^{[(f_i - f_{i+1})/n]}),$$

and

$$\tau_s^2(\pi^{\mathbf{f}}, \pi^{\mathbf{f}_1}) = \varepsilon(-1, \pi)^{f_i f_{i+1}}\, q^{-2}\, g^{(f_{i+1} - f_i - 1)}.$$

Remark. — This lemma clearly determines $\tau(s, \omega', h, h^*)$ completely.

The proof is an application of the formula derived above, and we shall postpone it to the end of this section.

Let $\boldsymbol{\rho} = (1, 2, \ldots, r) \in \mathbf{Z}^r$. We define an action of W on $\mathbf{Z}^r$ by

$$w[\mathbf{m}] = w(\mathbf{m} - \boldsymbol{\rho}) + \boldsymbol{\rho} \qquad (\mathbf{m} \in \mathbf{Z}^r, w \in W)$$

where if $\quad \mathbf{m} = (m_1, m_2, \ldots, m_r)$

we set $\quad w(\mathbf{m}) = (m_{w^{-1}(1)}, m_{w^{-1}(2)}, \ldots, m_{w^{-1}(r)})$.

Note that if $|\mathbf{m}| = \Sigma\, m_j$ then

$$|w[\mathbf{m}]| = |\mathbf{m}|.$$

The action of W on $\mathbf{Z}^r$ induces one on $\mathbf{Z}^r/n\mathbf{Z}^r$.

Corollary **I.3.4.** — *Suppose that* F *is such that* $|n|_F = 1$ *and that* ω *is exceptional. Let* $\mathbf{c}$ *be a function on* $\widetilde{H}$ *satisfying*

$$\mathbf{c}(\eta h) = (\omega'\,\mu)(\eta)^{-1}\,\mathbf{c}(h) \qquad (\eta \in \widetilde{H}_*, h \in \widetilde{H})$$

for which $\lambda(\mathbf{c})$ *induces an element of* $\mathrm{Wh}(V_0(\omega'))$. *Then*

$$\mathbf{c}(\pi^{s[\mathbf{f}]}) = \theta q^{f_i - f_{i+1} + 1 - [(f_{i+1} - f_i - 2)/n]}.\,\varepsilon(-1, \pi)^{f_i f_{i+1}}\, g^{(f_{i+1} - f_i - 1)}\, \mathbf{c}(\pi^{\mathbf{f}}),$$

if s *is the simple reflection associated with the root* $(i, i+1)$. *Conversely, given a non-zero function* $\mathbf{c}$ *satisfying these conditions* $\lambda(\mathbf{c})$ *induces a non-zero element of* $\mathrm{Wh}(V_0(\omega'))$.

Proof. — We have seen that we must have

$$\sum_{\mathbf{f}} \tau(s, {}^s\omega', \pi^{\mathbf{f}}, \pi^{\mathbf{f}^*})\, \mathbf{c}(\pi^{\mathbf{f}}) = 0,$$

where $\mathbf{f}$ is summed over a set of representatives for $\mathbf{Z}^r/\Lambda$, in order that $\lambda(\mathbf{c})$ should induce an element of $\mathrm{Wh}(V_0(\omega'))$. We show now that this is equivalent to the stated condition for each s. Suppose s is associated with the root $(i, i+1)$. By Lemma I.3.3 the condition above is

$$\tau(s, {}^s\omega', \pi^{\mathbf{f}}, \pi^{\mathbf{f}})\ \mathbf{c}(\pi^{\mathbf{f}}) + \tau(s, {}^s\omega', \pi^{\mathbf{f}_1}, \pi^{\mathbf{f}})\ \mathbf{c}(\pi^{\mathbf{f}_1}) = 0.$$

This is, as ω' is exceptional,

$$-\ \theta q^{f_i - f_{i+1} - 1 + [(f_i - f_{i+1})/n]}\ \mathbf{c}(\pi^{\mathbf{f}}) \\ + q^{-2}\,\varepsilon(-1, \pi)^{(f_i+1)(f_{i+1}-1)}\, g^{(f_i - f_{i+1}+1)}\ \mathbf{c}(\pi^{\mathbf{f}_1}) = 0.$$

In this we replace $\mathbf{f}_1$ by $\mathbf{f}$ and note that $\mathbf{f}_1 = s[\mathbf{f}]$; the quoted condition follows immediately.

This leads at once the following theorem, which is the objective on this section.

Theorem **I.3.5.** — *Suppose that* F *is such that* $|n|_F = 1$ *and that* ω *is exceptional. Let* N *be the number of images of free orbits of* W *acting on* $\mathbf{Z}^r/n\mathbf{Z}^r$ *(orbits of size* $r!$*) in* $\mathbf{Z}^r/\Lambda$. *Then*

$$\dim(\mathrm{Wh}(V_0(\omega'))) = N.$$

As an immediate corollary one has:

Corollary **I.3.6.** — *With the above notations one has that* $\dim(\mathrm{Wh}(V_0(\omega'))) = 1$ *if and only if either*

$$r = n - 1, \qquad 2(c+1) \equiv 0 \pmod{n},$$

or

$$r = n.$$

Remarks. — 1) Although we shall not dwell on it here it is interesting to observe that $\mathrm{Wh}(V_0(\omega'))$ is a $\widetilde{Z}^0$-module on which $\widetilde{Z}$ acts by $(\omega\mu)^{-1}$. For example, if $r = n$ then $\widetilde{Z}^0$ is abelian but $\widetilde{Z}^0 \neq \widetilde{Z}$. Thus $\mathrm{Wh}(V_0(\omega'))$ determines a distinguished extension of $(\omega\mu)^{-1}$ to $\widetilde{Z}^0$, which could be computed. The significance of this extension is unclear. When $r = n$ or $n - 1$ Theorem I.3.5 implies that $\mathrm{Wh}(V_0(\omega'))$ is an irreducible $\widetilde{Z}^0$-module.

2) Although Theorem I.3.5 does not hold if $|n|_F = 1$ is not assumed, a weaker version does. Moreover this assumption is not needed for Corollary I.3.6. Such results will be derived by global methods—see Corollary II.2.6.

Proofs. — The corollary is a simple combinatorial exercise given the theorem and we shall leave it as such for the reader. We shall therefore just prove the theorem.

As $\widetilde{H}_*$ is standard $\mathbf{c}$ is determined by the values $\mathbf{c}(\pi^{\mathbf{f}})$ ($\mathbf{f} \in A$). Moreover from Corollary I.3.4 one sees that if $\mathbf{c}(\pi^{w[\mathbf{f}]})$ is non-zero then it is of the form $t(w, \mathbf{f})\ \mathbf{c}(\pi^{\mathbf{f}})$ where $t(w, \mathbf{f})$ is computable. Hence $\mathbf{c}(\pi^{\mathbf{f}})$ determines the $\mathbf{c}(\pi^{w[\mathbf{f}]})$. Suppose that $\mathbf{f}$ lies in a non-free orbit under W. Then, as one can easily see, there exists a $w \in W$ and a simple reflection s so that

$$s[w[\mathbf{f}]] \equiv w[\mathbf{f}] \pmod{n}.$$

Replace $\mathbf{f}$ by $w[\mathbf{f}]$. Then if s corresponds to the simple reflection $(i, i+1)$ one has that

$$f_i \equiv f_{i+1} - 1 \pmod n$$

and, by Corollary I.3.4,

$$\mathbf{c}(\pi^{s[\mathbf{f}]}) = -q^{f_i - f_{i+1} + 1 - [(f_{i+1} - f_i - 2)/n]} \mathbf{c}(\pi^{\mathbf{f}}).$$

But also

$$\mathbf{c}(\pi^{s[\mathbf{f}]}) = (\omega' \mu)(\pi^{s[\mathbf{f}]}.(\pi^{\mathbf{f}})^{-1})^{-1}.\mathbf{c}(\pi^{\mathbf{f}})$$

and so, if $\mathbf{c}(\pi^{\mathbf{f}}) \neq 0$ we must have

$$\omega' \mu(\pi^{\mathbf{f}}.(\pi^{s[\mathbf{f}]})^{-1})^{-1} = -q^{f_i - f_{i+1} + 1 - [(f_{i+1} - f_i - 2)/n]}.$$

But as ω is exceptional the absolute value of the left hand side is $q^{F(1+1/n)}$, with $F = f_i - f_{i+1} + 1$, and that of the right-hand side is $q^{F + F/n + 1}$. This is a contradiction; thus $\mathbf{c}(\pi^{\mathbf{f}}) = 0$ if $\mathbf{f}$ lies in a non-free orbit.

Suppose now that $\mathbf{f}$ lies in a free orbit. With the notations above one can easily verify that

a) if s is a simple reflection then $t(s, \mathbf{f}).t(s, s[\mathbf{f}]) = 1$, and
b) if s, s' are "neighbouring" simple reflections then

$$t(s, s' s[\mathbf{f}]).t(s', s[\mathbf{f}]).t(s, \mathbf{f}) = t(s', ss'[\mathbf{f}])\, t(s, s'[\mathbf{f}])\, t(s', \mathbf{f}).$$

Thus given $\mathbf{c}(\pi^{\mathbf{f}})$ we can inductively define $\mathbf{c}(\pi^{w[\mathbf{f}]})$ for all $w \in W$. If for some $w, w' \in W$ we have

$$\pi^{w[\mathbf{f}]} = h.\pi^{w'[\mathbf{f}]}$$

with $h \in \widetilde{H}_*$ then, as $|w[\mathbf{f}]| = |w'[\mathbf{f}]|$, we would have $w[\mathbf{f}] \equiv w'[\mathbf{f}] \pmod n$ and, by assumption, this would imply that $w = w'$. Hence $h = 1$. Thus the prescription

$$\mathbf{c}(h\pi^{w[\mathbf{f}]}) = (\omega' \mu(h))^{-1} \mathbf{c}(\pi^{w[\mathbf{f}]}) \quad (h \in \widetilde{H}_*)$$

$$\mathbf{c}(\eta) = 0 \quad (\eta \notin \bigcup_{w \in W} \widetilde{H}_*.\pi^{w[\mathbf{f}]}),$$

defines a solution to the system of linear equations which $\mathbf{c}$ had to satisfy. If

$$\pi^{\mathbf{f}'} \in \bigcup_{w \in W} \widetilde{H}_* \pi^{w[\mathbf{f}]}$$

then, for some $w \in W$, $\mathbf{f}' - w[\mathbf{f}]$ is of the form $(a, a, \ldots, a)$ where

$$a.(r - 1 + 2rc) \equiv 0 \pmod n.$$

From this the theorem follows at once.

It now remains to prove Lemma I.3.3.

Proof of Lemma **I.3.3.** — We have

$$\tau_s(\eta, \eta') = \langle \lambda_\eta, I_s f_{\eta'}^{(m)} \rangle / \text{meas}((K_m^+)^*)$$

for sufficiently large m. This then takes the form

$$\frac{1}{\text{meas}((K_m^+)^*)} \int_{N_*^{\ddagger}} \left\{ \int_{N_*^{\ddagger}(s)} f_{\eta'}^{(m)}(sn_1\,\eta w_0^{-1}\,n_2)\,dn_1 \right\} \bar{e}(n_2)\,dn_2$$

where the integrals are to be understood as their regularized values. We shall first evaluate the inner integral. To do so we first determine when $sn_1\,\eta w_0^{-1}\,n_2$ can lie in $\widetilde{B}_*\,\eta'\,K_m^*\,w_0 = \widetilde{B}_*\,\eta'\,w_0^{-1}(K_m^+)^*$. Suppose that

$$sn_1\,\eta w_0^{-1}\,n_2 = b\eta'\,w_0^{-1}\,k \quad (b \in \widetilde{B}_*,\, k \in (K_m^+)^*),$$

or

$$p(s)\,p(n_1) = p(b)\,p(\eta')\,p(w_0^{-1}\,kn_2^{-1}\,w_0)\,p(\eta)^{-1}.$$

We shall express matrices in G in $(r_1 + 2 + r_2) \times (r_1 + 2 + r_2)$ block form so that $p(s)$ is

$$\begin{bmatrix} I & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & 0 & I \end{bmatrix}.$$

We can then write $p(n_1)$ as

$$\begin{bmatrix} I & 0 & 0 \\ 0 & \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & I \end{bmatrix}$$

with $\xi \in F$, $p(b)$ as

$$\begin{bmatrix} b_1 & x_1 & x_2 \\ 0 & b_2 & x_3 \\ 0 & 0 & b_3 \end{bmatrix}$$

with b_1, b_2, b_3 upper-triangular, $p(\eta)$ (resp. $p(\eta')$) as

$$\begin{bmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{bmatrix} \quad \left(\text{resp.} \begin{bmatrix} \eta_1' & 0 & 0 \\ 0 & \eta_2' & 0 \\ 0 & 0 & \eta_3' \end{bmatrix} \right)$$

with η_j (resp. η_j') diagonal, and $p(w_0^{-1}\,kn_2^{-1}\,w_0)$ as

$$\begin{bmatrix} \bar{n}_1 & 0 & 0 \\ u_1 & \bar{n}_2 & 0 \\ u_2 & u_3 & \bar{n}_3 \end{bmatrix}$$

where $\bar{n}_1, \bar{n}_2, \bar{n}_3$ are lower-triangular matrices. From

$$p(s)\,p(n_1) = p(b)\,p(\eta')\,p(w_0^{-1}kn_2^{-1}w_0)\,p(\eta)^{-1}$$

one obtains after a short computation that

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0,$$
$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0,$$
$$\bar{n}_1 = I, \quad \bar{n}_3 = I$$
$$b_1 = \eta_1\eta_1'^{-1}, \quad b_3 = \eta_3\eta_3'^{-1}$$

and
$$\begin{pmatrix} 0 & 1 \\ 1 & \xi \end{pmatrix} = b_2\,\eta_2'\,\bar{n}_2\,\eta_2^{-1}.$$

One also demands that

$$\begin{bmatrix} \eta_1\eta_1^{-1} & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & \eta_3\eta_3^{-1} \end{bmatrix}$$

should belong to $\widetilde{B}_*$. From the last of the equations above

$$b_2 = \begin{pmatrix} -\xi^{-1} & 1 \\ 0 & \xi \end{pmatrix} \eta_2'\eta_2^{-1}$$

and
$$\bar{n}_2 = \begin{pmatrix} 1 & 0 \\ \nu\xi^{-1} & 1 \end{pmatrix}$$

where
$$\nu = p(\eta)^{\alpha}.$$

Let now
$$b(\xi) = \begin{bmatrix} I & 0 & 0 \\ 0 & \begin{pmatrix} -\xi^{-1} & 1 \\ 0 & \xi \end{pmatrix} & 0 \\ 0 & 0 & I \end{bmatrix} p(\eta)\,p(\eta')^{-1}$$

and
$$\widetilde{h}_\alpha(\xi) = \mathbf{s}\left(\begin{bmatrix} I & 0 & 0 \\ 0 & \begin{pmatrix} -\xi^{-1} & 0 \\ 0 & \xi \end{pmatrix} & 0 \\ 0 & 0 & I \end{bmatrix}\right).$$

The integral is then

$$\sum_{z \in F/P_F^m} \left\{\int_{\Xi'(z,\eta,\eta')} (\omega'\,\mu)(\widetilde{h}_\alpha(\xi)\,\eta\eta'^{-1})\,d\xi\right\} \bar{e}_0(z)$$

where
$$\Xi'(z,\eta,\eta') = \{\xi \in F : \nu\xi^{-1} + z \in P_F^m,\ \widetilde{h}_\alpha(\xi)\,\eta\eta'^{-1} \in \widetilde{H}_*\}.$$

It must still be regarded as a regularized value, first of the integral inside the braces and then of the sum.

If z does not represent the zero class of F/P_F^m then the inner integral is simply

$$(\omega'\mu)(\tilde{h}_\alpha(-\nu z^{-1})\eta\eta'^{-1})\bar{e}_0(z)q^{-m}|z|^{-2},$$

if $\tilde{h}_\alpha(-\nu z^{-1})\eta\eta'^{-1}\in\tilde{H}_*$ and is zero otherwise.

The integral over $\Xi'(0,\eta,\eta')$ is

$$A(m)=\int\omega'\mu(\tilde{h}_\alpha(\xi)\eta\eta'^{-1})\,d\xi$$

integrated over those ξ in νP_F^{-m} such that $\tilde{h}_\alpha(\xi)\eta\eta'^{-1}\in\tilde{H}_*$. This integral " is " a geometric series if we regard it as a sum over subintegrals of the form $|\xi|=R$. The regularization process simply means that we take the formal sum. It proves however more convenient to compute the integral by an indirect method.

Consider now the sum over z; this sum we also arrange as the sum of the integral over $\Xi'(0,\eta,\eta')$ and the sum of the subsums over $|z|=q^j$ $(j>-m)$. In these subsums we replace z by $z\theta$ with $\theta\in R_F$, $\theta\equiv 1 \pmod{P_F}$, and sum over all such θ modulo P_F^m. Since the conductor of e_0 is R_F this sum vanishes unless $|z|\le q$.

Since the sum in question converges in an open set of ω this argument is valid there and the consequence remains valid after regularization. The terms remaining are those with $|z|=q$, which yield

$$\sum_{\substack{z\in P_F^{-1}/R_F\\ z\notin R_F/R_F}}(\omega'\mu)(\tilde{h}_\alpha(-\nu z^{-1})\eta\eta'^{-1})\bar{e}_0(z)q^{-2}$$

and those with $|z|\le 1$, namely

$$B(m)=\sum_{\substack{z\in R_F/P_F^m\\ z\notin P_F^m/P_F^m}}(\omega'\mu)(\tilde{h}_\alpha(-\nu z^{-1})\eta\eta'^{-1})q^{-m}.|z|^{-2}.$$

In these last two sums there is an additional restriction on the z, viz. that the argument of $(\omega'\mu)$ should lie in $\tilde{H}_*$.

Hence $\tau(s,\omega',\eta,\eta')$ is the sum of

$$\sum_{\substack{z\in P_F^{-1}/R_F\\ z\notin R_F/R_F}}(\omega'\mu)(\tilde{h}_\alpha(-\nu z^{-1})\eta\eta'^{-1})\bar{e}_0(z)q^{-2}$$

and $A(m)+B(m)$ for m sufficiently large. Thus $A(m)+B(m)$ is constant, as a function of m, for m sufficiently large.

If in the integral defining $A(m+n)$ we replace ξ by $\pi^{-n}\xi_1$, we see that for all m

$$A(m+n)=\omega_\alpha^n(\pi)A(m).$$

Moreover, if $m\ge 0$, $m\equiv 0 \pmod n$ one has

$$B(m+n)=B(m)+\sum_{\substack{z\in P_F^m/P_F^{m+n}\\ z\notin P_F^{m+n}/P_F^{m+n}}}(\omega'\mu)(\tilde{h}_\alpha(-\nu z^{-1})\eta\eta'^{-1})q^{-m-n}|z|^{-2}.$$

In the second term here we replace z by $\pi^m z_1$ and obtain

$$B(m+n) = B(m) + \omega_\alpha^n(\pi^{m/n}).B(n).$$

From these if follows firstly that for large m, $m \equiv 0 \pmod n$, one has

$$\omega_\alpha^n(\pi)\,A(m) + B(m) + \omega_\alpha^n(\pi^{m/n})\,B(n) = A(m) + B(m),$$

and secondly that this equation then holds for all $m \geq 0$, $m \equiv 0 \pmod n$. Thus one has, for $m \equiv 0 \pmod n$,

$$A(m) = \omega_\alpha^n(\pi^{m/n})\,(1 - \omega_\alpha^n(\pi))^{-1}.B(n)$$

and so

$$A(m) + B(m) = A(0) + B(0)$$
$$= B(n)\,(1 - \omega_\alpha^n(\pi))^{-1}.$$

Therefore $\tau(s, \omega', \eta, \eta')$ is the sum of

$$\sum_{\substack{z \in P_F^{-1}/R_F \\ z \notin R_F/R_F}} (\omega'\,\mu)\,(\widetilde{h}_\alpha(-\nu z^{-1})\,\eta\eta'^{-1})\,\bar{e}_0(z).q^{-2}$$

and

$$(1 - \omega_\alpha^n(\pi))^{-1} \sum_{\substack{z \in R_F/P_F^n \\ z \notin P_F^n/P_F^n}} (\omega'\,\mu)\,(\widetilde{h}_\alpha(-\nu z^{-1})\,\eta\eta'^{-1})\,q^{-n}\,|z|^{-2}.$$

We are now in a position to deliver the coup de grâce. In the first term set $z = y/\pi$. Take η to be of the form π^f. Under the assumption made on ω' the sum becomes

$$\sum_{\substack{y \in R_F/P_F \\ y \notin P_F/P_F}} (\omega'\,\mu)\,(\widetilde{h}_\alpha(-\nu\pi)\,\eta\eta'^{-1})\,\varepsilon(\nu\pi, y)^{-1}\,\bar{e}_0(y/\pi).q^{-2}.$$

Clearly the class of η' in $\widetilde{H}_*\backslash\widetilde{H}$ is determined and we see that it is the class of π^{f_1}. If now

$$J = f_{i+1} - f_i - 1$$

then the sum is

$$q^{-2}(\omega'\,\mu)\,(\widetilde{h}_\alpha(-\nu\pi)\,\eta\eta'^{-1})\,g^{(J)},$$

which after a short computation can be seen to be

$$\theta q^{-2}\,g^{(J)}\,\varepsilon(-1, \pi)^{f_i f_{i+1}}.$$

Now consider the second term. It is invariant under replacing z by $z\theta$ $(\theta \in R_F^\times)$ and from this one sees that the only terms which contribute are those with $\mathrm{ord}(\nu z^{-1}) \equiv 0 \pmod n$. But the class of η' in $\widetilde{H}_*\backslash\widetilde{H}$ has to be that of η if the sum is not to vanish. Thus we can restrict our attention to the case $\eta' = \eta$, and here we obtain

$$(1 - \omega_\alpha^n(\pi))^{-1}(1 - q^{-1})\,(\omega'\,\mu)\,(\widetilde{h}_\alpha(-\nu z^{-1}))\,|z|^{-1}$$

where z is chosen with $0 \leq \mathrm{ord}(z) < n$. Since $\nu = \pi^{f_i - f_{i+1}}$ we choose

$$z = \pi^{f_i - f_{i+1} - n[(f_i - f_{i+1})/n]}$$

which yields

$$\theta(1-\omega_\alpha^n(\pi))^{-1}(1-q^{-1})\, q^{f_i-f_{i+1}}\, \omega_\alpha^n(\pi^{[(f_i-f_{i+1})/n]});$$

with this the proof of the lemma is complete.

I.4. Hecke Theory

In the last section we constructed the space $\mathrm{Wh}(\mathrm{V}_0(\omega'))$; in this section we shall study the function

$$\widetilde{\mathrm{H}} \to \mathbf{C} : h \mapsto \langle \lambda, \pi_0(h)\, v \rangle$$

for $\lambda \in \mathrm{Wh}(\mathrm{V}_0(\omega'))$ and $v \in \mathrm{V}_0(\omega')$.

To describe the results we regard $\mathrm{V}_0(\omega')$ as a subrepresentation of $\mathrm{V}({}^{w_0}\omega')$. As such it is the image of I_{w_0}. Let $\mathbf{c} : \widetilde{\mathrm{H}} \to \mathbf{C}$ be a function satisfying

$$\mathbf{c}(\eta h) = \omega'\, \mu(\eta)^{-1}.\, \mathbf{c}(h) \qquad (\eta \in \widetilde{\mathrm{H}}_*,\, h \in \widetilde{\mathrm{H}})$$

and which satisfies the conditions discussed in § I.3, namely

$$\Sigma\, \tau(s, {}^s\omega', \eta, \eta')\, \mathbf{c}(\eta) = 0$$

for all simple reflections s and $\eta' \in \widetilde{\mathrm{H}}$. If $v \in \mathrm{V}_0(\omega')$ and $\bar{v} \in \mathrm{V}(\omega')$ is chosen so that

$$\mathrm{I}_{w_0}(\bar{v}) = v$$

then $\Lambda(\mathbf{c}) : \mathrm{V}_0(\omega') \to \mathbf{C}$ can be defined by

$$\langle \Lambda(\mathbf{c}), v \rangle = \langle \lambda(\mathbf{c}), \bar{v} \rangle$$

and yields a Whittaker functional on $\mathrm{V}_0(\omega')$.

With these notations we can state our first main result.

Theorem **I.4.1.** — *With the notations above, let* $\mathbf{c}$ *be fixed and* $v \in \mathrm{V}_0(\omega')$. *Then there exist numbers* A, $\mathrm{A}' > 0$ *so that*

a) *if* $h \in \widetilde{\mathrm{H}}$ *and for some positive simple root* α *we have*

$$|p(h)^\alpha| > \mathrm{A}$$

then $\langle \Lambda(\mathbf{c}), \pi_0(h)\, v \rangle = 0,$

b) *if* $h \in \widetilde{\mathrm{H}}$ *and for all positive simple roots* α *we have*

$$|p(h)^\alpha| \leq \mathrm{A}'$$

then $\langle \Lambda(\mathbf{c}), \pi_0(h)\, v \rangle = \Sigma\, \mathbf{c}(\eta)\, \mu(\eta)^2\, v(\eta\, {}^{w_0}h)$

the sum being taken over $\widetilde{\mathrm{H}}_* \backslash \widetilde{\mathrm{H}}$.

Proof. — Part *a)* of the theorem is a triviality as $\mathrm{V}_0(\omega')$ is admissible so that we need only give a proof for part *b)*.

To do this first let $h_1 \in \widetilde{H}_n \widetilde{Z}$ and form

$$u = \pi_0(h_1)\, v - {}^{w_0}\omega\mu(h_1)\, v;$$

from Theorem I.2.9 *e)* it follows that $\varphi_0(u) = 0$, where φ_0 is the coinvariants functor of § I.2. Hence there exists an open compact subgroup N_0 of N_+^* so that

$$\int_{N_0} \pi_0(n)\, u\, dn = 0$$

(cf. [46], Lemma 2.2.1). Thus for $h \in \widetilde{H}$, $\lambda \in \mathrm{Wh}(V_0(\omega'))$ we have

$$\int_{N_0} \langle \lambda, \pi_0(h)\, \pi_0(n)\, u \rangle\, dn = 0.$$

But for a suitable $A_1 > 0$

$$|p(h)^\alpha| \leq A_1$$

for all positive simple α implies that

$$e \mid h N_0\, h^{-1} = 1.$$

Thus, under this condition we have that the integrand is constant and hence

$$\langle \lambda, \pi_0(h)\, u \rangle = 0.$$

Thus we obtain

$$\langle \lambda, \pi_0(hh_1)\, v \rangle = ({}^{w_0}\omega\mu)\,(h_1)\, \langle \lambda, \pi_0(h)\, v \rangle.$$

Since $V_0(\omega')$ is admissible

$$\widetilde{H}_n \widetilde{Z} / \{h \in \widetilde{H}_n \widetilde{Z} \mid \pi_0(h)\, v = v\}$$

is finitely generated and from this one deduces that given $v \in V_0(\omega')$ $h \in \widetilde{H}$ there exists $A_2 > 0$ so that the function

$$h_1 \mapsto {}^{w_0}\omega\mu(h_1)^{-1}\, \langle \lambda, \pi_0(h_1\, h)\, v \rangle$$

defined on

$$\{h_1 \in \widetilde{H}_n \widetilde{Z} : |p(h_1)^\alpha| < A_2 \text{ for all positive simple } \alpha\}$$

is constant. We next compute the right-hand side if $\lambda = \Lambda(\mathbf{c})$.

Replace n by $h_1\, n h_1^{-1}$. Then we have that this expression is equal to

$$\Sigma\, \mathbf{c}(\eta) \, . \int_{N_+^*} \bar{e}(h_1\, n h_1^{-1})\; \bar{v}(\eta w_0^{-1}\, nh)\; dn.$$

As $h_1 \to 0$

$$\bar{e}(h_1\, n h_1^{-1}) \to 1$$

and as the integral is absolutely convergent we see, by Lebesgue's dominated convergence theorem, that

$$\lim_{h_1 \to 0} {}^{w_0}\omega\mu(h_1)^{-1} \langle \Lambda(\mathbf{c}), \pi_0(h_1 h)\, v \rangle = \Sigma\, \mathbf{c}(\eta) \int_{N_+^*} \bar{v}(\eta w_0^{-1} nh)\, dn$$
$$= \Sigma\, \mathbf{c}(\eta)\, \mu(\eta)^2 \int_{N_+^*} \bar{v}(w_0^{-1} n \eta\, {}^{w_0}h)\, dn$$

The integral is however simply $I_{w_0}(\bar{v})(\eta\, {}^{w_0}h)$, or $v(\eta\, {}^{w_0}h)$. This proves the theorem.

Let us now assume that $|n|_F = 1$ and that e_0 has conductor R_F. If ω is unramified and ω' normalized we let $v_0 \in V_0(\omega')$ be that K^*-invariant element for which one has $v_0(I) = 1$. Call $h \in \widetilde{H}$ *weakly dominant* if $|p(h)^\alpha| \leq 1$ for all positive roots α.

Theorem **I.4.2.** — *With the notations above*

$$\langle \Lambda(\mathbf{c}), \pi_0(h)\, v_0 \rangle = \mathbf{c}((h^{w_0})^{-1})\, \mu(h)^2$$

if $h \in \widetilde{H}$ *is weakly dominant and* 0 *otherwise.*

Note that Theorem I.4.1 is the " asymptotic " form of this theorem.

It will be a consequence of the following theorem, of interest in itself:

Theorem **I.4.3.** — *With the notations above, if* $h \in \widetilde{H}_n \widetilde{Z}$ *is weakly dominant we have*

$$\int_{N_+^* \cap K^*} \pi_0(nh)\, v_0\, dn = {}^{w_0}\omega\mu(h) \,.\, v_0$$

Reminder. — The assumption that ω' was normalized carried with it the assumption that K^* is canonical. Hence $N_+^* \cap K^*$ is a lift of $N_+ \cap K$.

Deduction of Theorem **I.4.2** *from Theorem* **I.4.3.** — Let $h \in \widetilde{H}$, $h_1 \in \widetilde{H}_n \widetilde{Z}$ be weakly dominant. Then by Theorem I.4.3 one has

$$\int_{N_+^* \cap K^*} \pi_0(hnh_1)\, v_0\, dn = {}^{w_0}\omega\mu(h_1)\, \pi_0(h)\, v_0.$$

Apply $\Lambda(\mathbf{c})$ to both sides of this; as

$$h(N_+^* \cap K^*)\, h^{-1} \subset N_+^* \cap K^*$$

we obtain

$$\langle \Lambda(\mathbf{c}), \pi_0(hh_1)\, v_0 \rangle = {}^{w_0}\omega\mu(h_1)\, \langle \Lambda(\mathbf{c}), \pi_0(h)\, v_0 \rangle.$$

We choose now h_1 so that hh_1 lies in the region where Theorem I.4.1 *b)* is valid. This theorem yields for the left-hand side the expression

$$\Sigma\, \mathbf{c}(\eta)\, \mu(\eta)^2\, v_0(\eta\, {}^{w_0}(hh_1))$$

and, from the construction of v_0 this is simply

$$\mathbf{c}(((hh_1)^{w_0})^{-1}) \,.\, \mu(hh_1)^2.$$

Thus if h is weakly dominant we see that

$$\langle \Lambda(\mathbf{c}), \pi_0(h)\, v_0 \rangle = ({}^{w_0}\omega\mu(h_1))^{-1}.\, \mathbf{c}(((hh_1)^{w_0})^{-1}).\, \mu(hh_1)^2 = \mathbf{c}((h_{w}^{\circ})^{-1})\, \mu(h)^2.$$

This proves the required formula when h is weakly dominant. If h is not weakly dominant then it is easy to show that $\langle \Lambda(\mathbf{c}), \pi_0(h)\, v_0 \rangle = 0$.

We now turn to the proof of Theorem I.4.3. For this we make use of some auxillary considerations. Let $\kappa^* : K \to K^*$ be the canonical lift. Let $K_1 \subset K$ be the Iwahori subgroup of K, *i.e.* that subgroup of matrices which are, modulo π, upper-triangular; let $K_1^* = \kappa^*(K_1)$.

Proposition **I.4.4.** — *One has, with the notations above,*

$$\{v \in V_0(\omega') : \pi_0(k)\, v = v,\ k \in K_1^*\} = \mathbf{C}.v_0.$$

We shall first assume this and deduce Theorem I.4.3. Let N_- be the group of lower-triangular unipotent matrices and choose $n' \in \kappa^*\ (N_- \cap K_1)$. If now $n \in N_+^* \cap K^*$ it follows from the Bruhat decomposition that we can write $n'\, n$ in the form

$$n'\, n = u(n, n').\, a(n, n').\, u'(n, n')$$

where

$$u(n, n') \in N_+^* \cap K^*$$
$$a(n, n') \in \widetilde{H} \cap K^*$$
$$u'(n, n') \in \kappa^*(N_- \cap K_1^*).$$

Moreover one sees that $n \mapsto u(n, n')$ is a measure-preserving map, by first proving this statement for $SL_2(F)$ and then applying induction. We shall now verify that, for $h \in \widetilde{H}_n \widetilde{Z}$ dominant,

$$\int_{N_+^* \cap K^*} \pi_0(nh)\, v_0\, dn$$

is invariant under K_1^*. To prove this it is sufficient to prove that it is invariant under $\kappa^*(N_- \cap K_1)$, since it is clearly invariant under $N_+^* \cap K^*$ and $K^* \cap H$ and these groups together generate K_1^*. Thus, we let $n' \in \kappa^*(N_- \cap K_1)$ and we shall verify that the vector above is left fixed by n'. One has

$$\pi_0(n') \int_{N_+^* \cap K^*} \pi_0(nh)\, v_0\, dn = \int_{N_+^* \cap K^*} \pi_0(u(n, n')\, a(n, n')\, u'(n, n')\, h)\, v_0\, dn.$$

Since v_0 is K^*-invariant and since h and $a(n, n')$ commute the right-hand side is

$$\int_{N_+^* \cap K^*} \pi_0(u(n, n')\, h)\, v_0\, dn.$$

As $n \mapsto u(n, n')$ is measure-preserving and is clearly injective it is also surjective. Thus this last integral can be written as

$$\int_{N_+^* \cap K^*} \pi_0(uh)\, v_0\, dn,$$

which demonstrates the invariance claimed. Hence, by Proposition I.4.4 there exists a scalar $\theta(h)$ so that

$$\int_{N_\ddagger^* \cap K^*} \pi_0(nh)\, v_0\, dn = \theta(h)\, v_0.$$

It is easy to see that $\theta(h) = {}^{w_0}\omega\mu(h)$ by applying the functor φ_0 to this equation. This completes the proof of Theorem I.4.3, once Proposition I.4.4 has been proved.

Proof of Proposition **I.4.4.** — We continue to regard $V_0(\omega')$ as the image of $V(\omega')$ by I_{w_0} in $V({}^{w_0}\omega')$. Then if $v_1 \in V_0(\omega')$ is K_1^*-invariant there exists $\bar{v}_1 \in V(\omega')$ which is also K_1^*-invariant and such that $I_{w_0}(\bar{v}_1) = v_1$; this follows by the familiar averaging argument since K_1^* is compact. We shall define a basis f_w $(w \in W)$ of

$$Y = \{v \in V(\omega') : \pi_0(k)\, v = v \ (k \in K_1^*)\},$$

and it will suffice to show that all the $I_{w_0}(f_w)$ are equal. To that effect it will suffice to show that

$$I_s(f_w - f_{sw}) = 0$$

for all simple reflections s. The element f_w of $V(\omega')$ is defined to be that K_1^*-invariant element of $V(\omega')$ with

$$\mathrm{Supp}(f_w) = \widetilde{B}_*\, wK_1^*$$

and

$$f_w(w) = 1.$$

It is easy to verify that this exists. To see that these form a basis of such functions we observe that $\{\eta w : \eta \in T, w \in W\}$, where T is set of representatives in $\widetilde{H}$ for $\widetilde{H}_*\backslash\widetilde{H}$, is a set of representatives for $\widetilde{B}_*\backslash\widetilde{G}/K_1^*$. But it is easy to verify that if φ is K_1^*-invariant then $\varphi(\eta w) = 0$ when $\eta \notin \widetilde{H}_*$. Hence the f_w $(w \in W)$ do form a base. Moreover we see that if φ is K_1^*-invariant then $\varphi(w) = 0$ $(w \in W)$ implies that $\varphi = 0$.

Now let us verify that

$$I_s(f_w - f_{sw}) = 0$$

for a simple reflection s. By the principle which we have just explained, it suffices to show that

$$I_s(f_w - f_{sw})\,(w') = 0$$

for all $w' \in W$. However, if $w' \neq w, sw$, then $I_s(f_w) = 0$, $I_s(f_{sw}) = 0$ as the integrand of the integral defining I_s is zero. It will therefore suffice to show that

$$I_s(f_w - f_{sw})\,(w) = 0,$$

since the other condition, that with $w' = sw$, arises from replacing w by sw.

Thus we have to show that

$$\int_{N_\ddagger^*(s)} f_w(snw)\, dn = \int_{N_\ddagger^*(s)} f_{sw}(snw)\, dn.$$

Let α be the root corresponding to s. Then one finds that in order that $s(1+\xi\mathbf{e}_\alpha)^* w$ should lie in $\widetilde{B}_* wK_1^*$ one requires that $|\xi|>1$ and then

$$s(1+\xi\mathbf{e}_\alpha)^* w = \widetilde{h}_\alpha(\xi)(1+\xi^{-1}\mathbf{e}_{-\alpha})^* w$$

where $\widetilde{h}_\alpha(\xi)$ has the same meaning as in § I.3. Since ω' is exceptional it follows that

$$\begin{aligned} f_w(s(1+\xi\mathbf{e}_\alpha)^* w) &= |\xi|^{-\frac{n+1}{n}} \quad \text{if } \operatorname{ord}(\xi)\equiv 0 \pmod n \text{ and } |\xi|>1, \\ &= 0 \text{ otherwise.}\end{aligned}$$

Assume now that $\operatorname{meas}(R_F)=1$.

Then the left-hand side in the equality which we have to prove is

$$(1-q^{-1})\sum_{k>0} q^{+nk}.q^{-k(n+1)} = q^{-1}.$$

Consider next the right-hand side of the proposed equality. The condition that $(1+\xi\mathbf{e}_\alpha)^*$ should lie in $\widetilde{B}_* swK_1^*$ is $|\xi|<1$. But the integrand is then 1 and the integral is again q^{-1}. This proves the equality and the proposition.

I.5. Characters and Whittaker Models.

Our objective in this section is to establish a connection between a numerical invariant derived from the character χ_V of an irreducible, admissible representation V of $\widetilde{G}$, and dim(Wh(V)). This result could be used with a knowledge of the character of $V_0(\omega)$, if this were available, to compute $\dim(\mathrm{Wh}(V_0(\omega)))$ when $|n|_F<1$. We shall give a simple example of this (Theorem I.5.7 below), based on Flicker's formulation of the Shimura Correspondence in [7].

Before we can formulate the main result of this section we have to recall some standard facts about characters. Thus let (π, V) be an admissible representation of $\widetilde{G}$, and let f be a smooth function of compact support on $\widetilde{G}$. Then the map

$$\pi(f): V\to V; \quad v\mapsto \int f(x).\pi(x)\, v\, dx$$

has finite rank, and has therefore a trace, which we denote by $\operatorname{Tr}(\pi(f))$. Since

$$f\mapsto \operatorname{Tr}(\pi(f))$$

is a distribution, we can write it as

$$\int \chi_V(x) f(x)\, dx$$

where χ_V is a generalized function (or distribution) on $\widetilde{G}$. It is invariant in the sense that

$$\chi_V(x^y)=\chi_V(x) \qquad (y\in G).$$

Let G_{reg} be the set of regular elements of G, and let $\widetilde{G}_{reg}=p^{-1}(G_{reg})$.

Let now $\alpha = (r_1, r_2, \ldots, r_s)$ be a partition of r and let Π_α be a standard parabolic subgroup of G with Levi component isomorphic to $\mathrm{GL}_{r_1}(\mathrm{F}) \times \ldots \times \mathrm{GL}_{r_s}(\mathrm{F})$. Let Ω_α be the character of the representation of G induced from the identity representation of Π_α. Then one can compute the Ω_α explicitly (cf. [14] § 20.4, § 21.5, [12] Ch. 2, § 5). We shall not give the explicit formulae here, but we note the following consequences:

1. Ω_α does not depend on the choice of Π_α;
2. let U be a neighbourhood of I in G; then the $\Omega_\alpha \mid \mathrm{U}$ are linearly independent, and
3. let $\alpha_0 = (1, 1, \ldots, 1)$; then

$$\lim_{\substack{h \to \mathrm{I} \\ h \text{ regular}}} \Omega_\alpha(h)/\Omega_{\alpha_0}(h) = 0 \qquad (\alpha \neq \alpha_0).$$

Let $\widetilde{\Omega}_\alpha = \Omega_\alpha \circ p$.

Theorem **I.5.1.** — *Suppose that* $\mathrm{Char}(\mathrm{F}) = 0$, *and that* V *is an irreducible admissible representation of* $\widetilde{\mathrm{G}}$. *Then one has*:

(i) χ_V *can be represented by a locally constant function on* $\widetilde{\mathrm{G}}_{\mathrm{reg}}$;

(ii) *there exists a neighbourhood* U *of* I *so that* χ_V *is represented on* U *by a locally integrable function*; *more precisely there exist* $c_\alpha \in \mathbf{C}$ *(*α *a partition of* r*) such that*

$$\chi_\mathrm{V} \mid \mathrm{U} = \Sigma\, c_\alpha . \Omega_\alpha \mid \mathrm{U}.$$

Proof. — Part (i) follows by writing the proof of [19], Cor. to Theorem 4 in the context of metaplectic groups. Part (ii) will follow from [18], Theorem 20 once we have verified:

a) there exist G-invariant open neighbourhoods $\widetilde{\mathrm{Y}}$ of I in $\widetilde{\mathrm{G}}$, and Y of I in G such that

$$p \mid \widetilde{\mathrm{Y}} : \widetilde{\mathrm{Y}} \to \mathrm{Y}$$

is a homeomorphism, and

b) the $\Omega_\alpha \mid \mathrm{U}$ have the same span as the " ν_ξ " of [18].

Since the $\Omega_\alpha \mid \mathrm{U}$ are linearly independent, and since the Ω_α actually appear as the germs of characters, b) follows easily.

We shall now verify *a)*. Let $\mathrm{K}_1 \subset \mathrm{GL}_r(\mathrm{R_F})$ be an open normal subgroup such that

(i) there exists a $\mathrm{GL}_r(\mathrm{R_F})$-invariant section κ_1 of p over K_1 (*i.e.* $\kappa_1(x^\gamma) = \kappa_1(x)^\gamma$ if $x \in \mathrm{K}_1, \gamma \in \mathrm{GL}_r(\mathrm{R_F})$), and

(ii) if $\mathrm{K}_2 = \{x^n : x \in \mathrm{K}_1\}$ then K_2 is a subgroup of K_1,

$$\mathrm{K}_1 \to \mathrm{K}_2; \qquad x \mapsto x^n$$

is a bijection and the inverse map is given by the binomial series for $(\mathrm{I} + (y - \mathrm{I}))^{1/n}$.

It is easy to see that such a K_1 exists.

We now let

$$\widetilde{\mathrm{Y}} = \kappa_1(\mathrm{K}_2)^\mathrm{G}, \quad \mathrm{Y} = \mathrm{K}_2^\mathrm{G};$$

we have to verify that p is injective. To do this suppose that $x, x' \in K_2$, $g, g' \in G$ are such that $(x)^g = (x')^{g'}$; we have to verify that

$$\kappa_1(x)^g = \kappa_1(x')^{g'}.$$

To do this we can clearly assume that $g' = I$. Let $x = \xi^n$, $x' = \eta^n$. Since the binomial series for $(I + (\xi^n - I)^g)^{1/n}$ converges and is equal to that for $((I + (\xi^n - I))^{1/n})^g$ we obtain $\xi^g = \eta$. But then $\kappa_1(x)^g = \kappa_1(\xi^n)^g = (\kappa_1(\xi)^g)^n$; as $\kappa_1(\xi)^g = \kappa_1(\eta).i(\zeta)$ for some $\zeta \in \mu_n(F)$, this yields

$$\kappa_1(x)^g = \kappa_1(\eta)^n = \kappa_1(\eta^n) = \kappa_1(x').$$

This completes the proof of the theorem.

We now choose a non-degenerate character $e : N^*_+ \to \mathbf{C}^\times$ as in § I.3 and define Wh(V) as there for any admissible representation V of $\widetilde{G}$.

Theorem **I.5.2.** — (i). *Let* V *be an irreducible, admissible representation of* $\widetilde{G}$. *Then* Wh(V) *is finite-dimensional.*

(ii) *Suppose that* V *is cuspidal* (i.e. $\varphi_P(V) = \{0\}$ *for all* $P \neq G$). *Then* $Wh(V) \neq \{0\}$.

Here φ_P is the functor of coinvariants associated with the parabolic subgroup P of G as introduced in § I.2.

Proofs. — (i) This seems to be well-known but we could not find a reference; therefore we sketch a proof here. It suffices to prove the statement for V cuspidal, since, by the obvious analogue to [3] 4.5 for the metaplectic group, the general case can be deduced from this one. Under the assumption that V is admissible and irreducible there exists a finite subset V_0 of V such that $V = \langle \widetilde{B}.V_0 \rangle$.

Thus if $\lambda \in Wh(V)$, λ is determined by the functions

$$\widetilde{H} \to \mathbf{C}; \quad h \mapsto \lambda(\pi(h)\, v) \quad (v \in V_0).$$

Since V is admissible there exists $c_1 > 0$ so that, if for some $\alpha \in \Phi^+$ one has

$$|p(h)^\alpha|_F > c_1,$$

then $$\lambda(\pi(h)\, v) = 0 \quad (v \in V_0).$$

We shall show that likewise there exists $c_2 > 0$ so that, if for some $\alpha \in \Phi^+$ one has

$$|p(h)^\alpha|_F < c_2$$

then $$\lambda(\pi(h)\, v) = 0 \quad (v \in V_0).$$

Granted this, it follows that the functions above are determined by a finite collection of their values.

Let now α be a simple positive root and let $P_\alpha \supset B$ be that maximal parabolic subgroup of G whose unipotent radical N_α contains the unipotent subgroup of G corresponding to α. Then as

$$\varphi_{P_\alpha}(V) = 0$$

there exists a compact open subgroup Y of N_α such that

$$\int_Y \pi(n)\, v\, dn = 0 \qquad (v \in V_0).$$

Let $c_2(\alpha)$ be such that, if $|p(h)^\alpha|_F < c_2(\alpha)$, then

$$hYh^{-1} \subset \mathrm{Ker}(e);$$

this being so we have

$$0 = \lambda\left(\int_Y \pi(h)\, \pi(n)\, v\, dn\right) = \lambda(\pi(h)\, v) . \int_Y dn.$$

Thus $\qquad \lambda(\pi(h)\, v) = 0,$

for all h with $|p(h)^\alpha|_F < c_2(\alpha)$. This clearly implies the assertion we needed.

We now come to the central theorem of this section. To formulate it we must first introduce some notations. If $g \in G$ let $P(g)$ be the discriminant of the characteristic polynomial of g and let

$$\Delta(g) = |P(g)|_F^{1/2}\, |\det(g)|_F^{-(r-1)/2}.$$

If g is split with eigenvalues $\lambda_1, \ldots, \lambda_r$ then one has

$$\Delta(g) = \prod_{i<j} |\lambda_i - \lambda_j|_F . |\det(g)|_F^{-(r-1)/2}.$$

Naturally, Δ is a class-function on G; we call it the *Weyl factor.*

Theorem **I.5.3.** — *Suppose that* $\mathrm{Char}(F) = 0$. *Let* V *be an irreducible, admissible representation of* $\widetilde{G}$ *with character* χ_V. *Then*

$$\dim(\mathrm{Wh}(V)) = \frac{1}{r!} \lim_{h \to 1} \Delta(h)\, \chi_V(\mathbf{s}(h)),$$

where $h \in H$ *with h regular.*

Remark. — It is not difficult to show using Kubota's formula (following Proposition 0.1.2) that $\mathbf{s} \mid H$ is a continuous section of $p \mid \widetilde{H}$ (although it is not a homomorphism).

This theorem is a variant of one of Rodier's [41].

Proof. — The proof of this theorem is based on the relationship between $\widetilde{G}$ and the group $\widetilde{P}_r = p^{-1}(P_r)$ where $P_r = \{g \in G : (0, \ldots, 0, 1)\, g = (0, 0, \ldots, 0, 1)\}$. We shall make use of the structure theory of representations of $\widetilde{P}_r$, and, in particular, the notion of derived representations as discussed in [3] § 3. These are considered there only on P but the results apply also to $\widetilde{P}$.

We shall paraphrase those results of [3] § 3 which we need. Let W be an algebraic representation of $\widetilde{P}_r$. Then there exist representations (derived representations) $W^{(k)}$ of $\widetilde{G}_{r-k}$ and a filtration $0 \subset W_r \subset W_{r-1} \subset \ldots \subset W_1 = W$ of W such that

$$W_k/W_{k+1} \cong \mathrm{ind}_{\widetilde{G}_{r-k}.N_k^*}^{\widetilde{P}} (W^{(k)} \otimes e_{k-1})$$

where

$$N_k^* = \{n \in N^* : p(n)_{ij} = 0 \quad \text{if } i < j \leqslant r - k\},$$

and
$$e_{k-1}(n) = e_0(\sum_{r-k<l<r} p(n)_{l,l+1}) \qquad (n \in N_k^*),$$

here the subscripts denote the relevant matrix entries. Note also that $W^{(k)} \otimes e_{k-1}$ is well-defined on $\widetilde{G}_{r-k}.N_k^*$. We shall here consider the $W^{(k)}$ as the coinvariants defined without the intervention of the modulus function which is used in [3] 1.8 (b); as we have used the usual induction functor we have also used the corresponding modification in 1.8 (a) (1). Thus this leads merely to a slight difference in definition.

Let φ be a locally constant function of compact support on $\widetilde{P}$. We shall call φ *cuspidal* if for every parabolic subgroup $\widetilde{\Pi}$ of $\widetilde{G}$, such that $\widetilde{\Pi} \subset \widetilde{P}$ one has

$$\int_{U_\Pi^*} \varphi(ug)\, du = 0$$

where U_Π is the unipotent radical of Π and U_Π^* is the canonical lift of U_Π to $\widetilde{P}$. Examples of such functions are given by the matrix coefficients of cuspidal representations of $\widetilde{G}$ restricted to $\widetilde{P}$. Denote the space of cuspidal functions on $\widetilde{P}$ by $C_0(\widetilde{P})$.

Proposition **I.5.4.** — *Let* $\varphi \in C_0(\widetilde{P})$ *and let* W *be an algebraic representation of* $\widetilde{P}$. *Suppose moreover that* $W^{(r)}$ *is finite dimensional. Let* τ *(resp.* τ_j*) be the action of* $\widetilde{P}$ *on* W *(resp.* W_j*). Then* $\tau(\varphi)$ *has finite rank and*

$$\mathrm{Tr}(\tau(\varphi)) = \mathrm{Tr}(\tau_r(\varphi)).$$

Proof. — Since φ is continuous and of compact support one has that $\tau(\varphi)$ exists and that it respects the filtration $0 \subset W_r \subset W_{r-1} \subset \ldots \subset W_1 = W$. We shall show that

$$\tau(\varphi)\, W_1 = \tau(\varphi)\, W_r \subset W_r.$$

This will be proved by an inductive argument, of which the k-th step consists of the statements

a_k) $\tau(\varphi)\, W_j \subset W_{j+k} \quad (1 \leqslant j+k < r)$, and,

b_k) if $\overline{\tau(\varphi)} : W_j/W_{j+1} \to W_{j+k}/W_{j+k+1} \quad (1 \leqslant j+k < r)$ is the induced map, then $\overline{\tau(\varphi)} = 0$.

Clearly b_k) implies a_{k+1}). On the other hand a_k) implies that $\overline{\tau(\varphi)}$ is well-defined. If $k = 0$ b_k) also is meaningful and a_k) is a triviality. Now note that b_0) is true since $\varphi \in C_0(\widetilde{P})$, and so is b_k) $(k \geqslant 1)$ by [3] 3.2. Thus a_k) is true for all k. In particular $\tau(\varphi)\, W_1 \subset W_r$. The filtration gives $0 \subset \tau(\varphi)\, W_{r-1} \subset \ldots \subset \tau(\varphi)\, W_1 \subset W_r$. Observe that the natural induced maps

$$W_j/W_{j+1} \to \tau(\varphi)\, W_j/\tau(\varphi)\, W_{j+1} \qquad (1 \leqslant j < r)$$

are surjective, and o (by b_1)); hence it follows that

$$\tau(\varphi)\, W_1 = \tau(\varphi)\, W_2 = \ldots = \tau(\varphi)\, W_r$$

as required.

Thus it remains to prove that $\tau(\varphi)$ has finite rank in the case that (τ, W) is

$$(\tau_0, W_0) = \operatorname{ind}_{N^* \times \mu_n(F)}^{\widetilde{P}}(e \times \varepsilon).$$

The representation space for this latter representation consists of locally constant functions f on $\widetilde{P}$ such that

a) $f(g.i(\zeta)) = \varepsilon(\zeta) f(g) \quad (\zeta \in \mu_n(F), g \in \widetilde{P})$
b) $f(ng) = e(n) f(g) \quad (n \in N^*, g \in \widetilde{P})$
c) $|f|$, which is defined on $N\backslash P$, has compact support.

Then $\quad (\tau_0(\varphi) f)(g) = \int_{\widetilde{P}} f(\gamma)\, \varphi(g^{-1}\gamma)\, d\gamma,$

and $d\gamma$ is a left-invariant Haar measure on $\widetilde{P}$.

Since there is a compact subgroup $K_1(\varphi)$, which depends only on φ, such that $\tau(\varphi) f$ is $K_1(\varphi)$-invariant, it will suffice to show that there exists a compact subset $K(\varphi)$, which again depends only on φ, such that

$$\operatorname{Supp}(\tau_0(\varphi) f) \subset \widetilde{N}.K(\varphi).$$

To demonstrate this we shall use induction on r.

Let us write matrices in P in $((r-1)+1) \times ((r-1)+1)$-block form; let N_r be the subgroup of matrices of the form $\begin{pmatrix} I & x \\ 0 & 1 \end{pmatrix}$ $(x \in F^{r-1})$, and G_{r-1} the subgroup of the form $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ $(g \in GL_{r-1}(F))$. Let $n \mapsto n^*$ be the canonical lift of N_r to $\widetilde{P}$ and let $\widetilde{G}_{r-1} = p^{-1}(G_{r-1})$. We have a bijection

$$F^{r-1} \times \widetilde{G}_{r-1} \to \widetilde{P}; \quad (x, g) \mapsto \begin{pmatrix} I & x \\ 0 & 1 \end{pmatrix}^* .g.$$

We can therefore parameterize $\widetilde{P}$ by this map, the multiplication becomes

$$(x, g).(x', g') = (x + gx', gg'),$$

where gx' is, by definition, $p(g)\, x'$.

In the above integral we now replace g and γ by $(0, g_1)$ and (ξ, γ_1). The left-invariant measure can be taken to be

$$d\gamma = |\det(\gamma_1)|_F^{-1}.|d\xi|.d\gamma_1$$

where $d\gamma_1$ is a bi-invariant Haar measure on $\widetilde{G}_{r-1}$. Thus

$$(\tau_0(\varphi) f)(0, g_1) = \int_{\widetilde{G}_{r-1}} f(0, \gamma_1)\, |\det(\gamma_1)|_F^{-1} \int_{F^{r-1}} e_0(\xi_{r-1}).\varphi(g_1^{-1}\xi, g_1^{-1}\gamma_1)\, |d\xi|.d\gamma_1.$$

It is now convenient to replace ξ by $g_1 \xi$; then the right-hand side of the above expression becomes

$$\int_{\widetilde{G}_{r-1}} f(0, \gamma_1) \,|\det(g_1^{-1} \gamma_1)|_F^{-1} \int_{F^{r-1}} e_0(\sum_{j=1}^{r-1} g_{r-1,j} \xi_j) \,\varphi(\xi, g_1^{-1} \gamma_1) \,|d\xi| . d\gamma_1 .$$

In this expression $\xi = (\xi_1, \ldots, \xi_{r-1})$ and g_{ij} is the ij-th matrix coefficient of $p(g_1)$. Note that there exist compact sets $K_1 \subset F^{r-1}$, $K_2 \subset \widetilde{G}_{r-1}$ such that

$$\operatorname{Supp} \varphi \subset K_1 \times K_2$$

and also that φ is locally constant on $N_r^* \times \widetilde{G}_{r-1}$.

It follows from the last fact that there exists $c_1 > 0$ such that the integral over ξ_j is zero unless $|g_{r-1,j}|_F \leqslant c_1$. Secondly, as φ has compact support in the above sense, there exists a constant $c_2 > 0$ such that, if $|g_{r-1\,j}|_F \leqslant c_2$ for all j, then

$$\int_{F^{r-1}} e_0(\sum_{j=1}^{r-1} g_{r-1,j} \xi_j) \,\varphi(\xi, g_1^{-1} \gamma_1) \,|d\xi| = \int_{F^{r-1}} \varphi(\xi, g_1^{-1} \gamma_1) \, d\xi = 0$$

since φ is cuspidal.

It follows now that in order that $(\tau(\varphi) f)(0, g_1)$ should be non-zero we need that $|g_{r-1,j}|_F \leqslant c_1$ for all j, and that there should exist at least one j such that $|g_{r-1,j}|_F > c_2$. It follows easily from this that there is a compact subset K_3 of $\widetilde{G}_{r-1}$ such that $\tau(\varphi) f(0, g_1) = 0$ unless $g_1 \in \widetilde{P}_{r-1} . K_3$, where $\widetilde{P}_{r-1}$ is the subgroup of $\widetilde{G}_{r-1}$, defined as $\widetilde{P}$ was in $\widetilde{G}$. Moreover, in the integral representation of $\tau(\varphi) f(0, g_1)$ above $g_1^{-1} \gamma_1 \in K_2$; thus the integral need only be extended over $\widetilde{P}_{r-1} . K_3 K_2$. Observe here that $K_3 . K_2$ is also compact.

We have now to show that there exists a compact set $K_4 \subset \widetilde{P}_{r-1}$ such that, if $g_1 \in K_3$, then the map

$$\widetilde{P}_{r-1} \to \mathbf{C}; \quad p \mapsto \tau(\varphi) f(0, p g_1)$$

has support contained in $(N^* \cap \widetilde{P}_{r-1}) . K_4$. However, since $\tau(\varphi) f(0, g)$ is, as a function of g, right-invariant under an open subgroup which depends only on φ, we see that it suffices to prove this latter statement for any fixed $g_1 \in K_3$. This we shall do by reducing the question to $\widetilde{P}_{r-1}$, when it will be justified by the induction hypothesis.

Suppose that $\varphi(\xi, g)$ is, as a function of g, right-invariant under a compact open subgroup $K_5 \subset \widetilde{G}_{r-1}$. Then we can assume that $f(0, g)$ is too, since we can average it over K_5. Thus there exists a finite set $\gamma_{1,j} \in \widetilde{G}_{r-1}$ $(1 \leqslant j \leqslant N)$, and positive constants c_j $(1 \leqslant j \leqslant N)$, so that $\tau_0(\varphi)(0, p g_1)$ (where $p \in \widetilde{P}_{r-1}$) is given by

$$\sum_{1 \leqslant j \leqslant n} c_j \int_{\widetilde{P}_{r-1}} f(0, q \gamma_{1,j}) \,|\det(p^{-1} q)|_F^{-1} \,|\det(g_1^{-1} \gamma_{1,j})|_F^{-1} .$$
$$\int_{F^{r-1}} e_0(\sum_{k=1}^{r-1} g_{r-1,k} \xi_k) \,\varphi(\xi, g_1^{-1} p^{-1} q \gamma_{1,j}) \,|d\xi| \, dq$$

where dq is a left-invariant Haar measure on $\widetilde{P}_{r-1}$. Thus our contention will be proved, by virtue of the induction hypothesis, if we verify that the function on $\widetilde{P}_{r-1}$ given by

$$p \mapsto |\det(p)|_F^{-1} \int_{F^{r-1}} e_0\Big(\sum_{1 \leqslant k \leqslant r-1} g_{r-1,k}\,\xi_k\Big)\, \varphi(\xi, g_1^{-1} p \gamma_{1,j})\, |d\xi|$$

satisfies the same conditions on $\widetilde{P}_{r-1}$ as φ did on $\widetilde{P}_r$; for then the integral above can be interpreted as a " $\tau_0(\varphi) f(p)$ " and the conclusion follows.

This function is clearly of compact support and locally constant. That it is cuspidal is almost immediate from the corresponding condition on φ. Hence the induction is completed, and Proposition I.5.4 is proved.

We next need the existence of sufficiently many φ. We let (τ_0, W_0) be the representation of $\widetilde{P}$:

$$(\tau_0, W_0) = \operatorname{ind}_{N^* \times \mu_n(F)}^{\widetilde{P}} (e \times \varepsilon).$$

Lemma **I.5.5.** — *Let* $L \subset \widetilde{P}$ *be an open compact subgroup of* $\widetilde{P}$. *Then there exists* $\varphi \in C_0(\widetilde{P})$ *such that*

$$\operatorname{Supp}(\varphi) \subset L$$

and

$$\operatorname{Tr}(\tau_0(\varphi)) \neq 0.$$

Proof. — We shall first show that there exists such a $\varphi \neq 0$ but without the restriction that $\operatorname{Tr}(\tau_0(\varphi)) \neq 0$. We can also assume that L is of the form $K_m^* \cap \widetilde{P}$ where K_m^* is the lift of

$$K_m = \{k \in GL_r(R_F) : k \equiv I \pmod{\pi^m}\},$$

the lift exists for m large enough.

It will now suffice to show that there exists a non-zero locally constant function φ_1 on $\widetilde{G}$ with $\operatorname{Supp}(\varphi_1) \subset K_m^*$ such that for every parabolic subgroup $\Pi \subset G$ one has

$$\int_{U_\Pi^*} \varphi_1(ug)\, du = 0.$$

This is so since, for some $k \in K$ $(= GL_r(R_F))$, one will have that the function $p \mapsto \varphi_1(p^k)$ on $\widetilde{P}$ is non-zero.

Let now $h \in K$ be such that its image under the reduction map

$$GL_r(R_F) \to GL_r(R_F/(\pi))$$

is elliptic. We let $\alpha \in F^\times$, to be chosen later. Then define

$$\begin{aligned} \varphi_1(x) &= e_0(\alpha \operatorname{Tr}(xh)) \qquad (x \in K_m^*) \\ &= 0 \text{ (otherwise).} \end{aligned}$$

Here we are identifying K_m^* with K_m in writing $\operatorname{Tr}(xh)$. We have clearly only to show that φ_1 is cuspidal. To do this let Π be parabolic in G. Then Π is conjugate to a standard parabolic subgroup Π_1, and U_Π is then also conjugate, by the same element, to U_{Π_1}.

However, Π_1 normalizes U_{Π_1} and $G = \Pi_1 K$ so that there exists $k \in K$ such that $U_\Pi = (U_{\Pi_1})^k$. Thus it suffices to show that

$$\int_{(U_\Pi)^*} \varphi_1(u^k g)\, du = 0$$

for all $k \in K$ and all standard parabolic subgroups $\Pi \subset G$. Now since K_m^* is (for m large enough) normalized by K it follows that one can assume that $k = 1$. It then follows from the Iwasawa decomposition that if $(U_\Pi)^* g \cap K_m^* \neq \varnothing$ then $g \in U_\Pi^* . K_m^*$. Thus we need only show that

$$\int_{U_\Pi^* \cap K_m^*} \varphi_1(uk)\, du = 0$$

where $k \in K_m^*$. However, written out explicitly this is

$$\int_{U_\Pi^* \cap K_m^*} e_0(\alpha \operatorname{Tr}(ukh))\, du = 0.$$

Let Π' be the opposite parabolic subgroup to Π; then since kh has elliptic reduction (mod π)

$$p(kh) \notin K_1 . (\Pi' \cap K).$$

Thus
$$\operatorname{Tr}((kh)\, p(U_\Pi^* \cap K_m^*)) = \pi^m R_F$$

and so, for a suitable fixed α (depending only on m) the above integral is 0. Thus φ_1 is cuspidal.

Now let φ be as constructed. We observe that there exists an algebraic representation (ρ, Y) of $\widetilde{P}$ such that $\rho(\varphi) \neq 0$ (for example the regular representation). Thus from the discussion in the proof of Proposition I.5.4 one has $\rho_r(\varphi) \neq 0$. Since ρ_r is of the form $\tau_0 \otimes X$, where the action of $\widetilde{P}_r$ on X is trivial, it follows also that $(\tau_0)_r(\varphi) \neq 0$. Note that

$$(\tau_0, W_0) = \operatorname{ind}_{N^* \times \mu_n(F)}^{\widetilde{P}} (e \times \varepsilon)$$

is unitarizable. Hence if $\varphi' = \varphi * \bar{\varphi}$ one has

a) $\tau_0(\varphi') \neq 0$ and $\operatorname{Tr}(\tau_0(\varphi')) \neq 0$
b) $\operatorname{Supp}(\varphi') \subset L$, and
c) $\varphi' \in C_0(\widetilde{P})$,

as one sees at once. Thus the lemma is proved.

We are now in a position to prove Theorem I.5.3. We shall consider the restriction of V to $\widetilde{P}$. This is algebraic and the multiplicity of $(\tau_0, W_0)_r$ in V_r, or, for that matter V, is $\dim(\operatorname{Wh}(V))$; this we write as N(V), which is finite by Theorem I.5.2. Hence by Proposition I.5.4 one has, if π denotes the action of $\widetilde{G}$ on V and if $\varphi \in C_0(\widetilde{P})$,

$$\operatorname{Tr}(\pi(\varphi)) = N(V) . \operatorname{Tr}(\tau_0(\varphi)).$$

However, let $L \subset \widetilde{G}$ be a sufficiently small compact open subgroup, and let χ_L be the normalized characteristic function of L. Here "sufficiently small" means that the image of $\pi(\varphi)$ is to be L-invariant. Then $\chi_L * \varphi$ is locally constant and of compact support on $\widetilde{G}$, and $\pi(\varphi) = \pi(\chi_L * \varphi)$. Thus we see that

$$\langle \chi_V, \gamma_L * \varphi \rangle = N(V)\ \mathrm{Tr}(\tau_0(\varphi)).$$

From Theorem I.5.1 we have that there exists an open neighbourhood $\mathcal{O}$ of I in $\widetilde{G}$ and an expansion of the following form:

$$\chi_V \mid \mathcal{O} = \sum_\alpha c_\alpha(V)\ \widetilde{\Omega}_\alpha \mid \mathcal{O}.$$

We now choose $L \subset \mathcal{O}$ and φ such that

$$\mathrm{Supp}(\varphi) \subset \mathcal{O}.$$

Then we verify at once that

$$\langle \widetilde{\Omega}_\alpha, \chi_L * \varphi \rangle = 0$$

for all $\alpha \neq \alpha_0$, where $\Pi_{\alpha_0} = B$, since $\chi_L * \varphi$ is cuspidal (or since $\dim(\mathrm{Wh}(\mathrm{ind}_{\Pi_\alpha}^G(1))) = 0$). Thus

$$\langle \chi_V, \chi_L * \varphi \rangle = c_{\alpha_0}(V)\ \langle \widetilde{\Omega}_{\alpha_0}, \chi_L * \varphi \rangle;$$

hence there exists a constant γ_1 such that

$$N(V) = \gamma_1 \cdot c_{\alpha_0}(V).$$

This involves χ_V only in a neighbourhood of I; since the representation of G of which Ω_{α_0} is the character has a unique Whittaker model it follows that $\gamma_1 = 1$. Thus one has

$$c_{\alpha_0}(V) = \lim_{h \to I} \Omega_{\alpha_0}(h)^{-1}\ \chi_V(\mathbf{s}(h))$$

where h is taken to be regular. It then follows that

$$N(V) = (r!)^{-1} \lim_{h \to I} \Delta(h)\ \chi_V(\mathbf{s}(h))$$

since, in a neighbourhood of I on H

$$\Omega_{\alpha_0}(h) = r!/\Delta(h),$$

as one can see from Proposition I.5.6 below. This proves the theorem.

Let $V(\omega)$ be the representation defined in § I.1. Then one has

Proposition **I.5.6.** — *The character of* $V(\omega)$ *is represented by a locally integrable function* $\chi_{V(\omega)}$ *which is supported on* $(\widetilde{H}_n \widetilde{Z})^G$ *and whose value at a regular element h of* $\widetilde{H}_n \widetilde{Z}$ *is given by*

$$[\widetilde{H} : \widetilde{H}^*]\ \Delta(h)^{-1} . \sum_{w \in W} {}^w\omega(h).$$

Remarks. — 1. This proposition is a generalization of the computation of characters of principal series representations, which have been known for a long time (cf. [14] § 20.4, [12] Ch. 2, § 5).

2. If $n = 1$, $\omega = 1$ we obtain the formula for Ω_{α_0} used above.

3. From Theorem I.5.3 it would follow that

$$\dim \mathrm{Wh}(\mathrm{V}(\omega)) = [\widetilde{\mathrm{H}} : \widetilde{\mathrm{H}}_*];$$

this agrees with Lemma I.3.2.

4. In the case $r = 2$ this formula was given by Flicker [7] p. 141, but the factor $[\widetilde{\mathrm{H}} : \widetilde{\mathrm{H}}_*]$ has been inadvertently omitted.

Proof. — This is a generalization of [7] Lemma 2.1, and the proof is essentially the same. We shall merely note the modifications which have to be made. Firstly, if $h \in \mathrm{H}$ then the Jacobian of $\mathrm{N}_+ \to \mathrm{N}_+$; $n \mapsto nhn^{-1}h^{-1}$ is $\Delta(h)/\mu(h)$; thus the computation at the bottom of p. 141 of [7] can easily be generalized. The other point to note is that in [7] p. 142 the author has omitted the integration (in his terminology) over $\mathrm{A}_0\backslash\mathrm{A}$ and this accounts for the missing factor $[\widetilde{\mathrm{H}} : \widetilde{\mathrm{H}}_*]$.

Theorem **I.5.7.** — *Let* $n = 3$, $r = 2$ *and* $c \equiv -1 \pmod 3$. *Then*

$$\dim \mathrm{Wh}(\mathrm{V}_0(\omega)) = 1.$$

Proof. — In the sense of [7], Definition 1, p. 172 and Theorem 5.2 the representation $\mathrm{V}_0(\omega)$ corresponds to a one-dimensional representation of G, which, by computing its N_+^*-coinvariants by [7] Theorem 5.2 is seen to be χ where

$$\chi(h) = \omega(\mathbf{s}(h^n))\, \mu(h)^{-1} \qquad (h \in \mathrm{H}).$$

It is then easy to compute $\dim(\mathrm{Wh}(\mathrm{V}_0(\omega)))$ by Theorem I.5.3. This yields the result quoted.

Remarks. — Theorem I.5.7 is also a special case of Corollary II.2.6. The proof sketched here is quite different and appears to us to give different insights. We plan to return to the discussion of the generalization of the Shimura Correspondence as formulated in [7] and its consequences in a later publication.

I.6. Archimedean fields

In this section we shall discuss the analogues of the results obtained in the previous sections of this chapter for archimedean fields. We have three distinct cases to consider. These are

1. $n = 1$, $\mathrm{F} = \mathbf{R}, \mathbf{C}$
2. $n = 2$, $\mathrm{F} = \mathbf{R}$
3. $n > 1$, $\mathrm{F} = \mathbf{C}$.

In cases 1 and 3 $\widetilde{G}$ is isomorphic to $G \times \mu_n(F)$, and we need only review some well-known results. In case 2 the situation is a little more involved, but since $\widetilde{G}$ is a real reductive Lie group, locally isomorphic to G, it is also covered by the general theory.

We shall begin by defining the analogues of the $V(\omega')$ and discussing the intertwining operators. Then we shall consider the irreducibility of the representations, as in Theorem I.2.9. In cases 1 and 2 the results are analogous to those of § I.2; in case 3 the " relevant " $V(\omega')$ is actually irreducible. Finally we shall derive those results concerning Whittaker models which we shall later need.

We define H, N_+ as before, likewise $\widetilde{H}, \widetilde{H}_*, N_+^*$. In cases 1, 3 $\widetilde{H}_* = \widetilde{H} \cong H \times \mu_n(F)$, whereas in case 2 the index of $\widetilde{H}_*$ in $\widetilde{H}$ is $2^{[r/2]}$. We let $\widetilde{H}_n = p^{-1}\{h^n : h \in H\}$; this is $\widetilde{H}$ in cases 1 and 3, and of index 2^r in case 2. Let $\widetilde{Z}$ be the centre of $\widetilde{G}$ (and of $\widetilde{B}$); it is $p^{-1}\{\lambda I : \lambda^{r-1} \in F^{\times n}\}$. Thus in cases 1 and 3, $\widetilde{Z} = p^{-1}\{\lambda I : \lambda \in F^\times\}$, and in case 2

$$\widetilde{Z} = p^{-1}\{\lambda I : \lambda \in \mathbf{R}^\times\} \quad (r \text{ odd})$$
$$= p^{-1}\{\lambda I : \lambda \in \mathbf{R}_+^\times\} \quad (r \text{ even}).$$

The function μ can be defined as before; note, however, that we shall always take $|z|_{\mathbf{C}} = z\bar{z}$.

Now let ω be a quasicharacter of $\widetilde{H}_n \widetilde{Z}$, extending $\varepsilon : \mu_n(F) \to \mathbf{C}^\times$, and let ω' be an extension of ω as a quasicharacter to $\widetilde{H}_*$. We let $V(\omega')$ be the space of functions $f : \widetilde{G} \to \mathbf{C}$ which satisfy

(i) $$f(bg) = (\omega' \mu)(b) f(g) \quad (b \in \widetilde{B}_*(= \widetilde{H}_* N_+^*))$$

where we have extended ω' to $\widetilde{B}_*$ by requiring that $\omega' \mid N_+^* = 1$,

(ii) $g \mapsto f(g)$ is a smooth function.

Now we define, as before, $I_w : V(\omega') \to V({}^w\omega')$ by

$$(I_w f)(g) = \int_{N_+^*(w)} f(w^{-1} ng)\, dn$$

which, as we shall see below, converges when ω is dominant.

Let us also define ω_α^n and $\sigma(\omega)$ as before. We shall define, as usual, the L-function associated with a quasicharacter χ of $F^\times$ as follows:

$$\begin{aligned} L(\chi) &= \pi^{-\frac{s}{2}}\,\Gamma(s/2) && F = \mathbf{R}, && \chi(x) = |x|_{\mathbf{R}}^s, \text{ or} \\ & && && \quad\;\; = |x|_{\mathbf{R}}^s x^{-1} \\ &= (2\pi)^{1-s}\,\Gamma(s) && F = \mathbf{C} && \chi(x) = |x|_{\mathbf{C}}^s x^{-m}, \text{ or} \\ & && && \quad\;\; |x|_{\mathbf{C}}^s \bar{x}^{-m} \quad (m \geqslant 0). \end{aligned}$$

We can also define ω_t, f_t exactly as in § I.2.

The analogue to Proposition I.2.3 (a) is clearly valid. Consequently, as in § I.2, most of the problems involving I_w can be reduced to the corresponding problems in the case of $r = 2$. In particular, one has the following analogue to Proposition I.2.3 (b).

Proposition **I.6.1.** — *If* $w \in W$ *then the integral defining* I_w *converges if* $\langle \alpha, \sigma(\omega) \rangle > 0$ *for all* $\alpha > 0$ *such that* $w\alpha < 0$ *and the function* $t \mapsto I_w f_t$ *is analytic in the corresponding region. Moreover the function*

$$t \mapsto \prod_{\substack{\alpha > 0 \\ w\alpha < 0}} L((\omega_t\, \omega)^n_\alpha)^{-1}\, I_w(f_t)$$

can be extended from its original domain of definition, viz.

$$\{t :\ \operatorname{Re}(\langle \alpha, t \rangle) > -\langle \alpha, \sigma(\omega) \rangle, \quad \text{for } \alpha > 0,\ w\alpha < 0\},$$

to a holomorphic function on $\Phi(\mathbf{C})$.

We shall not give the proof in detail here; we remark merely that it reduces to the case where $w = s$, a simple reflection, in which case we can treat the integral directly in order to determine its poles. They are amongst those of $L((\omega_t\, \omega)^n_\alpha)$ and the holomorphy follows from this. One should note that the holomorphic function above has zeros.

We shall now formulate a version of Lemma I.2.5. As in § I.2 we can define $\alpha_s : V({}^s\omega') \to V(\omega')$, in the case that $\omega^n_\alpha = 1$ where α is the positive simple root corresponding to s.

Lemma **I.6.2.** — *Suppose that* s *is a simple reflection associated with the root* α. *Then, for* $f \in V(\omega')$, *one has*

$$\alpha_s(\lim_{t \to 0} L(|\ |_F(\omega_t\, \omega)^n_\alpha)\, L((\omega_t\, \omega)^n_\alpha)^{-1}\, I_s f_t) = A f$$

where

$$A = n^{-1}\, |n|_F\, A_0$$

and A_0 *is the measure of* $\widetilde{H}_*/\widetilde{H}^s_* \cap \widetilde{H}_*$ *used in the definition of* α_{s_0}.

Remark. — As usual here, we take the measure associated with dx on $\mathbf{R}$, and with $dz \wedge d\bar{z}$ on $\mathbf{C}$.

Proof. — We write ξ for the complex conjugate of ξ when $F = \mathbf{C}$, and $\bar{\xi} = \xi$ when $F = \mathbf{R}$. Then one readily verifies that

$$\lim_{t \to 0} L(|\ |_F(\omega_\alpha)^n)\, L((\omega_\alpha)^n)^{-1}\, I_s f_t(g)$$

is equal to

$$\lim_{t \to 0} L(|\ |_F(\omega_t)^n_\alpha)\, L((\omega_\alpha)^n)^{-1} \int_{|\xi|_F > 1} (\omega_t\, \mu)_\alpha (1 + \xi\bar{\xi})^{-1/2} f(\widetilde{h}^*_\alpha(\xi)\, \eta_0\, g)\, d\xi$$

where η_0 is as in § I.2 and $h^*_\alpha(\xi) = \widetilde{h}_\alpha(\xi/(\xi\bar{\xi})^{\frac{1}{2}})$, with $\widetilde{h}_\alpha$ as before.

When $F = \mathbf{R}$, we set $R_0 = \{1, -1\}$ and the limit becomes

$$n \sum_{x \in R_0} f(\widetilde{h}_\alpha(x)\, \eta_0\, g)$$

from which it follows that

$$\alpha_s(\lim_{t\to 0} \mathrm{L}(|\ |_{\mathrm{F}}(\omega_\alpha)^n)\,\mathrm{L}((\omega_\alpha)^n)^{-1}\,\mathrm{I}_s f_t)\,(g)$$

is equal to $\mathrm{A}_0 f(g)$ both when $n = 1$ and $n = 2$. This proves the lemma in this case.

When $\mathrm{F} = \mathbf{C}$ then $f(h_\alpha^*(\xi)\,\eta_0 g) = f(\eta_0 g)$ (as $\omega_\alpha^n = 1$) and then the limit becomes $nf(\eta_0 g)$. This yields, for $\alpha_s(\lim_{t\to 0} \mathrm{L}(|\ |_{\mathrm{F}}(\omega_t)^n)\,\mathrm{L}((\omega_t)^n)^{-1}\,\mathrm{I}_s f_t)\,(g)$, the value $n.f(\eta_0 g)$, as required.

As a consequence of Proposition I.6.1, $\prod_{\substack{\alpha>0\\ w\alpha<0}} \mathrm{L}(\omega_\alpha^n)^{-1}\,\mathrm{I}_w$ can be given a meaning for all ω; this we understand to be the regularization of I_w. Note that it involves a process of analytic continuation.

Since $\prod_{\substack{\alpha>0\\ w\alpha<0}} \mathrm{L}(\omega_\alpha^n)^{-1}\,\mathrm{I}_w$ is sometimes zero we need some finer information, which will be given by the analogue of Theorem I.2.6. We define

$$\mathrm{I}'_w = \prod_{\substack{\alpha>0\\ w\alpha<0}} (|n|_{\mathrm{F}}^{-\frac{1}{2}}\,\mathrm{L}(|\ |_{\mathrm{F}}\,\omega_\alpha^n)\,\mathrm{L}(\omega_\alpha^n)^{-1})\,\mathrm{I}_w$$

which is defined at generic ω but which does have singularities.

Theorem **I.6.3.** — $\mathrm{I}'_{w_1 w_2} = \mathrm{I}'_{w_1}\,\mathrm{I}'_{w_2}$.

Proof. — We begin by making some simplifications. First of all, as in the proof of Theorem I.2.6 we can restrict ourselves to proving that $\mathrm{I}'^2_s = \mathrm{I}$, where s is a simple reflection. Next, we note that when $\mathrm{F} = \mathbf{C}$

$$|n|_{\mathrm{F}}^{-\frac{1}{2}}\,\mathrm{L}(|\ |_{\mathrm{F}}\,\omega_\alpha^n)\,\mathrm{L}(\omega_\alpha^n)^{-1} = \mathrm{L}(|\ |_{\mathrm{F}}\,\omega_\alpha)\,\mathrm{L}(\omega_\alpha)^{-1}$$

where $\omega_\alpha(x) = \omega(h_\alpha(x))$; thus in this case we can assume $n = 1$.

It would be possible to mimic the proof of Theorem I.2.6, but it is easier here to exploit the theory of Eisenstein series as we did at the end of the proof of Theorem I.2.6.

Let us first deal with the case $\mathrm{F} = \mathbf{C}$. Since we can take $n = 1$ we choose an imaginary quadratic field k (eg. $\mathbf{Q}(\sqrt{-1})$). Then, since we did not need the theory of Eisenstein series to prove the functional equation at those places w with $|n|_w = 1$, we can assume that Theorem I.2.6 is valid for all finite places of k. Since the global functional equation is also known, its validity at the one archimedean place of k follows immediately.

The same proof, with $n = 1$, $k = \mathbf{Q}$, shows that the assertion is also valid for $\mathrm{F} = \mathbf{R}$, $n = 1$.

Now let k' be an imaginary quadratic field with $k' \otimes_{\mathbf{Q}} \mathbf{Q}_2 \cong \mathbf{Q}_2 \oplus \mathbf{Q}_2$. The proof of Theorem I.2.6 applies to this case and so we derive the validity of Theorem I.2.6 for $\mathrm{F} = \mathbf{Q}_2$, with $n = 2$. Now finally we can apply the argument above to $\mathbf{Q}$ to derive the validity of Theorem I.6.3 in the case $F = \mathbf{R}$, $n = 2$. The remaining cases of Theorem I.2.6 now follow.

We next call an ω as above *exceptional* if for all positive simple α one has $\omega_\alpha^n = |\ |_F$. We also let, as before, $V_0(\omega')$ be the image in $V({}^{w_0}\omega')$ of $V(\omega')$ under I'_{w_0}. We suppose here that ω' is exceptional. Then one has

Theorem **I.6.4.** — *The representation* $V_0(\omega')$ *is the unique irreducible subrepresentation of* $V({}^{w_0}\omega')$ *and the unique irreducible quotient representation of* $V(\omega')$. *Moreover*

a) *if* $n = 1$ *then* $V_0(\omega')$ *is the one-dimensional representation the restriction of which to* H *is* ${}^{w_0}\omega\mu$,

b) *if* $n = 2$, $F = \mathbf{R}$ *then* $V_0(\omega')$ *is a proper subrepresentation of* $V({}^{w_0}\omega')$ *and a proper quotient representation of* $V(\omega')$,

c) *if* $r \leqslant n$, $F = \mathbf{C}$, $V_0(\omega') \cong V(\omega') \cong V({}^{w_0}\omega')$.

Proof. — The initial statement follows from the general principles of [31]. Statement a) is well-known. To prove c) regard $\widetilde{G}$ as $G \times \mu_n(k)$. Then $V(\omega) \cong (W \otimes \chi) \times \varepsilon$ where χ is a one-dimensional representation and W is induced from the representation $\mu^{\frac{1}{n}}.\mu$ of H. Since, if $r \leqslant n$ there is no root α for which $(\mu^{\frac{1}{n}})_\alpha = |\ |_F$; it follows that W is a non-degenerate representation in a complementary series.

To prove b) we check that if s is a simple reflection associated with the simple root α, then the composite map

$$V({}^s\omega') \xrightarrow{\text{“}L(\omega_\alpha^n)^{-1} I_s\text{”}} V(\omega') \xrightarrow{I'_s} V({}^s\omega')$$

is 0. Thus, by Theorem I.6.3, the image of $V({}^s\omega')$ under I_s lies in the kernel of I'_{w_0}. (One observes that, by Proposition I.6.1, both maps are defined, that the composite is 0 follows from Theorem I.6.3.) On the other hand on examining the proof of Proposition I.6.1 one sees that “ $L(\omega_\alpha^n)^{-1} I_s$ ” is non-zero. Hence the kernel of I'_{w_0} is non-zero. From [31] it follows also that the image of $V(\omega')$ under I'_{w_0} is non-zero. Hence $V(\omega')$ is reducible.

Now we can discuss the Whittaker models of the $V_0(\omega')$. Let e be a non-degenerate character of N_+^*. Then if ω is dominant, $\eta \in \widetilde{H}$, we can define the linear functional λ_η on $V(\omega')$ by

$$\langle \lambda_\eta, f \rangle = \int_{N_+^*} \bar{e}(n) f(\eta w_0^{-1} n)\, dn.$$

If $h \in \widetilde{H}_*$ then one has

$$\lambda_{h\eta} = (\omega' \mu)(h)\, \lambda_\eta.$$

These define Whittaker functionals. Let, as before Wh(V) be the space of all Whittaker functionals on V.

Theorem **I.6.5.** — a) *When* $n = 1$, $\mathrm{Wh}(V_0(\omega')) = 0$.

b) *When* $n = 2$, $r = 2$, $F = \mathbf{R}$

$$\dim(\mathrm{Wh}(V_0(\omega'))) = 1.$$

c) *When* $r \leqslant n$, $F = \mathbf{C}$, *then* $V_0(\omega')$ *is isomorphic to* $V(\omega')$ *and*

$$Wh(V(\omega')) = \langle \lambda_I \rangle.$$

Proof. — *a)* is obvious in view of Theorem I.6.4.

b) is proved in [8] § 4.

c) is well-known (see, e.g. [44], Corollary to Theorem 2.2 which is much stronger).

Remark. — It would be possible to show, as in [44], that the λ_η can be regularized, and that these give rise to all of $Wh(V(\omega'))$. Then, by studying the effect to the intertwining operators I'_w on the λ_η (as in § I.3, in the p-adic case) one could show that, when $n = 2$, $r > 2$, $F = \mathbf{R}$, $Wh(V_0(\omega')) = 0$. We shall not need this result. More generally, if $r > n$ then $Wh(V_0(\omega')) = 0$.

We shall need however one particular result of this type when $F = \mathbf{C}$ and ω is arbitrary. This result is a special case of the functional equation proved by Jacquet in [22] (see also [43], [44]), so that we shall merely reformulate these known results in our terminology.

We let $\lambda = \lambda_I$. Then the function $t \mapsto \langle \lambda, f_t \rangle$ has a meromorphic extension to $\Phi(\mathbf{C})$; hence λ can be defined on a generic $V(\omega')$. If $w \in W$ then $f \mapsto \langle \lambda, I'_w f \rangle$ is also a Whittaker functional on $V(\omega')$. Since we know that $\dim(Wh(V(\omega'))) = 1$ this must be proportional to λ. We shall determine the factor of proportionality.

If χ is a quasicharacter of $\mathbf{C}^\times$, then $\chi(x)$ can be written either as $x^{-m} |x|_{\mathbf{C}}^t$ or $\bar{x}^{-m} |x|_{\mathbf{C}}^t$ where m is a positive integer. We define then

$$\varepsilon(\chi) = i^{-m}.$$

We fix the additive character e of $\mathbf{C}$ to be

$$e(z) = e^{2\pi i(z+\bar{z})}.$$

If ω is a quasicharacter of H we let, for $\alpha \in \Phi$,

$$\omega_\alpha(x) = \omega(h_\alpha(x)).$$

Moreover we define, for $\alpha \in \Phi$, $\eta_\alpha \in H$ to be $\mathrm{diag}(1, 1, \ldots, 1, -1, 1, \ldots, 1)$ where, if $\alpha = (ij)$, the -1 is in the i-th position. Then the functional equation is given by:

Theorem **I.6.6.** — *With the notations above,* $f \in V(\omega')$

$$\langle \lambda, I_w f \rangle = \prod_{\substack{\alpha > 0 \\ w\alpha < 0}} (\omega(\eta_\alpha)\, \varepsilon(\omega_\alpha)^{-1}\, L(\omega_\alpha)\, L(|\ |\, \omega_\alpha^{-1})^{-1})\, \langle \lambda, f \rangle.$$

Remark. — We could also deduce this from the " global " functional equation of Theorem II.1.3, the procedure being the inverse of that which we shall use in the proof of Corollary II.2.4. This suffices to determine the factor of proportionality up to a constant. (To complete the proof one reduces the evaluation of this constant to the case $r = 2$ and then makes a special choice of f.)

II. — GLOBAL THEORY

In this chapter we shall apply the local theory of Chapter I and the theory of Eisenstein series to construct an automorphic representation of the global metaplectic group $\widetilde{G}_{A}$ over a global field k with $\text{Card}(\mu_n(k)) = n$. We shall show that this representation is, in an appropriate sense, a tensor product of the " exceptional " representations of Theorem I.2.9 and § I.6.

In § II.1 we shall recall those aspects of the theory of Eisenstein series which we need. The automorphic representation referred to above is then described in Theorem II.2.1. Although this is of interest in itself one can only derive consequences of arithmetic interest after some further work. The automorphic forms have " Fourier coefficients " which may be described in two different ways.

Firstly, as we have represented the representation as a restricted tensor product, these Fourier coefficients are described by the local Whittaker models as in Theorem II.2.2. One consequence of this is that one can use global methods to complete the local theory in certain points. Thus, for example, one can show that if $r = n$ or if $r = n - 1$ and $2(c + 1) \equiv 0 \pmod n$ in our customary notations then all the local factors of the representation have a unique Whittaker model whereas we could only deduce the corresponding statement for almost all local factors from the local theory at our disposal. For this see Theorem II.2.5 and Corollary II.2.6.

The other approach is to observe that the Fourier coefficients can be expressed as residues of the corresponding Fourier coefficients of Eisenstein series. These, in their turn, are Dirichlet series, here denoted by Ψ or Ψ_f whose coefficients are typically Gauss sums.

Although we shall not discuss these functions in general here we shall investigate the special case $r = 2$ in § II.3. Thus our treatment of the arithmetic implications of the theory of metaplectic forms will be rather cursory, but our intention has been to stress the representation theoretic aspects here. Moreover we shall restrict our attention to the case of a number field k, although the case of a function field is also of interest, and is in some senses simpler since $|n|_v = 1$ for all places v of k.

The arithmetically interesting consequences follow from a comparison of these two approaches and are given in Theorem II.2.3, Corollary II.2.4 and in the consequences of Theorem II.3.3.

We conclude this introduction with a notational remark; we shall write $\widetilde{G}_{A}$ in place of $\widetilde{G}_{A}^{(c)}$, but the dependence on c should not be forgotten.

II.1. Eisenstein series on the metaplectic group

Let k be a global field satisfying $\mathrm{Card}(\mu_n(k)) = n$, and let $\widetilde{G}_{\mathbf{A}}$ be constructed as in § 0.2. Let $\widetilde{H}_{n,\mathbf{A}}$ (resp. $\widetilde{Z}_{\mathbf{A}}$) be the subgroup each of whose local components lies in $\widetilde{H}_{n,k_v}$ (resp. $\widetilde{Z}_{k_v}$). Let G_k^*, H_k^* be the images of G_k, H_k under the lifting described in § 0.2.

Lemma **II.1.1.** — *$H_k^* \widetilde{H}_{n,\mathbf{A}} \widetilde{Z}_{\mathbf{A}}$ is a maximal abelian subgroup of $\widetilde{H}_{\mathbf{A}}$.*

Proof. — Recall that, relative to the n-th order Hilbert symbol on $k_{\mathbf{A}}^\times$ the subgroup $k^\times k_{\mathbf{A}}^{\times n}$ is maximal isotropic; cf. [50] XIII-5, Prop. 8. The assertion then follows from Proposition 0.3.1.

Next for each place v of k we choose a maximal abelian subgroup $\widetilde{H}_{*,v}$ of $\widetilde{H}_v$ which, at all but a finite number of places, is the one appearing in the discussion preceeding Lemma I.1.1. Then let $\widetilde{H}_{*,\mathbf{A}}$ be the subgroup of $\widetilde{H}_{\mathbf{A}}$ all of whose components lie in the corresponding $\widetilde{H}_{*,v}$. This $\widetilde{H}_{*,\mathbf{A}}$ is a second maximal abelian subgroup of $\widetilde{H}_{\mathbf{A}}$; the material in § 0.3 was introduced chiefly in order to compare it with $H_k^* \widetilde{H}_{n,\mathbf{A}} \widetilde{Z}_{\mathbf{A}}$.

Let ω be a quasicharacter of $\widetilde{H}_{n,\mathbf{A}} \widetilde{Z}_{\mathbf{A}}$ which is trivial on $H_k^* \cap (\widetilde{H}_{n,\mathbf{A}} \widetilde{Z}_{\mathbf{A}})$, and let ω_0 be the extension of ω to $H_k^* \widetilde{H}_{n,\mathbf{A}} \widetilde{Z}_{\mathbf{A}}$ which is trivial on H_k^*. Let ω_* be any extension of ω to $\widetilde{H}_{*,\mathbf{A}}$. As in the local case we shall fix an injective homomorphism $\varepsilon : \mu_n(k) \to \mathbf{C}^\times$, and we shall only consider those ω for which $\omega \circ i = \varepsilon$.

We remark also that there exist $\sigma(\omega)$ in $\Phi(\mathbf{R})$, $\tau \in \mathbf{R}$ so that, if $\sigma(\omega) = \Sigma\, \sigma_\alpha(\omega).\alpha$ then

$$|\omega(h)| = \prod_\alpha \| p_{\mathbf{A}}(h)^\alpha \|_{\mathbf{A}}^{\sigma_\alpha(\omega)} . \| \det(p_{\mathbf{A}}(h)) \|_{\mathbf{A}}^\tau$$

where $\| \ \|_{\mathbf{A}} : k_{\mathbf{A}}^\times \to \mathbf{R}_+^\times$ is the idele norm.

We observe next that, as the map $p_{\mathbf{A}}$ in

$$\begin{array}{ccccccccc} (1) & \longrightarrow & \mu_n(k) & \xrightarrow{i} & \widetilde{G}_{\mathbf{A}} & \xrightarrow{p_{\mathbf{A}}} & G_{\mathbf{A}} & \longrightarrow & (1) \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & G_k & & \end{array}$$

is a finite covering map, then the usual consequences of reduction theory ([4], [15]) hold also for the discrete subgroup G_k^* of $\widetilde{G}_{\mathbf{A}}$; see for example the remark in the proof of Theorem II.1.3. In particular the assumptions necessary for the development of the theory of Eisenstein series in [16], [29], [36] are fulfilled.

Let $\mu_{\mathbf{A}}(h) = \| p_{\mathbf{A}}(h)^\rho \|_{\mathbf{A}}$ where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

The theory of Eisenstein series begins by constructing (analogously to the $(\pi(\omega), V(\omega))$ of Chapter I) representations (π_ω, V_ω) of $\widetilde{G}_{\mathbf{A}}$ which are realized in a space of slowly increasing functions on $G_k^* \backslash \widetilde{G}_{\mathbf{A}}$. The Eisenstein series form the intertwining operator from the standard representation into this space. They are first defined when $\sigma(\omega) - \rho$ lies in the dominant Weyl chamber, this definition is then extended to almost all ω by analytic continuation (due to Roelcke, Selberg and Langlands).

To be able to express these results we must first describe the analytic structure on the set of all ω. To do this we define for $s \in \Phi(\mathbf{C})$ the quasicharacter ω_s of $\widetilde{\mathrm{H}}$ by

$$\omega_s(h) = \prod_{\alpha \in \Phi} ||p_{\mathbf{A}}(h)^\alpha||_{\mathbf{A}}^{s(\alpha)} \quad \text{if } s = \sum_{\alpha \in \Phi} s(\alpha)\,\alpha.$$

This is trivial on $i(\mu_n(k))$ and H_k^*. We let $\Omega_\omega = \{\omega_s\, \omega : s \in \Phi(\mathbf{C})\}$ where ω is as above, then the map $\Phi(\mathbf{C}) \to \Omega_\omega$; $s \mapsto \omega_s\, \omega$ defines an analytic structure on Ω_ω. We regard Ω_ω as the component of ω in the set Ω of all quasicharacters $\omega : \widetilde{\mathrm{H}}_{n,\mathbf{A}}\, \widetilde{\mathrm{Z}}_{\mathbf{A}} \to \mathbf{C}^\times$, with $\omega \circ i = \varepsilon$, ω trivial on $\mathrm{H}_k^* \cap (\widetilde{\mathrm{H}}_{n,\mathbf{A}}\, \widetilde{\mathrm{Z}}_{\mathbf{A}})$.

Let $\widetilde{\mathrm{K}}_{\mathbf{A}} = p_{\mathbf{A}}^{-1}(\mathrm{K}_{\mathbf{A}})$ where $\mathrm{K}_{\mathbf{A}}$ is the standard maximal compact subgroup of $\mathrm{G}_{\mathbf{A}}$.

Let $\mathrm{K}_{\mathbf{A},\infty}$ (resp. $\mathrm{K}_{\mathbf{A},f}$) be the product of the archimedean (resp. non-archimedean) components of $\mathrm{K}_{\mathbf{A}}$. Then there is a natural product decomposition

$$\mathrm{K}_{\mathbf{A}} = \mathrm{K}_{\mathbf{A},\infty} \times \mathrm{K}_{\mathbf{A},f}.$$

Corresponding to this one obtains a product decomposition of $\widetilde{\mathrm{K}}_{\mathbf{A}}$,

$$\widetilde{\mathrm{K}}_{\mathbf{A}} = \widetilde{\mathrm{K}}_{\mathbf{A},\infty} \times_{\mu_n(k)} \widetilde{\mathrm{K}}_{\mathbf{A},f},$$

where $\widetilde{\mathrm{K}}_{\mathbf{A},\infty}$ (resp. $\widetilde{\mathrm{K}}_{\mathbf{A},f}$) is a $\mu_n(k)$-covering of $\mathrm{K}_{\mathbf{A},\infty}$ (resp. $\mathrm{K}_{\mathbf{A},f}$). We therefore have a map from the usual product to $\widetilde{\mathrm{G}}_{\mathbf{A}}$,

$$\widetilde{\mathrm{K}}_{\mathbf{A},\infty} \times \widetilde{\mathrm{K}}_{\mathbf{A},f} \to \widetilde{\mathrm{G}}_{\mathbf{A}}; \qquad (k_\infty, k_f) \mapsto (k_\infty \times k_f)$$

(this defines the notation to be used).

We call a function $f : \widetilde{\mathrm{G}}_{\mathbf{A}} \to \mathbf{C}$ *right* $\widetilde{\mathrm{K}}_{\mathbf{A}}$*-smooth* if

a) for all $g \in \widetilde{\mathrm{G}}_{\mathbf{A}}$ the function

$$\widetilde{\mathrm{K}}_{\mathbf{A},\infty} \to \mathbf{C}; \qquad k \mapsto f(g(k_\infty \times \mathrm{I}))$$

is smooth, and

b) there exists an open subgroup L of $\widetilde{\mathrm{K}}_{\mathbf{A},f}$ so that for all $g \in \widetilde{\mathrm{G}}_{\mathbf{A}}$ one has

$$f(g(\mathrm{I} \times k_f)) = f(g) \qquad (k_f \in \mathrm{L}).$$

Naturally this definition does not depend on the choice of $\mathrm{K}_{\mathbf{A}}$.

For each $\omega \in \Omega$ let $\mathscr{F}_0(\omega)$ be the space of right $\widetilde{\mathrm{K}}_{\mathbf{A}}$-smooth functions f_0 on $\widetilde{\mathrm{G}}_{\mathbf{A}}$ which satisfy

$$f_0(bg) = (\omega_0\, \mu_{\mathbf{A}})(b)\, f_0(g) \qquad (b \in (\mathrm{H}_k^*\, \widetilde{\mathrm{H}}_{n,\mathbf{A}}\, \widetilde{\mathrm{Z}}_{\mathbf{A}})\, \mathrm{N}_{+,\mathbf{A}}^*)$$

(where $\omega_0\, \mu_{\mathbf{A}}$ is taken to be trivial on $\mathrm{N}_{+,\mathbf{A}}^*$). There is a trivial holomorphic vector bundle $\mathscr{F}_0$ over Ω with fibre $\mathscr{F}_0(\omega)$ at ω. We shall also have occasion to speak of the space $\mathscr{F}_0(\mathrm{U})$ of holomorphic sections over an open subset $\mathrm{U} \subset \Omega$.

Let now ω_* be an extension of ω to $\widetilde{\mathrm{H}}_{*,\mathbf{A}}$; as

$$((\mathrm{H}_k^*\, \widetilde{\mathrm{H}}_{n,\mathbf{A}}\, \widetilde{\mathrm{Z}}_{\mathbf{A}}) \cap \widetilde{\mathrm{H}}_{*,\mathbf{A}})/\widetilde{\mathrm{H}}_{n,\mathbf{A}}\, \widetilde{\mathrm{Z}}_{\mathbf{A}}$$

is finite we can assume that $\omega_* = \omega_0$ on $(H_k^* \widetilde{H}_{n,A} \widetilde{Z}_A) \cap \widetilde{H}_{*,A}$. Let $\mathscr{F}_*(\omega_*)$ be the space of right $\widetilde{K}_A$-smooth functions f_* on $\widetilde{G}_A$ which satisfy

$$f_*(bg) = (\omega_* \mu_A)(b) f_*(g) \qquad (b \in \widetilde{H}_{*,A}.N^*_{+,A}),$$

just as above one can form a vector bundle $\mathscr{F}_*$ and so on if we choose ω_* so that $(\omega_s \omega)_* = \omega_s \omega_*$.

From § 0.3 one sees that there are inverse isomorphisms $\mathscr{F}_0(\omega_0) \to \mathscr{F}_*(\omega_*)$, $\mathscr{F}_*(\omega_*) \to \mathscr{F}_0(\omega_0)$. To simplify notations we let f_0, f_* be functions in $\mathscr{F}_0(\omega_0)$ and $\mathscr{F}_*(\omega_*)$ respectively which correspond under these isomorphisms. We recall that

$$f_*(g) = \int_{\widetilde{H}_{n,A} \widetilde{Z}_A \backslash \widetilde{H}_{*,A}} (\omega_* \mu_A)(h)^{-1} f_0(hg)\, dh$$

and

$$f_0(g) = \sum_{h \in H^*_{n,k} Z^*_k \backslash H^*_k} f_*(hg).$$

Notice that from its definition, f_* can be written as a finite sum of functions of the form $\bigotimes f_{*,v}(g_v)$ where $g = (g_v) \in \widetilde{G}_A$ in the sense discussed in § 0.3, and $f_{*,v}(g_v)$ satisfies

$$f_{*,v}(b_v g_v) = (\omega_{*,v} \mu_v)(b_v) f_{*,v}(g_v)$$

(*i.e.* $f_{*,v} \in V(\omega_{*,v})$). Note that at almost all v, $f_{*,v}$ is the K_v^*-invariant vector with $f_{*,v}(I) = 1$. Also, $\omega_{*,v}$ is the v-th component of ω_*, and μ_v is μ as in Chapter I with $F = k_v$.

Now we can define the Eisenstein series themselves; if $f_0 \in \mathscr{F}_0(\omega)$ let

$$E(g, \omega, f) = \sum_{\gamma \in B_k^* \backslash G_k^*} f_0(\gamma g).$$

This converges absolutely if $\sigma(\omega) - \rho$ lies in the dominant Weyl chamber ([16], [29]). The convergence is locally uniform and if $U \subset \{\omega \in \Omega : \sigma(\omega) - \rho$ lies in the dominant Weyl chamber$\}$, $f_0 \in \mathscr{F}_0(U)$, then

$$E(\cdot, \cdot, f_0) : U \to \text{Slowly increasing functions on } G_k^* \backslash \widetilde{G}_A$$

is analytic.

One of the major achievements of the theory of Eisenstein series is the proof that E has an analytic continuation as a meromorphic function, in the sense that E is defined as a meromorphic function on any $U \times \mathscr{F}_0(U)$ (with values in the space of slowly increasing functions on $G_k^* \backslash \widetilde{G}_A$) with the obvious compatibility under restriction from U to $U_1 \subset U$ and which agrees with the original class of U. The analytic properties of E can be described quite accurately. As we shall need a few of the intermediate results we shall sketch some of the important features of the theory. The first concept which we need is that of the constant term.

Normalize the additive Haar measure on k_A so that it is selfdual with respect to a character e_0 trivial on k, [50] VII-2); then $\text{meas}(k \backslash k_A) = 1$ (*loc. cit.* Cor. 2 to Thm. 1). We can assume that this measure is a product measure formed from the local Haar measure self-dual with respect to the v-th component $e_{0,v}$ of e_0. Other measures on adelic groups will be those given be the Tamagawa construction.

Proposition **II.1.2.** — *Let notations be as above. Then*

$$\int_{N^*_{+,k}\backslash N^*_{+,A}} E(ng, \omega, f_0)\, dn = \sum_{w \in W} I^0_w(f_0)(g)$$

where $\quad I^0_w : \mathscr{F}_0(\omega) \to \mathscr{F}_0({}^w\omega)$

is defined by

$$I^0_w(f_0)(g) = \int_{N^*_+(w)_A} f_0(w^{-1} ng)\, dn.$$

Moreover if $I^*_w : \mathscr{F}_*(\omega_*) \to \mathscr{F}_*({}^w\omega_*)$ *is the corresponding map, and if* $f = \bigotimes f_{*,v}$, *then* $I^*_w f = \bigotimes (I_{w,v} f_{*,v})$, *where* $I_{w,v}$ *is, with* $F = k_v$, *the operator introduced in* § I.2 *or* § I.6.

Proof. — The proof is based on the Bruhat decomposition of G^*_k, viz.

$$G^*_k = \bigcup_{w \in W} B^*_k\, w^{-1} N^*_+(w)_k.$$

Since $\quad E(g, \omega, f_0) = \sum_{\gamma \in B^*_k \backslash G^*_k} f_0(\gamma g)$

we have immediately that

$$\int_{N^*_{+,k}\backslash N^*_{+,A}} E(ng, \omega, f_0)\, dn = \sum_w \int_{N^*_{+,k}\backslash N^*_{+,A}} \sum_{n' \in N^*_+(w)_k} f_0(w^{-1} n' ng)\, dn.$$

On factoring the integral into two parts we see that the right-hand side is equal to

$$\sum_{w \in W} \int_{N^*_+[w]_k \backslash N^*_+[w]_A} \int_{N^*_+(w)_A} f_0(w^{-1} n_1 n_2 g)\, dn_2 . dn_1$$

where $N^*_+[w]$ corresponds to those $\alpha \in \Phi^+$ such that $w^{-1}\alpha > 0$ (as $N_+(w)$ corresponds to those $\alpha \in \Phi^+$ such that $w^{-1}\alpha < 0$). Since f_0 is left $N^*_{+,A}$-invariant, and as dn_1 is a product of self-dual measures, this is

$$\sum_{w \in W} \int_{N^*_+(w)_A} f_0(w^{-1} ng)\, dn.$$

This proves the first assertion of the proposition. The second statement, the translation in terms of $\mathscr{F}_*$, is formal. This proves the proposition.

Now let $\alpha \in \Phi$, $\alpha = (ij)$; we let $H^n_\alpha : k^\times_A \to \widetilde{H}_A$ be the homomorphism such that $H^n_\alpha(\xi) = \mathbf{s}(\mathrm{diag}(1, \dots, 1, \xi^n, \dots, \xi^{-n}, \dots, 1))$ where ξ^n sits at the i-th place, ξ^{-n} at the j-th. Then we let $\omega^n_\alpha = \omega \circ H^n_\alpha$; this is a Grössencharacter, which is to say, a quasi-character of $k^\times_A$ trivial on $k^\times$. If v is any place of k we let $L_v(\omega^n_\alpha)$ be the corresponding L-function, and $L(\omega^n_\alpha) = \prod L_v(\omega^n_\alpha)$.

We now redefine I_w by setting, when $f_* = \bigotimes_v f_{*,v}$,

$$I^*_w(f_*) = \{ \prod_{\substack{\alpha > 0 \\ w\alpha < 0}} L(\omega^n_\alpha)\, L(\|\cdot\|_A\, \omega^n_\alpha)^{-1} \} \bigotimes_v [L_v(\|\ \|_A\, \omega^n_\alpha)\, L_v(\omega^n_\alpha)^{-1} . I_{w,v}(f_{*,v})].$$

The expression in [ ] is, by Proposition I.2.4, the canonical K^*-invariant vector at almost all places. Thus this operator has an analytic continuation as a meromorphic operator since the L-functions have analytic continuations as meromorphic functions.

Let

$$\Omega_+ = \{\omega \in \Omega : \langle \sigma(\omega), \alpha \rangle \geqslant 0 \quad (\alpha \in \Phi^+)\}$$

and

$$\Omega_+^0 = \{\omega \in \Omega : \langle \sigma(\omega), \alpha \rangle > 0 \quad (\alpha \in \Phi^+)\}.$$

Note that the only singularities of $I_w^*(f_*)$ in Ω_+^0 lie on the "hyperplanes"

$$T_\alpha = \{\omega : \omega_\alpha^n = \|\ \|_A\} \quad (\alpha \in \Phi^+).$$

Theorem **II.1.3.** — *The operator* E *can be analytically continued to a meromorphic operator on* Ω *in the sense explained above, and satisfies the following functional equations*

$$E(g, \omega, f_0) = E(g, {}^w\omega, I_w^0(f_0)) \quad (w \in W),$$

and

$$I_{ww'}^0 = I_w^0 I_{w'}^0.$$

Let U *be an open subset of* Ω, $U \supset \Omega_+$ *and let* $f_0 \in \mathscr{F}_0(U)$. *Then the function*

$$\omega \mapsto \prod_{\alpha > 0} L(\|\ \|_A \omega_\alpha^n)\, E(g, \omega, f_0)$$

in analytic on $\Omega_+ - \bigcup_{\alpha > 0} T_\alpha$.

Proof. — This follows from the general results of [16], [29], [36]; we shall first discuss the analytic continuation and functional equation.

In the case that k is an algebraic number field we need only remark that if K is a compact open subgroup of $p_A^{-1}(\prod_{v \nmid \infty} G_v)$ then $G_k^* \backslash \widetilde{G}_A / \widetilde{Z}_A K$ can be covered by a finite number of spaces of the form $\Gamma \backslash \prod_{v | \infty} (\widetilde{G}_v / \widetilde{Z}_v)$, where Γ is arithmetic: thus the assumptions of [29] are satisfied. When k is a function field the arguments of [36] can be modified, as L. Morris indicated to us. We do observe, however, that since in this case the analytic continuation of the I_w^0 is known, the analytic continuation follows from the "principle of the constant term" ([29], Lemma 6.2). This leads to substantially simpler proofs in our case. The functional equations follow from the Maass-Selberg relations, which will be discussed in the course of the proof of Theorem II.1.4. They are essentially equivalent to the local functional equations (Theorems I.2.6 and I.6.3), along with the functional equation of Hecke-Tate.

It will be useful to make this a little more explicit. Let us recall the functional equation of Hecke-Tate; it is

$$L(\omega) = \varepsilon(\omega)\, L(\|\ \|_A \omega^{-1})$$

where, with a choice of e_0 as above, $\varepsilon(\omega)$ has the form $\prod_v \varepsilon_v(\omega_v, e_{0,v})$ where the ε-functions are monomials and are given in [23] pp. 104-5, 194-5, or, essentially, [50] VII-7.

We consider instead of the I_w^0, the I_w^*. Then if $f_* = \bigotimes_v f_{*,v} \in \mathcal{F}_*(\omega_*)$, we can write

$$I_w^* f_* = \prod_{\substack{\alpha>0\\ w\alpha<0}} \{L(\omega_\alpha^n)\, L(|\ |_A\, \omega_\alpha^n)^{-1}\} \bigotimes_v [L_v(|\ |_A\, \omega_\alpha^n)\, L_v(\omega_\alpha^n)^{-1} I_{w,v}(f_{*,v})]$$

$$= \prod_{\substack{\alpha>0\\ w\alpha<0}} \{L(|\ |_A(\omega_\alpha^n)^{-1})\, L(|\ |_A\, \omega_\alpha^n)^{-1}\}$$

$$\bigotimes_v [|n|_v^{-\frac{1}{2}} \varepsilon_v(\omega_\alpha^n, e_{0,v})\, L_v(|\ |_A\, \omega_\alpha^n)\, L_v(\omega_\alpha^n)^{-1} I_{w,v}(f_{*,v})].$$

If we write $I''_{w,v}(f_v)$ for

$$\prod_{\substack{\alpha>0\\ w\alpha<0}} (|n|_v^{-\frac{1}{2}} \varepsilon_v(\omega_\alpha^n, e_{0,v})\, L_v(|\ |_A\, \omega_\alpha^n)\, L_v(\omega_\alpha^n)^{-1})\, I_{w,v}(f_{*,v})$$

then one verifies easily that, equivalently to Theorems I.2.6 and I.6.3, one has

$$I''_{w_1 w_2, v} = I''_{w_1, v}\, I''_{w_2, v}$$

and the global functional equation follows from these local ones. This, however, is not a proof since we used the global functional equations in the proofs of the local ones, but it serves to demonstrate the equivalence.

Now that we have made these remarks we can return to the study of the singularities of

$$\prod_{\alpha>0} L(|\ |_A\, \omega_\alpha^n)\, E(g, \omega, f_0).$$

By the principle of the constant term these are determined by the singularities of the constant term. We shall write the constant term of the above function in terms of f_*, which we assume to have the form $\bigotimes_v f_{*,v}$; it is

$$\sum_{w\in W} \prod_{\substack{\alpha>0\\ w\alpha>0}} L(|\ |_A\, \omega_\alpha^n) \prod_{\substack{\alpha>0\\ w\alpha<0}} L(|\ |_A\, (\omega_\alpha^n)^{-1}) \bigotimes_v (I''_{w,v} f_{*,v}).$$

Since $L(\chi)$ is holomorphic except where $\chi = 1, |\ |_A$, we see that the singularities in Ω_+ lie in the sets $\omega_\alpha^n = |\ |_A, 1$ $(\alpha \in \Phi^+)$. The content of the assertion of the theorem is that there is (essentially) no singularity along $\omega_\alpha^n = 1$.

We shall first show that it suffices to prove that the constant term is analytic at those $\omega \in \Omega_+ - \bigcup_\alpha T_\alpha$ for which there exists exactly one $\beta \in \Phi^+$ with $\omega_\beta^n = 1$; such a β is necessarily simple. Consider a singular point

$$\omega_1 \in \Omega_+ - (\Omega_+^0 \cup \bigcup_\alpha T_\alpha).$$

It follows from the functional equation and Proposition I.2.3 *b)* that the divisor of the singularity is $\mathrm{Stab}_W(\omega_1)$-invariant. Since the divisor is disjoint from $\Omega_+^0 - \bigcup_\alpha T_\alpha$ it follows that it must be a union of some of the hyperplanes

$$\{\omega : \omega_\alpha^n = 1\} \quad (\alpha \in \Phi^+)$$

which pass through ω_1. It therefore suffices to show that the constant term is analytic at those points of $\Omega_+ - \Omega_+^0$ described above.

We shall therefore consider a positive simple root α_1 with associated reflection s_1 and show that at $\omega \in \Omega_+ - \bigcup\limits_{\alpha} T_\alpha$ with $\omega_{\alpha_1}^n = 1$ but $\omega_\alpha^n \neq 1$ $(\alpha \in \Phi^+ - \{\alpha_1\})$ the constant term is analytic. The constant term consists of a sum over W; we shall show that the sum of the terms corresponding to w and ws_1 is analytic at ω. To see this observe that $\{\alpha > 0; ws_1\alpha < 0\}$ differs from $\{\alpha > 0; w\alpha < 0\}$ by exactly α_1. Thus, as $\omega_{\alpha_1}^n \to 1$ the products of L-functions differ by one factor, which has a pole, and in the two terms the residue of one is just the negation of the residue of the other. Thus we have to show that

$$\bigotimes_v I''_{ws_1,v} f_{*,v} - \bigotimes_v I''_{w,v} f_{*,v} = 0,$$

or, equivalently,

$$\bigotimes_v I''_{s_1,v} f_{*,v} = \bigotimes_v f_{*,v}.$$

Now recall that ω_* was an extension of ω; the assumption that $\omega_{\alpha_1}^n = 1$ is equivalent to ${}^{s_1}\omega = \omega$. Thus ${}^{s_1}\omega_*$ is a second extension of ω_*. As we have seen there is locally an isomorphism $\alpha_{s_1,v} : V({}^{s_1}\omega_{*,v}) \to V(\omega_{*,v})$ (see the discussion leading up to Lemma I.2.5). These yield an isomorphism $\alpha_{s_1} : \mathscr{F}_*({}^{s_1}\omega_*) \to \mathscr{F}_*(\omega_*)$, and one verifies from the construction of the maps involved that the diagram

$$\begin{array}{ccc} \mathscr{F}_0(\omega) & \xleftarrow{\ \mathrm{Id}\ } & \mathscr{F}_0({}^{s_1}\omega) = \mathscr{F}_0(\omega) \\ \Big\downarrow & & \Big\downarrow \\ \mathscr{F}_*(\omega_*) & \xleftarrow{\ \alpha_{s_1}\ } & \mathscr{F}_*({}^{s_1}\omega_*) \end{array}$$

commutes. Thus we have to show that

$$\alpha_s(\otimes(I''_{s_1,v} f_{*,v})) = \bigotimes_v f_{*,v}$$

where now the equation takes place inside $\mathscr{F}_*(\omega_*)$. But by Lemmas I.2.5 and I.6.2 the left-hand side is a positive multiple of the right-hand side. Moreover, as $I_{s,v}^2 = 1$ it follows that this multiple can only be 1. Hence we have proved the equality, and with it, the theorem.

The final result of this section is a simple, but important consequence of the Maass-Selberg relations. We call ω *exceptional* if $\omega_\alpha^n = \| \ \|_A$ for all positive, simple roots α. At such points the Eisenstein series have their greatest singularity. Let ω be exceptiona and let

$$\theta(g, f_0) = \lim_{\omega' \to \omega} \left(\prod_{\alpha > 0} L((\omega')_\alpha^n)^{-1}\right) E(g, \omega', f_0).$$

This depends on the choice of ω, and we shall write $\theta(g,\omega,f_0)$ if it is necessary to make the choice of ω explicit.

By Proposition II.1.2

$$\int_{N^*_{+,k}\backslash N^*_{+,A}} \theta(ng,f_0)\,dn = \widetilde{I}_{w_0}(f_0,\omega)(g)$$

where $$\widetilde{I}_{w_0}(f_0,\omega) = \lim_{\omega'\to\omega}\big(\prod_{\alpha>0} L((\omega')^n_\alpha)^{-1}\big)\, I^0_{w_0}(f_0,\omega').$$

If we express this in terms of f_*, as usual taken to be in the form $\bigotimes_v f_{*,v}$, then

$$\widetilde{I}_{w_0}(f,\omega') = Z_k(2)^{-(r-1)} Z_k(3)^{-(r-2)} \dots Z_k(r)^{-1} \bigotimes_v I''_{w_0,v}(f_{*,v}),$$

where $$Z_k(s) = L(|\,|\,|^s_A).$$

Note that if $\omega^{(1)}$ and $\omega^{(2)}$ are exceptional then $\omega^{(1)}.(\omega^{(2)})^{-1}$ is of the form $\chi\circ\det\circ p_A$ where χ is a Grössencharakter. Thus we can alter a given ω by such a character so that $\omega\mid\widetilde{Z}_A$ is unitary. This we shall do, and fix such an exceptional ω^1 such that $\omega^1\mid\widetilde{Z}_A$ is unitary. Then the result which we referred to above is the inner product formula in the following theorem.

Theorem **II.1.4.** — *With the notations above, if* $\omega^1\mid\widetilde{Z}_A$ *is unitary*

$$\int_{G^*_k\backslash\widetilde{G}_A/\widetilde{Z}_A} \theta(g,f_{1,0})\,.\,\overline{\theta(g,f_{2,0})}\,dg$$

is equal to

$$c\int_{H^*_k\backslash\widetilde{H}_A/\widetilde{Z}_A}\int_{N^*_{+,A}} f_{1,0}(\eta w_0 n)\,\overline{\widetilde{I}_{w_0}(f_{2,0})(\eta w_0 n)}\,dn\,d\eta,$$

where c *is a non-zero constant which depends on the choices of Haar measures.*

Remark. — We observe, before beginning the proof, that under the assumptions of the theorem,

$$^{w_0}\omega(h)^{-1} = \overline{\omega(h)} \qquad (h\in H^*_k\,\widetilde{H}_{n,A}\,\widetilde{Z}_A)$$

as one easily verifies. Thus the integral is well-defined.

Proof. — The proof is an adaption of the technique used by Langlands in [30]. We must begin by developing some general principles. Let ω_s $(s\in\Phi(\mathbf{C}))$ be as above. If $\alpha_1,\dots,\alpha_{r-1}$ are the positive simple roots in the usual order we can write $s=\sum s_j\alpha_j$. We then let ds be the differential form

$$ds_1\wedge ds_2\wedge\dots\wedge ds_{r-1}$$

on $\Phi(\mathbf{C})$. From now on, we take k to be a number field, leaving the modifications for the function field case to the reader.

Fix now a quasicharacter χ on $\widetilde{Z}_A$, trivial on $\widetilde{Z}_A \cap H_k^*$, and consider the space Ψ of functions on $\widetilde{G}_A$ which satisfy the following conditions:

if $\psi \in \Psi$ then

(i) $\psi(hng\, z) = \chi(z)\, \psi(g) \quad (h \in H_k^*,\, n \in N_A^*,\, z \in \widetilde{Z}_A)$
(ii) ψ is right $\widetilde{K}_A$-finite,
(iii) ψ has a representation of the form

$$\psi(g) = \sum_{\widetilde{\omega}} (2\pi i)^{-(r-1)} \int_{(\sigma)} \hat{\psi}(g, \widetilde{\omega}\omega_s)\, ds \quad (\sigma \in \Phi(\mathbf{R}))$$

where (a) $\widetilde{\omega}$ is a quasicharacter of $H_k^* \widetilde{H}_{n,A} \widetilde{Z}_A$, trivial on H_k^* and

$$\hat{\psi}(hng, \widetilde{\omega}) = \widetilde{\omega}\mu_A(h)\, \hat{\psi}(g, \widetilde{\omega}) \quad (h \in H_k^* \widetilde{H}_{n,A} \widetilde{Z}_A,\, n \in N_{+,A}^*),$$

b) if Ω is the set of quasicharacters satisfying *a)* and $\Omega_C = \{\omega_s : s \in \Phi(\mathbf{C})\}$ then (noting that Ω is a Ω_C-homogenous space), the $\widetilde{\omega}$-sum is over a set of representatives of $\Omega_C \backslash \Omega$,

c) the function $s \mapsto \psi(g, \widetilde{\omega}\omega_s)$ is analytic and there exists $B > r$ such that, for any norm $||\cdot\cdot||$ on $\Phi(\mathbf{C})$, given a compact set $C \subset \Phi(\mathbf{R})$ one has for $\mathrm{Re}(s)\ (= \mathrm{Re}(\omega_s)) \in C$

$$\hat{\psi}(g, \widetilde{\omega}\omega_s) \ll (1 + ||s||)^{-B};$$

moreover $\hat{\psi}(\cdot, \widetilde{\omega}) = 0$ for $\widetilde{\omega}$ lying in all but finitely many elements of $\Omega_C \backslash \Omega$.

d) $\int_{(\sigma)}$ indicates that the integral is taken over the product of lines $\mathrm{Re}(s) = \sigma$ given the usual orientation.

Before we continue, some remarks on this rather lengthy definition are in order. The fundamental part of the definition is given in (i), (ii). Parts *a)* and *b)* of (iii) state that ψ is a Fourier transform with respect to $H_k^* \backslash H_k^* \widetilde{H}_{n,A} \widetilde{Z}_A / \widetilde{Z}_A$ and *c)* imposes fairly stringent regularity conditions which, although they could be relaxed, suffice for our purposes. Note that, if the Haar measures are suitably chosen

$$\hat{\psi}(g, \omega) = \int_{H_k^* \backslash H_k^* \widetilde{H}_{n,A} \widetilde{Z}_A / \widetilde{Z}_A} (\omega\mu_A)^{-1}(h)\, \psi(hg)\, dh.$$

Given $\psi \in \Psi$ we define

$$E(g, \psi) = \sum_{\gamma \in B_k^* \backslash G_k^*} \psi(\gamma g);$$

under the assumptions above this converges absolutely and locally uniformly. One has, by (iii) above,

$$E(g, \psi) = (2\pi i)^{-(r-1)} \int_{(\sigma)} E(g, \omega\omega_s, \hat{\psi})\, ds$$

where σ satisfies $\sigma(\omega) + \sigma > \rho$. Moreover, if χ is unitary, as we shall henceforth assume, then

$$\int_{G_k^* \backslash \widetilde{G}_A / \widetilde{Z}_A} |E(g, \psi)|^2\, dg < \infty.$$

Let $\hat{\Psi}$ be the space of $\hat{\psi}(g, \omega)$ as above. We define for $\hat{\psi}_1, \hat{\psi}_2 \in \hat{\Psi}$

$$[\hat{\psi}_1, \hat{\psi}_2](\omega) = \int_{H_k^*\backslash \widetilde{H}_A/\widetilde{Z}_A} \int_{N_{+,A}^*} \hat{\psi}_1(\eta w_0 n, \omega) \overline{\hat{\psi}_2(\eta w_0 n, \bar{\omega}^{-1})} \, dn \, d\eta.$$

Then the Maass-Selberg relation is, in this case, the inner-product formula

$$\int_{G_k^*\backslash \widetilde{G}_A/\widetilde{Z}_A} E(g, \psi_1) \overline{E(g, \psi_2)} \, dg = c_1 \sum_{w \in W} \sum_{\omega} (2\pi i)^{-(r-1)} \int_{(\sigma)} [\hat{\psi}_1, I_w^0 \hat{\psi}_2]({}^w(\omega\omega_s)) \, ds$$

where c_1 is a constant depending on the choice of Haar measure and $-(\sigma(\omega) + \sigma + \rho)$ is dominant. This is a variant of [29] Lemma 4.6 and is proved in the same way.

The right-hand side of the equation above thus defines an inner product which gives Ψ the structure of a pre-Hilbert space.

Let now ω satisfy ${}^w\omega = \omega$ $(w \in W)$ and let ω be unitary; for example, if ω^1 is as in the statement of the theorem then

$$\omega = \omega^1 \mu_A^{-1/n}$$

satisfies these conditions. Let $\Psi(\omega)$ be the subspace of those ψ of the form

$$\psi(g) = (2\pi i)^{-(r-1)} \int_{(\sigma)} \hat{\psi}(g, \omega\omega_s) \, ds.$$

Recall that $\langle \; , \; \rangle$ denotes the Killing form on $\Phi(\mathbf{C})$.

We let Ψ^{cl} be the Hilbert space completion of Ψ, and $\Psi(\omega)^{cl}$ the closure of $\Psi(\omega)$ in it. We define an operator

$$\Delta : \Psi(\omega) \to \Psi(\omega)^{cl}$$

by
$$(\Delta\psi)^\wedge(g, \omega\omega_s) = \langle s, s \rangle \hat{\psi}(g, \omega\omega_s)$$

which is a symmetric operator with an essentially self-adjoint extension.

Let now $L^2(G_k^*\backslash \widetilde{G}_A, \chi)$ be the space of functions f on $\widetilde{G}_A$ which satisfy

$$f(\gamma g z) = \chi(z) f(g) \qquad (\gamma \in G_k^*, \; z \in \widetilde{Z}_A)$$

and such that

$$\|f\|^2 = \int_{G_k^*\backslash \widetilde{G}_A/\widetilde{Z}_A} |f(g)|^2 \, dg$$

is finite. Let, for $f_1, f_2 \in L^2(G_k^*\backslash \widetilde{G}_A, \chi)$ be the inner product

$$(f_1, f_2) = \int_{G_k^*\backslash \widetilde{G}_A/\widetilde{Z}_A} f_1(g) \overline{f_2(g)} \, dg.$$

Thus E (the Eisenstein series function) can be regarded as a continuous map $E : \Psi \to L^2(G_k^*\backslash \widetilde{G}_A/\widetilde{Z}_A, \chi)$. Let L_ω be the image, under E, of $\Psi(\omega)$; then Δ induces a map (which will also be denoted by Δ) from L_ω to its completion L_ω^{cl} in $L^2(G_k^*\backslash \widetilde{G}_A, \chi)$.

Let $\Theta(\omega^1)$ be the subspace of $L^2(G_k^*\backslash \widetilde{G}_A, \chi)$ (where $\chi = \omega^1 \mid \widetilde{Z}_A$ and $\omega^1 = \omega\mu_A^{1/n}$) generated by the $\theta(g, f_0)$ $(f_0 \in F(\omega^1))$.

Let $\mathscr{E}$ be the spectral resolution of Δ on L^{cl}_{ω}. Then the argument of Langlands is based on the following facts:

a) $(\mathrm{E}(\psi), \theta(f_0)) = c_1[\hat{\psi}, \tilde{\mathrm{I}}_{w_0}(f_0)](\omega\mu_{\mathbf{A}}^{1/n})$

where c_1 is as above. This is proved by the same techniques as are used to prove the Maass-Selberg relations.

From this it follows that

$$(\Delta\mathrm{E}(\psi), \theta(f_0)) = n^{-2}\langle\rho, \rho\rangle \quad (\mathrm{E}(\psi), \theta(f_0))$$

and hence $\Theta(\omega^1)$ lies in the range of

$$\mathscr{E}_\theta = \mathscr{E}(a) - \mathscr{E}(a-) \quad \left(a = \frac{1}{n^2}\langle\rho, \rho\rangle\right).$$

b) For a certain constant c_2

$$(\mathscr{E}_\theta\, \mathrm{E}(\psi_1), \mathscr{E}_\theta\, \mathrm{E}(\psi_2)) = c_2[\hat{\psi}_1, \tilde{\mathrm{I}}_{w_0}\, \hat{\psi}_2]\,(\omega\mu_{\mathbf{A}}^{1/n}).$$

This is proved by a transcription of the argument on pp. 145-148 of [30], using the version of the Maass-Selberg relations above instead of the corresponding result used by Langlands.

Now we choose ψ_0 such that

$$\hat{\psi}_0(\omega\mu_{\mathbf{A}}^{1/n}) = f_0.$$

Then, combining *a)* and *b)* we see that

$$(\mathscr{E}_\theta\, \mathrm{E}(\psi_1), \mathscr{E}_\theta\, \mathrm{E}(\psi_0) - (c_2/c_1)\, \theta(f_0)) = 0$$

and hence $\quad \mathscr{E}_\theta\, \mathrm{E}(\psi_0) = (c_2/c_1)\, \theta(f_0)$

and, in particular, the range of $\mathscr{E}_\theta$ is precisely $\Theta(\omega^1)$. Also

$$\begin{aligned}(\mathrm{E}(\psi), \theta(f_0)) &= \mathscr{E}_\theta(\psi), \theta(f_0)) \\ &= (c_1/c_2)(\theta(\psi(\omega')), \theta(f_0))\end{aligned}$$

and so, writing $f_0' = \hat{\psi}(\omega^1)$ we have, by *a)*,

$$(\theta(f_0'), \theta(f_0)) = c_1^2/c_2[f_0', \tilde{\mathrm{I}}_{w_0}(f_0)]\,(\omega\mu_{\mathbf{A}}^{1/n})$$

which is the assertion of the theorem.

II.2. The Main Theorems

We are now in a position to state and prove our main results. These involve the construction of an automorphic representation (π_θ, V_θ) of $\tilde{G}_{\mathbf{A}}$, which we shall identify locally.

We first remark that if (π, V) is an admissible irreducible representation of $\widetilde{G}_A$ then there exist representations (π_v, V_v) of $\widetilde{G}_v$, and, for almost all v, a K_v^*-invariant $\xi_v^0 \in V_v$ such that (π, V) is the restricted tensor product (over $\mathbf{C}[\mu_n(k)]$) of the (π_v, V_v) with respect to the family (ξ_v^0). Compare [23], Proposition 9.1. The (π_v, V_v) are irreducible and admissible.

Let ω be as in § II.1, and suppose that ω is exceptional (which we could express as $\omega_\alpha^n = \mu_{A,\alpha}$ for all $\alpha \in \Phi$). Then $\theta(f_0)$ is defined. Let V_θ be the space spanned by the $\theta(f_0)$ as f_0 runs through $\mathscr{F}_0(\omega)$, and let π_θ be the representation of $\widetilde{G}_A$ by right translation on V_θ. Since $\theta(\gamma g, f_0) = \theta(g, f_0)$ $(\gamma \in G_k^*)$ the representation (π_θ, V_θ) is automorphic. (One sees easily that $\theta(g, f_0)$ is slowly increasing, and even that (π_θ, V_θ) is unitary when $\omega\mu_A^{-1/n}$ is unitary (by Theorem II.1.4).)

We write $\omega = \bigotimes_v \omega_v$, where the tensor product is with respect to $\mathbf{C}[\mu_n(k)]$. Then it is possible to describe the local structure of (π_θ, V_θ).

Theorem **II.2.1.** — *With the notations above*

$$(\pi_\theta, V_\theta) \cong \bigotimes_v {}_{\mathbf{C}[\mu_n(k)]} (\pi_{0,v}, V_0(\omega_v))$$

where $(\pi_{0,v}, V_0(\omega_v)) \cong \operatorname{Im}(I_{w_0} : V(\omega_v') \to V({}^{w_0}\omega_v'))$ *is the representation discussed in* § I.2.

Proof. — This is an immediate consequence of Theorem II.1.4.

We come now to the discussion of Whittaker models of these representations. Let e_0 be an non-trivial character of k_A, trivial on k. We can write $e_0 = \bigotimes e_{0,v}$. We define the character e on $N_{+,A}^*$ by $e(n) = e_0(\sum_{1 \leq i < r} p(n)_{i,i+1})$, and an analogous character e_v on $N_{+,v}^*$. Let us denote the v-th factor of (π_θ, V_θ) by $(\pi_{0,v}, V_{0,v})$. Recall that a Whittaker model of $V_{0,v}$ is a linear map $\lambda_v : V_v \to \mathbf{C}$ such that $\lambda_v(\pi_0(n)\, v) = e(n)\, \lambda_v(v)$. We let Wh_v be the space of such functionals. At almost all places we have a preferred vector ξ_v^0 of $V_{0,v}$, as was described above.

Now, the same definition holds for (π_θ, V_θ). However we can construct a Whittaker functional λ_A on V_θ as follows:

$$\lambda_A(\theta(f_0)) = \int_{N_{+,k}^* \backslash N_{+,A}^*} \theta(n, f_0)\, \bar{e}(n)\, dn.$$

The space $\mathscr{F}_*(\omega_*)$ can be regarded as the restricted tensor product of the $V(\omega_{*v})$. Thus we can form

$$\bigotimes_{\mathbf{C}[\mu_n(k)]} V(\omega_v') \xrightarrow{J} F_0(\omega) \xrightarrow{\theta} V_\theta \xrightarrow{\lambda_A} \mathbf{C}.$$

This is a Whittaker functional on the left-hand space which factors through I_{w_0}. Let S be a finite set of places of k containing all the archimedean places and all the places v with $|n|_v < 1$ and such that if $v \notin S$ then $V(\omega_v')$ has a K_v^*-invariant vector. We can assume that $\omega_v'(v \notin S)$ is canonical in the sense of § I.1.

Let V(S) be the subspace of $\bigotimes_{\mathbf{C}[\mu_n(k)]} V(\omega'_v)$ which is the image of

$$(\bigotimes_{v \in S} V(\omega'_v)) \otimes (\bigotimes_{v \notin S} v_0),$$

where $v_0 \in V(\omega'_v)$ is the K_v^*-invariant vector with $v_0(I) = 1$, in $V(\omega'_v)$. All the tensor products over $V(\omega'_v)$ are to be taken with respect to $\mathbf{C}[\mu_n(k)]$.

The space V(S) is a $\widetilde{G}_{\mathbf{A}}(S)$-representation and $\lambda_{\mathbf{A}}$ is a Whittaker functional on V(S). Let $\widetilde{H}_{\mathbf{A}}(S) = \widetilde{H}_{\mathbf{A}} \cap \widetilde{G}_{\mathbf{A}}(S)$ and $\widetilde{H}_{*,\mathbf{A}}(S) = \widetilde{H}_{*,\mathbf{A}} \cap \widetilde{G}_{\mathbf{A}}(S)$. Then from § I.3 we see that there exists a function $\mathbf{c}_S$ on $\widetilde{H}_{\mathbf{A}}(S)$ satisfying

$$\mathbf{c}_S(\eta h) = (\omega_* \, \mu_{\mathbf{A}})(\eta)^{-1} \, \mathbf{c}_S(h)$$

if $\eta \in \widetilde{H}_{*,\mathbf{A}}(S)$ and $h \in \widetilde{H}_{\mathbf{A}}(S)$, such that

$$\lambda_{\mathbf{A}}(\theta(J(\otimes f_v))) = \sum_{\eta \in \widetilde{H}_{*,\mathbf{A}}(S) \backslash \widetilde{H}_{\mathbf{A}}(S)} \mathbf{c}_S(\eta) \prod_{v \in S} \langle \lambda_{\eta_v}, f_v \rangle$$

where $(\eta_v)_{v \in S}$ projects to η in $\widetilde{H}_{\mathbf{A}}(S)$. Here λ_η is as defined in § I.3. Moreover as J factors through I_{w_0} we have that $\mathbf{c}_S$ has to satisfy various relations. Let $v \in S$, v non-archimedean. Then there is a projection α of $\widetilde{H}_v \times \prod_{\substack{w \in S \\ w \neq v}} \widetilde{H}_w$ into $\widetilde{H}_{\mathbf{A}}(S)$.

Let $\eta^* \in \prod_{\substack{w \in S \\ w \neq v}} \widetilde{H}_w$; then we require that for each simple reflection s the following holds:

$$\sum_{\eta \in \widetilde{H}_{*,v} \backslash \widetilde{H}_v} \mathbf{c}_S(\alpha(\eta \times \eta^*)) \, \tau({}^s\omega'_v, s, \eta, \eta') = 0 \qquad (*)$$

where τ is as in § I.3. Moreover since

$$\langle \lambda_1, v_{0,v} \rangle = 1$$

and $\langle \lambda_\eta, v_{0,v} \rangle = 0 \quad (\eta \notin \widetilde{H}_{*,v})$

for almost all v it follows that if S is large enough and if $S' \supset S$ then

$$\mathbf{c}_S = \mathbf{c}_{S'} \mid \widetilde{H}_{\mathbf{A}}(S).$$

Thus the family of $\mathbf{c}_S$ defines a function $\mathbf{c}$ on $\widetilde{H}_{\mathbf{A}}$.

In this sense we can write

$$\lambda_{\mathbf{A}}(\theta(J(\otimes f_v))) = \sum_{\eta \in \widetilde{H}_{*,\mathbf{A}} \backslash \widetilde{H}_{\mathbf{A}}} \mathbf{c}(\eta) \, . \prod_v \langle \lambda_{\eta_v}, f_v \rangle.$$

Let $U(\omega)$ be the space of functions $f : \widetilde{H}_{\mathbf{A}} \to \mathbf{C}$ which satisfy

$$f(\eta h) = (\omega_* \, \mu_{\mathbf{A}})(\eta)^{-1} f(h)$$

when $\eta \in \widetilde{H}_{*,\mathbf{A}}$. Let $U^0(\omega)$ be the subspace satisfying the relations $(*)$ adumbrated above. Then we have shown:

Theorem **II.2.2.** — *There exists* $\mathbf{c} \in \mathrm{U}^0(\omega)$ *so that, when* $f_0 \in \mathscr{F}_0(\omega)$ *is such that* $f_* = \bigotimes f_{*,v}$ *then*

$$\int_{\mathrm{N}^*_{+,k}\backslash \mathrm{N}^*_{+,\mathbf{A}}} \bar{e}(n)\, \theta(n, f_0)\, dn = \sum_{\eta \in \widetilde{\mathrm{H}}_{*,\mathbf{A}}\backslash \widetilde{\mathrm{H}}_{\mathbf{A}}} \mathbf{c}(\eta) \,.\, \prod_v \langle \lambda_{\eta_v}, f_{*,v} \rangle.$$

Remark. — This result is particularly strong if we know that $\mathrm{U}^0(\omega)$ is 1-dimensional. We see that this is so precisely when $\dim(\mathrm{Wh}(\mathrm{V}_0(\omega'_v))) = 1$ for all v. By Corollary I.3.6 this is so if k is a function-field and

$$r = n \quad \text{or} \quad r = n - 1 \quad \text{and} \quad 2(c+1) \equiv 0 \ (\mathrm{mod}\ n).$$

We shall see in Theorem II.2.5 that the same conclusion holds when k is a number field.

Note that we have not shown that $\mathbf{c} \neq 0$. If $r > n$ then, as $\dim(\mathrm{Wh}(\mathrm{V}_0(\omega'_v))) = 0$ when $|n|_v = 1$, $\mathbf{c} = 0$. However, we shall also show in Theorem II.2.5 that, if $r \leqslant n$, then $\mathbf{c} \neq 0$.

Theorem II.2.2 is one of the major results of this work. In principle the space $\mathrm{U}^0(\omega)$ can be fully described by the methods of § I.3.

When $n = 2$ the theorem has content only when $r = 2$; in this case it is easy to see that $(\pi_\theta, \mathrm{V}_\theta)$ is the Weil representation r_1 discussed in [11].

When $n = 3$, $r = 2$ we obtain the results of [6] and [37].

There is another formulation of this theorem which is frequently useful. We observe that, for $\omega \in \mu_{\mathbf{A}}\, \Omega^0_+$ and $f_0 \in \mathscr{F}_0(\Omega)$,

$$\int_{\mathrm{N}^*_{+,k}\backslash \mathrm{N}^*_{+,\mathbf{A}}} \bar{e}(n)\, \mathrm{E}(n, \omega, f_0)\, dn = \int_{\mathrm{N}^*_{+,k}\backslash \mathrm{N}^*_{+,\mathbf{A}}} \bar{e}(n) \sum_{\gamma \in \mathrm{B}^*_k\backslash \mathrm{G}^*_k} f_0(\gamma n)\, dn.$$

Using the non-degeneracy of e and the Bruhat decomposition the right-hand side can be written as

$$\int_{\mathrm{N}^*_{+,\mathbf{A}}} \bar{e}(n)\, f_0(w_0\, n)\, dn.$$

If we write this in terms of $f_* = \bigotimes_v f_{*,v}$ we see that it is

$$\sum_{\eta \in \mathrm{H}^*_{n,k}\, \mathrm{Z}^*_k\backslash \mathrm{H}^*_k} \prod_v \int_{\mathrm{N}^*_{+,v}} \bar{e}_v(n)\, f_{*,v}(\eta w_0\, n)\, dn.$$

This we write as $\Psi^*(\omega, f_*)$ which is essentially a Dirichlet series in several variables whose coefficients involve Gauss sums. Now define

$$\Psi(\omega, f_*) = \prod_{\alpha > 0} \mathrm{L}(|\ |_{\mathbf{A}}\, \omega^n_\alpha)\, \Psi^*(\omega, f_*).$$

Also define I''_w by

$$\mathrm{I}''_w(\bigotimes f_{*,v}) = \bigotimes_v \mathrm{I}''_{w,v}(f_{*,v})$$

where $\mathrm{I}''_{w,v}$ was defined in § II.1. Then one has:

Theorem **II.2.3.** — $\Psi(\omega, f_*)$ *can be continued to a meromorphic function in* ω *whose singularities in* Ω_+ *lie in* $\bigcup_{\alpha>0}\{\omega \in \Omega_+ : \omega_\alpha^n = \|\ \|_{\mathrm{A}}\}$. *It satisfies the functional equations*

$$\Psi(\omega, f_*) = \Psi({}^w\omega, \mathrm{I}''_w f_*) \qquad (w \in \mathrm{W}).$$

Moreover, there exists a constant $a > 0$ *so that if* $f_* = \bigotimes f_{*,v}$ *then one has*

$$\lim_{\omega \to \omega^1} (\Psi(\omega, f_*) (\prod_{\alpha>0} \mathrm{L}(\omega_\alpha^n)^{-1})) = a \,.\sum_{\eta \in \widetilde{\mathrm{H}}_{*,\mathrm{A}} \backslash \widetilde{\mathrm{H}}_{\mathrm{A}}} \mathbf{c}(\eta) \prod_v \langle \lambda_{\eta_v}, f_{*,v} \rangle.$$

Proof. — The analytic properties of $\Psi(\omega, f_*)$ follow directly from the corresponding properties of the Eisenstein series given in Theorem II.1.3. The description of the singularity follows from Theorem II.2.2. The value of a is

$$\mathrm{Z}_k(2)^{r-1}\,\mathrm{Z}_k(3)^{r-2}\,\mathrm{Z}_k(4)^{r-3} \ldots \mathrm{Z}_k(r).$$

When k is a number field this formulation has the unattractive feature that the function $\Psi(\omega, f_*)$ involves factors which are Whittaker functions associated with the archimedean places. Since there are transcendental functions of a not particularly well understood type, and since they can be factored out, it is desirable to do this.

We assume now that k is a totally imaginary number field. We let

$$\omega_\infty = \prod_{v|\infty} \omega_v \circ \mathbf{s}_v, \qquad \mu_\infty = \prod_{v|\infty} \mu_v$$

which are quasicharacters of $\prod_{v|\infty} \mathrm{H}_v$. We define

$$\Psi_f^*(\omega, f_*) = \sum_{\eta \in \mathrm{H}_{n,k}^* \mathrm{Z}_k^* \backslash \mathrm{H}_k^*} \Big\{ \prod_{v \nmid \infty} \int_{\mathrm{N}_{+,v}^*} \bar{e}_v(n) f_{*,v}(\eta w_0 n)\, dn \Big\}\, \omega_\infty\, \mu_\infty(p_{\mathrm{A}}(\eta))$$

$$\mathrm{L}_f(\chi) = \prod_{v \nmid \infty} \mathrm{L}_v(\chi)$$

and

$$\Psi_f(\omega, f_*) = \prod_{\alpha>0} \mathrm{L}_f(\|\ \|_{\mathrm{A}}\, \omega_\alpha^n)\, \Psi_f^*(\omega, f_*).$$

The function $\Psi_f^*(\omega, f_*)$ is a Dirichlet series and it appears to be a fundamental function in this connection. Its coefficients are Gauss sums.

Let now ω_1 be a quasicharacter of $\widetilde{\mathrm{H}}_{\mathbf{C}}$; we define

$$\Gamma_{\mathbf{C}}^n(\omega_1) = \prod_{\alpha>0} \prod_{j=1}^{n-1} \mathrm{L}_{\mathbf{C}}(\omega_{1,\alpha} \,|\ |_{\mathbf{C}}^{j/n}).$$

We now observe that if we make use of the Gauss-Legendre multiplication formula, viz.,

$$\Gamma(ns) = n^{ns-\frac{1}{2}}(2\pi)^{-(n-1)/2}\, \Gamma(s)\, \Gamma\Big(s + \frac{1}{n}\Big) \ldots \Gamma\Big(s + \frac{n-1}{n}\Big),$$

where Γ is the usual gamma-function, then we can verify that the formula of Theorem I.6.6 can be expressed as

$$\prod_{\alpha>0} \mathrm{L_C}(|\ |_{\mathbf{C}}^{w}\ \omega_\alpha^n)\ \langle \lambda, \mathrm{I}_w'' f\rangle = \prod_{\substack{\alpha>0\\ w\alpha<0}} \varepsilon(\omega_\alpha)^{n-1}\ \omega(\eta_\alpha)\ \omega_\alpha^n(n)^{-1} \cdot \frac{\Gamma_{\mathbf{C}}^n({}^w\omega)}{\Gamma_{\mathbf{C}}^n(\omega)} \prod_{\alpha>0} \mathrm{L_C}(|\ |_{\mathbf{C}}\ \omega_\alpha^n)\ \langle \lambda, f_*\rangle$$

where $f_* \in \mathrm{V}(\omega)$ and η_α is as in Theorem I.6.6.

We now let k be as above and

$$\Gamma_\infty^n(\omega) = \prod_{v\mid\infty} \Gamma_{\mathbf{C}}^n(\omega_v).$$

Then if f_* is of the form $\bigotimes_{v\nmid\infty} f_{*,v}$ we see that the following corollary holds.

Corollary **II.2.4.** — *Let k be a totally imaginary number field. Then* $\Gamma_\infty^n(\omega)\ \Psi_f(\omega, f_*)$ *can be continued to a meromorphic function in* ω. *It satisfies the functional equations*

$$\Gamma_\infty^n({}^w\omega)\ \Psi_f({}^w\omega, \mathrm{I}_w'' f_*) = \prod_{v\mid\infty} \{\varepsilon(\omega_{\alpha,v})^{-(n-1)}\ \omega_v(\eta_\alpha)\ \omega_\alpha^n(n)\}\,.\,\Gamma_\infty^n(\omega)\ \Psi_f(\omega, f_*).$$

The singularities of $\Gamma_\infty^n(\omega)\ \Psi_f(\omega, f_*)$ *in* Ω_+ *lie in* $\bigcup_{\alpha>0} \{\omega \in \Omega_+ : \omega_\alpha^n = \|\ \|_{\mathrm{A}}\}$. *Also, if* ω' *is exceptional there exists a constant* $a' > 0$ *so that, if* $f_* = \bigotimes_v f_{*,v}$ *then*

$$\lim_{w\to w'} \Gamma_\infty^n(\omega)\ \Psi_f(\omega, f_*) \prod_{\alpha>0} \mathrm{L}_f(\omega_\alpha^n)^{-1} = a' \sum_{\eta \in \widetilde{\mathrm{H}}_{*,\mathrm{A}} \backslash \widetilde{\mathrm{H}}_{\mathrm{A}}} \mathbf{c}(\eta) \prod_{v\nmid\infty} \langle \lambda_{\eta_v}, f_{*,v}\rangle \cdot \prod_{v\mid\infty} \omega_v\ \mu_v(\eta_v).$$

Proof. — The only statement which remains to be proved is that the singularities are as described. To do this it suffices to consider $\sigma(\omega)$ lying in the closure of the dominant Weyl chamber in $\Phi(\mathbf{R})$. Thus, in this region we have to show that the functional (when $\mathrm{F} = \mathbf{C}$)

$$f_* \to \Gamma_{\mathbf{C}}^n(\omega)^{-1} \prod_{\alpha>0} \mathrm{L_C}(|\ |_{\mathbf{C}}\ \omega_\alpha^n)\ \langle \lambda_1, f_*\rangle$$

on $\mathrm{V}(\omega)$ is non-zero. For this see [44], § 2.

The following theorem closes our investigation.

Theorem **II.2.5.** — a) *Suppose that for each place v of k for which* $|n|_v = 1$ *one has* $\dim(\mathrm{Wh}(\mathrm{V}_{0,v})) > 0$. *Then* $\mathbf{c} \neq 0$.

b) *Suppose that for each place v of k for which* $|n|_v = 1$ *one has* $\dim(\mathrm{Wh}(\mathrm{V}_{0,v})) = 1$. *Then one has* $\dim(\mathrm{Wh}(\mathrm{V}_{0,v})) = 1$ *for every place v of k.*

Proof. — This is based on the ideas of [13]. We must first make some remarks about representations of local groups, concentrating on those over non-archimedean fields. Thus we fix a non-archimedean local field F and consider an exceptional representation (π_0, v_0) of $\widetilde{\mathrm{G}}$.

Let P be the subgroup of G defined by

$$P = \{g \in G : (0, 0, \ldots, 0, 1)\, g = (0, \ldots, 0, 1)\}$$

and let $\widetilde{P} = p^{-1}(P)$. Let Z^0 be the centre of G and $\widetilde{Z}^0 = p^{-1}(Z^0)$. Let e be a non-degenerate character of N^*_+. Let us now regard V_0 as a $\widetilde{P}$-representation. Then there exists a filtration

$$V_0 \supset V_1 \supset V_2 \supset \ldots \supset V_{r-1} \supset 0$$

of V_0 as a $\widetilde{P} \times_{\mu_n(F)} \widetilde{Z}^0$-module where

$$V_{r-1} = \mathrm{ind}_{N^*_+ \times \mu_n(F)}^{\widetilde{P}} (e \times \varepsilon) \otimes_{\mathbf{C}[\mu_n(F)]} r(V_0)$$

and $$r(V_0) = V_0 / \langle \pi_0(n)\, v - e(n)\, v \mid v \in V_0,\, n \in N^*_+ \rangle.$$

This is analogous to [3] 3.5. Here $r(V_0)$ is to be regarded as a $\mathbf{C}[\mu_n(F)]$-module. The group $\widetilde{P}$ acts on this through the first and $\widetilde{Z}_0$ through the second factor. From the discussion of § I.3 it follows that $r(V_0)$ is finite dimensional, and is non-zero if $\mathrm{Wh}(V_0) \neq \{0\}$.

Let us observe that V_{r-1} is cuspidal in the sense that if $N_1 \neq \{I\}$ is a unipotent subgroup of P and is also the unipotent radical of a parabolic subgroup of G then, letting N_1^* be the lift of N_1 to $\widetilde{P}$,

$$V_{r-1} / \langle v - \pi_0(n)\, v \mid v \in V_{r-1},\, n \in N_1^* \rangle = \{0\}.$$

If F is archimedean similar considerations can be applied—see [45]. These we shall pass over in silence. Note however that it follows from our assumptions, Theorem I.3.5 and Theorem I.6.5 that

$$\dim(\mathrm{Wh}(V_{0,v})) = 1$$

for archimedean v.

Let us now return to the global situation. We shall reformulate assertion *a)* in a suitable fashion. Let S be a finite set of places which

a) contains all archimedean places,
b) contains at least one non-archimedean place, and
c) is sufficiently large in a sense to be made precise later.

For $v \notin S$ let K_v be a compact open subgroup of G_v over which p_v splits, and such $K_v = GL_r(r_v)$ if $|n|_v = 1$. If $|n|_v < 1$ we assume K_v has the form

$$K_v = \{k \in GL_r(r_v) : k \equiv I \pmod{p_v^{m_v}}\}$$

where p_v is the maximal ideal of r_v, and m_v is some suitable integer. Let $G^S = \widehat{\prod_{v \notin S}} G_v$, the restricted topological product with respect to this family. Let $K^S = \prod_{v \notin S} K_v$.

Consider the diagonal embedding $G_k \hookrightarrow G^S$ and let $\Gamma_S = G_k \cap K^S$. Let now $G_S = \prod_{v \in S} G_v$ and

$$\widetilde{G}_S = (\prod_{v \in S} \widetilde{G}_v)/\langle i_v(\zeta) . i_{v'}(\zeta)^{-1} \mid v, v' \in S, \ \zeta \in \mu_n(k) \rangle.$$

Let $\kappa^* : K^S \to \widetilde{G}_{\mathbf{A}}$ and $\mathbf{s}_k : G_k \to \widetilde{G}_{\mathbf{A}}$ be the liftings which we have discussed. Let $\widetilde{G}_{\mathbf{A}}(S) \subset \widetilde{G}_{\mathbf{A}}$ be the subgroup $\widetilde{G}_S \times (K^S)^*$ discussed in § 0.2 where $(K^S)^* = \kappa^*(K^S)$. Let $\mathbf{s}_\Gamma : \Gamma_S \to \widetilde{G}_S$ be the map defined by $\mathbf{s}_\Gamma \times \kappa^* = \mathbf{s}_k$ in $\widetilde{G}_S \times (K^S)^* = \widetilde{G}_{\mathbf{A}}(S)$.

Now let f be a function on $\widetilde{G}_{\mathbf{A}}$ such that

a) $\qquad f(\gamma g) = f(g) \quad (\gamma \in G_k^*)$, and

b) $\qquad f(gk) = f(g) \quad (k \in (K^S)^*)$.

Let $\alpha_S : \widetilde{G}_S \to \widetilde{G}_{\mathbf{A}}$ be the composite

$$\widetilde{G}_S \to \widetilde{G}_S \times (K^S)^* \hookrightarrow \widetilde{G}_{\mathbf{A}}$$

where the first map is the injection into the first factor. Let f_S be the function on $\widetilde{G}_S$ defined by $f_S(g) = f(\alpha_S(g))$; then f_S satisfies

a') $\qquad f_S(\mathbf{s}_\Gamma(\gamma)\, g) = f_S(g) \quad (\gamma \in \Gamma_S)$.

Conversely, given f_S satisfying *a')* we can construct f on $\widetilde{G}_{\mathbf{A}}$ satisfying *a)* and *b)*, and which has support $G_k^* . \widetilde{G}_{\mathbf{A}}(S)$. Moreover it follows from Kneser's Strong Approximation Theorem [24] that if S is large enough then

$$G_k^* . \widetilde{G}_{\mathbf{A}}(S) = \widetilde{G}_{\mathbf{A}};$$

if this is so then f is uniquely determined by f_S and *a)*, *b)*. We shall henceforth demand that S is so large that this holds, and also that for each $w \in S$ one has $|n|_w = 1$.

The theory developed above for $\widetilde{G}_{\mathbf{A}}$, G_k^* can now be developed without essential change for the pair $\widetilde{G}_S$, Γ_S^* $(= : \mathbf{s}_\Gamma(\Gamma_S))$. Indeed this new theory is nothing other than the theory of $(K^S)^*$-invariant vectors of the original theory.

Denote by r^S the ring of S-integers of k and let

$$r' = \{ x \in r^S : \mathrm{ord}_v(x) \geq m_v \quad \text{if } |n|_v < 1 \}.$$

Let $e_0 = \prod_{v \in S} e_{0,v}$ be a character of $\prod_{v \in S} k_v$ which is trivial on r^S. Let $\mathfrak{D}$ be the fractional ideal of r^S defined by

$$\mathfrak{D}^{-1} = \{ x \in k : e_0(xy) = 1 \quad \text{for all } y \in r' \}.$$

We shall assume that S is so large that r^S is a principal ideal domain. We can then write $\mathfrak{D}$ as $\langle \delta \rangle$, $\delta \in k^\times$.

For $\qquad d = (d_{1,2}, d_{2,3}, \dots, d_{r-1,r})$

with $\qquad d_{i,i+1} \in \mathfrak{D}^{-1} - \{0\} \quad (1 \leq i < r)$

we set $\qquad e_d(n) = e_0(\sum_{1 \leq i < r} d_{i,i+1}\, n_{i,i+1})$

where $\qquad n = (n_{i,j}) \in N_+(\prod_{v \in S} k_v)$.

We regard e_d also as a character of $(N_+^*)_S = (N_+^*)_A \cap \widetilde{G}_S$. We write e for e_d with $d = (1, 1, \ldots, 1)$.

Now, if $v \in S$ let $T_v \subset V_{0,v}$ be the sub-space V_{r-1} discussed above, $T_S = \bigotimes_{v \in S} T_v$, the product being taken with respect to $\mathbf{C}[\mu_n(k)]$. Let now $\widetilde{P}_S \subset \widetilde{G}_S$ be the group whose local factors are the $p_v^{-1}(P(k_v))$ $(v \in S)$. Let, for $g \in \widetilde{G}_S$,

$$\beta_g : \widetilde{P}_S \to \widetilde{G}_S; \quad p \to p^g$$

be an embedding of $\widetilde{P}_S$ in $\widetilde{G}_S$. Let

$$T(g) = \pi_0(g^{-1})(T_S).$$

Then $T(g) \subset V_{0,S}$ is a $\beta_g(\widetilde{P}_S)$-module. Both $\beta_g(P_S)$ and $T(g)$ depend only on the class of g in $\widetilde{P}_S \widetilde{Z}_S^0 \backslash \widetilde{G}_S$, where the notations used should be self-explanatory.

Let $\theta : V_{0,S} \to \mathbf{C}$ be the Γ_S^*-invariant linear form given by the theory of Eisenstein series. We shall show that if

$$\int_{N_{+,S}^* \cap \Gamma_S^* \backslash N_{+,S}^*} \overline{e_d(n)}\, \theta(\pi_0(n) f)\, dn = 0$$

for all d as above and for all $f \in V_{0,S}$, then $\theta \mid T(\gamma) = \{0\}$ for all $\gamma \in \Gamma_S^*$. If this is so then, as $\Gamma_S^*.\widetilde{P}_S.\widetilde{Z}_S^0$ is dense in $\widetilde{G}_S$, we deduce that $\theta \mid T(g) = \{0\}$ for all $g \in \widetilde{G}_S$ and hence θ is zero on the space spanned by all the $T(g)$ $(g \in \widetilde{G}_S)$. But this latter space is $\widetilde{G}_S$-invariant and hence $\theta = 0$. But this contradicts the fact that

$$\int_{N_{+,S}^* \cap \Gamma_S^* \backslash N_{+,S}^*} \theta(\pi_0(n) f)\, dn$$

is not identically zero.

Hence it will follow that there exists at least one d and one $f \in V_{0,S}$ such that

$$\int_{N_{+,S}^* \cap \Gamma_S^* \backslash N_{+,S}^*} \overline{e_d(n)}\, \theta(\pi_0(n) f)\, dn \neq 0.$$

On translating this back into terms of the group $\widetilde{G}_A$ one sees that the " e-th Fourier coefficient " is not identically zero. This is what we had to prove.

It therefore remains to prove that, given the assumption above, $\theta \mid T(\gamma) = \{0\}$ for all $\gamma \in \Gamma_S^*$. It will suffice to show that $\theta \mid T(I) = \{0\}$ since θ is Γ_S^*-invariant. To do this consider $f \in T(I)$; as S contains at least one non-archimedean place, f is then cuspidal with respect to any unipotent subgroup of $\widetilde{P}$. The function $p \mapsto \theta(\pi_0(p) f)$ can then be expanded in a " Fourier series "

$$\theta(f) = \sum_d \sum_{p \in (N_{+S}^*, \cap \Gamma_S^*) \backslash (\widetilde{\Gamma}_S \cap \Gamma_S^*)} \int_{(N_{+,S}^* \cap \Gamma_S^*) \backslash N_{+,S}^*)} \bar{e}_d(n)\, \theta(\pi_0(np) f)\, dn$$

by following step for step the proof of [40] Theorem 1. Here we have assumed that the measure on $N_{+,S}^*$ is so normalised that the measure of $(N_{+,S}^* \cap \Gamma_S^*) \backslash N_{+,S}^*$ is one.

But, by assumption, all the terms in this sum are zero. Hence $\theta(f) = 0$, as we required. This completes the proof of *a)*.

We shall now prove *b)*. Let S be the finite set of places

$$S = \{v : v \text{ a place of } k, \ |n|_v < 1\}.$$

Under the assumptions made the group $\widetilde{Z}_v^0$ is abelian for all v. Moreover, since $\mathrm{Wh}(V_{0,v})$ is one-dimensional when $v \notin S$ we see that there exists a quasicharacter χ'_v of $\widetilde{Z}_v^0$ so that $\widetilde{Z}_v^0$ acts on $\mathrm{Wh}(V_{0,v})$ by χ'_v. It follows easily from *a)* that the χ'_v are local components of a quasicharacter χ' of $\widetilde{Z}_{\mathbf{A}}^0$ trivial on Z_k^{0*}.

We can factor

$$\widetilde{G}_{\mathbf{A}} = \widetilde{G}_S \times \widetilde{G}^S$$

$$\widetilde{Z}_{\mathbf{A}}^0 = \widetilde{Z}_S \times \widetilde{Z}^S$$

with the products being taken over $\mu_n(k)$. Likewise one has

$$V_\theta = V_{0,S} \otimes V^S$$

where the tensor product is with respect to $\mathbf{C}[\mu_n(k)]$.

For a non-archimedean place w of k define T_w as above and form

$$V_{\mathrm{cusp}}^S = \sum_{\substack{w \notin S \\ w \text{ non-arch}}} (T_w \otimes V^{S \cup \{w\}}).$$

Again the argument of [40] Theorem 1 yields the expansion

$$\theta(I, J(x \otimes y)) = \sum_{\gamma \in N_k^* \backslash P_k^*} \lambda_{\mathbf{A}}(\gamma(x \otimes y)).$$

If we now choose λ^S to be a generator of $\mathrm{Wh}(V^S)$ then we see that there exists a $\lambda_S \in \mathrm{Wh}(V_S)$ such that for $x' \in V_S$, $y' \in V^S$ one has

$$\lambda_{\mathbf{A}}(x' \otimes y') = \lambda_S(x')\, \lambda^S(y').$$

The local results recalled at the beginning of the proof of this theorem can easily be extended to the semi-local case and we obtain an embedding

$$j : \mathrm{ind}_{N_{+,S}^* \times \mu_n(k)}^{P_S}(e \times \varepsilon) \times \mathrm{Wh}(V_S)' \to V_S$$

where we have extended some of our notations in what we hope is a self-evident fashion. Here $\mathrm{Wh}(V_S)'$ is the dual space to $\mathrm{Wh}(V_S)$. We shall suppose that the induced representation here is realised by functions on $\widetilde{P}_S$ in the usual way. If $\lambda \in \mathrm{Wh}(V_S)$ then one has for f in the induced representation and for $l \in \mathrm{Wh}(V_S)'$

$$\lambda(j(f \otimes l)) = f(I) \langle \lambda, l \rangle.$$

Since $\widetilde{Z}_S^0$ acts on $\mathrm{Wh}(V_S)'$ we can consider j as a homomorphism of $\widetilde{P}_S\, \widetilde{Z}_S^0$-modules.

Suppose now that

$$\dim(\mathrm{Wh}(V_S)) > 1.$$

Then there exists $l \in \mathrm{Wh}(V_S)'$, $l \neq 0$ such that $\langle \lambda_S, l \rangle = 0$. It follows that there exists a $\widetilde{P}_S \widetilde{Z}_S^0$-subspace U_S of $V_{0,S}$ so that

$$\lambda_S(u) = 0 \qquad (u \in U_S).$$

We now construct $U_A = U_S \otimes V^S_{cusp} \subset V_\theta$; it is a $\widetilde{P}_A \widetilde{Z}_A^0$-subspace. From the Fourier expansion given above we also have

$$\theta(I, J(u)) = 0 \qquad (u \in U_A).$$

For $u \in U_A$ one can therefore regard

$$g \mapsto \theta(g, J(u))$$

as a function on $G_k^* \backslash \widetilde{G}_A$ which is zero on the subset $P_k^* Z_k^{0,*} \backslash \widetilde{P}_A \widetilde{Z}_A$. This subset is dense in the sense above and it follows that for $u \in U_A$ one has $\theta(g, J(u)) = 0$ for all $g \in \widetilde{G}_A$. Thus U_A generates a proper $\widetilde{G}_A$-subspace of V_θ. This is however impossible since V_θ is irreducible. This shows that the assumption that $\dim(\mathrm{Wh}(V_S)) > 1$ is untenable. Thus

$$\dim(\mathrm{Wh}(V_S)) = 1$$

by *a)*, and the assertion *b)* follows at once.

We record here a corollary of this theorem, an immediate consequence of it and Theorem I.3.5.

Corollary **II.2.6.** — *Let* F *be a non-archimedean local field with* $\mathrm{Card}(\mu_n(k)) = n$. *With the notations of Theorem* I.3.5 *one has*

a) *if* $N > 0$ *then* $\dim(\mathrm{Wh}(V_0(\omega'))) > 0$,

b) *if* $N = 1$ *then* $\dim(\mathrm{Wh}(V_0(\omega'))) = 1$.

Remarks. — 1) We are indebted to I. I. Piatetski-Shapiro for pointing out Theorem II.2.5 *b)* and Corollary II.2.6 *b)* and their proofs to us.

2) Examples based on Theorem I.5.3 and [7] show that one does *not* always have

$$\dim(\mathrm{Wh}(V_0(\omega'))) = N.$$

3) This corollary completes the local theory in an essential point. Its significance for the global theory is that it shows **c** to be determined up to a constant by local considerations. Whereas it is possible to give a purely local proof of Corollary II.2.6 *a)* we have not found one for Corollary II.2.6 *b)*. Another approach to this question can be given by the techniques of § I.5 and a generalisation of the Shimura Correspondence. We hope to return to this in a later publication.

II.3. Examples

In this section we give some examples to show how the theorems of the previous section relate to more familiar concepts. We restrict our attention to the case $r = 2$. What we shall do is to choose $(f_{*,v})$ conveniently and to make Corollary II.2.4 explicit.

Let k be a totally imaginary number field. Let $\mathfrak{A}$ be an integral ideal of k such that if v is a place of k for which $|n|_v < 1$ holds then $|\mathfrak{A}|_v < 1$ also holds. We shall also assume that for each v for which $|\mathfrak{A}|_v < 1$ holds we have furthermore

$$(1 + \mathfrak{A}r_v) \subset k_v^{\times n}.$$

Let $$S(\mathfrak{A}) = \{v : v \text{ a place of } k, \ |\mathfrak{A}|_v < 1\}.$$

If $v \in S(\mathfrak{A})$ we let

$$L_v(\mathfrak{A}) = \{k \in GL_2(r_v) : k \equiv I(\mathfrak{A}r_v)\}$$

and we let $L_v^*(A)$ be the Kubota lift of this to $\widetilde{G}$ ([26], Theorem 2). For convenience we shall take $c = 0$.

We shall next define the $\widetilde{H}_{*,v}$. If v is a finite place of k, $v \notin S(\mathfrak{A})$, then we take $\widetilde{H}_{*,v}$ to be the standard group $p^{-1}(H_{*,v})$ where

$$H_{*,v} = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : a_j \in k_v^\times, \ \mathrm{ord}_v(a_j) \equiv 0 \ (\mathrm{mod}\ n) \quad (j = 1, 2) \right\}.$$

If, however, $v \in S(\mathfrak{A})$ then we can take $\widetilde{H}_{*,v}$ to be $p^{-1}(H_{*,v})$ where

$$H_{*,v} = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : a_1 \in k_v^{\times n}, \ a_2 \in k_v^\times \right\}.$$

That this is a maximal isotropic subgroup is special feature of the case $r = 2$.

For our applications the set $S(\mathfrak{A})$ can be taken to be large, and to simplify our discussion we shall assume that it is sufficiently large. Recall that we have already assumed that

$$S(\mathfrak{A}) \supset \{v : |n|_v < 1\};$$

we supplement this with:

(A) *the ring of* $S(\mathfrak{A})$*-integers is a principal ideal domain.*

Let $\varepsilon : \mu_n(k) \to \mathbf{C}^\times$ be an injective character. We shall next denote by $\Omega_\varepsilon(\mathfrak{A})$ the group of quasicharacters of $\widetilde{H}_{n,\mathbf{A}}$, which restrict to ε on $i(\mu_n(k))$, and whose ramification is restricted by the following conditions on the local components ω_v of an $\omega \in \Omega_\varepsilon(\mathfrak{A})$ at the non-archimedean places of k:

$$\omega_v \mid (\widetilde{H}_{n,v} \cap K_v^*) = 1 \qquad (v \notin S(\mathfrak{A})),$$

and $$\omega_v \mid \widetilde{H}_{n,v} \cap L_v^*(\mathfrak{A}) = 1 \qquad (v \in S(\mathfrak{A})).$$

We are assuming here that $K_v = GL_2(r_v)$ and K_v^* is the canonical lift of K_v, which is defined whenever v is a non-archimedean place of k, $v \notin S(\mathfrak{A})$.

Note that if $\omega \in \Omega_\varepsilon(\mathfrak{A})$ then $\omega\omega_s \in \Omega_\varepsilon(\mathfrak{A})$ $(s \in \Phi(\mathbf{C}) \cong \mathbf{C})$.

Let $U(\mathfrak{A})$ be the group of units of the ring of $S(\mathfrak{A})$-integers. If S is any finite set of non-archimedean places of k let r^S be the ring of S-integers of k.

We now observe that $\omega \in \Omega_\varepsilon(\mathfrak{A})$ can be extended to $\widetilde{H}_{*,A}$ in such a fashion that

$$\omega_{*,v} \mid K_v^* \cap \widetilde{H}_{*,v} = 1 \quad (v \notin S(\mathfrak{A})),$$

$$\omega_{*,v} \mid L_v^*(\mathfrak{A}) \cap \widetilde{H}_{*,v} = 1 \quad (v \in S(\mathfrak{A})),$$

and

$$\omega_* \mid H_k^* \cap \widetilde{H}_{*,A} = 1.$$

To see this we first note that

$$\{x \in k^\times : \mathrm{ord}_v(x) \equiv 0 \ (\mathrm{mod}\ n) \quad (v \in S(\mathfrak{A}))\}$$

is, by (A), $k^{\times n}.U_n(\mathfrak{A})$, where

$$U_n(\mathfrak{A}) = \{u \in U(\mathfrak{A}) : u \in k_v^{\times n} \quad (v \in S(\mathfrak{A}))\}.$$

Then

$$p(H_k^* \cap \widetilde{H}_{*,A}) = H_{n,k}\cdot\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in U_n(\mathfrak{A}),\ b \in U(\mathfrak{A})\right\}.$$

We seek therefore $\omega_{*,v}$ on $\widetilde{H}_{*,v}$ such that

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \prod_{v \in S_\infty(\mathfrak{A})} \omega_{*,v}\left(\mathbf{s}_v\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right), \quad (a \in U_n(\mathfrak{A}),\ b \in U(\mathfrak{A}))$$

where

$$S_\infty(\mathfrak{A}) = S(\mathfrak{A}) \cup \{v : v \text{ archimedean}\},$$

should be trivial. Since this quasicharacter has trivial restriction to

$$\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in U^n(\mathfrak{A})\right\},$$

where

$$U^n(\mathfrak{A}) = \{u \in U(\mathfrak{A}) : u \in k^{\times n}\};$$

the existence of such $\omega_{*,v}$ is easy to see. That $\omega_{*,v} \mid L_v^*(\mathfrak{A}) \cap \widetilde{H}_{*,v} = 1 \quad (v \in S(\mathfrak{A}))$ is automatic.

Now that we have constructed ω_* we shall construct an element $f_*^0 \in V(\omega_*)$. This is $\bigotimes f_{*,v}^0$ where

(i) if $v \notin S(\mathfrak{A})$ but v is non-archimedean then $f_{*,v}^0$ is the element of $V(\omega_{*,v})$ which is K_v^*-invariant and such that $f_{*,v}^0(I) = 1$;

(ii) if $v \in S(\mathfrak{A})$ we let $f_{*,v}^0 \in V(\omega_{*,v})$ be the $L_v^*(\mathfrak{A})$-invariant vector with support $\widetilde{B}_{*,v}\, L_v^*(\mathfrak{A})\, w_0$ and such that $f_{*,v}^0(w_0) = 1$, and

(iii) if v is archimedean then $f_{*,v}^0$ is arbitrary.

As in Corollary II.2.4 we shall remove the archimedean places from our discussion.

From f_*^0 and $h \in \widetilde{H}_A$ we form f_* by

$$f_*(g) = f_*^0(gh).$$

We shall mainly have to consider $h \in H_k^*$.

Let e_0 be a character of k_A such that the conductor of the local factor $e_{0,v}$ is r_v at all non-archimedean $v \notin S(\mathfrak{A})$. It follows from assumption (A) that such an e_0 can be found. Let $\mathfrak{D}(e_0)$, or simply $\mathfrak{D}$, be the fractional ideal of k associated with the differential idele of e_0 as in [50], VII-2.

Let α be the unique positive root of GL_2.

If we define, for $\eta_v, h_v \in \widetilde{H}_v$,

$$G_v(\omega_*, \eta_v, h_v) = \int_{N^*_{+,v}} \bar{e}_v(n) f_{*,v}(\eta_v w_0 n)\, dn,$$

then we are to investigate

$$\Psi_f(\omega, f_*) = L_f(\|\ \|_{\mathbf{A}}\, \omega_\alpha^n) \cdot \sum_{\eta \in H^*_{n,k}/H^*_k} \omega_\infty\, \mu_\infty(\eta) \prod_{v \nmid \infty} G_v(\omega_{*,v}, \mathbf{s}_v(\eta), h_v)$$

where

$$\omega_\infty\, \mu_\infty(\eta) = \prod_{v \mid \infty} \omega_v\, \mu_v(\mathbf{s}_v(\eta)).$$

More precisely we shall show that $\Psi_f(\omega; f_*)$ is a Dirichlet series the coefficients of which are Gauss sums. In the obvious sense $\Psi_f(\omega, f_*)$ has an analytic continuation to $\Omega_\varepsilon(\mathfrak{A})$ as a meromorphic function. There is a pole at $\omega_\alpha^n = \|\ \|_{\mathbf{A}}$ and (using the isomorphism $\mathbf{C} \cong \Phi(\mathbf{C})$; $s \to s.\alpha$) the residue at ω' with $(\omega')_\alpha^n = \|\ \|_{\mathbf{A}}$ is given by

$$\sum_{\eta \in \widetilde{H}_{*,\mathbf{A}} \backslash \widetilde{H}_{\mathbf{A}}} \mathbf{c}(\eta) \cdot \prod_{v \nmid \infty} \langle \lambda_{\eta_v}, f_{*,v} \rangle \prod_{v \mid \infty} \omega_v\, \mu_v(\eta_v)$$

as in Corollary II.2.4, where $\mathbf{c} \in U^0(\omega)$.

To formulate the results we let

$$\psi_{\mathfrak{A}}(\omega, h) = \Psi_f(\omega, f_*);$$

let ω' be exceptional, *i.e.* $(\omega')_\alpha^n = \|\ \|_{\mathbf{A}}$, and

$$\rho_{\mathfrak{A}}(\omega', h) = \operatorname*{Res}_{\omega = \omega'} \psi_{\mathfrak{A}}(\omega, h).$$

Let $\widetilde{H}_{f,\mathbf{A}}$ (resp. $\widetilde{H}_{n,f,\mathbf{A}}$) be the group for which

$$\widetilde{H}_{\mathbf{A}} \cong \prod_{v \mid \infty} \widetilde{H}_v \times \widetilde{H}_{f,\mathbf{A}} \quad (\text{resp. } \widetilde{H}_{n,\mathbf{A}} \cong \prod_{v \mid \infty} \widetilde{H}_{n,v} \times \widetilde{H}_{n,f,\mathbf{A}})$$

where all the products are with respect to $\mu_n(k)$. Then $\widetilde{H}_{f,\mathbf{A}}$ (resp. $\widetilde{H}_{n,f,\mathbf{A}}$) is naturally a subgroup of $\widetilde{H}_{\mathbf{A}}$ (resp. $\widetilde{H}_{n,\mathbf{A}}$). In particular, we can make use of the restrictions of ω and $\mu_{\mathbf{A}}$ to $\widetilde{H}_{n,f,\mathbf{A}}$ as these are well-defined.

Theorem **II.3.1.** — *There exist* $c_v, c'_v \in \mathbf{R}_+^\times$ *for each non-archimedean place* v *of* k *such that* $c_v = c'_v = 1$ *for* $v \notin S(\mathfrak{A})$ *and*

a) *if, for one place* v *of* k *one has*

$$|p(h_v)^\alpha|_v > c'_v,$$

then

$$\rho_{\mathfrak{A}}(\omega', h) = 0,$$

b) *if* $h^{(1)}, h^{(2)} \in \widetilde{H}_{f,\mathbf{A}}$, $h^{(1)} h^{(2)-1} \in \widetilde{H}_{f,n,\mathbf{A}}$ *and*

$$|p(h_v^{(j)})^\alpha|_v \leq c_v \quad (j = 1, 2)$$

for all non-archimedean places v of k then one has

$$^{w_0}\omega\mu_{\mathbf{A}}(h^{(1)}\,h^{(2)-1})^{-1}\,\rho_{\mathfrak{A}}(\omega', h^{(1)}) = \rho_{\mathfrak{A}}(\omega', h^{(2)}).$$

Remark. — All the other consequences of Theorems I.4.1 and I.4.2 can be deduced from this one, as we shall explain in more detail later. Note that the corresponding identity for the $\psi_{\mathfrak{A}}(\omega, h)$ *does not* hold.

Proof. — Part *a)* of the theorem is an immediate consequence of Theorem I.4.1. Likewise part *b)* follows from Theorems I.4.1 and I.4.2.

Let $\varepsilon_1 : \mu_n(k) \to \mathbf{C}^\times$ be a homomorphism. Let $c \in r^{S(\mathfrak{A})}$, $c \neq 0$, and define the restricted Legendre symbol $\left(\frac{d}{c}\right)_{\mathfrak{A}}$ by

$$\left(\frac{d}{c}\right)_{\mathfrak{A}} = \prod_{\substack{v \mid c \\ v \notin S(\mathfrak{A})}} (c, d)_v$$

for d coprime to c in $r^{S(\mathfrak{A})}$. Let, for $x \in k$,

$$e_\infty(x) = \prod_{v \mid \infty} e_{0,v}(x),$$

where e_0 is the character of $k_{\mathbf{A}}$ defined above. Then we define, for $X \in r^{S(\mathfrak{A})}$

$$g_{\mathfrak{A}}(\varepsilon_1, X, c) = \sum_d \varepsilon_1\left(\left(\frac{d}{c}\right)_{\mathfrak{A}}\right) . \, e_\infty\left(\frac{X\,d}{c}\right)$$

where d is summed over a set of representatives for $(r^{S(\mathfrak{A})}/cr^{S(\mathfrak{A})})^\times$ which also satisfy

$$|d|_v \leq |c\,\mathfrak{D}^{-1}\,X^{-1}|_v \quad (v \in S(\mathfrak{A})).$$

We define the quasicharacter $\omega^{(1)}$ of $\prod_{v \in S_\infty(\mathfrak{A})} k_v^{\times n}$ by

$$\omega^{(1)}((c_v)) = \prod_{v \in S_\infty(\mathfrak{A})} \omega_v\,\mu_v\left(\mathbf{s}_v\begin{pmatrix} c_v^{-1} & 0 \\ 0 & c_v \end{pmatrix}\right).$$

Regarding $U_n(\mathfrak{A})$ as a subgroup of $\prod_{v \in S_\infty(\mathfrak{A})} k_v^{\times n}$ we see that $\omega^{(1)}(u) = 1$ $(u \in U_n(\mathfrak{A}))$.

Conversely it is easy to see that if a quasicharacter φ of $\prod_{v \in S_\infty(\mathfrak{A})} k_v^{\times n}$ is given such that $\varphi(u) = 1$ $(u \in U_n(\mathfrak{A}))$, then there exists a quasicharacter $\omega \in \Omega_\varepsilon(\mathfrak{A})$ of $\widetilde{H}_{n,\mathbf{A}}$ trivial on H_k^* such that $\omega^{(1)} = \varphi$.

Let φ be as above and define, for $X \in r^{S(\mathfrak{A})}$, $\varepsilon_1 : \mu_n(k) \to \mathbf{C}^\times$ injective,

$$\psi^0_{\mathfrak{A}}(\varphi, \varepsilon_1, X) = (\sum_c g_{\mathfrak{A}}(\varepsilon_1, X, c)\,\varphi(c))\,L_f(|\;|_{\mathbf{A}}\,\varphi_1^n),$$

where the sum is taken over $c \in r^{S(\mathfrak{A})}$ such that $c \in k_v^{\times n}$ for $v \in S_\infty(\mathfrak{A})$, and modulo $U_n(\mathfrak{A})$ (multiplicatively). Here φ_1^n is the Grössencharakter of $k_{\mathbf{A}}^\times$ unramified outside $S_\infty(\mathfrak{A})$ and whose $S_\infty(\mathfrak{A})$-factor is

$$(x_v) \to \varphi((x_v^n)).$$

Note that the series is well-defined, and, as one can easily verify, that it converges absolutely if $\sigma(\varphi) > 3/2$, where $\sigma(\varphi) \in \mathbf{R}$ satisfies

$$|\varphi((c_v))| = \prod_v |c_v|_v^{-\sigma(\varphi)} \quad ((c_v) \in \prod_{v \in S_\infty(\mathfrak{A})} k_v^{\times n}).$$

That $\sigma(\varphi)$ exists follows from Dirichlet's Unit Theorem.

Observe that whereas ω involves a choice of character $\varepsilon : \mu_n(k) \to \mathbf{C}^\times$, the quasi-character φ involves no such choice.

It is convenient here to record a proposition which will be of use to us later. Let $\mathfrak{A}$, φ be as above.

Proposition **II.3.2.** — *Let* $\mathfrak{p}$ *be a prime ideal of* k, *coprime to* $\mathfrak{A}$. *Let* $\pi \in r^{S(\mathfrak{A})}$ *be such that* $\pi r^{S(\mathfrak{A})} = \mathfrak{p}.r^{S(\mathfrak{A})}$. *Let* v *be the place of* k *associated with* $\mathfrak{p}$. *Let* φ' *be the quasicharacter of*

$$\prod_{w \in S_\infty(\mathfrak{A})} k_w^{\times n} \times k_v^{\times n}$$

which is φ *on the first factor, is unramified on the second factor and is trivial on* $U_n(\mathfrak{A}\mathfrak{p})$. *Let* $X \in r^{S(\mathfrak{A})}$ *and let* $l = \operatorname{ord}_v(X)$, *and* $X_0 = X\pi^{-l}$. *Suppose that*

$$\text{(i)}\ \pi^{l+1} \in k_w^{\times n} \quad (w \in S_\infty(\mathfrak{A})),$$

and (ii) *the natural map* $U_n(\mathfrak{A}) \to r_v^\times / r_v^{\times n}$ *is surjective.*

Suppose that ε_1 *is injective. Let* N *be the absolute norm. Then we have*

$$\psi_{\mathfrak{A}}^0(\varphi, \varepsilon_1, X)$$
$$= \left\{ \frac{1 - N(\pi)^{n-1}\varphi(\pi^n) - (1 - N(\pi)^{-1})(N(\pi^n)\varphi(\pi^n))^{1+[l/n]}}{1 - N(\pi^n)\varphi(\pi^n)} \right\} \psi_{\mathfrak{A}\mathfrak{p}}^0(\varphi', \varepsilon_1, X_0\pi^l)$$
$$+ g_{\mathfrak{A}}(\varepsilon_1^{l+1}, X_0, \pi)\, N(\pi)^l \varphi(\pi^{l+1})\, \psi_{\mathfrak{A}\mathfrak{p}}^0(\varphi', \varepsilon_1, X_0\pi^{n-l-2}).$$

Proof. — In the series defining $\psi_{\mathfrak{A}}^0(\varphi, \varepsilon_1, X)$ we replace c by $\pi^k c_0$, where $|c_0|_v = 1$ and $c_0 \in r^{S(\mathfrak{A})}$. It is summed over a set of representatives modulo $U_n(\mathfrak{A})$. Since

$$g_{\mathfrak{A}}(\varepsilon_1, X_0\pi^l, \pi^k c_0) = \varepsilon_1\left(\left(\frac{c_0}{\pi^k}\right)_{\mathfrak{A}}^{-1}\right) . \varepsilon_1\left(\left(\frac{\pi^k}{c_0}\right)_{\mathfrak{A}}^{-1}\right) g_{\mathfrak{A}}(\varepsilon_1, X_0\pi^l, \pi^k)\, g_{\mathfrak{A}}(\varepsilon_1, X_0\pi^l, c_0)$$

and since, as we shall see, we may assume that

$$\pi^k, c_0 \in \prod_{w \in S_\infty(\mathfrak{A})} k_w^{\times n}$$

for otherwise the term which we are computing will be zero, we obtain from the reciprocity law

$$g_{\mathfrak{A}}(\varepsilon_1, X, \pi^k c_0) = g_{\mathfrak{A}}(\varepsilon_1, X_0\pi^l, \pi^k) . g_{\mathfrak{A}}(\varepsilon_1, X_0\pi^{l-2k}, c_0).$$

Then

$$\psi^0_{\mathfrak{A}}(\varphi, \varepsilon_1, X)$$
$$= L_f(\| \ \|_A \varphi_1^n) \{ \sum_{k \geq 0} \sum_{c_0} g_{\mathfrak{A}}(\varepsilon_1, X_0 \pi^l, \pi^k) \, g_{\mathfrak{A}}(\varepsilon_1, X_0 \pi^{l-2k}, c_0) \, \varphi(c_0 \pi^k) \}.$$

Observe that

$$g_{\mathfrak{A}}(\varepsilon_1, X_0 \pi^l, \pi^k) = 0$$

unless $k \leq l$, $k \equiv 0 \pmod n$ or $k = l + 1$. As

$$c_0 \pi^k \in \prod_{w \in S_\infty(\mathfrak{A})} k_w^{\times n}$$

it follows from assumption (i) that

$$c_0 \in \prod_{w \in S_\infty(\mathfrak{A})} k_w^{\times n}.$$

It is easy to verify that the summand does not change if c_0 is replaced by $c_0 u$ ($u \in U_n(\mathfrak{A})$); hence by assumption (ii) we can assume that

$$c_0 \in r_v^{\times n}.$$

Thus the summation is now carried out over $c_0 \in r^{S(\mathfrak{A})}$, $c_0 \in \prod_{w \in S_\infty(\mathfrak{A})} k_w^{\times n} \times r_v^{\times n}$ and modulo

$$\{u \in U_n(\mathfrak{A}) : u \in r_v^{\times n}\}$$

multiplicatively. Since

$$U_n(\mathfrak{A}\mathfrak{p}) = \{u \in U_n(\mathfrak{A}) : u \in r_v^{\times n}\}.\pi^{n\mathbf{Z}}$$

this set of elements is also a set of representatives for those $c_1 \in r^{S(\mathfrak{A}\mathfrak{p})}$, $c_1 \in \prod_{w \in S_\infty(\mathfrak{A}\mathfrak{p})} k_w^{\times n}$ taken modulo $U_n(\mathfrak{A}\mathfrak{p})$. Hence we obtain

$$\psi^0_{\mathfrak{A}}(\varphi, \varepsilon_1, X)$$
$$= L_f(\| \ \|_A \varphi_1^n) . \{ \sum_{k \geq 0} g_{\mathfrak{A}}(\varepsilon_1, \pi^l X_0, \pi^k) \, \varphi(\pi^k) . \psi^0_{\mathfrak{A}\mathfrak{p}}(\varphi', \varepsilon_1, X_0 \pi^{l-2k}) \}.$$

However
$$\begin{aligned} g_{\mathfrak{A}}(\varepsilon_1, \pi^l X_0, \pi^k) &= \Phi(\pi^k) \quad (k \leq l, k \equiv 0 \pmod n), \\ &= N(\pi)^l g_{\mathfrak{A}}(\varepsilon_1^{l+1}, X_0, \pi) \quad (k = l + 1), \\ &= 0 \ \text{(otherwise)}, \end{aligned}$$

where Φ is the Euler totient function in k. Hence the term in braces becomes

$$\{ \sum_{\substack{0 \leq k \leq 1 \\ k \equiv 0 \pmod n}} \Phi(\pi^k) . \varphi(\pi^k) \} \psi^0_{\mathfrak{A}\mathfrak{p}}(\varphi', \varepsilon_1, X_0 \pi^l)$$
$$+ g_{\mathfrak{A}}(\varepsilon_1^{l+1}, X_0, \pi) \, N(\pi^l) . \varphi(\pi^{l+1}) . \psi^0_{\mathfrak{A}\mathfrak{p}}(\varphi', \varepsilon_1, X_0 \pi^{n-l-2}).$$

On carrying out the summation this gives the expression of the statement.

Let D_k be the discriminant of k. The central point in these computations is that $\psi_{\mathfrak{A}}(\omega, h)$ can be expressed in terms of $\psi^0_{\mathfrak{A}}$, at least if $h \in H_k^*$.

Theorem **II.3.3.** — *Suppose that* $h \in \mathrm{H}_k^*$ *and let* $\mathrm{X} = p(h)^\alpha$. *Let* $\omega \in \Omega_\varepsilon(\mathfrak{A})$ *and let* $\omega^{(1)}$ *be defined as above. Then*

$$\psi_{\mathfrak{A}}(\omega, h) = [\mathrm{U}(\mathfrak{A}) : \mathrm{U}_n(\mathfrak{A})].\mathrm{N}(\mathfrak{A})^{-1} |\mathrm{D}_k|^{-1/2}({}^{w_0}\omega\mu)_\infty(h)^{-1} \psi^0_{\mathfrak{A}}(\omega^{(1)}, \varepsilon^{-1}, \mathrm{X}) \quad \textit{if } \mathrm{X} \in \mathfrak{A}^{-1}\mathrm{D}^{-1},$$

$$= 0 \ \textit{otherwise}.$$

The proof of this theorem is based on a fairly long computation, so that it will be more convenient for us to discuss its significance now and to postpone its proof to the end of this section.

The first conclusion that one can draw is that $\psi^0_{\mathfrak{A}}(\varphi, \varepsilon, \mathrm{X})$ (ε injective) has an analytic continuation to the group of all φ, that it has at most a simple pole in $\sigma(\varphi) > 1$ and this where $\varphi = \varphi_0$ with

$$\varphi_0((x_w)) = \prod_{w \in \mathrm{S}_\infty(\mathfrak{A})} |x_w|_w^{1+\frac{1}{n}}.$$

Let

$$\rho^0_{\mathfrak{A}}(\varepsilon, \mathrm{X}) = \operatorname*{Res}_{\varphi=\varphi_0} \psi^0_{\mathfrak{A}}(\varphi, \varepsilon, \mathrm{X}).$$

From Proposition II.3.2 we see that if X, X_0, $\mathfrak{p}$, π, l are as in the enunciation of that proposition then one has

$$\rho^0_{\mathfrak{A}}(\varepsilon, \mathrm{X}) = \left\{\frac{1 - \mathrm{N}(\pi)^{-2}}{1 - \mathrm{N}(\pi)^{-1}}\right\} \rho^0_{\mathfrak{A}\mathfrak{p}}(\varepsilon, \mathrm{X}) + \{g_{\mathrm{A}}(\varepsilon^{l+1}, \mathrm{X}_0, \pi)\, \rho^0_{\mathfrak{A}\mathfrak{p}}(\varepsilon, \mathrm{X}_0\, \pi^{n-l-2})\, \mathrm{N}(\pi)^{-\left(\frac{l+1}{n}\right)-1} - \mathrm{N}(\pi)^{-\left[\frac{l}{n}\right]-1} \rho^0_{\mathfrak{A}\mathfrak{p}}(\varepsilon, \mathrm{X})\}.$$

If we assume, as we may, that $\pi \in \mathfrak{A}$ then we can deduce from Theorem II.3.1 that $\rho^0_{\mathfrak{A}}(\varepsilon, \mathrm{X}_0\, \pi^l)$ is ultimately periodic with period n in l for large l, as $\pi^n \in k^\times \cap k_{\mathbf{A}}^{\times n}$. Hence we can deduce that

$$\rho^0_{\mathfrak{A}}(\varepsilon, \mathrm{X}) = (1 + \mathrm{N}(\pi)^{-1}).\rho^0_{\mathfrak{A}\mathfrak{p}}(\varepsilon, \mathrm{X})$$

and, if $0 \leq l \leq n-1$, that

$$\rho^0_{\mathfrak{A}\mathfrak{p}}(\varepsilon, \mathrm{X}_0\, \pi^l) = g_{\mathfrak{A}}(\varepsilon^{l+1}, \mathrm{X}_0, \pi)\, \mathrm{N}(\pi)^{-\frac{l+1}{n}} \rho^0_{\mathfrak{A}\mathfrak{p}}(\varepsilon, \mathrm{X}\pi^{n-l-2}).$$

These represent the arithmetic form of our results. The restrictions on π represented by conditions (i) and (ii) of Proposition II.3.2 are quite stringent, but other variants can be given. Those quoted in [39], p. 180, are such; we shall not give a proof for those formulae here. Nevertheless the formal similarity between those derived here and those in [39] should convince the reader that the latter formulae are a consequence of the techniques developed here. The variations lie in the choice of $\widetilde{\mathrm{H}}_{*,w}$ and $f^0_{*,w}$ if $w \in \mathrm{S}(\mathfrak{A})$, and these lead to variations on Proposition II.3.2.

Observe that, if $n = 3$, then

$$\rho^0_{\mathfrak{A}}(\varepsilon, X_0\,\pi^2) = 0$$

and if $\pi \in \prod_{w \in S_\infty(\mathfrak{A})} k_w^{\times n}$

$$\rho^0_{\mathfrak{A}}(\varepsilon, X_0\,\pi) = g_{\mathfrak{A}}(\varepsilon^{-1}, X_0, \pi)\, N(\pi)^{-2/3}\, \rho^0_{\mathfrak{A}}(\varepsilon, X_0).$$

Moreover $\qquad \rho^0_{\mathfrak{A}}(\varepsilon, X_0\, m^3) = \rho^0_{\mathfrak{A}}(\varepsilon, X_0)$

if $m \in r^{S(\mathfrak{A})}$, by Theorem II.3.1. Hence $\rho^0_{\mathfrak{A}}(\varepsilon, X_0\,\pi^k)$ $(k \geq 0)$ is determined essentially by $\rho^0_{\mathfrak{A}}(\varepsilon, X_0)$. This observation is that of Corollary II.2.4 in a different guise.

The arithmetic significance of these $\rho^0_{\mathfrak{A}}(\varepsilon, X)$ for general n has been discussed in [39]. The significance of these results with $n = 3$ for the construction of the " cubic theta function " in [37] should also be clear, since the " Fourier coefficients " of that function are essentially $\rho_{(\sqrt{-3})}(\varepsilon, X)$. The results discussed here would not determine this function completely, but explain the " multiplicative " relations between different Fourier coefficients. In this connection the reader should also consult [6].

We turn finally to the proof of Theorem II.3.3. This is based on the evaluation of the $G_v(\omega_{*,v}, \eta_v, h_v)$ given by the following lemma. The additive measure on k_v will be taken to be the self-dual measure with respect to $e_{0,v}$.

Lemma **II.3.4.** — a) *Suppose that* $v \in S(\mathfrak{A})$; *then*

$$G_v(\omega_{*,v}, \eta_v, h_v) = |\mathfrak{A}|_v\, |\mathfrak{D}|_v^{1/2}\, \mu_v(h_v)^2 (\omega_*\, \mu)_v(\eta_v\, h_v^{w_0})$$

$$\textit{if } \eta_v\, h_v^{w_0} \in \widetilde{H}_{*,v} \quad \textit{and} \quad |p(h_v)^\alpha|_v \leq |\mathfrak{A}^{-1}\mathfrak{D}^{-1}|_v$$

$$= 0 \textit{ otherwise.}$$

b) *Suppose that v is a non-archimedean place of k, $v \notin S(\mathfrak{A})$. Let π_v be a uniformizer of k_v and let q_v be the modulus of k_v. Let*

$$\check{\pi}_v = \mathbf{s}_v\left(\begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v^{-1} \end{pmatrix}\right)$$

and let $\qquad p(h_v) = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}.$

Let $l = \mathrm{ord}_v(h_1/h_2)$, *and*

$$g_v^{(J)}(\pi_v) = \sum_{\substack{c\ (\mathrm{mod}\ \pi_v) \\ \langle c, \pi_v\rangle = 1}} \varepsilon((\pi_v, c_v))^J\, \bar{e}_{0,v}(c/\pi_v).$$

Then

$$G_v(\omega_{*,v}, \eta_v, h_v) = (\omega_*\,\mu)_v(\eta_v\, h_v\, \check{\pi}_v)\, g_v^{(-l-1)}(\pi_v)\,.\,\varepsilon(\pi_v, h_1/h_2)_v^{-1}\, \varepsilon(-1, h_2)_v$$

$$+ (\omega_*\,\mu)_v(\eta_v\, h_v^{w_0})\, \mu_v(h_v)^2 . \left\{\frac{1 - q^{-1}\omega^n_{\alpha,v}(\pi_v) - (1 - q_v^{-1})\, \omega^n_{\alpha,v}(\pi_v)^{1+[l/n]}}{1 - \omega^n_{\alpha,v}(\pi_v)}\right\}$$

if $l \geq 0$; *if the argument of* $(\omega_* \mu)_v$ *in either term does not lie in* $\widetilde{H}_{*,v}$ *then that term is taken to be zero. If* $l < 0$ *then*

$$G_v(\omega_{*,v}, \eta_v, h_v) = 0.$$

Proof. — Let us write $\eta_v = \mathbf{s}\begin{pmatrix}\eta_1 & 0 \\ 0 & \eta_2\end{pmatrix} i(\zeta)$ and $h_v = \mathbf{s}\begin{pmatrix}h_1 & 0 \\ 0 & h_2\end{pmatrix} i(\zeta')$.

Let $n = \mathbf{s}\begin{pmatrix}1 & \xi \\ 0 & 1\end{pmatrix}$. Then

$$\eta_v w_0 n h_v = \mathbf{s}\begin{pmatrix}0 & \eta_1 h_2 \\ \eta_2 h_1 & \eta_2 h_2 \xi\end{pmatrix} i((\eta_1 h_2, h_1)_v \zeta\zeta')$$

and this is equal to both

$$\mathbf{s}\begin{pmatrix}\eta_1 h_2 & 0 \\ 0 & \eta_2 h_1\end{pmatrix} i((\eta_1 h_2, h_1)_v . \zeta\zeta') \, \mathbf{s}\begin{pmatrix}0 & 1 \\ 1 & h_2 \xi / h_1\end{pmatrix}$$

and
$$\mathbf{s}\begin{pmatrix}-\eta_1 h_1 \xi^{-1} & h_2 \eta_1 \\ 0 & h_2 \eta_2 \xi\end{pmatrix} i((\eta_1 h_2, h_2 \xi)_v \zeta\zeta') \, \mathbf{s}\begin{pmatrix}1 & 0 \\ h_1/h_2 \xi & 1\end{pmatrix}.$$

a) In this case, using the Iwasawa decomposition given above, one sees that one must have

$$\eta_1 h_2 \in k_v^{\times n}$$

and
$$|h_2 \xi / h_1|_v \leq |\mathfrak{A}|_v.$$

The integral defining $G_v(\omega_{*,v}, \eta_v, h_v)$ is then simply

$$(\omega_* \mu)_v \left(\mathbf{s}\begin{pmatrix}\eta_1 h_2 & 0 \\ 0 & \eta_2 h_1\end{pmatrix} i((\eta_1 h_2, h_1)_v \zeta\zeta')\right) . \int_{|\xi|_v \leq |\mathfrak{A}|_v |h_1/h_2|_v} \bar{e}_{0,v}(\xi) \, d\xi.$$

The integral here is non-zero if and only if

$$|\mathfrak{A}|_v \, |h_1/h_2|_v \leq |\mathfrak{D}|_v^{-1};$$

when this condition is satisfied the integral is

$$|\mathfrak{A}|_v \, |h_1/h_2|_v \, |\mathfrak{D}|_v^{1/2}$$

by [50] VII-2, Cor. 3 to Prop. 2. Since

$$p(h)^\alpha = h_1/h_2$$

the statement in the lemma follows at once.

b) We split the integral into two parts, that over $\{\xi : |\xi|_v \leq |h_1/h_2|_v\}$ and that over $\{\xi : |\xi|_v > |h_1/h_2|_v\}$. The first of these can be treated as we treated *a)*; we find

$$(\omega_* \mu)_v (\eta_v h_v^{w_0}) \, \mu_v(h_v)^2 \quad \text{if } l \geq 0.$$

Now we turn to other integral, which is

$$\int_{|\xi|_v > |h_1/h_2|_v} (\omega_* \mu)_v \left(\mathbf{s} \begin{pmatrix} -\eta_1 h_1 \xi^{-1} & 0 \\ 0 & \eta_2 h_2 \xi \end{pmatrix} i((\eta_1 h_2, h_2 \xi)_v \zeta\zeta') \right) . \bar{e}_{0,v}(\xi) \, d\xi$$

from the second of the two formulae above. To compute this integral we let $\xi = \pi^{-t}.x$ with $t > -\operatorname{ord}_v(h_1/h_2)$ and $x \in r_v^\times$. Thus we obtain

$$\sum_{t > -\operatorname{ord}_v(h_1/h_2)} (\omega_* \mu)_v \left(\mathbf{s} \begin{pmatrix} -\eta_1 h_1 \pi_v^t & 0 \\ 0 & \eta_2 h_2 \pi_v^{-t} \end{pmatrix} i((\eta_1 h_2, \pi^{-t} h_2)_v \zeta\zeta') \right)$$

$$. q_v^t . \int_{r_v^\times} \varepsilon(\eta_1 h_2, x)_v \, \bar{e}_{0,v}(x/\pi_v^t) \, dx$$

since $\omega_{*,v}$ is unramified in the sense discussed above. Observe that with our conventions the summand is non-zero only when

$$\operatorname{ord}_v(\eta_1 h_1) + t \equiv 0 \pmod n$$

and

$$\operatorname{ord}_v(\eta_2 h_2) - t \equiv 0 \pmod n.$$

Moreover the integral over $r_v^\times$ is non-zero only when $t \leq 0$, $\operatorname{ord}_v(\eta_1 h_2) \equiv 0 \pmod n$ or $t = 1$. In the former case it has the value $(1 - q_v^{-1})$, and in the latter

$$q_v^{-1} . g_v^{(J)}(\pi_v)$$

with

$$J = \operatorname{ord}_v(\eta_1 h_2).$$

From these remarks it follows that our integral has become

$$\sum_{\substack{-\operatorname{ord}_v(h_1/h_2) < t \leq 0 \\ t \equiv -\operatorname{ord}_v(\eta_1 h_1) \pmod n}} (\omega_* \mu)_v \left(\mathbf{s} \begin{pmatrix} -\eta_1 h_1 \pi_v^t & 0 \\ 0 & \eta_2 h_2 \pi_v^{-t} \end{pmatrix} i((\eta_1 h_2, h_1)_v \zeta\zeta') \right) . (1 - q_v^{-1}) \, q_v^t$$

$$+ (\omega_* \mu)_v \left(\mathbf{s} \begin{pmatrix} -\eta_1 h_1 \pi_v & 0 \\ 0 & \eta_2 h_2 \pi_v^{-1} \end{pmatrix} i((\eta_1 h_2, \pi_v^{-1} h_2)_v \zeta\zeta') \right) g_v^{(J)}(\pi_v),$$

under the assumption that $l \geq 0$. In the first term we replace π_v^t by $h_1^{-1} h_2 \pi_v^{-u}$ where u is summed over

$$-\operatorname{ord}_v(h_1/h_2) \leq u < 0, \qquad u \equiv 0 \pmod n.$$

The u-th summand is then

$$(\omega_* \mu)_v (\mu_v h_v^{w_0}) . \omega_{\alpha,v}^n (\pi_v)^{-u/n} . |h_1/h_2|_v (1 - q_v^{-1}).$$

The sum is now a geometric series, which is easily computed. Combined with the first integral it yields the second term in the quoted formula.

In the second term we observe that

$$\text{ord}_v(\eta_1 h_1) \equiv -1 \pmod n$$

so that

$$J \equiv -l-1 \pmod n.$$

After a little rearrangement this yields the first term in the formula of the lemma.

Finally we observe that if $l<0$ the integral is zero as in Theorem I.4.1.

We are now in a position to prove Theorem II.3.3. This we shall do by bringing both expressions $\psi_{\mathfrak{A}}(\omega, h)$ and $\psi^0_{\mathfrak{A}}(\varphi, \varepsilon^{-1}, X)$ to forms which can be easily compared. Observe first that

$$\psi_{\mathfrak{A}}(\omega, h) = L_f(|\,|_{\mathbf{A}}\, \omega^n_\alpha) \sum_{\eta \in H_{n,k}\backslash H_k} \omega_\infty\, \mu_\infty(\eta) \prod_{v \nmid \infty} G_v(\omega_{*,v}, \mathbf{s}_v(\eta), \mathbf{s}_v(h))$$

and hence, by Lemma II.3.4, $\psi_{\mathfrak{A}}(\omega, h) = 0$ unless $X(=p(h)^\alpha) \in \mathfrak{A}^{-1}\mathfrak{D}^{-1}$. Hence we can make this assumption henceforth.

Let us examine the factors in the product. The factors corresponding to $v \in S(\mathfrak{A})$ yield

$$N(\mathfrak{A})^{-1}\,|\mathfrak{D}_k|^{-1/2} \prod_{v \in S(\mathfrak{A})} (\omega_{*,v}\,\mu_v)\,(\mathbf{s}_v(\eta)\,\mathbf{s}_v(h^{w_0}))\,\mu_v(h)^2.$$

It is now convenient to replace η by $\eta.(h^{w_0})^{-1}$.

Suppose that $\eta = \begin{pmatrix}\eta_1 & 0\\ 0 & \eta_2\end{pmatrix}$; we require that $\eta_1 \in \prod_{v \in S(\mathfrak{A})} k_v^{\times n}$. Let $p(h) = \begin{pmatrix}h_1 & 0\\ 0 & h_2\end{pmatrix}$.

We let $\Sigma(\mathfrak{A})$ be the set of non-archimedean places of k not in $S(\mathfrak{A})$. Then the product of those $G_v(\omega_{*,v}, \mathbf{s}_v(\eta(h^{w_0})^{-1}), \mathbf{s}_v(h))$ can be taken to be the sum over all *finite* subsets Σ' of $\Sigma(\mathfrak{A})$ of

$$\prod_{v \in \Sigma'} (\omega_*\,\mu)_v(\mathbf{s}_v(\eta(h^{w_0})^{-1})\,\mathbf{s}_v(h)\,\check{\pi}_v)\, g_v^{(-l_v-1)}(\pi_v).\varepsilon(\pi_v, h_1/h_2)_v^{-1}\,\varepsilon(-1, h_2)_v$$
$$\times \prod_{w \in \Sigma(\mathfrak{A})-\Sigma'} (\omega_*\,\mu)_w(\mathbf{s}_w(\eta(h^{w_0})^{-1})\,\mathbf{s}_w(h^{w_0}))\,\mu_w(h)^2$$
$$\times \left\{\frac{1-q_w^{-1}\,\omega^n_{\alpha,w}(\pi_w)-(1-q_w^{-1})\,\omega^n_{\alpha,w}(\pi_w)^{[l_w/n]+1}}{1-\omega^n_{\alpha,w}(\pi_w)}\right\}.$$

where

$$l_v = \text{ord}_v(h_1/h_2).$$

In order that this be non-zero we require that

$$\text{ord}_v(\eta_1) \equiv 0 \pmod n$$
$$\text{ord}_v(\eta_2) \equiv 0 \pmod n$$

if $v \in \Sigma(\mathfrak{A}) - \Sigma'$, and

$$\text{ord}_v(\eta_1 X) \equiv -1 \pmod n$$
$$\text{ord}_v(\eta_2 X^{-1}) \equiv 1 \pmod n,$$

if $v \in \Sigma'$. Recall that η_1 and η_2 are to be chosen modulo n-th powers, and that each summand is unchanged if either of these is replaced by a multiple of itself by an n-th power. Because $r^{S(\mathfrak{A})}$ is a principal ideal domain we can choose η_1 to be c^{-1} where

$$\begin{aligned}\mathrm{ord}_v(c) &= 0 \quad (v \in \Sigma(\mathfrak{A}) - \Sigma') \\ &= \mathrm{ord}_v(X) + 1 \quad (v \in \Sigma').\end{aligned}$$

This is then to be chosen modulo $U_n(\mathfrak{A})$. Let $\eta_2 = cu$, where $u \in U(\mathfrak{A})$, and u is to be taken modulo $U_n(\mathfrak{A})$. It is clear that if write η as

$$\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \cdot \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix}$$

(and noting that

$$\prod_v (\omega_* \mu)_v \left(\mathbf{s}_v \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}\right) = 1$$

by the assumption made on ω_* at the outset) the summation over u can be carried out and yields the factor

$$[U(\mathfrak{A}) : U_n(\mathfrak{A})].$$

Combining these, and using the reciprocity law, we see that

$$\begin{aligned}\psi_{\mathfrak{A}}(\omega, h) = {} & L_f(\|\cdot\|_{\mathbf{A}}\, \omega_\alpha^n) \cdot ({}^{u_0}\omega\mu)_\infty(h)^{-1}\, N(\mathfrak{A})^{-1}\, |\mathfrak{D}_k|^{-1/2} [U(\mathfrak{A}) : U_n(\mathfrak{A})] \\ & \cdot \sum_{c,\Sigma'} \prod_{v \in S_\infty(\mathfrak{A})} (\omega_{*,v}\, \mu_v) \left(\mathbf{s}_v \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix}\right) \cdot \prod_{v \in \Sigma'} g_v^{(-l_v-1)}(\pi_v)\, \mu_v(h)^{-2}\, \varepsilon(c, X\pi)_v \\ & \cdot \prod_{v \in \Sigma(\mathfrak{A}) - \Sigma'} \left\{\frac{1 - q_v^{-1}\, \omega_{\alpha,v}^n(\pi_v) - (1 - q_v^{-1})\, \omega_{\alpha,v}^n(\pi_v)^{[l_v/n]}}{1 - \omega_{\alpha,v}^n(\pi_v)}\right\}.\end{aligned}$$

Observe that here Σ' can be characterized as $\{v \in \Sigma(\mathfrak{A}) : |c|_v < 1\}$, so that the summation may be taken to be over c alone. The product over $S_\infty(\mathfrak{A})$ is $\omega^{(1)}(c)$.

In the product over Σ' we can take $\pi_v = c/X$. Hence we obtain

$$\begin{aligned}& L_f(\|\ \|_{\mathbf{A}}\, \omega_\alpha^n) \cdot N(\mathfrak{A})^{-1}\, |\mathfrak{D}_k|^{-1/2} [U(\mathfrak{A}) : U_n(\mathfrak{A})]\, N(X)^{-1} \\ & \cdot \sum_c \omega^{(1)}(c) \cdot \prod_{v \in \Sigma'} g_v^{(-l_v-1)}(c/X)\, |X|_v^{-1}\, \varepsilon(h_1, X)_v \\ & \cdot \prod_{v \in \Sigma(\mathfrak{A}) - \Sigma'} \left\{\frac{1 - q_v^{-1}\, \omega_{\alpha,v}^n(\pi_v) - (1 - q_v^{-1})\, \omega_{\alpha,v}^n(\pi_v)^{[l_v/n]}}{1 - \omega_{\alpha,v}^n(\pi_v)}\right\}.\end{aligned}$$

We shall now show that the sum over c is $\psi_{\mathfrak{A}}^0(\omega^{(1)}, \varepsilon^{-1}, X)\, L_f(\|\ \|_{\mathbf{A}}\, \omega_\alpha^n)^{-1}$. First observe that if ε_1 is injective $g_{\mathfrak{A}}(\varepsilon_1, X, c)$ is zero unless for each $v \in \Sigma(\mathfrak{A})$ one has either

$$\mathrm{ord}_v(c) \equiv 0 \pmod n, \qquad |X/c|_v \leq 1$$

or

$$\mathrm{ord}_v(X/c) = -1.$$

Thus, if we let

$$\Sigma'(c) = \{ v \in \Sigma(\mathfrak{A}) : \mathrm{ord}_v(X/c) = -1 \}$$

we obtain by a familiar transformation

$$g_{\mathfrak{A}}(\varepsilon^{-1}, X, c) = \prod_{v \in \Sigma'(c)} g_v^{(-l_v-1)}(c/X) . |X|_v^{-1} . \prod_{v \in \Sigma(\mathfrak{A}) - \Sigma'(c)} \mathrm{Card}((r_v/cr_v)^\times).$$

Consider next a fixed set Σ' and

$$\{ c : \Sigma'(c) = \Sigma' \}.$$

If this set is non-empty there is an element c_0 in it with $\mathrm{ord}_v(c_0) = 0$ for $v \in \Sigma(\mathfrak{A}) - \Sigma'$ since $r^{S(\mathfrak{A})}$ is a principal ideal domain. Then the sum of $g_{\mathfrak{A}}(\varepsilon^{-1}, X, c)\ \varphi(c)$ over this set is equal to

$$\varphi(c_0) \prod_{v \in \Sigma'} g_v^{(-l_v-1)}(c_0/X) . |X|_v^{-1}$$
$$\times \prod_{w \in \Sigma(\mathfrak{A}) - \Sigma'} \left\{ \frac{1 - q_w^{n-1}\varphi(\pi_w^n) - (1 - q_w^{-1})(q_w^n \varphi(\pi_w^n))^{1+[l_v/n]}}{1 - q_w^n \varphi(\pi_w^n)} \right\}$$

where $\pi_v \in k^\times$ is a uniformizer of k_v. Now notice that

$$\omega^{(1)}((c_v^n)) = \prod_{w \in S_\infty(\mathfrak{A})} \omega_{\alpha, w}^n(c_w)^{-1} |c_w|_w^{-n}$$

and so $$\omega_{\alpha, v}^n(\pi_v) = \omega^{(1)}(\pi_v) . q_v^n.$$

Finally observe that if c is as above

$$\prod_{v \in \Sigma'} (-1, c)_v = 1.$$

Hence, comparing the two expressions, we see that the identity asserted in the theorem is true. This completes the proof of the theorem.

BIBLIOGRAPHY

[1] H. Aritürk, On the composition series of principal series representations of a 3-fold covering group of SL (2, k), *Nagoya Math. J.*, **77** (1980), 177-196.

[2] H. Bass, J. Milnor and J.-P. Serre, Solution of the congruence subgroup problem for SL_n ($n \geq 3$) and Sp_{2n} ($n \geq 2$), *Publ. Math. IHES*, **33** (1967), 59-137.

[3] I. N. Bernstein and A. V. Zelevinski, Induced representations of reductive p-adic groups, *Ann. scient. Ec. Norm. Sup.*, **10** (1977), 441-472.

[4] A. Borel and Harish-Chandra, Arithmetic subgroups of algebraic groups, *Annals of Math.*, **75** (1962), 485-532.

[5] P. Deligne, *Notes on metaplectic groups* (Unpublished).

[6] P. Deligne, Sommes de Gauss cubiques et revêtements de SL(2). *Sém. Bourbaki*, Juin 1979, exposé n° 539; *Springer Lecture Notes*, **770** (1980), 244-277.

[7] Y. Flicker, Automorphic forms on covering groups of GL(2), *Inv. Math.*, **57** (1980), 119-182.

[8] S. Gelbart, Weil's representation and the spectrum of the metaplectic group, *Springer Lecture Notes*, **530** (1976).

[9] S. GELBART, R. HOWE and I. I. PIATETSKII-SHAPIRO, Existence and uniqueness of Whittaker models for the metaplectic group, *Israeli Math. J.*, **34** (1979), 21-37.

[10] S. GELBART and I. I. PIATETSKII-SHAPIRO, On Shimura's correspondence for modular forms of half-integral weight, *Proceedings, Colloquium on Automorphic Forms, Representation Theory and Arithmetic*, Bombay, 1979, 1-39.

[11] S. GELBART and I. I. PIATETSKII-SHAPIRO, Distinguished representations and modular forms of half-integral weight, *Inv. Math.*, **59** (1980), 145-188.

[12] I. M. GELFAND, M. I. GRAEV and I. I. PIATETSKII-SHAPIRO, *Representation Theory and Automorphic Functions*, Nauka (1966) (Russian), Saunders (1969) (English).

[13] I. M. GELFAND and D. A. KAZHDAN, Representations of the group GL(n, K) where K is a local field, *Lie Groups*, Budapest, Ed. I. M. Gelfand, 1974.

[14] I. M. GELFAND and M. I. NAIMARK, Unitary Representations of the Classical Groups, *Trudy Mat. Inst. Steklov*, **36**, 1950 (Russian), Akademie-Verlag Berlin, 1958 (German).

[15] G. HARDER, Minkowski'sche Reduktionstheorie über Funktionenkörper, *Inv. Math.*, **7** (1969), 107-149.

[16] G. HARDER, Chevalley groups over function fields and automorphic forms, *Annals of Math.*, **100** (1974), 249-306.

[17] HARISH-CHANDRA and VAN DIJK, Harmonic Analysis on Reductive p-adic Groups, *Springer Lecture Notes*, **162** (1970).

[18] HARISH-CHANDRA, *Admissible invariant distributions on reductive p-adic groups*, Princeton, Preprint, IAS.

[19] HARISH-CHANDRA, A submersion principle and its applications, *Geometry and Analysis, Papers dedicated to the Memory of V. K. Patodi*, Springer-Verlag, 1981, 95-102.

[20] E. HECKE, *Gesammelte Werke*, Vandenhoeck u. Ruprecht, 1970.

[21] R. HOWE, The Fourier transform and germs of characters (case of GL_n over a p-adic field), *Math. Ann.*, **208** (1974), 305-322.

[22] H. JACQUET, Fonctions de Whittaker associées aux groupes de Chevalley, *Bull. Soc. Math.*, France, **95** (1967), 243-309.

[23] H. JACQUET and R. P. LANGLANDS, Automorphic Forms on GL(2), *Springer Lecture Notes*, **114** (1970).

[24] M. KNESER, Strong Approximation, *Proc. Symp. Pure Math.*, IX (1966), 187-196.

[25] T. KUBOTA, Topological covering of SL(2) over a local field, *J. Math. Soc.* Japan, **19** (1967), 114-121.

[26] T. KUBOTA, *Automorphic Forms and the Reciprocity Law in a Number Field*, Kyoto Univ., 1969.

[27] T. KUBOTA, Some results concerning reciprocity and functional analysis, *Actes du Congr. Int. Math à Nice*, 1970, t. I, 395-399.

[28] T. KUBOTA, Some number-theoretical results on real-analytic automorphic forms, *Several Complex Variables* II, Maryland, 1970, *Springer Lecture Notes*, **185** (1971), 87-96.

[29] R. P. LANGLANDS, On the Functional Equations satisfied by Eisenstein Series, *Springer Lecture Notes*, **544** (1976).

[30] R. P. LANGLANDS, The volume of the fundamental domain for some arithmetical subgroups, *Proc. Symp. Pure Math.*, IX (1966), 143-148.

[31] R. P. LANGLANDS, On the Classification of Irreducible Representations of Real Algebraic Groups, *Lecture Notes*, Princeton, IAS.

[32] H. MATSUMOTO, Sur les sous-groupes arithmétiques des groupes semi-simples déployés, *Ann. scient., Ec. Norm. Sup.*, (1969), 1-62.

[33] J. MILNOR, An introduction to algebraic K-theory, *Annals of Mathematics Studies*, **72**, 1971.

[34] C. MOEN, *Ph. D. Thesis*, Univ. of Chicago, 1979.

[35] C. C. MOORE, Group extensions of p-adic and adelic linear groups, *Publ. Math. IHES*, **35** (1968), 157-222.

[36] L. E. MORRIS, Eisenstein series for reductive groups over global function fields, *Canad. Math. J.*, **34** (1982), 91-168.

[37] S. J. PATTERSON, A cubic analogue of the theta series, *J. für die r.u.a. Math.*, **296** (1977), 125-161, 217-220.

[38] S. J. PATTERSON, The constant term of the cubic theta series, *J. für die r.u.a. Math.*, **336** (1982), 185-190.

[39] S. J. PATTERSON, On the distribution of general Gauss sums at prime arguments, *Recent Progress in Analytic Number Theory*, Vol. 2, Academic Press, 1981.

[40] I. I. PIATETSKII-SHAPIRO, Euler subgroups, *Lie Groups*, Budapest, Ed. I. M. Gelfand, 1974.

[41] F. RODIER, Modèle de Whittaker et charactères de représentations. *Non-Commutative Harmonic Analysis, Proceedings*, 1976, *Springer Lecture Notes*, **466** (1977), 117-195.

[42] J.-P. Serre and H. Stark, Modular forms of weight 1/2, *Modular Forms in One Varible* VI, Bonn, 1976, *Springer Lecture Notes*, **627** (1977), 27-67.

[43] F. Shahidi, Functional equation satisfied by certain L-series, *Comp. Math.*, **37** (1978), 171-207.

[44] F. Shahidi, Whittaker models for real groups, *Duke Math. J.*, **47** (1980), 99-125.

[45] J. Shalika, The multiplicity one theorem for GL(n), *Annals of Math.*, **100** (1974), 171-193.

[46] A. J. Silberger, Introduction to Harmonic Analysis on Reductive p-adic Groups, *Princeton Math. Notes*, **23**, 1979.

[47] R. Steinberg, Lectures on Chevalley Groups, *Lectures Notes*, Yale Math. Dept., 1968.

[48] A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Hermann, 1940.

[49] A. Weil, Sur certains groupes d'opérateurs unitaires, *Acta Math.*, **111** (1964), 143-211.

[50] A. Weil, *Basic Number Theory*, Springer, 1967.

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