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# AN ANALYTIC PROOF OF NOVIKOV'S THEOREM ON RATIONAL PONTRJAGIN CLASSES

by D. SULLIVAN and N. TELEMAN <sup>(1)</sup>

We give here an *analytic* proof for the following:

*Theorem 1* (S. P. Novikov [3]). — *The rational Pontrjagin classes of any compact oriented smooth manifold are topological invariants.*

This problem was previously posed by I. M. Singer [4] and D. Sullivan [5]. Theorem 1 is a direct consequence of the following Theorems 2 and 3.

*Theorem 2* (D. Sullivan [5]). — *Any topological manifold of dimension  $\neq 4$  has a Lipschitz atlas of coordinates, and for any two such Lipschitz structures  $\mathcal{L}_i$ ,  $i = 1, 2$ , there exists a Lipschitz homeomorphism  $h: \mathcal{L}_1 \rightarrow \mathcal{L}_2$  close to the identity.*

*Remark 1.* — The proof of theorem 2 in general uses Kirby's annulus theorem to know that topological manifolds are stable <sup>(2)</sup>. The proof of Theorem 2 for stable manifolds is more elementary. Simply connected manifolds are stable and these <sup>(3)</sup> are sufficient for proving Novikov's theorem.

*Theorem 3* (N. Teleman [6]). — *For any compact oriented boundary free Riemannian Lipschitz manifold  $M^{2\mu}$ , and for any Lipschitz complex vector bundle  $\xi$  over  $M^{2\mu}$ , there exists a signature operator  $D_\xi^+$ , which is Fredholm, and its index is a Lipschitz invariant.*

Theorem 2 allows a strengthening of the statement of Theorem 3.

*Theorem 4.* — *For any simply-connected compact, oriented, boundary free topological manifold  $M^{2\mu}$  of dimension  $2\mu \neq 4$ , and for any complex continuous vector bundle  $\xi$  over  $M$ , there exists a class  $\mathcal{C}(M, \xi)$  of signature operators  $D_\xi^+$  which are Fredholm operators. The index*

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<sup>(2)</sup> See also P. TUKIA and J. VÄISÄLÄ [7] and [8].

<sup>(3)</sup> See remark in [3].

of any of these operators is the same and is a topological invariant of the pair  $(M, \xi)$ . When  $M$  and  $\xi$  are smooth, the smooth signature operators  $D_\xi^\pm$  (cf. [1]) belong to this class  $\mathcal{C}(M, \xi)$ .

*Proof.* — Pick a Lipschitz structure  $\mathcal{L}_1$  on  $M$  by Theorem 2, and regularize the bundle  $\xi$  up to a Lipschitz vector bundle  $\xi_1$ . Theorem 3 says that the class  $\mathcal{C}(M, \xi)$  is not void, and because the Lipschitz signature operators generalize the smooth signature operators, the last part of the theorem follows.

Suppose now that  $\mathcal{L}_i$ ,  $i = 1, 2$ , are two Lipschitz structures on  $M$  and that  $\xi_i$  are corresponding Lipschitz regularizations of  $\xi$ .

The Theorem 2 implies that there exists a Lipschitz homeomorphism  $h: \mathcal{L}_1 \rightarrow \mathcal{L}_2$  close to the identity (isotopic to the identity). As  $h$  is isotopic to the identity, the bundle  $h^*\xi_2$  is Lipschitz isomorphic to  $\xi_1$ ; let  $\bar{h}: \xi_1 \rightarrow \xi_2$  be such an isomorphism. Take any Lipschitz Riemannian metric [6]  $\Gamma_i$  on  $M$ ,  $i = 1, 2$ , and any connection  $\Delta_i$  in  $\xi_i$ ; the signature operators  $D_{\xi_i}^\pm$  are defined. From Theorem 3 we know that the index of  $D_{\xi_i}^\pm$ ,  $i$  fixed, is independent of the Riemannian metric  $\Gamma_i$  and the connection  $\Delta_i$  chosen. In order to compare Index  $D_{\xi_1}^\pm$  and Index  $D_{\xi_2}^\pm$  themselves, we chose  $\Gamma_2$  and  $\Delta_2$  arbitrarily, but we take

$$\Gamma_1 = h^*\Gamma_2, \quad \text{and} \quad \Delta_1 = \bar{h}^*\Delta_2.$$

From the very definition of the signature operators, we get that the homeomorphisms  $h, \bar{h}$  allow us to identify the corresponding domains and codomains of the operators  $D_{\xi_1}^\pm, D_{\xi_2}^\pm$ ; with these natural identifications,  $D_{\xi_1}^\pm$  and  $D_{\xi_2}^\pm$  coincide, and therefore, they have the same index.

*Proof of theorem 1.* — Suppose that  $M^{2u}$  is a smooth manifold, and  $\xi$  is a smooth complex vector bundle over  $M$ . The signature theorem due to F. Hirzebruch, and subsequently generalized by M. F. Atiyah and I. M. Singer [1], asserts that

$$\text{Index } D_\xi^\pm = \text{ch } \xi \cdot L(p_1, p_2, \dots, p_{u/2})[M]$$

where  $L$  is the Hirzebruch polynomial and  $p_1, p_2, \dots, p_{u/2}$  are the Pontrjagin classes of  $M$ . Theorem 4 implies that the right hand side of this identity is a topological invariant of the pair  $(M, \xi)$ . By letting  $\xi$  to vary,  $\text{ch } \xi$  generates over the rationals the whole even-cohomology subring of  $H^*(M, \mathbb{Q})$ . From the Poincaré duality we deduce further that the cohomology class  $L(p_1, \dots, p_{u/2})$  is a topological invariant. It is known that the homogeneous cohomology part  $L_i$  of degree  $4i$  of  $L(p_1, \dots, p_{u/2})$  is of the form (see e.g. [2])

$$L_i = a_i \cdot p_i + \text{polynomial in } p_1, p_2, \dots, p_{i-1}, \quad a_i \in \mathbb{Q}, \quad a_i \neq 0.$$

Therefore  $p_1, p_2, \dots, p_{u/2}$  are polynomial combinations with rational coefficients of  $L_1, L_2, \dots, L_{u/2}$ , which, as seen, are topological invariants.

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