PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

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Publications mathématiques de l'I.H.É.S., tome 58 (1983), p. 39-78
<http://www.numdam.org/item?id=PMIHES_1983__58__39_0>

THE INDEX OF SIGNATURE OPERATORS ON LIPSCHITZ MANIFOLDS

NICOLAE TELEMAN

Introduction

Before discussing the results of this paper we shall say a few words about its genesis.

In 1970, I. M. Singer [9] presents a comprehensive program aimed at extending the theory of elliptic operators and their index to more general situations: to " non-smooth manifolds, non-manifolds of special type and to a context where it is natural that *integer* (index) be replaced by *real number* ".

The author focused on the following problems from Singer's program: producing a Hodge theory and signature operators on circuits (pseudo-manifolds) [11] and on PL manifolds [12] looking for a possible analytic proof of Novikov's theorem about the topological invariance of the rational Pontryagin classes. A couple of years later, the author produced a Hodge theory and signature operators on PL manifolds, in [13] and [14].

In 1977, D. Sullivan [10] formulated the problem of constructing an index theory—which would lead to a new analytic proof of Novikov's theory about Pontryagin classes—on Lipschitz manifolds; the same problem was proposed to the author by D. Sullivan in a letter (1979).

The interest in studying Lipschitz manifolds derives from the following two desirable, but conflicting features [1] of the Lipschitz homeomorphisms in $\mathbf{R}^n$, see [10]: they preserve a rich analytic structure, whereas, from the topological point of view, they are very manageable. H. Whitney's [15] results show that any Lipschitz manifold has a complex of " flat forms " which satisfies some basic properties; the theory of Hausdorf measure and dimension on lower dimensional subsets—with important geometric consequences—is available.

The topological property we referred to, implies the following fundamental result due to D. Sullivan [10]: any topological manifold of dimension $\neq 4$ admits a Lipschitz structure which is unique, up to a Lipschitz homeomorphism close to the identity.

Partially supported by the NSF Grant # MCS 8102758.

(1) Unlike other known categories of manifolds.

The main result of this paper is the Theorem 13.1: for any closed Lipschitz Riemannian manifold M and for any Lipschitz complex vector bundle ξ over M, there exists a natural signature operator $D_\xi^+ : W_1^+(M, \xi) \to W_0^-(M, \xi)$, where $W_i^\pm(M, \xi)$ are Hilbert spaces of L_2-forms on M with values in ξ; this operator is a Fredholm operator, and its index is a Lipschitz invariant of the pair (M, ξ). (When M is a smooth Riemannian manifold, this operator is precisely the signature operator of M. F. Atiyah-I. M. Singer [3] defined on the Sobolev space of order 1.)

Although all techniques and results of the paper are necessary for the proof of this theorem, many of them are interesting by themselves. They include: Lipschitz Hodge theory (Theorem 4.1), a Rellich-type lemma (Theorem 7.1) and an excision property of the index of signature operators (Theorem 12.1).

The Lipschitz Hodge theory we present here is a slight modification of our combinatorial Hodge theory [14]. D. Sullivan [10] had pointed out that on any Lipschitz manifold L_2-forms, exterior derivatives and currents may be defined; all these objects are basic for the Hodge theory. In this construction, results due to H. Whitney [15] are involved.

The Hodge theory suffices for the computation of Index D_ξ^+ when ξ is a trivial bundle. The computation of this number when ξ is an arbitrary vector bundle requires substantially more analysis. In the second case the compactness of the inclusion $W_1^\pm(M, \xi) \hookrightarrow W_0^\pm(M, \xi)$ must be invoked. The proof of this result requires new ideas, see §§ 7, 8.

Another serious difficulty arises in proving that Index D_ξ^+ does not depend on the Riemannian metric on the base manifold. The starting difficulty consists of the fact that a change of the Riemannian metric produces a drastic change of the Sobolev space of order 1, $W_1^*(M, \xi)$. The desired result will be derived from the Excision Theorem 12.1.

The author thanks I. M. Singer and D. Sullivan for the problematic they created, and which has led to this paper, and for useful conversations.

The author thanks D. Sullivan and P. Trauber for their criticism on an earlier version of this paper.

The content of this paper was produced at the California Institute of Technology Pasadena, California, and while the author visited Scuola Normale Superiore di Pisa, Italy. Partial financial support was provided by the Rome University, Italy.

CONTENTS

LIPSCHITZ HODGE THEORY

All manifolds considered throughout this paper are compact, oriented topological manifolds without boundary, with Lipschitz structure.

A *Lipschitz structure* on a topological manifold M of dimension n is a maximal atlas $U = \{U_\alpha, \Phi_\alpha\}_{\alpha \in \Lambda}$, where $\Phi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbf{R}^n$ is a homeomorphism from an open set $U_\alpha \subset M$ onto an open subset V_α of $\mathbf{R}^n$, and the changes of coordinates $\Phi_\beta \circ \Phi_\alpha^{-1}$ are Lipschitz functions, *i.e.*

$$|\Phi_\beta \circ \Phi_\alpha^{-1}(x) - \Phi_\beta \circ \Phi_\alpha^{-1}(y)| \leq \kappa_{\alpha\beta}\, |x - y|$$

for any $x, y \in \Phi_\alpha(U_\alpha \cap U_\beta)$, with $\kappa_{\alpha\beta}$ a constant.

It is known, see D. Sullivan [10] that on any Lipschitz manifold, L_2-forms, exterior derivatives and currents can be defined. These objects are basic for this paper, so we shall define them from the beginning.

1. Differential forms

Let M be Lipschitz manifold as above.

An L_2-*form* ω *of degree* r on M is, by definition, a system

$$\omega = \{\omega_\alpha\}_{\alpha \in \Lambda},$$

where each ω_α is a complex L_2-differential form of degree r on the open subset $V_\alpha = \Phi_\alpha(U_\alpha)$ of $\mathbf{R}^n$, and they are required to satisfy the compatibility conditions:

$$(1.1) \qquad (\Phi_\beta \circ \Phi_\alpha^{-1})^* \omega_\beta = \omega_\alpha$$

(the pull-back $(\Phi_\beta \circ \Phi_\alpha^{-1})^*$ is defined component-wise).

This definition makes sense in view of the following result (see e.g. H. Whitney [15], p. 272):

Theorem 1.1 *(Rademacher). — Let* $f : U \to \mathbf{R}$ *be a Lipschitz function defined on an open subset* U *of* $\mathbf{R}^n$. *Then:*

(i) *the partial derivatives* $\partial f / \partial k^i$ *exist almost everywhere on* U, $1 \leq i \leq n$,

(ii) $\partial f / \partial k^i$ *are measurable and bounded.*

The space of all L_2-forms of degree r on M will be denoted by $L_2^r(M)$.

If σ is any L_2-form on an open subset U of $\mathbf{R}^n$, then σ is said (classically) to have *distributional exterior derivative* $d\sigma$ *in* L_2 if there exists an L_2-form denoted $d\sigma$ in $L_2^{r+1}(U)$ such that for any C^∞-form φ with compact support in U:

$$(1.2) \qquad \int \sigma \wedge d\varphi = (-1)^{\deg \sigma + 1} \int d\sigma \wedge \varphi$$

$$\deg \sigma + \deg \varphi + 1 = n.$$

If $\omega = \{\omega_\alpha\}_{\alpha \in \Lambda} \in L_2^r(M)$, and if $d\omega_\alpha \in L_2^{r+1}(\mathbf{R}^n)$ for any $\alpha \in \Lambda$, then we say that ω has distributional exterior derivative $d\omega = \{d\omega_\alpha\}_{\alpha \in \Lambda}$ in $L_2^{r+1}(M)$. Of course, in order to check that this definition is correct, it remains to verify that the forms $d\omega_\alpha$ satisfy the compatibility conditions:

$$(1.3) \qquad (\Phi_\beta \circ \Phi_\alpha^{-1})^* d\omega_\beta = d\omega_\alpha.$$

The relation (1.3) follows from the following:

Proposition 1.2. — *For any Lipschitz mapping* $f: \Omega_1 \to \Omega_2$, *where* Ω_1 *and* Ω_2 *are relatively compact open sets in* $\mathbf{R}^n$, *and for any form* $\omega \in L_2^r(\Omega_2)$, *the form* $f^*\omega$ *belongs to* $L_2^r(\Omega_1)$, *and*

$$(1.4) \qquad d(f^*\omega) = f^*(d\omega) \in L_2^{r+1}(\Omega_1).$$

Proof. — (i) It was proven by H. Whitney [15], Theorem 9C, p. 305, that (1.4) holds for Lipschitz mappings f and *flat* forms ω (a form ω is flat if ω and $d\omega$ have bounded measurable components).

(ii) A fortiori, (1.4) holds for Lipschitz mappings and smooth forms.

(iii) To prove (1.4) for an arbitrary L_2-form ω, we approximate ω by smooth forms, and we apply (ii). For a complete account of this last step, see the Appendix 14.0.

2. L_2 De-Rham Complex

We introduce the spaces

$$(2.1) \qquad \Omega_d^r(M) = \{\omega \mid \omega \in L_2^r(M), d\omega \in L_2^{r+1}(M)\}.$$

The exterior derivative d satisfies

$$(2.2) \qquad d^2 = 0;$$

therefore

$$(2.3) \qquad \Omega_d^*(M) \equiv \{\dots 0 \to \Omega_d^0(M) \xrightarrow{d} \Omega_d^1(M) \xrightarrow{d} \dots \to \Omega_d^r(M) \xrightarrow{d} \Omega_d^{r+1}(M) \to \dots\}$$

is a cohomology complex.

Let $L^r(M)$ denote the vector space of complex flat forms of degree r on M. The wedge product of any two flat forms is still a flat form, and hence $\bigoplus_r L^r(M)$ is a graded differential algebra. It follows from this that the wedge product induces a structure of complex algebra on the homology of $\bigoplus_r L^r(M)$. On the other hand, $L^*(M)$, the complex of flat forms on M, is a subcomplex of the complex $\Omega^*_d(M)$.

Theorem **2.1.** — (i) *If i denotes the inclusion*

$$i: L^*(M) \hookrightarrow \Omega^*_d(M), \tag{2.4}$$

then the induced homomorphism in homology:

$$i_*: H_*(L^*(M)) \to H_*(\Omega^*_d(M)) \tag{2.5}$$

is an isomorphism.

(ii) $H_*(L^*(M))$ *and* $H_*(\Omega^*_d(M))$ *are canonically isomorphic to* $H^*(M, \mathbf{C})$.

(iii) *If* ω *and* σ *are arbitrary cocycles of complementary degrees (say,* r *and* $\dim M - r$*) in* $\Omega^*_d(M)$, *then*

$$(\omega, \sigma) \mapsto \int_M \omega \wedge \sigma \tag{2.6}$$

induces a non-degenerate pairing:

$$H_r(\Omega^*_d(M)) \times H_{\dim M - r}(\Omega^*_d(M)) \to \mathbf{C} \tag{2.7}$$

(L_2-Poincaré duality).

Proof. — (i) and (ii) Notice that $L^r(M)$, $\Omega^r_d(M)$ are modules over the algebra of Lipschitz functions on M, and hence the associated differential sheaves of germs $\underline{L^*(M)}$, $\underline{\Omega^*_d(M)}$ are fine. These sheaves are resolutions of the constant sheaf $\mathbf{C}$ (Poincaré lemma). The proof of the Poincaré lemma for the sheaf $\underline{L^*(M)}$ can be performed as in the smooth case; for the case of the sheaf $\underline{\Omega^*_d(M)}$, see e.g. [14]. The generalized de Rham theorem applied to these sheaves and morphisms proves (i) and (ii).

(iii) If ω is a coboundary and σ is a cocycle in the complex $\Omega^*_d(M)$, then Lemma 4.1 below gives:

$$\int_M \omega \wedge \sigma = 0; \tag{2.8}$$

this shows that the pairing (2.7) is well defined.

We know from (i) that any cohomology class in $H_*(\Omega^*_d(M))$ may be represented by a flat cocycle. On the other hand, (2.6) defines also (for ω and σ flat forms) a pairing

$$H_r(L^*(M)) \times H_{\dim M - r}(L^*(M)) \to \mathbf{C}. \tag{2.9}$$

It is well known that the pairing (2.9) is non-degenerate, and hence the pairing (2.7) is non-degenerate. For the reader's benefit, we recall here, briefly, why (2.9) is non-degenerate. H. Whitney's theory [15] implies that, by means of the correspondence

$$\left\{\text{flat form} \rightleftarrows \text{flat cochain}, \int_M \rightleftarrows \text{evaluation on } [M], \quad \text{wedge product} \rightleftarrows \text{cup product}\right\}$$

the pairing (2.9) becomes the topological Poincare' pairing, which is known to be non-degenerate.

3. Riemannian metrics

A *Riemannian metric* on M is a collection $\Gamma = \{\Gamma_\alpha\}_{\alpha \in \Lambda}$, where Γ_α is a Riemannian metric on $V_\alpha = \Phi_\alpha(U_\alpha) \subseteq \mathbf{R}^n$, with measurable components, which satisfy the compatibility conditions

$$(\Phi_\beta \circ \Phi_\alpha^{-1})^* \Gamma_\beta = \Gamma_\alpha. \tag{3.1}$$

In addition, the Riemannian metrics Γ_α are required to define L_2-norms on V_α which are equivalent to the standard L_2-norm, i.e. there should exist two positive constants k_α, K_α such that, for any smooth form ω with compact support in V_α

$$k_\alpha \|\omega\| \leq \|\omega\|_\alpha \leq K_\alpha \|\omega\|; \tag{3.2}$$

here $\| \ \|$ and $\| \ \|_\alpha$ denote the usual L_2-norms:

$$\begin{aligned} \|\omega\|^2 &= \int \omega \wedge \overline{*\omega} \\ \|\omega\|_\alpha^2 &= \int \omega \wedge \overline{*_\alpha \omega}, \end{aligned} \tag{3.3}$$

where $*$ and $*_\alpha$ are the Hodge star operators of the Euclidean metric and of the metric Γ_α, respectively.

Any such Riemannian metric Γ will be called a *Lipschitz Riemannian metric* on M.

This norm derives from the scalar product

$$(\varphi, \sigma) = \int \varphi \wedge \overline{*\sigma}, \qquad \varphi, \sigma \in L_2^r(M). \tag{3.3'}$$

We show now that any compact Lipschitz manifold M has Lipschitz Riemannian metrics. For, we choose a finite Lipschitz subatlas $\{U_i, \Phi_i\}_{1 \leq i \leq N}$ on M such that all U_i have boundaries of measure zero. We partition M by the sets:

$$T_1 = U_1$$

$$T_i = U_i \setminus \bigcup_{r=1}^{i-1} U_r, \qquad 2 \leq i \leq N,$$

and we transfer on each T_i the standard Euclidean metric of $\mathbf{R}^n$ via the coordinate map Φ_i.

For any Lipschitz Riemannian metric

$$\Gamma = \{\Gamma_\alpha\}_{\alpha \in \Lambda}, \quad \text{and} \quad \text{for any} \quad \omega = \{\omega_\alpha\}_{\alpha \in \Lambda} \in L_2^r(M),$$

the form $*_\Gamma \omega$ defined by

$$*_\Gamma \omega = \{*_\alpha \omega_\alpha\}_{\alpha \in \Lambda} \tag{3.4}$$

is an L_2-form of complementary degree on M; we define by means of it the L_2-norm of ω:

$$\|\omega\|_\Gamma^2 = \int_M \omega \wedge \overline{*_\Gamma \omega}; \tag{3.5}$$

if there is no danger of confusion, the subscript Γ might be omitted in the sequel.

This is a norm on $L_2^r(M)$ which makes $L_2^r(M)$ a Hilbert space. Two different Lipschitz Riemannian metrics on M define equivalent norms on $L_2^r(M)$.

4. Hodge Theory on Lipschitz Manifolds

Let Γ be a Lipschitz Riemannian metric on the manifold M^m of dimension m, and let $*$ be the operator (3.4). This operator is an isometry:

$$*_r : L_2^r(M) \to L_2^{m-r}(M)$$

and

(4.0) $$*_{m-r} *_r = (-1)^{r(m-r)}.$$

The operator δ_r acting on forms of degree r is introduced formally as in the smooth case:

(4.1) $$\delta_r = (-1)^{m(r+1)+1} * d *.$$

Its domain of definition is the space

(4.2) $$\Omega_\delta^r(M) = *\Omega_d^{m-r}(M) \subset L_2^r(M).$$

The following spaces $W_1^r(M)$ are the Lipschitz analogues of the Sobolev spaces of forms of degree r and order one:

(4.3) $$W_1^r(M) = \Omega_d^r(M) \cap \Omega_\delta^r(M) = \{\omega \mid \omega \in L_2^r(M), \text{ with } d\omega, d*\omega \in L_2(M)\}.$$

Remark, as a new feature, that the spaces W_1^r depend [1], by means of the $*$-operator, on the Riemannian metric; this dependency is effective, as shown by Proposition 2.4 (ii), in [14].

It is easy to check that $W_1^r(M)$ is a Hilbert space under the *diagonal norm* $|| \ \ ||_1$:

(4.3') $$||u||_1^2 = ||u||^2 + ||du||^2 + ||\delta u||^2.$$

We define now the spaces of *harmonic forms*:

(4.4) $$H^r(M) = \{\omega \mid \omega \in W_1^r(M), \ d\omega = 0, \ \delta\omega = 0\}.$$

Any harmonic form is a cocycle in the complex $\Omega_d^*(M)$; the homomorphism:

$$x^r : H^r(M) \to \mathrm{H}^r(M, \mathbf{C})$$

(4.5) $$\omega \mapsto [\omega]$$

will be called *Hodge homomorphism.*

Theorem **4.1** *(Hodge Theory).*

For any closed, oriented, Lipschitz Riemannian manifold M^m, *and for any degree* r:

(i) *the Hodge homomorphism*

$$x^r : H^r(M) \to \mathrm{H}^r(M, \mathbf{C})$$

(1) In contrast with the smooth case.

is an isomorphism:

(ii) $* : H^r(M) \to H^{m-r}(M)$

is an isomorphism;

(iii) *there is a strong Hodge decomposition*

$$L_2^r = H^r(M) \oplus d\Omega_d^{r-1}(M) \oplus \delta\Omega_\delta^{r+1}(M), \tag{4.6}$$

which is a direct, orthogonal decomposition by closed subspaces in $L_2^r(M)$.

Proof. — The proof is based on the following Lemmas 4.1-4.4.

Lemma **4.1**. — *Suppose* $\omega \in L_2^r(M)$. *Then* $\omega \in \Omega_d^r(M)$ *if and only if there exists* $d\omega \in L_2^{r+1}(M)$ *such that, for any* $\varphi \in \Omega_d^{m-r+1}(M)$,

$$\int_M \omega \wedge d\varphi = (-1)^{r+1} \int_M d\omega \wedge \varphi. \tag{4.7}$$

Proof. — By a partition of unity argument via Lipschitz functions, checking (4.7) can be reduced to an analogue problem where ω is supported in an open set in $\mathbf{R}^m$. There the result is well known and can be proved by a little convolution argument.

Lemma **4.2**. — $\operatorname{Ker} d^r$ *is closed in* $L_2^r(M)$.

Proof. — Let $\omega_i \in \Omega_d^r(M)$, $i \in \mathbf{N}$, $d\omega^i = 0$, $\lim_{i\to\infty} \omega^i = \omega$. Let φ be any element in $\Omega_d^{m-r-1}(M)$; we have, using (4.7),

$$\begin{aligned} \int \omega \wedge d\varphi = \lim_{i\to\infty} \int \omega_i \wedge d\varphi &= (-1)^{r+1} \lim_{i\to\infty} \int d\omega_i \wedge \varphi \\ &= (-1)^{r+1} \lim_{i\to\infty} \int 0 \wedge \varphi = 0; \end{aligned} \tag{4.8}$$

from Lemma 4.1 we deduce that $d\omega = 0$.

Lemma **4.3**. — $\operatorname{Im} d^{r-1}$ is closed in $L_2^r(M)$.

Proof. — Suppose that $\omega_i = d\theta_i$, and $\lim_{i\to\infty} \omega_i = \omega$. Then ω_i are cocycles, and from Lemma 4.2, we deduce that ω is a cocycle. For any cocycle ξ we have, again using (4.7),

$$\int \omega \wedge \xi = \lim_{i\to\infty} \int \omega_i \wedge \xi = \lim_{i\to\infty} \int d\theta_i \wedge \xi = \pm \lim_{i\to\infty} \int \theta_i \wedge d\xi = \lim_{i\to\infty} 0 = 0, \tag{4.9}$$

which implies, in view of Theorem 2.1 (iii) that the cohomology class of ω is zero, or that ω is a coboundary.

Lemma **4.4**. — $h \in H^r(M) \Leftrightarrow \{h \in \operatorname{Ker} d^r, h \perp \operatorname{Im} d^{r-1}\}$.

Proof. — $h \perp \operatorname{Im} d^{r-1} \Leftrightarrow 0 = (d\varphi, h) = \int d\varphi \wedge *h$ for any $\varphi \in \Omega_d^{m-r-1}(M)$, which, by Lemma 4.1, is equivalent to $d(*h) = 0$.

The Lemmas 4.2-4.4 imply the statement (i) of Theorem 4.1. The statement (ii) is an immediate consequence of the definitions, and of (4.0).

For which regards the proof of (iii), we already know that $d\Omega_a^{r-1}(M)$ is closed (Lemma 4.3). Similarly, by (4.0),

$$\delta\Omega_\delta^{r+1}(M) = *d*(*\Omega_a^{m-r-1}(M)) = *(d\Omega_a^{m-r-1}(M)),$$

which is closed because $*$ is an isometry and $d\Omega_a^{m-r-1}(M)$ is a closed subspace, as seen before. Finally, $H^r(M)$ is closed in $L_2^r(M)$ because it is finite dimensional (Hodge isomorphism).

The fact that the three subspaces $H^r(M)$, $d\Omega_a^{r-1}(M)$, $\delta\Omega_\delta^{r+1}(M)$ are mutually orthogonal is a consequence of Lemma (4.1) along with (4.0). Any element in $L_2^r(M)$ which is orthogonal to $d\Omega_a^{r-1}(M) \oplus \delta\Omega_\delta^{r+1}(M)$ is, for the same reason, a harmonic form. The proof of Theorem 4.1 is completed.

For a proof of the combinatorial analogue of Theorem 4.1, the reader can refer to N. Teleman [14]. For a different approach and construction of Hodge theory on "admissible" pseudo-manifolds which are smoothly triangulated, see J. Cheeger [4]. See also J. Dodziuk [5] for Hodge theory on non-compact smooth manifolds.

Earlier abstract treatments for Hodge theory are due to G. Fichera [6] and M. Gaffney [7].

5. Signature Operators

In this section we present a first application of the Lipschitz Hodge theory.

For the sake of uniformity of notation we agree to denote $L_2^r(M)$ by $W_0^r(M)$. We introduce also, for $j = 1$ or 0,

$$W_j^*(M) = \bigoplus_{r=0}^{\dim M} W_j^r(M), \tag{5.1}$$

and

$$H^*(M) = \bigoplus_{r=0}^{\dim M} H^r(M). \tag{5.1'}$$

The operator

$$\begin{aligned} &D = d + \delta \\ &D : (W_1^*(M), \|\ \|_1) \to (W_0^*(M), \|\ \|) \end{aligned} \tag{5.2}$$

is clearly a continuous operator.

Proposition 5.1. — *The operator* D *is a Fredholm operator.*

Proof. — It is easy to check that

$$\operatorname{Ker} D = H^*(M). \tag{5.3}$$

Indeed, if $u \in \operatorname{Ker} D$, then the strong Hodge decomposition (4.6) gives:

$$0 = ((d+\delta)u, (d+\delta)u) = (du, du) + (\delta u, \delta u),$$

which shows that

$$du = 0, \quad \delta u = 0,$$

that is, u is harmonic. The reverse inclusion is obvious.

We are going to determine the range of D. We shall check that

$$\operatorname{Im} D = d\Omega_d^*(M) \oplus \delta\Omega_\delta^*(M), \tag{5.4}$$

where $\Omega_d^*(M) = \bigoplus_r \Omega_d^r(M)$,

and

$$\Omega_\delta^*(M) = \bigoplus_r \Omega_\delta^r(M). \tag{5.5}$$

Once (5.4) is established, the strong Hodge decomposition (4.6) will supply that D has closed range, and

$$\operatorname{Coker} D = H^*(M). \tag{5.6}$$

It is clear, from the very definition of D, that

$$\operatorname{Im} D \subseteq d\Omega_d^*(M) \oplus \delta\Omega_\delta^*(M); \tag{5.7}$$

conversely, let a be any element in $\Omega_d^*(M)$; then (4.6) tells that a may be decomposed:

$$a = h + d\alpha + \delta\beta, \tag{5.8}$$

where $h \in H^*(M), \quad \alpha \in \Omega_d^*(M), \quad \text{and} \quad \beta \in \Omega_\delta^*(M).$

As a, h, $d\alpha$ belong to $\Omega_d^*(M)$, it follows that $\delta\beta$ belongs to the same space. In addition, $\delta\beta \in \Omega_\delta^*(M)$, because $\delta^2 = 0$, and hence $\delta\beta \in W_1^*(M)$; then, from (5.8), we get

$$da = d\delta\beta = D(\delta\beta) \tag{5.9}$$

which shows

$$d\Omega_d^*(M) \subseteq \operatorname{Im} D. \tag{5.10}$$

We can prove similarly that

$$\delta\Omega_\delta^*(M) \subseteq \operatorname{Im} D. \tag{5.11}$$

From (5.10) and (5.11) we get

$$d\Omega_d^*(M) \oplus \delta\Omega_\delta^*(M) \subseteq \operatorname{Im} D, \tag{5.12}$$

which together with (5.7) implies (5.4).

Proposition 5.2. — *The space* $W_1^*(M)$ *is an infinite dimensional separable Hilbert space.*

Proof. — $W_0^*(M)$ is separable and infinite dimensional. The Proposition 5.1 along with the open map theorem completes the argument.

From now on we suppose that $\dim M = 4\mu$.

The operator D splits out [3] in a direct sum of two operators:

$$D = D^- \oplus D^+ \tag{5.13}$$

by means of the involution

$$\tau : W_0^r(M) \to W_0^{4\mu - r}, \qquad \tau = i^{r(r-1)+2\mu} *, \qquad i = \sqrt{-1}; \tag{5.14}$$

τ keeps the subspace $W_1^*(M)$ of $W_0^*(M)$ fixed and

$$\tau D = - D\tau. \tag{5.15}$$

Let $W_1^{\pm}(M)$, resp. $W_0^{\pm}(M)$, denote the eigenspaces in $W_1^*(M)$, resp. $W_0^*(M)$, corresponding to the eigenvalues ± 1 for the involution τ in $W_1^*(M)$, resp. $W_0^*(M)$. One defines $D^{\pm}$ to be the restriction of D to $W_1^{\pm}(M)$, and these operators act as in the smooth case [3]:

$$\begin{aligned} &D^{\pm} : W_1^{\pm}(M) \to W_0^{\mp}(M), \\ &D = D^- \oplus D^+. \end{aligned} \tag{5.15}$$

Theorem 5.3 (Signature Theorem). — *For any closed oriented Lipschitz manifold* M *of dimension* 4μ, *with Lipschitz Riemannian metric,* $D^{\pm}$ *are Fredholm operators and*

$$\operatorname{Index} D^+ \equiv \dim \operatorname{Ker} D^+ - \dim \operatorname{Coker} D^+ = \operatorname{Sig} M. \tag{5.16}$$

Proof. — D is a Fredholm operator by Proposition 5.1, and from (5.15) we get that its sumands D^-, D^+ should be Fredholm operators also.

The computation of Index D^+ follows from Hodge theory, see Atiyah-Singer [3].

The operator D^+ is called *signature operator.*

6. Signature operators with values in vector bundles

We keep all notations from the previous sections.

Let $\xi \to M$ be a Lipschitz complex vector bundle of finite rank over the closed, oriented Lipschitz Riemannian manifold M.

We introduce the following spaces:

$$L(M) = \text{algebra of complex valued Lipschitz functions on } M.$$

$$L(\xi) = L(M)\text{-module of Lipschitz sections in } \xi.$$

$$(6.1) \qquad \begin{aligned} W_j^r(M, \xi) &= W_j^r(M) \otimes_{L(M)} L(\xi), \quad j = 0 \text{ or } 1, \quad 0 \leq r \leq \dim M, \\ W_j^*(M, \xi) &= \bigoplus_r W_j^r(M, \xi), \\ W_j^{\pm}(M, \xi) &= W_j^{\pm}(M) \otimes_{L(M)} L(\xi), \quad j = 0 \text{ or } 1. \end{aligned}$$

$\tau_\xi = \tau \otimes \mathbf{1}_\xi$ is an involution in $W_j^*(M, \xi)$.

We intend to define signature operators with values in ξ, see M. F. Atiyah, R. Bott, V. K. Patodi [1] and M. F. Atiyah, I. M. Singer [3]. To this aim, we take a vector bundle embedding $\nabla : \xi \to \underline{N}$, where $\underline{N} \to M$ is the product bundle of rank N on M. Considering $\underline{N}$ endowed with the trivial Hermitian structure, ξ itself will become a Hermitian bundle by restriction, and we will refer to ∇ as a *linear connection* in ξ.

Let $\xi^\perp$ denote the orthogonal complement to ξ in $\underline{N}$.

We have for $j = 0$, or 1:

$$(6.2) \qquad \begin{aligned} \bigoplus^{N} W_j^*(M) &= W_j^*(M, \underline{N}) = W_j^*(M, \xi) \oplus W_j^*(M, \xi^\perp), \\ \bigoplus^{N} W_j^{\pm}(M) &= W_j^{\pm}(M, \underline{N}) = W_j^{\pm}(M, \xi) \oplus W_j^{\pm}(M, \xi^\perp). \end{aligned}$$

The relations (6.2), entitle us to introduce a Hilbert space structure on $W_j^*(M, \xi)$ and $W_j^{\pm}(M, \xi)$ given that they are closed subspaces of the direct sum of N copies of the Hilbert spaces: $W_j^*(M)$, $W_j^{\pm}(M)$.

Let p_ξ, i_ξ, $p_{\xi^\perp}$ and $i_{\xi^\perp}$ denote the obvious projections and inclusions:

$$(6.3) \qquad W_j^*(M, \xi) \underset{i_\xi}{\overset{p_\xi}{\rightleftarrows}} W_j^*(M, \underline{N}) \underset{i_{\xi^\perp}}{\overset{p_\xi}{\rightleftarrows}} W_j^*(M, \xi^\perp).$$

The operator

$$D_\xi^+ \equiv p_\xi \circ \underbrace{(D \oplus \ldots \oplus D)}_{N} \circ i_\xi : W_1^+(M, \xi) \to W_0^-(M, \xi) \qquad (6.4)$$

will be referred to as *signature operator with values in* ξ.

The *basic problem* of this paper consists in showing that the operator D_ξ^+ is a Fredholm operator and that its index depends only on the Lipschitz structure of the bundle ξ and its base manifold M. In the remainder of this paper we will provide all the necessary analysis for the proof of that statement.

Notice that the operator D_ξ^+ *depends* on two arbitrary structures, in addition to the Lipschitz structure on M: 1) a Riemannian metric on M, and 2) a connection ∇ in ξ.

7. The inclusion $W_1^\cdot(M, \xi) \hookrightarrow W_0^\cdot(M, \xi)$ is compact

In the forthcoming sections § 7-8 we shall prove the following Rellich-type result:

Theorem **7.1.** — *For any Lipschitz vector bundle* ξ *over the oriented closed Lipschitz Riemannian manifold* M^m *of any dimension* m, *the inclusion*:

$$(W_1^\cdot(M, \xi), \|\ \|_1) \hookrightarrow (W_0^\cdot(M, \xi), \|\ \|) \qquad (7.1)$$

is compact.

Corollary **7.1.** — *The signature operator* D_ξ^+ *is a Fredholm operator, and its index*

$$\operatorname{Ind} D_\xi^+ = \dim \operatorname{Ker} D_\xi^+ - \dim \operatorname{Coker} D_\xi^+$$

does not depend on the connection ∇.

Proof. — It is easy to check that

$$D_\xi^+ \oplus D_{\xi\perp}^+ = D_N^+ + A, \quad \text{where } D_N^+ = D^+ \oplus \ldots \oplus D^+ \ (N \text{ times}),$$

and A is a bundle homomorphism having, locally, bounded measurable coefficients; hence $\|A\omega\|_0 \leq \text{Const.}\, \|\omega\|_1$ for any $\omega \in W_1^\cdot(M, \xi)$. From the Proposition 7.3 below along with the Theorem 7.1 we deduce that the operator A is compact. By Theorem 5.3, D^+, and therefore also D_N^+, are Fredholm operators. Therefore $D_\xi^+ \oplus D_{\xi\perp}^+$ is a compact perturbation of a Fredholm operator, so it is a Fredholm operator itself, and so is D_ξ^+.

If ∇, ∇' are two connections in ξ, and D_ξ^+, $D_\xi'^+$ are the corresponding signature operators, for the same Lipschitz structure and Lipschitz Riemannian metric, then

$$D_\xi'^+ = D_\xi^+ + A,$$

where A is an operator of the same type as before, and so it is compact. Hence the index is unchanged.

Proof of Theorem **7.1**

By means of a Lipschitz partition of the unity on M, we may reduce the problem to a local one in $\mathbf{R}^m$; for the same reason we may suppose that the bundle ξ is trivial.

Let U denote a relatively compact open set in $\mathbf{R}^m$ with boundary of measure zero. We shall use on U two Lipschitz Riemannian metrics: the standard Euclidean metric Γ, which in some instances will be thought of as defined on all of $\mathbf{R}^m$, and an arbitrary Lipschitz Riemannian metric $\widetilde{\Gamma}$.

The metric Γ defines a star opertor $*$, a scalar product (3.3′) and norm (3.3). The coresponding objects deriving from $\widetilde{\Gamma}$ will be indicated by adding $\sim$.

Set $L_2^r(U) \subset L_2^r(\mathbf{R}^m)$ for the subspace of forms of degree r with support in $\overline{U}$. Both norms $\|\ \|$, $\|\ \|^\sim$ lead to the same spaces of L_2-forms on U, with equivalent norms.

For any r we introduce:

$$\widetilde{W}_1^r(U) = \{a \mid a \in L_2^r(U), \text{such that } da \in L_2^{r+1}(U) \text{ and } d\,\tilde{*}\,a \in L_2^{m-r+1}(U)\}, \tag{7.2}$$

d being the distributional exterior derivative, and we define on $\widetilde{W}_1^r(U)$ the *diagonal norm*:

$$\|a\|_1^2 = \|a\|^2 + \|da\|^2 + \|d\,\tilde{*}\,a\|^2 \tag{7.3}$$

which is equivalent to the $\|\ \|_1^\sim$-norm.

We shall check that $\{\widetilde{W}_1^r(U), \|\ \|_1^\sim\}$ is a Hilbert space. Notice then, that the obvious inclusion $\widetilde{W}_1^r(U) \subset L_2^r(U)$ is continuous.

For, let $\{a_n\}_n \subset \widetilde{W}_1^r(U)$ be a Cauchy sequence in the diagonal norm. Then $\{a_n\}_n$, $\{da_n\}_n$, $\{d\,\tilde{*}\,a_n\}_n$ are Cauchy sequences in $L_2(U)$ and so they converge here; say

$$\lim_{n\to\infty} a_n = a.$$

For any smooth form b with support in U,

$$\int_{\mathbf{R}^m} a \wedge db = \lim_{n\to\infty}\int_{\mathbf{R}^m} a_n \wedge db = (-1)^{k+1}\lim_{n\to\infty}\int_{\mathbf{R}^m} da_n \wedge b = (-1)^{k+1}\int_{\mathbf{R}^m}\lim_{n\to\infty} da_n \wedge b,$$

and
$$\int_{\mathbf{R}^m}\tilde{*}\,a \wedge db = \lim_{n\to\infty}\int_{\mathbf{R}^m}\tilde{*}\,a_n \wedge db = (-1)^{m-k+1}\int_{\mathbf{R}^m}\lim_{n\to\infty} d\,\tilde{*}\,a_n \wedge b,$$

which show that $a \in \widetilde{W}_1^r(U)$, and

$$da = \lim_{n\to\infty} da_n, \qquad d\,\tilde{*}\,a = \lim_{n\to\infty} d\,\tilde{*}\,a_n.$$

In the sequel $F : L_2^r(\mathbf{R}^m) \circlearrowleft$ will denote the Fourier transform, componentwise defined by

$$(Fa)(x) = (2\pi)^{-m/2}\int_{\mathbf{R}^m} e^{-ix.\xi} a(\xi)\,d\xi. \tag{7.4}$$

The proof of Theorem 7.1 is based on the following two propositions.

Proposition **7.2** (Basic Estimate). — *For any relatively compact open set* $U \subset \mathbf{R}^m$ *with boundary of measure zero, and any Lipschitz Riemannian metric* $\widetilde{\Gamma}$ *on* U, *there exists a positive constant* K *such that*

$$(7.5) \qquad \|a\|^2 \leq K \left\{ \left\| \frac{1}{r} F da \right\|^2 + \left\| \frac{1}{r} F d \,\tilde{*}\, a \right\|^2 \right\}$$

for any $a \in \widetilde{W}^r_1(U)$, $0 \leq r \leq m$, *where* $r : \mathbf{R}^m \to [0, \infty)$ *is the Euclidean distance to the origin* $(r^2(x^1, \ldots, x^m) = \sum_i x_i^2)$.

The constant K *may be taken in accordance with* (8.4).

If $\widetilde{\Gamma}$ *is the Euclidean metric, then the inequality in* (7.5) *becomes equality with* $K = 1$.

The proof of Proposition 7.2 will be deferred to Section 8.

Proposition **7.3**. — *Let* S *and* T *be two continuous linear operators from the separable Hilbert space* X *into the Hilbert spaces* Y_1, Y_2:

$$X \begin{array}{l} \overset{S}{\nearrow} Y_1 \\ \underset{T}{\searrow} Y_2 \end{array}$$

Suppose (i) T *is compact, and* (ii) *there exists a positive constant* K *such that, for any* $x \in X$, $\|Sx\| \leq K \|Tx\|$. *Then* S *is compact.*

Proof of Proposition **7.3**. — Take any bounded sequence $\{x_n\}_{n \in \mathbf{Z}} \subset X$; we shall prove that there exists a subsequence of $\{Sx_n\}_{n \in \mathbf{Z}}$ which converges in Y_1.

The operator T being compact, there exists a subsequence of $\{Tx_n\}_{n \in \mathbf{Z}}$ (which we suppose to be the sequence itself) converging in Y_2; this sequence is, a fortiori, a Cauchy sequence. The hypothesis (ii) implies that $\{Sx_n\}_{n \in \mathbf{Z}}$ is a Cauchy sequence in Y_1, and therefore it converges in Y_1. This proof is due to A. Schep.

Now, we can pass to the proof of Theorem 7.1. For, let us consider the following operators:

$$(7.6) \qquad \begin{aligned} &S \text{ and } T : \widetilde{W}^r_1(U) \to L^*_2(\mathbf{R}^m) \oplus L^*_2(\mathbf{R}^m), \\ &S : a \mapsto (a, a), \\ &T : a \mapsto \left(\frac{1}{r} F da, \frac{1}{r} F d \,\tilde{*}\, a \right). \end{aligned}$$

We will check that S and T verify the hypotheses of Proposition 7.3. We discuss first the case $m \geq 3$.

The operator S is, clearly, continuous.

In order to verify that T is continuous and compact, it is enough to show that the operator

$$Q: L_2^r(U) \to L_2^r(\mathbf{R}^m), \quad 0 \leq r \leq m,$$

$$Q: a \mapsto \frac{1}{r} Fa$$

is defined, continuous and compact because T can be factorized:

$$\widetilde{W}_1^r(U) \xrightarrow{(d, d\tilde{*})} L_2^{r+1}(U) \oplus L_2^{m-r+1}(U) \xrightarrow{(Q, Q)} L_2^*(\mathbf{R}^m) \oplus L_2^*(\mathbf{R}^m),$$

and the first operator is continuous.

In order to study Q, we decompose it in two operators $Q = Q_0 + Q_\infty$. These operators are

$$Q_0 = FQ, \quad \text{and} \quad Q_\infty = (1 - F)Q,$$

where $F: \mathbf{R}^m \to [0, 1]$ is a C^∞-bell function, with compact support, identically 1 on a neighborhood of 0. We show that Q_0 and Q_∞ are defined, continuous and compact.

The operator Q_0 is:

$$(Q_0 a)(x) = (2\pi)^{-m/2} \int_U \frac{F(x)}{r(x)} e^{-ix.\xi} . a(\xi) d\xi.$$

The L_2-norm of its kernel on the compact set $U \times \operatorname{Supp} F$ is:

$$(2\pi)^{-m} \int_{U \times \operatorname{Supp}F} \left| \frac{F(x)}{r(x)} e^{-ix.\xi} \right|^2 d\xi . dx \leq (2\pi)^{-m} \int_{\operatorname{Supp}F} \frac{dx}{r^2(x)} . \int_U d\xi.$$

If $m \geq 3$, $\int_{\operatorname{Supp}F} \frac{dx}{r^2(x)} < \infty$, and therefore Q_0 is a Hilbert-Schmidt operator. In consequence, it is compact.

Concerning Q_∞, notice that $F^{-1}Q_\infty$ is a pseudo-differential operator of order -1, and therefore it is compact. Hence Q and T are compact.

It remains to check (ii) from Proposition 7.3. We have:

$$\|Sa\|^2 = 2\|a\|^2$$

$$\|Ta\|^2 = \left\| \frac{1}{r} Fda \right\|^2 + \left\| \frac{1}{r} Fd \tilde{*} a \right\|^2.$$

The basic estimate (7.5) supplies the fulfillment of condition (ii) of Proposition 7.3. Therefore, Proposition 7.3 says that the operator S from (7.6) is compact; a fortiori, the inclusion of $\widetilde{W}_1^r(U)$ in $L_2^r(U)$ is compact, and Theorem 7.1 is proved for $m \geq 3$, once Proposition 7.2 will be established.

Before discussing the proof of Proposition 7.2, we intend to show that Theorem 7.1 proved only for $m \geq 3$ implies that same Theorem in any dimension.

For, let M be a Lipschitz Riemannian manifold of any dimension, and let T^3 denote the 3-dimensional torus endowed with the standard Riemannian metric. The space $W_j^*(M^m, \xi)$, $j = 1, 0$, may be thought of as a subspace of $W_j^*(M^m \times T^3, \xi \times T^3)$ through the isometric embedding

$$(7.7) \qquad \omega \mapsto \omega \otimes \mathbf{1}_{T^3}, \qquad \omega \in W_j^*(M, \xi).$$

Now, $\dim M \times T^3 \geq 3$ and so the inclusion:

$$W_1^*(M \times T^3, \xi \times T^3) \overset{i}{\hookrightarrow} W_0^*(M \times T^3, \xi \times T^3)$$

is compact; therefore, the restriction of the inclusion i to the subspace $W_1^*(M, \xi)$ will be still compact, which completes the discussion.

8. Proof of Proposition 7.2. (Basic Estimate)

Lemma **8.1.** — *If* H *is any Hilbert space, then:*

$$(8.1) \qquad ||a + b||^2 \geq \frac{1}{2}||a||^2 - \frac{3}{2}||b||^2, \quad \text{for any } a, b \in H.$$

Lemma **8.2.** — *Let* P_1, P_2 *be two complementary, orthogonal projectors in the Hilbert space* H, *i.e.*

$$(8.2) \qquad P_i^2 = P_i, \quad i = 1, 2, \quad P_1 + P_2 = 1_H$$

$$(P_1 a, P_2 b) = 0 \quad \text{for any } a, b \in H.$$

Let A *be any bounded, strictly positive operator in* H. *Then there exists a positive constant* K *such that*

$$(8.3) \qquad ||a||^2 \leq K\{||P_1 a||^2 + ||P_2 A a||^2\}, \quad \text{any } a \in H.$$

One can take

$$(8.4) \qquad K = (C^2 + 3||A||^2 + 2)/C^2,$$

where C *defines the positivity of* A:

$$(Aa, a) \geq C||a||^2, \quad 0 < C < 1.$$

Proof. — As $P_i H$, $i = 1, 2$, are complementary, orthogonal closed subspaces in H, any element $a \in H$ can be uniquely decomposed as $a = a_1 + a_2$, $a_i \in P_i H$, and $||a||^2 = ||a_1||^2 + ||a_2||^2$. For any real constant k, we have, using (8.1),

$$(8.5) \qquad k||P_1 a||^2 + ||P_2 A a||^2 = k||a_1||^2 + ||P_2 A a_1 + P_2 A a_2||^2$$

$$\geq k||P_1 a_1||^2 + \frac{1}{2}||P_2 A a_2||^2 - \frac{3}{2}||P_2 A a_1||^2$$

$$\geq \left(k - \frac{3}{2}||A||^2\right) . ||a_1||^2 + \frac{1}{2}||P_2 A a_2||^2.$$

This inequality proves the lemma for $a_2 = 0$. So, we may suppose $a_2 \neq 0$.

As $a_2 \in P_2H$, Parseval's equality, along with the positivity of A, give

$$\text{(8.6)} \qquad ||P_2 A a_2||^2 \geq \left|\left(A a_2, \frac{a_2}{||a_2||}\right)\right|^2 \geq C^2 ||a_2||^2.$$

Now, (8.5) becomes

$$\text{(8.7)} \qquad k\,||P_1 a||^2 + ||P_2 A a||^2 \geq \left(k - \frac{3}{2}||A||^2\right).||a_1||^2 + \frac{1}{2} C^2 ||a_2||^2.$$

If we take $k = \frac{1}{2} C^2 + \frac{3}{2} ||A||^2 + 1$ in (8.7), we get

$$k(||P_1 a||^2 + ||P_2 A a||^2) \geq k\,||P_1 a||^2 + ||P_2 A a||^2$$

$$\geq \frac{1}{2} C^2(||a_1||^2 + ||a_2||^2) = \frac{1}{2} C^2 ||a||^2,$$

which completes the proof.

Lemma **8.3**. — *Suppose* $a \in L_2^r(\mathbf{R}^m)$ *is compactly supported and has* $da \in L_2^{r+1}(\mathbf{R}^m)$. *Then:*

$$\text{(8.8)} \qquad F da = i r dr \wedge F a, \qquad i = \sqrt{-1}.$$

Proof. — By a convolution argument you can show that there exists a sequence $\{a_n\}_{n \in \mathbf{Z}}$ of smooth forms with compact support in $\mathbf{R}^m$, such that

$$\lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} da_n = da, \text{ in } L_2(\mathbf{R}^m).$$

Therefore, as F is continuous on L_2,

$$\text{(8.9)} \qquad F da = F \lim_{n\to\infty} da_n = \lim_{n\to\infty} F da_n = i \lim_{n\to\infty} \sum_{k=1}^{m} x^k dx^k \wedge F a_n = i \lim_{n\to\infty} r dr \wedge F a_n,$$

the third equality being easy to check. But, $\{F a_n\}_n$ is a sequence in $L_2^r(\mathbf{R}^m)$ whose limit is Fa. In consequence, (8.9) shows that

$$F da = i \lim_{n\to\infty} r dr \wedge F a_n = i r dr \wedge F a,$$

which proves the assertion.

Lemma **8.4**. — *Let* $\lrcorner$ *denote the Euclidean interior product. Then:*

(i) *the operators* $P_1, P_2 : L_2^r(\mathbf{R}^m) \to L_2^r(\mathbf{R}^m)$

$$\text{(8.10)} \qquad \begin{aligned} P_1 a &= dr \lrcorner (dr \wedge a), \\ P_2 a &= dr \wedge (dr \lrcorner a), \end{aligned}$$

are well-defined and continuous,

(ii) P_1, P_2 *are projectors,*

(iii) $P_1 + P_2 = 1$,

(iv) $(P_1 a, P_2 b) = 0$, *for any* $a, b \in L_2^r(\mathbf{R}^m)$.

Proof. — The Euclidean metric induces a scalar product $\langle, \rangle_x$ in $\Lambda^*(T_x^*\mathbf{R}^m)$, the exterior algebra over the cotangent space $T_x^*\mathbf{R}^m$. Notice also that dr is a C^∞-field of vectors of length 1 on $\mathbf{R}^m - \{0\}$.

We will say that a differential form b is "pointwise orthogonal to dr" if $\langle b(x), dr(x) \wedge \Lambda^*(T_x^*\mathbf{R}^m)\rangle_x = 0$ for any $x \in \mathbf{R}^m - \{0\}$.

Any differential form a on $\mathbf{R}^m$ may be uniquely decomposed:

$$a = dr \wedge b + c, \tag{8.11}$$

where b and c are pointwise orthogonal to dr. The operators P_1 and P_2 are differential operators of order zero, and they act on a form (8.11) in this way:

$$P_1 a = c, \quad P_2 a = dr \wedge b. \tag{8.12}$$

Notice that $dr \wedge b$ and c are point by point orthogonal.

We need some more elementary local considerations.

Any form which is pointwise orthogonal to dr, can be expressed, locally, as an exterior polynomial in $m - 1$ forms of order 1, which are pointwise orthogonal to dr, say $v_1, \ldots, v_{m-1}$; we may also suppose that these forms are orthonormal point by point and that

$$dr \wedge v_1 \wedge \ldots \wedge v_{m-1} = dx^1 \wedge \ldots \wedge dx^m.$$

We define a natural star operator $\circledast$ on the algebra of forms which are pointwise orthogonal to dr:

$$\circledast\, v_{i_1} \wedge \ldots \wedge v_{i_k} = \varepsilon v_1 \wedge \ldots \wedge \hat{v}_{i_1} \wedge \ldots \wedge \hat{v}_{i_k} \wedge \ldots \wedge v_{m-1} \tag{8.13}$$

where ε is the sign of the permutation of $m - 1$ indices $(i_1, \ldots, i_k, 1, \ldots, \hat{i}_1, \ldots, \hat{i}_k, \ldots, m-1)$. Obviously, $\circledast$ does not depend on the particular orthonormal base $v_1, \ldots, v_{m-1}$.

The operators $*$ and $\circledast$ are related by the formula:

$$*(dr \wedge b + c) = \circledast b \pm dr \wedge \circledast c, \tag{8.14}$$

where b and c are pointwise orthogonal to dr.

Lemma **8.5**. — *For any* $a \in L_2^r(\mathbf{R}^m)$, *we have*

$$P_2 * a = \pm\, dr \wedge \circledast P_1 a, \tag{8.15}$$

$$\|P_1 a\| = \|dr \wedge a\|, \tag{8.16}$$

$$\|\circledast P_1 a\| = \|P_1 a\|. \tag{8.17}$$

Proof. — (8.15) is an immediate consequence of (8.12) and (8.14). For which concerns (8.16) and (8.17), notice that for any form $a \in L_2(\mathbf{R}^m)$,

$$\|a\|^2 = \int_{\mathbf{R}_m} \langle a(x), a(x)\rangle_x dx^1 \ldots dx^m,$$

and that the equalities:

$$\langle P_1 a, P_1 a\rangle_x = \langle dr \wedge a, dr \wedge a\rangle_x$$

$$\langle \circledast P_1 a, \circledast P_1 a\rangle_x = \langle P_1 a, P_1 a\rangle_x$$

hold point by point in $\mathbf{R}^m - \{0\}$. The lemma is proved.

We are now in a position to prove the basic estimate.

Let $A : L_2^r(\mathbf{R}^m) \to L_2^r(\mathbf{R}^m)$ be the operator

(8.18) $$A(a) = (-1)^{r(m-r)} * F \tilde{*} F^{-1} a.$$

The operator A is continuous because F, $*$, $\tilde{*}$ are continuous.

We show that A is a strictly positive operator:

$$(Fa, A(Fa)) = (-1)^{r(m-r)}(Fa, *F\tilde{*}a)$$
$$= (-1)^{r(m-r)} \int_{\mathbf{R}_m} Fa \wedge \overline{**F\tilde{*}a} = \int_{\mathbf{R}_m} Fa \wedge \overline{F\tilde{*}a},$$

and as F is an isometry, we have further

$$(Fa, A(Fa)) = \int_{\mathbf{R}_m} a \wedge \overline{\tilde{*}a} = \|a\|_\sim^2 \geq C_1 \|a\|^2,$$

the last inequality being granted by (3.2). On the other hand, Lemma 8.4 says that the operators P_1, P_2 defined by (8.10), together with A given by (8.18), satisfy the hypothesis of Lemma 8.2. From this lemma we deduce that there exists a constant $K > 0$ such that

(8.19) $$\|a\|^2 = \|Fa\|^2 \leq K\{\|P_1 Fa\|^2 + \|P_2 AFa\|^2\}$$
$$= K\{\|P_1 Fa\|^2 + \|P_2 * F\tilde{*}a\|^2\}.$$

We compute, separately, the last two terms. By (8.16) and (8.8), we have

(8.20) $$\|P_1 Fa\| = \|dr \wedge Fa\| = \left\|\frac{1}{r} Fda\right\|,$$

and by (8.15), (8.16) (used twice), (8.17) and (8.8),

(8.21) $$\|P_2 * F\tilde{*}a\| = \|dr \wedge \circledast P_1 F\tilde{*}a\| = \|P_1 \circledast P_1 F\tilde{*}a\| = \|\circledast P_1 F\tilde{*}a\|$$
$$= \|P_1 F\tilde{*}a\| = \|dr \wedge F\tilde{*}a\| = \left\|\frac{1}{r} Fd\tilde{*}a\right\|.$$

From (8.19), (8.20) and (8.21), we get:

$$\|a\|^2 \leq K\left\{\left\|\frac{1}{r} Fda\right\|^2 + \left\|\frac{1}{r} Fd*a\right\|^2\right\},$$

which proves the basic estimate (7.5).

For which concerns the last assertion of Proposition 7.2, the new fact—which otherwise characterizes the Euclidean metrics—is that $F* = *F$. In consequence, if we represent $Fa = dr \wedge b + c$, with b and c pointwise orthogonal to dr, we have

$$
\begin{aligned}
&dr \wedge Fa = dr \wedge c, \\
(8.22) \qquad &dr \wedge F{*}a = dr \wedge {*}Fa = dr \wedge ((\circledast)b \pm dr \wedge (\circledast)c) = dr \wedge (\circledast)b,
\end{aligned}
$$

and further, using (8.22) and (8.8),

$$
\begin{aligned}
\|a\|^2 = \|Fa\|^2 = \|dr \wedge b\|^2 + \|c\|^2 &= \|dr \wedge (\circledast)b\|^2 + \|dr \wedge c\|^2 \\
&= \|dr \wedge F{*}a\|^2 + \|dr \wedge Fa\|^2 = \left\|\frac{1}{r} Fd{*}a\right\|^2 + \left\|\frac{1}{r} Fda\right\|^2.
\end{aligned}
$$

9. Weak maximum modulus property

This section is independent of the remainder of the paper. We present here an immediate application of the basic inequality (7.5).

Proposition **9.1.** — *Let Ω be a relatively compact domain in $\mathbf{R}^m$, and let Γ be a Lipschitz Riemannian metric on Ω. If ω is a Lipschitz harmonic form on $\mathbf{R}^m$ with support in Ω, then ω is the zero form.*

THE EXCISION THEOREM FOR THE INDEX OF SIGNATURE OPERATORS

10. Excisive triples

In the sequel we will show that, under certain circumstances, the indices of two signature operators on two different Lipschitz manifolds, having a common open part, are equal.

The following geometric situation is ever present in which follows, even if, later on, additional conditions will be required.

Let ξ_α be a vector bundle over the Lipschitz manifold M_α, $\alpha = 1, 2$. Let Γ_α be a Riemannian metric on M_α, and ∇_α a linear connection in ξ_α. We say that $\{(\nabla_1, \Gamma_1), (\nabla_2, \Gamma_2), U\}$ is an *excisive triple* if:

(i) the base manifolds M_1, M_2 have a common open Lipschitz submanifold U:

$$M_1 \overset{i_1}{\hookleftarrow} U \overset{i_2}{\hookrightarrow} M_2$$

(ii) $\Gamma_1 \mid U = \Gamma_2 \mid U$, and

(iii) a Lipschitz vector bundle isomorphism

$$\xi_1 \mid U \xrightarrow{\Lambda_{21}} \xi_2 \mid U$$

is given, which carries the connection $\nabla_1 \mid U$ into the connection $\nabla_2 \mid U$, and preserves the hermitian metrics.

We write Λ_{12} for Λ_{21}^{-1}. Note that since $\Lambda_{\alpha\beta}$ is hermitian on each fibre, $\Lambda^*_{\alpha\beta} = \Lambda_{\beta\alpha}$ (pointwise adjoint). This isomorphism will be called *identifying isomorphism* of the triple.

Given an excisive triple as above, let $f_\alpha : M_\alpha \to \mathbf{C}$, $\alpha = 1, 2$, be two Lipschitz functions so as

$$\text{(10.1)} \qquad \operatorname{supp} f_\alpha \subset U, \quad U \subset M_1 \cap M_2, \quad f_1 \mid U = f_2 \mid U = f.$$

For any $j \in \{0, 1\}$ and $\alpha, \beta \in \{1, 2\}$, let $\mu_j(f_\alpha, \Lambda_{\beta\alpha})$ be the multiplier

$$\text{(10.2)} \qquad \begin{aligned} \mu_j(f_\alpha, \Lambda_{\beta\alpha}) &: W^r_j(M_\alpha, \xi_\alpha) \to W^r_j(M_\beta, \xi_\beta), \\ \mu_j(f_\alpha, \Lambda_{\beta\alpha}) &: \omega \mapsto \Lambda_{\beta\alpha} f_\alpha \omega. \end{aligned}$$

Observe that $\mu_1(f_\alpha, \Lambda_{\beta\alpha})$ is well-defined because the Riemannian metrics agree on U (recall that W_1 depends on the metric).

If S and T are two continuous operators, we write $S \simeq T$ if $T - S$ is a compact operator. If T is a continuous operator, T^* denotes its adjoint.

Proposition **10.1.** — *For any* $\mu_j(f_\alpha, \Lambda_{\beta\alpha})$, $j = 0, 1$,

$$\mu_j(f_\alpha, \Lambda_{\beta\alpha})^* \simeq \mu_j(\bar{f}_\beta, \Lambda_{\alpha\beta}) = \mu_j(\bar{f}, \Lambda_{\alpha\beta}). \tag{10.3}$$

For this reason, the multiplier $\mu_j(f_\alpha, \Lambda_{\beta\alpha})$ will be simply denoted by $\Lambda_{\beta\alpha} f$, and $\mu_j(f_\alpha, \Lambda_{\beta\alpha})^*$ by $(\Lambda_{\beta\alpha} f)^*$.

Proof. — The proof of (10.3) for L_2-norms is immediate and, in fact,

$$\mu_0(f_\alpha, \Lambda_{\beta\alpha})^* = \mu_0(\bar{f}_\beta, \Lambda_{\alpha\beta}). \tag{10.4}$$

We discuss the case $j = 1$ now. Let $x \in W_1^r(M_\alpha, \xi_\alpha)$, and $y \in W_1^r(M_\beta, \xi_\beta)$ be arbitrary elements. A straightforward computation (in which the cancellation of some terms is involved), along with the Schwartz inequality give:

$$\begin{aligned} |(x, [\mu_1(f_\alpha, \Lambda_{\beta\alpha})^* - \mu_1(\bar{f}_\beta, \Lambda_{\alpha\beta})]y)_1| & \\ = |(\mu_1(f_\alpha, \Lambda_{\beta\alpha})x, y)_1 - (x, \mu_1(\bar{f}_\beta, \Lambda_{\alpha\beta})y)_1| & \\ \leq C\{\|x\|_0 . \|y\|_1 + \|x\|_1 . \|y\|_0\}, & \end{aligned} \tag{10.5}$$

where C is a constant depending on the Sup-norm of grad f.

In order to simplify the notation we introduce

$$K = \mu_1(f_\alpha, \Lambda_{\beta\alpha})^* - \mu_1(\bar{f}_\beta, \Lambda_{\alpha\beta}).$$

We take $x = K(y)$ in (10.5), and so we have, for any $y \in W_1^r(M_\beta, \xi_\beta)$,

$$\|Ky\|_1^2 \leq C\{\|Ky\|_0 . \|y\|_1 + \|Ky\|_1 . \|y\|_0\}. \tag{10.6}$$

Let $\{y_n\}_{n \in \mathbf{N}}$ be a bounded sequence in $W_1(M_\beta, \xi_\beta)$. As K is a continuous operator, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} &\|y_n\|_1 \leq C_1 \\ &\|Ky_n\|_1 \leq C_1, \quad \text{for any } n \in \mathbf{N}. \end{aligned} \tag{10.7}$$

The inclusion $W_1(M_\beta, \xi_\beta) \hookrightarrow W_0(M_\beta, \xi_\beta)$ being compact, there exists a subsequence—which we suppose to be the sequence itself—of the sequence $\{y_n\}_{n \in \mathbf{N}}$ with the property that it converges in $W_0(M_\beta, \xi_\beta)$. In the same way, we conclude that a subsequence—which we suppose again to be the sequence itself—of the sequence $\{Ky_n\}_{n \in \mathbf{N}}$, converges in $W_0(M_\alpha, \xi_\alpha)$.

The inequality (10.6) gives for $y = y_m - y_n$, in view of (10.7),

$$\|K(y_m - y_n)\|_1^2 \leq 2C . C_1\{\|K(y_m - y_n)\|_0 + \|y_m - y_n\|_0\}. \tag{10.8}$$

The right hand side of (10.8) converges to zero as m, n tend to infinity. Therefore

$$(10.9)\qquad \lim_{m,n\to\infty} \| K(y_m - y_n) \|_1^2 = 0,$$

which shows that the sequence $\{Ky_n\}_{n\in\mathbf{N}}$ converges in $W_1(M_\alpha, \xi_\alpha)$; therefore the operator $K = \mu_1(f_\alpha, \Lambda_{\beta\alpha})^* - \mu_1(\bar{f}_\beta, \Lambda_{\alpha\beta})$ is compact.

This completes the proof of Proposition 10.1.

Proposition **10.2.** — *Given an excisive triple as above, we have, for any* $\alpha, \beta \in \{1, 2\}$,

$$(10.10)\quad \begin{aligned} &\text{(i)} && D^{\pm}_{\xi_\beta}\Lambda_{\beta\alpha}f_\alpha \simeq \Lambda_{\beta\alpha}f_\alpha D^{\pm}_{\xi_\alpha} : W_1^{\pm}(M_\alpha, \xi_\alpha) \to W_0^{\mp}(M_\beta, \xi_\beta)\\ &\text{(ii)} && \Lambda_{\alpha\beta}\bar{f}_\beta(D^{\pm}_{\xi_\beta})^* \simeq (D^{\pm}_{\xi_\alpha})^*\Lambda_{\alpha\beta}\bar{f}_\beta : W_0^{\mp}(M_\beta, \xi_\beta) \to W_1^{\pm}(M_\alpha, \xi_\alpha).\end{aligned}$$

Proof. — (i) Taking into account that the operators $D^{\pm}_{\xi_\alpha}$, $D^{\pm}_{\xi_\beta}$ are first order differential operators (and hence, local) along with the fact that f_α is a Lipschitz function, we find that there exists a positive constant C (depending on $\operatorname{grad} f$) such that:

$$(10.11)\qquad \| (D^{\pm}_{\xi_\beta}\Lambda_{\beta\alpha}f_\alpha - \Lambda_{\beta\alpha}f_\alpha D^{\pm}_{\xi_\alpha})x \|_0 \leq C \| x \|_0,$$

for any $x \in W_1^{\pm}(M_\alpha, \xi_\alpha)$.

We know that the inclusion $i : W_1 \hookrightarrow W_0$ is compact (Theorem 7.1). The Proposition 7.3 applied to the operators

$$D^{\pm}_{\xi_\beta}\Lambda_{\beta\alpha}f_\alpha - \Lambda_{\beta\alpha}f_\alpha D^{\pm}_{\xi_\alpha},\ i,$$

which are related by the inequality (10.11), concludes the argument.

(ii) follows from (i) along with Proposition 10.1, by passing to adjoints.

11. The index of Skew-adjoint operators

If ξ is any bundle over M, we introduce (compare G. G. Kasparov [8]):

(i) the Hilbert space

$$(11.1)\qquad H_\xi = W_1^+(M, \xi) \oplus W_0^-(M, \xi);$$

(ii) the continuous operator

$$(11.2)\qquad D_\xi = \begin{pmatrix} 0 & -(D_\xi^+)^* \\ D_\xi^+ & 0 \end{pmatrix}$$

associated with the signature operator D_ξ^+;

(iii) the involution

$$(11.3)\qquad J_\xi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : H_\xi \to H_\xi.$$

We have

$$(11.4)\qquad D_\xi \circ J_\xi = -J_\xi \circ D_\xi.$$

Clearly,

(11.5) $$D_\xi^* = -D_\xi.$$

More generally, given an infinite dimensional separable Hilbert space H, we consider all pairs (D, J), where J is a Hermitian involution in H, and D is an operator which has the properties:

(i) $\mathrm{J} \circ D = -D \circ \mathrm{J}$

(11.6) (ii) $D^* \simeq -D$

(iii) D is a Fredholm operator.

The involution J splits out H in an orthogonal sum

(11.7) $$H = H^+ \oplus H^-,$$

where $H^\pm$ are the ± 1-eigenspaces of the involution J.

The requirement (i) implies that D, relative to the decomposition (11.7), has the following matricial description:

(11.8) $$D = \begin{pmatrix} 0 & \mathrm{D}^- \\ \mathrm{D}^+ & 0 \end{pmatrix},$$

where $\mathrm{D}^\pm : H^\pm \to H^\mp$,

while (11.6) (ii) gives that $(\mathrm{D}^+)^* \simeq -\mathrm{D}^-$.

Notice that for any D satisfying (11.6) (i)-(iii),

(11.9) $$\left(\frac{D - D^*}{2}\right)^* = -\frac{D - D^*}{2}$$

and

(11.9') $$\frac{D - D^*}{2} \simeq D.$$

D being a Fredholm operator, Ker D is finite dimensional, and (11.6) (i) ensures that J carries Ker D into itself.

Let $\mathrm{Ker}^\pm D$ denote the ± 1-eigenspaces of the involution J in Ker D. Then:

(11.10) $$\mathrm{Ker}^\pm D = \mathrm{Ker}\, \mathrm{D}^\pm.$$

If $D^* = -D$, as $(\mathrm{D}^\pm)^* = -\mathrm{D}^\mp$, we have

(11.11) $$\begin{aligned} \mathrm{Index}\, \mathrm{D}^+ &= \dim \mathrm{Ker}\, \mathrm{D}^+ - \dim \mathrm{Coker}\, \mathrm{D}^+ \\ &= \dim \mathrm{Ker}\, \mathrm{D}^+ - \dim \mathrm{Ker}\, \mathrm{D}^- \\ &= \dim \mathrm{Ker}^+ D - \dim \mathrm{Ker}^- D. \end{aligned}$$

These considerations suggest to introduce the following definition.

For any pair (D, J) satisfying (11.6) (i)-(iii), we define:

$$\text{(11.12)} \qquad \operatorname{Index}(D, \mathrm{J}) = \dim \operatorname{Ker}^{+} \frac{D - D^{*}}{2} - \dim \operatorname{Ker}^{-} \frac{D - D^{*}}{2},$$

and then:

$$\text{(11.12')} \qquad \operatorname{Index}(D_{\xi}, \mathrm{J}_{\xi}) = \operatorname{Index} \mathrm{D}_{\xi}^{+}.$$

With the pairs (D, J), the following basic operations may be performed:

$$\text{(11.13)} \qquad -(D, \mathrm{J}) \equiv (-D, -\mathrm{J}),$$

$$\text{(11.14)} \qquad (D_1, \mathrm{J}_1) + (D_2, \mathrm{J}_2) \equiv (D_1 \oplus D_2, \mathrm{J}_1 \oplus \mathrm{J}_2);$$

by the way, this would allow us to define K-homology groups [8], but the use of that terminology will be avoided in the sequel.

Theorem **11.1.** — *The index of skew-adjoint pairs* (11.6) *has the properties*:
(i) *if* K *is a compact operator which anticommutes with* J, *then*

$$\text{(11.15)} \qquad \operatorname{Index}(D + \mathrm{K}, \mathrm{J}) = \operatorname{Index}(D, \mathrm{J});$$

(ii) *if* $t \mapsto (D_t, \mathrm{J})$, $t \in [0, 1]$, *is a continuous homotopy of pairs* (11.6), *then*

$$\text{(11.16)} \qquad \operatorname{Index}(D_0, \mathrm{J}) = \operatorname{Index}(D_1, \mathrm{J});$$

(iii)

$$\text{(11.17)} \qquad \operatorname{Index}[(D_1, \mathrm{J}_1) + (D_2, \mathrm{J}_2)] = \operatorname{Index}(D_1, \mathrm{J}_1) + \operatorname{Index}(D_2, \mathrm{J}_2);$$

(iv)

$$\text{(11.18)} \qquad \operatorname{Index} - (D, \mathrm{J}) = - \operatorname{Index}(D, \mathrm{J}).$$

For (D, J) a pair (11.6), the homotopy [8]

$$\text{(11.19)} \qquad t \mapsto \left(\begin{pmatrix} D \cos t & -\sin t \\ \sin t & -D \cos t \end{pmatrix}, \mathrm{J} \oplus -\mathrm{J} \right), \qquad t \in \left[0, \frac{\pi}{2}\right]$$

connects the pair $(D, \mathrm{J}) + (-D, -\mathrm{J})$ with the *zero element*

$$\text{(11.20)} \qquad \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{J} & 0 \\ 0 & -\mathrm{J} \end{pmatrix} \right)$$

acting on $H \oplus H$. Everything is clear here, except for the fact that the homotopy is a homotopy by Fredholm operators. For that, it is enough to show that the operator:

$$\text{(11.21)} \qquad \begin{pmatrix} D \cos t & -\sin t \\ \sin t & -D \cos t \end{pmatrix}^{2} = \begin{pmatrix} D^2 \cos^2 t - \sin^2 t & 0 \\ 0 & D^2 \cos^2 t - \sin^2 t \end{pmatrix}$$

$$\simeq - \begin{pmatrix} D^{*}D \cos^2 t + \sin^2 t & 0 \\ 0 & D^{*}D \cos^2 t + \sin^2 t \end{pmatrix}$$

(cf. (11.6) (ii)) is Fredholm.

The fact that the last operator from (11.21) is a Fredholm operator is a particular case of Lemma 12.2 below.

12. The excision theorem for Index D_ξ^+

Theorem **12.1** (Excision). — *Let* M_α, $\alpha = 1, 2$, *be closed, oriented Lipschitz manifolds with Riemannian metrics* Γ_α, *and let* ξ_α, θ_α *be two Lipschitz vector bundles over* M_α *with linear connections* ∇_α, ∇'_α. *Suppose that* U *is an open Lipschitz submanifold of* M_1 *and* M_2, *and that* $V \subset U$ *is an open subset with* $\overline{V} \subset U$. *These objects are required to satisfy the following hypotheses:*

(a) $\{(\nabla_1, \Gamma_1), (\nabla_2, \Gamma_2), U\}$ *is an excisive triple with identifying isomorphism*

$$\xi_1 \mid U \xrightarrow{\Lambda_{21}} \xi_2 \mid U;$$

(b) $\{(\nabla'_1, \Gamma_1), (\nabla'_2, \Gamma_2), U\}$ *is an excisive triple with identifying isomorphism*

$$\theta_1 \mid U \xrightarrow{\Lambda'_{21}} \theta_2 \mid U;$$

(c) $\{(\nabla_\alpha, \Gamma_\alpha), (\nabla'_\alpha, \Gamma_\alpha), M_\alpha \backslash \overline{V}\}$, $\alpha = 1, 2$, *is an excisive triple with identifying isomorphism*

$$\xi_\alpha \mid (M_\alpha \backslash \overline{V}) \xrightarrow{\Sigma_\alpha} \theta_\alpha \mid (M_\alpha \backslash \overline{V});$$

(d) *the identifying isomorphisms are compatible*, i.e. *the diagram*

$$(12.1) \qquad \begin{array}{ccc} \xi_1 \mid (U\backslash\overline{V}) & \xrightarrow{\Lambda_{21}} & \xi_2 \mid (U\backslash\overline{V}) \\ \downarrow{\scriptstyle \Sigma_1} & & \downarrow{\scriptstyle \Sigma_2} \\ \theta_1 \mid (U\backslash\overline{V}) & \xrightarrow{\Lambda'_{21}} & \theta_2 \mid (U\backslash\overline{V}) \end{array}$$

is commutative.

Then,

$$(12.2) \qquad \text{Index } D^+_{\xi_1} - \text{Index } D^+_{\xi_2} = \text{Index } D^+_{\theta_1} - \text{Index } D^+_{\theta_2}.$$

If moreover,

(e) θ_1 *and* θ_2 *are trivial bundles, and*

(f) $\qquad \text{Sig } M_1 = \text{Sig } M_2,$

then

$$(12.3) \qquad \text{Index } D^+_{\xi_1} - \text{Index } D^+_{\xi_2} = 0.$$

(Compare with the Excision Axiom of M. F. Atiyah and I. M. Singer [2].)

Proof. — Notice first that (12.3) is a consequence of (12.2) along with Theorem 5.3; in fact, if (a)-(f) hold, then

$$(12.4) \qquad \begin{aligned} \text{Index } D^+_{\xi_1} - \text{Index } D^+_{\xi_2} &= \text{Index } D^+_{\theta_1} - \text{Index } D^+_{\theta_2} \\ &= \text{Rank } \theta_1 . \text{Sig } M_1 - \text{Rank } \theta_2 . \text{Sig } M_2 = 0. \end{aligned}$$

We are going to prove (12.2). For that purpose, we shall show that

$$\textbf{(12.5)} \qquad \operatorname{Index}\{[(D_{\xi_1}, J_{\xi_1}) - (D_{\xi_2}, J_{\xi_2})] - [(D_{\theta_1}, J_{\theta_1}) - (D_{\theta_2}, J_{\theta_2})]\} = 0$$

and then (11.12'), (11.17), (11.18) will complete the proof. Finally, in order to prove (12.5), we shall perform homotopies of the operators in the two brackets of (12.5), and ultimately, a homotopy of their difference.

Let

$$\varphi_\alpha, \psi_\alpha : M_\alpha \to [0, 1], \qquad \alpha = 1, 2$$

be Lipschitz functions having the following properties:

(12.6) (a) $\quad \operatorname{Supp} \varphi_1 = \operatorname{Supp} \varphi_2 \subset U,$

and for any $x \in U$, $\varphi_1(x) = \varphi_2(x)$,

(12.6) (b) $\quad$ for any $x \in M_\alpha$, $\quad \varphi_\alpha^2(x) + \psi_\alpha^2(x) = 1, \quad \alpha = 1, 2,$

(12.6) (c) $\quad \operatorname{Supp} \psi_\alpha \subset (M_\alpha \backslash V), \quad \alpha = 1, 2.$

A system of functions having these properties may be constructed as follows.

Let

$$\lambda : \overline{U} \to \left[0, \frac{\pi}{2}\right]$$

be a Lipschitz function with the properties:

$$\lambda \mid \partial\overline{U} = 0 \quad \text{and} \quad \lambda \mid V = \frac{\pi}{2}.$$

We extend λ by zero on $M_\alpha \backslash U$, and we get such a Lipschitz function λ_α, $\alpha = 1, 2$. The functions

$$\begin{matrix} \varphi_\alpha = \sin \lambda_\alpha \\ \psi_\alpha = \cos \lambda_\alpha \end{matrix} \qquad \alpha = 1, 2$$

have the required properties.

First step. — We will deal here with the first bracket of (12.5).

By means of the functions φ_α, ψ_α, we shall split out D_{ξ_α} into a sum of two operators: one with support in U and the other with support in $M_\alpha \backslash V$ in such a way that those with support in U agree, modulo compact operators. The homotopy (11.19) will serve in order to cancel out these two operators which agree on U with each other.

We will focus on the operators D_{ξ_α}. Proposition 10.2, with $M_1 = M_2 = M$, $\xi_1 = \xi_2$, $\Lambda_{12} = 1$, implies that:

$$\textbf{(12.7)} \qquad \begin{pmatrix} D_{\xi_1} & 0 \\ 0 & -D_{\xi_2} \end{pmatrix} \simeq \begin{pmatrix} \varphi_1 D_{\xi_1} \varphi_1 & 0 \\ 0 & -\varphi_2 D_{\xi_2} \varphi_2 \end{pmatrix} + \begin{pmatrix} \psi_1 . D_{\xi_1} . \psi_1 & 0 \\ 0 & -\psi_2 . D_{\xi_2} . \psi_2 \end{pmatrix}.$$

We consider the homotopy:

$$(12.8)\qquad \Delta_t(\xi_1, \xi_2) = \begin{pmatrix} \varphi_1 . D_{\xi_1} . \varphi_1 \cos t & -\Lambda_{12}\varphi_2^2 \sin t \\ \Lambda_{21}\varphi_1^2 \sin t & -\varphi_2 D_{\xi_2}\varphi_2 \cos t \end{pmatrix} + \begin{pmatrix} \psi_1 D_{\xi_1}\psi_1 & 0 \\ 0 & -\psi_2 . D_{\xi_2} . \psi_2 \end{pmatrix}, \qquad t \in \left[0, \frac{\pi}{2}\right]$$

where

$$\Delta_t(\xi_1, \xi_2) : H_{\xi_1} \oplus H_{\xi_2} \to H_{\xi_1} \oplus H_{\xi_2},$$

with H_{ξ_α} defined by (11.1).

It is easy to check that $(\Delta_t(\xi_1, \xi_2), J_{\xi_1} \oplus - J_{\xi_2})$ satisfies (11.6) (i), (ii). Now, we verify (11.6) (iii). We have:

$$(12.9)\qquad \Delta_t^2(\xi_1, \xi_2) = \left(\begin{array}{c|c} (\varphi_1 D_{\xi_1}\varphi_1 \cos t + \psi_1 . D_{\xi_1} . \psi_1)^2 - \varphi_1^2\varphi_2^2 \sin^2 t & -(\varphi_1 D_{\xi_1}\varphi_1 \cos t + \psi_1 D_{\xi_1}\psi_1)\Lambda_{12}\varphi_2^2 \sin t + \Lambda_{12}\varphi_2^2 \sin t(\varphi_2 D_{\xi_2}\varphi_2 \cos t + \psi_2 D_{\xi_2}\psi_2) \\ \hline \Lambda_{21}\varphi_1^2 \sin t(\varphi_1 D_{\xi_1}\varphi_1 \cos t + \psi_1 D_{\xi_1}\psi_1) - (\varphi_2 D_{\xi_2}\varphi_2 \cos t + \psi_2 D_{\xi_2}\psi_2)\Lambda_{21}\varphi_1^2 \sin t & (\varphi_2 D_{\xi_2}\varphi_2 \cos t + \psi_2 D_{\xi_2}\psi_2)^2 - \varphi_1^2 . \varphi_2^2 \sin^2 t \end{array}\right)$$

$$\simeq \left(\begin{array}{c|c} (\varphi_1^2 \cos t + \psi_1^2)^2 D_{\xi_1}^2 - \varphi_1^2\varphi_2^2 \sin^2 t & 0 \\ \hline 0 & (\varphi_2^2 \cos t + \psi_2^2)^2 D_{\xi_2}^2 - \varphi_1^2\varphi_2^2 \sin^2 t \end{array}\right) \qquad \text{(by (10.10))}\ (^1)$$

$$\simeq - \left(\begin{array}{c|c} (D_{\xi_1} f_1)^*(D_{\xi_1} f_1) + g_1^* . g_1 & 0 \\ \hline 0 & (D_{\xi_2} f_2)^*(D_{\xi_2} f_2) + g_2^* . g_2 \end{array}\right) \qquad \text{(by (10.10), (11.5))},$$

where

$$(12.10)\qquad f_\alpha = \varphi_\alpha^2 \cos t + \psi_\alpha^2 \qquad \alpha = 1, 2$$

$$g_\alpha = \varphi_\alpha^2 \sin t.$$

---

(1) Notice that, for any $x \in U$,

$$\varphi_1(x) = \varphi_2(x),$$
$$\varphi_1(x) . \psi_1(x) = \varphi_1(x) . \psi_2(x) = \varphi_2(x) . \psi_1(x) = \varphi_2(x) . \psi_2(x).$$

Lemma **12.2.** — *The operator*

$$U_\alpha = (D_{\xi_\alpha} f_\alpha)^*(D_{\xi_\alpha} f_\alpha) + g_\alpha^* . g_\alpha : H_{\xi_\alpha} \to H_{\xi_\alpha} \tag{12.11}$$

is a Fredholm operator for any $t \in \left[0, \frac{\pi}{2}\right]$.

That statement remains valid for any two Lipschitz functions f_α and g_α such that $_\alpha + g_\alpha$ does not vanish on M_α.

Proof. — Indeed, if $x \in \operatorname{Ker} U$, then

$$0 = (U_\alpha x, x)_{H_{\xi_\alpha}} = (D_{\xi_\alpha} f_\alpha x, D_{\xi_\alpha} f_\alpha x)_{H_{\xi_\alpha}} + (g_\alpha x, g_\alpha x)_{H_{\xi_\alpha}} \tag{12.12}$$

which shows that

$$\begin{aligned} D_{\xi_\alpha} f_\alpha x &= 0 \\ g_\alpha x &= 0. \end{aligned} \tag{12.13}$$

Then

$$D_{\xi_\alpha}(f_\alpha + g_\alpha) x = 0,$$

or

$$(f_\alpha + g_\alpha) x \in \operatorname{Ker} D_{\xi_\alpha}, \tag{12.14}$$

which says that

$$\operatorname{Ker} U_\alpha \subseteq \operatorname{Ker} D_{\xi_\alpha}(f_\alpha + g_\alpha). \tag{12.15}$$

But $\operatorname{Ker} D_{\xi_\alpha}$ is finite dimensional, and the operator defined by the function

$$f_\alpha + g_\alpha = \varphi_\alpha^2(\sin t + \cos t) + \psi_\alpha^2$$

is invertible (because this function is strictly positive on M_α); then (12.15) shows that

$$\dim \operatorname{Ker} U_\alpha \leq \dim \operatorname{Ker} D_{\xi_\alpha}(f_\alpha + g_\alpha).$$

The operator U_α is self-adjoint, and then $\operatorname{Coker} U_\alpha = \operatorname{Ker} U_\alpha$, provided that the range of U_α is closed in H_{ξ_α}, which we check here.

From (12.15) we get, by passing to orthogonal complements,

$$(\operatorname{Ker} U_\alpha)^\perp_{H_{\xi_\alpha}} \supseteq [\operatorname{Ker} D_{\xi_\alpha}(f_\alpha + g_\alpha)]^\perp_{H_{\xi_\alpha}}, \tag{12.15'}$$

where the second space of (12.15') is of finite codimension in the first one.

We have then:

$$\begin{aligned} U_\alpha(H_{\xi_\alpha}) = U_\alpha[(\operatorname{Ker} U_\alpha)^\perp_{H_{\xi_\alpha}}] &= U_\alpha([\operatorname{Ker} D_{\xi_\alpha}(f_\alpha + g_\alpha)]^\perp_{H_{\xi_\alpha}}) \\ &\quad + U_\alpha \text{ (finite dimensional space)}. \end{aligned} \tag{12.15''}$$

Therefore, in order to show that the range of U_α is closed, it is sufficient to show that

$$U_\alpha([\mathrm{Ker}\, D_{\xi_\alpha}(f_\alpha + g_\alpha)]^{\perp}_{H_{\xi_\alpha}})$$

is closed in H_{ξ_α}.

To do this, it is sufficient to prove that there exists a positive constant K such that

$$(12.16) \qquad \|U_\alpha x\| \geq K\,\|x\|, \quad \text{for any } x \in [\mathrm{Ker}\, D_{\xi_\alpha}(f_\alpha + g_\alpha)]^{\perp}_{H_{\xi_\alpha}}.$$

Suppose such a constant did not exist. This means that there would exist a sequence

$$(12.17) \qquad \{x_n\}_{n\in\mathbf{N}} \subset [\mathrm{Ker}\, D_{\xi_\alpha}(f_\alpha + g_\alpha)]^{\perp}_{H_{\xi_\alpha}}, \quad \|x_n\| = 1,$$

such that

$$(12.17') \qquad \lim_{n\to\infty} U_\alpha x_n = 0.$$

We then have, as in (12.12),

$$(12.18) \qquad 0 = \lim_{n\to\infty} (U_\alpha x_n, x_n) = \lim_{n\to\infty} \{\|D_{\xi_\alpha} f_\alpha x_n\|^2 + \|g_\alpha x_n\|^2\},$$

which implies, in view of (12.17) and (12.17') that

$$(12.19) \qquad \begin{aligned} &\lim_{n\to\infty} D_{\xi_\alpha} f_\alpha x_n = 0 \\ &\lim_{n\to\infty} g_\alpha x_n = 0. \end{aligned}$$

These last two relations, along with the continuity of D_{ξ_α} imply that

$$(12.20) \qquad \lim_{n\to\infty} D_{\xi_\alpha}(f_\alpha + g_\alpha) x_n = 0.$$

But (12.20) leads us to a contradiction. In fact, because the operator $D_{\xi_\alpha}(f_\alpha + g_\alpha)$ is a Fredholm operator, it is invertible on the orthogonal complement of its kernel. Therefore (12.17) and (12.20) imply that $\lim_{n\to\infty} x_n = 0$, which contradicts the second relation (12.17). Therefore, the lemma is proven.

Theorem 11.1 (i)-(iv) now implies, by means of the homotopy (12.8),

$$(12.21) \qquad \mathrm{Index}\, D^+_{\xi_1} - \mathrm{Index}\, D^+_{\xi_2} = \mathrm{Index}\,(\Delta_{\frac{\pi}{2}}(\xi_1, \xi_2), J_{\xi_1} \oplus -J_{\xi_2}),$$

where

$$(12.22) \qquad \Delta_{\frac{\pi}{2}}(\xi_1, \xi_2) = \begin{pmatrix} \psi_1 D_{\xi_1} \psi_1 & -\Lambda_{12}\varphi_2^2 \\ \Lambda_{21}\varphi_1^2 & -\psi_2 D_{\xi_2} \psi_2 \end{pmatrix}.$$

Second step. — Dealing with the second bracket of (12.5) in the same way we have dealt with the first one, we obtain

$$(12.23) \qquad \mathrm{Index}\, D^+_{\theta_1} - \mathrm{Index}\, D^+_{\theta_2} = \mathrm{Index}\,(\Delta_{\frac{\pi}{2}}(\theta_1, \theta_2), J_{\theta_1} \oplus -J_{\theta_2}),$$

where

$$(12.24) \qquad \Delta_{\frac{\pi}{2}}(\theta_1, \theta_2) = \begin{pmatrix} \psi_1 D_{\theta_1} \psi_1 & -\Lambda'_{12}\varphi_2^2 \\ \Lambda'_{21}\varphi_1^2 & -\psi_2 D_{\theta_2} \psi_2 \end{pmatrix}.$$

Third step. — We now pass to the proof of (12.5). To this aim, we consider the homotopy of the operator $(\Delta_{\frac{\pi}{2}}(\xi_1, \xi_2) \oplus - \Delta_{\frac{\pi}{2}}(\theta_1, \theta_2)$:

(12.25)

$$\Delta_t(\xi_1, \xi_2, \theta_1, \theta_2) = \begin{pmatrix} \psi_1 D_{\xi_1} \psi_1 \cos t & -\Lambda_{12}\varphi_2^2 & -\Sigma_1^{-1}\psi_1^2 \sin t & 0 \\ \Lambda_{21}\varphi_1^2 & -\psi_2 D_{\xi_2}\psi_2 \cos t & 0 & -\Sigma_2^{-1}\psi_2^2 \sin t \\ \Sigma_1\psi_1^2 \sin t & 0 & -\psi_1 D_{\theta_1}\psi_1 \cos t & \Lambda'_{12}\varphi_2^2 \\ 0 & \Sigma_2\psi_2^2 \sin t & -\Lambda'_{21}\varphi_1^2 & \psi_2 D_{\theta_2}\psi_2 \cos t \end{pmatrix},$$

which maps $H_{\xi_1} \oplus H_{\xi_2} \oplus H_{\theta_1} \oplus H_{\theta_2}$ into itself, for $t \in \left[0, \frac{\pi}{2}\right]$.

(Notice that this homotopy acts effectively on $H_{\xi_1} \oplus H_{\theta_1}$ and on $H_{\xi_2} \oplus H_{\theta_2}$).

We verify now that $\Delta_t(\xi_1, \xi_2, \theta_1, \theta_2)$ is a Fredholm operator. In fact, we have (see Appendix):

(12.26) $$\Delta_t(\xi_1, \xi_2, \theta_1, \theta_2)^2 \simeq \begin{pmatrix} \text{Fredholm Operator} & 0 & 0 & 0 \\ 0 & \text{Fredholm Operator} & 0 & 0 \\ 0 & 0 & \text{Fredholm Operator} & 0 \\ 0 & 0 & 0 & \text{Fredholm Operator} \end{pmatrix}.$$

A direct calculation shows that:

(12.27) $$\Delta_t(\xi_1, \xi_2, \theta_1, \theta_2) \circ (J_{\xi_1} \oplus - J_{\xi_2} \oplus - J_{\theta_1} \oplus J_{\theta_2}) = - (J_{\xi_1} \oplus - J_{\xi_2} \oplus - J_{\theta_1} \oplus J_{\theta_2}) \circ \Delta_t(\xi_1, \xi_2, \theta_1, \theta_2);$$

explicit computations are given in the Appendix.

We will show that

(12.28) $$\text{Index}\,(\Delta_{\frac{\pi}{2}}(\xi_1, \xi_2, \theta_1, \theta_2), J_{\xi_1} \oplus - J_{\xi_2} \oplus - J_{\theta_1} \oplus J_{\theta_2}) = 0,$$

and then (12.5) is proven.

We have, by Proposition 10.1,

$$\Delta_{\frac{\pi}{2}}(\xi_1, \xi_2, \theta_1, \theta_2) \simeq \Delta,$$

where

(12.29) $$\Delta = \begin{pmatrix} 0 & -(\Lambda_{21}\varphi_1^2)^* & -(\Sigma_1\psi_1^2)^* & 0 \\ \Lambda_{21}\varphi_1^2 & 0 & 0 & -\Sigma_2^{-1}\psi_2^2 \\ \Sigma_1\psi_1^2 & 0 & 0 & \Lambda'_{12}\varphi_2^2 \\ 0 & (\Sigma_2^{-1}\psi_2^2)^* & -(\Lambda'_{12}\varphi_2^2)^* & 0 \end{pmatrix}.$$

The operator Δ anticommutes with $J = J_{\xi_1} \oplus - J_{\xi_2} \oplus - J_{\theta_1} \oplus J_{\theta_2}$. To show this, it is sufficient to observe that

$$(12.30) \qquad J^*_{\xi_\alpha} = J_{\xi_\alpha}, \quad (J_{\theta_\alpha})^* = J_{\theta_\alpha}, \quad \alpha = 1, 2,$$

(see (11.2) (iii)), and

$$(12.31) \qquad \begin{aligned} \Lambda_{21}\varphi_1^2 J_{\xi_1} &= J_{\xi_2}\Lambda_{21}\varphi_1^2, \\ \Lambda'_{12}\varphi_2^2 J_{\theta_2} &= J_{\theta_1}\Lambda'_{12}\varphi_2^2, \\ \Sigma_1\psi_1^2 J_{\xi_1} &= J_{\theta_1}\Sigma_1\psi_1^2, \\ \Sigma_2^{-1}\psi_2^2 J_{\theta_2} &= J_{\xi_2}\Sigma_2^{-1}\psi_2^2; \end{aligned}$$

if we pass to adjoints in (12.31), and take into account (12.30), we get

$$(12.32) \qquad \begin{aligned} J_{\xi_1}(\Lambda_{21}\varphi_1^2)^* &= (\Lambda_{21}\varphi_1^2)^* J_{\xi_2}, \\ J_{\theta_2}(\Lambda'_{12}\varphi_2^2)^* &= (\Lambda'_{12}\varphi_2^2)^* J_{\theta_1}, \\ J_{\xi_1}(\Sigma_1\psi_1^2)^* &= (\Sigma_1\psi_1^2)^* J_{\theta_1}, \\ J_{\theta_2}(\Sigma_2^{-1}\psi_2^2)^* &= (\Sigma_2^{-1}\psi_1^2)^* J_{\xi_2}. \end{aligned}$$

The Theorem 11.1 (ii) tells us that

$$\begin{aligned} &\operatorname{Index}(\Delta_{\frac{\pi}{2}}(\xi_1, \xi_2, \theta_1, \theta_2), J_{\xi_1} \oplus - J_{\xi_2} \oplus - J_{\theta_1} \oplus J_{\theta_2}) \\ &\qquad = \operatorname{Index}(\Delta, J_{\xi_1} \oplus - J_{\xi_2} \oplus - J_{\theta_1} \oplus J_{\theta_2}). \end{aligned}$$

The operator Δ is skew-adjoint, and hence, by (11.12),

$$(12.33) \qquad \operatorname{Index}(\Delta, J) = \dim \operatorname{Ker}^+ \Delta - \dim \operatorname{Ker}^- \Delta.$$

We show that $\operatorname{Ker} \Delta = 0$, and then, a fortiori, $\operatorname{Index}(\Delta, J) = 0$, which ultimately will prove (12.28), and so (12.2) will be proven. Let $(x_1, x_2, y_1, y_2) \in \operatorname{Ker} \Delta$; then

$$(12.34) \qquad \begin{aligned} &(\mathrm{I}) \begin{cases} \Lambda_{21}\varphi_1^2 x_1 - \Sigma_2^{-1}\psi_2^2 y_2 = 0 \\ \Sigma_1\psi_1^2 x_1 + \Lambda'_{12}\varphi_2^2 y_2 = 0, \end{cases} \\ &(\mathrm{II}) \begin{cases} -(\Lambda_{21}\varphi_1^2)^* x_2 - (\Sigma_1\psi_1^2)^* y_1 = 0 \\ (\Sigma_2^{-1}\psi_2^2)^* x_2 - (\Lambda'_{12}\varphi_2^2)^* y_1 = 0. \end{cases} \end{aligned}$$

To solve the system (I), we multiply the first equation by $\Lambda_{12}\varphi_2^2$, and the second by $\Sigma_1^{-1}\psi_1^2$ to eliminate y_2. Indeed, this works because the commutativity of the diagram (12.1) is equivalent to

$$(12.35) \qquad (\Lambda_{12}\varphi_2^2)(\Sigma_2^{-1}\psi_2^2) = (\Sigma_1^{-1}\psi_1^2)(\Lambda'_{12}\varphi_2^2),$$

and then x_1 has to satisfy the equation

$$(12.36) \qquad [(\Lambda_{12}\varphi_2^2)(\Lambda_{21}\varphi_1^2) + (\Sigma_1^{-1}\psi_1^2)(\Sigma_1\psi_1^2)]x_1 = 0,$$

which implies $x_1 = 0$ (recall (12.6) *(b)*).

Then, we eliminate x_1 by multiplying the first equation by $\Sigma_2\psi_2^2$, and the second by $\Lambda'_{21}\varphi_1^2$ to get, in a similar fashion, that $y_2 = 0$.

The system (II) is solved in the same way, being aware of the extra complication due to the presence of the adjoints. For that purpose, we multiply the first equation (II) by $(\Sigma_1^{-1}\psi_1^2)^*$, and the second by $(\Lambda'_{21}\varphi_1^2)^*$. The equality

(12.37) $$(\Sigma_1^{-1}\psi_1^2)^*(\Lambda_{21}\varphi_1^2)^* = (\Lambda'_{21}\varphi_1^2)^*(\Sigma_2^{-1}\psi_2^2)^*$$

allows us to eliminate x_2, and then y_1 has to belong to the kernel of the operator

(12.38) $$(\Sigma_1^{-1}\psi_1^2)^*(\Sigma_1\psi_1^2)^* + (\Lambda'_{21}\varphi_1^2)^*(\Lambda'_{12}\varphi_2^2)^*,$$

which turns out to be invertible; therefore, $y_1 = 0$.

To check (12.37), notice that it follows from the commutativity of the diagram (12.1) by the passage to adjoints. On the other hand, the operator (12.38) is invertible because its adjoint is invertible.

Finally, we eliminate y_1 by multiplying the first equation (II) by $(\Lambda_{12}\varphi_2^2)^*$ and the second by $(\Sigma_2\psi_2^2)^*$; invoking the same kind of arguments, we get $x_2 = 0$.

INDEX D_ξ^+ IS A LIPSCHITZ INVARIANT

13. Independence of the Riemannian structure

Theorem **13.1.** — *For any Lipschitz complex vector bundle* ξ *over an oriented, closed Lipschitz Riemannian manifold* $M^{2\mu}$, Index D_ξ^+ *is a Lipschitz invariant.*

Proof. — Corollary 7.1 asserts that Index D_ξ^+ does not depend on the linear connection in ξ. It remains to show that Index D_ξ^+ does not depend on the Riemannian metric on $M^{2\mu}$.

We shall prove that Index D_ξ^+ remains unchanged when the Riemannian metric Γ is modified on an arbitrary small closed disc $D^{2\mu}$ in $M^{2\mu}$; let Γ' be the new Riemannian metric. We choose a linear connection ∇ in ξ, and then $\{(\nabla, \Gamma), (\nabla, \Gamma'), M\backslash D^{2\mu}\}$ is an excisive triple, with identifying isomorphism $\mathbf{1}_\xi$. Because $D^{2\mu}$ is contractible, the bundle ξ is trivial over it; we choose such a trivialization T, and we may choose ∇ so that $\nabla \mid D^{2\mu}$ be carried by T into the trivial connection ∇_0 in the product bundle θ, Rank θ = Rank ξ.

Then

$$\text{(13.1)}\qquad \begin{aligned} &\{(\nabla, \Gamma), (\nabla, \Gamma'), M\backslash \mathring{D}^{2\mu}\},\ \{(\nabla_0, \Gamma), (\nabla_0, \Gamma'), M\backslash \mathring{D}^{2\mu}\},\\ &\{(\nabla, \Gamma), (\nabla_0, \Gamma), D^{2\mu}\},\ \{(\nabla, \Gamma'), (\nabla_0, \Gamma'), D^{2\mu}\}\end{aligned}$$

are excisive triples with identifying isomorphisms, respectively,

$$\text{(13.2)}\qquad \begin{aligned} &\mathbf{1}_\xi \mid M\backslash D^{2\mu},\ \mathbf{1}_\theta \mid M\backslash D^{2\mu},\\ &T,\ T.\end{aligned}$$

The hypotheses (a)-(f) of the Excision Theorem 12.1 are verified for:

$$\begin{aligned} &M_1 = M_2 = M, \quad \xi_1 = \xi_2 = \xi, \quad \theta_1 = \theta_2 = \theta,\\ &\Gamma_1 = \Gamma, \quad \Gamma_2 = \Gamma', \quad \nabla_1 = \nabla_2 = \nabla, \quad \nabla_1' = \nabla_2' = \nabla_0,\\ &U = M\backslash D^{2\mu}, \quad V = M\backslash \tilde{D}^{2\mu},\end{aligned}$$

and the specified identifying isomorphisms; here $\tilde{D}^{2\mu} \subset M$ is a closed disc with $D^{2\mu} \subset$ Interior $\tilde{D}^{2\mu}$.

The Theorem 13.1 is completely proven.

Remark **13.2.** — Theorem 1.2 of [12] asserts that any smooth structure on a PL-manifold may be characterized by means of metric data derived from Riemannian metrics. Therefore, a (Lipschitz) Riemannian metric may be thought of as a generalized smoothing. This explaines why proving that Index D_ξ^+ is independent of the Riemannian metric is a delicate step toward the proof of the *topological* invariance of this number. The topological invariance of Index D_ξ^+ is proven in a joint paper with Dennis Sullivan in the same volume of this journal. In the same paper it is shown that the topological nvariance of Index D_ξ^+ implies the topological invariance of the rational Pontrjagin classes (S. P. Novikov).

Remark **13.3.** — The same scheme of proof may be used to show that Index D_ξ^+ is invariant under elementary cobordisms (surgery).

14. Appendix

14.0 We show here that if the relation (1.4) (see section 1)

$$d(f^*\omega) = f^*(d\omega) \tag{14.1}$$

holds for Lipschitz mappings f and smooth forms ω, then it holds for L_2-forms ω.

For, let $\omega \in L_2^r$; then, by a simple convolution argument, we can find a sequence of smooth forms $\{\omega_n\}_{n\in\mathbf{N}}$, such that:

$$\lim_{n\to\infty} \omega_n = \omega, \qquad \lim_{n\to\infty} d\omega_n = d\omega. \tag{14.2}$$

By (14.2) and (14.1), and recalling that f^* is a bounded operator between L_2-spaces, we have

$$f^*(d\omega) = f^*(\lim_{n\to\infty} d\omega_n) = \lim_{n\to\infty} f^*(d\omega_n) = \lim_{n\to\infty} d(f^*\omega_n). \tag{14.3}$$

These relations show that if we set $f^*\omega_n = \theta_n$, we have

$$\theta_n \in L_2^r, \qquad \lim_{n\to\infty} \theta_n = f^*\omega, \tag{14.4}$$

$$d\theta_n \in L_2^{r+1}, \qquad \lim_{n\to\infty} d\theta_n = f^* d\omega. \tag{14.4'}$$

Let ξ be any smooth testing form. We then have, successively by (14.4), (1.2) and (14.4'),

$$\int f^*\omega \wedge d\xi = \lim_{n\to\infty} \int \theta_n \wedge d\xi = \lim_{n\to\infty} (-1)^{r+1} \int d\theta_n \wedge \xi = (-1)^{r+1} \int f^* d\omega \wedge \xi; \tag{14.5}$$

in view of the very definition of the distributional exterior derivative, (14.5) says that

$$d(f^*\omega) = f^*(d\omega),$$

which completes the proof of Proposition 1.2.

14.1 We compute here $\Delta_t(\xi_1, \xi_2, \theta_1, \theta_2)^2$.

Hereafter, $\binom{i}{j}$ denotes the i-column and j-row entry of $\Delta_t(\xi_1, \xi_2, \theta_1, \theta_2)^2$. We have

$$\binom{1}{1} = (\psi_1 D_{\xi_1}\psi_1 \cos t)^2 - \varphi_1^2\varphi_2^2 - \psi_1^4 \sin^2 t$$
$$\simeq \text{Fredholm Operator} \quad \text{(Lemma 12.2)}$$

$$\binom{2}{1} = -\psi_1 D_{\xi_1}\psi_1\Lambda_{12}\varphi_2^2 \cos t + \Lambda_{12}\varphi_2^2\psi_2 D_{\xi_2}\psi_2 \cos t$$
$$\simeq -\psi_1^2\varphi_2 D_{\xi_1}\Lambda_{12}\varphi_2 \cos t + \Lambda_{12}\psi_2^2\varphi_2 D_{\xi_2}\varphi_2 \cos t \simeq 0$$

because $\psi_1^2\varphi_2 = \psi_2^2\varphi_2$ and by Proposition 10.2.

$$\binom{3}{1} = -\psi_1 D_{\xi_1}\psi_1\Sigma_1^{-1}\psi_1^2 \sin t \cos t$$
$$+ \Sigma_1^{-1}\psi_1^3 D_{\theta_1}\psi_1 \sin t \cos t \simeq 0 \quad \text{(Proposition 10.2)}.$$

$$\binom{4}{1} = \Lambda_{12}\Sigma_2^{-1}\varphi_2^2\psi_2^2 \sin t - \Sigma_1^{-1}\Lambda'_{12}\psi_1^2\varphi_2^2 \sin t = 0 \quad \text{by (12.1)}.$$

$$\binom{1}{2} = \Lambda_{21}\varphi_1^2\psi_1 D_{\xi_1}\psi_1 \cos t$$
$$-\psi_2 D_{\xi_2}\psi_2\Lambda_{21}\varphi_1^2 \cos t \simeq 0 \quad \text{(Proposition 10.2)}.$$

$$\binom{2}{2} = -\varphi_1^2\varphi_2^2 + (\psi_2 D_{\xi_2}\psi_2 \cos t)^2 - \psi_2^4 \sin^2 t$$
$$\simeq \text{Fredholm Operator} \quad \text{(Lemma 12.2)}.$$

$$\binom{3}{2} = -\Lambda_{21}\varphi_1^2\Sigma_1^{-1}\psi_1^2 \sin t + \Sigma_2^{-1}\psi_2^2 \sin t\, \Lambda'_{21}\varphi_1^2 = 0 \quad \text{by (12.1)}.$$

$$\binom{4}{2} = \psi_2 D_{\xi_2}\psi_2 \cos t\, \Sigma_2^{-1}\psi_2^2 \sin t$$
$$-\Sigma_2^{-1}\psi_2^3 D_{\theta_2}\psi_2 \sin t \cos t \simeq 0 \quad \text{(Proposition 10.2)}.$$

$$\binom{1}{3} = \Sigma_1\psi_1^3 \sin t\, D_{\xi_1}\psi_1 \cos t$$
$$-\psi_1 D_{\theta_1}\psi_1\Sigma_1\psi_1^2 \cos t \sin t \simeq 0 \quad \text{(Proposition 10.2)}.$$

$$\binom{2}{3} = -\Sigma_1\psi_1^2 \sin t\, \Lambda_{12}\varphi_2^2 + \Lambda'_{12}\varphi_2^2\Sigma_2\psi_2^2 \sin t = 0 \quad \text{by (12.1)}.$$

$$\binom{3}{3} = -\psi_1^2\psi_1^2 \sin^2 t + (\psi_1 D_{\theta_1}\psi_1 \cos t)^2 - \varphi_1^2\varphi_2^2$$
$$\simeq \text{Fredholm Operator} \quad \text{(Lemma 12.2)}.$$

$$\binom{4}{3} = -\psi_1 D_{\theta_1}\psi_1 \cos t\, \Lambda'_{12}\varphi_2^2$$
$$+ \Lambda'_{12}\varphi_2^2\psi_2 D_{\theta_2}\psi_2 \cos t \simeq 0 \quad \text{(Proposition 10.2)}.$$

$$\binom{1}{4} = \Sigma_2 \psi_2^2 \sin t\, \Lambda_{21} \varphi_1^2 - \Lambda'_{21} \varphi_1^2 \Sigma_1 \psi_1^2 \sin t = 0 \quad \text{by (12.1)}.$$

$$\binom{2}{4} = - \Sigma_2 \psi_2^3 D_{\xi_2} \psi_2 \sin t \cos t + \psi_2 D_{\theta_2} \psi_2 \Sigma_2 \psi_2^2 \sin t \cos t \simeq 0 \quad \text{(Proposition 10.2)}.$$

$$\binom{3}{4} = \Lambda'_{21} \varphi_1^2 \psi_1 D_{\theta_1} \psi_1 \cos t - \psi_2 D_{\theta_2} \psi_2 \cos t\, \Lambda'_{21} \varphi_1^2 \simeq 0 \quad \text{(Proposition 10.2)}.$$

$$\binom{4}{4} = - \psi_2^4 \sin^2 t - \varphi_1^2 \varphi_2^2 + (\psi_2 D_{\theta_2} \psi_2 \cos t)^2 \simeq \text{Fredholm Operator} \quad \text{(Lemma 12.2)}.$$

Therefore,

(14.1)
$$\Delta_t(\xi_1, \xi_2, \theta_1, \theta_2)^2 \simeq \begin{pmatrix} \text{Fredholm Operator} & 0 & 0 & 0 \\ 0 & \text{Fredholm Operator} & 0 & 0 \\ 0 & 0 & \text{Fredholm Operator} & 0 \\ 0 & 0 & 0 & \text{Fredholm Operator} \end{pmatrix}.$$

14.2 We now check the anticommutativity of $\Delta_t(\xi_1, \xi_2, \theta_1, \theta_2)$ with
$$J = J_{\xi_1} \oplus - J_{\xi_2} \oplus - J_{\theta_1} \oplus J_{\theta_2}.$$

We have:

(14.2)
$$\Delta_t(\xi_1, \xi_2, \theta_1, \theta_2) \circ J$$
$$= \begin{pmatrix} \psi_1 D_{\xi_1} \psi_1 \cos t\, J_{\xi_1} & \Lambda_{12} \varphi_2^2 J_{\xi_2} & \Sigma_1^{-1} \sin t\, \varphi_1^2 J_{\theta_1} & 0 \\ \Lambda_{21} \varphi_1^2 J_{\xi_1} & \psi_2 D_{\xi_2} \psi_2 \cos t\, J_{\xi_2} & 0 & - \Sigma_2^{-1} \sin t\, \psi_2^2 J_{\theta_2} \\ \Sigma_1 \sin t\, \psi_1^2 J_{\xi_1} & 0 & \psi_1 D_{\theta_1} \psi_1 \cos t\, J_{\theta_1} & \Lambda'_{12} \varphi_2^2 J_{\theta_2} \\ 0 & - \Sigma_2 \sin t\, \psi_2^2 J_{\xi_2} & \Lambda'_{21} \varphi_1^2 J_{\theta_1} & \psi_2 D_{\theta_2} \psi_2 \cos t\, J_{\theta_2} \end{pmatrix}$$

and

(14.3)
$$J \circ \Delta_t(\xi_1, \xi_2, \theta_1, \theta_2)$$
$$= \begin{pmatrix} J_{\xi_1} \psi_1 D_{\xi_1} \psi_1 \cos t & - J_{\xi_1} \Lambda_{12} \varphi_2^2 & - J_{\xi_1} \Sigma_1^{-1} \psi_1^2 \sin t & 0 \\ - J_{\xi_2} \Sigma_1 \varphi_1^2 & J_{\xi_2} \psi_2 D_{\xi_2} \psi_2 \cos t & 0 & \sin t\, J_{\xi_2} \Sigma_2^{-1} \psi_2^2 \\ - J_{\theta_1} \Sigma_1 \sin t\, \psi_1^2 & 0 & J_{\theta_1} \psi_1 D_{\theta_1} \psi_1 \cos t & - J_{\theta_1} \Lambda'_{12} \varphi_2^2 \\ 0 & \sin t\, J_{\theta_2} \Sigma_2 \psi_2^2 & - J_{\theta_2} \Lambda'_{21} \varphi_1^2 & J_{\theta_2} \psi_2 D_{\theta_2} \psi_2 \cos t \end{pmatrix}.$$

The operators J_{ξ_α}, J_{θ_α} $(\alpha = 1, 2)$ from (14.2) and (14.3) commute with φ_α, ψ_α as given by (12.32) and they anticommute with D_{ξ_α}, D_{θ_α} by virtue of (11.4), hence (12.27).

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Manuscrit reçu le 9 octobre 1981.