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PIERRE VOGEL

On the obstruction group in homology surgery

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ON THE OBSTRUCTION GROUP IN HOMOLOGY SURGERY

by Pierre VOGEL

0. Introduction

The theory of homology surgery has been introduced by Cappell and Shaneson [1]. This theory plays an important role in the theory of knots and codimension 2 embeddings.

Let $(X, \partial X)$ be a pair of finite complexes and f be a normal map from the normal bundle of a (Top, PL or Diff)-manifold V to a (Top, PL or Diff)-bundle over X and let M be a $\mathbf{Z}[\pi_1 X]$-module. The problem of homology surgery is to determine the obstruction to the existence of a normal cobordism, constant over ∂X, from f to an M-homology equivalence. Clearly we must suppose that f induces an M-homology equivalence from ∂V to ∂X and that the cap-product by $f_*[V]$ is an isomorphism from $H^*(X, \partial X; M)$ to $H_{n-*}(X; M^w)$, w being the first Stiefel-Whitney class of the bundle over X.

If $M = A$ is a quotient ring with involution of $\mathbf{Z}[\pi_1 X] = \mathbf{Z}\pi$, Cappell and Shaneson have solved the problem and have constructed an obstruction group $\Gamma_n(\mathbf{Z}\pi \to A)$ defined in terms of algebraic L-theory.

In many cases, this group was known to be the L_n-group of some ring Λ. For example, if there exists a classical localization $S^{-1}\mathbf{Z}\pi$ of $\mathbf{Z}\pi$, where S is the multiplicative subset $1 + \ker(\mathbf{Z}\pi \to A)$, Smith [7] has proved that $\Gamma_n(\mathbf{Z}\pi \to A)$ is the group $L_n(S^{-1}\mathbf{Z}\pi)$. An other example is given by Hausmann [3] who proves that $\Gamma_n(\mathbf{Z}\pi \to \mathbf{Z}[\pi/N])$ is the group $L_n(\mathbf{Z}[\pi/N])$ if N is a locally perfect normal subgroup of π.

My purpose is to show that the homology surgery is possible in a more general situation and that the obstruction group is always the L_n-group of a ring with involution Λ endowed with a subgroup of $\widetilde{K}_1(\Lambda)$.

For example, suppose that $\mathbf{Z}\pi \to A$ is a morphism of rings with involution (the involution of $\mathbf{Z}\pi$ is induced by w). Then we have a diagram of rings with involution

$$\begin{array}{ccc} & \Lambda & \\ \nearrow & & \searrow \\ \mathbf{Z}\pi & \longrightarrow & A \end{array}$$

well defined by the following properties:

i) For any matrix u with entries in $\mathbf{Z}\pi$, if $u \otimes \mathrm{A}$ is invertible then $u \otimes \Lambda$ is invertible too;

ii) Λ is universal with respect to the property i).

Theorem. — *Suppose the morphism* $\Lambda \to \mathrm{A}$ *is onto. Then any normal map* f *over a* n*-dimensional* A*-Poincaré complex* X *which is an* A*-homology equivalence over* $\partial \mathrm{X}$ *determines an element* $\sigma(f) \in \mathrm{L}_n^h(\Lambda)$, *and, if* $n \geq 5$, f *is normally cobordant to an* A*-homology equivalence if and only if* $\sigma(f)$ *vanishes.*

Corollary. — *If* A *is a quotient ring with involution of* $\mathbf{Z}\pi$, *the group* $\Gamma_n^h(\mathbf{Z}\pi \to \mathrm{A})$ *is isomorphic to* $\mathrm{L}_n^h(\Lambda)$.

Theorem. — *Let* D_{2n} *be the dihedral group of order* $2n$ *(n odd) and* $\mathrm{D}_{2n} \to \mathbf{Z}/2$ *be the non zero homomorphism. Then we have the following isomorphism:*

$$\Gamma_*(\mathbf{Z}\mathrm{D}_{2n} \to \mathbf{Z}) \xrightarrow{\sim} \Gamma_*(\mathbf{Z}[\mathbf{Z}/2] \to \mathbf{Z}) \simeq \mathrm{L}_*^h(\Lambda),$$

where Λ *is the pull back of rings:*

$$\begin{array}{ccc} \Lambda & \longrightarrow & \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathbf{Z}_{(2)}[\mathbf{Z}/2] & \longrightarrow & \mathbf{Z}_{(2)} \end{array}$$

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1. Statement of the main results

(1.1) Let A be a ring with involution $a \mapsto \bar{a}$. If M is a left A-module, it can be given a right A-module structure, by setting

$$ma = \bar{a}m, \quad \forall\, a \in \mathrm{A}, \quad \forall\, m \in \mathrm{M}.$$

Conversely any right A-module is a left A-module. From now on an A-module will mean a left or right A-module.

Denote by $\mathscr{C}(A)$ the category of $\mathbf{Z}$-graded complexes

$$\dots \to C_{n+1} \to C_n \to C_{n-1} \to \dots$$

such that each C_n is a finitely generated free A-module with fixed (unordered) basis and $\bigoplus_n C_n$ is finitely generated. Theses complexes will be called finite A-complexes.

We say that a sequence of finite A-complexes $0 \to C \to C' \to C'' \to 0$ is s-exact if, for any n, the complex $0 \to C_n \to C'_n \to C''_n \to 0$ is acyclic with torsion 0 in $\widetilde{K}_1(A)$; see [4] and [9].

Definition (**1.2**). — A class $\mathscr{W} \subset \mathscr{C}(A)$ is *exact* if $\mathscr{W}$ contains any acyclic finite A-complex with torsion 0, and if, for any s-exact sequence in $\mathscr{C}(A)$

$$0 \to C \to C' \to C'' \to 0,$$

one has the following property:

If two of the complexes C, C′, C″ lie in $\mathscr{W}$, then the third lies in $\mathscr{W}$ too.

Let C be a finite A-complex. Denote by $\hat{C}_n$ the dual module $\mathrm{Hom}(C_{-n}, A)$ endowed with the dual basis, and choose on $\hat{C}$ the differential so that the evaluation from $\hat{C} \otimes C$ to A is a cocycle. So we get a new finite A-complex $\hat{C}$.

Definition (**1.3**). — An exact class $\mathscr{W} \subset \mathscr{C}(A)$ is called *symmetric* if, for any $C \in \mathscr{W}$, $\hat{C}$ lies in $\mathscr{W}$.

Definition (**1.4**). — Let $\mathscr{W}$ be an exact class in $\mathscr{C}(A)$. A morphism f in $\mathscr{C}(A)$ is a *$\mathscr{W}$-equivalence* if the mapping cone of f is in $\mathscr{W}$.

Let f be a map from a finite CW-complex X to a finite connected CW-complex Y, with fundamental group π, and let $\mathscr{W}$ be an exact class in $\mathscr{C}(\mathbf{Z}\pi)$ containing any acyclic finite $\mathbf{Z}\pi$-complex with torsion in the image of $\pi \to \widetilde{K}_1(\mathbf{Z}\pi)$. Then f is a *$\mathscr{W}$-equivalence* if the chain map $C_*(X, \mathbf{Z}\pi) \to C_*(Y, \mathbf{Z}\pi)$ is a $\mathscr{W}$-equivalence.

Example (**1.5**). — Let $A \to B$ be a ring homomorphism and β be a subgroup of $\widetilde{K}_1(B)$. Let $\mathscr{W}$ be the class of finite A-complexes C such that $C \otimes_A B$ is acyclic with torsion in β. Then $\mathscr{W}$ is exact and the $\mathscr{W}$-equivalences are the B-homology equivalences with torsion in β.

If, in addition, $A \to B$ is a morphism of rings with involution and β is stable under the involution, $\mathscr{W}$ is symmetric.

Example (**1.6**). — Let M be an A-module. Then the class $\mathscr{W}$ of finite A-complexes C such that $H_*(C, M)$ (resp. $H^*(C, M)$) vanishes, is an exact class and the $\mathscr{W}$-equivalences are the M-homology (resp. M-cohomology) equivalences.

Notation (**1.7**). — Let $\mathscr{W}$ be an exact class in $\mathscr{C}(A)$. We denote by Σ the set of matrices u such that the direct sum of the complex $\ldots \to 0 \to A^p \xrightarrow{u} A^q \to 0 \to \ldots$ and its suspension is in $\mathscr{W}$.

In example (1.5), Σ is the set of matrices u with entries in A such that $u \otimes B$ is invertible.

Proposition (**1.8**). — *Let $\mathscr{W}$ be an exact class in $\mathscr{C}(A)$. Then there exists a ring homomorphism $A \to \Lambda$ unique up to isomorphism, which is universal with respect to the following property: For any matrix $u \in \Sigma$, $u \otimes \Lambda$ is invertible.*

If $\mathscr{W}$ is symmetric, $A \to \Lambda$ is a morphism of rings with involution.

Actually, the ring Λ is an inversive localization of A in the sense of Cohn [2].

Definition (**1.9**). — Let α be the subgroup of $\widetilde{K}_1(\Lambda)$ generated by the torsion of all complexes $C \otimes \Lambda$, such that $C \in \mathscr{W}$ and $C \otimes \Lambda$ is acyclic. The pair (Λ, α) will be called the *$\mathscr{W}$-localization* of A.

Let f be a normal map from the normal bundle of a compact n-dimensional (Top, PL or Diff)-manifold V to a (Top, PL or Diff)-bundle ξ over a pair $(X, \partial X)$ of finite complexes. Suppose X is connected. The first Stiefel-Whitney class of ξ induces an involution on the ring $A = \mathbf{Z}[\pi_1 X]$.

Let $\mathscr{W}$ be an exact symmetric class in $\mathscr{C}(A)$ containing any acyclic complex with torsion in the image of $\pi_1 X \to \widetilde{K}_1(A)$.

Suppose we have the following properties:

i) $(X, \partial X)$ is a $\mathscr{W}$-Poincaré complex; *i.e.* the cap-product by $f_*[V]$ is a $\mathscr{W}$-equivalence from $C^*(X; A)$ to $C_*(X, \partial X; A)$.

ii) The restricted map $f: \partial V \to \partial X$ is a $\mathscr{W}$-equivalence.

Theorem (**1.10**). — *Let (Λ, α) be the $\mathscr{W}$-localization of A. Suppose that any complex in $\mathscr{W}$ is Λ-acyclic. Then, the normal map f determines a well-defined element $\sigma(f) \in L_n^\alpha(\Lambda)$. And, if $n \geq 5$, f is normally cobordant, rel the boundary, to a $\mathscr{W}$-equivalence if and only if $\sigma(f)$ vanishes.*

Theorem (**1.11**). — *With the same hypothesis as above, if $n \geq 6$, and X is a product $M \times I$, M being a (Top, PL or Diff)-manifold, any element of $L_n^\alpha(\Lambda)$ is the obstruction $\sigma(f)$ of a normal map f restricting to an isomorphism over $M \times 0 \cup \partial M \times I$.*

Remark (**1.12**). — The condition of Λ-acyclicity of complexes in $\mathscr{W}$ is a very crucial point because, in the situation of (1.10), $\sigma(f)$ can be defined only if this condition is satisfied, or, more precisely, if the Poincaré duality on $(X, \partial X)$ is a Λ-homology equivalence and f restricts to a Λ-homology equivalence on the boundaries.

On the other hand, this condition is not always satisfied. For example, if $\mathscr{W}$

is the class of finite $\mathbf{Z}[t, t^{-1}]$-complex with finite homology, the ring Λ is $\mathbf{Z}[t, t^{-1}]$ and there exist many complexes in $\mathcal{W}$ which are not acyclic.

If the condition of Λ-acyclicity of complexes in $\mathcal{W}$ is not satisfied, denote by $\mathcal{W}'$ the class of Λ-acyclic complexes in $\mathcal{W}$. Then theorems (1.10) and (1.11) hold for the class $\mathcal{W}'$. Now, the last problem is to compare the surgery problems corresponding to classes $\mathcal{W}$ and $\mathcal{W}'$. But this question seems to be very difficult.

Let $A \to B$ be a ring homomorphism. Let Λ be the inversive localization of A in the sense of Cohn [2] obtained by formal inversion of the matrices u with entries in A such that $u \otimes B$ is invertible. The ring homomorphism $A \to \Lambda$ will be called the localization of $A \to B$.

Theorem **(1.13)**. — *Let* $A \to B$ *be a ring homomorphism and* β *be a subgroup of* $\widetilde{K}_1(B)$. *Denote by* $\mathcal{W}$ *the class of finite* A*-complexes which are* B*-acyclic with torsion in* β, *and by* (Λ, α) *the* $\mathcal{W}$*-localization of* A.

Then $A \to \Lambda$ *is the localization of* $A \to B$ *and* α *is the inverse image of* β *under the canonical morphism* $\varepsilon : \Lambda \to B$.

Moreover, if ε *is onto, any complex in* $\mathcal{W}$ *is* Λ*-acyclic.*

Remark **(1.14)**. — The ring Λ and the group $L_n^\alpha(\Lambda)$ are difficult to compute, but we have some interesting results.

Let $S \subset A$ be the set of elements in A invertible in B. Then, if there exists a classical localization $S^{-1}A$, Λ is the ring $S^{-1}A$. This holds, for example, if A is commutative or if $A \to B$ is the ring homomorphism $\mathbf{Z}\pi \to \mathbf{Z}\pi'$ induced by a group homomorphism $\pi \to \pi'$ with finitely generated nilpotent kernel onto a finite extension of a polycyclic group [7].

An other example is the following (see theorem (9.7)): Let $\pi \to G$ be a group-epimorphism with locally perfect kernel. Then the localization of $\mathbf{Z}\pi \to \mathbf{Z}G$ is $\mathbf{Z}\pi \to \mathbf{Z}G$ itself.

Anyway, the theorems (1.10), (1.11), (1.13) imply that the obstruction groups $\Gamma_n(A \to B)$ of Cappell and Shaneson [1] are always the L_n-groups of Λ (endowed with a subgroup of $\widetilde{K}_1(\Lambda)$), at least when the theory of Cappell and Shaneson holds, *i.e.* when $A \to B$ is locally epic. This was already proved in some particular cases by Cappell and Shaneson [1], Smith [7], Hausmann [3] and the author [8].

Nevertheless the condition of surjectivity of $\Lambda \to B$ holds in many other cases.

Proposition **(1.15)**. — *Let* $A \to B$ *be a ring homomorphism and* $A \to \Lambda$ *be the localization of* $A \to B$. *Let* $B_0 \subset B_1 \subset B_2 \subset \ldots$ *be subrings of* B *defined by*:

i) B_0 *is the image of* $A \to B$;

ii) *For any* $n \geq 0$, B_{n+1} *is generated by* B_n *and the inverses of the units of* B *contained in* B_n.

Then, the image of $\Lambda \to B$ *contains all the rings* B_n. *Therefore, if* B *is the union of the rings* B_n, *the morphism* $\Lambda \to B$ *is onto and the theorems* (1.10), (1.11), (1.13) *hold.*

In fact, the image of $\Lambda \to B$ can be strictly greater than the union of the rings B_n.

Example **(1.16)**. — Let F be the free group with p generators, $p > 1$, and let A be the group ring $\mathbf{Z}[F]$. Let $\mathscr{W}$ be the class of finite A-complexes C such that $H_*(C)$ is finitely generated over $\mathbf{Z}$ and let (Λ, α) be the $\mathscr{W}$-localization of A. Then the localization of $A \to \Lambda$ is $A \to \Lambda$ and the morphism $\Lambda \to \Lambda$ is the identity. One can prove that any square matrix with entries in A which is invertible in Λ, is invertible in A; hence $B_n = A$ for all n, but $A \to \Lambda$ is not surjective!

Remark **(1.17)**. — Let $A \to B$ be a ring homomorphism and β be a subgroup of $\widetilde{K}_1(B)$. Denote by $\mathscr{W}$ the class of finite A-complexes which are B-acyclic with torsion in β and by (Λ, α) the $\mathscr{W}$-localization of A.

If the morphism $\Lambda \to B$ is not onto, the condition of Λ-acyclicity of complexes in $\mathscr{W}$ is not always satisfied.

For example, this condition holds if $A \to B$ is the ring homomorphism $\mathbf{Z} \to \mathbf{R}$, but it does not hold if A is the ring $\mathbf{Z}[t, t^{-1}]$ and B is the product of the localizations of A with respect to the non zero principal prime ideals.

2. A first homology surgery obstruction group

In a first step, we will construct a surgery obstruction group $\Gamma_n(A, \mathscr{W})$ which looks like the group $\Gamma_n(A \to B)$ constructed by Ranicki [5], but from a dual point of view.

Throughout sections 2 and 3 we assume that A is a ring with involution and that $\mathscr{W}$ is an exact symetric class in $\mathscr{C}(A)$ (see (1.2) and (1.3)).

If C and C′ are finite A-complexes, we denote by Hom(C, C′) the set of A-homomorphisms from C to C′; Hom(C, C′) can be given a graded differential $\mathbf{Z}$-module structure by setting:

$$\partial^0 f(x) = \partial^0 f + \partial^0 x, \quad \text{for any } f \in \mathrm{Hom}(C, C'),\ x \in C$$
$$d(f(x)) = (df)(x) + (-1)^{\partial^0 f} f(dx), \quad \text{for any } f \in \mathrm{Hom}(C, C'),\ x \in C.$$

Moreover, by setting

$$\hat{f}(u) = (-1)^{\partial^0 f \partial^0 u} u \circ f, \quad \text{for any } f \in \mathrm{Hom}(C, C'),\ u \in \hat{C}',$$

we get a morphism $f \to \hat{f}$ from Hom(C, C′) to $\mathrm{Hom}(\hat{C}', \hat{C})$ which respects the degrees and the differentials.

Notation **(2.1)**. — If C is a finite A-complex, we denote by B(C) the graded differential $\mathbf{Z}$-module $\mathrm{Hom}(C, \hat{C})$. The composite map:

$$\mathrm{Hom}(C, \hat{C}) \to \mathrm{Hom}(\hat{\hat{C}}, \hat{C}) \xrightarrow{\sim} \mathrm{Hom}(C, \hat{C})$$

is an involution on B(C) and B(C) is a graded differential $\mathbf{Z}[\mathbf{Z}/2]$-module.

Definition **(2.2)**. — Let C be a finite A-complex. We use $Q_n(C)$ to denote the group $H_n(\mathbf{Z}/2, B(C))$. By a *quadratic n-form* over C, we mean an element of $Q_n(C)$ and by a *quadratic n-complex* we mean a pair (C, q), $q \in Q_n(C)$.

Let $C \to C'$ be an epimorphism of degree 0 between two finite A-complexes. We use $Q_n(C \to C')$ to denote the group $H_n(\mathbf{Z}/2, B(C)/B(C'))$. By a *quadratic n-form* over $C \to C'$, we mean an element of $Q_n(C \to C')$ and by a *quadratic n-pair*, we mean a pair $(C \to C', q)$, $q \in Q_n(C \to C')$.

Definition **(2.3)**. — Let (C, q) be a quadratic n-complex. We will say that q or (C, q) is *$\mathscr{W}$-non singular* if the image of q by the composite map

$$H_n(\mathbf{Z}/2, B(C)) \xrightarrow{\text{transfer}} H_n(1, B(C)) \simeq H_n(B(C))$$

is represented by a $\mathscr{W}$-equivalence from C to $\hat{C}$.

Let $(C \to C', q)$ be a quadratic n-pair. Let K be the kernel of $C \to C'$. We will say that q or $(C \to C', q)$ is *$\mathscr{W}$-non singular* if the image of q by the composite map

$$H_n(\mathbf{Z}/2, B(C)/B(C')) \xrightarrow{\text{transfer}} H_n(B(C)/B(C')) \to H_n(\mathrm{Hom}(K, \hat{C}))$$

is represented by a $\mathscr{W}$-equivalence from K to $\hat{C}$.

Remark **(2.4)**. — If C is zero except in dimension $-p$, a quadratic $2p$-form over C is exactly a $(-1)^p$-quadratic from over C_{-p} in the sense of Wall [11].

Remark **(2.5)**. — If $\mathscr{W}$ is the class of acyclic complexes with zero torsion, a $\mathscr{W}$-non singular quadratic n-form q over a finite A-complex C is an n-dimensional quadratic Poincaré structure on $\hat{C}$, in the sense of Ranicki [5], at least if $\hat{C}$ is (-1)-connected.

Definition **(2.6)**. — We will denote by $\Gamma_n(A, \mathscr{W})$ the set of $\mathscr{W}$-non singular quadratic n-complexes subject to the following cobordism relation: (C, q) is cobordant to (C', q') if there exists a $\mathscr{W}$-non singular quadratic $(n+1)$-pair $(\Sigma \to C \oplus C', u)$ such that $\partial u = q \oplus -q'$.

Let W be the standard free resolution of the $\mathbf{Z}[\mathbf{Z}/2]$-module $\mathbf{Z}$:

$$\mathbf{Z}[\mathbf{Z}/2]e_0 \xleftarrow{1-t} \mathbf{Z}[\mathbf{Z}/2]e_1 \xleftarrow{1+t} \mathbf{Z}[\mathbf{Z}/2]e_2 \xleftarrow{1-t} \cdots$$

Then $Q_n(C)$ is the n-th homology group of $W \otimes_{\mathbf{Z}/2} B(C)$.

Lemma **(2.7)**. — *Two $\mathscr{W}$-non singular quadratic n-complexes* (C, q) *and* (C', q') *are cobordant if and only if there exist two s-exact sequences*

$$0 \to K \to \Sigma \xrightarrow{\alpha} C \to 0$$

$$0 \to K' \to \Sigma \xrightarrow{\alpha'} C' \to 0$$

and an element $e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + \ldots$ *in* $W \otimes_{Z/2} B(\Sigma)$ *such that*:

i) *If* q *and* q' *are the homology classes of* φ *and* φ', *we have*

$$d(\Sigma e_i \otimes \psi_i) = \alpha^*(\varphi) - \alpha'^*(\varphi');$$

ii) $\psi_0 + \hat{\psi}_0$ *induces a* $\mathscr{W}$*-equivalence from* K *to* $\hat{K}'$.

Proof. — Suppose that q and q' are represented by $\varphi \in W \otimes_{Z/2} B(C)$ and $\varphi' \in W \otimes_{Z/2} B(C')$. If (C, q) and (C', q') are cobordant, there exists a s-exact sequence

$$0 \to \Sigma' \to \Sigma \xrightarrow{\alpha \oplus \alpha'} C \oplus C' \to 0$$

together with an element $\Sigma e_i \otimes \psi_i \in W \otimes B(\Sigma)$ such that:

(i) $d(\Sigma e_i \otimes \psi_i) = \alpha^*(\varphi) - \alpha'^*(\varphi')$;

(ii) $\psi_0 + \hat{\psi}_0$ induces a $\mathscr{W}$-equivalence from Σ' to $\hat{\Sigma}$.

Let K (respectively K′) be the kernel of α (respectively α'). We have a homotopy commutative diagram

$$\text{(I)} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma' & \longrightarrow & K & \longrightarrow & C' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & \hat{\Sigma} & \longrightarrow & \hat{K}' & \longrightarrow & \hat{C}' & \longrightarrow & 0 \end{array}$$

where the lines are homotopy s-exact and a and b are induced by $\psi_0 + \hat{\psi}_0$ and c is induced by the transfer of φ'.

Since a and c are $\mathscr{W}$-equivalences, b is a $\mathscr{W}$-equivalence too and the first part of the lemma is proved.

Conversely, suppose we have two s-exact sequences

$$0 \longrightarrow K \longrightarrow \Sigma \xrightarrow{\alpha} C \longrightarrow 0$$

$$0 \longrightarrow K' \longrightarrow \Sigma \xrightarrow{\alpha'} C' \longrightarrow 0$$

and an element $\Sigma e_i \otimes \psi_i$ satisfying the conditions (i) and (ii) of the lemma. Up to simple homotopy type, we may suppose that $\alpha \oplus \alpha'$ is onto with kernel $\Sigma' \in \mathscr{C}(A)$. Then we have the homotopy commutative diagram (I) where b and c are $\mathscr{W}$-equivalences and $\psi_0 + \hat{\psi}_0$ induces a $\mathscr{W}$-equivalence from Σ' to $\hat{\Sigma}$. Hence (C, q) and (C', q') are cobordant. ■

Lemma **(2.8)**. — *Let* (C, q) *be a* $\mathscr{W}$*-non singular quadratic n-complex and* $f: C' \to C$ *be a* $\mathscr{W}$*-equivalence. Then* $(C', f^*(q))$ *is a* $\mathscr{W}$*-non singular quadratic n-complex cobordant to* (C, q).

Proof. — We may suppose that f is epic with kernel $K \in \mathscr{C}(A)$. Then we have the s-exact sequences

$$0 \longrightarrow K \longrightarrow C' \xrightarrow{f} C \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow C' \xrightarrow{1} C' \longrightarrow 0$$

and the result is an easy consequence of lemma (2.7). ■

3. Algebraic surgery

In order to kill the homology of a $\mathscr{W}$-non singular quadratic n-complex, in low dimension, we need the following:

Lemma (3.1). — *Let* $0 \to I \xrightarrow{\alpha} C \xrightarrow{\beta} J \to 0$ *be an s-exact sequence of finite* A-*complexes. Let q be a* $\mathscr{W}$-*non singular quadratic n-form over* C *such that* $\alpha^* q = 0$. *Then, q is represented by a cycle* $\Sigma e_i \otimes f_i \beta$.

Moreover if q is represented by such a cycle, (C, q) *is cobordant to a* $\mathscr{W}$-*non singular quadratic n-complex* (C′, q') *where* C′ *is the mapping cone of* $\hat{\alpha} f_0$ (*the grading of* C′ *is chosen so that the map* $C' \to J$ *has degree* 0).

Proof. — Consider the following exact sequences of graded differential $\mathbf{Z}[\mathbf{Z}/2]$-modules:

$$0 \to B \to B(C) \xrightarrow{\alpha^*} B(I) \to 0$$

$$\mathrm{Hom}(C, \hat{J}) \oplus \mathrm{Hom}(J, \hat{C}) \to B \to 0.$$

If $\alpha^* q$ is zero, q is represented by a cycle in $W \otimes_{\mathbf{Z}/2} B$, and there exist morphisms f_i' and f_i'' in $\mathrm{Hom}(J, \hat{C})$ such that q is represented by

$$\Sigma e_i \otimes (f_i' \beta + \hat{\beta} \hat{f}_i'').$$

Now we have

$$d(e_{i+1} \otimes f_i'' \beta) = e_i \otimes f_i'' \beta + (-1)^{i+1} e_i \otimes \hat{\beta} \hat{f}_i'' + (-1)^{i+1} e_{i+1} \otimes df_i'' \beta.$$

Then there exist morphisms $f_i \in \mathrm{Hom}(J, \hat{C})$ such that q is represented by $\Sigma e_i \otimes f_i \beta$. Since $\Sigma e_i \otimes f_i \beta$ is a cycle, we have

$$\forall\, i \geq 0, \quad (-1)^i df_i \beta + f_{i+1} \beta + (-1)^{i+1} \hat{\beta} \hat{f}_{i+1} = 0,$$

whence $\quad d(\hat{\alpha} f_0) = 0, \quad \hat{\alpha} f_i = 0, \quad$ for any $i > 0$.

Let C′ be the mapping cone of $\hat{\alpha} f_0$. We have a split exact sequence

$$0 \longrightarrow \hat{I} \underset{r'}{\overset{\alpha'}{\rightleftarrows}} C' \xrightarrow{\beta'} J \longrightarrow 0$$

such that

$$\partial^0 \alpha' = -n - 1, \quad \partial^0 \beta' = 0, \quad dr' = \hat{\alpha} f_0 \beta', \quad r' \alpha' = 1$$

and $\quad 0 \to S^{-n-1} \hat{I} \to C' \to J \to 0$

is s-exact.

Let Σ be the pull-back of C and C′ over J:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\gamma} & C \\ {\scriptstyle \gamma'} \downarrow & & \downarrow {\scriptstyle \beta} \\ C' & \xrightarrow{\beta'} & J \end{array}$$

Let r be a retraction of α and let u be the element $e_0 \otimes \hat{\gamma}\hat{r}r'\gamma' \in W \otimes_{Z/2} B(\Sigma)$. We have

$$du = e_0 \otimes \hat{\gamma} d\hat{r}r'\gamma' + e_0 \otimes \hat{\gamma} f_0 \beta\gamma + e_0 \otimes \hat{\gamma}(\hat{r}\hat{\alpha} - 1) f_0 \beta'\gamma'$$

and it is easy to see that $\gamma^*(\Sigma e_i \otimes f_i \beta) - du$ has the form $\gamma'^*(\Sigma e_i \otimes \varphi'_i)$, $\varphi'_i \in B(C')$.

On the other hand, $\hat{\gamma}\hat{r}r'\gamma' + \hat{\gamma}'\hat{r}'r\gamma$ induces the identity from the kernel of γ' to the dual of the kernel of γ. Then $\Sigma e_i \otimes \varphi'_i$ represents a $\mathscr{W}$-non singular quadratic n-form q' over C′ and, by (2.7), (C, q) and (C′, q') are cobordant. ∎

Corollary **(3.2).** — *Any $\mathscr{W}$-non singular quadratic n-complex is cobordant to a $\mathscr{W}$-non singular quadratic n-complex* (C, q) *such that* C *is* $\left(\left[\frac{-n}{2}\right] - 1\right)$*-connected.*

Proof. — Just apply lemma (3.1), I being the $\left(\left[\frac{-n}{2}\right] - 1\right)$-skeleton of the complex. ∎

Lemma **(3.3).** — *Let* $0 \to I \xrightarrow{\alpha} C \xrightarrow{\beta} J \to 0$ *be an s-exact sequence of finite* A*-complexes and* $\gamma : J \to K$ *be an epimorphism of degree* 0 *which respects the differentials. Let q be a $\mathscr{W}$-non singular quadratic n-form over* $C \to K$ *such that* $\alpha^* q = 0$. *Then q is represented by* $\Sigma e_i \otimes f_i \beta$.

Moreover, if C′ *is the mapping cone of* $\hat{\alpha} f_0$ *(the grading being chosen as in lemma* (3.1)*), there exists a $\mathscr{W}$-non singular quadratic n-form q' over* $C' \to K$ *such that* ∂q *and* $\partial q'$ *coincide in* $Q_{n-1}(K)$.

Proof. — We have the following exact sequences of graded differential $\mathbf{Z}[\mathbf{Z}/2]$-modules:

$$0 \to B \to B(C)/B(K) \xrightarrow{\alpha^*} B(I) \to 0$$

$$\mathrm{Hom}(C, \hat{J}) \oplus \mathrm{Hom}(J, \hat{C}) \to B \to 0.$$

Then, as in lemma (3.1), we show that q is represented by an element $\Sigma e_i \otimes f_i \beta$ and we have

$$d(\hat{\alpha} f_0) = 0, \quad \hat{\alpha} f_i = 0, \quad \text{for any } i > 0.$$

Consider, as above, the diagram: $0 \to \hat{I} \underset{r'}{\overset{\alpha'}{\rightleftarrows}} C' \xrightarrow{\beta'} J \to 0$ and let s be a section of β. We have

$$ds = \alpha\delta, \quad \delta \in \mathrm{Hom}(J, I).$$

It is not difficult to see that the element

$$u = e_0 \otimes \hat{\beta}'\hat{\delta} r' + \Sigma e_i \otimes \hat{\beta}'\hat{s} f_i \beta'$$

represents a quadratic n-form q' over $C' \to K$ and that ∂q and $\partial q'$ coincide in $Q_{n-1}(K)$. Moreover, the transfer $\tilde{u}$ of u is:

$$\tilde{u} = \hat{\beta}'\hat{\delta} r' + (-1)^{n+1} \hat{r}'\delta\beta' + \hat{\beta}'\hat{s} f_0 \beta' + \hat{\beta}' \hat{f}_0 s \beta'$$

and we have

$$\widetilde{u}\alpha' = \widehat{\beta}'\widehat{\delta} \quad \text{and} \quad \widehat{\alpha}'\widetilde{u} = \delta\beta'.$$

Denote by $\overline{\mathrm{C}}$, $\overline{\mathrm{J}}$, $\overline{\mathrm{C}}'$ the kernels of the morphisms $\mathrm{C} \to \mathrm{K}$, $\mathrm{J} \to \mathrm{K}$ and $\mathrm{C}' \to \mathrm{K}$. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathrm{I}} & \xrightarrow{\alpha'} & \overline{\mathrm{C}}' & \xrightarrow{\beta'} & \overline{\mathrm{J}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\widehat{\delta}} & & \downarrow{\scriptstyle\widetilde{u}} & & \downarrow{\scriptstyle\delta} & & \\ 0 & \longrightarrow & \widehat{\mathrm{J}} & \xrightarrow{\widehat{\beta}'} & \widehat{\mathrm{C}}' & \xrightarrow{\widehat{\alpha}'} & \mathrm{I} & \longrightarrow & 0, \end{array}$$

and we obtain a s-exact sequence between the mapping cone of $\widehat{\delta}$, $\widetilde{u}$ and δ. Now the boundary of this s-exact sequence is homotopic to the morphism $(-1)^{n+1}(f_0\beta + \widehat{\beta}\widehat{f_0})$ from $\overline{\mathrm{C}}$ to $\widehat{\mathrm{C}}$, which is a $\mathscr{W}$-equivalence. Then the mapping cone of $\widetilde{u}: \overline{\mathrm{C}}' \to \widehat{\mathrm{C}}'$ is in $\mathscr{W}$ and q' is $\mathscr{W}$-non singular. ∎

Corollary **(3.4)**. — *Let* (C, q) *be a* $\mathscr{W}$*-non singular quadratic n-complex cobordant to zero. Then there exists a* $\mathscr{W}$*-non singular quadratic* $(n+1)$*-pair* $(\Sigma \to \mathrm{C}, u)$ *such that q is the boundary of u and the kernel of* $\Sigma \to \mathrm{C}$ *is* $\left(\left[\frac{-n-1}{2}\right]-1\right)$*-connected.*

Proof. — If (C, q) is cobordant to zero, there exists a $\mathscr{W}$-non singular quadratic $(n+1)$-pair $(\Sigma' \to \mathrm{C}, u')$ such that q is the boundary of u'. Then apply lemma (3.3), I being the $\left(\left[\frac{-n-1}{2}\right]-1\right)$-skeleton of the kernel of $\Sigma' \to \mathrm{C}$. ∎

Now, if we want to kill the homology of a $\mathscr{W}$-non singular quadratic n-form beyond the middle dimension, we must suppose that $\mathscr{W}$ satisfies some other properties. Actually, it is useful to consider the new class $\mathscr{W}'$ of all Λ-acyclic finite A-complexes.

Splitting lemma **(3.5)**. — *Let* C *be a complex in* $\mathscr{W}'$ *and let n be an integer. Then, there exist two finite* A*-complexes* L *and* L′ *concentrated in dimension n and a* $\mathscr{W}'$*-equivalence from* L *to the complex*

$$\mathrm{L}' \oplus (\ldots \to \mathrm{C}_{n+1} \to \mathrm{C}_n \to 0 \to \ldots).$$

This lemma will be proved in § 7.

Lemma **(3.6)**. — *Any* $\mathscr{W}'$*-non singular quadratic n-complex is cobordant to a* $\mathscr{W}'$*-non singular quadratic n-complex* (C, q) *where* C *vanishes except in dimension* $\left[\frac{-n}{2}\right]$ $\left(\text{and } \left[\frac{-n}{2}\right]+1\right.$ *if n is odd*).

Proof. — Let (C, q) be a $\mathscr{W}'$-non singular quadratic n-complex. By corollary (3.2), we may as well suppose that C_i vanishes for $i < \left[\frac{-n}{2}\right]$.

Suppose $n = -2p$. Since (C, q) is $\mathcal{W}'$-non singular, we have the following complex in $\mathcal{W}'$:

$$\ldots \to C_{p+1} \to C_p \to \hat{C}_p \to \hat{C}_{p+1} \to \ldots$$

and, by splitting lemma (3.5), there exist two complexes L and L′ concentrated in dimension p and a $\mathcal{W}'$-equivalence

$$f:\ L \to C \oplus L'.$$

Up to stabilization, we may suppose that L'_p is even dimensional. Let $q' \in Q_n(L')$ be a standard hyperbolic structure on L'_p.

Then (C, q) is cobordant to $(C \oplus L', q \oplus q')$ and by lemma (2.8), (C, q) is cobordant to $(L, f^*(q \oplus q'))$.

Suppose $n = -2p - 1$. Since (C, q) is $\mathcal{W}'$-non singular, we have the following complex in $\mathcal{W}'$:

$$\ldots \to C_{p+1} \to C_p \oplus \hat{C}_p \to \hat{C}_{p+1} \to \ldots,$$

and, by the splitting lemma (3.5), there exist two complexes L and L′ concentrated in dimension $p + 1$ and a $\mathcal{W}'$-equivalence

$$L \to L' \oplus (\ldots \to C_{p+2} \to C_{p+1} \to 0 \ldots).$$

We deduce a $\mathcal{W}'$-equivalence

$$f: (\ldots \to 0 \to L_{p+1} \to C_p \oplus L'_{p+1} \to 0 \ldots) \to C$$

and (C, q) is cobordant to $(\ldots \to 0 \to L_{p+1} \to C_p \oplus L'_{p+1} \to 0 \ldots, f^*q)$. ■

Lemma **(3.7)**. — *Let (C, q) be a $\mathcal{W}'$-non singular quadratic $(-2p)$-complex such that C_i vanishes for $i \neq p$. Then (C, q) is cobordant to zero if and only if there exists a $\mathcal{W}'$-non singular quadratic $(-2p + 1)$-pair $(\Sigma \to C, u)$ such that q is the boundary of u and Σ_i vanishes for $i \neq p, p - 1$.*

Proof. — Suppose (C, q) is cobordant to zero. By corollary (3.4), there exists a $\mathcal{W}'$-non singular quadratic $(-2p + 1)$-pair $(\Sigma' \to C, u')$ such that q is the boundary of u' and Σ'_i vanishes for $i < p - 1$. Let K′ be the kernel of $\Sigma \to C$.

Since u' is $\mathcal{W}'$-non singular, we have the following complex in $\mathcal{W}'$:

$$\ldots \to K'_{p+1} \to K'_p \to K'_{p-1} \oplus \hat{\Sigma}'_{p-1} \to \hat{\Sigma}'_p \to \hat{\Sigma}'_{p+1} \to \ldots$$

and, by the splitting lemma (3.5), there exist two complexes $L, L' \in \mathcal{C}(A)$ concentrated in dimension p and a $\mathcal{W}'$-equivalence

$$(\ldots \to 0 \to L_p \to K'_{p-1} \oplus L'_p \to 0 \to \ldots) \to K'.$$

Let K be the complex $\ldots 0 \to L_p \to K'_{p-1} \oplus L'_p \to 0 \to \ldots$ Since the $\mathscr{W}'$-equivalence $K \to K'$ is $(p-1)$-connected, the boundary $C \to K'$ lifts through K and we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \Sigma & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \| & & \\ 0 & \longrightarrow & K' & \longrightarrow & \Sigma' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where the lines are s-exact.

Then $(\Sigma \to C, f^* u')$ is the desired quadratic pair. ∎

Lemma **(3.8)**. — *Let* (C, q) *be a* $\mathscr{W}'$*-non singular quadratic* $(-2p-1)$*-form such that* C_i *vanishes for* $i \neq p,\ p+1$. *Then* (C, q) *is cobordant to zero if and only if there exists a* $\mathscr{W}'$*-non singular quadratic* $(-2p)$*-pair* $(\Sigma \to C, u)$ *such that* q *is the boundary of* u *and* $\Sigma_i \to C_i$ *is a simple isomorphism for* $i \neq p$.

Proof. — Suppose (C, q) is cobordant to zero. By corollary (3.4), there exists a $\mathscr{W}'$-non singular quadratic $(-2p)$-pair $(\Sigma' \to C, u')$ such that q is the boundary of u' and Σ'_i vanishes for $i < p$.

Let K′ be the kernel of $\Sigma' \to C$. We have a complex in $\mathscr{W}'$

$$\ldots \to \Sigma'_{p+1} \to \Sigma'_p \to \hat{K}'_p \to \hat{K}'_{p+1} \to \ldots$$

and, by the splitting lemma (3.5), there exist two complexes L and $L' \in \mathscr{C}(A)$ concentrated in dimension p and a $\mathscr{W}'$-equivalence

$$f\colon L \to \Sigma' \oplus L'.$$

Up to stabilization, we may suppose that L'_p is even dimensional. Let $v \in Q_{-2p}(L')$ be a standard hyperbolic structure on L'_p.

Let X be an acyclic finite A-complex with torsion zero concentrated in dimension $p-1$, p, $p+1$ and $X \to C$ be an epimorphism with kernel in $\mathscr{C}(A)$ such that $X_{p+1} \to C_{p+1}$ is an isomorphism. Let $(\Sigma'' \to C, u'')$ be the quadratic $(-2p)$-pair defined by $\Sigma'' = L \oplus X$, $u'' = f^*(u' \oplus v) \oplus 0$.

It is easy to see that u'' is $\mathscr{W}'$-non singular and that $\partial u'' = q$. Moreover the kernel K″ of $\Sigma'' \to C$ is concentrated in dimension $p-1$ and p.

Now, by lemma (3.3), we can kill the $p-1$ skeleton of K″ by surgery and we get a $\mathscr{W}'$-non singular $(-2p)$-pair $(\Sigma \to C, u)$ such that $\partial u = q$ and the kernel of $\Sigma \to C$ vanishes except in dimension p. ∎

Now, with the above lemmas, it is possible to give an interpretation of $\Gamma_n(A, \mathscr{W}')$ in term of special forms in the sense of Wall [10] and Cappell and Shaneson [1].

Definition **(3.9)**. — Let $\eta = \pm 1$ and $I_\eta = \{a - \eta\bar{a}, a \in A\}$. A *$\mathscr{W}'$-special η-form* is a triple (H, λ, μ) where H is a finitely generated free A-module, λ a **Z**-bilinear map from $H \otimes_{\mathbf{Z}} H$ to A and μ a map from H to A/I_η, and satisfying the following conditions:

Q_1 $\quad \lambda(ax, yb) = a\lambda(x,y)b, \quad \forall x, y \in H, \ \forall a, b \in A$

Q_2 $\quad \lambda(x,y) = \eta\overline{\lambda(y,x)}, \quad \forall x, y \in H$

Q_3 $\quad \mu(x) + \eta\mu(y) = \lambda(x,y), \quad \forall x, y \in H$

Q_4 $\quad \mu(x+y) \equiv \mu(x) + \mu(y) + \lambda(x,y) \bmod I_\eta, \quad \forall x, y \in H$

Q_5 $\quad \mu(xa) = \bar{a}\mu(x)a, \quad \forall x \in H, \ \forall a \in A$

Q_6 the morphism $\tilde{\lambda}$ induced by λ is a Λ-isomorphism (*i.e.* $\tilde{\lambda} \otimes \Lambda$ is an isomorphism).

Definition **(3.10)**. — Let (H, λ, μ) be a $\mathscr{W}'$-special η-form. A *$\mathscr{W}'$-subkernel* of (H, λ, μ) is a free A-module K endowed with a morphism $f: K \to H$ satisfying the following conditions:

S_1 $\quad f^*\lambda = 0, \quad f^*\mu = 0$

S_2 the following complex lies in $\mathscr{W}'$: $0 \to K \xrightarrow{f} H \xrightarrow{\hat{f}\tilde{\lambda}} \hat{K} \to 0$.

(3.11) Let $\eta = (-1)^p$ and let (H, λ, μ) be a $\mathscr{W}'$-special η-form. Since H is free, there exists a map $\varphi_0 : H \to \hat{H}$ such that

$$\lambda(x,y) = \varphi_0(x)(y) + \eta\overline{\varphi_0(y)(x)}, \quad \forall x, y \in H$$
$$\mu(x) \equiv \varphi_0(x)(x) \bmod I_\eta, \quad \forall x \in H.$$

And, if φ_0 and φ_0' are such two maps, $\varphi_0 - \varphi_0'$ has the form $\psi - \eta\hat{\psi}$.

Choose a basis for H and denote by H_* the finite A-complex defined by

$$H_i = \begin{cases} H, & i = -p \\ 0, & i \neq -p \end{cases}$$

Then $e_0 \otimes \varphi_0$ represents a $\mathscr{W}'$-non singular quadratic $2p$-form q over H_* and the cobordism class of (H_*, q) is a well defined element $\omega(H, \lambda, \mu) \in \Gamma_{2p}(A, \mathscr{W}')$.

(3.12) Let $\eta = (-1)^p$ and let $f: K \to B \oplus \hat{B}$ be a $\mathscr{W}'$-subkernel of a standard η-kernel $B \oplus \hat{B}$ (B is a finitely generated free A-module). The map f is induced by maps $d: K \to B$ and $\varphi_0 : K \to \hat{B}$. Since the quadratic form is trivial over K, there exists a map $\varphi_1 : K \to \hat{K}$ such that $\hat{\varphi}_0 \circ d = \varphi_1 - (-1)^p\hat{\varphi}_1$. Choose basis for K and B. Let C be the $-p$-dimensional complex

$$\ldots \to 0 \to K \xrightarrow{d} B \to 0 \to \ldots$$

Let $\varphi_0 \mid B = 0$. We get two bilinear forms φ_0 and φ_1 on C, and we have

$$d\varphi_0 = \varphi_1 - \hat{\varphi}_1.$$

Then, $e_0 \otimes \varphi_0 - e_1 \otimes \varphi_1$ is a cycle in $W \otimes_{Z/2} B(C)$ inducing a quadratic $(2p+1)$-form q over C.

It is easy to see that q is $\mathscr{W}'$-non singular. We denote by $\omega(f) \in \Gamma_{2p+1}(A, \mathscr{W}')$ the cobordism class of (C, q). This element depends *a priori* on the choice of φ_1.

On the other hand, the tensorization by Λ induces a map from $\Gamma_n(A, \mathscr{W}')$ to $\Gamma_n(\Lambda, \mathscr{W}_1)$ where $\mathscr{W}_1$ is the class of finite acyclic Λ-complexes. But the group $\Gamma_n(\Lambda, \mathscr{W}_1)$ is isomorphic to $L_n^h(\Lambda)$. Then we get a morphism ε from $\Gamma_n(A, \mathscr{W}')$ to $L_n^h(\Lambda)$ and $\varepsilon\omega(f)$ is the class of $f \otimes \Lambda$ in $L_n^h(\Lambda)$. We deduce that $\varepsilon\omega(f)$ does not depend on the choice of φ_1. But it will be proved in § 8 that ε is an isomorphism. Therefore $\omega(f)$ is well defined.

Proposition **(3.13)**. — *Any element of* $\Gamma_{2p}(A, \mathscr{W}')$ *has the form* $\omega(H, \lambda, \mu)$ *for some* $\mathscr{W}'$*-special* $(-1)^p$*-form* (H, λ, μ) *and any element of* $\Gamma_{2p+1}(A, \mathscr{W}')$ *has the form* $\omega(f)$ *for some* $\mathscr{W}'$*-subkernel* $f: K \to B \oplus \hat{B}$ *of a standard* $(-1)^p$*-kernel* $B \oplus \hat{B}$.

Proof. — In the even dimensional case, this is a trivial consequence of lemma (3.6).

In the odd dimensional case, we know by lemma (3.6) that any element of $\Gamma_{2p+1}(A, \mathscr{W}')$ is the cobordism class of a $\mathscr{W}'$-non singular $2p+1$-complexes (C, q) where C_i vanishes for $i \neq -p, -p-1$. It is not difficult to see that q is represented by $e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1$, where the morphism φ_0 is trivial over C_{-p-1}. Then the cobordism class of (C, q) is $\omega(f)$ where f is the map $d \oplus \varphi_0 : C_{-p} \to C_{-p-1} \oplus \hat{C}_{-p-1}$. ■

Proposition **(3.14)**. — *Let* (H, λ, μ) *be a* $\mathscr{W}'$*-special* $(-1)^p$*-form. Then* $\omega(H, \lambda, \mu)$ *is zero if and only if the direct sum of* (H, λ, μ) *and a standard kernel has a* $\mathscr{W}'$*-subkernel.*

Proof. — Suppose that (H, λ, μ) has a $\mathscr{W}'$-subkernel $f: K \to H$. Consider the quadratic $2p$-complex (H_*, q) constructed in (3.11). Choose a basis for K and denote by $K_* \in \mathscr{C}(A)$ the complex defined by

$$K_i = \begin{cases} K, & i = -p \\ 0, & i \neq -p. \end{cases}$$

Let $K_* \to H'_* \xrightarrow{g} H_*$ be a factorization of f such that g is a simple homotopy equivalence and $K_* \to H'_*$ is a monomorphism with free cokernel. After doing an algebraic surgery along $K_* \to H'_*$, we show that $(H'_*, g^* q)$ is cobordant to (H''_*, q'') where H''_* has the simple homotopy type of

$$\dots \to 0 \to K \to H \to \hat{K} \to 0 \to \dots$$

The complex H''_* is thus Λ-acyclic and (H_*, q) is cobordant to zero.

Now suppose that the direct sum of (H, λ, μ) and a standard kernel H' has a $\mathscr{W}'$-subkernel. We have

$$\omega(H, \lambda, \mu) = \omega(H, \lambda, \mu) + \omega(H') = 0.$$

Conversely suppose that $\omega(H, \lambda, \mu)$ vanishes. By lemma (3.7), there exists a $\mathscr{W}'$-non singular quadratic $(2p+1)$-pair $(\Sigma \xrightarrow{\alpha} H_*, u)$ such that q is the boundary of u and Σ_i vanishes for $i \neq -p, -p-1$.

The form u can be represented by $e_0 \otimes \psi_0 + e_1 \otimes \psi_1$, ψ_0 vanishing on Σ_{-p-1}. Let K be the kernel of $\Sigma_{-p} \to H$.

Since u is $\mathscr{W}'$-non singular, the following complex is Λ-acyclic:

$$0 \longrightarrow \Sigma_{-p} \xrightarrow{d \oplus (-1)^p \psi_0} \Sigma_{-p-1} \oplus \hat{\Sigma}_{-p-1} \xrightarrow{\hat{\psi}_0 + \hat{d}} \hat{K} \longrightarrow 0,$$

and since $\tilde{\lambda} : H \to \hat{H}$ is a Λ-isomorphism, we deduce that

$$\alpha \oplus d \oplus (-1)^p \psi_0 : \Sigma_{-p} \to H \oplus \Sigma_{-p-1} \oplus \hat{\Sigma}_{-p-1}$$

is a $\mathscr{W}'$-subkernel of the direct sum of (H, λ, μ) and the standard kernel $\Sigma_{-p-1} \oplus \hat{\Sigma}_{-p-1}$.

Proposition (**3.15**). — *Let* $f : K \to B \oplus \hat{B}$ *be a* $\mathscr{W}'$*-subkernel of the standard* $(-1)^p$*-kernel* $B \oplus \hat{B}$. *Then* $\omega(f)$ *is zero if and only if there exist a kernel* $C \oplus \hat{C}$ *endowed with its standard subkernel* $g : C \to C \oplus \hat{C}$ *and an isometry* h *of* $B \oplus \hat{B} \oplus C \oplus \hat{C}$ *leaving each element of* $B \oplus \hat{C}$ *fixed, such that the composite map*

$$K \oplus C \xrightarrow{h \circ (f \oplus g)} B \oplus \hat{B} \oplus C \oplus \hat{C} \longrightarrow B \oplus \hat{C}$$

is a Λ*-isomorphism.*

Proof. — Consider the "if" part first. If g is the standard subkernel of $C \oplus \hat{C}$, the complex associated to g (see (3.12)) is acyclic and then $\omega(g)$ vanishes.

The complex associated to $f \oplus g$ is

$$0 \to K \oplus C \to B \oplus C \to 0 \to \dots$$

If we perform a surgery along B, we get a new complex

$$\dots \to K \oplus C \to \hat{B} \oplus C \to 0 \to \dots$$

and $\omega(f)$ is equal to $\omega(f')$, f' being the new $\mathscr{W}'$-subkernel

$$K \oplus C \xrightarrow{f \oplus g} \hat{B} \oplus C \oplus (\widehat{\hat{B} \oplus C}).$$

It is easy to show that, for any isometry h of $\hat{B} \oplus C \oplus B \oplus \hat{C}$ leaving each element of $B \oplus \hat{C}$ fixed ($h \in UU_r(A)$ with the notations of [10]), the two $\mathscr{W}'$-subkernels f' and $h \circ f'$ represent the same quadratic $(2p+1)$-form over the same complex.

It suffices now to perform a surgery along $\hat{B} \oplus C$ to get a Λ-acyclic complex and $\omega(f)$ is zero.

Conversely, suppose $\omega(f)$ is zero. Let (C_*, q) be the quadratic complex associated to f (see (3.12)). By lemma (3.8), there exists a $\mathscr{W}'$-non singular quadratic $(2p+2)$-pair $(\Sigma_* \to C_*, u)$ such that q is the boundary of u and $\Sigma_i \to C_i$ is a simple isomorphism for $i \neq -p-1$.

The map $\Sigma_* \to C_*$ has the form

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & K & \xrightarrow{1} & K & & \\ & & \downarrow & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d} & & \\ 0 & \longrightarrow & X & \xrightarrow{\alpha} & \Sigma & \xrightarrow{\beta} & B & \longrightarrow & 0 \end{array}$$

where $K \xrightarrow{d'} \Sigma$ is the complex Σ_*.

If u is represented by $\Sigma e_i \otimes \psi_i$, ψ_0 is a homomorphism from Σ to $\hat{\Sigma}$ satisfying

$$\tilde{\psi} \circ d' + \hat{\beta} \circ \varphi_0 = 0 \quad \text{with } \tilde{\psi} = \psi_0 - (-1)^p \hat{\psi}_0$$

and the following complex is Λ-acyclic:

$$0 \longrightarrow K \xrightarrow{d'} \Sigma \xrightarrow{\hat{\alpha} \circ \tilde{\psi}} \hat{X} \longrightarrow 0.$$

By the splitting lemma (3.5), there exist two finitely generated free A-modules C and I and a homomorphism $\gamma : C \to \Sigma \oplus I$ such that $(\gamma \oplus d') \otimes \Lambda$ is an isomorphism. After adding a kernel to Σ_*, we may suppose that I is zero and γ is a homomorphism from C to Σ.

Then the morphism $\hat{\alpha} \circ \tilde{\psi} \circ \gamma : C \to \hat{X}$ is a Λ-isomorphism, and the morphism $\tilde{\psi} \circ \gamma \oplus \hat{\beta} : C \oplus \hat{B} \to \hat{\Sigma}$ is also a Λ-isomorphism. That implies that the composite map from $C \oplus K$ to $\hat{C} \oplus B$

$$(\hat{\gamma} \circ \hat{\tilde{\psi}} \oplus \beta) \circ (\gamma \oplus d') = -(-1)^p \hat{\gamma} \circ \tilde{\psi} \circ \gamma \oplus (-1)^p \hat{\gamma} \circ \hat{\beta} \circ \varphi_0 \oplus \beta \circ \gamma \oplus d$$

is a Λ-isomorphism.

Let h be the homomorphism from $B \oplus \hat{B} \oplus C \oplus \hat{C}$ to itself defined by

$$h = 1 \oplus (-1)^p \hat{\gamma} \circ \hat{\beta} \oplus (-1)^{p+1} \beta \circ \gamma \oplus (-1)^{p+1} \hat{\gamma} \circ \tilde{\psi} \circ \gamma.$$

It is easy to check that h is an isometry leaving each element of $B \oplus \hat{C}$ fixed and that the composite map

$$K \oplus C \xrightarrow{h \circ (f \oplus g)} B \oplus \hat{B} \oplus C \oplus \hat{C} \longrightarrow B \oplus \hat{C}$$

is a Λ-isomorphism. ■

4. Geometric surgery

Throughout this section, we will suppose that A is the group ring $\mathbf{Z}\pi$ with an involution induced by a morphism $w : \pi \to \pm 1$, and that $\mathscr{W}$ is an exact symmetric class in $\mathscr{C}(A)$ containing any acyclic complex with torsion in the image of $\pi \to \widetilde{K}_1(A)$.

We denote by (Λ, α) the $\mathscr{W}$-localization of A (1.9) and by $\mathscr{W}'$ the class of Λ-acyclic complexes in $\mathscr{C}(A)$. The class $\mathscr{W}'$ is exact and symmetric and the $\mathscr{W}'$-localization of A is $(\Lambda, \widetilde{K}_1(\Lambda))$. The fact that any element in $\widetilde{K}_1(\Lambda)$ is the torsion of a complex $C \otimes \Lambda$, $C \in \mathscr{W}'$, will be proved in § 7.

Let f be a degree one normal map from the normal bundle of a compact n-dimensional (Top, PL or Diff)-manifold V to a (Top, PL or Diff)-bundle ξ over a connected $\mathscr{W}$-Poincaré complex with fundamental group π, such that the first Stiefel-Whitney class of ξ is w. We assume that f induces a $\mathscr{W}$-equivalence on the boundaries.

Suppose that any complex in $\mathscr{W}$ is Λ-acyclic. Then f induces a Λ-homology equivalence with torsion in α between the boundaries. Then we can use Wall's technique [10] in order to define $\sigma(f) \in L_n^\alpha(\Lambda)$ and $\sigma(f)$ depends only on the normal cobordism class (relative the boundary) of f, and vanishes if f is normally cobordant to a $\mathscr{W}$-equivalence.

(4.1) *Proof of theorem* (1.10) *in the case* $\mathscr{W} = \mathscr{W}'$

Suppose $n = 2p$ or $2p + 1 \geq 5$ and $\sigma(f) = 0$. After performing surgeries, we may suppose that the normal map $f: V \to X$ is p-connected.

Denote by C_* the complex $\Sigma^{-1}C_*(X, V; \mathbf{Z}\pi)$. If g is a homotopy inverse of the cap product $C^*(V; \mathbf{Z}\pi) \to C_*(V, \partial V; \mathbf{Z}\pi)$, the composite map

$$C_* \to C_*(V; \mathbf{Z}\pi) \to C_*(V, \partial V; \mathbf{Z}\pi) \xrightarrow{g} C^*(V; \mathbf{Z}\pi) \to \hat{C}_*$$

is a $\mathscr{W}'$-equivalence.

a) *The even dimensional case*

If $n = 2p$, we have a complex in $\mathscr{W}'$

$$\dots \to C_{p+1} \to C_p \to \hat{C}_p \to \hat{C}_{p+1} \to \dots$$

and by the splitting lemma (3.5), there exist two complexes L and L′ concentrated in dimension p and a $\mathscr{W}'$-equivalence $L \to C_* \oplus L'$.

After performing trivial surgeries, we may suppose that L′ is zero. Then the intersection and self-intersection forms on $H_{p+1}(X, V; \mathbf{Z}\pi)$ induce forms λ and μ on L_p and (L_p, λ, μ) is a $\mathscr{W}'$-special $(-1)^p$-form. Clearly, $\omega(L_p, \lambda, \mu)$ is sent to $\sigma(f)$ by the canonical map: $\varepsilon: \Gamma_n(\mathbf{Z}\pi, \mathscr{W}') \to L_n^h(\Lambda)$.

But ε is an isomorphism. This will be proved in § 8.

Then $\omega(L_p, \lambda, \mu)$ is zero and by proposition (3.14), the direct sum of (L_p, λ, μ) and a $(-1)^p$-kernel has a $\mathscr{W}'$-subkernel. We can realize the direct sum by trivial surgeries. So we may as well suppose that (L_p, λ, μ) has a $\mathscr{W}'$-subkernel $K \to L_p$. Now it suffices to perform surgeries along a basis of K, via the map $K \to L_p \to C_p \to H_{p+1}(X, V; \mathbf{Z}\pi)$, to get a $\mathscr{W}'$-equivalence.

b) *The odd dimensional case*

If $n = 2p + 1$, we have a complex in $\mathscr{W}'$

$$\dots \to C_{p+2} \to C_{p+1} \to C_p \oplus \hat{C}_p \to \hat{C}_{p+1} \to \hat{C}_{p+2} \to \dots$$

and by the splitting lemma (3.5), there exist two complexes L and L′ concentrated in dimension $p+1$ and a $\mathscr{W}'$-equivalence

$$L \to (\ldots \to C_{p+2} \to C_{p+1} \to 0 \to \ldots) \oplus L'.$$

So we get a $\mathscr{W}'$-equivalence $(\ldots \to 0 \to L_{p+1} \to C_p \oplus L'_{p+1} \to 0 \to \ldots) \to C$.

Denote by $K \xrightarrow{d} B$ the map $L_{p+1} \to C_p \oplus L'_{p+1}$, and consider the composite map $B \to C_p \to \pi_{p+1}(X, V)$. The basis of B induces maps from S^p to V homotopic to zero in X. These maps are covered by fibered maps and we get immersions $\alpha_i : S^p \to V$, which we can suppose to be disjoint embeddings. Let U be a regular neighborhood of the images of these embeddings, connectified with 1-handles. The group $H_{p+1}(\mathrm{pt}, \partial U; \mathbf{Z}\pi)$ endowed with intersection and self-intersection forms is the standard $(-1)^p$-kernel $B \oplus \hat{B}$.

The morphisms $K \to B$ and $K \to C_{p+1}$ induce a morphism from K to the relative homology group

$$H_{p+2}\left(\begin{matrix} U & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{matrix}\; ; \; \mathbf{Z}\pi\right) = H_{p+2}\left(\begin{matrix} \partial U & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ V - \mathring{U} & \longrightarrow & X \end{matrix}\; ; \; \mathbf{Z}\pi\right)$$

and we get, upon composing with the boundary, a morphism h from K to

$$H_{p+1}(\partial U \to \mathrm{pt}; \mathbf{Z}\pi) = H_{p+1}(\mathrm{pt}, \partial U; \mathbf{Z}\pi) = B \oplus \hat{B}.$$

It is not difficult to see that the image under h of the basis of K can be represented by spheres immersed in ∂U with zero intersections and self-intersections. To prove that h is a $\mathscr{W}'$-subkernel, it suffices to show that the complex $\ldots \to 0 \to K \to B \oplus \hat{B} \to \hat{K} \to 0 \to \ldots$ lies in $\mathscr{W}'$; and this follows from the $\mathscr{W}'$-equivalences

$$(\ldots \to 0 \to K \to B \to \ldots) \to C_* \to \hat{C}_* \to (\ldots \to 0 \to \hat{B} \to \hat{K} \to 0 \to \ldots).$$

Then we get a $\mathscr{W}'$-subkernel h and an invariant $\omega(h) \in \Gamma_n(\mathbf{Z}\pi, \mathscr{W}')$. By construction, $\omega(h)$ is sent to $\sigma(f)$ by the isomorphism $\varepsilon : \Gamma_n(\mathbf{Z}\pi, \mathscr{W}') \to L^h_n(\Lambda)$. Hence $\omega(h)$ is zero. By proposition (3.15), there exist a standard $(-1)^p$-kernel $C \oplus \hat{C}$ endowed with its standard subkernel $g : C \to C \oplus \hat{C}$ and an automorphism φ on $B \oplus \hat{B} \oplus C \oplus \hat{C}$ leaving each element of $B \oplus \hat{C}$ fixed, such that the composite map

$$K \oplus C \xrightarrow{\varphi \circ (h \oplus g)} B \oplus \hat{B} \oplus C \oplus \hat{C} \longrightarrow B \oplus \hat{C}$$

is a Λ-isomorphism.

If we add trivial disjoint embeddings β_j, from S^p to V, corresponding to the basis of C, the new $\mathscr{W}'$-subkernel is $h \oplus g$. If we perform surgeries along the spheres α_i, the $\mathscr{W}'$-subkernel $h \oplus g$ is replaced by $T \circ (h \oplus g)$, where T exchanges the factors B and $\hat{B}$. The new embedded spheres are the duals $\bar{\alpha}_i$ of α_i and β_j.

Now we can choose a regular homotopy depending on φ (see [10]) to get new disjoint embeddings α'_i and β'_j and the $\mathscr{W}'$-subkernel $T \circ \varphi \circ (h \oplus g)$.

If we perform surgeries along the spheres α'_i and β'_j, we get the $\mathscr{W}'$-subkernel $T' \circ \varphi \circ (h \oplus g)$ where T' exchanges the factors C and $\hat{C}$.

So we obtain a new normal map $f' : V' \to X$ normally cobordant to f and a $\mathscr{W}'$-equivalence

$$(\ldots \to 0 \to K \oplus C \to B \oplus \hat{C} \to 0 \to \ldots) \to \Sigma^{-1} C_*(X, V'; \mathbf{Z}\pi).$$

Therefore f' is a $\mathscr{W}'$-equivalence. ■

(4.2) *Proof of theorem* (1.11) *in the case* $\mathscr{W} = \mathscr{W}'$

a) *The even dimensional case*

Suppose $n = 2p \geq 6$ and let $\sigma \in L^h_n(\Lambda)$. Since the morphism

$$\varepsilon : \Gamma_n(\mathbf{Z}\pi, \mathscr{W}') \to L^h_n(\Lambda)$$

is an isomorphism, σ is represented by a $\mathscr{W}'$-special $(-1)^p$-form (H, λ, μ) (3.13). Then we construct a normal map $f : W \to M \times I$ exactly as in ([10], p. 53). This normal map is an isomorphism over $M \times 0 \cup \partial M \times I$ and a $\mathscr{W}'$-equivalence over $M \times 1$ because λ is $\mathscr{W}'$-non singular. By construction, σ is the surgery invariant of f.

b) *The odd dimensional case*

Suppose $n = 2p + 1 \geq 7$ and let $\sigma \in L^h_n(\Lambda)$. We can represent σ by a trivial $(-1)^p$-kernel $B \oplus \hat{B}$ endowed with a $\mathscr{W}'$-subkernel $g : K \to B \oplus \hat{B}$ ((3.14)). After adding p-handles to $M \times I$ corresponding to the basis of B, we get a normal map $f_0 : W_0 \to M \times \left[0, \frac{1}{2}\right]$ which restricts to an isomorphism over $M \times 0 \cup \partial M \times \left[0, \frac{1}{2}\right]$. The inverse image M' of $M \times \frac{1}{2}$ is the connected sum of M and copies of $S^p \times S^p$ and the group $\pi_{p+1}\left(M \times \frac{1}{2}, M'\right)$ is the kernel $B \oplus \hat{B}$. Then we can perform surgeries along the image under g of the basis of K and we get a normal map

$$f_1 : W_1 \to M \times \left[\frac{1}{2}, 1\right].$$

These two normal maps induce a normal map $f : W \to M \times I$. It is easy to see that f restricts to an isomorphism over $M \times 0 \cup \partial M \times I$ and a $\mathscr{W}'$-equivalence over $M \times 1$. Moreover σ is the surgery obstruction $\sigma(f)$. ■

Actually this proof is almost identical with [10], p. 66.

Lemma **(4.3)**. — *Let* $\tau \in \widetilde{K}_1(\Lambda)$. *Then there exist two matrices* u *and* v *with entries in* A *such that* $u \otimes \Lambda$ *and* $v \otimes \Lambda$ *are invertible and* $\tau = \tau(u \otimes \Lambda) - \tau(v \otimes \Lambda)$.

This lemma will be proved in § 7.

Lemma **(4.4)**. — *Let* M *be a connected compact* (Top, PL *or* Diff)*-manifold,* $\dim M \geq 5$. *Let* φ *be an epimorphism from* $\pi_1 M$ *to* π *and* τ *be an element of* $\widetilde{K}_1(\Lambda)$. *Then, there exists a normal* *map* $f: \mathrm{V} \to \mathrm{M} \times \mathrm{I}$ *restricting to an isomorphism over* $\mathrm{M} \times 0 \cup \partial\mathrm{M} \times \mathrm{I}$ *and such that* f *is a* Λ*-homology equivalence with torsion* τ.

Proof. — By lemma (4.3), there exist two matrices

$$u: \mathbf{Z}\pi^p \to \mathbf{Z}\pi^q \quad \text{and} \quad v: \mathbf{Z}\pi^r \to \mathbf{Z}\pi^s$$

such that $u \otimes \Lambda$ and $v \otimes \Lambda$ are invertible and

$$\tau = \tau(u \otimes \Lambda) - \tau(v \otimes \Lambda).$$

After adding q 1-handles to $\mathrm{M} \times \mathrm{I}$, we get a normal map $f_1: \mathrm{V}_1 \to \mathrm{M} \times \mathrm{I}$ which is trivial on the handles. Now we add p 2-handles on V_1 along u and we get a normal map $f_2: \mathrm{V}_2 \to \mathrm{M} \times \mathrm{I}$ restricting to an isomorphism over $\mathrm{M} \times 0 \cup \partial\mathrm{M} \times \mathrm{I}$ and such that: $\tau(f_2) = \tau(u \otimes \Lambda) \in \widetilde{\mathrm{K}}_1(\Lambda)$.

Let M′ be the manifold $f_2^{-1}(\mathrm{M} \times 1)$. After adding s trivial 2-handles and r 3-handles along v, we construct a normal map $f_3': \mathrm{V}_3' \to \mathrm{M}' \times \mathrm{I}$ which restricts to an isomorphism over $\mathrm{M}' \times 0 \cup \partial\mathrm{M}' \times \mathrm{I}$, and f_3' is a Λ-homology equivalence with torsion $-\tau(v \otimes \Lambda)$.

Then after gluing f_2 and f_3' together, we get a normal map $f: \mathrm{V} \to \mathrm{M} \times \mathrm{I}$ which has the desired property. ■

(4.5) *Proof of theorem* (1.10) *in the general case*

Consider the Ranicki-Rothenberg exact sequence

$$\mathrm{L}^h_{n+1}(\Lambda) \xrightarrow{\partial} \mathrm{H}^n(\mathbf{Z}/2, \widetilde{\mathrm{K}}_1(\Lambda)/\alpha) \to \mathrm{L}^\alpha_n(\Lambda) \to \mathrm{L}^h_n(\Lambda).$$

Suppose that $\sigma(f)$ vanishes in $\mathrm{L}^\alpha_n(\Lambda)$. Then the surgery invariant of f is zero in $\mathrm{L}^h_n(\Lambda)$ and f is normally cobordant (relative the boundary) to a normal map $f_1: \mathrm{V}_1 \to \mathrm{X}$ which is a $\mathscr{W}'$-equivalence. Moreover f_1 is $\left[\frac{n}{2}\right]$-connected.

Let $\tau \in \widetilde{\mathrm{K}}_1(\Lambda)$ be the torsion of f_1. Since $\sigma(f)$ is zero, there exists an element $u \in \mathrm{L}^h_{n+1}(\Lambda)$ such that ∂u is represented by τ. But f_1 is 2-connected and $\pi_1\mathrm{V}_1 = \pi$. Then, by theorem (1.11) (proved in the case $\mathscr{W} = \mathscr{W}'$, $\mathrm{M} = \mathrm{V}_1$), there exists a normal map $g_1: \mathrm{W}_1 \to \mathrm{V}_1 \times \mathrm{I}$ restricting to an isomorphism over $\mathrm{V}_1 \times 0 \cup \partial\mathrm{V}_1 \times \mathrm{I}$ and such that $\sigma(g) = u$. This normal map induces a normal cobordism (relative the boundary) from f_1 to a normal map $f_2: \mathrm{V}_2 \to \mathrm{X}$ which is a $\mathscr{W}'$-equivalence. Moreover the torsion of f_2 is zero in $\mathrm{H}^n(\mathbf{Z}/2, \widetilde{\mathrm{K}}_1(\Lambda)/\alpha)$.

Then, there exists $\tau' \in \widetilde{\mathrm{K}}_1(\Lambda)$ such that: $\tau(f_2) \equiv \tau' + (-1)^n \bar{\tau}' \pmod{\alpha}$.

By lemma (4.4), there exists a normal map $g_2: \mathrm{W}_2 \to \mathrm{V}_2 \times \mathrm{I}$ restricting to an isomorphism over $\mathrm{V}_2 \times 0 \cup \partial\mathrm{V}_2 \times \mathrm{I}$ such that g_2 is a $\mathscr{W}'$-equivalence with torsion $-\tau'$. This normal map induces a normal cobordism from f_2 to $f_3: \mathrm{V}_3 \to \mathrm{X}$ and f_3 is a $\mathscr{W}'$-equivalence with torsion in $\alpha \subset \widetilde{\mathrm{K}}_1(\Lambda)$. Thus, theorem (1.10) is a trivial consequence of the following lemma (proved in § 7):

Lemma **(4.6)**. — *Any finite* A*-complex which is* Λ*-acyclic with torsion in* α *lies in* $\mathscr{W}$.

(4.7) *Proof of theorem* (1.11) *in the general case*

Consider again the Ranicki-Rothenberg exact sequence

$$H^n(\mathbf{Z}/2, \widetilde{K}_1(\Lambda)/\alpha) \to L_n^\alpha(\Lambda) \to L_n^h(\Lambda) \to H^{n-1}(\mathbf{Z}/2, \widetilde{K}_1(\Lambda)/\alpha).$$

Let σ be an element of $L_n^\alpha(\Lambda)$ and σ' be the image of σ in $L_n^h(\Lambda)$. By theorem (1.11) (proved in the case $\mathscr{W} = \mathscr{W}'$) there exists a normal map $f_1 : W_1 \to M \times I$ restricting to an isomorphism over $M \times 0 \cup \partial M \times I$ and such that the surgery obstruction of f_1 is σ' in $L_n^h(\Lambda)$. Let V_1 be the inverse image of $M \times 1$. Since σ' is sent to zero in $H^{n-1}(\mathbf{Z}/2, \widetilde{K}_1(\Lambda)/\alpha)$ the torsion of $f_1 : V_1 \to M$ is congruent to $\tau - (-1)^n \bar{\tau}$ (mod α) for some $\tau \in \widetilde{K}_1(\Lambda)$.

Then, by lemma (4.4), we can glue together f_1 and a normal map $f_1' : W_1' \to M \times I$ in order to construct a new normal map $f_2 : W_2 \to M \times I$ such that

(i) f_1 and f_2 have the same invariant in $L_n^h(\Lambda)$;
(ii) f_2 restricts over $M \times 1$ to a $\mathscr{W}'$-equivalence with torsion in α.

By construction, $\sigma(f_2) - \sigma$ is the image of an element of $H^n(\mathbf{Z}/2, \widetilde{K}_1(\Lambda)/\alpha)$ represented by $\tau' \in \widetilde{K}_1(\Lambda)$. By lemma (4.4), there exists a normal map

$$f_2' : W_2' \to f_2^{-1}(M \times 1) \times I$$

restricting to an isomorphism over $f_2^{-1}(M \times 1) \times 0 \cup \partial f_2^{-1}(M \times 1) \times I$ and such that f_2' is a $\mathscr{W}'$-equivalence with torsion $-\tau'$. Then, after gluing f_2 and f_2' together, we get a normal map $f : W \to M \times I$ with surgery obstruction σ. ■

5. Localization in the category of graded differential modules

Consider now the general case: A is a ring and $\mathscr{W}$ is an exact class in $\mathscr{C}(A)$. The $\mathscr{W}$-localization of A is (Λ, α).

Definition **(5.1)**. — A complex $C \in \mathscr{W}$ will be called *$\mathscr{W}$-splittable* if there exist, for any n, an n-dimensional complex $C' \in \mathscr{W}$ and an $(n-1)$-connected morphism from C' to C.

The class of $\mathscr{W}$-splittable complexes of $\mathscr{W}$ will be called $\mathscr{W}^s$.

Lemma **(5.2)**. — *The class $\mathscr{W}^s$ is exact.*

Proof. — The class $\mathscr{W}^s$ is clearly stable under simple homotopy equivalence and under any suspension.

Now let $0 \to C \to C' \to C'' \to 0$ be a s-exact sequence of finite A-complexes. Suppose that C and C' are $\mathscr{W}$-splittable.

Let n be an integer. There exists a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & \overline{C} & & \overline{C}' & & & & \end{array}$$

such that $\overline{C}$ (respectively $\overline{C}'$) is an $(n-1)$-dimensional (respectively n-dimensional) complex in $\mathscr{W}$ and the morphism $\overline{C} \to C$ (respectively $\overline{C}' \to C'$) is $(n-2)$-connected (respectively $(n-1)$-connected). The obstructions to factoring the morphism $\overline{C} \to C'$ through $\overline{C}'$ are in the groups $H^p(\overline{C}, H_p(C', \overline{C}'))$ which are all trivial. So we get a morphism $\overline{C} \to \overline{C}'$. It is easy to see that the mapping cone $\overline{C}''$ of $\overline{C} \to \overline{C}'$ is an n-dimensional complex in $\mathscr{W}$ and the induced morphism from $\overline{C}''$ to C'' is $(n-1)$-connected.

Then C'' is $\mathscr{W}$-splittable and, since $\mathscr{W}^s$ is stable under simple homotopy equivalence and suspension, it is easy to prove that $\mathscr{W}^s$ is exact. ■

Lemma **(5.3)**. — $\mathscr{W}^{ss} = \mathscr{W}^s$.

Proof. — The proof is by induction on the length of the complex. Clearly any complex in $\mathscr{W}^s$ of length two is $\mathscr{W}^s$-splittable. Suppose any complex in $\mathscr{W}^s$ of length $< p$ is $\mathscr{W}^s$-splittable, and let $C \in \mathscr{W}^s$ be a $\mathscr{W}$-splittable complex of length p. The complex C is n-dimensional and $(n-p)$-connected. Since C is $\mathscr{W}$-splittable, there exist an $(n-p+2)$-dimensional complex $C' \in \mathscr{W}$ and an $(n-p+1)$-connected morphism $C' \to C$.

The length of C' is 2 and C' lies in $\mathscr{W}^{ss}$. Then the mapping cone of $C' \to C$ is a complex in $\mathscr{W}^s$ of length $p-1$. By induction the mapping cone of $C' \to C$ lies in $\mathscr{W}^{ss}$ and $C \in \mathscr{W}^{ss}$. ■

We will work out a theory of localization in the category of graded differential modules. Unfortunately, the category $\mathscr{C}(A)$ is too small to do that and we must consider the category $\overline{\mathscr{C}}(A)$ of graded differential free A-modules bounded from below.

Notations **(5.4)**. — Denote by $\mathscr{W}_0$ the exact class of finite A-complexes C such that $C \oplus \Sigma C$ lies in $\mathscr{W}$ and by $\mathscr{W}_0^s$ the class $(\mathscr{W}_0)^s$. We use $\overline{\mathscr{W}}$ to denote the class of complexes $C \in \overline{\mathscr{C}}(A)$ such that any morphism from a finite A-complex to C factorizes through a complex in $\mathscr{W}_0^s$.

A morphism f in $\overline{\mathscr{C}}(A)$ is a $\overline{\mathscr{W}}$-equivalence if the mapping cone of f lies in $\overline{\mathscr{W}}$.

Definition **(5.5)**. — A complex $C \in \overline{\mathscr{C}}(A)$ will be called *local* if any morphism from a complex $C' \in \overline{\mathscr{W}}$ to C is null homotopic.

A morphism $f: C \to C'$ is a *localization* of C if f is a $\overline{\mathscr{W}}$-equivalence and C' is local.

Clearly, if C has a localization, this localization is unique up to homotopy.

Proposition (5.6). — *Any complex in* $\overline{\mathscr{C}}(A)$ *has a localization.*

Proof. — Let $C \in \overline{\mathscr{C}}(A)$. Suppose C is $(n-1)$-connected. Let $\mathscr{A}$ be the set of morphisms $K \to C$ such that K is a $(n-2)$-connected complex in $\mathscr{W}_0^s$. Let $\Phi(C)$ be the mapping cone of the morphism $\bigoplus_{\mathscr{A}} K \to C$.

Clearly $\Phi(C)$ is $(n-1)$-connected and we can carry on this process:

$$C \to \Phi(C) \to \Phi^2(C) \to \Phi^3(C) \to \dots$$

Denote by E(C) the limit of this system.

The complex $\Phi^{p+1}(C)/\Phi^p(C)$ is a direct sum of complexes in $\mathscr{W}_0^s$. Then, by induction, it is easy to show that $\Phi^p(C)/C$ lies in $\overline{\mathscr{W}}$. But, by construction, E(C) is $(n-1)$-connected and $E(C) \in \overline{\mathscr{C}}(A)$. Moreover $E(C)/C$ lies in $\overline{\mathscr{W}}$ and $C \to E(C)$ is a $\overline{\mathscr{W}}$-equivalence.

Now, let $\mathscr{C}$ be the class of complexes $C' \in \overline{\mathscr{C}}(A)$ such that any morphism from C' to E(C) is null homotopic. The class $\mathscr{C}$ is stable under homotopy equivalence and extension. The last problem is to prove that $\mathscr{C}$ contains $\overline{\mathscr{W}}$.

Let $K \in \mathscr{W}_0^s$. Since any complexe in $\mathscr{W}_0^s$ is $\mathscr{W}_0^s$-splittable ((5.3)), there exists a homotopy s-exact sequence $0 \to K' \to K \to K'' \to 0$ such that K' is a $n-1$-dimensional complex in $\mathscr{W}_0^s$ and K'' an $(n-2)$-connected complex in $\mathscr{W}_0^s$. Clearly $K' \in \mathscr{C}$. Let f be a morphism from K'' to E(C). Since K'' is finitely generated, the image of f is contained in some $\Phi^p(C)$ and f is homotopic to zero in $\Phi^{p+1}(C)$. Hence $K'' \in \mathscr{C}$ and $K \in \mathscr{C}$ too. Then $\mathscr{C}$ contains the class $\mathscr{W}_0^s$.

If $K \in \overline{\mathscr{C}}(A)$, denote by $\mathscr{H}^i(K)$ the group $[\Sigma^{-i}K, E(C)]$ of homotopy classes of morphisms from $\Sigma^{-i}K$ to E(C). The group $\mathscr{H}^i(K)$ vanishes for any $K \in \mathscr{W}_0^s$ and any $i \in \mathbf{Z}$, and we must prove that $\mathscr{H}^0(K)$ is zero for any $K \in \overline{\mathscr{W}}$.

If $K \in \overline{\mathscr{W}}$, K has the homotopy type of the limit of a directed system K_i, $K_i \in \mathscr{W}_0^s$, and we have a spectral sequence with the following E_2 term:

$$E_2^{pq} = \varprojlim{}^p \mathscr{H}^q(K_i).$$

The E_2 term is trivial and the spectral sequence converges to $\mathscr{H}^*(K)$. Then $\mathscr{H}^0(K)$ vanishes and $C \to E(C)$ is a localization of C. ∎

The localization plays an important role in view of the following propositions:

Proposition (5.7). — *Let* C *and* C' *be two complexes in* $\mathscr{C}(A)$, *with* $\dim C = n$. *Let* $C' \xrightarrow{\varepsilon} E(C')$ *be a localization of* C'. *Then, for any morphism* $f: C \to E(C')$, *there exist an n-dimensional complex* $\overline{C} \in \mathscr{C}(A)$ *and a homotopy commutative diagram*

$$\begin{array}{ccc} \overline{C} & \longrightarrow & C' \\ \downarrow & & \downarrow{\scriptstyle\varepsilon} \\ C & \xrightarrow{f} & E(C') \end{array}$$

such that $\overline{C} \to C$ *is a* $\mathscr{W}_0^s$*-equivalence.*

Proposition **(5.8)**. — *Let* C *and* C′ *be two complexes in* $\mathscr{C}(A)$ *with* $\dim C = n$. *Let* $C' \xrightarrow{\varepsilon} E(C')$ *be a localization of* C′. *Let* $f: C \to C'$ *be a map such that* $\varepsilon \circ f$ *is null homotopic. Then, there exists a* $\mathscr{W}_0^s$*-equivalence* $\overline{C} \to C$ *such that* $\overline{C} \in \mathscr{C}(A)$ *is n-dimensional and the composite map* $\overline{C} \to C \xrightarrow{f} C'$ *is null homotopic.*

Proof of (5.7). — Suppose ε is monic with free cokernel. We have an exact sequence

$$0 \to C' \to E(C') \to K' \to 0, \qquad K' \in \mathscr{W}.$$

Let us construct the homotopy commutative diagram

$$\begin{array}{ccccc} \overline{C} & & \longrightarrow & & C' \\ \downarrow & & & & \downarrow \\ C & & \longrightarrow & & E(C') \\ \downarrow & \searrow & & & \downarrow \\ K & \longrightarrow & L & \longrightarrow & K' \end{array}$$

in the following way: Since C is finitely generated, the map $C \to K'$ factorizes through a complex $L \in \mathscr{W}_0^s$ and by (5.3), there exist an $(n+1)$-dimensional complex $K \in \mathscr{W}_{0s}$ and an n-connected map $K \to L$. Then there is no obstruction to factorize the map $C \to L$ through K.

Let $\overline{C}$ be the homotopy kernel of $C \to K$. It is easy to check that $\overline{C}$ is n-dimensional and that the map $\overline{C} \to E(C')$ factorizes through C′. ■

Proof of (5.8). — Suppose ε is epic with kernel $K' \in \mathscr{W}$. Since the composite map $C \xrightarrow{f} C' \xrightarrow{\varepsilon} E(C')$ is null homotopic, f is homotopic to a map $f': C \to K'$. Then f' factorizes through a complex $L \in \mathscr{W}_0^s$. By (5.3), there exist an $(n+1)$-dimensional complex $K \in \mathscr{W}_0^s$ and an n-connected map $K \to L$. As before the map $C \to L$ retracts in K and the homotopy kernel of $C \to K$ has the desired properties. ■

6. The ring Λ

In this section, we will compute the homology groups of the localization of a complex $C \in \overline{\mathscr{C}}(A)$ in terms of the ring Λ defined in (1.8).

Let M be a (right) A-module. This module will be said local if any $q \times p$ matrix in Σ induces an isomorphism $\mathrm{Hom}(A^q, M) \to \mathrm{Hom}(A^p, M)$.

Lemma **(6.1)**. — *A module* M *is local if and only if* $H^n(C, M)$ *vanishes for any* $n \in \mathbf{Z}$ *and any* $C \in \overline{\mathscr{W}}$.

Proof. — Suppose that $H^n(C, M)$ vanishes for any $n \in \mathbf{Z}$ and any $C \in \overline{\mathscr{W}}$. If u is a matrix in Σ, denote by C the 1-dimensional complex

$$\ldots \to 0 \to A^p \xrightarrow{u} A^q \to 0 \to \ldots$$

Then $C \oplus \Sigma C$ lies in $\mathscr{W}$ (see (1.7)) and C is a complex of $\mathscr{W}_0^s \subset \overline{\mathscr{W}}$. Hence $H^*(C, M)$ vanishes and M is local.

Conversely, suppose M is local and denote by $\mathscr{C}$ the class of complexes $C \in \overline{\mathscr{C}}(A)$ such that $H^*(C, M) = 0$.

If C is a complex of length two in $\mathscr{W}_0^s$, C lies in $\mathscr{C}$ by definition.

If C is a complex in $\mathscr{W}_0^s$ of length $p > 2$, there exists a homotopy s-exact sequence

$$0 \to C' \to C \to C'' \to 0$$

such that C′ and C″ are complexes in $\mathscr{W}_0^s$ of length $< p$.

By induction, C is in $\mathscr{C}$ and $\mathscr{C}$ contains the class $\mathscr{W}_0^s$.

If $C \in \overline{\mathscr{W}}$, C is the limit of a directed system $C_i \in \mathscr{W}_0^s$ and we have a spectral sequence with E_2 term $E_2^{pq} = \varprojlim^p H^q(C_i, M)$. The E_2 term is zero and the spectral sequence converges to $H^*(C, M)$. Hence $H^*(C, M)$ vanishes and the lemma is proved.

Corollary **(6.2)**. — *A complex* $C \in \overline{\mathscr{C}}(A)$ *is local if and only if* $H_n(C)$ *is local for any* $n \in \mathbf{Z}$.

Proof. — If K is a complex, denote by $\mathscr{H}^i(K)$ the group of homotopy classes of maps $\Sigma^{-i}K \to C$. We have a spectral sequence with E_2 term

$$E_2^{pq} = H^p(K, H_{-q}(C))$$

and this spectral sequence usually converges to $\mathscr{H}^*(K)$.

Suppose C is local and let $K \in \mathscr{W}_0^s$ be a complex of length 2 defined by a matrix $u \in \Sigma$. Then the above spectral sequence collapses to exact sequences

$$0 \to H^n(K, H_{-i}(C)) \to \mathscr{H}^{n+i}(K) \to H^{n-1}(K, H_{-i-1}(C)) \to 0 \quad (n = \dim K).$$

Then all the groups $H^*(K, H_i(C))$ vanish and $H_i(C)$ is local for any $i \in \mathbf{Z}$.

Conversely suppose $H_*(C)$ is local. Then for any $K \in \overline{\mathscr{W}}$, the E_2 term of the above spectral sequence vanishes and the spectral sequence converges to $\mathscr{H}^*(K)$. Hence this last group vanishes and C is local. ■

Lemma **(6.3)**. — *Localization respects exact sequences.*

Proof. — Let $0 \to C \to C' \to C'' \to 0$ be a short exact sequence in $\overline{\mathscr{C}}(A)$. Take localizations $C \to E(C)$ and $C' \to E(C')$ of C and C′. We get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & E(C) & \longrightarrow & E(C) & & & & \end{array}$$

Let $E(C'')$ be the mapping cone of $E(C) \to E(C')$. We have a homotopy commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E(C) & \longrightarrow & E(C') & \longrightarrow & E(C'') & \longrightarrow & 0 \end{array}$$

Clearly $E(C'')$ is local and the map $C'' \to E(C'')$ is a $\overline{\mathscr{W}}$-equivalence. Then $C'' \to E(C'')$ is a localization of C'' and the result follows.

Lemma **(6.4)**. — *Localization respects direct sums.*

Proof. — Let $C_i \in \overline{\mathscr{C}}(A)$ be a class of complexes. Suppose that C_i is $(n-1)$-connected for any i, and take localizations $C_i \to E(C_i)$.

Clearly the mapping cone of $\bigoplus_i C_i \to \bigoplus_i E(C_i)$ lies in $\overline{\mathscr{W}}$ and, by (6.2), the sum $\bigoplus_i E(C_i)$ is local. Then the map $\bigoplus_i C_i \to \bigoplus_i E(C_i)$ is a localization of $\bigoplus_i C_i$. ■

Now if C is a complex in $\overline{\mathscr{C}}(A)$, denote by $\Phi_n(C)$ the group $H_n(E(C))$ where $C \to E(C)$ is a localization of C.

If M is a (right) A-module, we will also denote by $\Phi_n(M)$ the group $\Phi_n(C)$ where C is a free resolution of M. The Φ_n's are functors and we have a natural transformation $\eta : M \to \Phi_0(M)$.

Clearly, if M is local, a resolution of M is local ((6.2)). So η is bijective and $\Phi_i(M)$ vanishes for $i \neq 0$.

Lemma **(6.5)**. — *Let* M *be an* A*-module. Then, there is a natural homomorphism*

$$\varepsilon' : M \otimes_{\mathbf{Z}} \Phi_0(A) \to \Phi_0(M),$$

such that the following diagram commutes:

$$\begin{array}{ccc} M \otimes_{\mathbf{Z}} A & \longrightarrow & M \\ \big\downarrow{\scriptstyle 1 \otimes \eta} & & \big\downarrow{\scriptstyle \eta} \\ M \otimes_{\mathbf{Z}} \Phi_0(A) & \xrightarrow{\varepsilon'} & \Phi_0(M) \end{array}$$

Proof. — Let $m \in M$. Denote by $\varphi : A \to M$ the homomorphism $a \mapsto ma$. By setting $\varepsilon'(m, x) = \Phi_0(\varphi)(x)$, for any $x \in \Phi_0(A)$, we get a map $\varepsilon' : M \times \Phi_0(A) \to \Phi_0(M)$. Clearly, $\varepsilon'(m, x)$ is $\mathbf{Z}$-linear on x and, since Φ_0 respects direct sums, it is easy to see that $\varepsilon'(m, x)$ is $\mathbf{Z}$-linear on m. ■

Lemma **(6.6)**. — *The module* $\Phi_0(A)$ *is a ring and* ε' *induces a homomorphism*

$$\varepsilon : M \otimes_A \Phi_0(A) \to \Phi_0(M).$$

Proof. — Let $m \in M$ and $x, y \in \Phi(A)$. Denote by $\varphi : A \to M$ the map $a \mapsto ma$ and by $\psi : A \to \Phi_0(A)$ the map $a \to xa$.

We have a commutative diagram

$$\begin{array}{ccccc} & & A & \xrightarrow{\varphi} & M \\ & & \downarrow & & \downarrow \\ A & \xrightarrow{\psi} & \Phi_0(A) & \xrightarrow{\Phi_0(\varphi)} & \Phi_0(M) \\ \downarrow & & \sim \downarrow \eta & & \sim \downarrow \eta \\ \Phi_0(A) & \xrightarrow{\Phi_0(\psi)} & \Phi_0^2(A) & \xrightarrow{\Phi_0^2(\varphi)} & \Phi_0^2(M) \end{array}$$

and the following formulas:

$$\Phi_0^2(\varphi) \circ \Phi_0(\psi)(y) = \Phi_0^2(\varphi)(\varepsilon'(x,y)) = \eta\varepsilon'(m, \eta^{-1}\varepsilon'(x,y))$$

$$\Phi_0[\Phi_0(\varphi) \circ \psi](y) = \varepsilon'(\varepsilon'(m, x), y)$$

whence $$\eta\varepsilon'(m, \eta^{-1}\varepsilon'(x,y)) = \varepsilon'(\varepsilon'(m, x), y).$$

Then the map $\eta^{-1}\varepsilon'$ from $\Phi_0(A) \otimes_{\mathbf{Z}} \Phi_0(A)$ to $\Phi_0(A)$ induces a ring structure on $\Phi_0(A)$ and η is a ring homomorphism from A to $\Phi_0(A)$. Moreover ε' induces a homomorphism $\varepsilon : M \otimes_A \Phi_0(A) \to \Phi_0(M)$. ■

Lemma **(6.7)**. — *The ring homomorphism* $A \to \Phi_0(A)$ *is isomorphic to the homomorphism* $A \to \Lambda$.

Proof. — Let $A \to B$ be a ring homomorphism. The A-module B is local if and only if any $q \times p$ matrix $u \in \Sigma$ induces an isomorphism $u^* : \mathrm{Hom}(A^q, B) \to \mathrm{Hom}(A^p, B)$. But the matrix of u^* is the transpose of $u \otimes B$. Then, B is local if and only if, for any $u \in \Sigma$, $u \otimes B$ is invertible.

Hence, for any matrix $u \in \Sigma$, $u \otimes \Phi_0(A)$ is invertible and we will prove that $\Phi_0(A)$ is universal with respect to this property.

Let $A \to B$ be a ring homomorphism such that $u \otimes B$ is invertible for any $u \in \Sigma$. Let us choose free resolutions A_* and B_* of A and B and a localization $A_* \to E(A_*)$ of A_*. Since B is local, there exists an extension $E(A_*) \to B_*$ unique up to homotopy. Then there exists a unique extension $\Phi_0(A) \to B$ of $A \to B$.

Consider the following diagram:

$$\begin{array}{ccc} A & \longrightarrow & \Phi_0(A) \\ \downarrow & & \downarrow \\ B & \xrightarrow{\sim} & \Phi_0(B) \end{array}$$

All the morphisms of this diagram are ring homomorphisms and $B \xrightarrow{\sim} \Phi_0(B)$ is an isomorphism. Then the extension $\Phi_0(A) \to B$ is a ring homomorphism. So $A \to \Phi_0(A)$ satisfies the universal property of Λ and $A \to \Phi_0(A)$ is isomorphic to $A \to \Lambda$.

Lemma **(6.8).** — *For any module* M, *the morphism* $\varepsilon : M \otimes \Lambda \to \Phi_0(M)$ *is an isomorphism.*

Proof. — By lemma (6.4), the functor Φ_0 respects direct sums and ε is an isomorphism if M is free. Moreover, by lemma (6.5), Φ_0 is right exact and ε is an isomorphism for any M. ■

Corollary **(6.9).** — *If* M *is local, the canonical map* $M \to M \otimes \Lambda$ *is an isomorphism.*

Lemma **(6.10).** — *If* M *is local,* $\mathrm{Tor}_1(M, \Lambda)$ *is trivial.*

Proof. — Choose a free module L and an exact sequence

$$0 \to N \to L \to M \to 0.$$

By lemma (6.4), we have an exact sequence

$$\Phi_1(M) \to \Phi_0(N) \to \Phi_0(L) \to \Phi_0(M) \to 0.$$

If M is local, $\Phi_1(M)$ is zero and $\Phi_0(N) \to \Phi_0(L)$ is monic. But this map is isomorphic to the map $N \otimes \Lambda \to L \otimes \Lambda$ and its kernel is $\mathrm{Tor}_1(M, \Lambda)$. ■

Corollary **(6.11).** — *Let* $C \in \overline{\mathscr{C}}(A)$ *be an* $(n-1)$*-connected local complex. Then the canonical map* $H_i(C) \to H_i(C \otimes \Lambda)$ *is an isomorphism for* $i \leq n$ *and an epimorphism for* $i = n+1$.

Proof. — We have a spectral sequence with E^2 term $E^2_{pq} = \mathrm{Tor}_p(H_q(C), \Lambda)$ which converges to $H_*(C \otimes \Lambda)$. Since C is local, $H_*(C)$ is local and, by (6.9) and (6.10), we have

$$E^2_{0q} = \mathrm{Tor}_0(H_q(C), \Lambda) = H_q(C),$$

$$E^2_{1q} = \mathrm{Tor}_1(H_q(C), \Lambda) = 0.$$

The result follows.

Theorem **(6.12)**. — *Let* C *and* C′ *be two finite* A*-complexes and suppose that* $C' \otimes \Lambda$ *is* $(n-1)$*-connected. Then we have the following properties*:

(i) *If* $H^i(C, \Lambda)$ *vanishes for* $i > n+1$ *and* f *is a morphism from* $C \otimes \Lambda$ *to* $C' \otimes \Lambda$, *there exist a* $\mathscr{W}_0^s$*-equivalence* $\varepsilon : \overline{C} \to C$ *with* $\dim \overline{C} = \dim C$ *and a morphism* $g : \overline{C} \to C'$ *such that* $g \otimes \Lambda$ *is homotopic to* $f \circ (\varepsilon \otimes \Lambda)$.

(ii) *If* $H^i(C, \Lambda)$ *vanishes for* $i > n$ *and* f *is a morphism from* C *to* C′ *such that* $f \otimes \Lambda$ *is null homotopic, there exists a* $\mathscr{W}_0^s$*-equivalence* $\varepsilon : \overline{C} \to C$, *with* $\dim \overline{C} = \dim C$ *such that* $f \circ \varepsilon$ *is null homotopic.*

Proof. — Let $C' \to E(C')$ be a localization of C′ and consider the following diagram:

$$\begin{array}{ccc} C' & \longrightarrow & C' \otimes \Lambda \\ \downarrow & & \downarrow \\ E(C') & \longrightarrow & E(C') \otimes \Lambda \end{array}$$

If f is a morphism from $C \otimes \Lambda$ to $C' \otimes \Lambda$, f is defined by an A-homomorphism $f' : C \to C' \otimes \Lambda$.

The obstructions to lift the composite map $f'' : C \to C' \otimes \Lambda \to E(C') \otimes \Lambda$ through $E(C')$ lie in the groups $H^p(C, H_p(E(C') \otimes \Lambda, E(C')))$. Let H_p be the module $H_p(E(C') \otimes \Lambda, E(C'))$. Since $E(C')$ is local, H_p is a Λ-module and is trivial for $p \leq n+1$, by (6.11). But $H^i(C, \Lambda)$ vanishes for $i > n+1$ and the localization $E(\hat{C})$ of $\hat{C}$ is $(-n-2)$-connected. Then we have, for $p > n+1$,

$$H^p(C, H_p) = H_{-p}(\hat{C}, H_p) = H_{-p}(E(\hat{C}), H_p) = 0.$$

Then f'' lifts through $E(C')$ and, by (5.7), there exist a complex $\overline{C} \in \mathscr{C}(A)$ with $\dim \overline{C} = \dim C$, a $\mathscr{W}_0^s$-equivalence $\varepsilon : \overline{C} \to C$ and a morphism $g : \overline{C} \to C'$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} \overline{C} & \xrightarrow{g} & C' \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow \\ C & \xrightarrow{f''} & E(C') \otimes \Lambda \end{array}$$

On the other hand, any complex in $\mathscr{W}_0^s$ of length two is Λ-acyclic and, by induction, any complex in $\mathscr{W}_0^s$ is Λ-acyclic. This implies that any complex in $\overline{\mathscr{W}}$ is Λ-acyclic and $C' \otimes \Lambda \to E(C') \otimes \Lambda$ is a homotopy equivalence.

Then the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \overline{C} & \xrightarrow{g} & C' \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow \\ C & \xrightarrow{f'} & C' \otimes \Lambda \end{array}$$

and part (i) of the theorem is proved.

Suppose now f is a morphism from C to C′ with $\dim C = n$. If $f \otimes \Lambda$ is null homotopic, the composite map $C \to C' \to E(C') \otimes \Lambda$ is null homotopic and, by obstruction, the map $C \to E(C')$ is null homotopic. Then we may apply (5.8) and the theorem is proved.

7. The structure of $\mathscr{W}$

Lemma (7.1). — *The class* $\mathscr{W}_0^s$ *is the class* $\mathscr{W}'$ *of* Λ*-acyclic complexes in* $\mathscr{C}(A)$.

Proof. — If C is a complex in $\mathscr{W}_0$ of length two, it is Λ-acyclic by definition of Λ. Then, by induction, any complex in $\mathscr{W}_0^s$ is Λ-acyclic.

Conversely, let $C \in \mathscr{C}(A)$ be a Λ-acyclic complex and $C \to E(C)$ be a localization of C. Since C is Λ-acyclic, E(C) is Λ-acyclic too. Suppose E(C) is not acyclic and let H_n be the first non trivial homology group of E(C). The module H_n is local and

$$H_n \simeq H_n \otimes \Lambda \simeq H_n(E(C) \otimes \Lambda) = 0.$$

Hence E(C) is acyclic and $C \in \overline{\mathscr{W}}$. Since C is finite, the identity $C \to C$ factorizes through a complex $K \in \mathscr{W}_0^s$ and we get a split exact sequence

$$0 \to C' \to K \to C \to 0.$$

This implies that $C \oplus C'$ has the simple homotopy type of K and $C \oplus C'$ lies in $\mathscr{W}_0^s$.

On the other hand, ΣK has the simple homotopy type of the mapping cone of the zero map $C' \to \Sigma C$ and $C' \to \Sigma C$ is a $\mathscr{W}_0^s$-equivalence. Then $C \oplus \Sigma C$ lies in $\mathscr{W}_0^s$.

Now we will prove that C is in $\mathscr{W}_0^s$ by induction on the length of C.

If the length of C is two, $C \oplus \Sigma C$ is contained in $\mathscr{W}_0$ and $C \oplus \Sigma C \oplus \Sigma C \oplus \Sigma^2 C$ lies in $\mathscr{W}$. But $\Sigma(C \oplus \Sigma C \oplus \Sigma C \oplus \Sigma^2 C)$ is the mapping cone of the zero map $\Sigma C \oplus \Sigma C \oplus \Sigma^2 C \to \Sigma C$ which is a $\mathscr{W}$-equivalence. Then $C \oplus \Sigma C$ lies in $\mathscr{W}$ and C lies in $\mathscr{W}_0$. Since the length of C is two, C lies in $\mathscr{W}_0^s$.

If the length of C is $p > 2$, C is n-dimensional and $(n-p)$-connected. Since $C \oplus \Sigma C$ is $\mathscr{W}_0^s$-splittable, there exist an $(n-p+2)$-dimensional complex $K \in \mathscr{W}_0^s$ and an $(n-p+1)$-connected morphism $f \oplus g$ from K to $C \oplus \Sigma C$.

The morphism $f \oplus 0$ is clearly $(n-p+1)$-connected. Let M be the mapping cone of f. The complex $M \oplus \Sigma M$ is the mapping cone of $f \oplus \Sigma f$ and lies in $\mathscr{W}_0^s$. But the length of M is $p-1$. By induction, M lies in $\mathscr{W}_0^s$ and C lies in $\mathscr{W}_0^s$ too. ■

(7.2) *Proof of the splitting lemma* (3.5)

Let C be a complex in $\mathscr{W}'$ and let n be an integer. Since $\mathscr{W}' = \mathscr{W}_0^s$, C is $\mathscr{W}'$-splittable and there exist an n-dimensional complex $C' \in \mathscr{W}'$ and an $(n-1)$-connected morphism $C' \to C$.

Up to simple homotopy type, we may suppose that the map $C'_i \to C_i$ is bijective for $i < n-1$ and is epic with free kernel L'_n for $i = n-1$. Then we have the following complex in $\mathscr{W}'$:

$$\ldots \to C_{n+2} \to C_{n+1} \oplus C'_n \to C_n \oplus L'_n \to 0 \to \ldots$$

Now by setting

$$L = (\ldots \to 0 \to C'_n \to 0 \to \ldots)$$
$$L' = (\ldots \to 0 \to L'_n \to 0 \to \ldots),$$

we get a $\mathscr{W}'$-equivalence

$$L \to L' \oplus (\ldots \to C_{n+1} \to C_n \to 0 \to \ldots). \blacksquare$$

Lemma **(7.3).** — *For any complex* $C \in \mathscr{W}'$, *the complex* $C \oplus \Sigma C$ *lies in* $\mathscr{W}$.

Proof. — If C is Λ-acyclic, C lies in $\mathscr{W}_0^s \subset \mathscr{W}_0$ and then $C \oplus \Sigma C \in \mathscr{W}$. ■

(7.4) We use $K(\mathscr{W})$ to denote the class of complexes $C \in \mathscr{W}'$ fulfilling the following relation:

$$C \sim C' \Leftrightarrow C \oplus \Sigma C' \in \mathscr{W}.$$

By (7.3), this relation is an equivalence relation and $K(\mathscr{W})$ is a well defined set. Moreover the direct sum of complexes induces an abelian group structure on $K(\mathscr{W})$.

If C is a Λ-acyclic complex in $\mathscr{C}(A)$, the class of C in $K(\mathscr{W})$ will be denoted by $\theta(C)$.

Lemma **(7.5).** — *Let* $0 \to C \to C' \to C'' \to 0$ *be an s-exact sequence of* Λ-*acyclic complexes in* $\mathscr{C}(A)$. *Then* $\theta(C') = \theta(C) + \theta(C'')$.

Proof. — We have an s-exact sequence

$$0 \to C \oplus \Sigma C \to C' \oplus \Sigma C \oplus \Sigma C'' \to C'' \otimes \Sigma C'' \to 0$$

and, by lemma (7.3), $C' \oplus \Sigma C \oplus \Sigma C''$ is in $\mathscr{W}$. That proves the lemma.

Now if f is a Λ-homology equivalence between two finite A-complexes, we will define $\theta(f)$ as the class of the mapping cone of f in $K(\mathscr{W})$.

Lemma **(7.6).** — *Let* $f: C \to C'$ *and* $g: C' \to C''$ *be two* Λ-*homology equivalences between finite* A-*complexes. Then* $\theta(g \circ f) = \theta(f) + \theta(g)$.

Proof. — We have a short s-exact sequence between the mapping cones of f, g, $g\circ f\oplus 1_{C'}$. Then the result follows from (7.5).

(7.7) Let $f:\Lambda^p\to\Lambda^q$ be an isomorphism. Denote also by A the 0-dimensional complex $\ldots\to 0\to A\to 0\to\ldots$ Then f is a morphism from $A^p\otimes\Lambda$ to $A^q\otimes\Lambda$, and, by (6.12), there exist a $\mathcal{W}'$-equivalence $\varepsilon:\overline{C}\to A^p$ and a map $g:\overline{C}\to A^q$ such that $f\circ(\varepsilon\otimes\Lambda)$ is homotopic to $g\otimes\Lambda$.

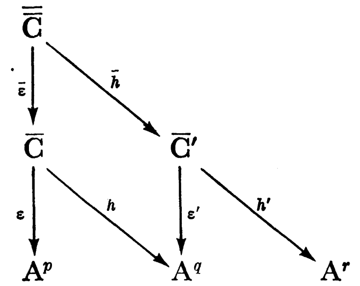
Since f is an isomorphism, g is a $\mathcal{W}'$-equivalence.

Then we define $\theta(f)$ as $\theta(g)-\theta(\varepsilon)$. By (6.12), it is easy to show that $\theta(f)$ does not depend on the choices.

Lemma **(7.8)**. — *Let* $f:\Lambda^p\to\Lambda^q$ *and* $g:\Lambda^q\to\Lambda^r$ *be two isomorphisms. Then we have*

$$\theta(g\circ f)=\theta(f)+\theta(g).$$

Proof. — By theorem (6.12), there exists a homotopy commutative diagram in $\mathscr{C}(A)$



such that the morphisms are Λ-homology equivalences and $h\otimes\Lambda$ and $h'\otimes\Lambda$ are homotopic to $f\circ(\varepsilon\otimes\Lambda)$ and $g\circ(\varepsilon'\otimes\Lambda)$. Then we have

$$\theta(g\circ f)=\theta(h'\circ\bar{h})-\theta(\varepsilon\circ\bar{\varepsilon})=\theta(h')+\theta(\bar{h})-\theta(\varepsilon)-\theta(\bar{\varepsilon})$$

whence $$\theta(g\circ f)=\theta(h')-\theta(\varepsilon)+\theta(h)-\theta(\varepsilon')=\theta(f)+\theta(g).\quad\blacksquare$$

Theorem **(7.9)**. — *The torsion homomorphism* $\varepsilon:K(\mathcal{W})\to\widetilde{K}_1(\Lambda)/\alpha$ *is an isomorphism.*

Proof. — If $x\in\widetilde{K}_1(\Lambda)/\alpha$ is represented by an isomorphism $f:\Lambda^p\to\Lambda^q$, we have

$$\varepsilon(\theta(f))\equiv\tau(f)\bmod\alpha\Rightarrow x=\varepsilon(\theta(f))$$

and ε is surjective.

Now let θ be an element of Ker ε, represented by a complex $C\in\mathcal{W}'$. Since $\varepsilon(\theta)$ vanishes, $\tau(C\otimes\Lambda)$ is in α and $\tau(C\otimes\Lambda)$ is the torsion of a complex $C'\otimes\Lambda$ where C' is a Λ-acyclic complex in $\mathcal{W}$. Then θ is represented by $C\oplus\Sigma C'$ and the torsion of $(C\oplus\Sigma C')\otimes\Lambda$ vanishes. Since $\mathcal{W}'$ is splittable, we can "split" $C\oplus\Sigma C'$ into complexes $C_i\in\mathcal{W}'$ of length 2. And we have

$$\theta=\Sigma\theta(C_i)\quad\text{and}\quad\Sigma\tau(C_i\otimes\Lambda)=0.$$

On the other hand, the suspension Σ^2 does not change the invariants θ and τ. So we may as well suppose that the complexes C_i are 1 or 2-dimensional.

Then there exist two 1-dimensional complexes in $\mathscr{W}'$

$$X = (\ldots \to 0 \to A^p \xrightarrow{f} A^q \to 0 \to \ldots)$$

$$Y = (\ldots \to 0 \to A^{p'} \xrightarrow{g} A^{q'} \to 0 \to \ldots)$$

such that $\quad \theta = \theta(X) - \theta(Y) \quad$ and $\quad \tau(X \otimes \Lambda) = \tau(Y \otimes \Lambda)$.

But the image of $\tau(X \otimes \Lambda) = \tau(f \otimes \Lambda)$ under the boundary $\widetilde{K}_1(\Lambda) \xrightarrow{\partial} K_0(\mathbf{Z})$ is $q - p$ [9]. Then, after stabilization on X and Y, we may suppose

$$p = p' \quad \text{and} \quad q = q'.$$

Let $\varphi \in GL_q(\Lambda)$ be the map for $(f \otimes \Lambda) \circ (g \otimes \Lambda)^{-1}$. Since $\tau(f \otimes \Lambda) - \tau(g \otimes \Lambda)$ is zero, the class of φ in $K_1(\Lambda)$ is in the image of $K_1(\mathbf{Z}) \to K_1(\Lambda)$. Then, after a permutation on the basis of A^q (in X) and after stabilization on X and Y, we may suppose that φ lies in the commutator subgroup of $GL_q(\Lambda)$:

$$\varphi = \prod_i [\varphi_i, \psi_i].$$

And we have

$$\theta = \theta(X) - \theta(Y) = \theta(f) - \theta(g) = \theta(f \otimes \Lambda) - \theta(g \otimes \Lambda) = \theta(\varphi)$$

whence $\quad \theta = \Sigma(\theta(\varphi_i) + \theta(\psi_i) - \theta(\varphi_i) - \theta(\psi_i)) = 0.$

This completes the proof.

Corollary **(7.10)**. — *The class of Λ-acyclic complexes in $\mathscr{W}$ is the class of Λ-acyclic complexes* C *such that the torsion of* $C \otimes \Lambda$ *is in* α.

Now we prove lemmas (4.3) and (4.6).

Lemma (4.6) is actually the corollary (7.10).

Let $\tau \in \widetilde{K}_1(\Lambda)$. By theorem (7.9), there exists a complex $C \in \mathscr{W}'$ such that τ is the torsion of $C \otimes \Lambda$. Since C is splittable ((7.1)), we can split C into Λ-acyclic complexes C_i of length two and we have $\tau = \Sigma\tau(C_i \otimes \Lambda)$. If C_i is $(n_i + 1)$-dimensional and the differential of C_i is u_i, we have:

$$\tau = \Sigma(-1)^{n_i}\tau(u_i \otimes \Lambda)$$

and lemma (4.3) follows.

8. The isomorphism theorem

Suppose now that A is a ring with involution and $\mathscr{W}$ is an exact symmetric class in $\mathscr{C}(A)$. The $\mathscr{W}$-localization of A is (Λ, α) and $A \to \Lambda$ is a morphism of rings with involution.

The class of Λ-acyclic complexes in $\mathscr{C}(A)$ is denoted by $\mathscr{W}'$ and the class of acyclic complexes in $\mathscr{C}(\Lambda)$ is denoted by $\mathscr{W}_\Lambda$.

We have a canonical map

$$\varepsilon : \Gamma_n(A, \mathscr{W}') \to \Gamma_n(\Lambda, \mathscr{W}_\Lambda) \simeq L_n^h(\Lambda).$$

In this section, we will prove that ε is an isomorphism.

Lemma **(8.1)**. — *Let* C (*respectively* Σ) *be a* p-*dimensional and* $(p-2)$-*connected complex in* $\mathscr{C}(A)$ (*respectively* $\mathscr{C}(\Lambda)$) *and* $f:\Sigma \to C\otimes\Lambda$ *be a map. Then there exist a* p-*dimensional complex* $\Sigma' \in \mathscr{C}(A)$, *a homotopy equivalence* $\varepsilon : \Sigma'\otimes\Lambda \to \Sigma$ *and a map* $g:\Sigma' \to C$ *such that* $f\circ\varepsilon$ *is homotopic to* $g\otimes\Lambda$.

Proof. — Let us consider the modules Σ_p, Σ_{p-1} as p-dimensional complexes $C'_p\otimes\Lambda$, $C'_{p-1}\otimes\Lambda$. The differential d on Σ is a map from $C'_p\otimes\Lambda$ to $C'_{p-1}\otimes\Lambda$. Then, by theorem (6.12), there exist a p-dimensional complex $\overline{C}\in\mathscr{C}(A)$, a $\mathscr{W}'$-equivalence $\bar\varepsilon:\overline{C}\to C'_p$ and a morphism $g:\overline{C}\to C'_{p-1}$ such that $g\otimes\Lambda$ is homotopic to $d\circ(\bar\varepsilon\otimes\Lambda)$.

Let M be the mapping cone of g. The $\mathscr{W}'$-equivalence $\bar\varepsilon$ induces a homotopy equivalence $\varepsilon' : M\otimes\Lambda\to\Sigma$. Moreover M is p-dimensional and $C\otimes\Lambda$ is $(p-2)$-connected. Then by (6.12), there exist a p-dimensional complex $\Sigma'\in\mathscr{C}(A)$, a $\mathscr{W}'$-equivalence $\varepsilon'':\Sigma'\to M$ and a morphism $g:\Sigma'\to C$ such that $f\circ\varepsilon'\circ(\varepsilon''\otimes\Lambda)$ is homotopic to $g\otimes\Lambda$. The result follows.

Lemma **(8.2)**. — *Let* C *be a finite* A-*complex such that* $H^i(C,\Lambda)$ *vanishes for* $i>p$ *and let* $\varphi\in B(C\otimes\Lambda)$ *be a bilinear form such that*

$$\partial^0\varphi \leq -2p+1, \qquad d\varphi = 0.$$

Then there exist a complex $C'\in\mathscr{C}(A)$ *with* $\dim C' = \dim C$, *a* $\mathscr{W}'$-*equivalence* $\varepsilon:C'\to C$ *and a bilinear form* $\varphi'\in B(C')$ *such that* $d\varphi'=0$ *and* $\varepsilon^*(\varphi)-\varphi'\otimes\Lambda$ *is a boundary.*

Proof. — By theorem (6.12), there exist a complex $C'\in\mathscr{C}(A)$ with $\dim C' = \dim C$, a $\mathscr{W}'$-equivalence $\varepsilon:C'\to C$ and a morphism $g:C'\to\hat{C}$ such that $\varphi\circ(\varepsilon\otimes\Lambda)$ is homotopic to $\Lambda\otimes g$. Then $\varphi'=\hat\varepsilon g$ is the desired form. ■

Lemma **(8.3)**. — *Let* C *be a finite* A-*complex such that* $H^i(C,\Lambda)$ *vanishes for* $i>p$ *and let* $\varphi\in B(C)$ *be a bilinear form such that*

$$\partial^0\varphi\leq -2p, \qquad d\varphi=0.$$

Then, if $\varphi\otimes\Lambda$ *is a boundary, there exist a complex* $C'\in\mathscr{C}(A)$ *with* $\dim C'=\dim C$ *and a* $\mathscr{W}'$-*equivalence* $\varepsilon:C'\to C$ *such that* $\varepsilon^*(\varphi)$ *is a boundary.*

Proof. — If $\varphi\otimes\Lambda$ is a boundary, $\varphi\otimes\Lambda$ is null homotopic and, by (6.12), there exist a complex $C'\in\mathscr{C}(A)$ with $\dim C'=\dim C$ and a $\mathscr{W}'$-equivalence $\varepsilon:C'\to C$ such that $\varphi\circ\varepsilon$ is null homotopic. Then $\varepsilon^*(\varphi)=\hat\varepsilon\circ\varphi\circ\varepsilon$ is a boundary. ■

Theorem **(8.4).** — *The morphism* $\varepsilon : \Gamma_n(A, \mathcal{W}') \to L_n^h(\Lambda)$ *is an isomorphism.*

Proof. — Suppose $n = -2p$ or $n = -2p + 1$, and let $\sigma \in L_n^h(\Lambda)$.

By lemma (3.6), σ is represented by a $\mathcal{W}_\Lambda$-non singular quadratic n-complex (C, q) where C is concentrated in dimension p (and $p-1$ if n is odd).

By lemma (8.1), there exist a p-dimensional complex $C' \in \mathcal{C}(A)$ and a homotopy equivalence from $C' \otimes \Lambda$ to C. Then σ is represented by $(C' \otimes \Lambda, q')$. Since C' is p-dimensional, q' is the class of $e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1$ and we have

$$d\varphi_0 + \varphi_1 - \hat{\varphi}_1 = 0, \qquad d\varphi_1 = 0.$$

By lemma (8.2), we may suppose that φ_1 has the form $\psi_1 \otimes \Lambda$, $\psi_1 \in B(C')$ and $d\psi_1$ is zero. Then $(\psi_1 - \hat{\psi}_1) \otimes \Lambda$ is a boundary and, by lemma (8.3), we may suppose that $\psi_1 - \hat{\psi}_1$ is a boundary $d\xi$.

Now, $\varphi_0 + \xi \otimes \Lambda$ is a cycle and, by (8.2), we may suppose that

$$\varphi_0 + \xi \otimes \Lambda = \varphi' \otimes \Lambda + d\eta$$

where φ' is a cycle in $B(C')$ and $\eta \in B(C' \otimes \Lambda)$. Then, we have

$$e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1 = (e_0 \otimes (\varphi' - \xi) + e_1 \otimes \psi_1) \otimes \Lambda + d(e_0 \otimes \eta).$$

Moreover $e_0 \otimes (\varphi' - \xi) + e_1 \otimes \psi_1$ is a cycle and represents a $\mathcal{W}'$-non singular quadratic n-form over C'. Then the morphism ε is surjective.

Now let $\sigma' \in \Gamma_n(A, \mathcal{W}')$ be an element in $\operatorname{Ker} \varepsilon$. By lemma (3.6), σ' is represented by a $\mathcal{W}'$-non singular quadratic n-complex (C, q) where C is a complex in $\mathcal{C}(A)$ concentrated in dimension p (and $p-1$ if n is odd).

Since $\varepsilon\sigma'$ is zero, $(C \otimes \Lambda, q \otimes \Lambda)$ is cobordant to zero and, by lemmas (3.7) and (3.8), there exists a $\mathcal{W}_\Lambda$-non singular quadratic $(n+1)$-pair $(\Sigma \to C \otimes \Lambda, u)$ such that q is the boundary of u and Σ_i vanishes for $i \neq p$, $p-1$.

By lemma (8.1), we may suppose that the morphism $\Sigma \to C \otimes \Lambda$ is the morphism $g \otimes \Lambda : \Sigma' \otimes \Lambda \to C \otimes \Lambda$, where Σ' is a p-dimensional complex in $\mathcal{C}(A)$. The quadratic form u is represented by

$$e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2, \qquad \psi_i \in B(\Sigma'),$$

and we have

$$\begin{aligned} d\psi_0 + \psi_1 - \hat{\psi}_1 &= \hat{g}\varphi_0 g \otimes \Lambda \\ -d\psi_1 + \psi_2 + \hat{\psi}_2 &= \hat{g}\varphi_1 g \otimes \Lambda \\ d\psi_2 &= 0 \end{aligned}$$

where $e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1$ represents q.

By lemma (8.2), we may suppose that

$$\psi_2 = \psi_2' \otimes \Lambda + d\xi_1, \qquad d\psi_2' = 0$$

and, after adding to $e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2$ the boundary of $e_2 \otimes \xi_1$, we have

$$\psi_2 = \psi_2' \otimes \Lambda, \qquad d\psi_2' = 0.$$

Then $(\hat{g}\varphi_1 g - \psi_2' - \hat{\psi}_2') \otimes \Lambda$ is a boundary and, by lemma (8.3), we may suppose that

$$\hat{g}\varphi_1 g = \psi_2' + \hat{\psi}_2' + d\eta_1.$$

Since $\psi_1 + \eta_1 \otimes \Lambda$ is a cycle, we may suppose, by lemma (8.2), that

$$\psi_1 + \eta_1 \otimes \Lambda = \psi_1' \otimes \Lambda + d\xi_0, \quad d\psi_1' = 0,$$

and, after adding to $e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2$ the boundary of $-e_1 \otimes \xi_0$, we may suppose that

$$\psi_1 + \eta_1 \otimes \Lambda = \psi_1' \otimes \Lambda, \quad d\psi_1' = 0.$$

Then, we have

$$d\psi_0 + (\psi_1' - \eta_1 - \hat{\psi}_1' + \hat{\eta}_1) \otimes \Lambda = \hat{g}\varphi_0 g \otimes \Lambda.$$

Let ψ be the form $\hat{g}\varphi_0 g - \psi_1' + \eta_1 + \hat{\psi}_1' - \hat{\eta}_1$. The bilinear form ψ is a cycle of degree n and $\psi \otimes \Lambda$ is a boundary. Moreover, by Poincaré duality, $H^i(\Sigma', \Lambda)$ vanishes for $i > -n-p$. Then lemma (8.3) holds and we may suppose that

$$\hat{g}\varphi_0 g - \psi_1' + \eta_1 + \hat{\psi}_1' - \hat{\eta}_1 = d\eta_0.$$

So $\psi_0 - \eta_0 \otimes \Lambda$ is a cycle and, by (8.2), we may suppose that

$$\psi_0 - \eta_0 \otimes \Lambda = \psi_0' \otimes \Lambda + d\xi_{-1}, \quad d\psi_0' = 0,$$

and, after adding to $e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2$ the boundary of $e_0 \otimes \xi_{-1}$, we may suppose that

$$\psi_0 - \eta_0 \otimes \Lambda = \psi_0' \otimes \Lambda.$$

Now it is easy to check that

$$e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2 = [e_0 \otimes (\eta_0 + \psi_0') + e_1 \otimes (-\eta_1 + \psi_1') + e_2 \otimes \psi_2'] \otimes \Lambda$$

and

$$d[e_0 \otimes (\eta_0 + \psi_0') + e_1 \otimes (-\eta_1 + \psi_1') + e_2 \otimes \psi_2'] = g^*(e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1).$$

Then $e_0 \otimes (\eta_0 + \psi_0') + e_1 \otimes (-\eta_1 + \psi_1') + e_2 \otimes \psi_2'$ represents a $\mathscr{W}'$-non singular quadratic $(n+1)$-form v over $\Sigma' \to C$ with boundary q. So σ' is zero and ε is injective. ∎

9. Some results about Λ and $L_n(\Lambda)$

Throughout this section, we assume that $A \to B$ is a ring homomorphism and β is a subgroup of $\tilde{K}_1(B)$.

The class of finite A-complexes C such that $C \otimes B$ is acyclic with torsion in β is denoted by $\mathscr{W}^\beta$, and the $\mathscr{W}^\beta$-localization of A is denoted by (Λ, α).

Proposition **(9.1)**. — *Let u be a matrix with entries in Λ. Then, if $u \otimes B$ is invertible, u is invertible too.*

Proof. — Let u be a matrix with entries in Λ. If we denote by A the 0-dimensional complex $\ldots \to 0 \to A \to 0 \to \ldots$, u is a morphism $A^p \otimes \Lambda \to A^q \otimes \Lambda$ and, by theorem (6.12), there exist a 0-dimensional complex $\overline{C} \in \mathscr{C}(A)$, a $(\mathscr{W}^\beta)_0^s$-equivalence $\varepsilon : \overline{C} \to A^p$ and a morphism $g : \overline{C} \to A^q$ such that $g \otimes \Lambda$ is homotopic to $u \circ (\varepsilon \otimes \Lambda)$.

Let K be the homotopy kernel of ε. Since K is $\mathscr{W}_0^\beta$-splittable, there exist a (-1)-dimensional complex $K' \in \mathscr{W}_0^\beta$ and a (-2)-connected morphism $f : K' \to K$. The composite map $K' \to K \to \overline{C}$ is (-2)-connected. Denote by C′ its mapping cone. The complex C′ lies in $\mathscr{W}_0^\beta$ and has the simple homotopy type of a complex C″ such that C''_i vanishes for $i \neq 0, -1$. Moreover ε and g factorize through C″ and we get two morphisms $\varepsilon' : C'' \to A^p$ and $g' : C'' \to A^q$ such that $g' \otimes \Lambda$ is homotopic to $u \circ (\varepsilon' \otimes \Lambda)$.

But $u \otimes B$ is invertible, then $g' \otimes B$ is a homotopy equivalence and the mapping cone of g' is B-acyclic and lies in $\mathscr{W}_0^\beta$. Since the length of this mapping cone is 2, g' is a $(\mathscr{W}^\beta)_0^s$-equivalence. Then, by (7.1), g' is a Λ-homology equivalence, and u is an isomorphism. ■

(9.2) *Proof of theorem* (1.13)

If u is a matrix with entries in A, denote by M(u) the 1-dimensional complex $\ldots \to 0 \to A^p \xrightarrow{u} A^q \to 0 \to \ldots$

The set Σ is the set of matrices u such that $(M(u) \oplus \Sigma M(u)) \otimes B$ is acyclic with torsion in β. But $M(u) \oplus \Sigma M(u)$ is B-acyclic if and only if M(u) is B-acyclic. Moreover if M(u) is B-acyclic, we have

$$\tau[M(u) \otimes B \oplus \Sigma M(u) \otimes B] = 0.$$

Then Σ is the set of matrices u such that $u \otimes B$ is invertible and $A \to \Lambda$ is the localization of $A \to B$.

Now let τ be an element of $\widetilde{K}_1(\Lambda)$. By lemma (4.3), there exists a finite A-complex C such that $C \otimes \Lambda$ is acyclic with torsion τ. Then, by lemma (7.10), τ lies in α if and only if C lies in $\mathscr{W}^\beta$. But the torsion of $C \otimes B$ is the image of τ by the morphism $\varepsilon : \Lambda \to B$. Hence α is the inverse image of β under ε.

Now suppose ε is onto, and let $C \in \mathscr{W}^\beta$. The complex $C \otimes B$ is acyclic and the identity is a homotopy: $1 = d \circ k + k \circ d$.

But $C \otimes \Lambda \to C \otimes B$ is onto and we can lift k in a map k' from $C \otimes \Lambda$ to itself. The morphism $d \circ k' + k' \circ d$ is invertible after tensorization by B. Then, by (9.1), $d \circ k' + k' \circ d$ is an isomorphism and $C \otimes \Lambda$ is acyclic. ■

(9.3) *Proof of Proposition* (1.15)

Let $B_0 \subset B_1 \subset B_2 \subset \ldots$ be subrings of B defined by:

(i) B_0 is the image of $A \to B$;

(ii) for any $n \geq 0$, B_{n+1} is generated by B_n and the inverses of the units of B contained in B_n.

Denote by B′ the image of $\Lambda \to B$. The subring B′ contains A and, by (9.1), any unit of B contained in B′ is a unit of B′. Then B′ contains all the rings B_n. ■

As a corollary of (9.1), we have:

Lemma **(9.4)**. — *If* $\Lambda \to B$ *is onto,* $\widetilde{K}_1(\Lambda) \to \widetilde{K}_1(B)$ *is onto.*

From now on, we will suppose that $A \to B$ is a morphism of rings with involution and that β is stable under the involution. Then $\mathscr{W}^\beta$ is symetric and Λ has an involution. We suppose also that $\Lambda \to B$ is onto.

Theorem **(9.5)**. — *If n is even, the morphism* $L_n^\alpha(\Lambda) \to L_n^\beta(B)$ *is epic. If n is odd, this morphism is monic.*

Proof. — By lemma (9.4), the relative group $L_n^{\alpha,\beta}(\Lambda \to B)$ does not depend on β. Then it suffices to prove the theorem in the case $\beta = \widetilde{K}_1(B)$.

Let $n = 2p$. An element $u \in L_{2p}^h(B)$ is represented by a hermitian $(-1)^p$-form (H, λ, μ) such that the induced map $\widetilde{\lambda} : H \to \hat{H}$ is an isomorphism. Since H is free over B and $\Lambda \to B$ is epic, there exists a hermitian $(-1)^p$-form (H', λ', μ') such that

$$H' \text{ is free over } \Lambda,$$
$$H' \otimes B = H, \quad \lambda' \otimes B = \lambda, \quad \mu' \otimes B = \mu.$$

Then, by lemma (9.1), λ′ induces an isomorphism from H′ to $\hat{H}'$ and (H', λ', μ') represents an element $v \in L_{2p}^h(\Lambda)$ such that $\varepsilon_*(v) = u$.

Let now $n = 2p + 1$. An element $v \in L_{2p+1}^h(\Lambda)$ is represented by an isometry between two standard kernel K and K′. If v is sent to zero in $L_{2p+1}^h(B)$, $K = K'$ and $g \otimes B$ is an element of $RU^h(B)$ (with the notations of [10]).

Consider the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & UU(\Lambda) & \longrightarrow & TU^h(\Lambda) & \longrightarrow & GL(\Lambda) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 1 & \longrightarrow & UU(B) & \longrightarrow & TU^h(B) & \longrightarrow & GL(B) & \longrightarrow & 1 \end{array}$$

By lemma (9.1), a and c are surjective. Then b is epic and the morphism $RU^h(\Lambda) \to RU^h(B)$ is epic too. Hence v can be represented by an isometry f such that $f \otimes B$ is the identity map.

Let $H \oplus \hat{H}$ be the standard kernel K. The isometry f is defined by

$$f(x, y) = (x + a(x) + b(y), y + c(x) + d(y)), \quad \forall\, x \in H,\ y \in \hat{H}$$

and $a \otimes B$, $b \otimes B$, $c \otimes B$, $d \otimes B$ vanish. By (9.1), $1 + a$ is invertible and, after composing f with an element of $GL(\Lambda)$, we may as well suppose that a is zero.

Since f is an isometry, it is easy to see that the map g defined by

$$g(x,y) = (x, y - c(x))$$

is an isometry leaving each element of $\hat{H}$ fixed and g lies in $RU^h(\Lambda)$. We have

$$g \circ f(x,y) = (x + b(y), y + d(y) - c \circ b(y)).$$

But $1 + d - c \circ d$ is invertible and there is an isometry $h \in RU^h(\Lambda)$ such that

$$h \circ g \circ f(x,y) = (x + a'(x) + b'(y), y).$$

It is easy to see that a' is zero and $h \circ g \circ f$ lies in $RU^h(\Lambda)$. Therefore V is zero. ∎

Theorem **(9.6)**. — *The relative group* $L^h_{2p+1}(\Lambda \to B)$ *is the group of equivalence classes of pairs* (H, K) *where* H *is a hermitian* $(-1)^p$*-form over* Λ *and* K *a subkernel of* $H \otimes B$, *subject to the following relation:*

(H, K) *is equivalent to* (H′, K′) *if there exist two* Λ*-kernels* H_0 *and* H'_0 *with subkernels* S_0 *and* S'_0 *and an isometry* $\varphi : H \oplus H_0 \to H' \oplus H'_0$ *such that*

$$\varphi(K \oplus S_0 \otimes B) = K' \oplus S'_0 \otimes B.$$

Proof. — By Wall ([10], p. 72), $L^h_{2p+1}(\Lambda \to B)$ is generated by such pairs. Moreover (H, K) and (H′, K′) represent the same element in $L^h_{2p+1}(\Lambda \to B)$ if there exist two kernels $\overline{H}_0$ and H'_0 with subkernels $\overline{S}_0$ and $\overline{S}'_0$ and an isometry

$$\bar{\varphi} : H \oplus \overline{H}_0 \oplus -H' \to H'_0$$

such that any automorphism $\bar{\psi}$ taking $\overline{S}'_0 \otimes B$ to $\bar{\varphi}(K \oplus \overline{S}_0 \otimes B \oplus K')$ lies in $RU^h(B)$. But the map $RU^h(\Lambda) \to RU^h(B)$ is epic (see the proof of (9.5)). Hence we can lift $\bar{\psi}$ to an automorphism ψ on H'_0.

Let S'_0 be the subkernel $\psi(\overline{S}'_0)$. We have an isometry

$$\varphi : H \oplus \overline{H}_0 \oplus -H' \oplus H' \to H' \oplus H'_0$$

taking $K \oplus \overline{S}_0 \otimes B \oplus K' \oplus K'$ to $K' \oplus S'_0 \otimes B$.

On the other hand, the diagonal $\overline{K}$ is a subkernel of $-H' \oplus H'$ and there exists an automorphism in $RU^h(B)$ taking $\overline{K} \otimes B$ to $K' \oplus K'$. By lifting this automorphism in $RU^h(\Lambda)$ we get an automorphism f and $f(\overline{K})$ is a subkernel of $-H' \oplus H'$ such that $f(\overline{K}) \otimes B = K' \oplus K'$. Let H_0 be the kernel $\overline{H}_0 \oplus -H' \oplus H'$ with subkernel $S_0 = \overline{S}_0 \oplus f(\overline{K})$. Then φ is an isometry taking $K \oplus S_0 \otimes B$ to $K' \oplus S'_0 \otimes B$. ∎

Now, consider the following question: Under what conditions is the map $\varepsilon : \Lambda \to B$ an isomorphism? To study this problem, it is convenient to use the following definitions:

An A-module M is called B-*perfect* if $M \otimes B$ is zero; it is called *locally* B-*perfect* if any element in M is contained in a finitely generated B-perfect submodule.

Theorem **(9.7)**. — *Suppose the kernel of* $A \to B$ *is locally* B*-perfect and* B *is the localization of* $\mathrm{Im}(A \to B)$ *with respect to a multiplicative subset of the center. Then the morphism* $\varepsilon : \Lambda \to B$ *is an isomorphism.*

Proof. — Let $a \in \mathrm{Ker}(A \to B)$ and suppose that a is contained in a finitely generated B-perfect submodule I. Let us choose a free resolution of I

$$C \xrightarrow{f} A^n \to I \to 0.$$

Since I is B-perfect, $f \otimes B$ is epic and has a section s. But $\Lambda \to B$ is epic and we can lift s to a morphism $g : \Lambda^n \to C \otimes \Lambda$. By (9.1), $f \otimes \Lambda \circ g$ is an isomorphism and $f \otimes \Lambda$ is epic. Hence I is Λ-perfect and the composite map $I \to A \to \Lambda$ is zero. Then $A \to B$ and $A \to \Lambda$ have the same kernel K.

Now it is easy to see that the maps $A/K \to B$ and $A/K \to \Lambda$ have the same universal property and $\varepsilon : \Lambda \to B$ is an isomorphism. ■

This theorem is in fact a generalization of a theorem of Hausmann [3] proved also in [6] and [8], theorem (1.4).

Finally, we will give an example of computation.

Let D_{2n} be the dihedral group of order $2n$ (n odd) and let $\mathbf{Z}D_{2n} \to \mathbf{Z}$ be the evaluation map. The group D_{2n} is not perfect and not nilpotent, then we cannot use the techniques of Hausmann or Smith in order to compute the group $\Gamma_*(\mathbf{Z}D_{2n} \to \mathbf{Z})$.

Theorem **(9.8)**. — *We have the isomorphisms*

$$\Gamma_*(\mathbf{Z}D_{2n} \to \mathbf{Z}) \xrightarrow{\sim} \Gamma_*(\mathbf{Z}[\mathbf{Z}/2] \to \mathbf{Z}) \xrightarrow{\sim} L^h_*(\Lambda)$$

where Λ *is the pull back of rings*

$$\begin{array}{ccc} \Lambda & \longrightarrow & \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathbf{Z}_{(2)}[\mathbf{Z}/2] & \longrightarrow & \mathbf{Z}_{(2)} \end{array}$$

Proof. — The group D_{2n} is generated by t and τ with the following relations:

$$t^n = 1, \quad \tau^2 = 1, \quad \tau t = t^{-1}\tau.$$

Let $\mathbf{Z}D_{2n} \to \Lambda$ be the localization of $\mathbf{Z}D_{2n} \to \mathbf{Z}$ and let x and y be the images of t and τ in Λ. We have

$$\left[\frac{1-n}{2}(1+\tau) + 1 + t + \ldots + t^{n-1}\right](1-\tau)(1-t) = 0.$$

But $\frac{1-n}{2}(1+\tau) + 1 + t + \ldots + t^{n-1}$ is sent to 1 in $\mathbf{Z}$ and

$$\frac{1-n}{2}(1+y) + 1 + x + \ldots + x^{n-1}$$

is invertible. This implies that

$$(1-y)(1-x) = 0.$$

On the other hand, $\mathbf{Z}D_{2n} \to \Lambda$ is a morphism of rings with involution. So we have:

$$(1-y)(1-x) = (1-x^{-1})(1-y) = 0 \Rightarrow (1-x)(1-y) = 0.$$

And x and y commute. Then:

$$yx = x^{-1}y = xy \Rightarrow x = 1.$$

Hence t is sent to 1 in Λ and Λ is the localization of $\mathbf{Z}[\mathbf{Z}/2] \to \mathbf{Z}$. But $\mathbf{Z}[\mathbf{Z}/2]$ is commutative and Λ is the localization $S^{-1}\mathbf{Z}[\mathbf{Z}/2]$ where S is the set of elements $a + b\tau \in \mathbf{Z}[\mathbf{Z}/2]$ with $a + b = 1$. Then it is easy to see that Λ is the subring of $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$ defined by

$$\Lambda = \{a + b\tau,\ a,\ b \in \mathbf{Z}_{(2)} \text{ and } a + b \in \mathbf{Z}\}.$$

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Université de Nantes,
Institut de Mathématiques et d'Informatique,
2, chemin de la Houssinière,
44072 Nantes Cedex (France).

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