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ON TRANSVERSE FOLIATIONS

by ITIRO TAMURA and ATSUSHI SATO

The structure of foliations displays a high degree of variability, and is generally far less rigid in contrast to complex structures. Thus, it is virtually impossible to give a precise description which characterizes effectively all foliations on a manifold, and the consequent lack of appropriate classification theorems seems to constitute a barrier to the derivation of precise results in foliation theory. However, if we fix some foliation on a manifold, and restrict our considerations to foliations having a definite relation with the given foliation (i.e. a structure of foliations on a foliated manifold), then a characterization of this class of foliations can often be obtained.

This paper deals with subfoliations of, and foliations transverse to, a given foliation. We shall establish classification theorems for codimension one foliations transverse to the Reeb component of $S^1 \times D^2$, and to the Reeb foliation of S^3 respectively (Theorems 1, 2, 3, 4 and 5).

Furthermore, as an application of Theorem 1, we shall prove that the foliations of codimension one of S^3 constructed from fibred knots do not admit any transverse foliation of codimension one (Theorem 6).

In Section 1, we define subfoliations, superfoliations, and transverse foliations. In Section 2, we consider a generalization of a result due to Reinhart, Davis and Wilson; this constitutes the starting point of our work. In Sections 3, 4 and 5, we study foliations of codimension one transverse to the Reeb component \mathcal{F}_R , the set of which is denoted by $t_1(\mathcal{F}_R)$. The existence of the half Reeb component and the TS components in $\mathcal{F}' \in t_1(\mathcal{F}_R)$ are proved in Sections 3 and 4, respectively. In Section 5, we give classification theorems for $t_1(\mathcal{F}_R)$. As a direct consequence of these theorems, the classification for foliations of codimension one transverse to the Reeb foliation of S^3 is derived in Section 6. In Section 7, we prove the non-existence of a foliation of codimension one transverse to a foliation of S^3 constructed from a fibred knot. The problems raised by the results of this paper are given in Section 8.

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1. Subfoliations, superfoliations and transverse foliations

Let M^n be an n -dimensional C^∞ manifold with or without boundary. Denote by $\mathcal{F}^{(k)}$ a C^r foliation of codimension k of M^n ($r \geq 0$), where, in case $\partial M^n \neq \emptyset$,

$$\mathcal{F}^{(k)} | \partial M^n = \{L \cap \partial M; L \in \mathcal{F}^{(k)}\}$$

is a C^r foliation of codimension $k-1$ or k of ∂M^n . Two C^r foliations $\mathcal{F}_1^{(k)}$ and $\mathcal{F}_2^{(k)}$ of codimension k of M^n are called *isomorphic* if there exists a C^r diffeomorphism $f: M^n \rightarrow M^n$ which preserves the leaves of $\mathcal{F}_1^{(k)}$ and $\mathcal{F}_2^{(k)}$.

Let $\mathcal{F}^{(k)}$ and $\mathcal{F}'^{(k')}$ be C^r foliations of codimensions k and k' of M^n respectively. Then $\mathcal{F}'^{(k')}$ is called a *subfoliation* of $\mathcal{F}^{(k)}$ and $\mathcal{F}^{(k)}$ is called a *superfoliation* of $\mathcal{F}'^{(k')}$, denoted by $\mathcal{F}'^{(k')} \prec \mathcal{F}^{(k)}$, if the following conditions hold:

(i) $k \leq k' \leq n$.

(ii) For any leaf L' of $\mathcal{F}'^{(k')}$, there exists a leaf L of $\mathcal{F}^{(k)}$ such that $L' \subset L$, and the restriction of $\mathcal{F}'^{(k')}$ on a leaf L of $\mathcal{F}^{(k)}$ is a C^r foliation of codimension $k' - k$ of L .

In case $r \geq 1$, it is obvious that, if $\mathcal{F}'^{(k')} \prec \mathcal{F}^{(k)}$ and $\mathcal{F}''^{(k'')} \prec \mathcal{F}'^{(k')}$, then $\mathcal{F}''^{(k'')} \prec \mathcal{F}^{(k)}$. Therefore the relation \prec is an order in the set of C^r foliations of M^n ($r \geq 1$).

Two subfoliations $\mathcal{F}_1'^{(k')}$ and $\mathcal{F}_2'^{(k')}$ of $\mathcal{F}^{(k)}$ are called *strongly isomorphic*, if there exists a C^r diffeomorphism $f: M^n \rightarrow M^n$ which preserves $\mathcal{F}^{(k)}$ and maps $\mathcal{F}_1'^{(k')}$ onto $\mathcal{F}_2'^{(k')}$.

Let $\tau(\mathcal{F}^{(k)})$ denote the subbundle of the tangent bundle $\tau(M^n)$ of M^n consisting of vectors tangent to leaves of $\mathcal{F}^{(k)}$. In order that $\mathcal{F}^{(k)}$ has a C^r subfoliation of codimension k' , it is necessary that $\tau(\mathcal{F}^{(k)})$ has a $(k' - k)$ -dimensional subbundle if $r \geq 1$.

A C^r foliation $\mathcal{F}'^{(k')}$ of codimension k' of M^n is called *transverse* to a C^r foliation $\mathcal{F}^{(k)}$ of M^n ($r \geq 1$), denoted by $\mathcal{F}'^{(k')} \bar{\cap} \mathcal{F}^{(k)}$, if the following conditions hold:

(i) $k + k' \leq n$.

(ii) Any leaves L of $\mathcal{F}^{(k)}$ and L' of $\mathcal{F}'^{(k')}$ intersect transversely in case $L \cap L' \neq \emptyset$.

Let $\mathcal{F}^{(k)} \cap \mathcal{F}'^{(k')}$ denote $\{L \cap L'; L \in \mathcal{F}^{(k)}, L' \in \mathcal{F}'^{(k')}\}$, then it is clear that $\mathcal{F}^{(k)} \cap \mathcal{F}'^{(k')}$ is a C^r foliation of codimension $k + k'$ which is a common subfoliation of $\mathcal{F}^{(k)}$ and $\mathcal{F}'^{(k')}$.

Two C^r foliations $\mathcal{F}_1'^{(k')}$ and $\mathcal{F}_2'^{(k')}$ which are transverse to $\mathcal{F}^{(k)}$ are called *strongly isomorphic*, if there exists a C^r diffeomorphism $f: M^n \rightarrow M^n$ which preserves $\mathcal{F}^{(k)}$ and maps $\mathcal{F}_1'^{(k')}$ onto $\mathcal{F}_2'^{(k')}$.

We note that the transversality $\mathcal{F}^{(k)} \bar{\cap} \mathcal{F}'^{(k')}$ is invariant under a small perturbation of $\mathcal{F}^{(k)}$ and $\mathcal{F}'^{(k')}$ respectively.

In order that $\mathcal{F}^{(k)}$ admits a transverse C^r foliation of codimension k' , it is necessary that $\tau(M^n)$ has an $(n - k')$ -dimensional subbundle which is transverse to $\tau(\mathcal{F}^{(k)})$ at each point of M^n if $r \geq 1$.

Example 1. — It is well known that a C^r foliation $\mathcal{F}^{(1)}$ of codimension 1 of M^n always admits a transverse C^r foliation of codimension $n - 1$ ($r \geq 1$).

Example 2. — Let $\mathcal{F}^{(2)}$ be a C^r foliation of codimension 2 of the 3-sphere S^3 consisting of compact leaves ($r \geq 1$). Then there exists no C^r foliation of codimension one which is transverse to $\mathcal{F}^{(2)}$. Because, if there exists a C^r foliation of codimension one transverse

to $\mathcal{F}^{(2)}$, say $\mathcal{F}^{(1)}$, then $\mathcal{F}^{(1)}$ contains a Reeb component by Novikov's theorem ([4]) which implies that $\mathcal{F}^{(2)}$ should contain non-compact leaves.

Example 3. — Let $\mathcal{F}^{(2)}$ be a C^r foliation of codimension 2 of S^3 which admits a superfoliation of codimension one, then $\mathcal{F}^{(2)}$ has a compact leaf. That is to say, the conjecture of Seifert holds in this case. Because a C^r foliation $\mathcal{F}^{(1)}$ of codimension one of S^3 having $\mathcal{F}^{(2)}$ as a subfoliation contains a Reeb component ([4]), and any subfoliation of a Reeb component has a compact leaf (see Proposition 2 of Section 2).

For a family $\{\mathcal{F}_\lambda^{(k)}\}_{\lambda \in \Lambda}$ of C^r foliations of codimension k of M^n , we denote by $t_j(M^n, \{\mathcal{F}_\lambda^{(k)}\})$ or simply by $t_j(\{\mathcal{F}_\lambda^{(k)}\})$ the family $\{\mathcal{F}_\sigma^{(k')}\}_{\sigma \in \Sigma}$ of C^r foliations of codimension k' such that $j = n - k - k'$ and that there exists $\mathcal{F}_\lambda^{(k)}$ transverse to $\mathcal{F}_\sigma^{(k')}$. Further we denote by $t_j^m(\{\mathcal{F}_\lambda^{(k)}\})$ the m fold iteration $t_j(t_j(\dots(t_j(\{\mathcal{F}_\lambda^{(k)}\})\dots))$ of t_j . It is obvious that the iterations of t_j have the property

$$t_j^m(\{\mathcal{F}_\lambda^{(k)}\}) \subset t_j^{m+2}(\{\mathcal{F}_\lambda^{(k)}\}) \quad (m \geq 1).$$

Now we give a sufficient condition for the existence of transverse plane fields for a C^r foliation:

Proposition 1. — Let M^n be a compact orientable n -dimensional C^∞ manifold and $\mathcal{F}^{(k)}$ a C^r foliation of codimension k ($r \geq 1$) such that, in case $\partial M^n \neq \emptyset$, $\mathcal{F}^{(k)}|_{\partial M^n}$ is a C^r foliation of codimension $k - 1$. Then, in order that M^n admits a $(k + 1)$ -plane field transverse to $\tau(\mathcal{F}^{(k)})$, it is sufficient that

$$H^j(M^n; \pi_{j-1}(S^{n-k-1})) = 0, \quad j = 1, 2, \dots, n.$$

In particular, any C^r foliation $\mathcal{F}^{(1)}$ of codimension one of S^3 admits a 2-plane field transverse to $\tau(\mathcal{F}^{(1)})$.

Proof. — The obstruction to construct a non-zero cross section of $\tau(\mathcal{F}^{(k)})$ lies in $H^j(M^n; \pi_{j-1}(S^{n-k-1}))$ ([1; Theorem (1.1)]). The $(k + 1)$ -plane field generated by the vector field of $\tau(\mathcal{F}^{(k)})$ and a k -plane field transverse to $\tau(\mathcal{F}^{(k)})$ has the required property.

2. Subfoliations of a foliation of codimension one defined by a fibering over S^1

In the following sections, we fix an orientation on the circle S^1 . The Reeb component of $S^1 \times D^2$ constructed by turbulizing ([4]) a collar of the boundary $S^1 \times \partial D^2$ in the minus (resp. plus) direction of S^1 is called the *plus Reeb component* (resp. the *minus Reeb component*) and denoted by $\mathcal{F}_R^{(+)}$ (resp. $\mathcal{F}_R^{(-)}$) (Fig. 1). That is, $S^1 \times \partial D^2$ has a contracting holonomy in the minus (resp. plus) direction of S^1 for $\mathcal{F}_R^{(+)}$ (resp. $\mathcal{F}_R^{(-)}$). We define the *plus Reeb component* $\overline{\mathcal{F}}_R^{(+)}$ (resp. the *minus Reeb component* $\overline{\mathcal{F}}_R^{(-)}$) of $S^1 \times D^1$ similarly (Fig. 2). We understand that $\mathcal{F}_R^{(\pm)}$, $\overline{\mathcal{F}}_R^{(\pm)}$ mean standard ones (i.e. leaves are "symmetric" with respect to an "axis" and $\{*\} \times D^2$ (resp. $\{*\} \times D^1$) is tangent to exactly one leaf of $\mathcal{F}_R^{(\pm)}$ (resp. $\overline{\mathcal{F}}_R^{(\pm)}$) at one point).

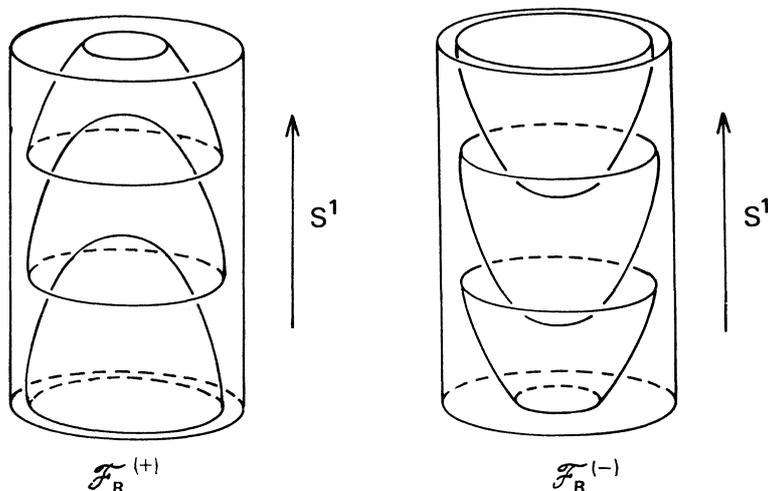


FIG. 1

A C^∞ foliation of codimension one of $S^1 \times D^1$ constructed by turbulizing a collar of $S^1 \times \{-1\}$ and $S^1 \times \{1\}$ in different directions is called a *slope component* and denoted by $\overline{\mathcal{F}}_S$ (Fig. 2). The set of vertices (i.e. maximal or minimal points) of leaves of $\overline{\mathcal{F}}_R^{(+)}$ (resp. $\overline{\mathcal{F}}_R^{(-)}$) is denoted by $\Sigma^{(+)}$ (resp. $\Sigma^{(-)}$) (Fig. 2).

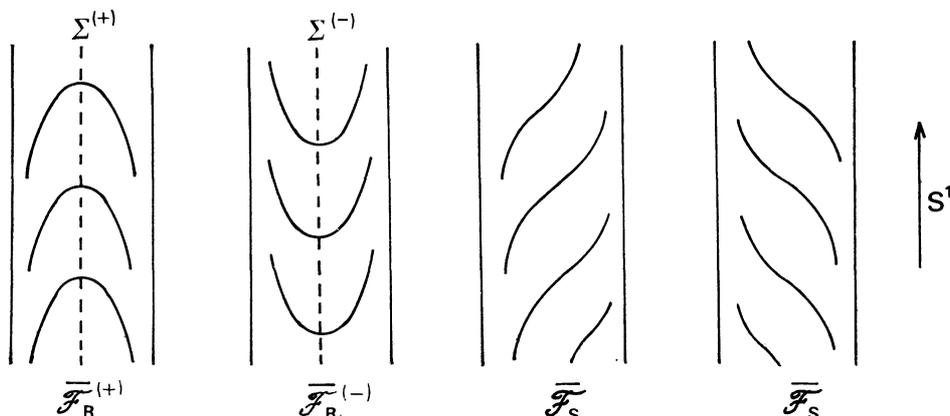


FIG. 2

Let E be a compact connected orientable 3-dimensional C^∞ manifold with boundary $\partial E = T^2$ (torus) and let $\pi : E \rightarrow S^1$ be a C^∞ fibering over S^1 with fibre $G - \text{Int } D^2$, where G is an orientable closed surface of genus g and D^2 is a 2-disc imbedded in G . The C^∞ foliation of codimension one of E constructed by turbulizing the fibers in a collar of the boundary ∂E in the minus (resp. plus) direction is denoted by $\mathcal{F}_\pi^{(+)}$ (resp. $\mathcal{F}_\pi^{(-)}$).

The following proposition is a generalization of a result of Reinhart, Davis and Wilson about tangent vector fields of the Reeb component ([1], [6]).

Proposition 2. — Let $\mathcal{F}^{(2)}$ be a C^∞ foliation of codimension 2 of E which is a subfoliation of $\mathcal{F}_\pi^{(+)}$. Denote by $\mathcal{F}^{(2)}|T^2$ the C^∞ foliation of codimension one of $\partial E = T^2$ which is the restriction of $\mathcal{F}^{(2)}$ to the compact leaf T^2 of $\mathcal{F}_\pi^{(+)}$. Then the following holds:

(i) $\mathcal{F}^{(2)}|T^2$ has a compact leaf.

(ii) $\mathcal{F}^{(2)}|T^2$ is isomorphic to a C^∞ foliation consisting of p copies of the plus Reeb component, q copies of the minus Reeb component (with respect to the orientation induced naturally from that of S^1), a countable number of slope components, and compact leaves (Fig. 3), for which, letting the homology class $[L_{\text{comp}}]$ of $H_1(T^2; \mathbf{Z})$ represented by a compact leaf L_{comp} of $\mathcal{F}^{(2)}|T^2$ with a suitable orientation be $a\alpha + b\beta$ ($a \geq 0$), the equation

$$a(p - q) = 2(1 - 2g)$$

holds, where α (resp. β) is the homology class represented by a cross section of π with the orientation compatible with that of the base space S^1 (resp. by $\partial(G - \text{Int } D^2)$).

(iii) In particular, if $G = S^2$, then we have

$$a = 1, \quad p - q = 2 \quad \text{or} \quad a = 2, \quad p - q = 1,$$

and $\tau(\mathcal{F}^{(2)})$ is orientable if and only if $a = 1$.

The number a in Proposition 2, (ii) is called the *longitudinal number* of $\mathcal{F}^{(2)}|T^2$.

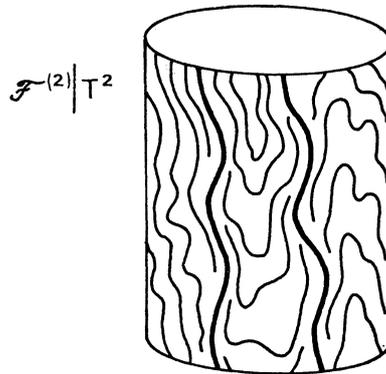


FIG. 3

Proof. — Let F be the line field on T^2 determined by $\mathcal{F}^{(2)}|T^2$, then F induces a homomorphism

$$F_* : H_1(T^2; \mathbf{Z}) \rightarrow H_1(P^1; \mathbf{Z})$$

([6]). If F_* is not a zero map, then $\mathcal{F}^{(2)}|T^2$ has a compact leaf ([6; Corollary 3]). Letting $c : T^2 \times I \rightarrow E$ be a collar of T^2 such that $c(x, 0) = x$ ($x \in T^2$), we define a projection $P : c(T^2 \times I) \rightarrow T^2$ by $P(c(x, t)) = x$. Let L be a leaf of $\mathcal{F}_\pi^{(+)}$ and let

$$\iota : G - \text{Int } D^2 \rightarrow L$$

be an imbedding such that

$$\iota(\partial(G - \text{Int } D^2)) \subset c(T^2 \times I).$$

Let F' be the line field on $\iota(G - \text{Int } D^2)$ determined by $\mathcal{F}^{(2)}|_{\iota(G - \text{Int } D^2)}$. If $F_*([P \circ \iota(\partial(G - \text{Int } D^2))]) = 0$, then the line field $F'|_{\iota(\partial(G - \text{Int } D^2))}$ should be homotopic to the line field tangent to $\iota(\partial(G - \text{Int } D^2))$. This implies that the Euler number $\chi(G)$ must be 1. This is a contradiction. Thus F_* is not a zero map and $\mathcal{F}^{(2)}|_{T^2}$ has a compact leaf.

As is easily verified ([1]), the existence of a compact leaf implies that $\mathcal{F}^{(2)}|_{T^2}$ is isomorphic to a C^∞ foliation consisting of p copies of the plus Reeb component, q copies of the minus Reeb component, a countable number of slope components, and compact leaves. Therefore we may choose the imbedding ι defined above so that it satisfies that $P \circ \iota(\partial(G - \text{Int } D^2))$ intersects $\mathcal{F}^{(2)}$ transversely except at $a(p+q)$ points corresponding to $\Sigma^{(+)}$ or $\Sigma^{(-)}$. Let \hat{G} be the double of $\iota(G - \text{Int } D^2)$, then F' defines a continuous line field on \hat{G} with ap singular points of plus type and aq singular points of minus type. Therefore, by computing the Euler number $\chi(G - \text{Int } D^2)$, we have

$$a(p-q) = 2(1-2g).$$

Thus (ii) is proved. The proof of (iii) is obvious.

3. Half Reeb components

Let D_+^2 denote the half 2-disc $\{(x, y) \in D^2; y \geq 0\}$. The restriction of the plus (resp. minus) Reeb component $\mathcal{F}_R^{(+)}$ (resp. $\mathcal{F}_R^{(-)}$) of $S^1 \times D^2$ to $S^1 \times D_+^2$ is called the *plus* (resp. *minus*) *half Reeb component* and denoted by $\mathcal{F}_{R/2}^{(+)}$ (resp. $\mathcal{F}_{R/2}^{(-)}$). Let \mathcal{F}'_+ (resp. \mathcal{F}'_-) denote the C^∞ foliation of codimension one of $S^1 \times D^2$ obtained from two copies of $\mathcal{F}_{R/2}^{(+)}$ (resp. $\mathcal{F}_{R/2}^{(-)}$) by identifying their compact leaves (Fig. 4).

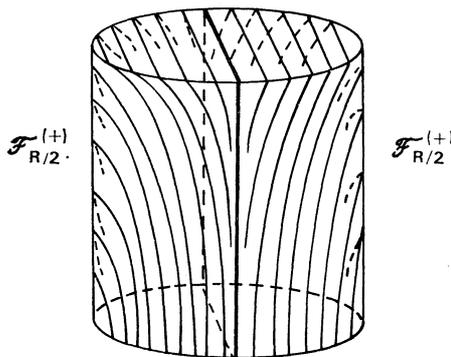


FIG. 4

It is well known that \mathcal{F}'_+ (resp. \mathcal{F}'_-) is transverse to $\mathcal{F}_R^{(+)}$ (resp. $\mathcal{F}_R^{(-)}$). (See, for example, [7].)

In Sections 3, 4 and 5, we let \mathcal{F}' be a C^∞ foliation of codimension one of $S^1 \times D^2$ transverse to $\mathcal{F}_R^{(+)}$:

$$\mathcal{F}' \in t_1(\mathcal{F}_R^{(+)}).$$

The C^∞ foliation $\mathcal{F}_R^{(+)} \cap \mathcal{F}'$ of codimension 2 of $S^1 \times D^2$ is a subfoliation of $\mathcal{F}_R^{(+)}$ (Section 1). Denote by $\overline{\mathcal{F}'}$ the restriction of $\mathcal{F}_R^{(+)} \cap \mathcal{F}'$ to $S^1 \times \partial D^2$. Then, by Proposition 2, the C^∞ foliation $\overline{\mathcal{F}'} = \{L \cap (S^1 \times S^1); L \in \mathcal{F}'\}$ is isomorphic to a C^∞ foliation $\overline{\mathcal{F}}$ consisting of p copies of the plus Reeb component, q copies of the minus Reeb component, a countable number of slope components, and compact leaves, for which Proposition 2, (iii) holds. Therefore there exists a C^∞ diffeomorphism $f: S^1 \times S^1 \rightarrow S^1 \times S^1$ isotopic to the identity such that f maps $\overline{\mathcal{F}}$ to $\overline{\mathcal{F}'}$ and that, for any $x \in S^1$, $f(\{x\} \times S^1)$ intersects $\overline{\mathcal{F}'}$ transversely except at $a(p+q)$ points $f(\{x\} \times S^1) \cap (\widehat{\Sigma}^{(+)} \cup \widehat{\Sigma}^{(-)})$, where $\widehat{\Sigma}^{(+)}$ (resp. $\widehat{\Sigma}^{(-)}$) denotes the union of $\Sigma^{(+)}$ (resp. $\Sigma^{(-)}$) of each plus (resp. minus) Reeb component contained in $\overline{\mathcal{F}}$ (Fig. 5).

We fix a natural product Riemannian metric on $S^1 \times D^2$. Let U be a neighborhood of $S^1 \times S^1$ in $S^1 \times D^2$ and let $V = \{V(z); z \in U\}$ be a C^∞ vector field on U satisfying the following conditions:

- (i) $|V(z)| = 1$;
- (ii) $V(z)$ is tangent to the leaf of \mathcal{F}' containing z ;
- (iii) for $z \in S^1 \times S^1$, $V(z)$ is inward and normal to the leaf of $\overline{\mathcal{F}'}$ containing z .

The existence of such a C^∞ vector field V is obvious.

For $z \in S^1 \times S^1$, let $\varphi(t, z)$ ($0 \leq t < \varepsilon_z$) denote the integral curve with the initial condition $\varphi(0, z) = z$. Let $\varepsilon > 0$ be sufficiently small and let $\bar{\varepsilon}: S^1 \rightarrow]0, \varepsilon[$ be a C^∞ function. Then, by a suitable choice of $\bar{\varepsilon}$, $\bigcup_{x \in S^1} \varphi(\bar{\varepsilon}(x), f(x, y_0))$ is transverse to $\mathcal{F}_R^{(+)}$. Denote by $L(x, y_0)$ ($(x, y_0) \in S^1 \times S^1$) the leaf of $\mathcal{F}_R^{(+)}$ containing $\varphi(\bar{\varepsilon}(x), f(x, y_0))$. Then there exists a unique C^∞ function

$$\gamma_x: S^1 \rightarrow]0, 1[$$

such that

$$\gamma_x(y_0) = \bar{\varepsilon}(x), \quad \varphi(\gamma_x(y), f(x, y)) \in L(x, y_0)$$

and that $\bigcup_{y \in S^1} \varphi(\gamma_x(y), f(x, y))$ is a simple closed curve in $L(x, y_0)$. Now we define

$$A = S^1 \times D^2 - \{\varphi(t, f(x, y)); 0 \leq t < \gamma_x(y), (x, y) \in S^1 \times S^1\}.$$

Then A is a 3-dimensional C^∞ manifold diffeomorphic to $S^1 \times D^2$, and $A \cap L$ is a closed 2-disk for each non-compact leaf L of $\mathcal{F}_R^{(+)}$. Let $\overline{\mathcal{F}''} = \{\partial A \cap L'; L' \in \mathcal{F}'\}$, then $\overline{\mathcal{F}''}$ is a C^∞ foliation of codimension one of ∂A . The C^∞ diffeomorphism $g: S^1 \times S^1 \rightarrow \partial A$ which maps (x, y) to $\varphi(\gamma_x(y), f(x, y))$ gives an isomorphism from $\overline{\mathcal{F}}$ to $\overline{\mathcal{F}''}$.

Denote by $A^{(x)}$ the intersection $A \cap L_x$, where L_x is the leaf of $\mathcal{F}_R^{(+)}$ containing $(x, 0) \in S^1 \times D^2$. By the construction above, $\partial A^{(x)}$ is a simple closed curve intersecting $\overline{\mathcal{F}''}$ transversely except at $a(p+q)$ points $\partial A^{(x)} \cap g(\widehat{\Sigma}^{(+)} \cup \widehat{\Sigma}^{(-)})$.

Obviously there exists a C^∞ diffeomorphism from A to $S^1 \times D^2$ which maps $A^{(x)}$ to $\{x\} \times D^2$. Thus, making use of the identification by this diffeomorphism, we may assume that

$$A = S^1 \times D^2, \quad A^{(x)} = \{x\} \times D^2,$$

and that the plus and the minus Reeb components in $\overline{\mathcal{F}}''$ of $\partial A = S^1 \times S^1$ are standard (as in Fig. 2). So we use the same notations $\widehat{\Sigma}^{(+)}, \widehat{\Sigma}^{(-)}$ for $\overline{\mathcal{F}}''$ as for $\overline{\mathcal{F}}$.

The intersection $A^{(x)} \cap L'$ ($L' \in \mathcal{F}'$) defines a family of C^∞ simple curves $\{\ell_\lambda^{(x)}\}_{\lambda \in \Lambda(x)}$ of $A^{(x)}$, where we understand that $\ell_\lambda^{(x)}$ is a closed set of $A^{(x)}$ and $\ell_\lambda^{(x)} \cap \text{Int } A^{(x)}$ is connected. We note that there exists a C^∞ vector field on the manifold (with corner) obtained by cutting $S^1 \times D^2$ at $\{x_0\} \times D^2$ such that integral curves are $\bigcup_x \{\ell_\lambda^{(x)}\}_{\lambda \in \Lambda(x)}$.

Lemma 1. — (i) $\ell_\lambda^{(x)}$ is tangent to $\partial A^{(x)}$ at (x, y) if and only if $y \in \partial A^{(x)} \cap \widehat{\Sigma}^{(-)}$, $y \in \ell_\lambda^{(x)}$.

(ii) $\ell_\lambda^{(x)}$ is reduced to a point at $(x, y) \in \partial A^{(x)} \cap \widehat{\Sigma}^{(+)}$.

(iii) $\{\ell_\lambda^{(x)}\}_{\lambda \in \Lambda(x)}$ forms a family of concentric half circles with (x, y) as center near $(x, y) \in \partial A^{(x)} \cap \widehat{\Sigma}^{(+)}$ and upper part of a family of confocal parabolas with (x, y) as focus near $(x, y) \in \partial A^{(x)} \cap \widehat{\Sigma}^{(-)}$ (Fig. 4 and 5).

This lemma is clear, because the situation of $\{\ell_\lambda^{(x)}\}_{\lambda \in \Lambda}$ near $(x, y) \in \partial A^{(x)} \cap \widehat{\Sigma}^{(+)}$ (resp. $(x, y) \in \partial A^{(x)} \cap \widehat{\Sigma}^{(-)}$) is similar as the situation of leaves of the plus (resp. minus) Reeb component $\overline{\mathcal{F}}_R^{(+)}$ (resp. $\overline{\mathcal{F}}_R^{(-)}$) near a point of $\widehat{\Sigma}^{(+)}$ (resp. $\widehat{\Sigma}^{(-)}$).

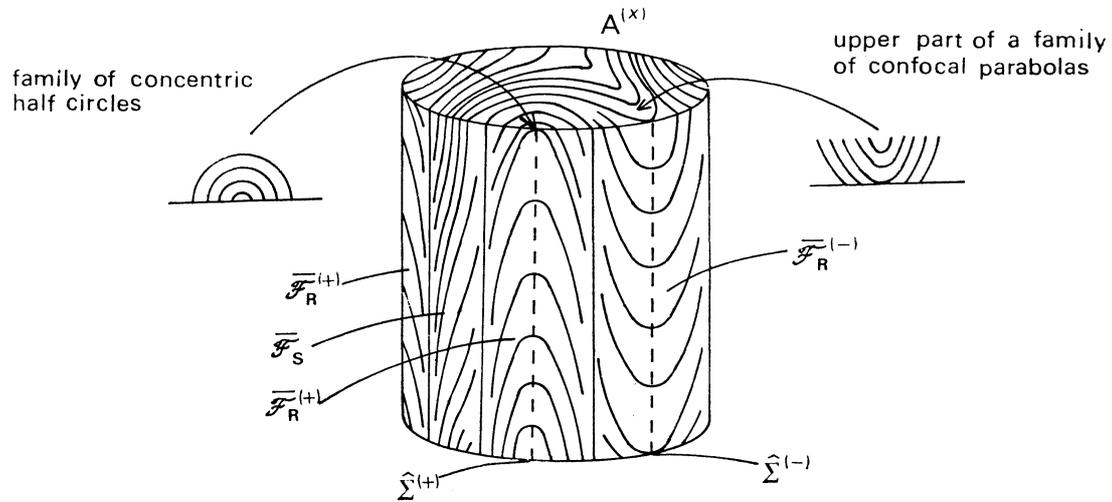


FIG. 5

For $y \in \partial D^2$, let $\ell^{(x)}(y)$ denote a simple curve of $\{\ell_\lambda^{(x)}\}_{\lambda \in \Lambda(x)}$ containing (x, y) . If $(x, y) \notin \widehat{\Sigma}^{(+)} \cup \widehat{\Sigma}^{(-)}$, $\ell^{(x)}(y)$ exists and is unique, and if $(x, y) \in \widehat{\Sigma}^{(-)}$ there exist two kinds of $\ell^{(x)}(y)$, say $\ell_1^{(x)}(y)$ and $\ell_2^{(x)}(y)$. The following lemma is an immediate consequence of the Poincaré-Bendixson theorem:

Lemma 2. — For $(x, y) \notin \widehat{\Sigma}^{(+)}$, the simple curve $\ell^{(2)}(y)$ (resp. $\ell_1^{(2)}(y)$) ($i = 1, 2$) intersects $\partial A^{(2)}$ at exactly two points.

We denote by $[\ell^{(2)}(y)]$ the intersection point different from (x, y) .

Let $X = \{X(z); z \in A\}$ be a C^∞ vector field on $A = S^1 \times D^2$ satisfying the following conditions:

- (i) $X(z) \neq 0$ if $z \in A - (\widehat{\Sigma}^{(+)} \cup \widehat{\Sigma}^{(-)})$; $X(z) = 0$ if $z \in \widehat{\Sigma}^{(+)} \cup \widehat{\Sigma}^{(-)}$.
- (ii) $X(z)$ ($z \notin \widehat{\Sigma}^{(+)} \cup \widehat{\Sigma}^{(-)}$, $z = (x, y) \in S^1 \times D^2$) is transverse to $A^{(2)} = \{x\} \times D^2$ and lies in the positive direction of S^1 .
- (iii) X is tangent to \mathcal{F}' and hyperbolic at each point of $\widehat{\Sigma}^{(-)}$.
- (iv) $X|_{\partial A}$ is tangent to $\partial A = S^1 \times \partial D^2$.

The existence of such a C^∞ vector field X is obvious.

Let $\tilde{\pi}: \mathbf{R} \times D^2 \rightarrow S^1 \times D^2$ denote the covering map such that $\tilde{\pi}^{-1}(\{*\} \times D^2) = \mathbf{Z} \times D^2$ and $\tilde{\pi}|_{(\mathbf{R} \times \{**\})}: \mathbf{R} \rightarrow S^1$ is orientation-preserving with respect to the natural orientation of \mathbf{R} . Let $\tilde{X} = \{\tilde{X}(\tilde{z}); \tilde{z} \in \mathbf{R} \times D^2\}$ be the C^∞ vector field on $\mathbf{R} \times D^2$ such that $\tilde{\pi}_* (\tilde{X}(\tilde{z})) = X(\tilde{\pi}(\tilde{z}))$, and $\tilde{\varphi}(t, \tilde{z})$ denote the integral curve of \tilde{X} with the initial condition $\tilde{\varphi}(0, \tilde{z}) = \tilde{z}$ (for $\tilde{z} \in \mathbf{R} \times D^2$).

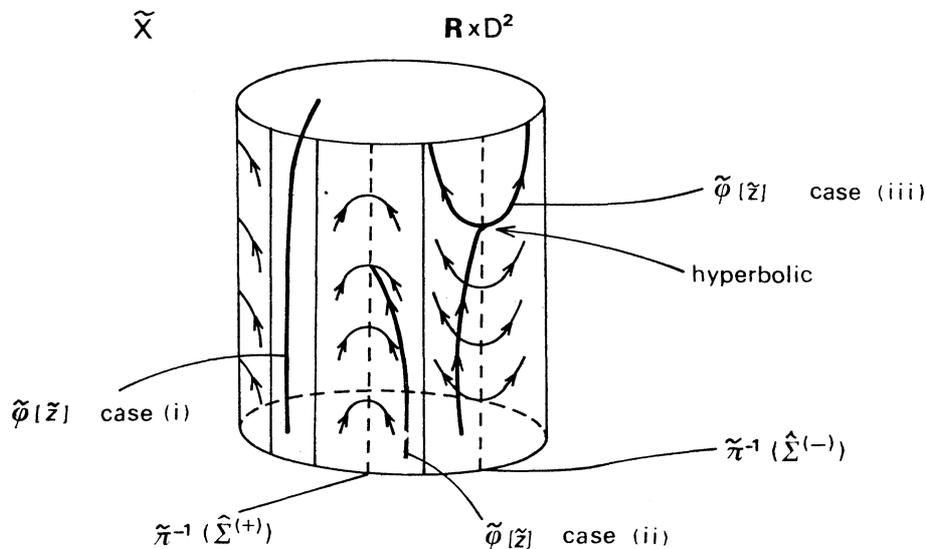


FIG. 6

For $\tilde{z} \in \mathbf{R} \times D^2 - \tilde{\pi}^{-1}(\widehat{\Sigma}^{(+)} \cup \widehat{\Sigma}^{(-)})$, we define a subset $\tilde{\varphi}[\tilde{z}]$ of $\mathbf{R} \times D^2$ as follows (Fig. 6):

- (i) if $\tilde{\varphi}(t, \tilde{z})$ does not approach to a point of $\tilde{\pi}^{-1}(\widehat{\Sigma}^{(+)} \cup \widehat{\Sigma}^{(-)})$ for $t \geq 0$, we define $\tilde{\varphi}[\tilde{z}] = \{\tilde{\varphi}(t, \tilde{z}); 0 \leq t < \infty\}$;

(ii) if $\tilde{\varphi}(t, \tilde{z})$ approaches to a point of $\tilde{\pi}^{-1}(\hat{\Sigma}^{(+)})$ for $t \geq 0$, we define

$$\tilde{\varphi}[\tilde{z}] = \{\tilde{\varphi}(t, \tilde{z}); 0 \leq t < \infty\} \cup \{\lim_{t \rightarrow \infty} \tilde{\varphi}(t, \tilde{z})\};$$

(iii) if $\tilde{\varphi}(t, \tilde{z})$ approaches to a point of $\pi^{-1}(\hat{\Sigma}^{(-)})$ for $t \geq 0$, we define

$$\tilde{\varphi}[\tilde{z}] = \{\tilde{\varphi}(t, \tilde{z}); 0 \leq t < \infty\} \cup \{\lim_{t \rightarrow \infty} \tilde{\varphi}(t, \tilde{z})\} \cup \{\tilde{z}'; \lim_{t \rightarrow -\infty} \tilde{\varphi}(t, \tilde{z}') = \lim_{t \rightarrow \infty} \tilde{\varphi}(t, \tilde{z})\}.$$

For $s \geq 0$, $z \in A = S^1 \times D^2$, we define a subset $\Phi_s(z)$ of A (possibly $\Phi_s(z) = \emptyset$) by

$$\Phi_s(z) = \tilde{\pi}(\tilde{\varphi}[\tilde{z}]) \cap (\{\tilde{x} + s\} \times D^2),$$

where $\tilde{\pi}(\tilde{z}) = z$ and $\tilde{z} \in \{\tilde{x}\} \times D^2$. $\Phi_s(z)$ consists of one or two points unless $\Phi_s(z) = \emptyset$.

If $\Phi_s(z) \subset \hat{\Sigma}^{(+)}$ for $z \in \ell^{(x)}(y)$, then, by Lemma 1, (iii), we have

$$\Phi_s(x, y) = \Phi_s([\ell^{(x)}(y)]) = \Phi_s(z) \quad (s > 0),$$

which implies that y and $[\ell^{(x)}(y)]$ belong to the interior of the same plus Reeb component. Thus, if one of the points $\ell_\lambda^{(x)} \cap \partial A^{(x)}$ is not contained in the interior of a plus Reeb component in $\overline{\mathcal{F}}''$, then we have

$$\Phi_s(z) \cap \hat{\Sigma}^{(+)} = \emptyset \quad (z \in \ell_\lambda^{(x)}, s \geq 0),$$

that is, $\Phi_s(z) \neq \emptyset$ for $z \in \ell_\lambda^{(x)}$, $s \geq 0$.

The image $\bigcup_{z \in \ell_\lambda^{(x)}} \Phi_s(z)$ of $\ell_\lambda^{(x)}$ with respect to Φ_s bifurcates at $\Phi_s(z)$ if and only if $\Phi_s(z) \subset A^{(\tilde{\pi}(\tilde{x} + s'))} \cap \hat{\Sigma}^{(-)}$ (Fig. 7). Thus, in general, the image of $\ell_\lambda^{(x)}$ with respect to Φ_s consists of a finite number of simple curves of $\{\ell_\lambda^{(\tilde{\pi}(\tilde{x} + s))}\}_{\lambda \in \Lambda(\tilde{\pi}(\tilde{x} + s))}$, because the number of values $s' \in [0, s]$ at which Φ_s bifurcates are finite (Fig. 7).

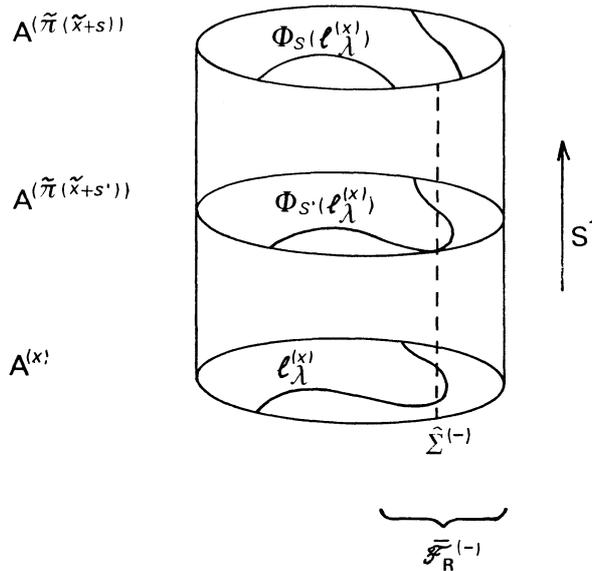


FIG. 7

Lemma 3. — Let \hat{L} be a compact leaf of $\overline{\mathcal{F}}''$ of ∂A and let $(x, \hat{y}) \in \hat{L} \cap \partial A^{(x)}$, then $[\ell^{(x)}(\hat{y})]$ is an intersection point of a compact leaf of $\overline{\mathcal{F}}''$ and $\partial A^{(x)}$.

Proof. — Let us consider the case where the longitudinal number a of $\overline{\mathcal{F}}''$ is 1. Thus $\{\ell_\lambda^{(x)}\}_{\lambda \in \Lambda(x)}$ is a family of integral curves of a C^∞ vector field on A .

First assume that the image of $\ell^{(x)}(\hat{y})$ with respect to Φ_s does not bifurcate for $0 \leq s \leq 1$. Then $\Phi_s(z)$ moves continuously for $0 \leq s \leq 1$, $z \in \ell^{(x)}(\hat{y})$. Since \hat{y} is a point of a compact leaf \hat{L} , we have $\Phi_1(x, \hat{y}) = \{(x, \hat{y})\}$. Thus, by the uniqueness of $\ell^{(x)}(\hat{y})$, the image of $\ell^{(x)}(\hat{y})$ with respect to Φ_1 agrees with itself:

$$\Phi_1([\ell^{(x)}(\hat{y})]) = \{[\ell^{(x)}(\hat{y})]\}.$$

This shows that $[\ell^{(x)}(\hat{y})]$ is contained in a compact leaf of $\overline{\mathcal{F}}''$.

Suppose that the image of $\ell^{(x)}(\hat{y})$ with respect to Φ_s bifurcates at a finite number of values of $0 \leq s \leq 1$, and the image of $\ell^{(x)}(\hat{y})$ with respect to Φ_1 is given by

$$\ell^{(x)}(\hat{y}_0) \cup \ell^{(x)}(\hat{y}_1) \cup \dots \cup \ell^{(x)}(\hat{y}_m),$$

where $\hat{y}_0 = \hat{y}$ and $\ell^{(x)}(\hat{y}_i)$ ($i = 0, 1, \dots, m$) are simple curves in $\{\ell_\lambda^{(x)}\}_{\lambda \in \Lambda(x)}$ such that $[\ell^{(x)}(\hat{y}_i)]$ and (x, \hat{y}_{i+1}) belong to the interior of the same minus Reeb component in $\overline{\mathcal{F}}''$ ($i = 0, 1, \dots, m-1$) (Fig. 7). Assume that $m \geq 1$. Then, $[\ell^{(x)}(\hat{y})] = [\ell^{(x)}(\hat{y}_0)]$ should belong to the interior of a minus Reeb component in $\overline{\mathcal{F}}''$. However, according to properties of the minus Reeb component, it is easy to see that $\{[\ell^{(x)}(\hat{y}_m)]\} = \Phi_1([\ell^{(x)}(\hat{y})])$ and $[\ell^{(x)}(\hat{y}_1)]$ belong to different connected components of $A^{(x)} - \ell^{(x)}(\hat{y}_0)$. On the other hand, $[\ell^{(x)}(\hat{y}_1)]$ and $[\ell^{(x)}(\hat{y}_m)]$ should be connected by a connected continuous curve in $A^{(x)}$ oriented by the following order

$$\bar{\ell}_1 \cup \ell^{(x)}(\hat{y}_2) \cup \bar{\ell}_2 \cup \ell^{(x)}(\hat{y}_3) \cup \dots \cup \bar{\ell}_{m-1} \cup \ell^{(x)}(\hat{y}_m)$$

such that $\bar{\ell}_i$ is contained in the interior of a minus Reeb component ($i = 1, 2, \dots, m-1$), and that, if $\bar{\ell}_i$ is contained in the minus Reeb component to which $\ell^{(x)}(\hat{y}_m)$ belongs, the orientation of $\bar{\ell}_i$ is consistent to $\overrightarrow{[\ell^{(x)}(\hat{y}_0)](x, \hat{y}_1)}$. This is a contradiction. Therefore $\ell^{(x)}(\hat{y})$ does not bifurcate for $0 \leq s \leq 1$. Thus this lemma is proved in case $a = 1$.

In case $a = 2$, the same arguments hold by considering the double covering of $A = S^1 \times D^2$. Thus Lemma 3 is proved. (See also [9; p. 61].)

Lemma 4. — Let \hat{L} be a compact leaf of $\overline{\mathcal{F}}''$ of $\partial A = S^1 \times \partial D^2$ and let L be the leaf of \mathcal{F}' containing \hat{L} . Then $L \cap A$ is compact, and it is an annulus in case $a = 1$ and is an annulus or a Möbius band in case $a = 2$, where a is the longitudinal number of $\overline{\mathcal{F}}''$. $L \cap A^{(x)}$ consists of a simple arc in case $a = 1$ and of one or two simple arcs in case $a = 2$.

Proof. — According to Lemma 3, it is easy to see that there exists a diffeomorphism from $\hat{L} \times I$ or Möbius band $\hat{L} \times I / \mathbf{Z}_2$ to $L \cap A$.

Lemma 5. — Let $\overline{\mathcal{F}}_R^{(+)}$ be a plus Reeb component in $\overline{\mathcal{F}}''$ of $\partial A = S^1 \times \partial D^2$ and let $|\overline{\mathcal{F}}_R^{(+)}|$ denote the underlying submanifold of $\overline{\mathcal{F}}_R^{(+)}$ in ∂A . Denote by \hat{L}, \hat{L}' the compact leaves of

$\mathcal{F}_R^{(+)}: \partial|\mathcal{F}_R^{(+)}| = \widehat{L} \cup \widehat{L}'$, where it may happen that $\widehat{L} = \widehat{L}'$ in case the longitudinal number of \mathcal{F}'' is 2. Then, for $\{c\} \in \widehat{L} \cap \partial A^{(x)}$, we have

$$[\ell^{(x)}(c)] \in \widehat{L}' \cap \partial A^{(x)}.$$

Proof. — Let $\sigma \in |\mathcal{F}_R^{(+)}| \cap \partial A^{(x)} \cap \widehat{\Sigma}^{(+)}$. We denote $\widehat{L}' \cap \partial A^{(x)}$ by c' . We fix an orientation on $\partial A^{(x)}$ so that the oriented arcs $\widehat{c'\sigma}$, $\widehat{\sigma c}$ have the orientation compatible with that of $\partial A^{(x)}$, where $\widehat{c'\sigma} \cap \widehat{\sigma c} = \{\sigma\}$ (Fig. 8).

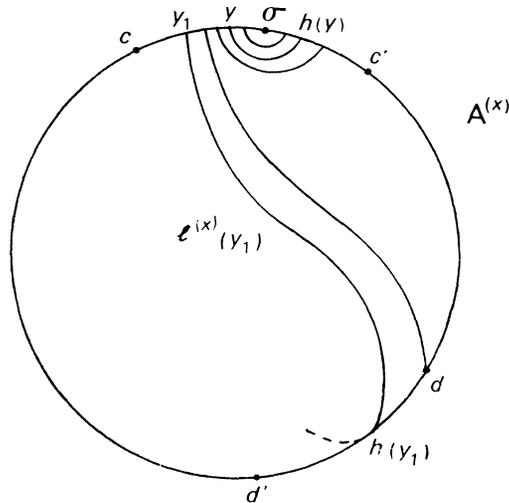


FIG. 8

Let $h: \widehat{\sigma c} \rightarrow \partial A^{(x)}$ be the map (not necessary continuous) defined by

$$h(y) = [\ell^{(x)}(y)] \quad (y \in \widehat{\sigma c}).$$

Then, by Lemma 1, (iii), there exists a neighborhood U_0 of σ in $\partial A^{(x)}$ such that h is continuous on $U_0 \cap \text{Int } \widehat{\sigma c}$.

Assume that there exists a point $y_1 \in \widehat{\sigma c}$ such that h is continuous on $\widehat{\sigma y_1} - \{y_1\}$ and is not continuous at y_1 . This is equivalent to that $\ell^{(x)}(y)$ intersects $\partial A^{(x)}$ transversely at $h(y) = [\ell^{(x)}(y)]$ for $y \in \widehat{\sigma y_1} - \{y_1\}$ and $\ell^{(x)}(y_1)$ is tangent to $\partial A^{(x)}$ at $h(y_1) = [\ell^{(x)}(y_1)]$ (Fig. 8, g). Thus we have $h(y_1) \in \widehat{\Sigma}^{(-)}$.

Denote by $\mathcal{F}_R^{(-)}$ the minus Reeb component such that $h(y_1) \in |\mathcal{F}_R^{(-)}|$, where $|\mathcal{F}_R^{(-)}|$ is the underlying submanifold of $\mathcal{F}_R^{(-)}$. Let d, d' denote the boundary points $\partial(|\mathcal{F}_R^{(-)}| \cap A^{(x)})$ such that oriented arcs $\widehat{d'h(y_1)}$, $\widehat{h(y_1)d}$ contained in $|\mathcal{F}_R^{(-)}| \cap A^{(x)}$ have the orientation compatible with that of $\partial A^{(x)}$ (Fig. 8).

Suppose that $\ell^{(x)}(y_1)$ is tangent to $\partial A^{(x)}$ at $h(y_1)$ in the inverse direction of $\partial A^{(x)}$, then, it is easy to see that

$$h|_{\widehat{\sigma y_1}}: \widehat{\sigma y_1} \rightarrow \widehat{h(y_1)\sigma}$$

is an onto homeomorphism (Fig. 8). Thus $h^{-1}(d)$ exists in $\widehat{\text{Int } \sigma y_1}$ which should be contained in a compact leaf of $\overline{\mathcal{F}}''$ by Lemma 3. This is a contradiction. Therefore $\ell^{(x)}(y_1)$ is tangent to $\partial A^{(x)}$ at $h(y_1)$ in the direction of $\partial A^{(x)}$ (Fig. 9).

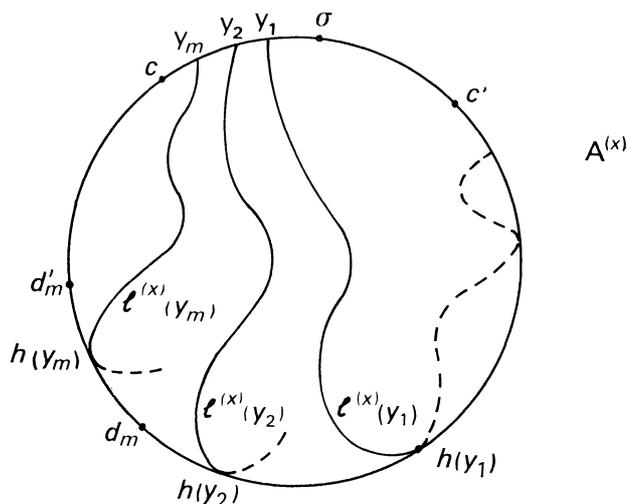


FIG. 9

Thus there exists a neighborhood U_1 of y_1 in $\partial A^{(x)}$ such that h is continuous on $U_1 \cap \widehat{\text{Int } y_1 c}$. If h is not continuous at a point of $\widehat{y_1 c}$, then there exists $y_2 \in \widehat{\text{Int } y_1 c}$ such that h is continuous on $\widehat{\text{Int } y_1 y_2}$ and is not continuous at y_2 . By the same argument used above, $\ell^{(x)}(y_2)$ is tangent to $\partial A^{(x)}$ at $h(y_2)$ in the direction of $\partial A^{(x)}$ (Fig. 9).

Since the number of minus Reeb components is finite, by repeating this process, there are a finite number of points y_1, y_2, \dots, y_m of $\widehat{\text{Int } \sigma c}$ situated in this order such that h is continuous on $\widehat{\sigma c} - \{\sigma\} - \bigcup_{i=1}^m y_i$ and discontinuous at y_i ($i=1, 2, \dots, m$) and that $\ell^{(x)}(y_i)$ is tangent to $\partial A^{(x)}$ at $h(y_i) \in \widehat{\Sigma}^{(-)}$ in the direction of $\partial A^{(x)}$ (Fig. 9). Suppose that $h(y_m)$ is contained in a minus Reeb component $\overline{\mathcal{F}}_R^{(-)}$ and let $d'_m d_m$ be the arc $|\overline{\mathcal{F}}_R^{(-)}| \cap A^{(x)}$ having the orientation compatible with that of $\partial A^{(x)}$:

$$h(y_m) \in \widehat{d'_m d_m}.$$

Then h maps $\widehat{y_m c}$ into $\widehat{ch(y_m) - \{c\}}$. If $h(\widehat{y_m c}) \subset \widehat{d'_m h(y_m) - \{d'_m\}}$, then $\widehat{d'_m h(y_m) - \{d'_m\}}$ must contain the point $h(c)$ of a compact leaf of $\overline{\mathcal{F}}''$ by Lemma 3. This is a contradiction. Further, if $h(\widehat{\text{Int } y_m c}) \supset \widehat{d'_m h(y_m)}$, then $\widehat{\text{Int } y_m c}$ must contain the point $h^{-1}(d'_m)$ of a compact leaf of $\overline{\mathcal{F}}''$ by Lemma 3. This is also a contradiction. Thus $h(c) = d'_m$ holds. This implies that c and d'_m lie on a compact leaf L of \mathcal{F}' by Lemma 4.

However, since c is a point of a compact leaf of the boundary of a plus Reeb

component, L has a contracting holonomy in the negative direction of S^1 in this side, and, on the other hand, since d'_m is a point of a compact leaf of the boundary of a minus Reeb component, L has a contracting holonomy in the direction of S^1 in the same side. This is a contradiction. Therefore there exists no discontinuous point of h on $\widehat{\sigma c}$ and

$$h|(\widehat{\sigma c} - \{\sigma\}) : \widehat{\sigma c} - \{\sigma\} \rightarrow \widehat{h(c)\sigma} - \{\sigma\}$$

is a C^∞ diffeomorphism. The point $h(c)$ must belong to a compact leaf of $\overline{\mathcal{F}}''$ by Lemma 3. Thus, making use of the same argument as above, we have

$$h(c) = c'.$$

This completes the proof of Lemma 5.

Proposition 3. — (i) Let \widehat{L} be a compact leaf of $\overline{\mathcal{F}}'$ of $\partial(S^1 \times D^2)$ and let L be the leaf of \mathcal{F}' containing \widehat{L} . Then L is compact and an annulus in case $a=1$ and an annulus or a Möbius band in case $a=2$ such that $\partial L = L \cap \partial(S^1 \times D^2)$ consists of two compact leaves of $\overline{\mathcal{F}}'$ in case $a=1$ and of one or two compact leaves of $\overline{\mathcal{F}}'$ in case $a=2$, where a is the longitudinal number of $\overline{\mathcal{F}}'$.

(ii) For a plus Reeb component $\overline{\mathcal{F}}_R^{(+)}$ in $\overline{\mathcal{F}}'$, there exists a plus half Reeb component $\mathcal{F}_{R/2}^{(+)}$ in \mathcal{F}' such that $\overline{\mathcal{F}}_R^{(+)}$ is the restriction of $\mathcal{F}_{R/2}^{(+)}$ to $|\overline{\mathcal{F}}_R^{(+)}|$, where it may happen that the compact leaf of $\mathcal{F}_{R/2}^{(+)}$ forms a Möbius band in $\overline{\mathcal{F}}'$ identified by a free \mathbf{Z}_2 action in case $a=2$. Let $A = \bigcup_x A^{(x)}$ be as in Section 2, then $\{A^{(x)} \cap L'; L' \in \overline{\mathcal{F}}_{R/2}^{(+)}\}$ consists of concentric half circles.

Proof. — There is a natural isomorphism from $\overline{\mathcal{F}}'$ to $\overline{\mathcal{F}}''$ of ∂A and the compact leaves corresponding by this isomorphism are the boundary of an annulus which is the restriction of a leaf of \mathcal{F}' to $S^1 \times D^2 - \text{Int } A$. Thus the first part of (i) is an immediate consequence of Lemma 4. For a compact leaf \widehat{L} of the boundary of $|\overline{\mathcal{F}}_R^{(+)}|$, there exists a compact leaf L containing \widehat{L} as above. According to Lemma 5, ∂L consists of the two compact leaves of $\overline{\mathcal{F}}_R^{(+)}$ in case $a=1$ and of one or two compact leaves in case $a=2$. Thus the second part of (i) is proved.

Now we prove (ii). Let L be the compact leaf of \mathcal{F}' containing a compact leaf of $\partial|\overline{\mathcal{F}}_R^{(+)}|$. Assume L is annular. Let R denote the closure of a connected component of $S^1 \times D^2 - L$ which contains $\text{Int}|\overline{\mathcal{F}}_R^{(+)}|$. Since, as was shown in the proof of Lemma 5, $R \cap A^{(x)}$ consists of concentric half circles, $\mathcal{F}'|R$ is a plus half Reeb component. Thus Proposition 3 is proved. In case L is a Möbius band, the same arguments hold by considering the double covering of $S^1 \times D^2$.

4. TS components

First we prove the following lemma.

Lemma 6. — Let \widehat{L} be a compact leaf of $\overline{\mathcal{F}}'$ which is a boundary of a minus Reeb component $\overline{\mathcal{F}}_R^{(-)}$ or a slope component $\overline{\mathcal{F}}_S$ and let L be the compact leaf of \mathcal{F}' containing \widehat{L} (Propo-

sition 3, (i)). Let B denote the closure of a connected component of $S^1 \times \partial D^2 - \partial L$ which contains $\text{Int}|\overline{\mathcal{F}}_R^{(-)}|$ or $\text{Int}|\overline{\mathcal{F}}_S|$, where $|\overline{\mathcal{F}}_R^{(-)}|$, $|\overline{\mathcal{F}}_S|$ denote underlying submanifolds. Then $\hat{L}' = \partial L - \hat{L}$ is a compact leaf of \mathcal{F}' which is a boundary of a minus Reeb component or a slope component contained in B , unless $\partial L = \hat{L}$.

Proof. — First assume that the longitudinal number a is 1. Suppose that there exists a family of compact leaves $\{\hat{L}'_i\}$ of $\mathcal{F}'|B$ which accumulates to \hat{L}' . Then, by Proposition 3, (i), we have a family of compact leaves $\{L_i\}$ of \mathcal{F}' such that

$$\partial L_i = \hat{L}_i \cup \hat{L}'_i = L_i \cap (S^1 \times \partial D^2).$$

Thus $\{\hat{L}_i\}$ accumulates to \hat{L} which contradicts the assumption on \hat{L} . Thus \hat{L}' is a boundary of a plus or minus Reeb component, or of a slope component. But \hat{L}' cannot be a boundary of a plus Reeb component by Proposition 3, (ii). In case $a = 2$, the same arguments hold by considering the double covering of $S^1 \times D^2$. Note that it may happen that $\partial L = \hat{L}$ in this case. Thus this lemma is proved.

In the following $\overline{\mathcal{F}}_i$ denotes a minus Reeb component or a slope component contained in \mathcal{F}' . $\overline{\mathcal{F}}_1$ and $\overline{\mathcal{F}}_2$ are called to be *connected by a compact leaf* L , denoted by $\overline{\mathcal{F}}_1 \underset{L}{\sim} \overline{\mathcal{F}}_2$, if there exists a compact annular leaf (resp. a Möbius band in case $a = 2$) L of \mathcal{F}' with $\partial L = \bar{L} \cup \bar{L}'$ (resp. $\partial L = \bar{L}$) such that $\bar{L} \subset |\overline{\mathcal{F}}_1|$, $\bar{L}' \subset |\overline{\mathcal{F}}_2|$ (resp. $\bar{L} \subset |\overline{\mathcal{F}}_1|$, $\bar{L} \subset |\overline{\mathcal{F}}_2|$) and that $\text{Int}|\overline{\mathcal{F}}_1|$ and $\text{Int}|\overline{\mathcal{F}}_2|$ are contained in the same connected component of $S^1 \times \partial D^2 - \bar{L} - \bar{L}'$ (resp. $S^1 \times \partial D^2 - \bar{L}$) (Fig. 10).

Further, $\overline{\mathcal{F}}_0$ and $\overline{\mathcal{F}}_m$ are called to be *connected* if there exists a sequence $\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_2, \dots, \overline{\mathcal{F}}_{m-1}$ such that

$$\overline{\mathcal{F}}_i \underset{L_i}{\sim} \overline{\mathcal{F}}_{i+1} \quad (i = 0, 1, \dots, m-1)$$

for some compact leaves L_i ($i = 0, 1, \dots, m-1$) of \mathcal{F}' .

By Lemma 6, q copies of the minus Reeb components in $\overline{\mathcal{F}'}$ are divided into connected components.

Lemma 7. — Let $\mathcal{C} = \{\overline{\mathcal{F}}_j^{(-)}; j = 1, 2, \dots, m\}$ be a connected component of q copies of the minus Reeb component in $\overline{\mathcal{F}'}$. Then there exist two slope components $\overline{\mathcal{F}}_S^{(1)}, \overline{\mathcal{F}}_S^{(2)}$ in $\overline{\mathcal{F}}^{(1)}$ such that $\{\overline{\mathcal{F}}_j^{(-)}; j = 1, 2, \dots, m\} \cup \{\overline{\mathcal{F}}_S^{(1)}, \overline{\mathcal{F}}_S^{(2)}\}$ is a connected component of the set of q copies of the minus Reeb component and slope components in $\overline{\mathcal{F}'}$. Further, $\overline{\mathcal{F}}_S^{(1)}$ and $\overline{\mathcal{F}}_S^{(2)}$ are connected by a compact leaf.

Proof. — First we assume that the longitudinal number a is 1. Let

$$\{\overline{\mathcal{F}}_j^{(-)}; j = 1, 2, \dots, m, m+1, \dots, m'\} \cup \{\overline{\mathcal{F}}_S^{(\delta)}; \delta \in \Delta\}$$

be a connected component of the set of q copies of the minus Reeb component and slope components in $\overline{\mathcal{F}'}$ containing \mathcal{C} . Let L be an arbitrary compact leaf of \mathcal{F}' , then

$L \cap A^{(x)}$ is a simple curve in $A^{(x)}$ and $L \cap A^{(x)}$ divides $A^{(x)}$ into two connected components, say $A_1^{(x)}$ and $A_2^{(x)}$. Since the Euler number $\chi(A_i^{(x)})$ is equal to 1 ($i = 1, 2$), we have

$$A_i^{(x)} \cap \widehat{\Sigma}^{(+)} \neq \emptyset \quad (i = 1, 2).$$

Since $\overline{\mathcal{F}'}$ contains only a finite number of plus Reeb components, this observation shows that Δ is a finite set, say $\Delta = \{\delta_i; i = 1, 2, \dots, r\}$.

Let L_k ($k = 0, 1, \dots, r + m' - 1$) be compact leaves which connect

$$\{\overline{\mathcal{F}}_j^{(-)}; j = 1, 2, \dots, m'\} \cup \{\overline{\mathcal{F}}_s^{(\delta_i)}; i = 1, 2, \dots, r\}$$

and let $Q^{(x)}$ denote the closure of a connected component of $A^{(x)} - \bigcup_{k=0}^{r+m'-1} L_k$ intersecting $\text{Int} |\overline{\mathcal{F}}_j^{(-)}|$ and $\text{Int} |\overline{\mathcal{F}}_s^{(\delta_i)}|$. Denote by $\widehat{Q}^{(x)}$ the double of $Q^{(x)}$ obtained by pasting $Q^{(x)} \cap \partial A^{(x)}$. Then the Euler number $\chi(\widehat{Q}^{(x)})$ is equal to $2 - (m' + r)$. On the other hand, a C^∞ vector field on $\widehat{Q}^{(x)}$ introduced naturally by $\{\ell_\lambda^{(x)}\}_{\lambda \in \Lambda^{(x)}}$ is tangent to $\partial \widehat{Q}^{(x)}$ and has exactly m' singular points of index -1 . Thus we have $r = 2$.

Further compact leaves having contracting holonomy in the negative direction of S^1 are only contained in $\overline{\mathcal{F}}_s^{(\delta_1)}$ and $\overline{\mathcal{F}}_s^{(\delta_2)}$. In order to be connected by a compact leaf, compact leaves in $\overline{\mathcal{F}}_j^{(-)}$ or in $\overline{\mathcal{F}}_s^{(\delta_i)}$ should have the same holonomy. Therefore $\overline{\mathcal{F}}_s^{(\delta_1)}$ and $\overline{\mathcal{F}}_s^{(\delta_2)}$ should be connected by a compact leaf which implies that $m = m'$.

In case $a = 2$, the same arguments hold by considering the double covering of $S^1 \times D^2$. Thus this lemma is proved.

Let \mathcal{C} be as in Lemma 7, then, by Lemma 7, there exists slope components $\overline{\mathcal{F}}_s^{(1)}$, $\overline{\mathcal{F}}_s^{(2)}$ and compact leaves L_k ($k = 0, 1, 2, \dots, m + 1$) of \mathcal{F}' such that

$$\begin{aligned} \overline{\mathcal{F}}_s^{(1)} &\underbrace{\quad}_{L_1} \overline{\mathcal{F}}_1^{(-)}, & \overline{\mathcal{F}}_i^{(-)} &\underbrace{\quad}_{L_{i+1}} \overline{\mathcal{F}}_{i+1}^{(-)} & (i = 1, 2, \dots, m-1), \\ \overline{\mathcal{F}}_m^{(-)} &\underbrace{\quad}_{L_{m+1}} \overline{\mathcal{F}}_s^{(2)}, & \overline{\mathcal{F}}_s^{(2)} &\underbrace{\quad}_{L_0} \overline{\mathcal{F}}_s^{(1)}. \end{aligned}$$

Let $Q(\mathcal{C})$ or simply Q denote the closure of a connected component of $S^1 \times D^2 - \bigcup_{i=0}^{m+1} L_i$ containing $\text{Int} |\overline{\mathcal{F}}_j^{(-)}|$ ($j = 1, 2, \dots, m$). The C^∞ foliation $\mathcal{F}'|_Q$ of codimension one which is the restriction of \mathcal{F}' to $Q(\mathcal{C})$ is called a *TS component of type m* with respect to \mathcal{C} and denoted by TS_m . We denote by $|\text{TS}_m|$ the underlying submanifold Q of TS_m .

Proposition 4. — *There exists a C^∞ foliation \mathcal{F}' of codimension one of $S^1 \times D^2$ transverse to the plus Reeb component such that \mathcal{F}' contains a TS component of type m .*

Proof. — Figure 5 and Figure 10 show the existence of a TS component of type 1. Similar a TS component of type m exists for any $m \geq 1$.

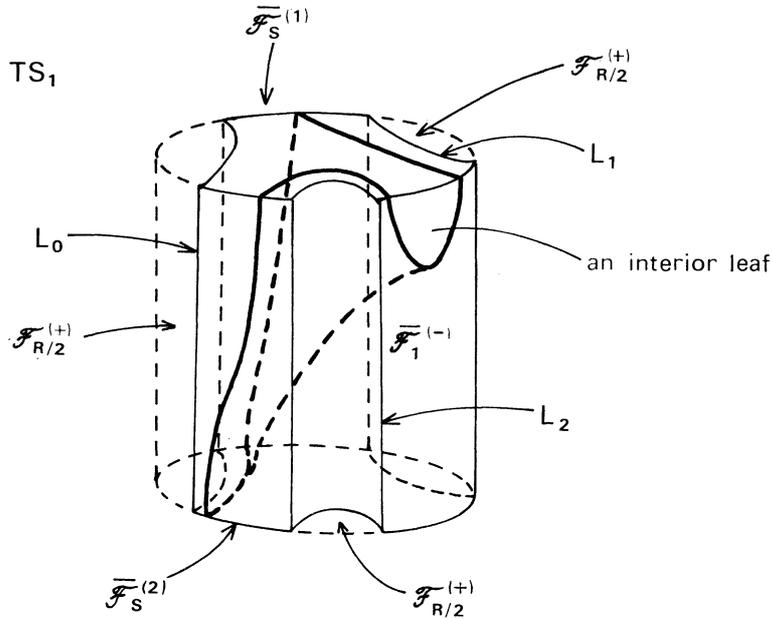


FIG. 10

The following proposition is an immediate consequence of Lemma 7.

Proposition 5. — For a minus Reeb component $\overline{\mathcal{F}}_R^{(-)}$ in $\overline{\mathcal{F}}'$, there exists a TS component TS_m of type m such that $|\text{TS}_m| \supset |\overline{\mathcal{F}}_R^{(-)}|$.

Proposition 6. — The TS component TS_m of type m with respect to

$$\mathcal{C} = \{\overline{\mathcal{F}}_j^{(-)}; j = 1, 2, \dots, m\}$$

has the following properties:

(i) The underlying submanifold $|\text{TS}_m|$ of $S^1 \times D^2$ is a compact connected 3-dimensional C^∞ manifold with corner, where the corner consists of the boundaries of compact leaves L_k ($k = 0, 1, \dots, m + 1$) and is C^∞ diffeomorphic to $S^1 \times D^2$ by straightening the corner. The set $|\text{TS}_m| \cap (S^1 \times \partial D^2)$ consists of minus Reeb components $\overline{\mathcal{F}}_j^{(-)}$ ($j = 1, 2, \dots, m$) and slope components $\overline{\mathcal{F}}_S^{(i)}$ ($i = 1, 2$).

(ii) The intersection $|\text{TS}_m| \cap (\{x\} \times D^2)$ ($x \in S^1$) is a polygon with $2(m + 2)$ vertices (resp. a polygon with $4m + 4$ vertices or two disjoint polygons with $2(m + 2)$ vertices) if the longitudinal number a is 1 (resp. 2).

(iii) The compact leaves in TS_m are exactly L_k ($k = 0, 1, \dots, m + 1$). They are annular in case $a = 1$ and one of them may be a Möbius band in case $|\text{TS}_m| \cap (\{x\} \times D^2)$ is a polygon with $4m + 4$ vertices. The compact leaf L_k has a contracting holonomy in the positive (resp. negative) direction of S^1 if $k = 1, 2, \dots, m + 1$ (resp. $k = 0$).

(iv) Every non-compact leaf L of TS_m meets $\text{Int } |\overline{\mathcal{F}}_j^{(-)}|$ ($j = 1, 2, \dots, m$) and $\text{Int } |\overline{\mathcal{F}}_S^{(1)}|$, $\text{Int } |\overline{\mathcal{F}}_S^{(2)}|$.

(v) Let H denote the subset of \mathbf{R}^2 defined by

$$H = [0, 2m + 1] \times \mathbf{R} \\ - \left\{ (x, y); y - c_k > \frac{1}{(x-k)(k+1-x)}, k < x < k+1, k = 1, 3, \dots, 2m-1 \right\},$$

where c_k is a constant; then every non-compact leaf in TS_m is C^∞ diffeomorphic to H (Fig. 11).

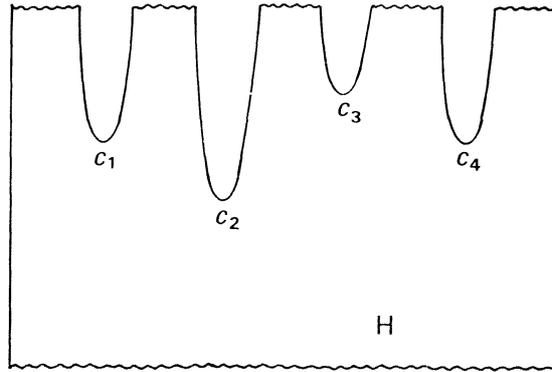


FIG. 11

Proof. — Properties (i), (ii), (iii) are obvious. So we prove (iv), (v) here. A non-compact leaf L of $\mathcal{F}'|Q$ meets $A^{(x)}$ for some $x \in S^1$. Then $L \cap A^{(x)}$ contains a simple curve $\ell_\lambda^{(x)}$ of $\{\ell_\lambda^{(x)}\}_{\lambda \in \Lambda(x)}$. Let (x, y) be an end point of $\ell_\lambda^{(x)}$, then (x, y) belongs to the interior of one of $|\overline{\mathcal{F}}_j^{(-)}|$ ($j = 1, 2, \dots, m$) or $|\overline{\mathcal{F}}_S^{(i)}|$ ($i = 1, 2$) by the identification of ∂A and $S^1 \times \partial D^2$. Assume that $(x, y) \in |\overline{\mathcal{F}}_j^{(-)}|$, then it is easy to see that $L \cap |\overline{\mathcal{F}}_j^{(-)}|$ contains a point (x', y') and (x'', y'') which lie near to L_j and L_{j+1} respectively. Since $\ell^{(x)}(y') \subset L$, $\ell^{(x)}(y'') \subset L$, we have $L \cap |\overline{\mathcal{F}}_{j-1}^{(-)}| \neq \emptyset$, $L \cap |\overline{\mathcal{F}}_{j+1}^{(-)}| \neq \emptyset$. By iterating this process for $|\overline{\mathcal{F}}_j^{(-)}|$ and $|\overline{\mathcal{F}}_S^{(i)}|$, (iv) is proved.

Let \bar{L}' be a non-compact leaf of $\overline{\mathcal{F}}_S^{(1)}$ and let $\bar{L}' \subset L'$ ($L' \in \mathcal{F}'$). For $(x, y) \in \bar{L}'$ ($x \in S^1, y \in \partial D^2$), making use of the identification $S^1 \times D^2 = A$, we consider a simple curve $\ell^{(x)}(y)$ in $A^{(x)}$. If (x, y) is near to L_1 , then $[\ell^{(x)}(y)]$ is a point of $|\overline{\mathcal{F}}_1^{(-)}|$ because $\ell^{(x)}(y)$ lies near to L_1 . Let (x, y_1) denote a point of $|\overline{\mathcal{F}}_1^{(-)}|$ which is symmetric to $[\ell^{(x)}(y)]$ with respect to $\widehat{\Sigma}^{(-)} \cap |\overline{\mathcal{F}}_1^{(-)}|$, then it is obvious that $(x, y_1) \in \bar{L}'$. Thus $\ell^{(x)}(y_1) \subset L'$. Therefore, in general, for a point $(x, y) \in \bar{L}'$, there exists a sequence $\ell^{(x)}(y_0), \ell^{(x)}(y_1), \dots, \ell^{(x)}(y_s)$ of simple curves in $A^{(x)}$ ($s \leq m$) such that

- 1) $(x, y_0) = (x, y) \in |\overline{\mathcal{F}}_S^{(1)}|$, $[\ell^{(x)}(y_s)] \in |\overline{\mathcal{F}}_S^{(2)}|$,
- 2) $\ell^{(x)}(y_k) \subset L'$ ($k = 0, 1, \dots, s$),
- 3) $[\ell^{(x)}(y_k)]$ and (x, y_{k+1}) are points of $|\overline{\mathcal{F}}_{i_k}^{(-)}|$ which are symmetric with respect to $\widehat{\Sigma}^{(-)} \cap |\overline{\mathcal{F}}_{i_k}^{(-)}|$.

Further if (x, y) is sufficiently near to L_0 , then $s=1$ and the above sequence consists of a simple curve $\ell^{(x)}(y)$ such that $[\ell^{(x)}(y)] \in |\overline{\mathcal{F}}_s^{(2)}|$.

By these observations, we can define a C^∞ diffeomorphism f from H onto $L' \cap A$ such that f maps $H \cap ([0, 2m+1] \times \{u\})$ onto $\bigcup_{k=0}^s \ell^{(x)}(y_k)$. Obviously L' is diffeomorphic to H . Thus (v) is proved.

Let TS_m be a TS component of type m with respect to \mathcal{C} as above, $|\widetilde{TS}_m|$ the universal covering of $|TS_m|$ and $\tilde{\pi}: |\widetilde{TS}_m| \rightarrow |TS_m|$ the projection. For the natural projection $p_1: |TS_m| \rightarrow S^1$ which is the restriction of the projection to the first factor $S^1 \times D^2 \rightarrow S^1$, there exist the covering map $\tilde{\pi}'$ and the natural projection \tilde{p} satisfying the following commutative diagram:

$$\begin{array}{ccc} |\widetilde{TS}_m| & \xrightarrow{\tilde{\pi}} & |TS_m| \\ \tilde{p} \downarrow & & \downarrow p_1 \\ \mathbf{R} & \xrightarrow{\tilde{\pi}'} & S^1. \end{array}$$

Denote by $\tilde{\mathcal{F}}'$ the C^∞ foliation of codimension one of $|\widetilde{TS}_m|$ defined by $\{\tilde{\pi}^{-1}(L'); L' \in TS_m\}$. Let $h^{(1)}$ (resp. $h^{(2)}$) be a C^∞ diffeomorphism from the open interval $]0, 1[$ onto a connected component of $\tilde{\pi}^{-1}(\text{Int } |\overline{\mathcal{F}}_s^{(1)}| \cap (\{x\} \times \partial D^2))$ (resp. $\tilde{\pi}^{-1}(\text{Int } |\overline{\mathcal{F}}_s^{(2)}| \cap (\{x\} \times \partial D^2))$), then $\{h^{(1)}(t)\}$ ($0 < t < 1$) is an index set for leaves of $\tilde{\mathcal{F}}' | \tilde{\pi}^{-1}(|\overline{\mathcal{F}}_s^{(1)}|)$. The leaf \tilde{L}_t of $\tilde{\mathcal{F}}'$ containing $h^{(1)}(t)$ intersects $\tilde{\pi}^{-1}(\hat{\Sigma}^{(-)} \cap |\overline{\mathcal{F}}_j^{(-)}|)$ (resp. $\tilde{\pi}^{-1}(|\overline{\mathcal{F}}_s^{(2)}| \cap h^{(2)}(]0, 1[))$ at one point, say $f_j(t)$ (resp. $h^{(2)}(\bar{f}(t))$) for $0 < t < 1$. Then it is easy to see that

$$\begin{aligned} \tilde{p} \circ f_j:]0, 1[&\rightarrow \mathbf{R} \quad (j=1, 2, \dots, m) \\ \bar{f}:]0, 1[&\rightarrow]0, 1[\end{aligned}$$

are C^∞ diffeomorphism. The maps f_j ($j=1, 2, \dots, m$) and \bar{f} are called *lag functions* for TS_m . The lag functions depend on the choice of $\{x\}$ and $h^{(i)}$ ($i=1, 2$).

Now we define a standard TS component of type m . Let P_{2m+4} denote the regular polygon of $2m+4$ vertices and let $\hat{Q}(m)$ be a compact connected orientable 3-dimensional C^∞ manifold with corner obtained from $P_{2m+4} \times I$ by identifying $P_{2m+4} \times \{0\}$ and $P_{2m+4} \times \{1\}$ after twisting of b times, where b is an integer. The boundary $\partial \hat{Q}(m)$ consists of $2m+4$ annuli, say $(S^1 \times I)_i$ ($i=0, 1, \dots, 2m+3$), whose boundaries are corners of $\hat{Q}(m)$.

By the turbulization of $\text{Int } \hat{Q}(m)$ in neighborhoods of $m+1$ annuli $(S^1 \times I)_{2i}$ ($i=1, \dots, m+1$) along $(S^1 \times I)_{2i}$ in the direction of S^1 for $i=1, 2, \dots, m+1$ and in the negative direction of S^1 for $i=0$, a C^∞ foliation $\hat{\mathcal{F}}(m)$ of codimension one of $\hat{Q}(m)$ is constructed. Compact leaves in $\hat{\mathcal{F}}(m)$ are $(S^1 \times I)_{2i}$ ($i=0, 1, \dots, m+1$) and

$\widehat{\mathcal{F}}(m)|(\mathbf{S}^1 \times \mathbf{I})_{2i+1}$ is the minus Reeb component if $i=1, 2, \dots, m$ and the slope component if $i=0, m+1$. The foliation $\widehat{\mathcal{F}}(m)$ is called the *standard TS component of type m* and denoted by $\widehat{\mathbf{TS}}_m$. Clearly the lag functions f_j of $\widehat{\mathbf{TS}}_m$ satisfy

$$f_1 = f_2 = \dots = f_m.$$

5. Classification theorems for foliations transverse to the Reeb component

Theorem 1. — Let \mathcal{F}' be a \mathbf{C}^∞ foliation of codimension one transverse to the plus Reeb component $\mathcal{F}_R^{(+)}$ of $\mathbf{S}^1 \times \mathbf{D}^2$: $\mathcal{F}' \in t_1(\mathbf{S}^1 \times \mathbf{D}^2, \mathcal{F}_R^{(+)})$. Then the following conditions hold:

(i) Let $[\mathbf{L}_{\text{comp}}] = a\alpha + b\beta$ be the homology class of $\mathbf{H}_1(\mathbf{S}^1 \times \partial\mathbf{D}^2; \mathbf{Z})$ represented by a compact leaf \mathbf{L}_{comp} in $\mathcal{F}' = \mathcal{F}'|(\mathbf{S}^1 \times \partial\mathbf{D}^2)$, where $\alpha = [\mathbf{S}^1 \times \{*\}]$ with the given orientation, $\beta = [\{*\} \times \partial\mathbf{D}^2]$ and $a \geq 0$. Then we have $a=1$ or 2 (the number a is the longitudinal number of \mathcal{F}').

(ii) \mathcal{F}' consists of p copies of the plus half Reeb component, s_m copies of the TS component of type m for $m=1, 2, \dots, u$, and a finite number of foliated \mathbf{I} -bundles over $\mathbf{S}^1 \times \mathbf{I}$ in case $a=1$ and over $\mathbf{S}^1 \times \mathbf{I}$ or the Möbius band in case $a=2$ such that

$$p - \sum_{m=1}^u m s_m = 2 \quad \text{if} \quad a=1,$$

$$p - \sum_{m=1}^u m s_m = 1 \quad \text{if} \quad a=2,$$

and that the foliated \mathbf{I} -bundles are trivial \mathbf{I} -bundles in case $a=1$.

Proof. — (i) and a part of (ii) concerning plus and minus half Reeb components and TS components are direct consequences of Proposition 2 and Proposition 5.

Let $(\mathcal{F}_{R/2}^{(+)})_i$ ($i=1, 2, \dots, p$) and $(\mathbf{TS}_m)_j$ ($j=1, 2, \dots, s_m$) be the plus Reeb components and the TS components of type m in \mathcal{F}' respectively, and let

$$\mathbf{M} = \mathbf{S}^1 \times \mathbf{D}^2 - \left(\bigcup_{i=1}^p \text{Int} | (\mathcal{F}_{R/2}^{(+)})_i | \right) - \left(\bigcup_{m=1}^u \left(\bigcup_{j=1}^{s_m} \text{Int} | (\mathbf{TS}_m)_j | \right) \right).$$

Let \mathbf{C} be a connected component of \mathbf{M} , then, by Proposition 3, (ii) and Proposition 5, the family of simple curves formed by the intersection of $\mathbf{A}^{(x)}$ and leaves of $\mathcal{F}'| \mathbf{C}$ are transverse to $\partial\mathbf{A}^{(x)}$. This implies that $\mathcal{F}'| \mathbf{C}$ is a foliated \mathbf{I} -bundle isomorphic to $\overline{\mathcal{F}}'_C \times \mathbf{I}$ or $\widetilde{\overline{\mathcal{F}}}'_C \times \mathbf{I}/\mathbf{Z}_2$, where $\overline{\mathcal{F}}'_C$ denotes the restriction of $\overline{\mathcal{F}}' = \mathcal{F}'| \partial\mathbf{A}$ to one of the connected components of $\mathbf{C} \cap (\mathbf{S}^1 \times \partial\mathbf{D}^2)$ and $\widetilde{\overline{\mathcal{F}}}'_C$ denotes its double covering. Since $\overline{\mathcal{F}}'_C$ is a foliated \mathbf{I} -bundle over \mathbf{S}^1 such that this bundle is trivial if $a=1$ and is trivial or a Möbius band if $a=2$. Thus this theorem is proved.

In order to state the classification theorem for $t_1(\mathcal{F}_R^{(+)})$, we introduce the concept of TS diagram. TS diagrams consist of finite number of smooth simple arcs $\widehat{\ell}$

($i = 1, 2, \dots, r$) in the 2-disc D^2 and symbols $+$, $-$, \times and \parallel on $2r$ arc intervals of ∂D^2 divided by $\widehat{\ell}_i$ ($i = 1, 2, \dots, r$) satisfying the following conditions (Fig. 12, 14):

(i) $\widehat{\ell}_i$ ($i = 1, 2, \dots, r$) are mutually disjoint smooth simple arcs in D^2 intersecting ∂D^2 transversely such that

$$\widehat{\ell}_i \cap \partial D^2 = \partial \widehat{\ell}_i \quad (i = 1, 2, \dots, r).$$

(ii) Let N_i ($i = 1, 2, \dots, r+1$) denote the closures of connected components of $D^2 - \bigcup_{i=1}^r \widehat{\ell}_i$. Then the symbols are given as follows:

- (a) if $N_i \cap \partial D^2$ consists of one connected component, then the symbol $(+)$ is given on this arc interval;
- (b) if $N_i \cap \partial D^2$ consists of two connected components, then the symbol \parallel is given on each arc interval of $N_i \cap \partial D^2$;
- (c) if the number k of connected components of $N_i \cap \partial D^2$ is ≥ 3 , then the symbol $(-)$ is given for $k-2$ arc intervals of $N_i \cap \partial D^2$ and the symbol \times is given for the rest two arc intervals. Further two arc intervals with symbol \times are contained in a connected component of

$$\partial D^2 - [(k-2) \text{ arc intervals with symbol } (-)].$$

(iii) Let p and q denote the numbers of arc intervals having the symbol $(+)$ and $(-)$ respectively. Then $p - q = 2$.

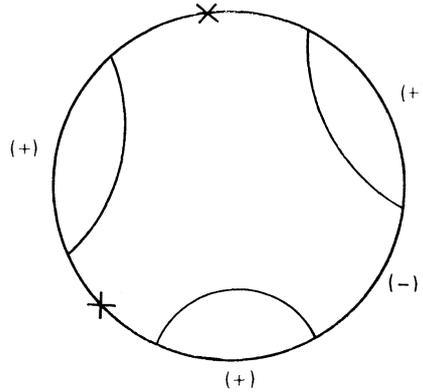


FIG. 12

The TS diagram of Figure 12 corresponds to \mathcal{F}' illustrated in Figure 5.

Two TS diagrams are *isomorphic* if and only if there exists a C^∞ diffeomorphism of D^2 preserving simple arcs $\{\widehat{\ell}_i\}$ and symbols. Let $A = \bigcup_x A^{(x)}$ be as in Section 2, then the TS diagram illustrates $A^{(x)} \cap L'$ ($L' \in \mathcal{F}'$) (see Fig. 5).

The following theorem is an immediate consequence of Theorem 1 and the definition of TS diagrams.

Theorem 2. — Let $\mathcal{F}' \in t_1(\mathbf{S}^1 \times \mathbf{D}^2, \mathcal{F}_R^{(+)})$. Then a pair (a, b) of integers and an isomorphism class of TS diagrams satisfying the following conditions correspond uniquely to \mathcal{F}' .

(i) $a = 1$ or 2 . The homology class $[\mathbf{L}_{\text{comp}}]$ of $\mathbf{H}_1(\mathbf{S}^1 \times \partial\mathbf{D}^2; \mathbf{Z})$ represented by a compact leaf \mathbf{L}_{comp} in $\widehat{\mathcal{F}'} = \mathcal{F}' | \mathbf{S}^1 \times \partial\mathbf{D}^2$ is $a\alpha + b\beta$. In case $a = 1$, \mathcal{F}' is transversely orientable. In case $a = 2$, \mathcal{F}' is transversely non-orientable. The TS diagram should be invariant under the action of order 2 if $a = 2$.

(ii) Arc intervals of $\partial\mathbf{D}^2$ with symbols $+$, $-$, \times and $||$ represent the plus, the minus Reeb components, the slope components and foliated I-bundles over \mathbf{S}^1 (i.e. union of slope components and compact leaves) contained in \mathcal{F}' respectively.

(iii) In case $a = 1$ (resp. $a = 2$), \mathbf{N}_i (resp. a pair of \mathbf{N}_i which is invariant under the action of order 2) represents the plus half Reeb component or the TS component of type m or a foliated I-bundle over $\mathbf{S}^1 \times \mathbf{I}$ (resp. $\mathbf{S}^1 \times \mathbf{I}$ or the Möbius band) if $\mathbf{N}_i \cap \partial\mathbf{D}^2$ consists of one or $m + 2$ or 2 connected components respectively.

(iv) Each simple arc $\widehat{\ell}_i$ represents a compact leaf diffeomorphic to $\mathbf{S}^1 \times \mathbf{I}$ in case $a = 1$ and to $\mathbf{S}^1 \times \mathbf{I}$ or the Möbius band in case $a = 2$.

In the following we consider the topological classification for $t_1(\mathbf{S}^1 \times \mathbf{D}^2, \mathcal{F}_R^{(+)})$.

Lemma 8. — Any TS component \mathbf{TS}_m of type m is topologically isomorphic to the standard TS component $\widehat{\mathbf{TS}}_m$ of type m if the longitudinal number is 1.

Proof. — Let $\mathbf{L}_0, \mathbf{L}_1, \dots, \mathbf{L}_{m+1}$ (resp. $\widehat{\mathbf{L}}_0, \widehat{\mathbf{L}}_1, \dots, \widehat{\mathbf{L}}_{m+1}$) be compact leaves of \mathbf{TS}_m (resp. $\widehat{\mathbf{TS}}_m$). We fix \mathbf{C}^∞ diffeomorphisms

$$\gamma_i : \mathbf{S}^1 \times \mathbf{I} \rightarrow \mathbf{L}_i \quad (\text{resp. } \widehat{\gamma}_i : \mathbf{S}^1 \times \mathbf{I} \rightarrow \widehat{\mathbf{L}}_i) \quad (i = 0, 1, \dots, m+1).$$

Then it is easy to see that there exists a collar c (resp. \widehat{c}) of $\bigcup_{i=0}^{m+1} \mathbf{L}_i$ (resp. $\bigcup_{i=0}^{m+1} \widehat{\mathbf{L}}_i$) in $|\mathbf{TS}_m|$ (resp. $|\widehat{\mathbf{TS}}_m|$):

$$c : \left(\bigcup_{i=0}^{m+1} \mathbf{L}_i \right) \times [0, 1] \rightarrow |\mathbf{TS}_m|, \quad c(z, 0) = z \quad \left(z \in \bigcup_{i=0}^{m+1} \mathbf{L}_i \right)$$

$$(\text{resp. } \widehat{c} : \left(\bigcup_{i=0}^{m+1} \widehat{\mathbf{L}}_i \right) \times [0, 1] \rightarrow |\widehat{\mathbf{TS}}_m|, \quad \widehat{c}(z, 0) = z \quad \left(z \in \bigcup_{i=0}^{m+1} \widehat{\mathbf{L}}_i \right))$$

such that

- (a) $c(\bigcup_{i=0}^{m+1} \mathbf{L}_i \times \{1\})$ (resp. $\widehat{c}(\bigcup_{i=0}^{m+1} \widehat{\mathbf{L}}_i \times \{1\})$) is transverse to \mathbf{TS}_m (resp. $\widehat{\mathbf{TS}}_m$),
- (b) $c(\{z\} \times [0, 1])$ (resp. $\widehat{c}(\{z\} \times [0, 1])$) is transverse to \mathbf{TS}_m (resp. $\widehat{\mathbf{TS}}_m$),
- (c) $c(\gamma_i(\{x\} \times \mathbf{I}), t)$ (resp. $\widehat{c}(\widehat{\gamma}_i(\{x\} \times \mathbf{I}), t)$) is contained in a leaf of \mathbf{TS}_m (resp. $\widehat{\mathbf{TS}}_m$), where $t \in [0, 1]$.

Since, by Proposition 6, the restriction of TS_m (resp. $\widehat{\text{TS}}_m$) to

$$Q' = |\text{TS}_m| - c\left(\prod_{i=0}^{m+1} L_i\right) \times [0, 1] \quad (\text{resp. } \widehat{Q}' = |\widehat{\text{TS}}_m| - \widehat{c}\left(\prod_{i=0}^{m+1} \widehat{L}_i\right) \times [0, 1])$$

is a C^∞ foliation of codimension one whose leaves are the regular polygon P_{2m+4} with $2m+4$ vertices, it follows from the Reeb stability theorem [5] that $\text{TS}_m|Q'$ (resp. $\widehat{\text{TS}}_m|\widehat{Q}'$) is a product foliation. Thus, as is easily verified, there exists a C^∞ diffeomorphism

$$h_{Q'} : Q' \rightarrow \widehat{Q}'$$

such that

- (i) $h_{Q'}$ preserves the foliations $\text{TS}_m|Q'$ and $\widehat{\text{TS}}_m|\widehat{Q}'$,
- (ii) $h_{Q'}(L_i \times \{1\}) = \widehat{L}_i \times \{1\}$.

Further, letting $h_Q(z, 1) = (z', 1)$ ($z \in L_i$), we define surjective homeomorphisms

$$h_i : c(L_i \times [0, 1]) \rightarrow \widehat{c}(\widehat{L}_i \times [0, 1]), \quad i = 0, 1, \dots, m+1,$$

by $h_i(z, t) = (z', \xi_z(t))$,

where $\xi_z : [0, 1] \rightarrow [0, 1]$ is a surjective homeomorphism depending continuously on z . By a suitable choice of ξ_z , the homeomorphisms h_i preserve the foliations $\text{TS}_m|c(L_i \times [0, 1])$ and $\widehat{\text{TS}}_m|\widehat{c}(\widehat{L}_i \times [0, 1])$.

Then the homeomorphism

$$h : |\text{TS}_m| \rightarrow |\widehat{\text{TS}}_m|$$

defined by $h|Q' = h_{Q'}$ and $h|c(L_i \times I) = h_i$ ($i = 0, 1, \dots, m+1$) is a surjective homeomorphism preserving foliations TS_m and $\widehat{\text{TS}}_m$. Thus this lemma is proved.

The following theorem is an immediate consequence of Theorem 2 and Lemma 8.

Theorem 3. — Let $\mathcal{F}'_1, \mathcal{F}'_2 \in t_1(S^1 \times D^2, \mathcal{F}_R^{(+)})$. Suppose that \mathcal{F}'_1 and \mathcal{F}'_2 satisfy the following conditions:

- (i) \mathcal{F}'_1 and \mathcal{F}'_2 have the same longitudinal number;
- (ii) there exists an isomorphism between their TS diagrams, say f_0 ;
- (iii) for foliated I-bundles in \mathcal{F}'_1 and in \mathcal{F}'_2 corresponding by f_0 , there exists an isomorphism between them compatible with f_0 .

Then \mathcal{F}'_1 and \mathcal{F}'_2 are topologically isomorphic.

Let us consider $\mathcal{F}'_1 \in t_1(S^1 \times D^2, \mathcal{F}_R^{(+)})$ consisting of 3 copies of the half Reeb component and one TS component of type 1 (Fig. 12). We represent \mathcal{F}'_1 by illustrating $\mathcal{F}'_1 \cap A^{(x)}$ and $\mathcal{F}'_1|(S^1 \times \partial D^2)$ by dotted curves in Figure 13.

Let \mathcal{F}'_1'' be an element of $t_1(S^1 \times D^2, \mathcal{F}_R^{(+)})$ represented by real curves illustrating $\mathcal{F}'_1'' \cap A^{(x)}$ and $\mathcal{F}'_1''|(S^1 \times \partial D^2)$ in Figure 13. The foliation \mathcal{F}'_1'' consists of 2 copies of the half Reeb component and a foliated I-bundle over $S^1 \times I$, and is transverse to \mathcal{F}'_1 :

$$\mathcal{F}'_1'' \pitchfork \mathcal{F}'_1.$$

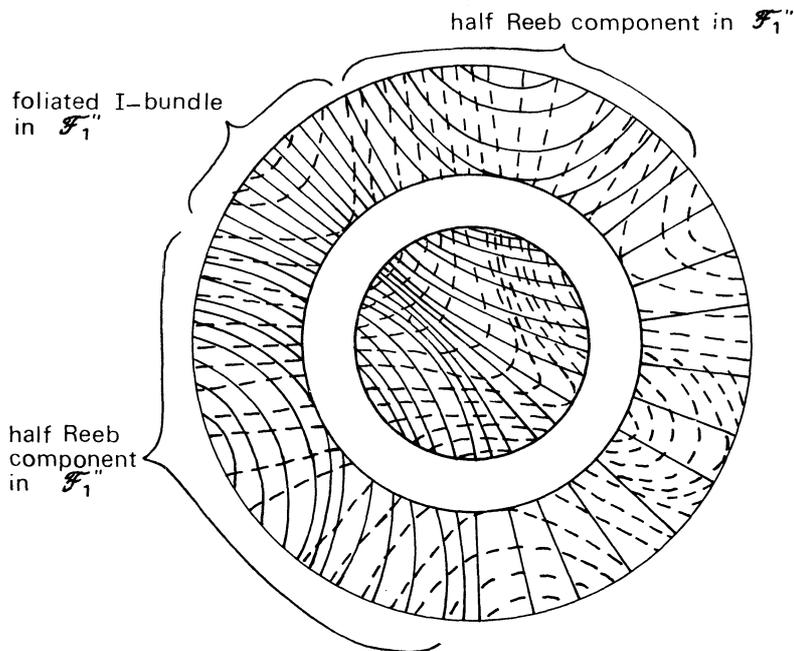


FIG. 13

For $\mathcal{F}' \in t_1(S^1 \times D^2, \mathcal{F}_R^{(+)})$ consisting of m copies of the half Reeb component and one TS component of type $m-1$, we can construct $\mathcal{F}'' \in t_1(S^1 \times D^2, \mathcal{F}_R^{(+)})$ such that $\mathcal{F}'' \bar{\cap} \mathcal{F}'$ by similar methods. It seems to us that, for any $\mathcal{F}' \in t_1(S^1 \times D^2, \mathcal{F}_R^{(+)})$, there exists always $\mathcal{F}'' \in t_1(S^1 \times D^2, \mathcal{F}_R^{(+)})$ such that $\mathcal{F}' \bar{\cap} \mathcal{F}''$. However, in general, \mathcal{F}'' is not unique, because \mathcal{F}_1'' above is also transverse to $\mathcal{F}_1''' \in t_1(S^1 \times D^2, \mathcal{F}_R^{(+)})$ consisting of 2 copies of the half Reeb component (Fig. 4).

6. Foliations transverse to the Reeb foliation of S^3

Let $S^3 = (S_1^1 \times D_1^2) \cup_h (D_2^2 \times S_2^1)$ be the decomposition of the 3-sphere into the union of two solid tori, where $h: S_1^1 \times \partial D_1^2 \rightarrow \partial D_2^2 \times S_2^1$ is given by $h(x, y) = (x, y)$.

Let \mathcal{F}_R denote the Reeb foliation of S^3 . We fix orientations on S_1^1 and S_2^1 so that $'\mathcal{F}_R = \mathcal{F}_R|_{(S_1^1 \times D_1^2)}$ and $''\mathcal{F}_R = \mathcal{F}_R|_{(D_2^2 \times S_2^1)}$ are the plus Reeb components.

Let \mathcal{F}' be a C^∞ foliation of codimension one of S^3 transverse to \mathcal{F}_R . Then we have

$$\mathcal{F}'|_{(S_1^1 \times D_1^2)}, \mathcal{F}'|_{(D_2^2 \times S_2^1)} \in t_1(S^1 \times D^2, \mathcal{F}_R^{(+)}).$$

Since \mathcal{F}' is transversely orientable, their longitudinal numbers are both 1. The restrictions $\mathcal{F}'|_{(S_1^1 \times D_1^2)}$ and $\mathcal{F}'|_{(D_2^2 \times S_2^1)}$ must be isomorphic by h on their boundaries. Thus it is obvious that the homology class $[L_{\text{comp}}]$ of $H_1(S_1^1 \times \partial D_1^2; \mathbf{Z})$ represented by

a compact leaf L_{comp} of $\widehat{\mathcal{F}}' = \mathcal{F}' | \partial(S^1 \times D^2)$ having the orientation compatible with the orientations of S^1 and S^2 is $\alpha + \beta$, where $\alpha = [S^1 \times \{*\}]$, $\beta = [\{\bullet\bullet\} \times S^2]$ ($\{*\}, \{\bullet\bullet\} \in \partial D^2$) with given orientations (Theorem 1, (i)).

Conversely from two elements $\mathcal{F}'_1, \mathcal{F}'_2 \in t_1(S^1 \times D^2, \mathcal{F}_R^{(+)})$ with the longitudinal number 1 such that $\mathcal{F}'_1 | \partial(S^1 \times D^2)$ is isomorphic to $\mathcal{F}'_2 | \partial(S^1 \times D^2)$ by the map $(x, y) \rightarrow (y, x)$, we obtain an $\mathcal{F}' \in t_1(S^3, \mathcal{F}_R)$ by identifying their boundaries. Thus the following theorem holds:

Theorem 4. — Let \mathcal{F}' be a C^∞ foliation of codimension one transverse to the Reeb foliation \mathcal{F}_R of S^3 : $\mathcal{F}' \in t_1(S^3, \mathcal{F}_R)$. Then \mathcal{F}' is obtained from two foliations $\widehat{\mathcal{F}}'_1, \widehat{\mathcal{F}}'_2 \in t_1(S^1 \times D^2, \mathcal{F}_R^{(+)})$ with longitudinal number 1 such that $\widehat{\mathcal{F}}'_1 | \partial(S^1 \times D^2)$ is isomorphic to $\widehat{\mathcal{F}}'_2 | \partial(S^1 \times D^2)$ by the map $(x, y) \rightarrow (y, x)$ identifying their boundaries.

Let D_i^2 ($i = 0, 1, \dots, m+1$) be 2-discs imbedded disjointly in the 2-sphere S^2 and let $C = (S^2 - \bigcup_{i=0}^{m+1} \text{Int } D_i^2) \times S^1$. We fix an orientation on S^1 . A full TS component of type $(m; r)$, denoted by $\overline{\text{TS}}_{(m; r)}$, is a C^∞ foliation \mathcal{F}_C of codimension one of C having the following properties:

- (i) Compact leaves of \mathcal{F}_C are $\partial D_i^2 \times S^1$ ($i = 0, 1, \dots, m+1$).
- (ii) Interior leaves of \mathcal{F}_C are transverse to $\{x\} \times S^1$ ($x \in S^2 - \bigcup_{i=0}^{m+1} D_i^2$).
- (iii) \mathcal{F}_C has a contracting holonomy in the negative direction of S^1 on $\partial D_i^2 \times S^1$ ($i = 0, 1, \dots, r$) and in the positive direction of S^1 on $\partial D_i^2 \times S^1$ ($i = r+1, r+2, \dots, m+1$), where $0 \leq r \leq m$. We note that a full TS component contains compact leaves having contracting holonomy in different directions of S^1 .

Example A. — Let us consider two copies of

$$\widehat{\mathcal{F}}' \in t_1(S^1 \times D^2, \mathcal{F}_R^{(+)}) \quad \text{with} \quad [L_{\text{comp}}] = \alpha + \beta.$$

We may suppose that h gives an isomorphism $\mathcal{F}_R^{(+)} | (S^1 \times \partial D^2) \rightarrow \mathcal{F}_R^{(+)} | (S^1 \times \partial D^2)$. Thus the C^∞ foliation \mathcal{F}' obtained from two copies of $\widehat{\mathcal{F}}'$ identifying their boundaries by h is an element of $t_1(S^3, \mathcal{F}_R)$. If $\widehat{\mathcal{F}}'$ consists of p copies of the plus half Reeb component, s_m copies of the TS component of type m ($m = 1, 2, \dots, u$), then \mathcal{F}' consists of p copies of the Reeb component and s_m copies of the full TS component of type $(m; 0)$.

Example B (Koichi Yano). — Let $\widehat{\mathcal{F}}'_1, \widehat{\mathcal{F}}'_2$ be elements of $t_1(S^1 \times D^2, \mathcal{F}_R^{(+)})$ with $[L_{\text{comp}}] = \alpha + \beta$ such that their TS diagrams are given by Figure 14, (a), (b) respectively and that h gives an isomorphism $\widehat{\mathcal{F}}'_1 | (S^1 \times \partial D^2) \rightarrow \widehat{\mathcal{F}}'_2 | (S^1 \times \partial D^2)$ (thus, the symbol \parallel in their TS diagrams represent slope components). The C^∞ foliation \mathcal{F}' obtained from $\widehat{\mathcal{F}}'_1$ and $\widehat{\mathcal{F}}'_2$ identifying their boundaries by h is an element of $t_1(S^3, \mathcal{F}_R)$ consisting of 7 copies of the Reeb component, a full TS component of type $(1; 1)$ and a full TS component of type $(3; 1)$.

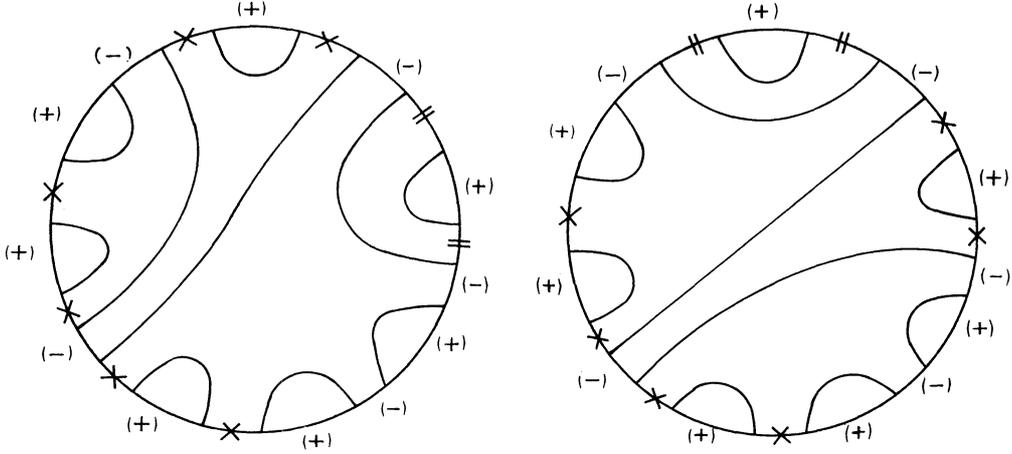


FIG. 14

Theorem 5. — Let $\mathcal{F}' \in t_1(\mathbb{S}^3, \mathcal{F}_R)$, then \mathcal{F}' consists of a finite number of Reeb components, full TS components and foliated I-bundles over $\mathbb{S}^1 \times \mathbb{S}^1$. Furthermore, let ℓ_R , etc. (resp. ℓ_{TS}) denote a closed curve in a Reeb component (resp. a full TS component) in \mathcal{F}' homotopic to the longitude and transverse to \mathcal{F}' , then ℓ_R and ℓ_{TS} are both unknotted and the linking number of ℓ_R and ℓ'_R is ± 1 .

Proof. — Let $\widehat{\mathcal{F}}'_1, \widehat{\mathcal{F}}'_2$ be as in Theorem 4: $\mathcal{F}' = \widehat{\mathcal{F}}'_1 \cup \widehat{\mathcal{F}}'_2$. First assume that the number of compact leaves contained in $\widehat{\mathcal{F}}'_1$ (thus also in $\widehat{\mathcal{F}}'_2$) is finite. This is equivalent to the assumption that foliated I-bundles in $\widehat{\mathcal{F}}'_1$ and in $\widehat{\mathcal{F}}'_2$ contain only a finite number of compact leaves. For a compact leaf L_1 in $\widehat{\mathcal{F}}'_1$, $L_1 \cap (\mathbb{S}^1 \times \partial D^2)$ consists of two compact leaves in $\widehat{\mathcal{F}}'_2 | (\mathbb{S}^1 \times \partial D^2)$, say \bar{L}_1, \bar{L}_1 . Then there exists a unique compact leaf in $\widehat{\mathcal{F}}'_2$, say L_2 (resp. L'_2), which contains \bar{L}_1 (resp. \bar{L}_1). Thus the set of compact leaves in $\widehat{\mathcal{F}}'_1$ and $\widehat{\mathcal{F}}'_2$ forms a finite number of compact leaves in \mathcal{F}' which are diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$. By Proposition 2, (iii), the numbers of plus half Reeb components in $\widehat{\mathcal{F}}'_1$ and in $\widehat{\mathcal{F}}'_2$ are the same and they form the same number of Reeb components in \mathcal{F}' . Further, as is easily verified, TS components and foliated I-bundles in $\widehat{\mathcal{F}}'_1$ and in $\widehat{\mathcal{F}}'_2$ form a finite number of full TS components and foliated I-bundles over $\mathbb{S}^1 \times \mathbb{S}^1$ with finite compact leaves in \mathcal{F}' (Examples A, B). Since $[L_{\text{comp}}] = \alpha + \beta$, it follows from the construction as above that ℓ_R and ℓ_{TS} are unknotted and the linking number of ℓ_R and ℓ'_R is ± 1 . Thus the theorem is proved in this case.

Now suppose that $\widehat{\mathcal{F}}'_1$ (thus also $\widehat{\mathcal{F}}'_2$) contains foliated I-bundles with infinite numbers of compact leaves. We denote $A, A^{(x)}$ and $\{\ell_\lambda^{(x)}\}_{\lambda \in \Lambda(x)}$ of Section 3 for $\widehat{\mathcal{F}}'_1$ (resp. $\widehat{\mathcal{F}}'_2$) by $A_1, A_1^{(x)}$ and $\{\ell_\lambda^{(x)}\}_{\lambda \in \Lambda'(x)}$ (resp. $A_2, A_2^{(x)}$ and $\{\ell'_\lambda^{(x)}\}_{\lambda \in \Lambda''(x)}$). Since $[L_{\text{comp}}] = \alpha + \beta$, the diffeomorphism h induces naturally a diffeomorphism

$$\bar{h}: \partial A_1^{(x_1)} \rightarrow \partial A_2^{(x_2)}$$

such that $\bar{h}(y) \in h(\bar{L})$ for $y \in \bar{L}$. Let \hat{S} be the 2-sphere obtained from $A_1^{(x_1)}$ and $A_2^{(x_2)}$ identifying their boundaries by \bar{h} . Then we may consider that

$$\mathcal{L} = \{ \ell_\lambda^{(x_1)} \}_{\lambda \in \Lambda'(x_1)} \cup \{ \ell_\lambda^{(x_2)} \}_{\lambda \in \Lambda''(x_2)}$$

is a family of integral curves of a C^∞ vector field Y on \hat{S} which is non-singular except $\partial A_1^{(x_1)} \cap (\hat{\Sigma}^{(+)} \cup \hat{\Sigma}^{(-)})$. Here \mathcal{L} represents the leaves of \mathcal{F}' . If there exists a sequence of an infinite number of compact leaves $L_1^{(1)}, L_2^{(2)}, L_1^{(3)}, L_2^{(4)}, \dots, L_1^{(2s-1)}, L_2^{(2s)}, \dots$ such that

$$\begin{aligned} L_1^{(2s-1)} \in \hat{\mathcal{F}}_1', \quad L_2^{(2s)} \in \hat{\mathcal{F}}_2', \\ L_1^{(2s-1)} \cap L_2^{(2s)} \neq \emptyset, \quad L_2^{(2s)} \cap L_1^{(2s+1)} \neq \emptyset \quad (s = 1, 2, 3, \dots), \end{aligned}$$

and that the union $(\bigcup_{s=1}^{\infty} L_1^{(2s-1)}) \cup (\bigcup_{s=1}^{\infty} L_2^{(2s)})$ is a non-compact leaf of \mathcal{F}' , then, letting $L_1^{(2s-1)} \cap A_1^{(x_1)} = \hat{\ell}^{(2s-1)}$, $L_2^{(2s)} \cap A_2^{(x_2)} = \hat{\ell}^{(2s)}$, the union $\hat{\ell} = \bigcup_{s=1}^{\infty} \hat{\ell}^{(s)}$ is a non-compact integral

curve of Y . By the Poincaré-Bendixson theorem, the ω -limit set of $\hat{\ell}$ is a circle, say ω . We denote by L_ω the leaf of \mathcal{F}' containing ω and by \bar{L}_ω a connected component of $L_\omega \cap \partial A_1$. Then, it is obvious that \bar{L}_ω cannot be a non-compact leaf (i.e. an interior leaf of a slope component) of $\hat{\mathcal{F}}_1' | \partial A_1$. This implies that L_ω is a compact leaf of \mathcal{F}' diffeomorphic to $S^1 \times S^1$, say $L_\omega = \omega \times S^1$. Let $\bar{L}^{(2s-1)}$ (resp. $\bar{L}^{(2s)}$) denote a connected component of $L_1^{(2s-1)} \cap \partial A_1$ (resp. $L_2^{(2s)} \cap \partial A_2$), then there exists a sequence of compact leaves $\bar{L}^{(s_1)}, \bar{L}^{(s_2)}, \dots, \bar{L}^{(s_q)}, \dots$ of $\hat{\mathcal{F}}_1' | \partial A_1 = \hat{\mathcal{F}}_2' | \partial A_2$ such that this sequence converges to \bar{L}_ω . If, for a given integer q' , there always exists an integer $q > q'$ such that a slope component of $\hat{\mathcal{F}}_1' | \partial A_1$ (maybe contained in a foliated I-bundle of $\hat{\mathcal{F}}_1' | \partial A_1$) situated between $\bar{L}^{(s_q)}$ and $\bar{L}^{(s_{q+1})}$, then L_ω has contracting holonomy in both $[\omega]$, $[\{*\} \times S^1]$ ($* \in \omega$). Since \mathcal{F}' is of class C^∞ , this contradicts to the Kopell's theorem [2]. Therefore, for a compact leaf \bar{L} of the boundary of a slope component of $\hat{\mathcal{F}}_1' | \partial A_1$, the saturation of \bar{L} in \mathcal{F}' is a compact leaf of \mathcal{F}' . Thus, as is easily verified, for a slope component $\bar{\mathcal{F}}_s'$ in $\hat{\mathcal{F}}_1' | \partial A_1$, one of the following occurs:

(i) The saturation of $|\bar{\mathcal{F}}_s'|$ in \mathcal{F}' contains a TS component of $\hat{\mathcal{F}}_1'$ or $\hat{\mathcal{F}}_2'$.

(ii) The saturation of $|\bar{\mathcal{F}}_s'|$ in \mathcal{F}' forms a foliated I-bundle over $S^1 \times S^1$ with two compact leaves.

Further, let $\bar{\mathcal{F}}$ be a subset of $\hat{\mathcal{F}}_1' | \partial A_1$ which satisfies the following:

- (a) $\bar{\mathcal{F}}$ consists of compact leaves;
- (b) the union of the leaves in $\bar{\mathcal{F}}$ is diffeomorphic to $S^1 \times I$;
- (c) the boundary of $|\bar{\mathcal{F}}|$ consists of two compact leaves which belong to the boundaries of slope components.

Then the saturation of $|\bar{\mathcal{F}}|$ in \mathcal{F}' is a foliated I-bundle over $S^1 \times S^1$.

It is obvious that the union of two foliated I-bundles over $S^1 \times S^1$ with a common

compact leaf forms a foliated I-bundle over $S^1 \times S^1$. Moreover, let F be the saturation in \mathcal{F}' of a sufficiently small subset of ∂A_1 such that the boundary of F consists of two compact leaves, then we can show that $\mathcal{F}'|F$ is a foliated I-bundle over $S^1 \times S^1$ by constructing a vector field transverse to $\mathcal{F}'|F$.

By the observation above, there exist foliated I-bundles $\mathcal{F}'_1, \mathcal{F}'_2, \dots, \mathcal{F}'_u$ over $S^1 \times S^1$ in \mathcal{F}' such that $S^3 - \bigcup_{i=1}^u \text{Int } |\mathcal{F}'_i|$ contains only a finite number of compact leaves. So the theorem reduces to the case above. Thus the theorem is proved.

Remark. — The properties of the full TS component in \mathcal{F}' depend mainly on the lag functions of two TS components contained in it. For example, see [10] for the Godbillon-Vey classes of TS components.

7. Foliations of codimension one of S^3 admitting no transverse foliation of codimension one

Let k be a fibred knot in S^3 . That is, letting $N(k)$ be a tubular neighborhood of k , $E = S^3 - \text{Int } N(k)$ is a C^∞ fibre bundle over S^1 , $\pi: E \rightarrow S^1$, with fibre $G - \text{Int } D^2$ where G is a closed surface of genus g and D^2 is a 2-disc imbedded in G . For example, the intersection k of $S^3 = \{(z_1, z_2); |z_1|^2 + |z_2|^2 = 1\}$ and $\{(z_1, z_2); z_1^p + z_2^q = 0\}$ is a fibred knot [3].

Let \mathcal{F} be a C^∞ foliation of codimension one of S^3 constructed by the spinnable structure having the fibred knot as the axis ([8]). That is, by choosing suitable orientations on S^1 of $S^1 \times D^2$ and on S^1 of $E \rightarrow S^1$, \mathcal{F} is the union of the plus Reeb component $\mathcal{F}_R^{(+)}$ of $N(k) = S^1 \times D^2$ and the C^∞ foliation $\mathcal{F}_\pi^{(+)}$ of Proposition 2: $\mathcal{F} = \mathcal{F}_R^{(+)} \cup \mathcal{F}_\pi^{(+)}$. Then we have the following theorem.

Theorem 6. — *Let \mathcal{F} be a C^∞ foliation of codimension one of S^3 defined from a fibred knot k as above, where the genus g is ≥ 1 . Then \mathcal{F} does not admit any transverse C^∞ foliation of codimension one:*

$$t_1(S^3, \mathcal{F}) = \emptyset.$$

Proof. — Suppose there exists $\mathcal{F}' \in t_1(S^3, \mathcal{F})$. Let α and β denote the homology classes of $H_1(\partial E; \mathbf{Z})$ represented by a meridian of $N(k)$ and $\partial(G - \text{Int } D^2)$ with orientations chosen as above respectively. Then, by Proposition 2, a compact leaf L_{comp} of $\mathcal{F}'| \partial E$ represents a homology class $\bar{a}\alpha + \bar{b}\beta$ ($\bar{a} \geq 0$), and $\mathcal{F}'| \partial E$ contains p copies of the plus Reeb component and q copies of the minus Reeb component, where

$$\bar{a}(p - q) = 2(1 - 2g).$$

On the other hand, also by Proposition 2, (iii), since \mathcal{F}' is transversely orientable, $\mathcal{F}'| \partial N(k)$ contains p' copies of the plus Reeb component and q' copies of the minus Reeb component with

$$(*) \quad p' - q' = 2$$

and $\bar{b} = \pm 1$. Since

$$p - q = \pm (p' - q'),$$

it follows that

$$(**) \quad p - q = -2, \quad \bar{a} = 2g - 1.$$

\mathcal{F}' has a Reeb component ([4]), say $\mathcal{F}'_R, |\mathcal{F}'_R| \subset S^3$. If $|\mathcal{F}'_R| \subset N(k)$, then obviously $|\mathcal{F}'_R| \subset \text{Int } N(k)$. Thus $\mathcal{F}' | |\mathcal{F}'_R|$ consists of compact leaves. On the other hand, since $\mathcal{F}' | |\mathcal{F}'_R| \in t_1(\mathcal{F}'_R)$, $\mathcal{F}' | |\mathcal{F}'_R|$ must contain non-compact leaves by Theorem 1. This contradiction implies that

$$|\mathcal{F}'_R| \not\subset N(k).$$

Similarly we have

$$|\mathcal{F}'_R| \not\subset E.$$

Since $\mathcal{F}' | N(k) \in t_1(N(k), \mathcal{F}'_R^{(+)})$, $\mathcal{F}' | N(k)$ satisfies the conditions of Theorem 1. Now we consider $\mathcal{F}'_R | (N(k) \cap |\mathcal{F}'_R|)$. If there exists a TS component in

$$\mathcal{F}'_R | (N(k) \cap |\mathcal{F}'_R|),$$

say TS_m , then the longitude of $|TS_m|$ and that of $|\mathcal{F}'_R|$ must coincide. This contradicts that TS_m contains compact leaves having the contracting holonomy in both positive and negative directions (Proposition 6). Thus $\mathcal{F}'_R | (N(k) \cap |\mathcal{F}'_R|)$ does not contain a TS component. Similarly, by the fact that $[L_{\text{comp}}] = (2g - 1)\alpha \pm \beta$, $\mathcal{F}'_R | (N(k) \cap |\mathcal{F}'_R|)$ does not contain a foliated I-bundle. Therefore, $\mathcal{F}'_R | (N(k) \cap |\mathcal{F}'_R|)$ should consist of half Reeb components.

Since $\mathcal{F}' | |\mathcal{F}'_R|$ is transverse to a Reeb component \mathcal{F}'_R , $\mathcal{F}' | (E \cap |\mathcal{F}'_R|)$ consists of half Reeb components, TS components and foliated I-bundles over $S^1 \times I$ by Theorem 1. If $\mathcal{F}' | (E \cap |\mathcal{F}'_R|)$ contains a TS component, say TS'_m , then, since compact leaves of TS'_m have contracting holonomy, they must be subsets of ∂E . However, \mathcal{F} has the contracting holonomy in one direction on $\partial N(k)$ in the side of E . This contradicts the property of TS components about contracting holonomy. Thus $\mathcal{F}' | (E \cap |\mathcal{F}'_R|)$ cannot contain a TS component. Similarly, by the fact that $[L_{\text{comp}}] = (2g - 1)\alpha \pm \beta$, $\mathcal{F}' | (E \cap |\mathcal{F}'_R|)$ cannot contain a foliated I-bundle. Thus $\mathcal{F}' | (E \cap |\mathcal{F}'_R|)$ must consist of half Reeb components. This shows that $E \cap |\mathcal{F}'_R|$ is diffeomorphic to the disjoint union of a finite number of copies of $S^1 \times D_+^2$. Further, since $\mathcal{F}'_R | (N(k) \cap |\mathcal{F}'_R|)$ consists of half Reeb components, $\mathcal{F}' | (E \cap |\mathcal{F}'_R|)$ is a half Reeb component, and, thus, $\mathcal{F}'_R | (N(k) \cap |\mathcal{F}'_R|)$ is also a half Reeb component.

Moreover, since $\mathcal{F}'_R | (\partial N(k) \cap |\mathcal{F}'_R|)$ is a plus Reeb component in $\mathcal{F}' | \partial N(k)$, it follows from (*), (**) that $\mathcal{F}'_R | (\partial E \cap |\mathcal{F}'_R|)$ is a minus Reeb component in $\mathcal{F}' | \partial E$.

Let $c(\partial E)$ be a sufficiently small collar of ∂E in E and denote $A_E = \overline{E - c(\partial E)}$. Then, for a non-compact leaf L of $\mathcal{F}' | (E \cap |\mathcal{F}'_R|)$, $L \cap A_E$ is a half-disc. Thus the Euler number $\chi(L \cap A_E) = 1$.

We may consider that the family of curves $\{L \cap A_E \cap L'; L' \in \mathcal{F}'\}$ is that of integral curves of a C^∞ vector field V_L on $L \cap A_E$ such that V_L is tangent to $L \cap A_E \cap \partial|\mathcal{F}'_R|$ and is non-singular except at one point of $L \cap A_E$, say \hat{p} . The singularity of V_L at \hat{p} is of minus type, because $\mathcal{F}'_R|(\partial E \cap |\mathcal{F}'_R|)$ is a minus Reeb component in $\mathcal{F}'|E$. Let $D(L \cap A_E)$ denote the double of $L \cap A_E$ obtained from two copies of $L \cap A_E$ by pasting at $L \cap \partial A_E$, then the Euler number of $D(L \cap A_E)$ is -1 . On the other hand, it follows from $\chi(L \cap A_E) = 1$ that $\chi(D(L \cap A_E)) = 1$. This is a contradiction. Thus there exists no \mathcal{F}' . This completes the proof.

As a corollary of Theorem 6, we have the following theorem.

Theorem 7. — Let $\bar{\mathcal{F}}$ be a C^∞ foliation of codimension 2 which is a subfoliation of \mathcal{F} of Theorem 6. Then $\bar{\mathcal{F}}$ does not admit any transverse C^∞ foliation of codimension one:

$$t_0(S^3, \bar{\mathcal{F}}) = \emptyset.$$

Proof. — Clearly $\bar{\mathcal{F}}$ exists (cf. Proposition 1). Suppose $\mathcal{F}' \in t_0(S^3, \bar{\mathcal{F}})$. Then it is obvious that $\mathcal{F}' \in t_1(S^3, \mathcal{F})$. This contradicts the result of Theorem 6.

8. Problems

The following are some problems raised by the results of this paper.

Problem 1. — Classify or characterize C^∞ subfoliations of codimension 2 of the Reeb component $(S^1 \times D^2, \mathcal{F}_R^{(+)})$. In case the restriction of the subfoliation to $S^1 \times \partial D^2$ consists of two copies of the half Reeb foliation, do they coincide with $\mathcal{F}_R^{(+)} \cap \mathcal{F}'$ ($\mathcal{F}' \in t_1(\mathcal{F}_R^{(+)})$ of Fig. 4)?

Problem 2. — Determine $t_1^m(S^1 \times D^2, \mathcal{F}_R^{(+)})$ for $m = 2, 3, \dots$. Does there exist a stability: $t_1^m(S^1 \times D^2, \mathcal{F}_R) = t_1^{m+2}(S^1 \times D^2, \mathcal{F}_R) = \dots = t_1^{m+2j}(S^1 \times D^2, \mathcal{F}_R) = \dots$?

Problem 3. — Classify or characterize C^∞ subfoliations of codimension 2 of the Reeb foliation (S^3, \mathcal{F}_R) .

Problem 4. — Characterize C^∞ foliations of codimension 2 of S^3 which have superfoliations of codimension one.

Problem 5. — Does there exist a C^∞ foliation \mathcal{F} of codimension one of S^3 such that $t_1(\mathcal{F}) \neq \emptyset$ and $t_1(\mathcal{F}) \cap t_1(\mathcal{F}') = \emptyset$ for some $\mathcal{F}' \in t_1(\mathcal{F})$.

Problem 6. — Characterize C^∞ foliations contained in the limit of the sequence $\{(S^3, \mathcal{F}_R)\} \subset t_1(S^3, \mathcal{F}_R) \subset \dots \subset t_1^m(S^3, \mathcal{F}_R) \subset \dots$. Does there exist a stability for this sequence? Is $t_1^2(S^3, \mathcal{F}_R) = t_1(S^3, \mathcal{F}_R)$?

Problem 7. — Does there exist a C^∞ foliation of codimension one of S^3 such that $t_1(S^3, \mathcal{F})$ (or the limit of the sequence $t_1(S^3, \mathcal{F}) \subset \dots \subset t_1^m(S^3, \mathcal{F}) \subset \dots$) is equal to the

set of C^∞ foliations of codimension one which admit transverse C^∞ foliations of codimension one.

Problem 8. — For $(S^3, \mathcal{F}') \in t_1(S^3, \mathcal{F}_R)$, is it true that “ the Godbillon-Vey number zero ” implies “ cobordant to zero ”?

Problem 9. — Consider the deformation classes in $t_1(S^3, \mathcal{F}_R)$.

Problem 10. — Find conditions for C^∞ foliated manifolds to admit transverse foliations.

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