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ON THE DE RHAM COHOMOLOGY OF ALGEBRAIC VARIETIES

by Robin Hartshorne

CONTENTS

INTRODUCTION

1. Historical.

The idea of using differential forms and their integrals to define numerical invariants of algebraic varieties goes back to Picard and Lefschetz. More recently, Atiyah and Hodge [2] and Grothendieck [16] showed that algebraic differential forms could be used to calculate the singular cohomology of a smooth scheme over **C**. This algebraic-topological comparison theorem has been generalized by Lieberman and Herrera [29] and by Deligne (unpublished) to include the case of singular schemes over **C**. Lieberman and Herrera also proved a duality theorem for first order differential operators, of which our Theorem (II.5.1) is a special case. See also Lieberman [49] for another comparison theorem.

In another context, algebraic differential forms have proved to be useful in the study of the monodromy of a family of complex varieties, using the Gauss-Manin connection. See Katz and Oda [34], Katz ([32] and [33]), Deligne [10], and Brieskorn [8].

In the purely analytic context, holomorphic differentials on a singular analytic space have been studied by Reiffen [44] and Bloom and Herrera [5].

Finally in the study of varieties in characteristic p, algebraic differential forms are important in Monsky's formal cohomology [38], and as motivation for Grothendieck's crystalline cohomology ([18] and [4]).

Our interest in De Rham cohomology dates from 1967, when we first found an algebraic proof of Poincaré duality for a smooth proper scheme over a field. Our purpose in the present paper is to lay the foundations of a purely algebraic theory of algebraic De Rham cohomology and homology for schemes with arbitrary singularities over a field of characteristic zero.

2. Main results.

The main results of this paper have been outlined in the announcement [27], to which we refer. See also [24, III, § 7.8], and [25]. Not mentioned in the announcement however, are the Thom-Gysin sequence of a vector bundle (II, § 7.9), the theory of relative cohomology and base change (III, § 4, 5), and the formal-analytic Poincaré lemma (IV, § 2).

Please note also some changes necessary in the results of the announcement. The theorem [27, 1.2 d] needs a slight further hypothesis (see (II.4.4)). The local finiteness theorem [27, 2.2] is now valid more generally (III.2.1). The field of representatives in [27, 2.4] must be assumed to be "good" (see III, § 6). The results of [27, § 3] as stated are valid only if Y is connected and rational over k (see (III.3.2)).

3. Comments.

Whereas the comparison theorems of Chapter IV have been proved before, our paper is original in that the properties of algebraic De Rham cohomology developed in Chapters II and III are proved by purely algebraic techniques. Our presentation is also original in the systematic use of a homology theory, analogous to the theory of homology with locally compact supports of Borel and Moore [6]. When dealing with smooth proper schemes, Poincaré duality eliminates the need for a homology theory. But in dealing with non-proper schemes, and with singularities, one needs something more. One can develop the theory of De Rham cohomology with compact supports, using the techniques of [26]. However we have found it preferable to develop a homology theory. The connection is that for any scheme Y of finite type over k, $H^i_c(Y)$ is the dual of $H_i(Y)$.

Using this homology theory, the Gysin sequence [24, III.8.3] of our earlier treatment of De Rham cohomology now reappears as the exact sequence of homology of a closed subset (II.3.3).

4. Applications.

The most important applications of the theory so far, which also provided the motivation for writing this paper, are contained in the work of Ogus [43]. He was able to find algebraic proofs of the striking theorems of Barth [3], while at the same time eliminating the hypotheses of non-singularity. He was also able to determine completely the cohomological dimension of projective space $\mathbf{P}^n$ minus a subscheme Y, in terms of the De Rham cohomology invariants of Y. This answers problems raised in [23] and [24, III, § 5].

We will discuss some of Ogus' results, in particular the Lefschetz theorem on hyperplane sections, in Chapter III, § 7.

Another application of our theory is to give some necessary conditions for a singular variety to be deformed into a smooth variety. These will be discussed in our forthcoming paper [28].

5. Problems.

Of the many questions which come to mind, perhaps the most persistent is "What about characteristic p?" Of course our results depend on characteristic zero, not only for the resolution of singularities which is not yet known in characteristic p, but also for the integration used in proving the invariance of our definitions, which is false in characteristic p. However, one can hope that the more formal aspects of our theory, in particular the systematic use of homology and its exact sequences, may be useful in characteristic p.

Another interesting topic for research is the algebraic treatment of monodromy and the cohomology of special fibres of a family. Here one can hope that the introduction of the sheaves of relative cohomology in the singular case (III, § 4, 5) will be useful. In particular, one can ask, do the De Rham cohomology groups of the fibres of a flat proper morphism obey any semi-continuity analogous to the semi-continuity theorems for coherent sheaves [EGA III, § 6, 7]?

A third problem is to study various filtrations on the De Rham cohomology. For instance, we do not know an algebraic proof of the degeneration of the first spectral sequence, which begins $E_1^{pq} = H^q(X, \Omega^p)$ for X smooth and proper over **C**. For a scheme Y with arbitrary singularities, since the complex $\hat{\Omega}_X^{\bullet}$ used to define the cohomology of Y, where Y is embedded in the smooth scheme X, is unique up to quasi-isomorphism, the sheaves $h^q(\hat{\Omega}_X^{\bullet})$ depend only on Y. Thus the second spectral sequence, which begins $E_2^{pq} = H^p(Y, h^q(\hat{\Omega}_X^{\bullet}))$ depends only on Y, and induces a well-defined filtration on $H^*(Y)$. It would be interesting to know more about it.

6. Writing style.

In presenting a subject with a great many technical details, one faces a difficult choice about how much generality to use where. Rather than state each result in its most general form, I have chosen to present the simplest case first, and more general cases later. This approach has two advantages: it presents the main ideas of the subject in a form which is not overburdened with details, and it achieves a certain economy of proofs, because in proving the more general form, one need only indicate what needs to be modified in the earlier proof. On the other hand, this approach demands more from the reader, who is asked to provide many details for himself. I hope he will be willing to accept this responsibility.

7. Acknowledgements.

This work owes so much to other people that it would be impossible to mention them all. I am especially indebted to A. Grothendieck, who awakened my interest in the subject, and who taught me most of the techniques used here, in the course of explaining his duality theory for my seminar at Harvard [RD]. I wish to thank P. Deligne and D. Lieberman, who first proved many results of this paper, for generously sharing their ideas on De Rham cohomology. I also wish to thank A. Ogus for a year of stimulating discussions, which culminated in the writing of this paper and his thesis [43]. Finally, I am grateful to the Alfred P. Sloan Foundation and the Research Institute for Mathematical Sciences of Kyoto University for their support during the preparation of this paper.

CHAPTER I

PRELIMINARIES

This chapter contains discussions of various technical matters which are logically necessary for the sequel but which can be omitted at first reading. Of special importance is § 4 on cohomology and inverse limits.

1. Cohomology theories.

In this paper, we will be dealing with cohomology of algebraic varieties, cohomology of complex analytic spaces, cohomology with constant coefficients, cohomology of formal schemes, and more. Historically these cohomology theories have been developed at different times, and by different methods. Since a main purpose of this paper is to prove comparison theorems, we should state clearly what cohomology theory we are using, and how it relates to the different theories which can be found in the literature.

Unless otherwise specified, we will always mean cohomology in the sense of derived functors: if X is a topological space, and F a sheaf of abelian groups on X, we define $H^i(X, F)$ to be the i-th right derived functor of the global section functor Γ, on the category $\boldsymbol{Ab}_X$ of all sheaves of abelian groups on X. (See for example [14, § 2.3] or [RD, Ch. II].) In particular, this cohomology can be calculated as follows. We take an injective resolution of F in the category $\boldsymbol{Ab}_X$, i.e., an exact sequence

$$0 \to F \to I^0 \to I^1 \to I^2 \to \dots$$

where the sheaves I^p are injective objects of $\boldsymbol{Ab}_X$. This gives a sequence of abelian groups

$$\Gamma(X, I^0) \xrightarrow{d^0} \Gamma(X, I^1) \xrightarrow{d^1} \Gamma(X, I^2) \to \dots,$$

and we have $H^i(X, F) \cong \ker d^i / \operatorname{im} d^{i-1}$.

If X or F has additional structure, for example if X is a scheme, or an analytic space, or a formal scheme, and if F has a structure of $\mathcal{O}_X$-Module, or is coherent, we forget the additional structure, and compute cohomology in the sense of topological spaces and sheaves of abelian groups. To compute cohomology with constant coefficients in a group G, we take the cohomology of the space with respect to the constant sheaf determined by G.

This is the cohomology theory which was introduced by Grothendieck in Tohoku [14], and is the theory which is used in EGA.

We now compare this theory to the other principal cohomology theories in use. The use of sheaf cohomology in algebraic geometry started with Serre's paper [FAC]. In that paper, and in the later paper [GAGA] Serre uses Čech cohomology for coherent sheaves on an algebraic variety with its Zariski topology. The equivalence of this theory with the derived functor theory follows from the "Theorem of Leray" (see [EGA III, 1.4.1]). The same argument, using Cartan's "Theorem B" shows that the Čech cohomology of a coherent analytic sheaf on a complex analytic space is equal to the derived functor cohomology. This ties in with the cohomology theory of the Cartan seminars. Gunning and Rossi [21] use a cohomology theory defined by using fine resolutions of a sheaf on a paracompact Hausdorff space. The equivalence of this theory with ours is shown in the book of Godement [12, Thm. 4.7.1, p. 181 and Ex. 7.2.1, p. 263], who shows at the same time that both theories coincide with his theory which is defined by a canonical flasque resolution. Godement also shows [12, Thm. 5.10.1, p. 228] that on a paracompact Hausdorff space, his theory coincides with Čech cohomology. This provides a bridge to the standard topological theories with constant coefficients, as developed in the book of Spanier [48]. He introduces Čech cohomology $H^*(X, G)$ on p. 327. He shows on p. 334 that for a paracompact Hausdorff space, it coincides with Alexander cohomology $\overline{H}^*(X, G)$. On the other hand, he has shown earlier (p. 314) that Alexander cohomology satisfies all of his axioms, and so coincides with singular cohomology.

If $(X, \mathcal{O}_X)$ is a locally ringed space, and F an $\mathcal{O}_X$-Module, then $H^i(X, F)$ can be computed also as the derived functors of Γ on the category of $\mathcal{O}_X$-Modules. This is because injective $\mathcal{O}_X$-Modules are flasque, and flasque sheaves are acyclic for cohomology [EGA, $\mathbf{0}_{\mathrm{III}}$, 12.1.1].

2. Direct images and inverse images.

Let $f: X \to Y$ be a continuous map of topological spaces. If F is a sheaf of abelian groups on X, we define the *direct image* f_*F to be the sheaf on Y, whose value on an open set $V \subseteq Y$ is given by $f_*F(V) = F(f^{-1}V)$. We denote by R^if_* the derived functors of f_*, considered as a functor from the category of abelian sheaves on X to the category of abelian sheaves on Y. As before, even if X and Y have additional structure, we mean R^if_* in this sense. Note that R^if_*F can be computed as the sheaf associated to the presheaf

$$V \mapsto H^i(f^{-1}V, F).$$

This is because the restriction of an injective sheaf to an open set is injective, and the operation "sheaf associated to a presheaf" is an exact functor.

The same remarks apply if f is a morphism of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. Thus we see that if F is an $\mathcal{O}_X$-Module, the sheaves R^if_*F can be calculated as derived functors of f_* on the category of $\mathcal{O}_X$-Modules, and hence they have a structure of $\mathcal{O}_Y$-Module.

Again let $f : X \to Y$ be a continuous map of topological spaces. Then for an abelian sheaf G on Y, we denote by $f^{-1}G$ the sheaf on X, whose value on an open set U is

$$f^{-1}G(U) = \varinjlim_{V \supseteq f(U)} G(V).$$

We call this the *topological inverse image* of G. If f is a morphism of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, then we define also the *inverse image* f^* in the sense of ringed spaces as $f^*G = f^{-1}G \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.

Let $f : X \to Y$ be a continuous map. Let F, G be abelian sheaves on X, Y respectively, and suppose given a map $\alpha : f^{-1}(G) \to F$. Then we obtain natural maps of cohomology

$$\alpha_i : H^i(Y, G) \to H^i(X, F)$$

for all i, with obvious functorial properties. Indeed, the map α gives maps

$$H^i(X, f^{-1}G) \to H^i(X, F),$$

so it is sufficient to define natural maps $H^i(Y, G) \to H^i(X, f^{-1}G)$. Let $G \to I^\bullet$ be an injective resolution of G. Now f^{-1} is an exact functor, so $f^{-1}G \to f^{-1}I^\bullet$ is a resolution, not necessarily injective. If $f^{-1}G \to J^\bullet$ is an injective resolution of $f^{-1}G$, then there is a natural map $f^{-1}I^\bullet \to J^\bullet$, whence maps of complexes $\Gamma(Y, I^\bullet) \to \Gamma(X, f^{-1}I^\bullet) \to \Gamma(X, J^\bullet)$, which give the desired maps on cohomology.

3. Hypercohomology.

Let $F^\bullet$ be a bounded below complex of sheaves of abelian groups on a topological space X. In other words, we have a collection of sheaves F^p, $p \in \mathbf{Z}$, with $F^p = 0$ for $p \ll 0$, and maps $d^p : F^p \to F^{p+1}$ such that $d^{p+1} d^p = 0$ for all p. We define the cohomology sheaves of $F^\bullet$ by $h^i(F^\bullet) = \ker d^i / \operatorname{im} d^{i-1}$. A map of complexes $\varphi : F^\bullet \to G^\bullet$ is a *quasi-isomorphism* if it induces an isomorphism $h^i(\varphi) : h^i(F^\bullet) \to h^i(G^\bullet)$ for all i.

We define the *hypercohomology* $\mathbf{H}^i(X, F^\bullet)$ of X with coefficients in the complex $F^\bullet$ to be the derived functors of Γ, in the sense of derived categories [RD, Ch. I]. In particular, the hypercohomology can be computed as follows: let $F^\bullet \to I^\bullet$ be an *injective resolution* of $F^\bullet$, i.e. a quasi-isomorphism of $F^\bullet$ to a complex $I^\bullet$, all of whose elements I^p are injective sheaves. Then

$$\mathbf{H}^i(X, F^\bullet) = h^i(\Gamma(X, I^\bullet)),$$

where $h^i = \ker d^i / \operatorname{im} d^{i-1}$.

One can also compute hypercohomology by using Cartan-Eilenberg resolutions [EGA, 0_{III}, 11.4]. We will often use the two spectral sequences of hypercohomology [EGA, *loc. cit.*] which have terms

(First) $\quad E_1^{pq} = H^q(X, F^p) \Rightarrow E^n = \mathbf{H}^n(X, F^\bullet)$

(Second) $\quad E_2^{pq} = H^p(X, h^q(F^\bullet)) \Rightarrow E^n = \mathbf{H}^n(X, F^\bullet).$

Similarly, if $f : X \to Y$ is a continuous map, and if $F^{\bullet}$ is a complex of abelian sheaves on X, we define the *hyper-direct image* sheaves $\mathbf{R}^i f_*(F^{\bullet})$ as the derived functors of f_*.

4. Inverse Limits.

In this section we gather together information on inverse limits and cohomology, including slight generalizations of results of Grothendieck [EGA, $\mathbf{0}_{\text{III}}$, § 13] and Roos [45]. I am indebted to Arthur Ogus for explaining the following elegant construction of the derived functors of $\varprojlim$, and for pointing out some deficiencies in an earlier write-up.

In order for inverse limits to exist in an abelian category $\mathscr{C}$, one needs to assume the existence of infinite direct products (this is axiom AB 3* of [14]). Furthermore, to get reasonable properties for $\varprojlim$, one wants the direct product functor to be exact (AB 4*). This is true for the category $\boldsymbol{Ab}$ of abelian groups, but fails in the category $\boldsymbol{Ab}(X)$ of abelian sheaves on a topological space X. However, we deal mostly with coherent sheaves on schemes or analytic spaces, in which case we can overcome this difficulty. We axiomatize this situation as follows.

Let $\mathscr{C}'$ be an abelian category with enough injectives (e.g. assume AB 5 and $\mathscr{C}'$ has a generator), and let $\mathscr{C}$ be a full subcategory of $\mathscr{C}'$. Assume

IL1. Arbitrary direct products exist in $\mathscr{C}'$.

IL2. There is a functor $\sigma : \mathscr{C}' \to \boldsymbol{Ab}$ such that

1) σ is exact on $\mathscr{C}$.

2) If $A \to B \to C$ is a sequence of objects in $\mathscr{C}'$, and if $\sigma(A) \to \sigma(B) \to \sigma(C)$ is exact, then $A \to B \to C$ is exact.

3) σ commutes with arbitrary direct products.

In applications we will have three cases where these conditions apply :

a) $\mathscr{C} = \mathscr{C}' = \boldsymbol{Ab}$ and $\sigma = \text{Id}$.

b) Let X be a noetherian scheme, $\mathscr{C}$ = the category of quasicoherent sheaves on X, $\mathscr{C}'$ = all abelian sheaves on X. Let $\mathfrak{B}$ be a base for the topology of X consisting of open affine subsets, and let $\sigma(F) = \prod_{U \in \mathfrak{B}} \Gamma(U, F)$, for any sheaf F.

c) Let X be a complex analytic space, $\mathscr{C}$ = the category of coherent analytic sheaves on X, $\mathscr{C}'$ = all abelian sheaves on X, $\mathfrak{B}$ = a base for the topology of X consisting of open Stein subsets of X, and $\sigma(F) = \prod_{U \in \mathfrak{B}} \Gamma(U, F)$ for any sheaf F.

Now we consider inverse systems $A = (A_n)_{n \geq 1}$ of objects of $\mathscr{C}'$, with maps $\varphi_{mn} : A_m \to A_n$ for $m \geq n$, where $\varphi_{mn} \circ \varphi_{\ell m} = \varphi_{\ell n}$ for $\ell \geq m \geq n$. The category of all such inverse systems is denoted by pro-$\mathscr{C}'$. It follows from IL1 that inverse limits exist, so $\varprojlim$ becomes a functor from pro-$\mathscr{C}'$ to $\mathscr{C}'$. It is always left exact. Since pro-$\mathscr{C}'$ also has enough injectives, we can consider the right derived functors of $\varprojlim$, denoted by $\varprojlim^{(i)}$, $i \geq 1$.

It follows from IL2 that $\varprojlim$ commutes with σ. It also follows that the direct product Π is exact on $\mathscr{C}$, and we deduce the useful consequence that if $A=(A_n)$ is an inverse system of objects of $\mathscr{C}$, and if all the maps φ_{mn} are surjective, then the natural map $\varprojlim A_n \to A_m$ is surjective for all m. Indeed, this is true in the category ***Ab***, so we can pull the result back with σ.

Now let $\mathbf{N}$ be the set of natural numbers, and define a topology on $\mathbf{N}$ by taking as open sets $\mathbf{N}$ itself, and $U_n=[1, n]$ for $n=1, 2, \ldots$. To an inverse system $A \in \text{pro-}\mathscr{C}$, we associate a sheaf $\widetilde{A}$ on $\mathbf{N}$ by $\widetilde{A}(U_n)=A_n$, $\widetilde{A}(\mathbf{N})=\varprojlim A_n$. This gives an equivalence of categories between pro-$\mathscr{C}$ and $\mathscr{C}$-valued sheaves on $\mathbf{N}$. Under this correspondence, $\varprojlim A_n = H^0(\mathbf{N}, \widetilde{A})$. Thus we can calculate $\varprojlim^{(i)} A_n$ as the cohomology $H^i(\mathbf{N}, \widetilde{A})$ for all $i \geq 0$.

Note that $\widetilde{A}$ is flasque if and only if the maps of the inverse system A are all surjective. In general, we can calculate cohomology by flasque resolutions. Given an inverse system $A \in \text{pro-}\mathscr{C}$, define $B \in \text{pro-}\mathscr{C}$ by $B_n = \prod_{i\leq n} A_i$ for each n. Define $A_n \to B_n$ by $\prod_{i \leq n} \varphi_{ni}$. Then $\widetilde{B}$ is flasque, and $A \to B$ is injective. Let Q be the quotient:

$$0 \to A \to B \to Q \to 0.$$

Then $Q_n = \prod_{i \leq n} (A_i/\varphi_{ni}(A_n))$. In particular, the maps of the inverse system Q are surjective, so $\widetilde{Q}$ is also flasque. Thus we have

Proposition (**4.1**) (Roos). — *Let* $A=(A_n)$ *be an inverse system in the abelian category* $\mathscr{C}$, *indexed by the natural numbers* $\mathbf{N}$. *Then* $\varprojlim^{(i)} A_n = 0$ *for* $i \geq 2$.

Furthermore, the above construction gives an explicit description of $\varprojlim^{(1)} A_n$. Indeed, for each i, let $\hat{A}_i = \varprojlim (A_i/\varphi_{ni}(A_n))$ be the completion of A_i with respect to the filtration by the subobjects $\{\varphi_{ni}(A_n)\}_{n \geq i}$. Then:

Proposition (**4.2**). — *With the above notation*

$$\varprojlim^{(1)} A_n = \prod_i (\hat{A}_i/\operatorname{Im}(A_i)).$$

This follows directly from the exact sequence

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim Q_n \to \varprojlim^{(1)} A_n \to 0.$$

Recall that A is said to satisfy the *Mittag-Leffler condition* (ML), if for every i, the filtration of A_i by $\{\varphi_{ni}(A_n)\}$ is eventually constant. In that case it is clear that $A_i \to \hat{A}_i$ is surjective, so we have

Corollary (**4.3**). — *If* A *satisfies* (ML), *then* $\varprojlim^{(1)} A_n = 0$.

Next we study the cohomology of an inverse system of complexes (which is the same thing as a complex of inverse systems).

Proposition (4.4). — *Let* $A^{\bullet} = (A^p_n)_{n\in\mathbf{N},\, p\in\mathbf{Z}}$ *be an inverse system of complexes, and assume for each* p *that the inverse system* $(A^p_n)_{n\in\mathbf{N}}$ *is surjective. Then for each* p *there is an exact sequence*

$$0 \to \varprojlim{}^{(1)} h^{p-1}(A^{\bullet}_n) \to h^p(\varprojlim A^{\bullet}_n) \to \varprojlim h^p(A^{\bullet}_n) \to 0.$$

Proof. — As usual, let Z denote cocycles and let B denote coboundaries. Then we have for each p, n

$$0 \to Z^p_n \to A^p_n \to B^{p+1}_n \to 0$$

$$0 \to B^p_n \to Z^p_n \to h^p(A^{\bullet}_n) \to 0.$$

Taking inverse limits, and noting that A^p_n and hence B^p are surjective systems, we have

$$0 \to \varprojlim Z^p_n \to \varprojlim A^p_n \to \varprojlim B^{p+1}_n \to \varprojlim{}^{(1)} Z^p_n \to 0,$$

$$0 \to \varprojlim B^p_n \to \varprojlim Z^p_n \to \varprojlim h^p(A^{\bullet}_n) \to 0,$$

and $\varprojlim{}^{(1)} Z^p_n \cong \varprojlim{}^{(1)} h^p(A^{\bullet}_n)$.

Now $\varprojlim$ is left exact, so $\varprojlim Z^p_n = Z^p(\varprojlim A^{\bullet}_n)$. Taking cohomology of the limit, we have

$$0 \to Z^p(\varprojlim A^{\bullet}_n) \to \varprojlim A^{\bullet}_n \to B^{p+1}(\varprojlim A^{\bullet}_n) \to 0$$

$$0 \to B^p(\varprojlim A^{\bullet}_n) \to Z^p(\varprojlim A^{\bullet}_n) \to h^p(\varprojlim A^{\bullet}_n) \to 0.$$

The conclusion follows by a simple diagram chase.

Note that in view of the preceding corollary, this is a slight generalization of [EGA, $\mathbf{0}_{\text{III}}$, 13.2.3].

Next we come to a basic result about the cohomology of an inverse limit of sheaves on a topological space. This result, which is a slight generalization of [EGA, $\mathbf{0}_{\text{III}}$, 13.3.1], will be frequently used in the sequel.

Let X be a topological space, and let $F = (F_n)_{n\in\mathbf{N}}$ be an inverse system of abelian sheaves on X. One sees immediately that the presheaf $U \mapsto \varprojlim F_n(U)$ is a sheaf, and that it is the inverse limit $\varprojlim F_n$ of the sheaves F_n.

Theorem (4.5). — *Let* $(F_n)_{n\in\mathbf{N}}$ *be an inverse system of abelian sheaves on the topological space* X. *Let* T *be a functor on the category of abelian sheaves on* X, *which commutes with arbitrary direct products. We assume that there is a base* $\mathfrak{B}$ *for the topology of* X *such that*:

a) *For each* $U \in \mathfrak{B}$, *the inverse system* $(F_n(U))$ *is surjective.*
b) *For each* $U \in \mathfrak{B}$, $H^i(U, F_n) = 0$ *for all* $i > 0$ *and all* n.

Then, for each i, *there is an exact sequence*

$$0 \to \varprojlim{}^{(1)} R^{i-1}T(F_n) \to R^iT(\varprojlim F_n) \xrightarrow{\alpha_i} \varprojlim R^iT(F_n) \to 0.$$

In particular, if for some i, $(R^{i-1}T(F_n))$ *satisfies* (ML), *then* α_i *is an isomorphism.*

Lemma **(4.6).** — *Let* $G \xrightarrow{\alpha} F$ *be a map of sheaves and let* $0 \to F \to I^\bullet$ *be a given injective resolution of* F. *Then there exists an injective resolution* $0 \to G \to J^\bullet$, *and a commutative diagram*

$$\begin{array}{ccccc} 0 & \longrightarrow & G & \longrightarrow & J^\bullet \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta^\bullet} \\ 0 & \longrightarrow & F & \longrightarrow & I^\bullet \end{array}$$

such that for every p, *there is an isomorphism* $J^p \cong I^p \oplus K^p$, *and* β^p *is the projection onto the first factor.* (It follows that K^p is also injective.)

Proof. — Let $G \to K^0$ be an injection of G into an injective sheaf K^0. Let $J^0 = I^0 \oplus K^0$. Define a map $G \to J^0$ by taking the sum of the maps $G \xrightarrow{\alpha} F \to I^0$ and $G \to K^0$. Let $\beta^0 : J^0 \to I^0$ be the projection. Then we have a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & G & \longrightarrow & J^0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta^0} \\ 0 & \longrightarrow & F & \longrightarrow & I^0 \end{array}$$

Taking the cokernels of the horizontal maps and proceeding inductively gives the result.

Proof of theorem. — Using the lemma, we construct for each n an injective resolution $0 \to F_n \to I_n^\bullet$ of F_n, together with compatible maps $\beta_n^\bullet : I_{n+1}^\bullet \to I_n^\bullet$, such that for each n, p, there is an isomorphism $I_{n+1}^p \cong I_n^p \oplus K_n^p$, and β_n^p is the projection onto the first factor. We claim that the natural map

$$\varprojlim F_n \to \varprojlim I_n^\bullet$$

makes the complex $\varprojlim I_n^\bullet$ into an injective resolution of $\varprojlim F_n$. In the first place, for each p, $\varprojlim I_n^p \cong I_1^p \oplus \prod_{n \geq 1} K_n^p$, and hence, being a direct product of injective sheaves, it is also an injective sheaf. Next, to see that we have a resolution of sheaves, it is sufficient to check for each $U \in \mathfrak{B}$ that the sequence of groups

$$0 \to \Gamma(U, \varprojlim F_n) \to \Gamma(U, \varprojlim I_n^0) \to \Gamma(U, \varprojlim I_n^1) \to \dots$$

is exact. This sequence is the inverse limit of the system of sequences

$$0 \to \Gamma(U, F_n) \to \Gamma(U, I_n^0) \to \Gamma(U, I_n^1) \to \dots,$$

which are exact because of the hypothesis *b)* of the theorem. Now by hypothesis *a)* the inverse system $(\Gamma(U, F_n))$ is surjective; by construction the inverse system $(\Gamma(U, I_n^p))$ is surjective for each p. Thus it follows from Proposition (4.4) above that the inverse limit is also an exact sequence.

Now we can use the above injective resolutions of F_n and $\varprojlim F_n$ to calculate the derived functors of T. We have

$$R^i T(F_n) = h^i(T(I_n^\bullet))$$

and

$$R^i T(\varprojlim F_n) = h^i(T(\varprojlim I_n^\bullet)).$$

Now since T commutes with direct products by hypothesis, we have $T(\varprojlim I_n^\bullet) = \varprojlim T(I_n^\bullet)$. Furthermore, the inverse system $T(I_n^p)$ is surjective for each p, since I_n^p is a direct summand of I_{n+1}^p, so we can apply Proposition (4.4) again, and get the result of the theorem.

Remarks. — In applications, we will apply this theorem to the functors Γ, Γ_Y, and f_*. Note that the hypothesis *a)* could be weakened to say that $(F_n(U))$ satisfies (ML), but we will not need this.

Next we include some special results about inverse limits of quasi-coherent sheaves. We say a sheaf is *countably quasi-coherent* if it is quasi-coherent and locally countably generated.

Proposition **(4.7).** — *Let* X *be a scheme of finite type over a field k, let* G *be a countably quasi-coherent sheaf on* X, *and let* $\{H_n\}$ *be a descending sequence of quasi-coherent subsheaves of* G. *Assume that the natural map* $G \to \varprojlim (G/H_n)$ *is a surjective map of sheaves. Then the sequence* $\{H_n\}$ *is eventually constant.*

Proof. — First of all, we can find a countable field $k_0 \subseteq k$ such that X, G, H_n all are defined over k_0. Then one sees easily that the hypothesis $G \to \varprojlim (G/H_n)$ surjective descends to k_0, and the conclusion ascends to k, so we reduce to the case k a countable field.

Next, covering X with a finite number of affine open sets, we reduce to the case X affine, say $X = \operatorname{Spec} A$. Let $M = H^0(X, G)$, $N_n = H^0(X, H_n)$. Then $\{N_n\}$ is a descending sequence of submodules of M, and it is sufficient to show it is stationary.

We cannot assert that the map $M \to \varprojlim (M/N_n)$ is surjective, because $\bigcap_n H_n$ might not be quasi-coherent, and so $H^1(X, \bigcap_n H_n)$ might not be zero. However, we can assert that $\varprojlim (M/N_n)$ is a countable set. Indeed, $\varprojlim (M/N_n) = H^0(X, \varprojlim (G/H_n))$. Since the map of sheaves $G \to \varprojlim (G/H_n)$ is surjective, for any section $s \in H^0(X, \varprojlim (G/H_n))$ there is a (finite) open cover $\{U_i\}$ of X, and sections $s_i \in H^0(U_i, G)$ which map to s, and hence determine s. Now the set of finite open covers of X is countable, and for each open set $U \subseteq X$, the set $H^0(U, G)$ is countable. So we find that $\varprojlim (M/N_n)$ is a countable set.

From this it follows that the sequence $\{N_n\}$ is stationary. For if not, one could construct distinct elements of $\varprojlim (M/N_n)$ corresponding to each infinite dyadic number, and these form an uncountable set.

Proposition **(4.8).** — *Let* X *be a scheme of finite type over a field k, let* $\{F_n\}$ *be an inverse system of quasi-coherent sheaves on* X, *and assume that* $\varprojlim F_n$ *is a quotient of a coherent* (resp. *countably quasi-coherent*) *sheaf* G. *Then* $\varprojlim F_n$ *itself is coherent* (resp. *countably quasi-coherent*).

Proof. — Let $F'_n \subseteq F_n$ be the image of G, and let H_n be the kernel, so we have

$$0 \to H_n \to G \to F'_n \to 0.$$

Taking inverse limits, we have maps

$$G \to \varprojlim F'_n \to \varprojlim F_n$$

whose composition is surjective. Hence $G \to \varprojlim F'_n$ is surjective, and $\varprojlim F'_n = \varprojlim F_n$.

Now by the previous proposition, the sequence $\{H_n\}$ must be stationary. Hence $\varprojlim F'_n = F'_m$ for sufficiently large m, so it is coherent (resp. countably quasi-coherent) as required.

Proposition **(4.9)**. — *Let* $\{F_n\}$ *be an inverse system of countably quasi-coherent sheaves on a scheme* X *of finite type over a field* k. *Then the following conditions are equivalent*:

(i) $\{F_n\}$ *satisfies* (ML);

(ii) $\varprojlim^{(1)} F_n = 0$;

(iii) $\varprojlim^{(1)} F_n$ *is countably quasi-coherent.*

Proof. — We need only prove (iii) ⇒ (i). So assume that $\varprojlim^{(1)} F_n$ is countably quasi-coherent. By (4.2) above, we have

$$\varprojlim^{(1)} F_n = \prod_n (\hat{F}_n / \mathrm{Im}\, F_n).$$

Thus for each n, $\hat{F}_n/\mathrm{Im}\, F_n$ is a quotient of a countably quasi-coherent sheaf. Since F_n is also countably quasi-coherent, it follows that $\hat{F}_n$ is a quotient of a countably quasi-coherent sheaf. Now by the previous proposition and its proof, it follows that $\hat{F}_n$ is countably quasi-coherent, and that the filtration on F_n is eventually stationary. But this is exactly the Mittag-Leffler condition.

Examples. — 1. In Proposition (4.7) it is not sufficient to assume that X is a noetherian scheme. For example, let $X = \mathrm{Spec}\, k[[t]]$, $G = \mathcal{O}_X$, $H_n = I_P^n$, where P is the closed point. Then $G \to \varprojlim (G/H_n)$ is surjective, but the sequence is not stationary, and the latter sheaf is not quasi-coherent. Note also in this example that $\varprojlim^{(1)} H_n = 0$ by (4.2), although $\{H_n\}$ does not satisfy (ML).

2. Even when $\varprojlim F_n$ is coherent, the inverse system may be bad. For example, let $X = \mathbf{A}^1_k$, let $P_1, P_2, \ldots$ be an infinite sequence of closed points, and let $F_n = I_{P_1} \ldots I_{P_n}$. Then $\varprojlim F_n = 0$, but the sequence does not satisfy (ML), and its $\varprojlim^{(1)}$ is huge.

Finally, we include one result about commutation of inverse limits and tensor products.

Proposition **(4.10)**. — *Let* $\{F_n\}$ *be an inverse system of countably quasi-coherent sheaves on the scheme* X *of finite type over* k, *and let* E *be a flat* $\mathcal{O}_X$*-Module. We consider the natural map*

$$\alpha : (\varprojlim F_n) \otimes E \to \varprojlim (F_n \otimes E).$$

If we assume that $\varprojlim F_n$ *is countably quasi-coherent, then* α *is injective. If we assume furthermore that* $\varprojlim^{(1)} F_n$ *is countably quasi-coherent, then* α *is an isomorphism.*

Proof. — As in the proof of (4.8), let F'_n be the image of $F = \varprojlim F_n$ in F_n. If F is countably quasi-coherent, then as above, we have $F = F'_n$ for $n \gg 0$. Hence $F \otimes E = F'_n \otimes E$ for $n \gg 0$. On the other hand, F'_n is a subsheaf of F_n, and E is flat, so we have

$$0 \to F'_n \otimes E \to F_n \otimes E.$$

It follows that we have an injection

$$\beta : \varprojlim (F'_n \otimes E) \to \varprojlim (F_n \otimes E).$$

But the first system is eventually constant, hence is equal to $F'_n \otimes E$ for $n \gg 0$, which is the same as $F \otimes E$, so α is injective.

Assuming furthermore that $\varprojlim^{(1)} F_n$ is countably quasi-coherent, we have by (4.9) that $\{F_n\}$ satisfies (ML). Let $Q_n = F_n/F'_n$. Then $\{Q_n\}$ also satisfies (ML), and $\varprojlim Q_n = 0$, so $\{Q_n\}$ is essentially zero. It follows that the inverse system $\{Q_n \otimes E\}$ is essentially zero, and so $\varprojlim (Q_n \otimes E) = 0$. Hence β is an isomorphism, and so α is also an isomorphism.

5. Completions.

Let X be a noetherian scheme or a complex analytic space, and let Y be a closed subspace, defined by a coherent sheaf of ideals I. We define the *formal completion* of X along Y, denoted by $X_{/Y}$ or $\hat{X}$, to be the ringed space whose underlying topological space is Y, and whose sheaf of rings is $\varprojlim (\mathcal{O}_X/I^r)$. There is a natural morphism of ringed spaces $i : \hat{X} \to X$. In the algebraic case, $\hat{X}$ is a *formal scheme* in the sense of [EGA I, § 10]. In the analytic case, $\hat{X}$ is what one might call a *formal analytic space*. However, we have no need to develop a theory of abstract formal analytic spaces.

If F is a coherent sheaf on X, there are two natural ways to define its completion $\hat{F}$: one is to take $i^*F = F \otimes_{\mathcal{O}_X} \mathcal{O}_{\hat{X}}$. The other is to take $\varprojlim (F/I^r F)$. The following proposition shows that these two definitions coincide, so that we may use the notation $\hat{F}$ for either of them, without ambiguity.

Proposition (5.1). — *Let* X *be a noetherian scheme or a complex analytic space, and let* Y *be a closed subspace defined by a coherent sheaf of ideals* I. *Let* $\hat{X}$ *be the formal completion as above. Then for any coherent sheaf* F *on* X, *the natural map*

$$i^*F \to \varprojlim (F/I^r F)$$

is an isomorphism. Furthermore, denoting this sheaf by $\hat{F}$, *the functor* $F \mapsto \hat{F}$ *is exact on the category of coherent sheaves on* X.

Proof. — (See [EGA I.10.8.8] for the algebraic case.) The question is local, so we may assume that F is the quotient of a free sheaf $L \cong \mathcal{O}_X^n$. Let G be the kernel, so that we have an exact sequence

$$0 \to G \to L \to F \to 0,$$

with G also coherent. For each r, we have an exact sequence

$$0 \to G/(G \cap I^r L) \to L/I^r L \to F/I^r F \to 0.$$

We claim that on sufficiently small open sets U, the inverse system $G/(G \cap I^r L)$ is cofinal with the inverse system $G/I^r G$. Indeed, in the algebraic case, this follows from the Artin-Rees theorem, as soon as U is affine. However, in the analytic case, the ring of global sections of $\mathcal{O}_X$ over an open set U is not noetherian in general. So we take an open set $U \subseteq X$ whose closure $\overline{U}$ is compact. For every point $x \in \overline{U}$, the local ring $\mathcal{O}_{x,X}$ is noetherian. So for each r, there is an $s = s(r, x)$ such that

$$(G \cap I^r L)_x \supseteq (I^r G)_x \supseteq (G \cap I^s L)_x.$$

The sheaves being coherent, the same is true in a neighborhood $V(r, x)$. Covering $\overline{U}$ with a finite number of these neighborhoods, we find an s such that $I^r G \supseteq G \cap I^s L$ on all of U. The same argument for each r shows that the two inverse systems are cofinal on U. Since the system $\{G/I^r G\}$ is surjective it satisfies (ML). It follows that the system $\{G/(G \cap I^r L)\}$ satisfies (ML). Thus we get an exact sequence of inverse limits. The functor i^* is in any case right exact, so we have exact sequences

$$\begin{array}{ccccccccc}
 & & i^*G & \longrightarrow & i^*L & \longrightarrow & i^*F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \varprojlim (G/I^r G) & \longrightarrow & \varprojlim (L/I^r L) & \longrightarrow & \varprojlim (F/I^r F) & \longrightarrow & 0
\end{array}$$

Now β is an isomorphism because L is free, and the map $i^* \mathcal{O}_X \to \varprojlim (\mathcal{O}_X / I^r \mathcal{O}_X)$ is an isomorphism by construction. Hence γ is surjective. But F was any coherent sheaf, and G is coherent, so α is surjective. Hence γ is an isomorphism.

This proves the first statement. The same argument applied to any short exact sequence of coherent sheaves shows that the functor $F \to \varprojlim (F/I^r F)$ is exact.

6. Associated analytic spaces.

For foundations of the theory of complex analytic spaces, we refer to Gunning and Rossi [21], except that we will allow nilpotent elements in the structure sheaves. If X is a scheme of finite type over **C**, there is a natural way to associate with X a complex analytic space X_h (see Serre [GAGA]). Roughly speaking, the construction is as follows. Cover X with open affine sets U_i. Embed U_i as a closed subscheme of a suitable affine space $\mathbf{A}^n_{\mathbf{C}}$, and let its ideal be generated by polynomials $f_1, \ldots, f_r$. These polynomials define a closed analytic subspace of $\mathbf{C}^n$, which we call $(U_i)_h$. Glue together the analytic spaces $(U_i)_h$ according to the original glueing data to obtain X_h. We call X_h the *associated analytic space* of X. There is a natural morphism of ringed spaces $j : (X_h, \mathcal{O}_{X_h}) \to (X, \mathcal{O}_X)$. The sheaf of rings $\mathcal{O}_{X_h}$ is faithfully flat over $\mathcal{O}_X$ [GAGA], and so the functor j^* is exact and faithful. For any sheaf of $\mathcal{O}_X$-modules F, we denote by F_h the sheaf j^*F. According to the general principles of § 2 above, there are natural maps of cohomology

$$\alpha_i : H^i(X, F) \to H^i(X_h, F_h).$$

A fundamental result, upon which the comparison theorems of this paper are based, is the following theorem of Serre.

Theorem (Serre [GAGA, Thm. 1, p. 19]). — *Let* X *be a projective variety over* **C**, *and let* F *be a coherent sheaf on* X. *Then the natural maps*

$$\alpha_i : H^i(X, F) \to H^i(X_h, F_h)$$

are isomorphisms, for all i.

Note that this is the easiest of the three main theorems in [GAGA]. Its proof, given in [GAGA, § 13] depends only on knowing that $H^i(\mathbf{P}^n_h, \mathcal{O}_h) = \mathbf{C}$ for $i=0$, $=0$ for $i>0$. It does not use Cartan's theorems A and B.

If Y is a closed subscheme of X, then we can consider the formal completion $\hat{X}$ and its natural map $i : \hat{X} \to X$. On the other hand, we can consider the completion $\hat{X}_h$ of X_h along Y_h, and its map $i' : \hat{X}_h \to X_h$. Then there is a natural map of ringed spaces $j' : \hat{X}_h \to \hat{X}$, making a commutative diagram.

$$\begin{array}{ccc} \hat{X}_h & \xrightarrow{i'} & X_h \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\ \hat{X} & \xrightarrow{i} & X \end{array}$$

We could call $\hat{X}_h$ the formal analytic space associated to the formal scheme X. For any coherent formal sheaf $\mathcal{F}$ on $\hat{X}$, we denote by $\mathcal{F}_h$ the sheaf $j'^*(\mathcal{F})$. Since we have a commutative diagram, we have for any coherent sheaf F on X

$$(F_h)\hat{} = i'^* j^*(F) = j'^* i^*(F) = (\hat{F})_h.$$

Thus we will denote both sheaves by $\hat{F}_h$, with no ambiguity.

A useful consequence of Serre's theorem is the following result about cohomology of formal completions.

Proposition **(6.1)**. — *Let* X *be a projective scheme over* **C**, *and let* Y *be a closed subscheme. Let* F *be a coherent sheaf on* X. *Then the natural maps of cohomology*

$$\alpha_i : H^i(\hat{X}, \hat{F}) \to H^i(\hat{X}_h, \hat{F}_h)$$

are isomorphisms for all i.

Proof. — By (5.1) above we have $\hat{F} = \varprojlim (F/I^r F)$, and $\hat{F}_h = \varprojlim (F_h/I_h^r F)$, where I is the sheaf of ideals of Y. The sheaves $F/I^r F$ are coherent, so by Serre's theorem, the maps

$$H^i(X, F/I^r F) \to H^i(X_h, F_h/I_h^r F_h)$$

are isomorphisms for all i and r. Now we apply (4.5) above. We take for $\mathfrak{B}$ the set of open affine subsets in the algebraic case, and the set of open Stein subsets in the analytic case. Then the hypotheses *a)* and *b)* are satisfied. On the other hand, the cohomology groups $H^i(X, F/I^rF)$ are all finite-dimensional, so the inverse systems all satisfy (ML). This gives our result.

We will also need a relative form of the comparison theorem, which generalizes Serre's theorem.

Theorem (Grothendieck [19, XII.4.2]). — *Let* $f: X \to Y$ *be a proper morphism of schemes of finite type over* **C**. *Let* F *be a coherent sheaf on* X. *Then the natural maps*

$$\alpha_i: (R^i f_* F)_h \to R^i f^h_*(F_h)$$

are isomorphisms.

The proof uses dévissage and natural generalizations of Serre's techniques. Note that this result implies that the sheaves $R^i f^h_*(F_h)$ are coherent. This is a special case of the theorem of Grauert [13] on the coherence of higher direct images of coherent sheaves under a proper morphism of analytic spaces. In our case, the proof is more elementary, since we are dealing with analytic spaces which come from algebraic varieties.

From this result we deduce the " Fundamental theorem of a proper morphism " in the case of analytic spaces which come from algebraic varieties (cf. [EGA III, 4.1.5] for the algebraic case).

Proposition **(6.2)**. — *Let* $f: X' \to X$ *be a proper map of schemes of finite type over* **C**. *Let* Y *be a closed subset of* X, *and let* $Y' = f^{-1}(Y)$. *Let* ^ *denote formal completion along* Y *or* Y′, *respectively. Then for any coherent sheaf* F *on* X′, *the natural maps*

$$(R^i f^h_* F_h)^\wedge \to R^i \hat{f}^h_* \hat{F}_h$$

are isomorphisms.

Proof. — Looking at the algebraic fundamental theorem [EGA, *loc. cit.*] and its proof, we will use the facts that the maps

$$(R^i f_* F)^\wedge \to \varprojlim R^i f_* F_r$$

are isomorphisms, that the system $\{R^i f_* F_r\}$ satisfies (ML), and that kernel and cokernel systems of the maps

$$(R^i f_* F) \otimes (\mathcal{O}_X / I^r) \to R^i f_* F_r$$

are both essentially zero (this is equivalent to the statement in [EGA III, 4.1.7] that a certain filtration is I-good).

The property of an inverse system of coherent sheaves satisfying (ML) or being essentially zero carries over under the exact functor h. We deduce that the inverse systems

$$(R^i f_* F)_h \otimes (\mathcal{O}_{X_h} / I^r_h) \to (R^i f_* F_r)_h$$

both satisfy (ML), and that their kernel and cokernel systems are essentially zero. It follows that their inverse limits are isomorphic, so we have

$$(R^i f_* F)^{\wedge}_h \xrightarrow{\approx} \varprojlim (R^i f_* F_r)_h.$$

Using the relative comparison theorem above on both sides, we have isomorphisms

$$(R^i f^h_* F_h)^{\wedge} \xrightarrow{\approx} \varprojlim R^i f^h_* F_{rh}.$$

Finally, we apply (4.5) above to the functor f^h_* to deduce isomorphisms

$$R^i \hat{f}^h_* \hat{F}_h \xrightarrow{\approx} \varprojlim R^i f^h_* F_{rh}.$$

Combining, we have the desired result.

Remark. — We do not know whether a similar result will hold for an arbitrary proper morphism of analytic spaces and a coherent analytic sheaf. To copy the proof of the algebraic version, it would be sufficient to prove an analytic analogue of [EGA III, 3.3.2]. This could be considered as a generalization of Grauert's coherence theorem.

7. Functorial maps on De Rham cohomology.

Let X be a smooth scheme over **C**, and let $\Omega^\cdot$ be the De Rham complex on X. If Y is a closed subset of X, we may wish to complete along Y. On the other hand, we may wish to consider the associated analytic space. The functors ^ and h are defined on the category of $\mathcal{O}_X$-Modules. The sheaves Ω^i are coherent $\mathcal{O}_X$-Modules, but the maps d are not $\mathcal{O}_X$-linear. So we adopt an *ad hoc* definition of $\hat{\Omega}^\cdot$ and $\Omega^\cdot_h$.

In the first case, let I be the sheaf of ideals of Y. Then $d(I^r) \subseteq I^{r-1}$, so that the maps $d : \Omega^i \to \Omega^{i+1}$ are continuous for the I-adic topology. This allows us to define $d : \hat{\Omega}^i \to \hat{\Omega}^{i+1}$, and hence the complex $\hat{\Omega}^\cdot$. If $i : \hat{X} \to X$ is the natural map of ringed spaces, we have a natural map of complexes

$$i^{-1}\Omega^\cdot \to \hat{\Omega}^\cdot.$$

Thus according to the general principles of § 2 above, we have natural maps on cohomology

$$H^i(X, \Omega^\cdot) \to H^i(\hat{X}, \hat{\Omega}^\cdot).$$

For the associated analytic space, Ω^i_h is just the sheaf of holomorphic i-forms on X_h. We know how to differentiate holomorphic functions, so we define $d : \Omega^i_h \to \Omega^{i+1}_h$ in the usual way. Thus we define $\Omega^\cdot_h$ to be the holomorphic De Rham complex on X_h. If $j : X_h \to X$ is the natural map of ringed spaces, we have a map of complexes

$$j^{-1}\Omega^\cdot \to \Omega^\cdot_h,$$

and hence as above, we have natural maps on cohomology

$$H^i(X, \Omega^\cdot) \to H^i(X_h, \Omega^\cdot_h).$$

Chapter II

GLOBAL ALGEBRAIC THEORY

This chapter is devoted to the purely algebraic development of the theory of algebraic De Rham cohomology and homology of a scheme Y of finite type over a field k of characteristic zero not necessarily algebraically closed. The comparison with singular cohomology in the complex case will be discussed in Chapter IV.

We consider a scheme Y which admits an embedding as a closed subscheme of a scheme X smooth over k. Let Ω_X^p denote the sheaf of p-differential forms on X over k, and let $\Omega_X^{\bullet}$ denote the complex $\mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \rightarrow \dots$ of sheaves of differential forms, which we call the *De Rham complex* of X. We refer to [EGA IV, § 16, 17] for general results on differentials. Then we define the De Rham cohomology of Y to be the hypercohomology of the formal completion $\hat{\Omega}_X^{\bullet}$ of the De Rham complex of X along Y. This is the same definition used by Deligne in his (unpublished) lectures at Harvard in 1969, and agrees at least in the proper case with the inverse limit definition used by Lieberman and Herrera [29]. If Y does not admit a global embedding into a smooth scheme, the definition can be generalized (see Remark at end of § 1), but for simplicity we will stick to the embeddable case.

In developing the cohomology theory, our first task is to show that this definition is independent of the choice of the embedding. This is accomplished using an algebraic analogue of the famous lemma 17 of Atiyah and Hodge [2]. Characteristic zero is essential.

Next we define the homology of Y by using the local cohomology of the De Rham complex of X with supports in Y. This definition was suggested by Grothendieck [16, footnote 9, p. 101]. Again we must show that the definition is independent of the embedding. This is more difficult than in the case of cohomology, because to define the covariant functorial map of homology we need something like the trace map used in duality theory. Thus we are led to introduce a canonical resolution of the De Rham complex, using the notion of Cousin complex of [RD].

Next we establish a number of functorial properties of cohomology and homology, and give exact sequences relating to closed subsets and birational morphisms. We also prove a duality theorem relating the cohomology and homology of a proper scheme over k. The duality theorem is proved by using spectral sequences to reduce to the duality theorem for coherent sheaves.

To establish the finite-dimensionality of our cohomology and homology groups, we apply Hironaka's resolution of singularities. Using all the functorial properties developed earlier, the proof reduces to the case of a smooth proper scheme over k, where the result follows from Serre's finiteness theorem for cohomology of coherent sheaves.

In the final section of this chapter we construct the cohomology class associated to a cycle on a smooth scheme, and show that it is compatible with intersection theory. In particular, it follows that our cohomology theory has the properties needed for Grothendieck's construction of the Chern classes of vector bundles [15].

1. Algebraic De Rham cohomology.

In this section we define the algebraic De Rham cohomology of a scheme Y of finite type over a field k of characteristic zero, which admits an embedding as a closed subscheme of a smooth scheme X over k. We call Y an *embeddable* scheme over k. We define the cohomology of Y by taking the formal completion of the De Rham complex on X along Y. Then we show that this definition is independent of the choice of embedding, and we show that this cohomology is a contravariant functor in Y.

Definition. — Let Y be an embeddable scheme over k. Let $Y \to X$ be a closed immersion of Y into a smooth scheme over k. Then we define the *algebraic De Rham cohomology* of Y by

$$H^q_{DR}(Y) = \mathbf{H}^q(\hat{X}, \hat{\Omega}^{\bullet}_X),$$

the hypercohomology of the formal completion of the De Rham complex $\Omega^{\bullet}_X$ along Y.

To show that this definition is independent of the choice of embedding, we will need the following proposition.

Proposition (**1.1**) (char $k = 0$). — *Let $f : X \to Y$ be either a smooth morphism or a closed immersion of smooth schemes over k. Let Z be a closed subscheme of X such that the restriction $f : Z \to Y$ is a closed immersion. Then the natural map*

$$f^* : \hat{\Omega}^{\bullet}_Y \to \hat{\Omega}^{\bullet}_X,$$

where ^ *denotes formal completion along Z, is a quasi-isomorphism of complexes of abelian groups on Z.*

Proof. — *Case 1.* Suppose $f : X \to Y$ is a closed immersion. The question is local, so we can factor f into a sequence of closed immersions of codimension one. By composition, we reduce to the case of codimension one. Furthermore, we may assume X and Y are affine, say $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, and $B = A/(x)$, where $x \in A$ is a local equation for X. After completing along Z, we may assume that A is complete with respect to the x-adic topology, and our problem is to show that the natural map of complexes

$$\Omega^{\bullet}_{A} \to \Omega^{\bullet}_{B}$$

is a quasi-isomorphism.

We need the following lemma of Grothendieck.

Lemma (**1.2**). — *Let* A *be a noetherian ring, containing a field* k, *and complete with respect to an* x*-adic topology, where* x *is a non-zero-divisor in* A. *Let* $B = A/xA$, *and assume that* B *is formally smooth over* k. *Then there is an isomorphism* $A \cong B[[x]]$.

Proof (see also [19, III, 5.6]). — The main point, as in the Cohen structure theorems for complete local rings, is to show that A contains a " ring of representatives " for B, i.e. a subring B_0 which maps isomorphically onto B. For this we use the infinitesimal lifting property of formally smooth morphisms [EGA IV, 17.1.1]. For each n, $A/x^{n+1} \to A/x^n$ is a surjective map defined by a nilpotent ideal. We have a map $B \xrightarrow{\cong} A/x$, and B is smooth over k. Hence we can lift step by step, and obtain maps $B \to A/x^n$, for each n, compatible with the projections. Passing to the limit, we have a map $B \to \varprojlim (A/x^n) = A$, which lifts the original map $B \to A/x$. Let the image be B_0.

Now we can map $B[[x]] \to A$ by sending B to B_0 and x to x. It is injective, since x is a non-zero-divisor in A. It is surjective, because the kernel of $A \to B$ is generated by x. Hence it is an isomorphism.

So in our case, we may assume $A \cong B[[x]]$. Now any element $\omega \in \Omega^p_A$ can be written uniquely in the form

$$\omega = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + (\beta_0 + \beta_1 x + \beta_2 x^2 + \ldots)\, dx,$$

where $\alpha_n \in \Omega^p_B$ and $\beta_n \in \Omega^{p-1}_B$, $n = 0, 1, 2, \ldots$ With ω written in this form, one finds that $d\omega = 0$ if and only if

$$d\alpha_n = 0, \qquad n = 0, 1, 2, \ldots$$

and $$n\alpha_n = (-1)^{p-1} d\beta_{n-1}, \qquad n = 1, 2, \ldots$$

The natural map $\Omega^p_A \to \Omega^p_B$ sends ω to α_0. This shows immediately that the cycles of $\Omega^{\bullet}_A$ map surjectively onto the cycles of $\Omega^{\bullet}_B$. Now suppose $d\omega = 0$. Let

$$\theta = (-1)^{p-1}\left(\beta_0 x + \frac{1}{2}\beta_1 x^2 + \ldots + \frac{1}{n}\beta_{n-1} x^n \ldots\right).$$

Then $\omega = \alpha_0 + d\theta$. Hence ω determines the same cohomology class as α_0. On the other hand, α_0 is a boundary in the complex $\Omega^{\bullet}_A$ if and only if it is a boundary in $\Omega^{\bullet}_B$. Hence we have an isomorphism on cohomology, as required.

Case 2. — Suppose that $f:X\to Y$ is an étale morphism. Then the formal schemes $\hat{X}$ and $\hat{Y}$, the completions along Z, are isomorphic. Since the natural maps $f^*\Omega^p_Y\to\Omega^p_X$ are isomorphisms, the morphism of complexes

$$\hat{\Omega}^{\bullet}_Y\to\hat{\Omega}^{\bullet}_X$$

is actually an isomorphism.

Case 3. — Suppose that $f:X\to Y$ is a smooth morphism. Again the question is local. Hence, using the lemma below, we may assume that Z is contained in a closed subscheme W of X, which is étale over Y. Then we have a commutative diagram of completions along Z

$$\begin{array}{ccc}\hat{\Omega}^{\bullet}_Y & \longrightarrow & \hat{\Omega}^{\bullet}_X\\ & \searrow \quad \swarrow & \\ & \hat{\Omega}^{\bullet}_W & \end{array}$$

Now the map from X to W is a quasi-isomorphism by Case 1. The map from Y to W is an isomorphism by Case 2. Hence the map from Y to X is a quasi-isomorphism, as required.

It remains only to prove the following lemma.

Lemma (**1.3**). — *Let* $f:X\to Y$ *be a smooth morphism of smooth schemes over* k. *Let* Z *be a closed subscheme of* X *such that the restriction* $f:Z\to Y$ *is a closed immersion. Then for any point* $z\in Z$, *one can find a neighborhood* U *of* z *in* X, *and a closed subscheme* W *of* U, *containing* $Z\cap U$, *and with* W *étale over* Y.

Proof. — Let n be the relative dimension of X over Y. Let I be the ideal of Z. First we will show that one can find elements $g_1, \ldots, g_n\in I_z$ such that $dg_1, \ldots, dg_n$ span $(\Omega^1_{X/Y})$. Indeed, we have exact sequences

$$\begin{array}{ccccccc} J/J^2 & \longrightarrow & \Omega^1_Y\otimes\mathcal{O}_Z & \longrightarrow & \Omega^1_Z & \longrightarrow & 0\\ \downarrow & & \downarrow & & \downarrow\wr\!\cong & & \\ I/I^2 & \longrightarrow & \Omega^1_X\otimes\mathcal{O}_Z & \longrightarrow & \Omega^1_Z & \longrightarrow & 0\\ & \searrow^{\beta} & \downarrow & & & & \\ & & \Omega^1_{X/Y}\otimes\mathcal{O}_Z & & & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

where J is the ideal of Z in Y. It follows that the diagonal arrow β is surjective. Now $\Omega^1_{X/Y}$ is free of rank n. Hence using Nakayama's lemma, we can find $g_1, \ldots, g_n\in I_z$, as required.

Let the functions $g_1, \dots, g_n$ be defined in a neighborhood U of z, and let W be the subscheme defined by the ideal $K = (g_1, \dots, g_n)$. W contains $Z \cap U$ by construction. To show that W is étale over Y, consider the exact sequence

$$K/K^2 \to \Omega^1_{X/Y} \otimes \mathcal{O}_W \to \Omega^1_{W/Y} \to 0.$$

Now K/K^2 is generated by n elements. The first arrow is surjective (shrinking U if necessary), and $\Omega^1_{X/Y} \otimes \mathcal{O}_W$ is locally free of rank n. We conclude that the first arrow is an isomorphism, and $\Omega^1_{W/Y} = 0$. It follows that W is smooth over Y [EGA IV, 17.12.1], that W is unramified over Y [EGA IV, 17.4.1] and hence étale [EGA IV, 17.6.1].

This concludes the proof of the proposition. Now we can establish that our definition is independent of the choice of embedding.

Theorem **(1.4)**. — *Let k be a field of characteristic zero. For each embeddable scheme* Y *over k, the algebraic De Rham cohomology as defined above is independent of the embedding chosen. Furthermore, these cohomology groups are contravariant functors in* Y.

Proof (see also the general remarks in the proof of (3.2) below). — Let $Y \to X_1$ and $Y \to X_2$ be two closed immersions of Y into smooth schemes X_1 and X_2 over k. Then we consider also the diagonal embedding $Y \to X_1 \times X_2$. The projections p_1 and p_2 onto the two factors are smooth morphisms, so by the Proposition above, they induce quasi-isomorphisms of complexes

$$\hat{\Omega}^{\bullet}_{X_1} \overset{p_1^*}{\to} \hat{\Omega}^{\bullet}_{X_1 \times X_2} \overset{p_2^*}{\leftarrow} \hat{\Omega}^{\bullet}_{X_2}$$

where $\hat{}$ is always completion along Y. These give rise to isomorphisms of hypercohomology, and so by composition we obtain an isomorphism

$$\mathbf{H}^q(\hat{X}_1, \hat{\Omega}^{\bullet}_{X_1}) \cong \mathbf{H}^q(\hat{X}_2, \hat{\Omega}^{\bullet}_{X_2}).$$

This shows that the two definitions of $H^q_{DR}(Y)$ are isomorphic. Furthermore, if $Y \to X_3$ is a third embedding, then these isomorphisms are compatible with each other.

Now let $g : Y' \to Y$ be a morphism of embeddable schemes, and let $Y' \to X'$, $Y \to X$ be closed immersions into smooth schemes over k. Replacing X′ by a suitable open subset of $X' \times X$, and taking the "diagonal" embedding of $Y' \to X' \times X$, we may assume that there is a smooth morphism $f : X' \to X$ whose restriction to Y′ is g. This induces a morphism of formal schemes $\hat{X}' \to \hat{X}$. The natural map $f^*\Omega^1_X \to \Omega^1_{X'}$ gives rise to a morphism of complexes

$$g^{-1}(\hat{\Omega}^{\bullet}_X) \to \hat{\Omega}^{\bullet}_{X'}$$

from which we deduce a natural map on cohomology

$$H^q_{DR}(Y) \to H^q_{DR}(Y').$$

We leave to the reader that this map is independent of the embeddings chosen, *via* the isomorphisms above, and is functorial in Y.

Remark. — In this paper, we consider only embeddable schemes, because the theory is technically simpler in that case. They should suffice for most applications. However, for the reader who is interested in the general case, there are two approaches. One is to use the theory of crystals [4] and [18], where the cohomology is intrinsically defined. According to an unpublished proof of Deligne, the crystalline cohomology for schemes of finite type over **C** is isomorphic to the complex cohomology, so it gives a good theory.

Another approach is to globalize our definitions by a Čech process which we will now describe. Note by the way that this globalization problem is much easier than the one encountered for dualizing complexes in [RD]. For here we are dealing with actual complexes and well-defined maps, whereas there we were trying to glue elements of the derived category, which is not always possible.

Here is the construction. Let Y be a scheme of finite type over k. We consider a *system of local embeddings* $\mathcal{U}=\{U_i, X_i\}$, where $\{U_i\}$ is an open cover of Y, and for each i, $U_i \to X_i$ is a closed immersion into a smooth scheme X_i over k. For each $(p+1)$-tuple $(i)=\{i_0<i_1<\ldots<i_p\}$ we consider the open set

$$U_{(i)}=U_{i_0}\cap\ldots\cap U_{i_p}$$

and the embedding

$$U_{(i)}\to X_{(i)}=X_{i_0}\times\ldots\times X_{i_p}.$$

Then we consider the complex of sheaves on Y

$$C_{(i)}=j_*(\hat{\Omega}^{\bullet}_{X_{(i)}}),$$

where $\hat{}$ denotes the completion along $U_{(i)}$ and where j is the inclusion of $U_{(i)}$ into Y.

Now for any $0\leq j\leq p$, let $(i')=\{i_0, \ldots, \hat{i_j}, \ldots, i_p\}$. Then we have a natural inclusion $U_{(i)}\to U_{(i')}$, and a projection $X_{(i)}\to X_{(i')}$ which is a smooth morphism. Hence there is a natural map

$$\hat{\Omega}^{\bullet}_{X_{(i')}|U_{(i)}}\to\hat{\Omega}^{\bullet}_{X_{(i)}},$$

and hence we have a map of complexes on Y

$$\delta_{j,(i)} : C_{(i')}\to C_{(i)}.$$

Note by construction that for two integers $0\leq j<k\leq p$, the corresponding four δ maps are compatible with each other. Hence we can define a double complex $\mathcal{C}(\mathcal{U})$ by

$$\mathcal{C}(\mathcal{U})^p=\prod C_{(i)}$$

and

$$\delta^{p-1}=\prod(-1)^j\delta_{j,(i)}.$$

Now we define the De Rham cohomology of Y to be the hypercohomology of the associated simple complex of $\mathcal{C}(\mathcal{U})$.

To show that this definition is independent of the system of local embeddings chosen, we proceed as follows. A *refinement* of the system of local embeddings $\mathcal{U}$ is another such system $\mathscr{V}=\{V_j, Z_j\}$, together with a mapping of index sets λ such that V_j is an open subset of $U_{\lambda(j)}$ for each j, and together with smooth morphisms $Z_j \to X_{\lambda(j)}$ compatible with these inclusions for each j. Now it is clear that any two systems of local embeddings have a common refinement. Furthermore, if $\mathscr{V}$ is a refinement of $\mathscr{U}$, there is a natural map of double complexes

$$\varphi : \mathscr{C}(\mathscr{U}) \to \mathscr{C}(\mathscr{V}).$$

Thus it will be sufficient to show that for any refinement of $\mathscr{U}$, the associated map of simple complexes to φ is a quasi-isomorphism.

The question is local on Y, so we may assume $Y=U_1$ and $Y=V_1$. Let $\mathscr{U}'=\{U_1, X_1\}$ and let $\mathscr{V}'=\{V_1, Z_1\}$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathscr{C}(\mathscr{U}') & \longrightarrow & \mathscr{C}(\mathscr{V}') \\ \big\downarrow & & \big\downarrow \\ \mathscr{C}(\mathscr{U}) & \longrightarrow & \mathscr{C}(\mathscr{V}) \end{array}$$

The top arrow is a quasi-isomorphism by the global case (1.1). Thus it is sufficient to show that the vertical arrows are quasi-isomorphisms. So we can forget $\mathscr{V}$, and need only show that

$$\mathscr{C}(\mathscr{U}') \to \mathscr{C}(\mathscr{U})$$

is a quasi-isomorphism. Indeed, we can write $\mathscr{C}(\mathscr{U})$ as the direct sum

$$\mathscr{C}(\mathscr{U}) = \mathscr{C}' + \mathscr{C}''$$

where
$$\mathscr{C}' = \prod_{i_0 \neq 1} C_{(i)}$$
$$\mathscr{C}'' = \prod_{i_0 = 1} C_{(i)}.$$

Now for each $(i)=\{i_0, \ldots, i_p\}$ with $i_0 \neq 1$, the map

$$\delta : C_{(i)} \to C_{(1, i_0, \ldots, i_p)}$$

is a quasi-isomorphism, by (1.1) again, since the open sets $U_{(i)}$ and $U_{(1, i_0, \ldots, i_p)}$ are equal. Hence the maps δ give a quasi-isomorphism of $\mathscr{C}'$ onto its image in $\mathscr{C}''$, which is everything except $C_1 = \mathscr{C}(\mathscr{U}')$. Thus the map

$$\mathscr{C}(\mathscr{U}') \to \mathscr{C}(\mathscr{U})$$

is a quasi-isomorphism as required.

Thus using the Čech process, our definition of De Rham cohomology can be generalized to arbitrary schemes of finite type over k. We leave the reader the task of carrying this generalization through the rest of the paper. (However, the corresponding globalization for homology may be more difficult—we have not looked into it carefully.)

2. The canonical resolution of the De Rham complex.

In defining the De Rham cohomology theory, we took completions of the De Rham complex along a closed subvariety. This worked well, partly because completion is an exact functor on the category of coherent sheaves. In defining the De Rham homology theory, we will take cohomology with supports along a closed subscheme. Thus we will need to take an injective resolution of the De Rham complex. Furthermore, to establish the covariant functorial properties of De Rham homology, we are led to questions closely related to duality and trace or residue maps. Therefore in this section we will introduce a canonical resolution of the De Rham complex, constructed out of the " residual complexes " of [RD]. For these canonical resolutions, we can define a covariant trace map, for any smooth morphism or any closed immersion of smooth sheaves. This formalism will be used in the next section to define De Rham homology.

We recall the notion of Cousin complex from [RD IV, § 2]. Let X be a noetherian topological space. For each $p \geq 0$ let Z^p be the set of points of codimension $\geq p$. For any point $x \in X$ and any abelian group M, let $i_x(M)$ be the sheaf on X which is the constant sheaf M on $\{x\}^-$, and zero elsewhere. Let F be an abelian sheaf on X. Then by [RD IV, 2.3] there is a unique augmented complex $F \to C^\bullet$, called the *Cousin complex* of F, with the following properties:

a) For each $p \geq 0$, there is an isomorphism $C^p \cong \sum_{x \in Z^p - Z^{p+1}} i_x(M_x)$ for suitable abelian groups M_x.

b) For each $p > 0$, $h^p(C^\bullet)$ has supports in Z^{p+2}.

c) The map $F \to h^0(C^\bullet)$ has kernel with supports in Z^1 and cokernel with supports in Z^2.

Furthermore, the formation of the Cousin complex is functorial in F. We will denote the Cousin complex of a sheaf F by $E^\bullet(F)$.

If X is a regular scheme, and if L is an invertible sheaf on X, then $E^\bullet(L)$ is an injective resolution of L [RD VI, § 2, Example, p. 239]. In fact, it is a *residual complex* in the sense of [RD VI, § 1]: namely, it is a complex $K^\bullet$ of quasi-coherent injective $\mathcal{O}_X$-Modules, bounded below, with coherent cohomology sheaves, and such that there is an isomorphism

$$\sum_{p \in Z} K^p \cong \sum_{x \in X} J(x).$$

Here $J(x)$ denotes the sheaf $i_x(I_x)$, where I_x is an injective hull of $k(x)$ over the local ring $\mathcal{O}_x$.

For our purposes, we will have to generalize slightly, by considering arbitrary locally free sheaves instead of just invertible sheaves. So we will define a *generalized residual complex (of rank r)* on a scheme X to be a complex $K^\bullet$ of quasi-coherent injective

$\mathcal{O}_X$-Modules, bounded below, with coherent cohomology sheaves, and such that there is an isomorphism

$$\sum_{p \in \mathbf{Z}} K^p \cong \Big(\sum_{x \in X} J(x)\Big)^r.$$

Now if F is a locally free sheaf on a regular scheme, $E^{\bullet}(F)$ will be a generalized residual complex. On the other hand, the construction of the functor $f^!$, and the construction of the trace map

$$\mathrm{Tr}_f : f_* f^! \to 1$$

for residual complexes in [RD VI] carries over immediately to generalized residual complexes.

Now we come to the construction of the canonical resolution of the De Rham complex. Fix a base field k (of arbitrary characteristic). Let X be a smooth scheme over k. For each i we consider the Cousin complex $E^{\bullet}(\Omega^i_{X/k})$, which is an injective resolution of $\Omega^i_{X/k}$. Since $E^{\bullet}$ is a functor on the category of abelian sheaves on X, not just $\mathcal{O}_X$-Modules, the k-linear map $d : \Omega^i_{X/k} \to \Omega^{i+1}_{X/k}$ gives rise to a map of complexes

$$d : E^{\bullet}(\Omega^i_{X/k}) \to E^{\bullet}(\Omega^{i+1}_{X/k}),$$

compatible with the original map d. As i varies, we obtain a double complex.

Definition. — We denote by $E(\Omega^{\bullet})$ the simple complex associated to the double complex above. Thus $E(\Omega^{\bullet})$ is a complex of quasi-coherent injective $\mathcal{O}_X$-Modules, and k-linear maps, and we have a natural map of complexes

$$\Omega^{\bullet}_{X/k} \to E(\Omega^{\bullet})$$

making the latter into an injective resolution of the former (i.e. the m p is a quasi-isomorphism). We call $E(\Omega^{\bullet})$ the *canonical resolution* of the De Rham complex of X. (Note that $E(\Omega^{\bullet})$ is *not* the Cousin complex of the complex $\Omega^{\bullet}$ in the sense of [RD IV, § 3]. Thus our notation is different from that introduced in [RD, p. 241].)

Next we come to the construction of the trace map on the canonical resolutions. Let $f : X \to Y$ be a smooth map of smooth schemes over k, of relative dimension n. We have an exact sequence

$$0 \to f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0,$$

and hence for each i we deduce a natural map

$$\Omega^{i+n}_X \to f^*\Omega^i_Y \otimes \omega_{X/Y}$$

where $$\omega_{X/Y} = \Omega^n_{X/Y}.$$

On the other hand, the definition of $f^!$ for a smooth morphism [RD VI, 3.1; p. 313 for f^z; p. 145 for $f^{\#}$] shows that

$$f^! E^{\bullet}(\Omega^i_Y) = E^{\bullet}(f^*\Omega^i_Y \otimes \omega_{X/Y})[n].$$

Thus the trace map for residual complexes [RD VI, 4.2] gives a map of graded sheaves

$$\mathrm{Tr}_f : f_*\mathrm{E}^\bullet(f^*\Omega^i_{\mathrm{Y}}\otimes\omega_{\mathrm{X/Y}})[n] \to \mathrm{E}^\bullet(\Omega^i_{\mathrm{Y}}).$$

Combining with the map on differentials above, applying f_*, $\mathrm{E}^\bullet$, and $[n]$, we get a map of graded sheaves

$$f_*\mathrm{E}^\bullet(\Omega^{i+n}_{\mathrm{X}})[n] \to \mathrm{E}^\bullet(\Omega^i_{\mathrm{Y}}).$$

Doing this for all i gives a map of graded sheaves, which we call the *trace map* for the canonical resolutions

$$\mathrm{Tr}_f : f_*\mathrm{E}(\Omega^\bullet_{\mathrm{X}})[2n] \to \mathrm{E}(\Omega^\bullet_{\mathrm{Y}}).$$

Note the shift of $2n$, arising on the one hand from the shift in exponent i, and on the other hand, from the shift in the trace map for residual complexes. Note also we have said nothing about whether the map Tr_f commutes with the boundary maps of the complexes $\mathrm{E}(\Omega^\bullet_{\mathrm{X}})$ and $\mathrm{E}(\Omega^\bullet_{\mathrm{Y}})$. So far it is only a map of graded sheaves.

Now let $f:\mathrm{X}\to\mathrm{Y}$ be a closed immersion of smooth schemes over k, of codimension m. Then we have an exact sequence

$$0\to\mathrm{I}/\mathrm{I}^2\to f^*\Omega^1_{\mathrm{Y}}\to\Omega^1_{\mathrm{X}}\to 0,$$

and hence natural maps

$$\wedge^m(\mathrm{I}/\mathrm{I}^2)\otimes\Omega^i_{\mathrm{X}}\to f^*\Omega^{i+m}_{\mathrm{Y}}$$

or $\quad \Omega^i_{\mathrm{X}}\to f^*\Omega^{i+m}_{\mathrm{Y}}\otimes\omega_{\mathrm{X/Y}},$

where $\quad \omega_{\mathrm{X/Y}}=(\wedge^m(\mathrm{I}/\mathrm{I}^2))^\vee.$

If we set $n=-m=\dim\mathrm{X}-\dim\mathrm{Y}$, the " relative dimension of X over Y ", we can write these maps as

$$\Omega^{i+n}_{\mathrm{X}}\to f^*\Omega^i_{\mathrm{Y}}\otimes\omega_{\mathrm{X/Y}}.$$

Now the definition of $f^!$ for a finite morphism [RD VI 3.1; p. 311 for $f^\natural$, p. 165 for $f^\flat$], together with the fundamental local isomorphism [RD III 7.2] shows that

$$f^!\mathrm{E}^\bullet(\Omega^i_{\mathrm{Y}})=\mathrm{E}^\bullet(f^*\Omega^i_{\mathrm{Y}}\otimes\omega_{\mathrm{X/Y}})[n].$$

Thus the trace map for residual complexes gives

$$\mathrm{Tr}_f : f_*\mathrm{E}^\bullet(f^*\Omega^i_{\mathrm{Y}}\otimes\omega_{\mathrm{X/Y}})[n] \to \mathrm{E}^\bullet(\Omega^i_{\mathrm{Y}}).$$

Combining with the maps on differentials above, and doing this for all i, we have a map of graded sheaves, the *trace map* for a closed immersion

$$\mathrm{Tr}_f : f_*\mathrm{E}(\Omega^\bullet_{\mathrm{X}})[2n] \to \mathrm{E}(\Omega^\bullet_{\mathrm{Y}}).$$

Summing up, we have

Proposition **(2.1)**. — *Let* $f:\mathrm{X}\to\mathrm{Y}$ *be a smooth morphism of smooth schemes over* k, *or a closed immersion of smooth schemes over* k. *Let* $n=\dim\mathrm{X}-\dim\mathrm{Y}$. *Then there is a natural trace map of graded sheaves*

$$\mathrm{Tr}_f : f_*\mathrm{E}(\Omega^\bullet_{\mathrm{X}})[2n] \to \mathrm{E}(\Omega^\bullet_{\mathrm{Y}}).$$

Furthermore, if $g : Y \to Z$ *is another such map (either smooth or a closed immersion) for which* gf *is also either smooth or a closed immersion, then we have*

$$\mathrm{Tr}_{gf} = (\mathrm{Tr}_g)(g_* \mathrm{Tr}_f).$$

Proof. — We gave the construction above. The compatibilities follow from the compatibilities of the trace map on residual complexes [RD VI 4.2].

So far, the trace map has been constructed only as a map of graded sheaves. It will be important to know when it is a morphism of complexes. As a matter of notation, we use d to denote the map deduced from the exterior differential in the De Rham complex. So

$$d : E^{\boldsymbol{\cdot}}(\Omega^i) \to E^{\boldsymbol{\cdot}}(\Omega^{i+1})$$

is a map of complexes, for each i. On the other hand, we use δ to denote the coboundary map in the residual complexes. So we have

$$\delta : E(\Omega^i)^j \to E(\Omega^i)^{j+1}$$

for each i, j. To show that the trace map on $E(\Omega^{\boldsymbol{\cdot}})$ is a map of complexes, is to show that trace commutes with the total differential on $E(\Omega^{\boldsymbol{\cdot}})$, i.e., trace commutes with both d and δ. We will see that trace always commutes with d; it commutes with δ under suitable properness hypotheses.

The following is the main result of this section. It is also the key point which allows us to extend the duality theory for coherent sheaves to duality for De Rham cohomology. It is probably also an essential step in constructing a suitable duality theory for complexes of coherent sheaves with differential operators.

Proposition **(2.2)**. — *Let* $f : X \to Y$ *be either a smooth morphism or a closed immersion of smooth schemes over* k, *and let* $n = \dim X - \dim Y$. *Then* d *commutes with the trace map: for each* i *we have a commutative diagram*

$$\begin{array}{ccc} f_* E^{\boldsymbol{\cdot}}(\Omega_X^{i+n})[n] & \xrightarrow{\mathrm{Tr}_f} & E^{\boldsymbol{\cdot}}(\Omega_Y^i) \\ \Big\downarrow d & & \Big\downarrow d \\ f_* E^{\boldsymbol{\cdot}}(\Omega_X^{i+n+1})[n] & \xrightarrow{\mathrm{Tr}_f} & E^{\boldsymbol{\cdot}}(\Omega_Y^{i+1}) \end{array}$$

Proof. Case 1. — First suppose f is a closed immersion. The question is local on X and Y, so we can factor f into a sequence of closed immersions of smooth schemes of codimension one. The trace maps compose, so we are reduced to the case of codimension one. The trace map sends

$$f_* E^{\boldsymbol{\cdot}}(\Omega_X^{i-1})[-1] \to E^{\boldsymbol{\cdot}}(\Omega_Y^i).$$

The image lands in the subcomplex $\Gamma_X(E^\bullet(\Omega_Y^i))$, so it will be sufficient to check the compatibility with d there, i.e., to show that

$$\begin{array}{ccc} f_* E^\bullet(\Omega_X^{i-1})[-1] & \xrightarrow{\mathrm{Tr}_f} & \Gamma_X(E^\bullet(\Omega_Y^i)) \\ \downarrow d & & \downarrow d \\ f_* E^\bullet(\Omega_X^{i})[-1] & \xrightarrow{\mathrm{Tr}_f} & \Gamma_X(E^\bullet(\Omega_Y^{i+1})) \end{array}$$

is commutative. Since X is of codimension one, and Ω_Y^i is locally free, there is only one non-zero local cohomology sheaf, namely $H_X^1(\Omega_Y^i)$. Hence we have

$$\Gamma_X(E^\bullet(\Omega_Y^i)) = E^\bullet(H_X^1(\Omega_Y^i))[-1]$$

where the second $E^\bullet$ denotes the Cousin complex on X. With this identification, the trace map is simply obtained by applying the functor $E^\bullet$ to the map of sheaves

$$\Omega_X^{i-1} \to H_X^1(\Omega_Y^i)$$

defined by

$$\eta \mapsto \eta \wedge z^{-1} dz$$

where z is a local equation for X, and where we have identified locally

$$H_X^1(\Omega_Y^i) \cong \Omega_Y^i[z^{-1}]/\Omega_Y^i.$$

Since $E^\bullet$ is a functor on abelian sheaves, to prove our compatibility, it will be sufficient to show that the diagram of sheaves on X

$$\begin{array}{ccc} \Omega_X^{i-1} & \longrightarrow & H_X^1(\Omega_Y^i) \\ \downarrow d & & \downarrow d \\ \Omega_X^{i} & \longrightarrow & H_X^1(\Omega_Y^{i+1}) \end{array}$$

is commutative. This follows from the fact that for any local section $\eta \in \Omega_X^{i-1}$, we have

$$d(\eta \wedge z^{-1} dz) = d\eta \wedge z^{-1} dz.$$

Case 2. — Next we will consider the case of an étale morphism $f : X \to Y$. In this case the relative dimension is zero, and the trace map

$$\mathrm{Tr}_f : f_* E^\bullet(\Omega_X^p) \to E^\bullet(\Omega_Y^p)$$

is deduced from the " classical " trace map on sheaves

$$\mathrm{Tr} : f_* \Omega_X^p \to \Omega_Y^p.$$

Since f is étale, the natural map $f^* \Omega_Y^p \to \Omega_X^p$ is an isomorphism. Thus we have a natural identification

$$f_* \Omega_X^p \cong f_* f^* \Omega_Y^p \cong \Omega_Y^p \otimes f_* \mathcal{O}_X.$$

The trace map is defined as follows: choose a local basis $e_1, \ldots, e_n$ of $f_* \mathcal{O}_X$ as a free $\mathcal{O}_Y$-Module. For any $\eta \in f_* \Omega^p_X$ expand ηe_i in terms of the e_j:

$$\eta e_i = \sum_j \zeta_{ij} e_j$$

with $\zeta_{ij} \in \Omega^p_Y$. Then

$$\mathrm{Tr}_f(\eta) = \sum_i \zeta_{ii}.$$

To see that trace commutes with d, we apply d to the equation above. We find

$$(d\eta)e_i + (-1)^p \eta \wedge de_i = \sum_j (d\zeta_{ij})e_j + (-1)^p \sum_j \zeta_{ij} \wedge de_j.$$

Let $de_i = \sum_k \theta_{ik} e_k$ with $\theta_{ik} \in \Omega^1_Y$. Then we have

$$(d\eta)e_i = \sum_j (d\zeta_{ij})e_j + (-1)^p \sum_{j,k} \zeta_{ij} \wedge \theta_{jk} e_k - (-1)^p \sum_{j,k} \zeta_{jk} \wedge \theta_{ij} e_k.$$

Hence

$$\begin{aligned}\mathrm{Tr}_f(d\eta) &= \sum_i d\zeta_{ii} + (-1)^p \sum_{i,j} \zeta_{ij} \wedge \theta_{ji} - (-1)^p \sum_{i,j} \zeta_{ji} \wedge \theta_{ij} \\ &= d(\mathrm{Tr}_f(\eta)).\end{aligned}$$

Case 3. — Now let $f: X \to Y$ be a smooth morphism. It can be factored locally into an étale morphism followed by an affine n-space over Y. Using the previous case, we reduce to affine n-space. This in turn factors into a sequence of affine 1-space morphisms, so we are reduced to the case $X = \mathbf{A}^1_Y$. Now the trace map is a sum of maps $f_* J(x) \to J(y)$ for $x \in X$, $y = f(x)$, and x closed in its fibre (see the construction of the trace map [RD VI § 4]). Hence it is sufficient to verify our commutativity at points $x \in X$ which are closed in their fibres. Any such point is contained in a subscheme $Z \subseteq X$ which is étale over Y. Making the base extension $Z \to Y$, the point x lifts to $Z \times_Y X$, where it is contained in the "diagonal section" $Z \to Z \times_Y X$. $J(x)$ is the same for $x \in X$ and $x \in Z \times_Y X$, so we reduce to studying the morphism $Z \times_Y X \to Z$. Now $Z \times_Y X = \mathbf{A}^1_Z$, and by an automorphism, the given section can be brought to the zero-section. Thus, changing Z to Y, we have reduced to showing that Trace commutes with d for points contained in the zero-section of $X = \mathbf{A}^1_Y$. So we must show that

$$\begin{array}{ccc} f_* E^\bullet(\Omega^{i+1}_X)[1] & \xrightarrow{\mathrm{Tr}_f} & E^\bullet(\Omega^i_Y) \\ \Big\downarrow d & & \Big\downarrow d \\ f_* E^\bullet(\Omega^{i+2}_X)[1] & \xrightarrow{\mathrm{Tr}_f} & E^\bullet(\Omega^{i+1}_Y) \end{array}$$

is commutative, for points in the zero section. Let $W \subseteq X$ be the zero-section. Then it is sufficient to verify the commutativity on the subcomplex

$$\Gamma_W(E^\bullet(\Omega^{i+1}_X)[1]).$$

But W has codimension 1 in X. As above in Case 1, we find that

$$\Gamma_W(E^\bullet(\Omega_X^{i+1})[1]) = E^\bullet(H_W^1(\Omega_X^{i+1})),$$

where the second $E^\bullet$ is the Cousin complex on W. With this identification, the trace map is obtained by applying the functor $E^\bullet$ to the map of sheaves

$$H_W^1(\Omega_X^{i+1}) \to \Omega_Y^i$$

defined by

$$\sum_j \theta_j z^j + \sum_j \eta_j z^j\, dz \mapsto \eta_{-1},$$

where z is the parameter of $\mathbf{A}^1$, and we have identified

$$H_W^1(\Omega_X^{i+1}) \cong \Omega_Y^{i+1}[z^{-1}] \oplus \Omega_Y^i[z^{-1}]dz.$$

To finish the proof, we have only to show that d commutes with this trace map on sheaves. Indeed

$$d(\sum_j \theta_j z^j + \sum_j \eta_j z^j\, dz) = \sum_j d\theta_j z^j + (-1)^{i+1} \sum_j j\theta_j z^{j-1}\, dz + \sum_j d\eta_j \wedge z^j\, dz.$$

Applying trace, we get just $d\eta_{-1}$, because there is no $z^{-1}dz$ term in the other sum. q.e.d.

Proposition **(2.3)**. — *Let $f : X \to Y$ be either a smooth map or a closed immersion of smooth schemes over k. Let $X' \subseteq X$ and $Y' \subseteq Y$ be closed subschemes such that $f(X') \subseteq Y'$, and the induced map $f' : X' \to Y'$ is proper. Then the induced map*

$$\mathrm{Tr}_f : f_* \Gamma_{X'}(E(\Omega_X^\bullet)[2n]) \to \Gamma_{Y'}(E(\Omega_Y^\bullet))$$

on the subcomplexes with supports in X' and Y', commutes with δ. Hence, by the previous result, it is a morphism of complexes.

Proof. — If the map f itself is proper, this follows immediately from the Residue Theorem [RD VII 2.1] which says that the trace map on residual complexes for a proper morphism is a morphism of complexes. Examining the proof of the Residue Theorem [*loc. cit.*], it shows that trace commutes with δ at every point $x \in X$ whose closure $\{x\}^-$ is proper over Y. And this is just what we need for this result.

3. Algebraic De Rham homology.

Throughout this section we fix a field k of characteristic zero. We will define the De Rham homology of an embeddable scheme, and prove that it is independent of the embedding. The De Rham homology is a covariant functor for proper morphisms, and a contravariant functor for open immersions. We will establish a long exact sequence relating the homology of a closed subset to the whole space and the open complement.

Definition. — Let Y be an embeddable scheme over k. Let $Y \to X$ be a closed immersion of Y into a smooth scheme X over k, of dimension n. Then we define the *algebraic De Rham homology* of Y by

$$H_q^{DR}(Y) = \mathbf{H}_Y^{2n-q}(X, \Omega^{\bullet}_{X/k}),$$

the hypercohomology, with supports in Y, of the De Rham complex on X.

Using the canonical resolution introduced in the last section, we can give a more concrete interpretation of this cohomology. Expressing $\Gamma_Y = \Gamma \circ \mathit{\Gamma}_Y$, and recalling that $E(\Omega^{\bullet})$ is an injective resolution of $\Omega^{\bullet}$, we can write

$$H_q^{DR}(Y) = \mathbf{H}^{2n-q}(Y, \mathit{\Gamma}_Y(E(\Omega^{\bullet}_{X/k}))).$$

Furthermore, since the complex $\mathit{\Gamma}_Y(E(\Omega^{\bullet}))$ is a complex of injective $\mathcal{O}_X$-Modules, which are thus flasque sheaves, this hypercohomology is just the cohomology of the complex of global sections:

$$H_q^{DR}(Y) = h^{2n-q}(\Gamma(Y, \mathit{\Gamma}_Y(E(\Omega^{\bullet}_{X/k})))).$$

Our first task is to show that this definition is independent of the embedding, and is functorial in Y. For this we will need the following lemma, which makes essential use of characteristic zero.

Lemma **(3.1)** (char. $k=0$). — *Let* $f: X \to Y$ *be either a smooth morphism or a closed immersion of smooth schemes over k. Let Z be a closed subscheme of X such that the induced map* $f: Z \to Y$ *is also a closed immersion. Then the trace map gives a quasi-isomorphism of complexes*

$$\mathrm{Tr}_f : f_* \mathit{\Gamma}_Z(E(\Omega^{\bullet}_X))[2n] \to \mathit{\Gamma}_Z(E(\Omega^{\bullet}_Y)),$$

where $n = \dim X - \dim Y$.

Proof. — We know at least from Proposition (2.3) above that Tr_f is a morphism of complexes. To show that it is a quasi-isomorphism, we separate cases.

Case 1. — Suppose f is a closed immersion. The question is local, so we can factor f into a sequence of closed immersions of smooth schemes of codimension one, and thus we reduce to the case of codimension one. We have $Z \subseteq X \subseteq Y$, so it will be sufficient to show that the map

$$\mathrm{Tr}_f : E(\Omega^{\bullet}_X)[-2] \to \mathit{\Gamma}_X(E(\Omega^{\bullet}_Y))$$

is a quasi-isomorphism. As in the proof of Proposition (2.2) above, we have

$$\mathit{\Gamma}_X(E(\Omega^{\bullet}_Y)) = E(H^1_X(\Omega^{\bullet}_Y))[-1],$$

where $H^1_X(\Omega^{\bullet}_Y)$ denotes the complex formed of the sheaves $H^1_X(\Omega^p_Y)$. The trace map is obtained by applying the functor E, term by term, and taking the associated simple complex, of the map of complexes

$$\Omega^{\bullet}_X[-1] \to H^1_X(\Omega^{\bullet}_Y).$$

This map of complexes is defined by $\eta \mapsto \eta \wedge z^{-1}dz$, where $\eta \in \Omega_X^p$, and z is a local equation of X in Y.

To simplify the discussion, we may assume that X and Y are affine, say X=Spec B, Y=Spec A, and B=A/(z). Furthermore, we may complete A with respect to the z-adic topology without changing the situation. Then by Lemma (1.2), there is an isomorphism $A \cong B[[z]]$. Finally, with these identifications, we have an isomorphism

$$H_X^1(\Omega_Y^p) \cong \Omega_Y^p[z^{-1}]/\Omega_Y^p$$

for each p. Thus our map of complexes is

$$\Omega_B^{\cdot}[-1] \to \Omega_A^{\cdot}[z^{-1}]/\Omega_A^{\cdot}$$

defined by $\eta \mapsto \eta \wedge z^{-1}dz$

for $\eta \in \Omega_B^p$. It will be sufficient to show that this is a quasi-isomorphism of complexes.

Let $\gamma \in \Omega_A^{p+1}[z^{-1}]/\Omega_A^{p+1}$.

Then γ can be written uniquely in the form

$$\gamma = \alpha z^{-1} + \ldots + \alpha_s z^{-s} + (\beta_1 z^{-1} + \ldots + \beta_s z^{-s})dz$$

for suitable $s \geq 0$ and $\alpha_i \in \Omega_B^{p+1}$ and $\beta_i \in \Omega_B^p$ for each i. Now $d\gamma = 0$ if and only if

$$d\alpha_i = 0, \quad i = 1, \ldots, s$$
$$d\beta_1 = 0,$$
$$d\beta_{i+1} = (-1)^{p+1} i\alpha_i, \quad i = 1, \ldots, s.$$

Letting

$$\theta = (-1)^{p+1}(\beta_2 z^{-1} + (1/2)\beta_3 z^{-2} + \ldots + (1/(s-1))\beta_s z^{-s+1}),$$

we find $\gamma = d\theta + \beta_1 z^{-1}dz$, and $d\beta_1 = 0$.

Thus our map of complexes is surjective for cycles modulo boundaries. Secondly, note that $\beta_1 z^{-1}dz = d\psi$ for some ψ if and only if $\beta_1 = d\rho$ for some ρ. Indeed, ψ must be of the form $\rho z^{-1}dz$ for some ρ. Thus our map of complexes is a quasi-isomorphism, as required.

Case 2. — Suppose f is an étale morphism. Then the trace map

$$\mathrm{Tr}_f : f_* \Gamma_Z(E(\Omega_X^{\cdot})) \to \Gamma_Z(E(\Omega_Y^{\cdot}))$$

is actually an isomorphism of complexes. Indeed, for each $x \in Z$, the injective hulls of $k(x)$ over the local rings $\mathcal{O}_{x,X}$ and $\mathcal{O}_{x,Y}$ are the same, and the trace map is an isomorphism between them. This follows from the construction of trace for residual complexes [RD VI § 4].

Case 3. — Suppose f is a smooth morphism. The question is local on Z, so by Lemma (1.3), we can find a closed subscheme W of X, containing Z, and which is étale over Y. Let $j : W \to X$ be the inclusion, and $g : W \to Y$ the restriction of f to W. Then we have a commutative diagram of trace maps

$$\begin{array}{ccc} g_* \Gamma_Z(E(\Omega^{\bullet}_W)) & \xrightarrow{f_* \mathrm{Tr}_j} & f_* \Gamma_Z(E(\Omega^{\bullet}_X))[2n] \\ & {\scriptstyle \mathrm{Tr}_g} \searrow \quad \swarrow {\scriptstyle \mathrm{Tr}_f} & \\ & \Gamma_Z(E(\Omega^{\bullet}_Y)) & \end{array}$$

Now $f_* \mathrm{Tr}_j$ is a quasi-isomorphism by Case 1; Tr_g is an isomorphism by Case 2. Hence Tr_f is a quasi-isomorphism, as required.

Now we come to the main result of this section.

Theorem **(3.2)**. — *Let k be a field of characteristic zero. For each embeddable scheme* Y *over k, the algebraic De Rham homology groups* $H_q^{DR}(Y)$ *defined above are independent of the embedding used in the definition. For each proper morphism* $Y' \to Y$ *of embeddable schemes there is a natural transformation* $H_q(Y') \to H_q(Y)$ *making* H_q *into a covariant functor on the category of embeddable schemes and proper morphisms. For each open immersion* $Y_0 \to Y$ *there is a natural transformation* $H_q(Y) \to H_q(Y_0)$ *making* H_q *into a contravariant functor with respect to open immersions. For compositions of proper morphisms and open immersions, there are commutative diagrams of these maps.*

Proof. — Actually the phrase " H_q is independent of the embedding " is somewhat inexact. To give a completely precise statement of what we mean is rather tedious, and may be left to the reader. Such questions were treated at some length in [RD], so need not be repeated here. What we will prove is that for each embedding $Y \to X_1$, and for each q, we have a group

$$H_1 = H_q^{DR}(Y),$$

depending on X_1. Given another embedding $Y \to X_2$, we have an isomorphism $\alpha_{12} : H_1 \to H_2$. Given a third embedding $Y \to X_3$, we have a compatibility $\alpha_{13} = \alpha_{23}\alpha_{12}$. Then, given a proper morphism $Y' \to Y$, and embeddings $Y' \to X'_1$, $Y \to X_1$ we have a homomorphism $\beta_1 : H'_1 \to H_1$. Given another choice of embeddings $Y' \to X'_2$ and $Y \to X_2$, we have a compatibility $\beta_2 \alpha_{12} = \alpha'_{12} \beta_1$. There will be further compatibilities for a composition of two proper morphisms, for open immersions, and for combinations of proper morphisms and open immersions.

In our situation, the compatibilities will all follow from the analogous compatibilities for the trace map on residual complexes, which were spelled out in [RD VI, 4.2]. So we will say no more about the compatibilities here, but will confine ourselves to defining the isomorphism comparing two embeddings, and the functorial maps for proper morphisms and open immersions.

Let $Y \to X_1$ and $Y \to X_2$ be two closed immersions of Y into smooth schemes over k. Then the diagonal map $Y \to X_1 \times X_2$ is also a closed immersion. We will compare the two original embeddings to this one. Let $n_1 = \dim X_1$, $n_2 = \dim X_2$, and let p_1, p_2 be the projections onto the two factors. According to the lemma, the trace maps for p_1 and p_2 induce quasi-isomorphisms of complexes, which in turn induce isomorphisms of hypercohomology as follows:

$$\begin{array}{ccc} & \mathbf{H}_Y^{2n_1+2n_2-q}(X_1 \times X_2, \Omega^\bullet_{X_1 \times X_2}) & \\ {\scriptstyle \mathrm{Tr}_{p_1}}\ \swarrow \cong & & \cong \searrow\ {\scriptstyle \mathrm{Tr}_{p_2}} \\ \mathbf{H}_Y^{2n_1-q}(X_1, \Omega^\bullet_{X_1}) & & \mathbf{H}_Y^{2n_2-q}(X_2, \Omega^\bullet_{X_2}) \end{array}$$

Composing these two isomorphisms gives the required isomorphism α_{12} between the two definitions of $H_q^{DR}(Y)$. We let the reader verify the compatibility for three embeddings.

Now let $g : Y' \to Y$ be a proper morphism of embeddable schemes. Let $Y' \to X'$ and $Y \to X$ be embeddings into smooth schemes over k. Replacing $Y' \to X'$ by the "diagonal" embedding $Y' \to X' \times X$, which is a closed immersion because g is proper, we may assume there is a smooth morphism $f : X' \to X$ restricting to g on Y'. Then by Proposition (2.3), the trace map for f induces a morphism of complexes

$$\mathrm{Tr}_f : f_* \Gamma_{Y'}(E(\Omega^\bullet_{X'})[2n'-2n]) \to \Gamma_Y(E(\Omega^\bullet_X)),$$

where $n = \dim X$, $n' = \dim X'$. Taking hypercohomology, we get a homomorphism

$$\mathrm{Tr}_f : \mathbf{H}_{Y'}^{2n'-q}(\Omega^\bullet_{X'}) \to \mathbf{H}_Y^{2n-q}(\Omega^\bullet_X),$$

in other words, a map

$$H_q^{DR}(Y') \to H_q^{DR}(Y).$$

Again we leave to the reader the verification that this map is independent of the embeddings chosen, and is functorial in g.

Finally, let $Y_0 \to Y$ be an open immersion. Let $Y \to X$ be an embedding into a smooth scheme, and let $X_0 \subseteq X$ be an open subset such that $X_0 \cap Y = Y_0$. Then the restriction map to X_0 gives maps

$$\mathbf{H}_Y^{2n-q}(X, \Omega^\bullet_X) \to \mathbf{H}_{Y_0}^{2n-q}(X_0, \Omega^\bullet_{X_0})$$

hence maps $\quad H_q^{DR}(Y) \to H_q^{DR}(Y_0)$.

These are clearly functorial, and compatible with the maps for proper morphisms defined above.

Theorem **(3.3)** (Exact sequence of a closed subset). — *Let* Y *be an embeddable scheme over k, and let* Z *be a closed subscheme. Then there is a long exact sequence of algebraic De Rham homology*

$$\dots \to H_q^{DR}(Z) \to H_q^{DR}(Y) \to H_q^{DR}(Y-Z) \to H_{q-1}^{DR}(Z) \to \dots$$

Furthermore, the formation of this exact sequence is compatible with H_q *as a covariant functor on proper morphisms and as a contravariant functor on open immersions.*

Proof. — Let $Y \to X$ be an embedding of Y into a smooth scheme of dimension n over k. Then this sequence is just the long exact sequence of local cohomology

$$\ldots \to \mathbf{H}_Z^{2n-q}(X, \Omega_X^\bullet) \to \mathbf{H}_Y^{2n-q}(X, \Omega_X^\bullet) \to \mathbf{H}_{Y-Z}^{2n-q}(X, \Omega_X^\bullet) \to \mathbf{H}_Z^{2n-q+1}(X, \Omega_X^\bullet) \to \ldots$$

together with the excision isomorphism

$$\mathbf{H}_{Y-Z}^{2n-q}(X, \Omega_X^\bullet) \cong \mathbf{H}_{Y-Z}^{2n-q}(X-Z, \Omega_{X-Z}^\bullet).$$

(See [LC 1.3].)

Proposition **(3.4)**. — *If* Y *is smooth over* k, *of dimension* n, *then there are natural isomorphisms*

$$H_q^{DR}(Y) \cong H_{DR}^{2n-q}(Y).$$

Proof. — Immediate from the definitions: take $Y = X$ in both cases.

4. Mayer-Vietoris and the exact sequence of a proper birational morphism.

In this section we establish a Mayer-Vietoris sequence for algebraic De Rham cohomology and homology, and we also establish exact sequences of cohomology and homology for a proper birational morphism. These sequences, used later in conjunction with the resolution of singularities, will be essential for proving the finite-dimensionality of our cohomology and homology groups, and for proving the comparison theorems with the analytic and topological theories.

Proposition **(4.1)** (Mayer-Vietoris sequence for cohomology). — *Let* Y *be an embeddable scheme over* k, *which is a union of two closed subschemes* Y_1 *and* Y_2. *Then there is an exact sequence of algebraic De Rham cohomology*

$$\ldots \to H^q(Y) \to H^q(Y_1) \oplus H^q(Y_2) \to H^q(Y_1 \cap Y_2) \to H^{q+1}(Y) \to \ldots$$

Proof. — Embed Y in a smooth scheme X over k. Let I_1 and I_2 be sheaves of ideals defining Y_1 and Y_2 respectively. Then for every n, we have an exact sequence

$$0 \to \mathcal{O}_X/(I_1^n \cap I_2^n) \to (\mathcal{O}_X/I_1^n) \oplus (\mathcal{O}_X/I_2^n) \to \mathcal{O}_X/(I_1^n + I_2^n) \to 0.$$

Now the $\{I_1^n \cap I_2^n\}$ topology on $\mathcal{O}_X$ is equal to the $\{(I_1 \cap I_2)^n\}$ topology. Indeed, $I_1^n \cap I_2^n \supseteq (I_1 \cap I_2)^n$. Conversely

$$(I_1 \cap I_2)^n \supseteq (I_1 I_2)^n = I_1^n I_2^n.$$

Now by Krull's theorem, $I_1^n I_2^n \supseteq I_1^n \cap I_2^m$ for some $m \geq n$. This in turn contains $I_1^m \cap I_2^m$, so we are done. On the other hand, the $\{I_1^n + I_2^n\}$ topology is easily seen to be equal to the $\{(I_1+I_2)^n\}$ topology. Therefore, in taking the inverse limit over n of the above sequence, we obtain an exact sequence of sheaves

$$0 \to \hat{\mathcal{O}}_{X/Y} \to \hat{\mathcal{O}}_{X/Y_1} \oplus \hat{\mathcal{O}}_{X/Y_2} \to \hat{\mathcal{O}}_{X/(Y_1 \cap Y_2)} \to 0.$$

The same argument applies to any locally free sheaf on X. In particular, we can apply it to the sheaves Ω^p_X of p-differential forms. The resulting maps are compatible with the derivation d of the De Rham complex. So we have an exact sequence of complexes of sheaves

$$0 \to \hat{\Omega}^{\cdot}_{X/Y} \to \hat{\Omega}^{\cdot}_{X/Y_1} \oplus \hat{\Omega}^{\cdot}_{X/Y_2} \to \hat{\Omega}^{\cdot}_{X/(Y_1 \cap Y_2)} \to 0.$$

The resulting long exact sequence of cohomology is the Mayer-Vietoris sequence of the proposition.

Proposition (4.2) (Mayer-Vietoris sequence for homology). — *With the same hypotheses as the previous proposition, there is an exact sequence of algebraic De Rham homology*

$$\dots \to H_q(Y_1 \cap Y_2) \to H_q(Y_1) \oplus H_q(Y_2) \to H_q(Y) \to H_{q-1}(Y_1 \cap Y_2) \to \dots$$

Proof. — As above, embed Y in a smooth scheme X over k. Let $n = \dim X$. Then our sequence can be written

$$\dots \to \mathbf{H}^{2n-q}_{Y_1 \cap Y_2}(\Omega^{\cdot}_X) \to \mathbf{H}^{2n-q}_{Y_1}(\Omega^{\cdot}_X) \oplus \mathbf{H}^{2n-q}_{Y_2}(\Omega^{\cdot}_X) \to \mathbf{H}^{2n-q}_{Y}(\Omega^{\cdot}_X) \to \mathbf{H}^{2n-q+1}_{Y_1 \cap Y_2}(\Omega^{\cdot}_X) \to \dots$$

This is just the Mayer-Vietoris sequence of local cohomology, obtained as follows: For any sheaf of abelian groups F on X, there is an exact sequence

$$0 \to \Gamma_{Y_1 \cap Y_2}(F) \to \Gamma_{Y_1}(F) \oplus \Gamma_{Y_2}(F) \to \Gamma_Y(F).$$

If F is flasque, in particular if F is injective, the last map is surjective. Indeed, given $s \in \Gamma_Y(F)$, consider the section $s' \in \Gamma(X-(Y_1 \cap Y_2), F)$ whose restriction to $Y_1-(Y_1 \cap Y_2)$ is s, and which is zero elsewhere. Then s' extends to an element of $\Gamma(X, F)$, which necessarily has support in Y_1. Furthermore, $s-s'$ has support in Y_2, so our map is surjective. Now taking derived functors gives the required exact sequence of local cohomology.

Next we come to the exact sequences related to a proper birational map. We include a statement for coherent sheaves, which will be used in the proof of the following result, and which may be useful in its own right.

Proposition (4.3) [26, Prop. 4.1]. — *Let* $f: X' \to X$ *be a proper morphism of schemes. Let* Y *be a closed subscheme of* X, *and let* $Y' = f^{-1}(Y)$. *Assume that* f *maps* $X'-Y'$ *isomorphically onto* $X-Y$. *Suppose given coherent sheaves* F *on* X *and* F′ *on* X′, *and an injective* *map* $F \to f_* F'$, *whose restriction to* $X - Y$ *is an isomorphism. Then there is a long exact sequence of cohomology*

$$\ldots \to H^i(X, F) \to H^i(X', F') \oplus H^i(\hat{X}, \hat{F}) \to H^i(\hat{X}', \hat{F}') \to H^{i+1}(X, F) \to \ldots$$

where $\hat{}$ *denotes completion along* Y *or* Y', *respectively.*

Proof. — Take an injective resolution $F' \to I^\bullet$ of F', and an injective resolution $\hat{F}' \to J^\bullet$, and fix a map $I^\bullet \to J^\bullet$ compatible with the map $F' \to \hat{F}'$. Then we have natural inclusions $F \to f_* I^\bullet$ and $\hat{F} \to \hat{f}_* J^\bullet$ induced by the given inclusion $F \to f_* F'$. Thus we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & f_* I^\bullet & \longrightarrow & Q^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{F} & \longrightarrow & \hat{f}_* J^\bullet & \longrightarrow & R^\bullet & \longrightarrow & 0 \end{array}$$

where $Q^\bullet$ and $R^\bullet$ are the respective quotient complexes.

We will show that the map $Q^\bullet \to R^\bullet$ is a quasi-isomorphism, i.e., the maps $h^i(Q^\bullet) \to h^i(R^\bullet)$ of cohomology sheaves are isomorphisms, for all i. On the h^0 level, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & f_* F' & \longrightarrow & h^0(Q^\bullet) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{F} & \longrightarrow & \hat{f}_* \hat{F}' & \longrightarrow & h^0(R^\bullet) & \longrightarrow & 0 \end{array}$$

Now the map $F \to f_* F'$ was assumed to be an isomorphism when restricted to $X - Y$; f is proper, so $f_* F'$ is coherent, hence $h^0(Q^\bullet)$ is a coherent sheaf, whose support is contained in Y. By the "fundamental theorem of a proper morphism" [EGA III 4.1.5], $\hat{f}_* \hat{F}' = (f_* F')\hat{}$. Hence $h^0(R^\bullet) = h^0(Q^\bullet)\hat{}$. But since $h^0(Q^\bullet)$ is coherent, with support in Y, it is equal to its completion. So we have $h^0(Q^\bullet) \cong h^0(R^\bullet)$.

For $i > 0$, we have

$$\begin{array}{ccc} R^i f_* F' & \xrightarrow{\approx} & h^i(Q^\bullet) \\ \downarrow & & \downarrow \\ R^i \hat{f}_* \hat{F}' & \xrightarrow{\approx} & h^i(R^\bullet). \end{array}$$

Again by [EGA III, 4.1.5], $R^i \hat{f}_* \hat{F}' = (R^i f_* F')\hat{}$. Since f is an isomorphism outside of Y', the sheaves $R^i f_* F'$ for $i > 0$ are coherent and have support contained in Y. So they are equal to their completions, and we have $h^i(Q^\bullet) \cong h^i(R^\bullet)$. Hence $Q^\bullet \to R^\bullet$ is a quasi-isomorphism, as claimed.

Now we take the long exact sequences of hypercohomology of the two short exact sequences of complexes above. Note that in the middle, we get

$$\mathbf{H}^i(X, f_* I^\bullet) = H^i(X', F')$$

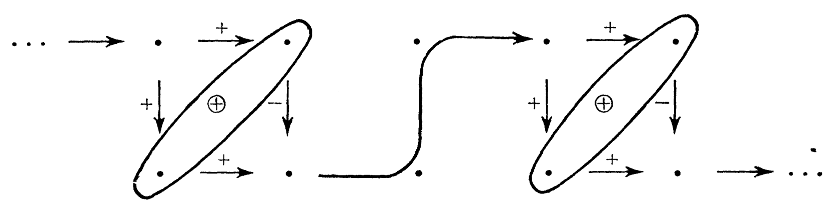
and

$$\mathbf{H}^i(X, \hat{f}_* J^\bullet) = H^i(\hat{X}', \hat{F}'),$$

since $\Gamma(X', \cdot)$ is equal to the composite functor $\Gamma(X, f_*(\cdot))$ (cf. [RD II, 5.1]). On the other hand, the quasi-isomorphism $Q^\bullet \to R^\bullet$ gives rise to isomorphisms of hypercohomology. So we have exact sequences

$$\begin{array}{ccccccccccc} \ldots \longrightarrow & H^i(X, F) & \longrightarrow & H^i(X', F') & \longrightarrow & \mathbf{H}^i(Q^\bullet) & \longrightarrow & H^{i+1}(X, F) & \longrightarrow & H^{i+1}(X', F') & \longrightarrow \ldots \\ & \downarrow & & \downarrow & & \downarrow \approx & & \downarrow & & \downarrow & \\ \ldots \longrightarrow & H^i(\hat{X}, \hat{F}) & \longrightarrow & H^i(\hat{X}', \hat{F}') & \longrightarrow & \mathbf{H}^i(R^\bullet) & \longrightarrow & H^{i+1}(\hat{X}, \hat{F}) & \longrightarrow & H^{i+1}(\hat{X}', \hat{F}') & \longrightarrow \ldots \end{array}$$

Now we deduce the exact sequence of the proposition by an elementary diagram chase according to the following pattern:



Theorem (4.4) (Exact sequence of cohomology for a proper birational morphism). *Let* $f: X' \to X$ *be a proper morphism of schemes. Let* Y *be a closed subscheme of* X, *and let* $Y' = f^{-1}(Y)$. *Assume that* f *maps* $X' - Y'$ *isomorphically to* $X - Y$. *Assume furthermore that there exist closed immersions* $X \to Z$ *and* $X' \to Z'$ *of* X *and* X' *into smooth schemes over* k, *and a proper morphism* $g: Z' \to Z$ *such that* $g|_{X'} = f$ *and* g *maps* $Z' - g^{-1}(Y)$ *isomorphically to* $Z - Y$. *Then there is an exact sequence of algebraic De Rham cohomology*

$$\ldots \to H^q(X) \to H^q(X') \oplus H^q(Y) \to H^q(Y') \to H^{q+1}(X) \to \ldots$$

Remark. — The extra hypothesis about the existence of the embeddings into Z and Z' will automatically be satisfied in many applications. For example, if X is a variety, if $f: X' \to X$ is a resolution of singularities of X, and if $X \to Z$ is any embedding of X into a smooth scheme over k, then a Z' will exist as above. Indeed, the morphism f is obtained as a sequence of monoidal transformations with respect to non-singular centers. If we apply the same transformations to Z, we obtain a smooth scheme Z', and a birational map $g: Z' \to Z$, and X' appears as the proper transform of X.

Proof of theorem: *Case 1.* — X and X' smooth. Then Z and Z' are unnecessary. We will follow the pattern of the previous proof. For each p, take an injective resolution (in the category of abelian sheaves on X'), $\Omega^p_{X'} \to I^{p\bullet}$, and extend the maps of the complex $\Omega^\bullet_{X'}$ to get a double complex $I^{\bullet\bullet}$. Similarly, take an injective resolution (in the category of abelian sheaves on Y') $\hat{\Omega}^p_{X'} \to J^{p\bullet}$, where ^ denotes completion along Y', and form the double complex $J^{\bullet\bullet}$. Fix also a map of double complexes $I^{\bullet\bullet} \to J^{\bullet\bullet}$ compatible with the completion map $\Omega^\bullet_{X'} \to \hat{\Omega}^\bullet_{X'}$.

We have natural inclusions $\Omega^\bullet_X \to f_* \Omega^\bullet_{X'}$ and $\hat{\Omega}^\bullet_X \to f_* \hat{\Omega}^\bullet_{X'}$, where ^ on X denotes the completion along Y. Thus we obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^\bullet_X & \longrightarrow & f_* I^{\bullet\bullet} & \longrightarrow & Q^{\bullet\bullet} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{\Omega}^\bullet_X & \longrightarrow & f_* J^{\bullet\bullet} & \longrightarrow & R^{\bullet\bullet} & \longrightarrow & 0 \end{array}$$

where $Q^{\bullet\bullet}$ and $R^{\bullet\bullet}$ are the quotient double complexes.

Note that for each p, the natural map $\Omega^p_X \to f_* \Omega^p_{X'}$ is injective, and is an isomorphism on $X-Y$. The previous proposition and its proof show that for each p, the map of complexes $Q^{p\bullet} \to R^{p\bullet}$ is a quasi-isomorphism. It follows that the map of associated simple complexes $s(Q^{\bullet\bullet}) \to s(R^{\bullet\bullet})$ is also a quasi-isomorphism.

To complete the proof, we take the associated simple complexes of the diagram above, then form the long exact sequences of hypercohomology, and deduce the exact sequence of the theorem by a diagram chase just as in the proof of the previous result.

Case 2. — General case. With the hypotheses of the theorem, let $Y'' = g^{-1}(Y)$. Then we necessarily have $g^{-1}(X) = X' \cup Y''$, and $Y' = X' \cap Y''$. We will make a three-step constructions as in Case 1, applied to the spaces $Z \supseteq X \supseteq Y$ and $Z' \supseteq X' \cup Y'' \supseteq Y''$. Then, with a slight change of notation, we have exact sequences of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^\bullet_Z & \longrightarrow & \mathbf{R}f_* \Omega^\bullet_{Z'} & \longrightarrow & Q^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^\bullet_{Z/X} & \longrightarrow & \mathbf{R}f_* \Omega^\bullet_{Z'/(X' \cup Y'')} & \longrightarrow & R^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^\bullet_{Z/Y} & \longrightarrow & \mathbf{R}f_* \Omega^\bullet_{Z'/Y''} & \longrightarrow & S^\bullet & \longrightarrow & 0 \end{array}$$

where we have written Z/X to denote the formal completion of Z along X, etc., and $\mathbf{R}f_*$ to denote the process of taking an injective resolution followed by f_*, and where $Q^\bullet$, $R^\bullet$, $S^\bullet$ denote the quotient simple complexes.

Now the hypotheses of Case 1 apply to the situations $(Z \supseteq X, Z' \supseteq X' \cup Y'')$ and $(Z \supseteq Y, Z' \supseteq Y'')$. We find that $Q^\bullet \to R^\bullet$ and $Q^\bullet \to S^\bullet$ are quasi-isomorphisms. It follows that $R^\bullet \to S^\bullet$ is a quasi-isomorphism.

This result, together with the Mayer-Vietoris sequence

$$0 \to \Omega^\bullet_{Z'/(X' \cup Y'')} \to \Omega^\bullet_{Z'/X'} \oplus \Omega^\bullet_{Z'/Y''} \to \Omega^\bullet_{Z'/Y'} \to 0$$

and an interesting diagram-chase show that in the situation

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{\bullet}_{Z/X} & \longrightarrow & \mathbf{R}f_*\Omega^{\bullet}_{Z'/X'} & \longrightarrow & T^{\bullet} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^{\bullet}_{Z/Y} & \longrightarrow & \mathbf{R}f_*\Omega^{\bullet}_{Z'/Y'} & \longrightarrow & U^{\bullet} & \longrightarrow & 0, \end{array}$$

the map $T^{\bullet} \to U^{\bullet}$ is a quasi-isomorphism. (This diagram-chase, if carried out in the derived category, gives a good opportunity to use the octohedral axiom.) Then, as before, taking the long exact sequence of cohomology gives the result of the theorem.

Proposition **(4.5)** (Exact sequence of homology for a proper birational morphism). — *Let* X, X′, Y, Y′ *be as in the previous theorem. Then there is an exact sequence of algebraic De Rham homology*

$$\ldots \to H_q(Y') \to H_q(Y) \oplus H_q(X') \to H_q(X) \to H_{q-1}(Y') \to \ldots$$

Proof. — We write the exact sequences associated to the closed subsets Y in X and Y′ in X′ (Theorem (3.3) above) and the functorial maps for the proper morphism $X' \to X$:

$$\begin{array}{ccccccccccccc} \ldots & \longrightarrow & H_q(Y') & \longrightarrow & H_q(X') & \longrightarrow & H_q(X'-Y') & \longrightarrow & H_{q-1}(Y') & \longrightarrow & H_{q-1}(X') & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & H_q(Y) & \longrightarrow & H_q(X) & \longrightarrow & H_q(X-Y) & \longrightarrow & H_{q-1}(Y) & \longrightarrow & H_{q-1}(X) & \longrightarrow & \ldots \end{array}$$

The middle vertical arrow is an isomorphism because $X'-Y'$ is isomorphic to $X-Y$. So the required exact sequence follows by the diagram chase of Proposition (4.3) above.

As an application of these results, we will consider the cohomology of an affine scheme. If X is a smooth affine scheme of dimension n, then we can calculate $H^i(X)$ using the De Rham complex of $X^{\bullet}$ itself. Since X is affine, the spectral sequence of hypercohomology degenerates, and we have $H^i(X) = h^i\Gamma(X, \Omega^{\bullet}_X)$. Since $\Omega^p_X = 0$ for $p > n$, we find that $H^i(X) = 0$ for $i > n$.

The same argument in the complex case, shows that if X is a Stein manifold of dimension n, then $H^i(X, \mathbf{C}) = 0$ for $i > n$. This prompted Serre to ask whether the same is true for a Stein space with singularities, and this question has been answered affirmatively by Kaup [35], R. Narasimhan [42], and Bloom and Herrera [5].

Here we prove a slightly weaker result in the algebraic case.

Theorem **(4.6)**. — *Let* X *be an affine scheme of dimension* n. *Then* $H^i(X) = 0$ *for* $i > n+1$.

Proof. — We use induction on n, the case $n = 0$ being trivial. By the Mayer-Vietoris sequence and the induction hypothesis, we reduce to the case X integral. Let

Y be the singular locus of X. Then by resolution of singularities we can find a smooth scheme X' and a proper birational map $f: X' \to X$ which satisfies the hypotheses of (4.4). Thus we have an exact sequence of cohomology

$$\ldots \to H^i(X) \to H^i(X') \oplus H^i(Y) \to H^i(Y') \to \ldots$$

By the induction hypothesis we may assume that $H^i(Y) = 0$ for $i > n$. Thus to prove our result, it will be sufficient to show that the restriction map

$$H^i(X') \to H^i(Y')$$

is an isomorphism for $i > n$.

Since X' is smooth, we can calculate $H^i(X')$ and $H^i(Y')$ using the De Rham complex on X', and its completion along Y'. Since X and Y are affine, we can write the spectral sequences of hypercohomology as follows:

$$E_1^{pq} = H^0(X, R^q f_*(\Omega^p_{X'})) \Rightarrow H^i(X')$$
$$'E_1^{pq} = H^0(X, R^q \hat{f}_*(\hat{\Omega}^p_{X'})) \Rightarrow H^i(Y').$$

Furthermore, there is a natural morphism of the first spectral sequence into the second. By the fundamental theorem of a proper morphism, we have

$$R^q \hat{f}_*(\hat{\Omega}^p_{X'}) = R^q f_*(\Omega^p_{X'})\hat{}.$$

On the other hand, since f is an isomorphism of X'—Y' onto X—Y, the support of $R^q f_*(\Omega^p_{X'})$ is contained in Y for $q > 0$. It is a coherent sheaf, therefore it is equal to its completion.

We conclude that $E_1^{pq} \to {}'E_1^{pq}$ is an isomorphism for $q > 0$ and all p, and for $p > n$ and all q (in the latter case it is zero). Hence in the abutment we have an isomorphism for $i > n$, as required.

Remarks: 1. According to the comparison theorem (IV, 1.1) below, and the analytic result about Stein spaces, it is also true that $H^{n+1}(X) = 0$. However we do not know an algebraic proof of this fact.

2. One can also ask about the homology of an affine scheme. In the smooth case, we have $H_i(X) = 0$ for $i < n$, by (3.4). This result does not hold in the singular case without further restrictions, as one can see by considering the union of two affine planes in four-space, which meet at a point. In that case $H_1(X) \neq 0$. However, if one imposes the condition of Ogus [43] that DR-depth $X = n$, then one has $H_i(X) = 0$ for $i < n$ also in the singular case. (See Ch. III, § 7 below for a discussion of this situation.)

5. Duality.

Theorem **(5.1)** (Duality). — *Let* Y *be an embeddable scheme, proper over k. Then there is a natural isomorphism for each q*

$$H^q(Y) \cong (H_q(Y))',$$

where ′ denotes the dual k-vector space. These isomorphisms are compatible with the functorial maps of H^q *as a contravariant functor, and* H_q *as a covariant functor, on the category of proper schemes over k.*

Proof. — Embed Y in a smooth scheme X of dimension n over k. Then we wish to establish an isomorphism

$$\mathbf{H}^q(\hat{X}, \hat{\Omega}^{\bullet}_X) \cong (\mathbf{H}^{2n-q}_Y(X, \Omega^{\bullet}_X))'.$$

Our technique is to deduce this from a similar "formal duality" for coherent sheaves on X, which in turn follows from the usual "Serre duality" for coherent sheaves. The essential point is to show that Serre duality for the coherent sheaves Ω^p_X is compatible with the exterior differential d. For this we will use the fact (Proposition (2.2) above) that d commutes with the trace map.

Proposition **(5.2)** (Formal duality). — *Let* X *be a smooth scheme of dimension n over k, let* Y *be a closed subscheme, proper over k, let* F *be a locally free sheaf on* X, *and let* $\omega = \Omega^n_{X/k}$. *Then there are natural isomorphisms*

$$H^q(\hat{X}, \hat{F}) \cong (H^{n-q}_Y(X, \check{F} \otimes \omega))'.$$

Note that these vector spaces need not be finite-dimensional.

Proof. (See also [24, III, 3.3], where the same result is proved with the unnecessary additional hypothesis that X be proper over k.) — With the needs of our theorem in mind, we will take some care in defining the map which gives rise to this duality isomorphism. We will use the notation of § 2: E denotes the Cousin complex of a locally free sheaf, and the trace map will be

$$\mathrm{Tr}_f : \Gamma(E(\omega))[n] \to k.$$

This is only a map of graded sheaves, but since Y is proper over k, the induced map

$$\mathrm{Tr}_f : \Gamma_Y(E(\omega))[n] \to k$$

is a morphism of complexes (see Proposition (2.3) above).

To establish the map of our proposition, we first define a map of complexes of sheaves

$$\varphi : F \to \mathcal{H}om^{\bullet}_{\mathcal{O}_X}(\mathit{\Gamma}_Y(E(\check{F} \otimes \omega)), \mathit{\Gamma}_Y(E(\omega))).$$

(See [RD, p. 63] for the notation $\mathcal{H}om^{\bullet}$.) Because of the nature of the Cousin complex, it is sufficient to define for each $x \in Y$, a map

$$\hat{F}_x \to \mathrm{Hom}_{\mathcal{O}_x}(\check{F} \otimes \omega \otimes I_x, \omega \otimes I_x).$$

But this is immediate, using the fact that I_x, the injective hull of $k(x)$ over $\mathcal{O}_x$, has a natural structure of $\mathcal{O}_x$-module, and using the natural map $F \to \mathrm{Hom}(\check{F} \otimes \omega, \omega)$.

Applying the functor Γ to the complex above, and composing with the trace map, we have natural maps of complexes

$$\begin{array}{c} \Gamma(\mathcal{H}om^{\bullet}_{\mathcal{O}_X}(\Gamma_Y(E(\check{F}\otimes\omega)), \Gamma_Y(E(\omega))) \\ \downarrow \\ \mathrm{Hom}^{\bullet}_k(\Gamma_Y(E(\check{F}\otimes\omega)), \Gamma_Y(E(\omega))) \\ \downarrow{\scriptstyle \mathrm{Tr}_f} \\ \mathrm{Hom}^{\bullet}_k(\Gamma_Y(E(\check{F}\otimes\omega)), k[-n]). \end{array}$$

Now the map φ defined above induces maps on cohomology; the complex on the right is composed of flasque sheaves, so composing with the maps just defined, and noting that

$$h^q(\Gamma_Y(E(\check{F}\otimes\omega))) = H^q_Y(X, \check{F}\otimes\omega),$$

we have maps

$$H^q(\varphi) : H^q(\hat{X}, \hat{F}) \to (H^{n-q}_Y(X, \check{F}\otimes\omega))'.$$

This is the duality map of the proposition.

To show it is an isomorphism, we consider the scheme Y_r defined by the sheaf of ideals I^r_Y. Expressing the cohomology of $\hat{X}$ as an inverse limit (see Ch. I, § 4) and the local cohomology as a direct limit of Ext's, we reduce to showing that

$$H^q(Y_r, F\otimes\mathcal{O}_{Y_r}) \cong (\mathrm{Ext}^{n-q}_{\mathcal{O}_X}(\mathcal{O}_{Y_r}\otimes F, \omega))'.$$

Now $E(\omega)[n]$ is a residual complex for X. Hence

$$K^{\bullet}_r = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{Y_r}, E(\omega)[n])$$

is a residual complex for Y_r. Rewriting, we must show that

$$H^q(Y_r, F\otimes\mathcal{O}_{Y_r}) \cong (\mathrm{Ext}^{-q}_{\mathcal{O}_{Y_r}}(F\otimes\mathcal{O}_{Y_r}, K^{\bullet}_r))'$$

is an isomorphism. But this is just the duality theorem for Y_r [RD, Ch. VII].

Proof of theorem (continued). — To define the map of the theorem, we follow the pattern of the previous proof, using always homomorphisms which commute with d, i.e. homomorphisms of graded $\Omega^{\bullet}$-modules. So we have a natural map

$$\varphi : \Omega^{\bullet}_X \to \mathcal{H}om^{\bullet}_{\Omega^{\bullet}_X}(\Gamma_Y(E(\Omega^{\bullet}_X)), \Gamma_Y(E(\Omega^{\bullet}_X))).$$

The trace map is

$$\mathrm{Tr}_f : \Gamma_Y(E(\Omega^{\bullet}_X))[2n] \to k,$$

which is a morphism of complexes, by Propositions (2.2) and (2.3) above. So applying Γ and composing with Tr_f, we have natural maps of complexes

$$\begin{array}{c}\Gamma(\mathscr{H}om^{\bullet}_{\Omega^{\bullet}}(\mathit{\Gamma}_Y(E(\Omega^{\bullet}_X)), \mathit{\Gamma}_Y(E(\Omega^{\bullet}_X)))\\ \downarrow \\ \mathrm{Hom}^{\bullet}_k(\Gamma_Y(E(\Omega^{\bullet}_X)), \Gamma_Y(E(\Omega^{\bullet}_X)))\\ \downarrow \\ \mathrm{Hom}^{\bullet}_k(\Gamma_Y(E(\Omega^{\bullet}_X)), k[-2n]).\end{array}$$

Now φ induces maps on hypercohomology

$$\mathbf{H}^q(\hat{X}, \hat{\Omega}^{\bullet}_X) \to (\mathbf{H}^{2n-q}_Y(X, \Omega^{\bullet}_X))'$$

which are the maps of the theorem.

On the other hand, for each Ω^p_X, we can consider the maps of the previous proposition, noting that $\mathscr{H}om_{\mathcal{O}_X}(\Omega^p_X, \omega) = \Omega^{n-p}_X$. So for each p, we have maps

$$\varphi^p : \hat{\Omega}^p_X \to \mathscr{H}om^{\bullet}_{\mathcal{O}_X}(\mathit{\Gamma}_Y(E(\Omega^{n-p}_X)), \mathit{\Gamma}_Y(E(\omega))).$$

Summing over p, we have

$$\sum_p \varphi^p : \hat{\Omega}^{\bullet}_X \to \mathscr{H}om^{\bullet}_{\mathcal{O}_X}(\mathit{\Gamma}_Y(E(\Omega^{\bullet}_X)), \mathit{\Gamma}_Y(E(\omega))).$$

Furthermore, the trace maps are compatible *via* the projection

$$\begin{array}{c}\mathscr{H}om^{\bullet}_{\Omega^{\bullet}}(\mathit{\Gamma}_Y(E(\Omega^{\bullet}_X)), \mathit{\Gamma}_Y(E(\Omega^{\bullet}_X)))\\ \downarrow \\ \mathscr{H}om^{\bullet}_{\mathcal{O}_X}(\mathit{\Gamma}_Y(E(\Omega^{\bullet}_X)), \mathit{\Gamma}_Y(E(\omega)))\end{array}$$

defined by " taking the component which lands in $\omega = \Omega^n$ ".

This shows that the map of the theorem is compatible with the maps of the proposition for the sheaves Ω^p_X. Thus using the spectral sequence of hypercohomology for $\Omega^{\bullet}_X$, the isomorphisms of the proposition show that the map of the theorem is also an isomorphism.

The functoriality of our duality isomorphisms as Y varies follows from the functoriality of the construction.

Corollary **(5.3)** (Poincaré duality). — *Let* Y *be smooth and proper of dimension n over k. Then there are natural isomorphisms*

$$H^q(Y) \cong (H^{2n-q}(Y))'.$$

Proof. — Combine the theorem with Proposition (3.4) above.

6. Finiteness.

In this section we will prove the finite-dimensionality of the algebraic De Rham cohomology and homology groups, as vector spaces over the base field. We have delayed this result, because it uses the resolution of singularities of Hironaka, and we wished to show how far the general theory can be developed without using resolution. On the other hand, it seems that resolution is essential for proving finite-dimensionality, and it will also be essential for proving the comparison theorem with complex cohomology of the associated analytic spaces.

Theorem **(6.1)**. — *Let* Y *be an embeddable scheme of finite type over a field k of characteristic zero. Then the cohomology groups* $H^q(Y)$ *and the homology groups* $H_q(Y)$ *are finite-dimensional k-vector spaces, for all q.*

Proof. — We proceed in several steps. First suppose that Y is smooth and proper over k. Then we can calculate $H^q(Y)$ as the hypercohomology of the De Rham complex $\Omega^{\bullet}_Y$ on Y. The first spectral sequence of hypercohomology for $\mathbf{H}^q(Y, \Omega^{\bullet}_Y)$ has initial terms $E_1^{pq} = H^q(Y, \Omega^p)$. These are finite-dimensional over k since Y is proper, and Ω^p is coherent, by the finiteness theorem of Serre and Grothendieck [EGA III, § 3]. Hence the abutment terms $H^q(Y)$ are also finite-dimensional. Since Y is smooth, we have $H_q(Y) \cong H^{2n-q}(Y)$, where n is the dimension of Y, so the homology groups are also finite-dimensional.

For the rest of the proof, we will use induction on the dimension of Y, the case $\dim Y = 0$ being trivial.

For the next step, suppose that Y is smooth over k, but not necessarily proper. Since Y is smooth, we have $H^q(Y) \cong H_{2n-q}(Y)$, so it will be sufficient to consider the homology groups. We will give two slightly different proofs in this case. For the first, we use the embedding theorem of Nagata [40] to deduce that Y can be embedded as an open, dense subset of a scheme $\overline{Y}_0$ which is proper over k. By resolution of singularities, we can find a smooth proper scheme $\overline{Y}$ over k, which also contains Y as an open dense subset. Let $Z = \overline{Y} - Y$. Then the exact sequence of homology for a closed subset gives

$$\dots \to H_q(Z) \to H_q(\overline{Y}) \to H_q(Y) \to H_{q-1}(Z) \to \dots$$

By the induction hypothesis, we may assume that the homology of Z is finite-dimensional. The scheme $\overline{Y}$ is proper and smooth over k, so its homology is finite-dimensional, as shown above. We conclude that the homology of Y is finite-dimensional.

Our second proof of this step avoids the use of Nagata's embedding theorem. We first reduce to the case Y smooth and affine, by chopping off a closed subset of lower dimension, and using the exact sequence of homology of a closed subset. Now Y, being affine, can be embedded in an affine space $\mathbf{A}_k^N$ for some N. Its closure $\overline{Y}_0$ in $\mathbf{P}_k^N$ is a proper scheme over k, containing Y as an open dense subset. Now we complete the proof as above.

For the last step, we consider an arbitrary embeddable scheme Y over k. Using the Mayer-Vietoris sequences for cohomology and homology, and the induction hypothesis, we reduce to the case of an irreductible scheme Y. Its cohomology and homology is the same as the reduced schemes Y_{red}, so we may assume Y is reduced and irreducible. Let $Y \to Z$ be an embedding of Y in a smooth scheme Z over k. By resolution of singularities, we can find a proper, birational morphism $g : Z' \to Z$ of smooth schemes over k, such that the proper transform Y′ of Y is smooth over k, and the fundamental locus W of g is properly contained in Y. Let $f = g|_{Y'}$, and let $W' = f^{-1}(W)$. Then by Theorem (4.4) above, there is an exact sequence of cohomology

$$\ldots \to H^q(Y) \to H^q(Y') \oplus H^q(W) \to H^q(W') \to H^{q+1}(Y) \to \ldots$$

Since W and W′ have smaller dimension, we can apply the induction hypothesis. By the previous step, the cohomology of Y′ is finite-dimensional. We conclude that the cohomology of Y is finite-dimensional. The same argument for homology, using Proposition (4.5), shows that the homology of Y is finite-dimensional. This completes the proof of the theorem.

Remark. — For a smooth affine scheme Y over a field of characteristic zero, Monsky [39] has given an entirely different proof of the finiteness of De Rham cohomology, without using resolution of singularities. It would be interesting to know if this proof would extend also to the singular case.

7. Further Developments.

We have now established enough of the general theory so that a number of further developments, common to many cohomology theories, will follow quite easily, provided they are approached in the right order. Therefore in this section we will treat a number of topics in a briefer style, trusting that the reader can supply any missing details.

The principal result of this section is the definition of the cohomology class of a cycle on a smooth variety, the fact that it depends only on the rational equivalence class of the cycle, and transforms intersection of cycle classes into cup-product of cohomology classes. We also calculate the homology and cohomology of a vector bundle, and establish a " Thom isomorphism ". This shows in particular that our cohomology theory has the properties needed for the development of a theory of Chern classes [15], and the phrasing of the Riemann-Roch theorem [7].

For a scheme Y, we first define the fundamental homology class $\eta(Y) \in H_{2r}(Y)$, where $r = \dim Y$. Then if Y is a closed subscheme of a scheme X, we define the homology class $\eta(Y) \in H_{2r}(X)$ of Y on X by the natural map on homology. Then, if X is smooth of dimension n, we obtain the cohomology class of Y, $\eta(Y) \in H^{2n-2r}(X)$, by the natural isomorphism of homology with cohomology on a smooth scheme (3.4). If Y is a Cartier divisor, we also give another construction of $\eta(Y)$ using Čech cohomology.

We do not know how to define the cohomology class of a cycle on a singular scheme, and hence we do not know how to define Chern classes for vector bundles on a singular scheme. It would be interesting to know to what extent these results could be generalized to the singular case.

7.1. Explicit calculations.

First we note that if Y is any scheme, the homology and cohomology of Y are the same as for the reduced scheme Y_{red}. If Y is a disjoint union of schemes Y_i, then the homology and cohomology of Y are the direct sum of that of the Y_i. If Y is any scheme of finite type over k, then $H^0(Y)$ is a finite-dimensional commutative k-algebra with 1. In particular, if Y is connected, it is an integral domain, hence a field. Thus if k is algebraically closed, and Y is connected, we have $H^0(Y)=k$.

Next we compute the cohomology of affine n-space over k. Let $Y=\mathbf{A}^n_k$. Then $H^0(Y)=H_{2n}(Y)=k$, and all other cohomology and homology groups are zero. This follows from the " Poincaré lemma ", whose proof we include for completeness.

Proposition **(7.1)** (Poincaré lemma). — *Let k be a field of characteristic zero. Let* $R=k[x_1, \ldots, x_n]$. *Then the sequence*

$$0 \to k \to R \xrightarrow{d} \Omega^1_{R/k} \xrightarrow{d} \ldots \to \Omega^n_{R/k} \to 0$$

is an exact sequence of k-vector spaces.

Proof. — By induction on n, the case $n=0$ being trivial. Let $\omega \in \Omega^p$, with $d\omega=0$. Then we must show $\omega=d\eta$ for some $\eta \in \Omega^{p-1}$. We separate out the part of ω which depends on dx_1, i.e., we write

$$\omega = \omega' \, dx_1 + \omega''$$

where ω' and ω'' do not involve dx_1. Then we define

$$\eta' = \int \omega' \, dx_1$$

where of course " integration " simply means the algebraic process of replacing x_1^r by $x_1^{r+1}/(r+1)$ wherever it occurs. Then $\eta' \in \Omega^{p-1}$, and

$$d\eta' = \omega' \, dx_1 + \omega''',$$

where ω''' does not involve dx_1. Now, replacing ω by $\omega - d\eta'$, we reduce to the case where ω does not involve dx_1.

Assuming that ω does not involve dx_1, we can write

$$\omega = \Sigma f_{i_1 \ldots i_p} \, dx_{i_1} \ldots dx_{i_p}, \quad i_j > 1.$$

Since $d\omega=0$, we have $\partial f_{i_1 \ldots i_p}/\partial x_1=0$ for all $i_1, \ldots, i_p$. Thus the polynomials $f_{i_1 \ldots i_p}$ do not involve x_1 at all. So we have reduced to the case of the polynomial ring $k[x_2, \ldots, x_n]$, for which the result is true, by induction.

Remark. — This proof can be generalized in two directions. We can replace the field k by any ring A containing the rational numbers. Secondly, instead of taking R to be a polynomial ring, we can take R to be formal power series, or germs of holomorphic functions at the origin, in case $k=\mathbf{C}$. We need only observe that in each case, the process of " integration " gives again power series of the same type.

Using this remark, one can show easily that if Y is any scheme of finite type over k, then the projection of $\mathbf{A}^n_Y$ onto Y induces an isomorphism on cohomology.

We will also need one further case of the Poincaré lemma, which requires a more subtle proof.

Proposition **(7.1.1)** (Ogus). — *Let k be a field of characteristic zero, and let* R *be the completion of $k[x_1, \ldots, x_n]$ with respect to an* I*-adic topology, where* I *is a homogeneous ideal. Then the sequence of the proposition above is still exact.*

Proof. — This is proved for $I=\mathfrak{m}$ in Ogus' thesis [43, Prop. 1.1]. The same proof works, once one observes that for a homogeneous ideal I, the homotopy operators R of *loc. cit.* are continuous for the I-adic topology, hence pass to the I-adic completions.

The cohomology of projective space $\mathbf{P}^n_k$ can also be found explicitly. We have

$$H^q(\mathbf{P}^n_k)=\begin{cases} k & \text{for} \quad 0 \leq q \leq 2n, \quad q \text{ even}, \\ 0 & \text{otherwise}, \end{cases}$$

and similarly for the homology. One way to show this is to show that the coherent sheaf cohomology $H^q(\mathbf{P}^n, \Omega^p)$ is equal to k for $0 \leq q=p \leq n$ and zero otherwise. We leave to the reader the extension of this result to show that for a smooth scheme X, if $\pi : \mathbf{P}^n_X \to X$ is the projection, then π^* gives an isomorphism

$$H^{\cdot}(X)[\xi]/(\xi^{n+1}) \to H^{\cdot}(\mathbf{P}^n_X),$$

where ξ goes to the " class of a hyperplane ".

7.2. Dimension.

Theorem **(7.2)**. — *Let* Y *be a scheme of finite type of dimension n over k. Then* $H^q(Y)=0$ *and* $H_q(Y)=0$ *for* $q<0$ *and* $q>2n$.

Proof. — If Y is non-singular, we have $H_q(Y)=H^{2n-q}(Y)$, and $H^{\cdot}(Y)$ is the abutment of the spectral sequence of hypercohomology beginning with $E_1^{pq}=H^q(Y, \Omega^p)$. Thus the result follows from a theorem of Grothendieck [14], stating that for a topological space Y of combinatorial dimension n, and for any abelian sheaf F, $H^q(Y, F) = 0$ for $q < 0$ and $q > n$.

If Y is singular, we use induction on the dimension of Y. Furthermore, we may assume that Y is reduced, and using the Mayer-Vietoris sequences, we reduce to the case Y integral.

To show that the homology is zero in the desired range, let Y′ be the singular locus of Y. Then Y′ has smaller dimension, Y—Y′ is smooth, so that we have the result by the exact sequence of a closed subset (Theorem (3.3) above).

To show that the cohomology is zero in the desired range, we let $f: X \to Y$ be a resolution of singularities of Y. This is a birational map which is an isomorphism outside the singular locus Y′ of Y. In this case we have the result by induction on the dimension, using the exact sequence of cohomology for a birational morphism (Theorem (4.4) above).

7.3. Čech cohomology.

Theorem **(7.3)**. — *Let* Y *be a closed subscheme of a smooth scheme* X. *Let* $\mathfrak{A}$ *be an open affine cover of* Y. *Then the natural map*

$$\check{H}^q(\mathfrak{A}, \hat{\Omega}^{\cdot}_X) \to H^q(Y)$$

is an isomorphism for all q.

Indeed, there is always a natural map from Čech cohomology to derived functor cohomology. It is compatible with the spectral sequences of hypercohomology. Hence the result of the theorem follows from the fact that Čech cohomology of coherent sheaves on schemes (and formal schemes, for the same reason) computes the derived functor cohomology (see Ch. I, § 1).

7.4. Products.

To define products, it seems most convenient to use the canonical flasque resolutions of Godement [12, Ch. II, § 6.6]. Since his cohomology theory is equivalent to ours (see Ch. I, § 1 above) we obtain for any ringed space $(X, \mathcal{O}_X)$ and any $\mathcal{O}_X$-Modules F, G, natural cup-products

$$H^i(X, F) \times H^j(X, G) \to H^{i+j}(X, F \otimes_{\mathcal{O}_X} G).$$

Furthermore, if V and W are closed subsets of X, then we have the cup-product for cohomology with supports

$$H^i_V(X, F) \times H^j_W(X, G) \to H^{i+j}_{V \cap W}(X, F \otimes G).$$

In the case of De Rham cohomology, we take the canonical flasque resolutions of the sheaves Ω^i (or $\hat{\Omega}^i$ on a formal scheme). Since the canonical resolution is a functor, we get a double complex, and can form the associated simple complex. Then the exterior algebra structure on the De Rham complex, together with the above process, allows us to construct products in De Rham cohomology.

Let Y be a closed subscheme of a smooth scheme X. Then the cup-product for the De Rham complex $\hat{\Omega}^{\bullet}_{X}$ on the formal scheme $\hat{X}$ gives us a *cup-product* for algebraic De Rham cohomology

$$H^i(Y) \times H^j(Y) \to H^{i+j}(Y).$$

Furthermore, if V and W are closed subsets of Y, then we have the cup-product for algebraic De Rham cohomology with supports

$$H^i_V(Y) \times H^i_W(Y) \to H^i_{V \cap W}(Y).$$

For homology, recall that the homology of Y is computed using the complex $\Gamma_Y(E(\Omega^{\bullet}_X))$. The exterior product for differential forms induces maps

$$\hat{\Omega}^i_X \times \Gamma_Y(E(\Omega^j_X)) \to \Gamma_Y(E(\Omega^{i+j}_X)).$$

So the corresponding cup-product gives us a *cap-product* for algebraic De Rham cohomology and homology

$$H^i(Y) \times H_j(Y) \to H_{j-i}(Y).$$

If Z is a closed subscheme of Y, then taking $V=Z$, $W=Y$, the cup-product with supports gives us a cap-product with supports

$$H^i_Z(Y) \times H_j(Y) \to H_{j-i}(Z).$$

These products are functorial with respect to change of scheme. If Y is smooth, the cup-product and cap-product are compatible *via* the isomorphisms $H^i(Y) \cong H_{2n-i}(Y)$. If Y is proper over k, the cup-product and cap-product are compatible with the duality isomorphisms $H^i(Y) \cong (H_i(Y))'$. Indeed, looking back at the construction of the duality map, it is just the cap-product

$$H^i(Y) \times H_i(Y) \to H_0(Y)$$

followed by the trace map $H_0(Y) \to k$.

7.5. Projection formula.

The projection formula expresses the relation of the cap-product to the functors f^* and f_* on cohomology and homology.

Theorem (7.5). — *Let* $f: X \to Y$ *be a proper morphism of schemes. Let* $x \in H_q(X)$, *and* $y \in H^r(Y)$. *Then we have*

$$f_*(x . f^*(y)) = f_*(x) . y.$$

Here the dot represents cap-product on X *and* Y, *respectively, and the equality takes place in* $H_{q-r}(Y)$.

Basically this follows from the fact that f_* is defined by means of the trace map on the canonical resolution of the De Rham complex. The trace map of a morphism is linear over the base. Also, trace is compatible with the Čech process for computing cohomology, and hence with cup-product. We will leave details of the proof to the reader.

7.6. Homology class of a cycle.

Let X be a scheme of dimension n, and let Y be a cycle of dimension r on X, i.e., $Y=\sum_i n_i Y_i$ where $n_i \in \mathbf{Z}$, and the Y_i are integral closed subschemes of X, of dimension r. Then we will define the homology class $\eta(Y)\in H_{2r}(X)$.

First we need to define the fundamental class of an integral scheme Y of dimension r. Let Y′ be the singular locus of Y. The exact sequence of a closed subset gives us

$$\ldots \to H_{2r}(Y') \to H_{2r}(Y) \to H_{2r}(Y-Y') \to H_{2r-1}(Y') \to \ldots$$

But Y′ has dimension less than r, so by the dimension theorem $H_{2r}(Y')=H_{2r-1}(Y')=0$ and we have $H_{2r}(Y)\cong H_{2r}(Y-Y')$. On the other hand, since $Y-Y'$ is smooth, we have $H_{2r}(Y-Y')\cong H^0(Y-Y')$, which is a finite field extension of k. We define the *fundamental class* of Y

$$\eta(Y)\in H_{2r}(Y)$$

to be the image of $1\in H^0(Y-Y')$ under these isomorphisms.

Now let X be any scheme, and let $Y=\sum_i n_i Y_i$ be a cycle of dimension r on X. Let $j: Y_i \to X$ be the inclusion map. Then we define the *homology class of the cycle* Y by

$$\eta(Y)=\sum_i n_i j_*(\eta(Y_i))\in H_{2r}(X).$$

It is clear from the definition that if $U\subseteq X$ is an open subset, then the class of the cycle $Y\cap U$ in $H_{2r}(U)$ is the restriction of $\eta(Y)$. The other functorial aspect of η is less obvious.

Proposition (7.6). — *Let* $f: X'\to X$ *be a proper morphism of schemes, and let* Y′ *be a cycle on* X′. *Then*

$$f_*(\eta(Y'))=\eta(f_*Y').$$

Proof. — Since both f_* and η are linear, we may assume that Y′ is a closed integral subscheme of X′, of dimension r. Let $Y=f(Y')$. Recall by definition of f_* for cycles that if $\dim Y<r$, then $f_*(Y')=0$, and if $\dim Y=r$, then $f_*(Y')=mY$, where m is the degree of the morphism $Y'\to Y$, i.e., m is the dimension of the function field $K(Y')$ as a vector space over $K(Y)$.

Since the homology classes in X′ and X are direct images of the fundamental classes of Y′ and Y, respectively, it is enough to study f_* of the fundamental class $\eta(Y')\in H_{2r}(Y')$.

If $\dim Y < r$, then $H_{2r}(Y) = 0$, so there is nothing to prove. If $\dim Y = r$, we must show that

$$f_*(\eta(Y')) = m.\eta(Y)$$

in $H_{2r}(Y)$. By removing proper closed subsets of Y and Y′, as above, we may assume that Y and Y′ are both smooth over k, and that the map $f: Y' \to Y$ is finite. Then $H_{2r}(Y') = H^0(Y')$, and $H_{2r}(Y) = H^0(Y)$, and f_* is given by the trace map from $E(\Omega^\bullet_{Y'}) \to E(\Omega^\bullet_Y)$. Now $\eta(Y')$ is just $1 \in K(Y')$, $\eta(Y)$ is $1 \in K(Y)$, and the trace map from $K(Y') \to K(Y)$ sends 1 to m. So we are done.

7.7. Cohomology class of a Cartier divisor.

Let X be a smooth scheme, and let Y be a Cartier divisor on X. We will define the cohomology class of Y, $\gamma(Y) \in H^2(X)$. We will show that it depends only on the linear equivalence class of Y. Also we will show that it is equal to the homology class of Y considered as a cycle, via the isomorphism $H^2(X) \cong H_{2n-2}(X)$. Finally we will show that it has good functorial properties.

Given X and the Cartier divisor Y, let $\mathfrak{U} = (U_i)$ be an open affine cover of X, and let Y be defined by the rational function f_i on U_i. Then on $U_i \cap U_j$ the quotient f_i/f_j is regular. We define the *cohomology class* of Y, $\gamma(Y) \in H^2(X)$ to be the class of the Čech cocycle

$$\{d \log(f_i/f_j)\} \in \mathscr{C}^1(\mathfrak{U}, \Omega^1_X).$$

One checks easily that this is a cycle, and that the cohomology class $\gamma(Y)$ is independent of the choice of f_i. It is also clear that γ is an additive map from the group of Cartier divisors to $H^2(X)$. If Y is linearly equivalent to zero, that means that we can choose all the f_i equal to a single rational function f, in which case clearly $\gamma(Y) = 0$. Thus γ defines a group homomorphism

$$\gamma: \operatorname{Pic} X \to H^2(X).$$

If $f: X' \to X$ is a morphism of smooth schemes, it is clear from the construction that $f^*(\gamma(Y)) = \gamma(f^*Y)$. Somewhat less evident is the fact that this construction is compatible with the definition of the homology class of a cycle in the previous section.

Proposition (7.7.1). — *Let* X *be a smooth scheme, and let* Y *be a Cartier divisor on* X, *also considered as a cycle of codimension one. Then* $\gamma(Y) = \eta(Y)$ *via the natural isomorphism* $H^2(X) \cong H_{2n-2}(X)$.

Proof. — One is tempted to say, since both constructions are natural, how could they fail to give the same result? The difficulty is that $\gamma(Y)$ is defined using Čech cohomology, and $\eta(Y)$ is defined by the trace map on the canonical resolution of the De Rham complex. So we must chase through the identification of these two cohomology theories.

To compare the two, we apply the Čech process to the canonical resolution, and consider the triple complex $\mathscr{C}(\mathfrak{U}, E(\Omega^\cdot))$. We will use d for the exterior differentiation in $\Omega^\cdot$, δ for the maps in the Cousin complexes $E^\cdot$, and ∂ for the Čech coboundary map. We have natural quasi-isomorphisms of associated simple complexes

$$\mathscr{C}(\mathfrak{U}, \Omega^\cdot) \xrightarrow{\varphi} \mathscr{C}(\mathfrak{U}, E(\Omega^\cdot)) \xleftarrow{\psi} E(\Omega^\cdot)$$

which express the equivalence of the two cohomology theories.

Both γ and η are additive, so we may assume Y is an integral subscheme of codimension one. Now $\gamma(Y)$ is represented by the Čech cocycle

$$a = \{d \log(f_i/f_j)\} \in \mathscr{C}^1(\mathfrak{U}, \Omega^1),$$

and $\eta(Y)$ is represented by the image of $1 \in \mathcal{O}_Y$ under the trace map, which gives a section

$$b \in E^1(\Omega^1).$$

Consider the class in the triple complex

$$c = \{d \log f_i\} \in \mathscr{C}^0(\mathfrak{U}, E^0(\Omega^1)).$$

This makes sense because $E^0(\Omega^1) \cong \Omega^1 \otimes K(X)$, and the functions f_i are in $K(X)$. Note that $dc = 0$ by construction. Clearly $\partial c = \varphi(a)$. And δc is the class given by the images of df_i/f_i in $E^1(\Omega^1)$, which are all the same, and are equal to $\psi(b)$, as one sees by going back to the definition of the trace map.

Thus the total differential of c is the difference of the images of $\varphi(a)$ and $\psi(b)$ (up to a sign which we will let the reader straighten out), and hence these two define the same cohomology class.

We now generalize this result in a way which will be useful in the next section.

Proposition (**7.7.2**). — *Let* $f : X' \to X$ *be a morphism of integral schemes, with* X *smooth, and let* Y *be a Cartier divisor on* X *such that* $f(X') \not\subseteq \operatorname{Supp} Y$. *Then*

$$\eta(X') . f^*(\gamma(Y)) = \eta(cf^*Y).$$

Here c denotes the cycle associated to a Cartier divisor.

Proof. — If X′ is smooth, then we can consider the cohomology class $\gamma(f^*Y)$. This result then is a combination of the previous proposition with the fact that γ commutes with f^*, and that $\eta(X')$ is the identity element in the cohomology ring of X′, via the isomorphism of homology with cohomology.

If X′ is normal, we can reduce to the smooth case by removing closed subsets of X′ of codimension at least two. Our equality takes place in $H_{2n'-2}(X')$, where $n' = \dim X'$, which according to the dimension theorem and the exact sequence of a closed subset is not changed by removing these subsets.

In the general case, let $g : X'' \to X'$ be the normalization of X′. Then we have the result for X″, which says

$$\eta(X'') . g^*f^*(\gamma(Y)) = \eta(cg^*f^*(Y)).$$

Applying g_* and the projection formula, and the fact that g_* and η commute, we have

$$\eta(X') . f^*(\gamma(Y)) = \eta(g_* c g^* f^*(Y)).$$

Now the result follows from the easy result that for a finite birational morphism $g : X'' \to X'$, and a Cartier divisor Z on X′, we have $g_* c g^*(Z) = c(Z)$.

7.8. Intersection Theory.

Let X be a smooth scheme of dimension n. Let Y be a cycle on X of dimension r. We have already constructed the homology class of Y, $\eta(Y) \in H_{2r}(X)$. Via the isomorphism $H_{2r}(X) \cong H^{2n-2r}(X)$ we obtain the *cohomology class of the cycle* Y, which we denote also by

$$\eta(Y) \in H^{2n-2r}(X).$$

We now assume that X is quasi-projective, so that we can apply the theory of rational equivalence and intersection theory of rational equivalence classes of cycles [9]. We will show that $\eta(Y)$ depends only on the rational equivalence class of Y, and that $\eta(Y.Z) = \eta(Y) . \eta(Z)$ for any two cycle classes Y and Z.

Proposition (**7.8.1**). — *Let* X *be a smooth scheme, and let* Y *be a cycle on* X. *Then the cohomology class* $\eta(Y)$ *depends only on the rational equivalence class of* Y.

Proof. — To define rational equivalence, one considers a cycle Z on $X \times \mathbf{P}^1$. Let $X_0 = X \times \{0\}$, $X_1 = X \times \{1\}$. If Z meets X_0 and X_1 properly, then we say that the cycles $Z_0 = Z.X_0$ and $Z_1 = Z.X_1$ on X are rationally equivalent, and these special equivalences generate the relation of rational equivalence. Thus to prove our result, we must have $\eta(Z_0) = \eta(Z_1)$ in $H_\bullet(X)$.

Let $\sigma_0 : X_0 \to X \times \mathbf{P}^1$ and $\sigma_1 : X_1 \to X \times \mathbf{P}^1$ be the injections. Then one verifies easily that σ_{0*} and σ_{1*} define the same map of $H_\bullet(X) \to H_\bullet(X \times \mathbf{P}^1)$, and this map is injective. Hence it will be sufficient to show that

$$\eta(Z.X_0) = \eta(Z.X_1)$$

in $H_\bullet(X \times \mathbf{P}^1)$.

Now X_0 and X_1 are linearly equivalent divisors on $X \times \mathbf{P}^1$. Hence they have the same cohomology class: $\gamma(X_0) = \gamma(X_1)$. So our result will follow from the special case of the intersection theorem which says

$$\eta(Z.X_0) = \eta(Z) . \gamma(X_0).$$

To prove this, we apply Proposition (7.7.2) above to the morphism $f : Z \to X \times \mathbf{P}^1$. (We may assume of course that Z is an integral subscheme of $X \times \mathbf{P}^1$.) Thus we have

$$\eta(Z) . f^*(\gamma(X_0)) = \eta(c f^* X_0)$$

in $H_\bullet(Z)$. Applying the projection formula for f, we have

$$\eta(Z).\gamma(X_0)=\eta(Z.X_0)$$

in $H_\bullet(X\times\mathbf{P}^1)$, as required.

Remark. — It seems reasonable to expect that $\eta(Y)$ depends only on the algebraic equivalence class of Y, but we don't know how to prove this.

Theorem (**7.8.2**). — *Let* X *be a smooth quasi-projective scheme, and let* Y, Z *be cycles on* X. *Then*

$$\eta(Y.Z)=\eta(Y).\eta(Z).$$

Here Y.Z *denotes the rational equivalence class intersection, and the multiplication on the right is cup-product in* $H^\bullet(X)$.

Proof. — By linearity we may assume that Y and Z are integral subschemes of X. On the other hand, we can replace Z by a rationally equivalent cycle, so that Y and Z intersect properly, and we may even require that Z intersect the singular locus of Y properly also. Thus we may assume that $Y\cap Z=\bigcup_i W_i$, where W_i are subvarieties of the correct codimension, and the generic points of W_i are smooth on Y.

The cohomology group of X in which this equality takes place is not affected by removing subsets of X of dimension less than W (using the dimension theorem and the exact sequence of a closed subset). So by throwing away closed subsets of W, we may assume that the W_i are disjoint, that each W_i is contained as a closed subset of an open affine subset U_i of X, and that $Y\cap U_i$ is a complete intersection in U_i, for each i.

For the continuation of the proof it will be convenient to use local cohomology. For any closed subset $V\subseteq X$ we define $H^i_V(X)=\mathbf{H}^i_V(X,\Omega^\bullet_X)$. Now $\eta(Y)$ lies naturally in $H^\bullet_Y(X)$, $\eta(Z)$ lies in $H^\bullet_Z(X)$. The cup-product applies also to cohomology with supports, and gives us a product

$$\eta(Y).\eta(Z)\in H^\bullet_{Y\cap Z}(X),$$

whose image in $H^\bullet(X)$ is the usual cup-product. To finish our proof, we will establish the apparently stronger result that

$$\eta(Y.Z)=\eta(Y).\eta(Z) \text{ in } H^\bullet_{Y\cap Z}(X).$$

Now $Y\cap Z$ is the disjoint union of closed subsets W_i. Hence

$$H^\bullet_{Y\cap Z}(X)=\sum_i H^\bullet_{W_i}(X).$$

And by excision for local cohomology, $H^\bullet_{W_i}(X)=H^\bullet_{W_i}(U_i)$. Under this decomposition the components of $\eta(Y.Z)$ and $\eta(Y).\eta(Z)$ are obtained by restricting to the U_i.

Thus we have reduced to the case X affine, and Y a complete intersection, and we must show

$$\eta(Y.Z)=\eta(Y).\eta(Z) \text{ in } H^\bullet_W(X)$$

where $W = Y \cap Z$. If Y is a divisor, this follows from Proposition (7.7.2) and the projection formula, as in the previous proof, except that one needs to refine the projection formula in an obvious way so as to take into account the cohomology with supports. The case Y is a complete intersection of arbitrary codimension follows by an easy induction on the codimension.

7.9. Cohomology and homology of a vector bundle.

As a further illustration of the functorial properties of De Rham cohomology, we calculate the cohomology and homology of a vector bundle. Let Y be a scheme of dimension n, and let E be a vector bundle of rank r over Y. We denote the projection by $\pi : E \to Y$, and the zero-section by $\sigma : Y \to E$.

Proposition (7.9.1). — *The natural maps*

$$H^i(Y) \underset{\sigma^*}{\overset{\pi^*}{\rightleftarrows}} H^i(E)$$

are isomorphisms, inverse to each other, for all i.

Proof. — It is sufficient to show that π^* is an isomorphism, since $\pi . \sigma = \mathrm{Id}$. Using Čech cohomology, the question becomes local on Y. But E is locally isomorphic to $\mathbf{A}^r_Y$, whose cohomology is equal to that of Y, as we saw earlier (§ 7.1 above).

In order to define the Thom isomorphism associated to a vector bundle, we need to have a cohomology class of the zero-section. In general, we do not know how to define cohomology classes of cycles on singular schemes, so we will consider the following condition.

(*) *Assume that there exists an embedding* $Y \to X$ *into a smooth scheme* X, *and a vector bundle* F *on* X *such that* $E \cong F|_Y$.

In particular, this condition will be satisfied if Y is smooth. Assuming (*), we can consider the cohomology class $\xi \in H^{2r}_X(F)$ of the zero-section $\sigma(X)$ of F. We identify X with $\sigma(X)$. We can also consider the restriction of ξ to Y, which we denote also by $\xi \in H^{2r}_Y(E)$.

Theorem (7.9.2) (Thom isomorphism). — *Let* E *be a rank r vector bundle on a scheme* Y, *satisfying* (*), *and let* $\xi \in H^{2r}_Y(E)$ *be the element defined above. Then the cup-product map*

$$H^i(E) \xrightarrow{\cup \xi} H^{i+2r}_Y(E)$$

and the cap-product map

$$H^i(E) \xrightarrow{\cap \xi} H_{i-2r}(Y)$$

are isomorphisms for all i.

Proof. — By using resolution of singularities, induction on the dimension of Y, and the exact sequences of cohomology and homology for a proper birational morphism, we reduce to the case Y smooth. In that case

$$H_Y^{i+2r}(Y) = H_{2n-i}(Y) = H^i(Y)$$

and the composed map $H^i(E) \to H^i(Y)$ is just σ^* which is an isomorphism by the previous result. For the homology we have

$$H_i(E) = H^{2n+2r-i}(E)$$

and

$$H_{i-2r}(Y) = H^{2n-i+2r}(Y)$$

which are isomorphic for the same reason.

Remark. — We have used the embedding of Y in X to define ξ, and this ξ apparently depends on the choice of X. Following the terminology of Spanier [48, p. 259], if E is a rank r bundle over a scheme Y, not necessarily satisfying (*), then we can define an *orientation* of E to be a cohomology class $\eta \in H_Y^{2r}(E)$, such that for each closed point $y \in Y$, $\eta_Y \in H_Y^{2r}(E_Y) = k(y)$ is non-zero. We say E is *orientable* if it has an orientation.

Now the theorem will apply to any oriented vector bundle (E, η), the maps in question depending on the orientation. In this context it is natural to ask whether every vector bundle has an orientation, and if so, whether it is unique up to a scalar multiple. Presumably this question is related to the problem of defining Chern classes for vector bundles on singular schemes.

Corollary **(7.9.3)** (Thom-Gysin sequences). — *Let* Y, E *be as in the preceding theorem, and let* $\zeta = \sigma^*(\xi)$ *be the "self-intersection" class in* $H^{2r}(Y)$. *Then there are exact sequences*

$$\ldots \to H^{i-2r}(Y) \xrightarrow{\cup\zeta} H^i(Y) \to H^i(E-Y) \to H^{i-2r+1}(Y) \to \ldots$$

and

$$\ldots \to H_i(Y) \xrightarrow{\cap\zeta} H_{i-2r}(Y) \to H_i(E-Y) \to H_{i-1}(Y) \to \ldots .$$

Proof. — This follows from the exact sequences

$$\ldots \to H_Y^i(E) \to H^i(E) \to H^i(E-Y) \to H_Y^{i+1}(E) \to \ldots$$

and

$$\ldots \to H_i(Y) \to H_i(E) \to H_i(E-Y) \to H_{i-1}(Y) \to \ldots$$

and the isomorphisms of the theorem.

Chapter III

LOCAL AND RELATIVE THEORY

In this chapter, we generalize the theory of the previous chapter in two different directions. On the one hand, we define the De Rham cohomology and homology of the spectrum of a complete local ring. This gives a technique for purely local investigation of a singularity. On the other hand, we define the relative cohomology and homology sheaves of a morphism, which gives a technique for studying the variation of cohomology in an algebraic family.

To define the local invariants, we write the given complete local ring as a quotient of a complete regular local ring, i.e. we embed the spectrum Y in the spectrum X which is a regular scheme. Then as in Chapter II, we define cohomology and homology by using the (continuous) differentials on X, and taking their formal completion (resp. local cohomology) along Y.

The elementary aspects of the theory then proceed as in Chapter II. However, in order to prove the duality and finiteness theorems, we need to use resolution of singularities. Recall that Hironaka's resolution theorem [30] is actually stated for schemes of finite type over a certain class of rings **B**, which includes not only all fields of characteristic zero, but also formal power series or convergent power series rings over such a field. In that case, resolution must be understood in the absolute sense: if X is an integral scheme of finite type over a ring $A \in \mathbf{B}$, then resolution guarantees the existence of a birational morphism $f: X' \to X$, where X' is a regular scheme, not necessarily smooth over Spec A. Of course the theorem is more precise than this (see [30]).

Now when we apply resolution to the spectrum of a complete local ring, the result is no longer local. So even though our interest is mainly in the local case, we are forced to consider a more general class of schemes. Therefore we consider the category $\mathscr{C}$ of schemes which admit a finite type, quasi-projective morphism to the spectrum S of a complete local ring over k. Of course we could consider schemes which are " embeddable over S ", following the pattern of Chapter II, but this is not necessary, since all the schemes which will arise in our proofs for the local case will lie in $\mathscr{C}$.

This being said, the local theory proceeds as in the global case. One curious divergence from the previous theory is that in the global case, we were able to prove duality without using resolution. In the local case, however, we were not able to do this. Instead we use the inelegant method of applying resolution to reduce to the global case.

In section 3 we explain the relation between the global invariants of a scheme in projective space and the local invariants of the vertex of the cone over it. This is useful in applications; also it gives motivation for studying the local invariants.

In the second part of this chapter we define the relative cohomology and homology sheaves of a morphism $f: Y \to S$ of schemes. We define the Gauss-Manin connection and the Leray spectral sequence as suggested by Grothendieck [16, footnote 13], and using the method of Katz [32]. We also give some elementary results about monodromy and the relation with the cohomology of a fibre. Our main new result here is that the coherence of the sheaves $R^i f_*(Y)$ at a point s of S is a sufficient condition for the monodromy around s to be trivial and for the cohomology of the fibre at s to be isomorphic to the cohomology of nearby fibres. One can hope that a deeper study of the sheaves $R^i f_*(Y)$ in the non-coherent case will lead to a better understanding of the cohomology of the closed fibre in relation to nearby fibres.

As an application of the theory so far we give a new proof of the Lefschetz theorem on cohomology of hyperplane sections. This is closely related to questions of cohomological dimension which were raised in [23] and [24, Ch. III], and recently answered by Ogus [43]. Since Ogus' work contains a thorough discussion of the cohomological dimension questions, we will not go into them more here.

1. Local invariants.

For a fixed ground field k of characteristic zero, we will consider the category $\mathscr{C}$ of schemes X which admit a finite type, quasi-projective morphism to the spectrum S of a complete local k-algebra A, whose residue field $A/\mathfrak{m}$ is a finite extension of k. If $X \in \mathscr{C}$, the local ring A can be assumed to be a regular local ring with residue field k. We make the following conventions for schemes in $\mathscr{C}$. If X is irreducible, we define the *dimension* of X as follows. Since X is of finite type over S, it can be covered by open sets U which are closed subschemes $U \subseteq \mathbf{A}^N_S$ of suitable affine spaces over S. Then we define

$$\dim X = \dim A + N - \operatorname{codim}(U, \mathbf{A}^N_S).$$

This dimension may be different from the combinatorial dimension of the topological space X, but it is consistent with the usual notion of dimension for schemes of finite type over k, if X arises from a morphism $X_0 \to Y_0$ of schemes of finite type over k, by taking a closed point $P \in Y_0$, and making the base extension $\operatorname{Spec} \widehat{\mathcal{O}}_{P, Y_0} \to Y_0$. Indeed, this special case provides motivation for our treatment of schemes in $\mathscr{C}$.

Again let $X \in \mathscr{C}$. We define the sheaf of continuous differential forms $\Omega^1_{X/k}$ of X over k as follows. For each open affine set $U = \operatorname{Spec} B$, we consider B as an A-module, and give it the *strong topology*, namely the strongest topology on B which makes it a topological A-module, when A has its $\mathfrak{m}$-adic topology. Then we consider differential forms for B/k which are continuous in this topology. Glueing together defines the sheaf Ω^1_X. Now one can check that if $X \in \mathscr{C}$, and X is a regular scheme (i.e. all its local rings are regular local rings) and if $\dim X = n$ (as defined above), then $\Omega^1_{X/k}$ is a locally free sheaf of rank n. Furthermore, if X arises from a morphism $X_0 \to Y_0$ of schemes of finite type over k, then Ω^1_X is obtained from $\Omega^1_{X_0/k}$ by base extension.

Now we are in a position to define cohomology and homology for schemes in $\mathscr{C}$. For a scheme $Y \in \mathscr{C}$, we can always find an embedding of Y as a closed subscheme of a scheme $X \in \mathscr{C}$ which is smooth and finite type over $S = \operatorname{Spec} A$, where A is a complete regular local ring containing k, and with residue field k. We will call such a triple (Y, X, S) an *embedding* of Y.

Definition. — Let $Y \in \mathscr{C}$, let (Y, X, S) be an embedding of Y, and let Z be a closed subset of Y. Then we define the *local De Rham cohomology* of Y *with supports in* Z by

$$H^i_Z(Y) = \mathbf{H}^i_Z(\hat{X}, \hat{\Omega}^\bullet_X),$$

where $\hat{X}$ is the formal completion of X along Y, $\Omega^\bullet_X$ is the complex of continuous differential forms of X over k, and $\hat{\Omega}^\bullet_X$ is its completion along Y.

Definition. — With $Y \in \mathscr{C}$ as above, we define the *De Rham homology* of Y by

$$H_i(Y) = \mathbf{H}^{2n-i}_Y(X, \Omega^\bullet_X),$$

where $n = \dim X$.

Note that the category $\mathscr{C}$ contains all schemes which are quasi-projective and finite type over k. In that case, we recover the global definitions given in Ch. II. On the other hand, $\mathscr{C}$ also contains purely local schemes of the form $Y = \operatorname{Spec} B$, where B is any complete local ring containing k, and with residue field finite over k. In that case, we have defined the invariants which will be of interest to us, namely the local cohomology $H^i_P(Y)$, where P is the closed point, and the homology $H_i(Y)$.

Proposition (**1.1**). — *For any scheme* $Y \in \mathscr{C}$, *the cohomology and homology groups defined above are independent of the embedding* (Y, X, S).

Proof. — We proceed in two steps. First, keeping S fixed, we consider different embeddings of Y in schemes X which are smooth and finite type over S. Then by straightforward adaptation of the methods of Ch. II, § 1, 2, 3, we show that the cohomology and homology is independent of X. We leave details to the reader.

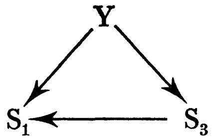
Concerning the choice of S, suppose Y admits finite type maps to

$$S_1 = \operatorname{Spec} k[[t_1, \ldots, t_r]]$$

and to

$$S_2 = \operatorname{Spec} k[[u_1, \ldots, u_s]].$$

Then considering $S_3 = \operatorname{Spec} k[[t_1, \ldots, t_r, u_1, \ldots, u_s]]$ and the diagonal map, we reduce to the case of a commutative diagram Y with Y, S_1, and S_3.



Now we consider the intermediate scheme $S' = \operatorname{Spec} k[[t_1, \ldots, t_r]][u_1, \ldots, u_s]$, and choose an embedding $Y \to X'$ where X' is smooth and finite type over S'. Then X' is smooth over S_1, so we may take (Y, X', S_1) as one embedding, and (Y, X_3, S_3) as the other, where X_3 is the completion of X' along the inverse image of the locus $u_1 = \ldots = u_s = 0$. Now the formal schemes $X'_{/Y}$ and $X_{3/Y}$ are actually isomorphic, so that the cohomology and homology with respect to these two embeddings is naturally isomorphic. Now that we have defined the De Rham cohomology and homology invariants for schemes in $\mathscr{C}$, the basic properties follow exactly as they were developed in Chapter II. Instead of making formal statements, we will just list those results here. One word of caution however: these functorial properties apply only to morphisms of finite type among schemes in $\mathscr{C}$. When it comes to considering morphisms which are not of finite type, some more work will be necessary.

First of all, cohomology is a contravariant functor. More precisely, if $f: Y_1 \to Y_2$ is a morphism of finite type of schemes in $\mathscr{C}$, and if $Z_1 \subseteq Y_1$ and $Z_2 \subseteq Y_2$ are closed subsets such that $f^{-1}(Z_2) \subseteq Z_1$, then there is a natural map of cohomology

$$f^* : H^i_{Z_2}(Y_2) \to H^i_{Z_1}(Y_1)$$

for all i. Homology is a covariant functor for proper (finite type) morphisms, and it is a contravariant functor for open immersions. If Z is a closed subset of Y, and $U = Y - Z$, then we have long exact sequences

$$\ldots \to H^i_Z(Y) \to H^i(Y) \to H^i(U) \to H^{i+1}_Z(Y) \to \ldots$$

and

$$\ldots \to H_i(Z) \to H_i(Y) \to H_i(U) \to H_{i-1}(Z) \to \ldots .$$

If Y is a regular scheme of dimension n in $\mathscr{C}$, then $H_i(Y) \cong H^{2n-i}(Y)$ for all i.

Finally, we have exact Mayer-Vietoris sequences of cohomology and homology just like the earlier ones (II (4.1) and II (4.2)), and we have exact sequences of a proper birational morphism just like the earlier ones (II (4.4) and II (4.5)). We will need one extension of these results, namely, an exact sequence of cohomology with supports for a proper birational morphism:

Proposition **(1.2)**. — *Let* X, X′, Y, Y′ *be schemes in* $\mathscr{C}$ *satisfying the hypotheses of* (II, 4.4); *suppose further that a closed subset* $V \subseteq X$ *is given, and let* $V' = f^{-1}(V)$. *Then there is an exact sequence of cohomology with supports*

$$\ldots \to H^q_V(X) \to H^q_{V'}(X') \oplus H^q_{V \cap Y}(Y) \to H^q_{V' \cap Y'}(Y') \to H^{q+1}_V(X) \to \ldots$$

Proof. — The proof is the same as the proof of (II, 4.4) except in the last step, where instead of applying the functor $H^q(X, \cdot)$ to the two exact sequences of complexes, one uses the functor $H^q_V(X, \cdot)$.

We will use these results freely from now on without further comment.

2. Finiteness and Duality Theorems.

In this section we will prove finiteness and duality theorems for local De Rham cohomology. In each case the proof will use resolution of singularities and the various functorial properties of De Rham cohomology to reduce to the case of schemes of finite type over a field, already treated in Chapter II. As before, we denote by $\mathscr{C}$ the category of schemes which admit a finite type quasi-projective morphism to the spectrum of a complete local ring with residue field k.

Theorem **(2.1)** (Finiteness). — *For any scheme* $Y \in \mathscr{C}$ *and for any closed subset* $Z \subseteq Y$, *the groups* $H^i_Z(Y)$ *and* $H_i(Y)$ *are finite-dimensional k-vector spaces, for all i. Fuıthermore, they are zero for* $i<0$ *and* $i>2n$, *where* $n=\dim Y$.

Proof. — To prove the finiteness, we use induction on the dimension of Y. First of all, the local cohomology $H^i_Z(Y)$ fits in an exact sequence with $H^i(Y)$ and $H^i(Y-Z)$, so it will be sufficient to consider cohomology without restricted supports. Secondly, using Mayer-Vietoris sequences and the induction hypothesis for cohomology and homology, we reduce to the case Y irreducible. Since nilpotent elements don't affect the definition, we may assume Y is integral. Next, using resolution of singularities and the sequence of a proper birational morphism, and the induction hypothesis, we reduce to the case Y regular. In that case $H_i(Y) \cong H^{2n-i}(Y)$, so it is sufficient to prove that either the homology or the cohomology is finite-dimensional.

Now Y was assumed to be quasi-projective over the spectrum S of a complete local ring with residue field k, so we can embed Y as an open dense subset of a scheme $\overline{Y}$, projective over S, which by resolution of singularities we may assume to be regular. Using the exact sequence of homology for the closed subset $\overline{Y}-Y$ and the induction hypothesis, we reduce to the case Y regular, and proper over S. Let $P \in S$ be the closed point, and let $Y_0 = f^{-1}(P)$, where $f: Y \to S$ is the projection. Then, according to the lemma below, $H^i(Y) = H^i(Y_0)$, and Y_0 is a scheme of finite type over k, so this cohomology is finite-dimensional by (II, 6.1).

As for the vanishing in case $i<0$ or $i>2n$, this can be checked easily by following through the proof given above, and reducing to the case Y of finite type over k, which was treated in (II, 7.2). One must be a little careful with the case of cohomology with supports, which was not explicitly treated above, but this follows from the observation that if Y is irreducible of dimension n, then $H^{2n}(Y)=0$ except in case Y is proper and finite type over k.

It remains to prove the following lemma.

Lemma (**2.2**). — *Let* $\pi : \mathrm{Y} \to \mathrm{S}$ *be a proper map of schemes in* $\mathscr{C}$, *where* S *is the spectrum of a complete local ring with residue field* k. *Let* $\mathrm{P} \in \mathrm{S}$ *be the closed point, and let* $\mathrm{Y}_0 = \pi^{-1}(\mathrm{P})$. *Then the natural map*

$$\mathrm{H}^i(\mathrm{Y}) \to \mathrm{H}^i(\mathrm{Y}_0)$$

is an isomorphism for all i.

Proof. — We may assume that S is regular. Embed Y in a scheme X smooth over S. Then we can use the embeddings (Y, X, S) and $(\mathrm{Y}_0, \mathrm{X}, \mathrm{S})$ to calculate the cohomology of Y and of Y_0. We have

$$\mathrm{H}^i(\mathrm{Y}) = \mathbf{H}^i(\hat{\mathrm{X}}, \hat{\Omega}^{\cdot}_{\mathrm{X}})$$

and

$$\mathrm{H}^i(\mathrm{Y}_0) = \mathbf{H}^i(\tilde{\mathrm{X}}, \tilde{\Omega}^{\cdot}_{\mathrm{X}})$$

where $\hat{}$ denotes completion along Y, and $\sim$ denotes completion along Y_0.

Using the first spectral sequence of hypercohomology, it will be sufficient to show that for each p, q, the natural map

$$\mathrm{H}^q(\hat{\mathrm{X}}, \hat{\Omega}^p_{\mathrm{X}}) \to \mathrm{H}^q(\tilde{\mathrm{X}}, \tilde{\Omega}^p_{\mathrm{X}})$$

is an isomorphism. In fact, we will show more generally that for any coherent sheaf F on X, the natural map

$$\mathrm{H}^i(\hat{\mathrm{X}}, \hat{\mathrm{F}}) \to \mathrm{H}^i(\tilde{\mathrm{X}}, \tilde{\mathrm{F}})$$

is an isomorphism for all i.

For each r, let Y_r be the subscheme of X defined by $\mathrm{I}^r_{\mathrm{Y}}$, and let $\mathrm{F}_r = \mathrm{F} \otimes \mathcal{O}_{\mathrm{Y}_r}$. Then $\hat{\mathrm{F}} \cong \varprojlim \mathrm{F}_r$. Applying the fundamental theorem of a proper morphism [EGA III, § 4] to the morphism $\pi : \mathrm{Y}_r \to \mathrm{S}$, and remembering that S is the spectrum of a complete local ring, we have isomorphisms

$$\mathrm{H}^i(\mathrm{Y}_r, \mathrm{F}_r) \to \mathrm{H}^i(\tilde{\mathrm{Y}}_r, \tilde{\mathrm{F}}_r)$$

for each r, where as before $\sim$ defines completion along Y_0. On the other hand, we have $\hat{\mathrm{F}} = \varprojlim \mathrm{F}_r$ and $\tilde{\mathrm{F}} = \varprojlim \tilde{\mathrm{F}}_r$, so we can apply the theorem about cohomology of an inverse limit of sheaves (I.4.5) to each. We obtain exact sequences and natural maps for each i

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim^{(1)}(\mathrm{H}^{i-1}(\mathrm{Y}_r, \mathrm{F}_r)) & \longrightarrow & \mathrm{H}^i(\hat{\mathrm{X}}, \hat{\mathrm{F}}) & \longrightarrow & \varprojlim \mathrm{H}^i(\mathrm{Y}_r, \mathrm{F}_r) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varprojlim^{(1)}(\mathrm{H}^{i-1}(\tilde{\mathrm{Y}}_r, \tilde{\mathrm{F}}_r)) & \longrightarrow & \mathrm{H}^i(\tilde{\mathrm{X}}, \tilde{\mathrm{F}}) & \longrightarrow & \varprojlim \mathrm{H}^i(\tilde{\mathrm{Y}}_r, \tilde{\mathrm{F}}_r) & \longrightarrow & 0
\end{array}$$

Since we have seen that $\mathrm{H}^i(\mathrm{Y}_r, \mathrm{F}_r) \to \mathrm{H}^i(\tilde{\mathrm{Y}}_r, \tilde{\mathrm{F}}_r)$ is an isomorphism for all i and r, the two outside arrows are isomorphisms, and we conclude that the middle one is also, as required.

Remark. — If we apply the lemma to the case $Y = S$, we find that $H^0(Y) = k$, $H^i(Y) = 0$ for $i \neq 0$. This gives a slightly different proof of the Poincaré lemma (II, 7.1) in the case of a complete local ring.

Theorem **(2.3)** (Duality). — *Let* $\pi : Y \to S$ *be a proper morphism of finite type of schemes in* $\mathscr{C}$, *where* S *is the spectrum of a complete local ring with residue field k. Let* $P \in S$ *be the closed point, and let* $Y_0 = \pi^{-1}(P)$. *Then we have natural duality isomorphisms*

$$H^i_{Y_0}(Y) \cong (H_i(Y))'$$

of k-vector spaces for each i.

Proof. — As usual, we may assume that S is regular. To define the map, let (Y, X, S) be an embedding of Y. We follow the method of the proof of (II, 5.1). As in that case, we have a natural map

$$\varphi : \Gamma_{Y_0}(\hat{\Omega}^{\bullet}_X) \to \mathscr{H}om^{\bullet}_{\Omega^{\bullet}_X}(\Gamma_Y(E(\Omega^{\bullet}_X)), \Gamma_{Y_0}(E(\Omega^{\bullet}_X)))$$

except that here we have introduced the supports Y_0. Now Y_0 is proper over k, so we have the trace map

$$\mathrm{Tr}_f : \Gamma_{Y_0}(E(\Omega^{\bullet}_X))[2n] \to k.$$

Then, proceeding as before, we obtain the natural map

$$\mathbf{H}^i_{Y_0}(\hat{X}, \hat{\Omega}^{\bullet}_X) \to (\mathbf{H}^{2n-i}_Y(X, \Omega^{\bullet}_X))'$$

which is the one we want.

To show that it is an isomorphism we use the functorial properties of cohomology and homology and the five-lemma. We proceed by induction on the dimension of Y. By Mayer-Vietoris sequences we may assume that Y is integral. Then by resolution of singularities, using Proposition (1.2) above, we may assume that Y is regular. Now in the case Y regular of dimension n, we have

$$H^i_{Y_0}(Y) \cong H_{2n-i}(Y_0).$$

On the other hand

$$H_i(Y) \cong H^{2n-i}(Y) \cong H^{2n-i}(Y_0)$$

by the Lemma above. So we have reduced to proving the duality theorem for Y_0, which is proper over k, and this has been done already (II, 5.1).

Remark. — We are most interested in the case when Y itself is the spectrum of a complete local ring. The theorem then states that $H^i_P(Y)$ is the dual vector space of $H_i(Y)$. One can ask whether it is possible to give a purely local proof of this result, based on Grothendieck's local duality theorem for modules over a local ring [LC and RD], without resorting to global methods.

Embed Y in the spectrum X of a complete regular local ring A of dimension n. Then

$$H_P^i(Y) = \mathbf{H}_P^i(\hat{X}, \hat{\Omega}_X^\bullet)$$

and

$$H_i(Y) = \mathbf{H}_Y^{2n-i}(X, \Omega_X^\bullet).$$

These groups are the abutments of the first spectral sequences of hypercohomology beginning

$$E_1^{pq} = H_P^q(\hat{X}, \hat{\Omega}_X^p)$$

and

$$'E_1^{pq} = H_Y^{n-q}(X, \Omega_X^{n-p}).$$

On the other hand, if F is any locally free sheaf on X, the local duality theorem of Grothendieck [LC, § 6] says that

$$H_P^q(Y_r, F_r) \cong D(\mathrm{Ext}_X^{n-q}(\mathcal{O}_{Y_r}, \check{F} \otimes \omega))$$

where Y_r is the subscheme defined by I_Y^r, and where D is a dualizing functor for A-modules, namely $D = \mathrm{Hom}_A(\cdot, I)$ where I is an injective hull of k over A. Taking an inverse limit on the left and a direct limit inside the parenthesis on the right, we find that

$$H_P^q(\hat{X}, \hat{F}) \cong D(H_Y^{n-q}(\check{F} \otimes \omega)).$$

Applying this to $F = \Omega^p$ (in which case $\check{F} \otimes \omega = \Omega^{n-p}$), we find that in the spectral sequences above

$$E_1^{pq} \cong D('E_1^{n-p, n-q}).$$

So the initial terms of the spectral sequence are dual as A-modules, and we would like to show that the abutments are dual as k-vector spaces. We do not know how to deduce the latter from the former. What seems to be needed is a duality theory for a suitable abelian category of k-vector spaces (perhaps with additional structure), which should include all A-modules, and certain differential operators between them, and which should be a simultaneous generalization of the known duality theorems for A-modules and for k-vector spaces. (Cf. Macdonald [37], who has given a duality theory for topological A-modules.)

3. Relations between local and global cohomology.

In this section we will prove a strong excision theorem for local cohomology. If Z is a closed subset of a topological space Y, then the usual excision theorem says that $H_Z^i(Y, F) = H_Z^i(U, F)$, where U is any open set containing Z, and F is any abelian sheaf. Our theorem for De Rham cohomology says that if P is a closed point of a scheme Y of finite type over k, then $H_P^i(Y) = H_P^i(Y')$, where $Y' = \mathrm{Spec}\, \hat{\mathcal{O}}_{P,Y}$. In fact, we give a slightly more general statement below.

We use this result to calculate the local cohomology and homology of the vertex of the cone over a projective scheme, in terms of the global cohomology and homology of the projective scheme.

Proposition (3.1) (Strong excision). — *Let* $f: Y \to S$ *be a projective morphism of schemes of finite type over* k. *Let* P *be a closed point of* S, *and let* $Z = f^{-1}(P)$. *Let* $S' = \operatorname{Spec} \hat{\mathcal{O}}_{P,S}$, *and let* $Y' = Y \times_S S'$. *Then we have natural isomorphisms for all* i

$$H^i_Z(Y) \xrightarrow{\cong} H^i_Z(Y').$$

Here we identify Z *with the isomorphic scheme* $Z' = Z \times_S S'$.

Proof. — We may assume S is affine and regular. Let $Y \to X$ be an embedding of Y into a scheme X smooth over S, and let $X' = X \times_S S'$. Then there is a morphism of schemes $X' \to X$, and so we obtain the maps above as the natural maps of cohomology

$$\mathbf{H}^i_Z(\hat{X}, \hat{\Omega}^{\bullet}_X) \to \mathbf{H}^i_Z(\hat{X}', \hat{\Omega}^{\bullet}_{X'})$$

where the completions are along Y and Y′, respectively.

Using the spectral sequence of hypercohomology, it is sufficient to show that the corresponding map is an isomorphism for each Ω^i. In fact, we will prove that for any coherent sheaf F on X, the natural map

$$H^i_Z(\hat{X}, \hat{F}) \to H^i_Z(\hat{X}', \hat{F}')$$

is an isomorphism, where F′ is the pull-back of F to X′.

We write $\hat{F} = \varprojlim F_r$, and $\hat{F}' = \varprojlim F'_r$, where $F_r = F/I^r_Y F$ and $F'_r = F'/I^r_{Y'} F'$. Then F'_r is the pull-back of F_r. Using (I.4.5) applied to the functors $H^i_Z(Y, \cdot)$ and $H^i_Z(Y', \cdot)$, it will be sufficient to show for each F_r, or more generally for any coherent sheaf G on X with support in Y, that the natural map

$$H^i_Z(Y, G) \to H^i_Z(Y', G')$$

is an isomorphism.

Since Z is the total inverse image of P, we have Leray spectral sequences

$$E^{pq}_Z = H^p_P(S, R^q f_*(G)) \Rightarrow H^n_Z(Y, G)$$

and

$$'E^{pq}_Z = H^p_P(S', R^q f'_*(G')) \Rightarrow H^n_Z(Y', G').$$

But f and f' are proper, so $R^q f_*(G)$ and $R^q f'_*(G')$ are coherent sheaves. Furthermore, since $S' \to S$ is a flat affine base extension, we have $R^q f'_*(G') = R^q f_*(G) \otimes_S S'$.

The local cohomology with supports in P can now be calculated over the local rings $A = \mathcal{O}_{P,S}$ and $\hat{A}$. To complete the proof, we have only to recall [LC, 5.9] that if M is a finite-type A-module, then

$$H^i_P(M) \to H^i_P(\hat{M})$$

is an isomorphism.

Now we will calculate the local cohomology of the vertex of a cone over a projective variety. Let $V^n \subseteq \mathbf{P}^N_k$ be a projective variety. Let $Y = C(V) \subseteq \mathbf{A}^{N+1}$ be the affine cone over V. Let $P \in Y$ be the vertex, and let $Y' = \operatorname{Spec} \hat{\mathcal{O}}_{P,Y}$.

Proposition **(3.2)**. — *With the above notations, we have* $H^0_P(Y') = H_0(Y') = 0$, *and exact sequences*

$$0 \to k \to H^0(V) \to H^1_P(Y') \to 0$$

$$0 \to H_1(Y') \to H_0(V) \to k \to 0$$

and

$$\dots \to H^{i-2}(V) \xrightarrow{\cup \zeta} H^i(V) \to H^{i+1}_P(Y') \to H^{i-1}(V) \to \dots$$

$$\dots \to H_{i+1}(Y') \to H_i(V) \xrightarrow{\cap \zeta} H_{i-2}(V) \to H_i(Y') \to \dots$$

for $i \geq 1$, *where* $\zeta \in H^2(V)$ *is the class of a hyperplane section.*

Proof. — The homology statements follow immediately from the cohomology statements by applying local and global duality, so it is sufficient to prove the latter.

So we need to calculate $H^i_P(Y')$ for all i. Using the strong excision theorem, this is the same as $H^i_P(Y)$. Now we use the exact sequence of local cohomology

$$\dots \to H^i_P(Y) \to H^i(Y) \to H^i(Y - P) \to H^{i+1}_P(Y) \to \dots$$

First we note that $H^i(Y) = 0$ except for $i = 0$, when $H^0(Y) = k$. (This corresponds to the fact that Y is topologically contractible.) To prove this, we take $X = \mathbf{A}^{N+1}$. Then $H^i(Y)$ can be calculated as the cohomology of the De Rham complex of modules over the completion of the polynomial ring $k[x_0, \dots, x_N]$ with respect to the I_Y-adic topology. But then the Poincaré Lemma applies to show that this complex is acyclic (II.7.1.1).

Next we note that $Y - P$ is fibred over V with fibre $\mathbf{A}^1 - \{0\}$. In fact, if $E = \mathbf{V}(\mathcal{O}_V(-1))$, then $Y - P$ is isomorphic to the vector bundle E minus its zero-section V. Now applying the Thom-Gysin sequence of cohomology (II.7.9.3) to E, and combining with the above observations, we have the result.

4. Relative cohomology.

In this section we consider a morphism of schemes $f : Y \to S$. We will define relative De Rham cohomology $R^i f_*(Y)$ and homology $R_i f_*(Y)$ as sheaves on S. Furthermore, if S is smooth, we will construct a Leray spectral sequence, abutting to the cohomology of Y, and beginning with $E_2^{pq} = H^p(S, R^q f_*(Y))$, where the latter is a suitably defined notion of cohomology with local coefficients.

For the definition of the relative cohomology and homology, we proceed as in the absolute case (Ch. II). Let $f : Y \to S$ be a morphism of finite type of noetherian schemes, such that Y admits a closed immersion (over S) into a scheme X which is smooth over S. Then let $\Omega^\bullet_{X/S}$ be the complex of relative differential forms of X over S, and let $\hat{\Omega}^\bullet_{X/S}$ be its formal completion along Y.

Definition. — With the above hypotheses and notation, we define the *relative algebraic De Rham cohomology*

$$R^i_{DR}f_*(Y) = R^i f_*(\hat{X}, \hat{\Omega}^\bullet_{X/S})$$

and *homology* $$R^{DR}_i f_*(Y) = R^{2n-i}_Y f_*(X, \Omega^\bullet_{X/S}),$$

where n is the relative dimension of X over S. Here we write $R^i_Y f_*$ to denote the derived functors of the composite functor $f_*.\Gamma_Y$.

Note that the relative De Rham cohomology and homology are $\mathcal{O}_S$-Modules, because the maps in the complex $\Omega^\bullet_{X/S}$ are $\mathcal{O}_S$-linear. Using exactly the same methods as in the absolute case, one verifies that these definitions are independent of the choice of X and they enjoy analogous functorial properties on the category of schemes over S (II.1.4, II.3.2). Furthermore, the same proofs show that the exact sequence of homology of a closed subset (II.3.3), the relation between cohomology and homology in the smooth case (II.3.4), the Mayer-Vietoris sequences (II.4.1, II.4.2), and the exact sequences of a proper birational morphism (II.4.4, II.4.5) all hold in the relative case. On the other hand, one does not have direct analogues of the duality and finiteness theorems, so we will come back to them later.

Now, to define a Leray spectral sequence, we need a suitable notion of De Rham cohomology with local coefficients. If S is smooth, then the notion of an $\mathcal{O}_S$-Module with an integrable connection provides a good definition. If S is not smooth, then we would need to envisage equivalence classes of sheaves with integrable connections on formal schemes $\hat{T}$, where $S \to T$ is a closed immersion into a smooth scheme T, and $\hat{T}$ is its formal completion along S. As we have no applications in mind for this more general case, we will stick to the case S smooth.

So let S be a smooth scheme of finite type over k. We consider the category ***MIC***(S) of Modules with integrable connection on S. In other words, an element of ***MIC***(S) is an $\mathcal{O}_S$-Module E, together with a k-linear mapping $\nabla : E \to E \otimes \Omega^1_S$ which satisfies the connection rule

$$\nabla(se) = s\nabla(e) + e \otimes ds$$

for all local sections $s \in \mathcal{O}_S$ and $e \in E$, and such that ∇ is integrable, i.e. $\nabla^1.\nabla = 0$, where $\nabla^1 : E \otimes \Omega^1 \to E \otimes \Omega^2$ is the map defined by

$$\nabla^1(e \otimes \omega) = e \otimes d\omega + \nabla(e) \wedge \omega.$$

For background on integrable connections, see [1], [10], [32].

If $(E, \nabla) \in$ ***MIC***(S), then we can define the De Rham cohomology of S with coefficients in (E, ∇) in the following way. For each i, we define

$$\nabla^i : E \otimes \Omega^i \to E \otimes \Omega^{i+1}$$

by $$\nabla^i(e \otimes \omega) = e \otimes d\omega + \nabla(e) \wedge \omega.$$

Then because ∇ is integrable, $\nabla^{i+1}.\nabla^i = 0$ for all i, so we can consider the complex

$$E \xrightarrow{\nabla} E\otimes\Omega^1 \xrightarrow{\nabla^1} E\otimes\Omega^2 \longrightarrow \dots \longrightarrow E\otimes\Omega^n.$$

We denote this complex by $E\otimes\Omega^\bullet$ (although it also depends on ∇). Then we define the *De Rham cohomology of* S *with coefficients in* (E, ∇) by

$$H^i_{DR}(S; E, \nabla) = \mathbf{H}^i(S, E\otimes\Omega^\bullet).$$

Following Katz and Oda [34] we will define the canonical Gauss-Manin connection on $R^if_*(Y)$. The exact sequence

$$0 \to f^*\Omega^1_S \to \Omega^1_X \to \Omega^1_{X/S} \to 0$$

of sheaves on X gives rise to a filtration of the complex $\Omega^\bullet_X$, whose quotients are the complexes $f^*\Omega^p_S\otimes\Omega^\bullet_{X/S}$. Taking completions, we obtain a filtration of the complex $\hat{\Omega}^\bullet_X$. We consider the spectral sequence of the derived functors of f_* applied to this filtered complex. It begins with

$$E_1^{pq} = R^qf_*(f^*\Omega^p_S\otimes\hat{\Omega}^\bullet_{X/S})$$

and abuts to the hypercohomology $R^nf_*(\hat{\Omega}^\bullet_X)$. Since Ω^p_S is a locally free sheaf on S, we have

$$E_1^{pq} \cong R^qf_*(\hat{\Omega}^\bullet_{X/S})\otimes\Omega^p_S.$$

Now the map

$$d_1^{0q} : R^qf_*(\hat{\Omega}^\bullet_{X/S}) \to R^qf_*(\hat{\Omega}^\bullet_{X/S})\otimes\Omega^1_S$$

is easily seen to be an integrable connection, and it is the one we want.

Thus the sheaf $R^if_*(Y)$ with its Gauss-Manin connection becomes in a natural way an element of $\boldsymbol{MIC}(S)$.

Theorem **(4.1)** (Leray spectral sequence). — *Let* $f: Y \to S$ *be a morphism of finite type, with* S *smooth over* k. *Then there is a spectral sequence*

$$E_2^{pq} = H^p_{DR}(S; R^qf_*(Y)) \Rightarrow E^n = H^n_{DR}(Y).$$

The Leray spectral sequence was constructed in the case S affine and f smooth by Katz and Oda [34]; it was proved more generally in case f is smooth by Deligne and Katz [32]. We will indicate how Katz' latter proof can be adapted to work in our case as well. The main ideas of this proof are also in the paper [31].

Keeping the above notation, we consider the category $\boldsymbol{MIC}(X)$ of modules with integrable connections on X. (Of course we use only the continuous differentials $\Omega^1_{X/k}$ in the definition.) For $(E, \nabla)\in\boldsymbol{MIC}(X)$, we consider the S-connection $E \to E\otimes\Omega^1_{X/S}$ obtained by composing ∇ with the projection $\Omega^1_X \to \Omega^1_{X/S}$. Then we can define the relative cohomology with coefficients in E by

$$R^i_{DR}f_*(E, \nabla) = R^if_*(E\otimes\hat{\Omega}^\bullet_{X/S}).$$

The construction of the Gauss-Manin connection applies also in this case, and shows that $R^i_{DR}f_*(E, \nabla)$ is naturally an element of $\boldsymbol{MIC}(S)$.

Proposition **(4.2)**. — *In the situation above,* $R^i_{DR}f_*(E, \nabla)$ *is the i-th right derived functor of the functor*

$$R^0_{DR}f_* : \boldsymbol{MIC}(X) \to \boldsymbol{MIC}(S).$$

Proof. — First, we consider the sheaf $\mathscr{D}_S$ of differential operators on S. This is a sheaf of $\mathscr{O}_S$-algebras by left multiplication. If $s_1, \dots, s_n$ are local parameters on S, then elements of $\mathscr{D}$ are represented locally as finite sums $\sum_I f_I D^I$ where $I = (i_1, \dots, i_n)$, $f_I \in \mathscr{O}_S$, and

$$D^I = \frac{\partial^{i_1}}{\partial s_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial s_n^{i_n}}.$$

Note that the category $\boldsymbol{MIC}(S)$ is equivalent to the category of left $\mathscr{D}$-Modules, by the map which associated to a $\mathscr{D}$-Module E the sheaf E, as an $\mathscr{O}_S$-Module, with the connection ∇ given locally by

$$\nabla(e) = \sum_i \frac{\partial e}{\partial s_i} ds_i.$$

(The integrability of ∇ corresponds to the fact that $\frac{\partial}{\partial s_i}$ and $\frac{\partial}{\partial s_j}$ commute with each other.)

Similarly we consider the sheaf $\mathscr{D}_X$ of differential operators on X, and let $\hat{\mathscr{D}}_X = \hat{\mathscr{O}}_X \otimes_{\mathscr{O}_X} \mathscr{D}_X$. Then the category $\boldsymbol{MIC}(X)$ becomes equivalent to the category of left $\hat{\mathscr{D}}_X$-Modules.

With those identifications, the functor $R^0_{DR}f_*$ mentioned above can be expressed as the functor

$$f_* \mathscr{H}om_X(f^* \mathscr{D}_S, \cdot)$$

from the category of left $\hat{\mathscr{D}}_X$-Modules to the category of left $\mathscr{D}_S$-Modules. Here $f^*\mathscr{D}_S$ is given a structure of left $\hat{\mathscr{D}}_X$-Module by composing the differential operators.

To compute the derived functors of this functor, we use a projective resolution of $f^*\mathscr{D}_S$, namely

$$0 \to \hat{\mathscr{D}}_X \otimes \Lambda^n T \to \dots \to \hat{\mathscr{D}}_X \otimes T \to \hat{\mathscr{D}}_X \to f^* \mathscr{D}_S \to 0$$

where $T = T_{X/S}$ is the relative tangent bundle. Locally, this is just the Koszul complex of the elements $\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}$ in $\hat{\mathscr{D}}_X$, where $x_1, \dots, x_n$ are the " vertical " parameters. Globally, this sequence makes sense, because T is locally free on X. Note that in writing the Koszul complex, we should put the $\frac{\partial}{\partial X_i}$ terms on the right.

Now the derived functor of our functor can be written

$$\mathbf{R}f_* \mathbf{R}\, \mathscr{H}om_{\hat{\mathscr{D}}_X}(f^* \mathscr{D}_S, \cdot)$$

in the derived category. Substituting the projective resolution above, this is equal to

$$\mathbf{R}f_* \mathscr{H}om_{\hat{\mathscr{D}}_X}(\hat{\mathscr{D}}_X \otimes \Lambda^\bullet T, \cdot).$$

For any $\hat{\mathscr{D}}_X$-Module E, we now find that

$$\mathscr{H}om_{\hat{\mathscr{D}}_X}(\hat{\mathscr{D}}_X \otimes \Lambda^\bullet T, E) = E \otimes \hat{\Omega}^\bullet_{X/S}.$$

Thus we have as derived functor $\mathbf{R}f_*(E \otimes \hat{\Omega}^\bullet_{X/S})$, as required.

By the way, the observation that $f_* \mathscr{H}om_{\hat{\mathscr{D}}_X}(f^* \mathscr{D}_S, \cdot)$ has a natural structure of $\mathscr{D}_S$-Module gives another construction of the Gauss-Manin connection.

Proof of theorem. — The Leray spectral sequence now appears as the spectral sequence of the composite functor

$$R^0_{DR} f_* : \boldsymbol{MIC}(X) \to \boldsymbol{MIC}(S)$$

followed by

$$R^0_{DR} g_* : \boldsymbol{MIC}(S) \to \boldsymbol{MIC}(k)$$

where $g : S \to k$ is the projection, applied to the sheaf $\mathcal{O}_X$. Note that $\boldsymbol{MIC}(k)$ is just the category of k-vector-spaces, and $R^i_{DR} g_* = H^i_{DR}$.

Remarks. — A number of variations are possible in the construction of the Leray spectral sequence.

1. We can replace k by a scheme T. Then if $f : X \to S$ and $g : S \to T$ are morphisms, with g smooth, we have a Leray spectral sequence

$$E_2^{pq} = R^p g_*(S, R^q f_*(X)) \Rightarrow E^n = R^n(gf)_*(X).$$

2. With Y, S as in the theorem, let $Z \subseteq Y$ be a closed subset, and let $T \subseteq S$ be a closed subset. Then using cohomology with supports, we have a Leray spectral sequence

$$E_2^{pq} = H^p_T(S, R^q_Z f_*(Y)) \Rightarrow E^n = H^n_{Z \cap f^{-1}(T)}(Y).$$

Here we denote by $R^q_Z f_*$ the derived functors of the functor $f_* . \Gamma_Z$.

3. The same method shows that the relative De Rham homology of Y over S has a Gauss-Manin connection. Thus we obtain a Leray spectral sequence for homology

$$E^2_{pq} = H_p(S, R_q f_*(Y)) \Rightarrow E_n = H_n(Y).$$

where H_p means H^{2s-p} on S, s being the dimension of S.

5. Cohomology and Base Extensions.

In this section we consider the behavior of the relative cohomology and homology sheaves of a morphism $f : Y \to S$ with respect to base changes $g : S' \to S$. In particular, we are interested in the relationship between the cohomology of the general fibre and the special fibre of f. In general this is a very interesting and difficult problem, and we will present only the most elementary results here. One can hope that a deeper systematic study of the sheaves $R^i f_*(Y)$ and $R_i f_*(Y)$ will lead to a better understanding of the monodromy transformation, in a purely algebraic context.

Theorem **(5.1)**. — *Let* $f : Y \to S$ *be a morphism of finite type of reduced schemes over* k. *Then there is an open dense subset* $U \subseteq S$ *such that the sheaves of relative De Rham cohomology* $R^i f_*(Y)$ *and homology* $R_i f_*(Y)$ *are coherent and locally free on* U.

Proof. — The idea of the proof is to repeat the usual proof of finiteness of cohomology for the generic fibre, and then show that this method can be spread out over an open set.

First of all, we may assume S is integral. Then once we show the sheaves $R^i f_*(Y)$ are coherent they will automatically be locally free on a dense open set. Next, using the Mayer-Vietoris sequence, we may assume that Y is also integral. Let $\sigma \in S$ be the generic point of S, and consider the generic fibre Y_σ of f, as a scheme over $k(\sigma)$. Let $Y'_\sigma \to Y_\sigma$ be a resolution of singularities of Y_σ. This is obtained as a finite succession of monoidal transformations, hence it can also be thought of as a single blowing up with respect to a suitable closed subscheme Z_σ of Y_σ. Let Z be the closure of Z_σ in Y, and let $Y' \to Y$ be the blowing-up of Z. Let $W \subseteq Y'$ be the set of points which are not smooth over S. Then W is closed, and does not meet the generic fibre. Hence its image in S is contained in a proper closed subset. Thus, replacing S by a dense open set, we may assume that Y' is smooth over S. In other words, relative resolution of singularities is possible, provided you are willing to replace the base space by a dense open subset.

Now we proceed as in the usual proof of finiteness (Ch. II, § 6). We use induction on the dimension of the generic fibre, resolution of singularities, the exact sequence of a proper birational morphism, relative homology, and embedding into a proper morphism. In the case of relative dimension zero, we may assume Y is finite étale over S, in which case the relative cohomology is just $f_* \mathcal{O}_Y$, which is coherent.

Proposition **(5.2)**. — *Let* $f : Y \to S$ *be a morphism of finite type of schemes of finite type over* k. *Let* q *be an integer such that* $R^i f_*(Y)$ *is coherent for* $i = q$ *and* $i = q+1$. *Then for any flat base extension* $g : S' \to S$ *(which need not be of finite type), the natural map*

$$g^*(R^q f_*(Y)) \to R^q f'_*(Y')$$

is an isomorphism, where $'$ *denotes base extension.*

Proof. — To calculate $R^i f_*(Y)$, we choose an embedding $Y \to X$ where X is smooth over S. Let f also denote the map from X to S. Then by definition

$$R^i f_*(Y) = R^i f_*(\hat{\Omega}^\bullet_{X/S}),$$

where ^ denotes the formal completion along Y. For each $r \geq \dim X/S$, we consider the complex $F^\bullet_r$ defined to be the complex

$$\mathcal{O}_X / I_Y^r \xrightarrow{d} \Omega^1_{X/S} / I_Y^{r-1} \xrightarrow{d} \Omega^2_{X/S} / I_Y^{r-2} \to \dots$$

Then $\hat{\Omega}^{\bullet}_{X/S}$ is the inverse limit of the complexes $F^{\bullet}_r$, so by (I.4.5) we have exact sequences for each i

$$(1)\qquad 0 \to \varprojlim{}^{(1)} R^{i-1}f_*(F^{\bullet}_r) \to R^i f_*(Y) \to \varprojlim R^i f_*(F^{\bullet}_r) \to 0.$$

Similarly, considering the map $f' : Y' \to S'$, we may take the base extension X' as our embedding of Y', and then $\hat{\Omega}^{\bullet}_{X'/S'} = \varprojlim F'_r$ so we have also

$$(2)\qquad 0 \to \varprojlim{}^{(1)} R^{i-1}f'_*(F'^{\bullet}_r) \to R^i f'_*(Y') \to \varprojlim R^i f'_*(F'^{\bullet}_r) \to 0.$$

Furthermore, cohomology of coherent sheaves commutes with flat base extension, so we have natural isomorphisms

$$(3)\qquad g^*(R^i f_*(F^{\bullet}_r)) \xrightarrow{\cong} R^i f'_*(F'^{\bullet}_r)$$

for each i and r.

Next, we observe that for each r, the sheaves $R^i f_*(F^{\bullet}_r)$ are the abutment of a spectral sequence beginning with $E_1^{pq} = R^q f_*(F^p_r)$, and these sheaves are countably quasi-coherent. Hence the sheaves $R^i f_*(F^{\bullet}_r)$ are countably quasi-coherent for each i and r.

Now the hypothesis that $R^q f_*(Y)$ is coherent implies by (I.4.8) that

a) $\varprojlim R^q f_*(F^{\bullet}_r)$ is coherent.

So by the exact sequence (1) we have $\varprojlim{}^{(1)} R^{q-1}f_*(F^{\bullet}_r)$ is coherent, and so by (I.4.9) we have

b) $R^{q-1}f_*(F^{\bullet}_r)$ satisfies (ML).

Similarly the hypothesis $R^{q+1}f_*(Y)$ coherent implies

c) $R^q f_*(F^{\bullet}_r)$ satisfies (ML).

Applying (I.4.10) and using *a)* and *c)*, we get an isomorphism

$$(4)\qquad g^*(\varprojlim R^i f_*(F^{\bullet}_r)) \cong \varprojlim (g^* R^i f_*(F^{\bullet}_r)).$$

On the other hand, *b)* implies that

$$(5)\qquad \varprojlim{}^{(1)} R^{q-1}f_*(F^{\bullet}_r) = 0.$$

It also implies that the inverse system $g^* R^{q-1}f_*(F^{\bullet}_r)$ satisfies (ML), and so by (3) we have also

$$(6)\qquad \varprojlim{}^{(1)} R^{q-1}f'_*(F'^{\bullet}_r) = 0.$$

Finally, applying g^* to (1) and combining with (2)-(6), we find that the natural map

$$g^*(R^q f_*(Y)) \to R^q f'_*(Y')$$

is an isomorphism.

Of course the reader will note that the complexity of this proof arises from the fact that in general, tensor product does not commute with formal completions and inverse limits.

Next, we study the cohomology of the closed fibres of a proper morphism.

Proposition (5.3). — *Let* $f : Y \to S$ *be a proper morphism, with* S *smooth and of finite type over* k. *Let* P *be a point of* S, *and let* $g : S' \to S$ *be the base extension to* $S' = \operatorname{Spec} \widehat{\mathcal{O}}_{P,S}$. *Fix a field of representatives* $k(P) \subseteq \widehat{\mathcal{O}}_{P,S}$, *so that* Y' *and* S' *are schemes over* $k(P)$. *Let* σ *be the generic point of* S, *with function field* K; *let* σ' *be the generic point of* S', *with function field* K'. *Then*

a) *For each* i *we have isomorphisms*

$$H^i(Y_\sigma) \otimes_K K' \xrightarrow{\cong} H^i(Y'_{\sigma'}).$$

b) *The* K'*-vector space* $H^i(Y'_{\sigma'})$ *has a canonical integrable* $k(P)$*-connection* ∇, *which we may call the "monodromy" around* P.

c) *For each* i *there is a natural map*

$$\alpha^i : H^i(Y_P) \to H^i(Y'_{\sigma'})^\nabla$$

where the exponent ∇ *denotes the kernel of* ∇, i.e. *the "invariant cocycles"*.

d) *For each* i *there is a natural map*

$$\beta^i : H^i(Y_P) \otimes_{k(P)} K' \to H^i(Y'_{\sigma'}).$$

Proof. — By (5.1) and (5.2) there is a dense open set $U \subseteq S$ such that all $R^i f_*(Y)$ are coherent, locally free on U, and commute with flat base extensions. In particular, considering the base extensions to Spec K and Spec K' gives *a)*.

Since S' is formally smooth over $k(P)$, we have the Gauss-Manin connection on $R^i f'_*(Y')$ for all i. Localizing at σ' gives the connection ∇ on $H^i(Y'_{\sigma'})$.

Next we consider the Leray spectral sequence (4.1) of the morphism $f' : Y' \to S'$ over $k(P)$. It begins with

$$E_2^{pq} = H^p(S'; R^q f'_*(Y'), \nabla)$$

and abuts to $E^n = H^n(Y')$.

By (2.2) above since f is proper, the natural maps

$$H^n(Y') \to H^n(Y_P)$$

are isomorphisms. Thus the edge homomorphism of the Leray spectral sequence gives a map, for each i

$$H^i(Y_P) \to H^0(S; R^i f'_*(Y'), \nabla).$$

Composing with the natural map of global sections to the generic stalk, which is just $H^i(Y'_{\sigma'})$, we obtain the map

$$\alpha^i : H^i(Y_P) \to H^i(Y'_{\sigma'})^\nabla.$$

Forgetting about ∇ and tensoring with K' gives the map β^i.

Proposition (5.4). — *With the hypotheses of the previous proposition, let* r *be an integer such that* $R^i f'_*(Y')$ *is coherent on* S' *for* $i \leq r$. *Then for each* $i \leq r$, $R^i f'_*(Y')$ *is locally free, its connection is trivial, hence the "monodromy" connection on* $H^i(Y'_{\sigma'})$ *is trivial, and the maps* α^i *and* β^i *defined above are isomorphisms.*

Proof. — First we use the well-known result that a coherent sheaf with integrable connection over a formal power series ring in characteristic zero is locally free, and the connection trivial. It follows that the Leray spectral sequence degenerates for $i \leq r$: we have $E_2^{pq}=0$ for $q \leq r$ and $p>0$, so that the edge homomorphism above is an isomorphism for $i \leq r$. Since the connection on $R^i f'_*(Y')$ is trivial, $\ker \nabla$ is a constant sheaf, so its global sections are the same as its generic stalk. Thus α^i is an isomorphism, and it follows that β^i is an isomorphism.

Corollary **(5.5)**. — *Let $f: Y \to S$ be a smooth proper morphism of smooth schemes of finite type over k. Then for each i the function*

$$\varphi(P) = \dim_{k(P)} H^i(Y_P)$$

is locally constant on S.

Proof. — Since f is smooth, the sheaves $R^i f_*(Y)$ are automatically coherent. So for any $P \in S$, we can apply (5.2) and (5.4), and find $\dim_{k(P)} H^i(Y_P) = \dim_K H^i(Y_\sigma)$ where σ is the generic point of the component containing P.

Remark. — This last result could also be proved using the theory of cohomology and base extension for coherent sheaves [EGA, III, § 7].

Next we consider the relative homology sheaves and their behavior with respect to base change. Here the results are somewhat simpler, because there are no inverse limits involved.

Proposition **(5.6)**. — *Let $f: Y \to S$ be a morphism of finite type of noetherian schemes. Then the relative homology sheaves $R_i f_*(Y)$ are quasi-coherent, and commute with all flat base extensions.*

Proof. — Considering an embedding $Y \to X$ with X smooth over S, we have

$$R_i f_*(Y) = R_Y^{2n-i} f_*(X, \Omega^{\bullet}_{X/S}).$$

Now using the exact sequence of local cohomology for Y in X, and recalling that higher direct images of coherent sheaves are quasi-coherent and commute with flat base extension, we have the result.

Proposition **(5.7)**. — *Let $f: Y \to S$ be a morphism of finite type, where* S *is the spectrum of a complete regular local ring of dimension* 1 *with residue field k. Let $P \in S$ be the closed point, let $\sigma \in S$ be the generic point, and let $K = k(\sigma)$. Then:*

a) *The* K*-vector space $H_i(Y_\sigma)$ has a canonical integrable k-connection ∇, which we may call the "monodromy" around* P.

b) *For each i there is a natural map*

$$\alpha_i : H_i(Y_\sigma)_\nabla \to H_i(Y_P)$$

where the subscript ∇ denotes the cokernel of ∇.

Proof. — Of course the connection on $H_i(Y_\sigma)$ comes from the Gauss-Manin connection on $R_i f_*(Y)$ by base extension. To define α_i, we consider an embedding of $Y \to X$, with X smooth over S, and use the Leray spectral sequence with supports on X (see Remark 2 at end of § 4 above). We have

$$E_2^{pq} = H_P^p(S, R_Y^q f_*(\Omega^\bullet_{X/S})) \Rightarrow E^m = \mathbf{H}^m_{Y_P}(X, \Omega^\bullet_{X/k}).$$

Now $E_2^{pq} = 0$ for $p > 2$, so we have an edge homomorphism

$$H_P^2(S; R_Y^q f_*(\Omega^\bullet_{X/S})) \to \mathbf{H}^{q+2}_{Y_P}(\Omega^\bullet_X).$$

On the other hand, since $S - \{P\} = \{\sigma\} = \operatorname{Spec} K$, we have a natural map

$$H^1(\operatorname{Spec} K; R_Y^q f_*(\Omega^\bullet_{X/S})_\sigma) \to H_P^2(S; R_Y^q f_*(\Omega^\bullet_{X/S})).$$

Now let $q = 2n - i$, where n is the relative dimension of X over S. Then

$$R_Y^q f_*(\Omega^\bullet_{X/S})_\sigma = H_i(Y_\sigma),$$

and $$H^{q+2}_{Y_P}(\Omega^\bullet_X) = H_i(Y_P).$$

Combining, we obtain the map

$$\alpha_i : H_i(Y_\sigma)_\nabla \to H_i(Y_P).$$

Remark. — If $\dim S > 1$, the situation is a bit more complicated. We must interpret $H_i(Y_\sigma)_\nabla$ to mean the cokernel of ∇^*, where ∇^* is the connection considered as a map

$$\nabla^* : H^i(Y_\sigma) \otimes T \to H^i(Y_\sigma)$$

where T is the relative tangent space of K over k. Furthermore, the map α_i is more difficult to define in an invariant way.

Proposition (**5.8**). — *With the hypotheses of the previous proposition, let r be an integer such that $R_i f_*(Y)$ is coherent for $i \leq r$. Then for each $i \leq r$, $R_i f_*(Y)$ is locally free, its connection is trivial, the monodromy connection on $H_i(Y_\sigma)$ is trivial, and the map α_i is an isomorphism.*

Proof. — A coherent sheaf with integrable connection on S is necessarily trivial, because S is complete. Hence the spectral sequence above degenerates: for $q \geq 2n - r$, and $p \neq 2$ we have $E_2^{pq} = 0$. Thus the edge homomorphism above is an isomorphism, and we conclude that α_i is an isomorphism.

Corollary (**5.9**). — *Let $f : Y \to S$ be a morphism of finite type, where S is a finite type curve over k. Then there exists a non-empty open set $U \subseteq S$ such that the functions*

$$\varphi^i(P) = \dim_{k(P)} H^i(Y_P)$$

and $$\varphi_i(P) = \dim_{k(P)} H_i(Y_P)$$

are locally constant on U.

Proof. — We use induction on the dimension of the generic fibre of f. We resolve the singularities of Y generically over S, use the exact sequence of a proper birational morphism, and thus reduce to the smooth case, where it is sufficient to consider homology. Then using (5.1), (5.6) and (5.8) we have the result.

Remark. — Surely this result holds for $\dim S > 1$ also. We would only need to generalize (5.7).

6. Cohomology at a non-closed point.

As an application of the results of the preceding section, we will prove a theorem about the local cohomology at a non-closed point.

Let Y be a scheme of finite type over k, and let $Q \in Y$ be a (not necessarily closed) point of Y. Let $k(Q)$ be the residue field of Q, and consider a choice

$$\theta : k(Q) \to \hat{\mathcal{O}}_{Q,Y}$$

of a field of representatives for the complete local ring $\hat{\mathcal{O}}_{Q,Y}$. Having chosen θ, we can consider the local De Rham cohomology

$$H^i_Q(\operatorname{Spec} \hat{\mathcal{O}}_{Q,Y}; \theta)$$

which is a finite-dimensional $k(Q)$-vector space. We will say θ is a *good* field of representatives if it makes $k(Q)$ a finite extension of the field $\theta(k(Q)) \cap \mathcal{O}_{Q,Y}$. Since we can lift algebraically independent elements of $k(Q)$ into $\mathcal{O}_{Q,Y}$, it is clear that good fields of representatives exist.

Proposition **(6.1)**. — *In the above situation, the dimension of*

$$H^i_Q(\operatorname{Spec} \hat{\mathcal{O}}_{Q,Y}; \theta)$$

is independent of the good field of representatives chosen.

Hence by abuse of notation we will denote this space $H^i_Q(Y)$, and consider it as a $k(Q)$-vector space, even though strictly speaking the vector space itself depends on θ. Note that this notation does not conflict with the previously introduced notion of local cohomology with respect to a closed subset. For if Q is a closed point, then the two definitions coincide because of the strong excision theorem (3.1).

Theorem **(6.2)**. — *Still in the above situation, let* $Z = \{Q\}^-$. *Then there is a non-empty open subset* $U \subseteq Z$ *such that for all closed points* $P \in U$, *and for all* i, *we have*

$$\dim_{k(P)} H^i_P(Y) = \dim_{k(Q)} H^{i-2r}_Q(Y)$$

where $r = \dim Z$.

We will prove both propositions at the same time. Fix a good field of representatives θ. Then it will be sufficient to prove the second proposition using that θ, and the first will follow. Pick algebraically independent elements $z_1, \ldots, z_r \in \theta(k(Q)) \cap \mathcal{O}_{Q,Y}$.

Using them, and replacing Y by a suitable neighborhood of Q, we can define a map of Y to $S = \mathbf{A}_k^r$ such that Z is generically finite over S. Then shrinking S, we may assume Z is étale over S. Finally, we make the base change $Z \to S$, and choose a section $Z \to Z \times_S Y$ lying over the original Z in Y. Now an étale base change does not affect our problem, because everything is computed on complete local rings. Thus we have reduced to the case where Y admits a map f to a scheme S (namely Z) which induces an isomorphism $f : Z \to S$.

Next, we consider the relative De Rham cohomology sheaves of f with supports in Z, which we denote $R^i_Z f_*(Y)$. Applying (5.1) and (5.2) to Y and $Y - Z$, and using the exact sequence of local cohomology, we find that there is a dense open set $U \subseteq S$ such that all $R^i_Z f_*(Y)$ are coherent, locally free on U, and commute with flat base changes. We may also assume U is smooth over k.

Now let $P \in U$ be a closed point, and consider the base extension $g : S' \to S$ where $S' = \operatorname{Spec} \hat{\mathcal{O}}_{P,S}$. We consider Y′, Z′, S′ as schemes over $k(P)$. The Leray spectral sequence with supports (§ 4, Remark 2) gives us a spectral sequence

$$E_2^{pq} = H_P^p(S'; R^q_{Z'} f'_*(Y')) \Rightarrow E^n = H_P^n(Y').$$

Since the sheaves $R^q_{Z'} f'_*(Y')$ are coherent, they are locally free, and their connection is trivial, so the spectral sequence degenerates. Indeed, for a free rank one module E with trivial connection we have

$$H^i_P(S'; E, \nabla) = H^i_P(S') = \begin{cases} 0 & i \neq 2r \\ k(P) & i = 2r. \end{cases}$$

So from the spectral sequence we find

$$\dim_{k(P)} H^i_P(Y') = \operatorname{rank} R^{i-2r}_{Z'} f'_*(Y') = \dim_{k(Q)} H^{i-2r}_Q(Y).$$

Finally, by strong excision we have $H^i_P(Y') = H^i_P(Y)$, so we are done.

Remarks. — 1. This statement is slightly weaker than the one in the announcement [27], because of the restriction to "good" residue fields. However, it should be sufficient for most purposes.

2. Since we have shown that the cohomology of $\operatorname{Spec} \hat{\mathcal{O}}_{Q,Y}$ is independent of the choice of field of representatives $k(Q)$, one might be tempted to ask whether the analytic isomorphism class of $\hat{\mathcal{O}}_{Q,Y}$ as a $k(Q)$-algebra is also independent of the choice of field of representatives. However, this is not so, as we will show by example.

Let $A = k[[x, y]]$ be a power series ring in two variables over a field k. An algebroid curve defined by an equation $g = 0$, $g \in A$, is said to have an *ordinary* r-fold point if $g = f_1 \dots f_r$ where $f_i \in \mathfrak{m} - \mathfrak{m}^2$ for each i, and where the tangent directions $\bar{f}_i \in \mathfrak{m}/\mathfrak{m}^2$ are all distinct. Now allowing automorphisms of A which leave k fixed, one can easily show that any two ordinary double points are equivalent, and any two ordinary triple points are equivalent. If $g = 0$ has an ordinary four-fold point, then g can be written in the form

$$g = xy(x + y)(x + ay)$$

with $a \in k$, $a \neq 0, 1$, and a is uniquely determined once one fixes the order of the branches. Thus there is a one-dimensional family of ordinary four-fold points, parametrized by a, which is the cross-ratio of the tangent directions.

Using Schlessinger's theory of deformations of singularities, one can show that there are two two-dimensional families of ordinary five-fold points, up to analytic isomorphism. There are the " straight " ones, which can be written in the form

$$g = xy(x+y)(x+ay)(x+by)$$

with $a, b \in k$, $a, b \neq 0, 1$, $a \neq b$, and where a and b are uniquely determined, if the order of the branches is fixed. Then there are the " curly " ones, which can be written

$$g' = g + h$$

where g is as above, and h is anything which is not in the ideal generated by the partial derivatives g_x, g_y. Here the isomorphism class is determined by a and b, and is independent of h. Finally, by considering families of the form $g + \varepsilon h$, as $\varepsilon \to 0$, one sees that the curly ones specialize to the straight ones. This exhibits a jump phenomenon, because for $\varepsilon \neq 0$ the singularities $g + \varepsilon h$ are all isomorphic to each other.

Now for our example. Let

$$\mathrm{Y} = \operatorname{Spec} k[x, y, t]/(x^5 + x^4 y + t y^5)$$

and let Q be the generic point of the subvariety $x = y = 0$. Then $k(\mathrm{Q}) = k(t)$. For the obvious choice of field of representatives, we have

$$\hat{\mathcal{O}}_{\mathrm{Q,Y}} \cong k(t)[[x, y]]/(x^5 + x^4 y + t y^5).$$

However, if we take $t' = t + y$ as our representative of t, then we find

$$\hat{\mathcal{O}}_{\mathrm{Q,Y}} \cong k(t')[[x, y]]/(x^5 + x^4 y + t' y^5 - y^6).$$

Now y^6 is not the ideal generated by the two partial derivatives of $x^5 + x^4 y + t' y^5$, so the latter is a curly ordinary five-fold point, which cannot be made isomorphic to the earlier straight one by any isomorphism extending the isomorphism $k(t) \to k(t')$ which sends t to t'.

7. The Lefschetz Theorems.

As an application of the theory developed so far, we will give a proof of the Lefschetz theorems on the cohomology of a hyperplane section of a projective variety. An earlier version of these results appeared in [24, Ch. III, Theorems 7.4 and 8.6]. Now we give a purely algebraic proof also in the case of a scheme with singularities. Recall that the *cohomological dimension* of a scheme U is the least integer $\mathrm{cd}(\mathrm{U}) = n$ such that $\mathrm{H}^i(\mathrm{U}, \mathrm{F}) = 0$ for all $i > n$ and all coherent sheaves F.

Theorem (7.1). — *Let* X *be a proper scheme over* k, *of dimension* n, *and let* Y *be a closed subscheme. Assume that* $X-Y$ *is smooth, and that* $\mathrm{cd}(X-Y)<r$ *for some integer* r. *Then the natural map of De Rham cohomology*

$$H^i(X) \to H^i(Y)$$

is an isomorphism for $i<n-r$, *and is injective for* $i=n-r$.

Proof. — We first show that $H^i(X-Y)=0$ for $i\geq n+r$. Indeed, $X-Y$ is smooth, so we can use the differentials on $X-Y$ to calculate the De Rham cohomology:

$$H^i(X-Y)=\mathbf{H}^i(X-Y, \Omega^{\cdot}_{X-Y}).$$

This hypercohomology is the ending of a spectral sequence which begins with

$$E_1^{pq}=H^q(X-Y, \Omega^p_{X-Y}).$$

Now $X-Y$ has dimension n and cohomological dimension $<r$, so $E_1^{pq}=0$ if $p>n$ or $q\geq r$. Hence the abutment $H^i(X-Y)$ is zero for $i\geq n+r$.

Now since $X-Y$ is smooth, we can interpret this in terms of homology: $H_j(X-Y)=0$ for $j\leq 2n-(n+r)=n-r$. Then by the exact sequence of homology of the closed subset Y, we find that the natural maps

$$\alpha_j : H_j(Y) \to H_j(X)$$

are isomorphisms for $j<n-r$, and surjective for $j=n-r$. Finally, since X and Y are both proper over k, we obtain the conclusion of the theorem by duality.

Corollary (7.2) (Lefschetz). — *Let* X *be a projective variety of dimension* n, *and let* Y *be a hyperplane section of* X. *Assume that* $X-Y$ *is smooth. Then*

$$H^i(X) \to H^i(Y)$$

is an isomorphism for $i<n-1$, *and is injective for* $i=n-1$.

Proof. — In this case $X-Y$ is affine, so has cohomological dimension zero.

Corollary (7.3). — *Let* Y *be a subscheme of dimension* s *in* $\mathbf{P}^n_k$ *which is a set-theoretic complete intersection, and which may have arbitrary singularities. Then*

$$H^i(Y)=\begin{cases} k \text{ for } i \text{ even} \\ 0 \text{ for } i \text{ odd}\end{cases}$$

for all $0\leq i<s$.

Proof. — In this case $\mathrm{cd}(\mathbf{P}^n-Y)=n-s-1$, and the cohomology of $\mathbf{P}^n$ is known to be k for i even, 0 for i odd, for $0\leq i\leq 2n$.

In the local case, we have the following analogous result.

Theorem **(7.4).** — *Let* X *be the spectrum of a complete local ring containing k, of dimension n. Let* Y *be a closed subscheme of* X, *and let* P *be the closed point. Assume* X—Y *is smooth, and* $\mathrm{cd}(X-Y)<r$ *for some integer r. Then the natural maps*

$$H^i_P(X) \to H^i_P(Y)$$

are isomorphisms for $i<n-r$ *and injective for* $i=n-r$.

Proof. — Reread the proof of Theorem (7.1), and use local duality at the end.

Corollary **(7.5).** — *Let* $X=\mathrm{Spec}\,A$, *where* A *is a complete local ring of dimension n containing k. Let* $t\in A$ *be a non-zero-divisor, and let* Y *be the locus* $t=0$. *Assume* X—Y *is smooth. Then*

$$H^i_P(X) \to H^i_P(Y)$$

is an isomorphism for $i<n-1$, *and is injective for* $i=n-1$.

Corollary **(7.6).** — *Let* $Y=\mathrm{Spec}\,B$, *where* B *is a complete local complete intersection of dimension s. Then*

$$H^i_P(Y)=0 \quad \textit{for} \quad i<s.$$

Proof. — Embed Y in a regular local scheme X of dimension *n*. Then $\mathrm{cd}(X-Y)=n-s-1$, so we apply the theorem, together with the observation that since X is regular, $H^i_P(X)=0$ for $i<2n$.

Remark. — Ogus [43] has recently shown that one can replace the hypothesis " non-singular " in the above results by a suitable local topological condition. Let Y be a scheme of finite type over *k*. For each (not necessarily closed) point $P\in Y$ we denote by $H^i_P(Y)$ the local De Rham cohomology of the complete local ring of P, which was considered in the last section. Then Ogus [43, 2.12] defines the *De Rham depth* of Y by

$$\text{DR-depth}\,Y\geq d \Leftrightarrow \text{for all } P\in Y, \quad H^i_P(Y)=0 \text{ for all } i<d-\dim\{P\}^-.$$

Note that one always has DR-depth $Y\leq\dim Y$. If Y is non-singular, or more generally is a local complete intersection, then DR-depth $Y=\dim Y$.

Then Ogus proves the following fundamental result relating DR-depth to local cohomology of coherent sheaves:

Theorem (Ogus [43, 2.13]). — *Let* Y *be a closed subscheme of a smooth scheme* X *of dimension* N. *Then*

DR-depth $Y\geq d \Leftrightarrow \mathcal{H}^i_Y(F)=0$ *for all coherent sheaves* F *on* X, *and all* $i>N-d$.

From this result, using the spectral sequence for De Rham cohomology, one sees easily that if U is a scheme with $\mathrm{cd}(U)\leq r$ and DR-depth $U\geq d$, then $H_i(U)=0$ for $i<d-r$. In particular, if U is affine, then $H_i(U)=0$ for $i<d$.

Thus in (7.1), (7.2) and (7.4) above, the hypothesis " X—Y smooth " can be replaced by " DR-depth X—Y$=n$ " [43, 4.10 and 4.11].

CHAPTER IV

COMPARISON THEOREMS

In this chapter we consider the case $k=\mathbf{C}$, and show that the algebraic De Rham cohomology groups studied in this paper are isomorphic with well-known topological invariants of the corresponding complex-analytic space. These results generalize the original comparison theorem of Grothendieck [16], which was for the case of a smooth scheme of finite type over $\mathbf{C}$. The global comparison theorem (1.1) was proved by Lieberman and Herrera [29] in the proper case, and by Deligne (unpublished) in the general case. Deligne also proved a local comparison theorem and a relative comparison theorem in the smooth case. As a by-product of our methods, we prove the " formal-analytic Poincaré lemma " which implies that the singular cohomology of complex-analytic subspace Y of a complex manifold X can be calculated using the formal completion of the holomorphic De Rham complex on X.

1. The Global Case.

Let Y be a scheme of finite type over $\mathbf{C}$, which admits a closed immersion $Y\to X$ to a scheme X which is smooth over $\mathbf{C}$. Let Z be a closed subset of Y. Then we have defined the algebraic De Rham cohomology of Y with supports in Z to be

$$H^i_{Z,\mathrm{DR}}(Y)=\mathbf{H}^i_Z(\hat{X},\hat{\Omega}^{\bullet}_X).$$

Let Y_h, X_h be the corresponding complex-analytic spaces, and Z_h the corresponding closed subset (see Ch. I, § 6). Then we can define the analytic De Rham cohomology of Y_h in a similar way. Let $\Omega^{\bullet}_{X_h}$ be the complex of sheaves of holomorphic differential forms on X_h, let $\hat{\Omega}^{\bullet}_{X_h}$ be its formal completion along Y_h. Then we define

$$H^i_{Z_h,\mathrm{DR}}(Y_h)=\mathbf{H}^i_{Z_h}(\hat{X}_h,\hat{\Omega}^{\bullet}_{X_h}),$$

and according to general principles (Ch. I, § 7) there are functorial maps

$$\alpha^i:\ H^i_{Z,\mathrm{DR}}(Y)\to H^i_{Z_h,\mathrm{DR}}(Y_h).$$

On the other hand, the natural map $\mathbf{C}\to\Omega^{\bullet}_{X_h}$ induces a map $\mathbf{C}_{Y_h}\to\hat{\Omega}^{\bullet}_{X_h}$, and hence maps of cohomology

$$\beta^i:\ H^i_{Z_h}(Y_h,\mathbf{C})\to H^i_{Z_h,\mathrm{DR}}(Y_h).$$

Theorem (**1.1**) (Global comparison theorem). — *With the hypotheses and notations above, the natural maps* α^i *and* β^i *are isomorphisms for all* i.

Corollary (**1.2**). — *With the same hypotheses, we have natural isomorphisms*

$$H_i^{DR}(Y) \cong H_i^{BM}(Y_h, \mathbf{C})$$

of the algebraic De Rham homology of Y *with the Borel-Moore homology of* Y_h.

Proof of Corollary. — Indeed, if $Y \to X$ is an embedding of Y into a smooth scheme of dimension n, we have

$$H_i^{DR}(Y) \cong H_{Y, DR}^{2n-i}(X)$$

and

$$H_i^{BM}(Y_h, \mathbf{C}) \cong H_{Y_h}^{2n-i}(X_h, \mathbf{C})$$

so the result follows immediately from the theorem. See [6] for definition and properties of the Borel-Moore homology, also called " homology with locally compact supports ".

Before proving the theorem, we need to make some general remarks about analytic De Rham cohomology. If V is any complex analytic space which admits a global embedding $V \to W$ into a complex manifold, we define its *analytic De Rham cohomology* by

$$H_{DR}^i(V) = \mathbf{H}^i(\hat{W}, \hat{\Omega}_W^\bullet).$$

Proposition (**1.3**). — *If* V *is an embeddable complex-analytic space, its De Rham cohomology is independent of the choice of embedding. Analytic De Rham cohomology is a contravariant functor in* V.

Proof. — We copy the proof of (II.1.4) with a few modifications. As in (II.1.1), the important case is when $V \subseteq W \subseteq U$ where U is another complex manifold, and W is a submanifold defined locally by a single equation $z = 0$. The question is local, so we may assume W is an open subset of $\mathbf{C}^n$, and U is open in $\mathbf{C}^{n+1}$. Then taking $A = \Gamma(\hat{U}, \hat{\mathcal{O}}_U)$, $B = \Gamma(\hat{W}, \hat{\mathcal{O}}_W)$, we have $A \cong B[[z]]$. Thus (II.1.2) is unnecessary, and it doesn't matter that A and B may not be noetherian. The rest of the proof is the same.

Of course we can define analytic De Rham cohomology with supports, and we have the long exact sequence of local cohomology. We also have a Mayer-Vietoris exact sequence.

Proposition (**1.4**) (Mayer-Vietoris sequence). — *Let* Y *be an embeddable analytic space, which is a union of two closed subspaces* Y_1 *and* Y_2. *Then there is an exact sequence of analytic De Rham cohomology*

$$\ldots \to H^i(Y) \to H^i(Y_1) \oplus H^i(Y_2) \to H^i(Y_1 \cap Y_2) \to H^{i+1}(Y) \to \ldots$$

Proof. — Embed Y in a complex manifold X. The proof is practically the same as for the algebraic case (II.4.1), but we must be careful, because the rings of holomorphic functions on open sets $U \subseteq X$ need not be noetherian, so that we cannot apply Krull's theorem there. Let I_1, I_2 be coherent sheaves of ideals defining Y_1, Y_2. For each n, we have an exact sequence, as before,

$$0 \to \mathcal{O}_X/(I_1^n \cap I_2^n) \to (\mathcal{O}_X/I_1^n) \oplus (\mathcal{O}_X/I_2^n) \to \mathcal{O}_X/(I_1^n + I_2^n) \to 0.$$

Now $(I_1+I_2)^n \supseteq I_1^n + I_2^n \supseteq (I_1+I_2)^{2n}$, so the two corresponding topologies on $\mathcal{O}_X$ are equal, as before. On the left, we will show that for any open subset $U \subseteq X$, whose closure $\overline{U}$ is *compact*, the $\{I_1^n \cap I_2^n\}$ and $\{(I_1 \cap I_2)^n\}$ topologies are equal in the sheaf $\mathcal{O}_X|_U$. This will be sufficient to give the exact sequence of sheaves in the completion, as above.

For each point $x \in X$, the local ring $\mathcal{O}_x$ is noetherian. Hence we can apply Krull's theorem, and we find that for each n, there is an $m = m(n, x)$ such that

$$(I_1 \cap I_2)^n \supseteq I_1^m \cap I_2^m$$

is the local ring $\mathcal{O}_x$. Since these are coherent sheaves of ideals, the same is true in a neighborhood V_x of x. We do this for each $x \in \overline{U}$. Since $\overline{U}$ is compact, a finite number of the V_x cover $\overline{U}$. Hence there is an $m = m(n)$ which works for all $x \in \overline{U}$, and *a fortiori*, for all $x \in U$.

The remainder of the proof is the same as in the algebraic case.

Next we need the analytic analogue of (II.4.4), namely an exact sequence of a proper birational morphism. Unfortunately, we do not know how to prove an analytic analogue of Grothendieck's "fundamental theorem of a proper morphism" [EGA, III, 4.1.5], concerning the compatibility of higher direct images of coherent sheaves with formal completion along a subspace. Hence we cannot establish the analytic analogue of (II.4.4) in general. However, for our purposes it will be sufficient to prove the result for analytic spaces which come from schemes.

Proposition (**1.5**). — *Let $f: X' \to X$ be a proper morphism of schemes of finite type over* $\mathbf{C}$, *let Y be a closed subscheme of X, let $Y' = f^{-1}(Y)$, and assume that they satisfy the hypotheses of* (II.4.4). *Then there is an exact sequence of analytic De Rham cohomology*

$$\dots \to H^i_{DR}(X_h) \to H^i_{DR}(X'_h) \oplus H^i_{DR}(Y_h) \to H^i_{DR}(Y'_h) \to H^{i+1}_{DR}(X_h) \to \dots$$

Proof. — Using (I.6.2) in place of [EGA, III.4.1.5], the proof proceeds exactly as in the algebraic case (II.4.4).

We are now ready to proceed with the proof of Theorem (1.1). In the course of the proof, we will use various functorial properties of usual singular cohomology. These results are all well known, except possibly for the exact sequence of a proper birational morphism, so we will prove it.

Proposition (**1.6**). — *Let $f: X' \to X$ be a proper map of polyhedral topological spaces. Let $Y \subseteq X$ be a closed subset, $Y' = f^{-1}(Y)$, and assume that f maps $X' - Y'$ isomorphically onto $X - Y$. Then there is a long exact sequence*

$$\dots \to H^i(X, \mathbf{C}) \to H^i(X', \mathbf{C}) \oplus H^i(Y, \mathbf{C}) \to H^i(Y', \mathbf{C}) \to H^{i+1}(X, \mathbf{C}) \to \dots$$

Proof. — Let $0 \to \mathbf{C}_{X'} \to I^\bullet$ be an injective resolution of the constant sheaf $\mathbf{C}$ on X'; let $0 \to \mathbf{C}_{Y'} \to J^\bullet$ be a resolution of $\mathbf{C}$ on Y', and let $I^\bullet \to J^\bullet$ be a map compatible with the restriction $\mathbf{C}_{X'} \to \mathbf{C}_{Y'}$. Then we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{C}_X & \longrightarrow & f_* I^\bullet & \longrightarrow & Q^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{C}_Y & \longrightarrow & f_* J^\bullet & \longrightarrow & R^\bullet & \longrightarrow & 0 \end{array}$$

To copy the proof of (II.4.4), we need only know that for all i, $R^i f_*(\mathbf{C}_{X'})|_Y = R^i f_*(\mathbf{C}_{Y'})$. To prove this, it is sufficient to show that for any point $y \in Y$, the stalks $R^i f_*(\mathbf{C}_{X'})_y$ and $R^i f_*(\mathbf{C}_{Y'})_y$ are equal. But since $f^{-1}(y)$ is a compact polyhedron, both of these are isomorphic to $H^i(f^{-1}(y), \mathbf{C})$. (See Spanier, [48], pp. 281, 291, where he shows $\overline{H}^q = H^q$ for compact polyhedra.)

Proof of theorem. — Using the exact sequence of local cohomology and the five lemma, it will be sufficient to consider cohomology $H^i(Y)$ without supports. We use induction on the dimension of Y. We may assume Y is reduced, and by the Mayer-Vietoris sequence, we may assume Y is integral. Let $f: Y' \to Y$ be a resolution of singularities of Y. Then using the exact sequence of a proper birational morphism, and the induction hypothesis again, we reduce to the case Y smooth.

If Y is smooth, then the analytic Poincaré lemma says that $\Omega^\bullet_{Y_h}$ is a resolution of the constant sheaf $\mathbf{C}$ on Y_h. Thus the map β^i is automatically an isomorphism. In fact, for any closed subset $Z_h \subseteq Y_h$, the map

$$\beta^i : H^i_{Z_h}(Y_h, \mathbf{C}) \to H^i_{Z_h, DR}(Y_h)$$

is an isomorphism. So letting $\gamma^i = (\beta^i)^{-1}\alpha^i$, we have maps

$$\gamma^i : H^i_{Z, DR}(Y) \to H^i_{Z_h}(Y_h, \mathbf{C}).$$

We will now prove that for any smooth scheme Y, and any closed subset Z, the maps γ^i are isomorphisms. Note that for this proof, we no longer need functorial properties of analytic De Rham cohomology: we deal only with the algebraic cohomology of Y, and singular cohomology of Y_h.

First, we use induction on the dimension of Y. Next, if $Z = Y$ (i.e. for the case of arbitrary supports), we consider an embedding of Y as an open dense subset of a smooth proper scheme Y', and let $Z' = Y' - Y$. Now for Y' we can apply Serre's [GAGA] to the sheaves $\Omega^p_{Y'}$, and we deduce immediately that $\gamma^i(Y')$ is an isomorphism for all i. Thus by the exact sequences of local cohomology, it is sufficient to consider $H^i_{Z'}(Y')$.

In other words, we have reduced to proving γ^i is an isomorphism for Y smooth and $Z \neq Y$. In this case, keeping dim Y fixed, we use induction on dim Z. If Z is smooth, we have

$$H^i_Z(Y) \cong H^{i-2r}(Z),$$

where $r = \operatorname{codim}(Z, Y)$, so the result follows by the first induction. If Z is not smooth, let $W = \operatorname{Sing} Z$. Then we have an exact sequence of local cohomology

$$\dots \to H^i_W(Y) \to H^i_Z(Y) \to H^i_{Z-W}(Y-W) \to \dots$$

Now $\dim W < \dim Z$, and $Z - W$ is smooth, so the result follows by the five lemma and the second induction.

Remark. — One can also ask the purely analytic question: let V be a complex analytic space, and let Z be a closed subset. Are the maps

$$\beta^i : H^i_Z(V, \mathbf{C}) \to H^i_{Z, DR}(V)$$

always isomorphisms? Our proof shows that it is true if there are schemes $X \supseteq Y$ such that $V = X_h$ and $Z = Y_h$. On the other hand, Liebermann and Herrera [29] have proved this result if V is compact and $Z = V$. In the next section, we will show that this result is true in general, by establishing the formal analytic Poincaré lemma.

2. The Formal Analytic Poincaré Lemma.

Theorem (**2.1**). — *Let* X *be a complex manifold, let* Y *be a closed analytic subspace, let* $\Omega^{\bullet}_X$ *be the complex of sheaves of holomorphic differential forms on* X, *and let* $\hat{\Omega}^{\bullet}_X$ *be its formal completion along* Y. *Then* $\hat{\Omega}^{\bullet}_X$ *is a resolution of the constant sheaf* $\mathbf{C}_Y$ *on* Y.

Corollary (**2.2**). — *With the hypotheses above, let* Z *be any closed subset of* Y. *Then we have isomorphisms*

$$H^i_Z(Y, \mathbf{C}) \to \mathbf{H}^i_Z(\hat{X}, \hat{\Omega}^{\bullet}_X)$$

for all i.

The general idea of the proof is to resolve the singularities of Y locally. This is a purely algebraic operation, so using suitable GAGA-type comparison theorems for coherent sheaf cohomology, we can apply the algebraic techniques developed earlier in this paper.

First we need a notion of relative scheme in analytic geometry. Let S be an analytic space, and $P \in S$ a point. Consider the category $\mathscr{C}_P$ of schemes of finite type over $\operatorname{Spec} \mathcal{O}_{P,S}$. If $X_P \in \mathscr{C}_P$ is such a scheme, it is defined locally by polynomial equations with coefficients in $\mathcal{O}_{P,S}$. These coefficients are all holomorphic in a suitable neighborhood U of P, so using the same local equations and patching data, we can construct an analytic space X^h_P over U. It may not be unique, but given two such, their restrictions to smaller neighborhoods of P become isomorphic.

Definition. — A morphism $f : X \to S$ of analytic spaces is a *relative scheme at* $P \in S$ if there exists a scheme $X_P \in \mathscr{C}_P$, and a neighborhood $P \in U$ such that X^h_P is defined on U, and $X|_U \cong X^h_P$. Note that if $f : X \to S$ is a relative scheme at P, then it is also such for all points Q in a neighborhood of P. We say $f : X \to S$ is a *relative scheme* if it is such at all points $P \in S$. Similarly, we say a coherent sheaf F on X is *relatively* *algebraic* over S if for each $P \in S$, it comes from a suitable coherent algebraic sheaf on the scheme X_P.

We will also need to use the following result about some special subsets of an analytic space, which was announced by Hironaka [30, footnote, p. 136], and has been proved by Frisch [11, Thm. I, 9] (see also [36]).

Theorem. — *Let* K *be a compact subset of an analytic space* S, *which is semi-analytic* (i.e. *which can be defined by a finite number of real-analytic equalities and inequalities), and which has a fundamental system of open neighborhoods which are Stein spaces. Then the ring* $A = \Gamma(K, \mathcal{O}_S)$ *of germs of holomorphic functions in a neighborhood of* K *is a noetherian ring. Furthermore, the map* $F \mapsto \Gamma(K, F)$ *gives an equivalence of categories between germs of coherent sheaves in a neighborhood of* K, *and finite type* A-*modules. Finally, for each* $P \in K$, *the local ring* $\mathcal{O}_{P,S}$ *is a flat* A-*module.*

We will call the compact sets of the theorem *special* compact subsets of S. If K is a special compact subset of S, we consider the category $\mathscr{C}_K$ of schemes of finite type over Spec A. As in the case K = a point, treated above, if $X_K \in \mathscr{C}_K$ is such a scheme, we can define its associated analytic space X_K^h in a suitable neighborhood U of K. If $f: X \to S$ is a relative scheme, then for any point $P \in S$ we can find a special compact neighborhood K of P, and a scheme $X_K \in \mathscr{C}_K$, and a neighborhood U of K such that $X_K^h \cong X|_U$.

Now we have the following result of GAGA-type.

Proposition (**2.3**). — *Let* S *be an analytic space, let* $f: X \to Y$ *be a proper morphism of relative schemes over* S, *and let* F *be a relatively algebraic coherent sheaf on* X. *Let* K *be a special compact subset of* S *such that* $f: X \to Y$ *comes from a morphism of schemes* $f: X_K \to Y_K$ *in* $\mathscr{C}_K$, *and* F *comes from a coherent sheaf* F_K *on* X_K. *Then there is a neighborhood* U *of* K *and isomorphisms*

$$R^i f_*(F_K)^h \cong R^i f_*(F)|_U$$

for all i.

Proof. — Since we have the functor *h* from $\mathscr{C}_K$ to analytic spaces over a neighborhood of K, the proof proceeds exactly as the proof of the normal relative GAGA theorem [19, XII, Thm. 4.2, p. 327]. Note that for any particular sheaf F, the dévissage of the proof involves only finitely many other sheaves, so the whole proof extends to a suitable neighborhood of K.

Proposition (**2.4**) (Fundamental theorem of a proper morphism of relative schemes). — *Let* S *be an analytic space, let* $f: X' \to X$ *be a proper morphism of relative schemes over* S, *let* Y *be a closed relative subscheme of* X, *let* $Y' = f^{-1}(Y)$, *and let* F *be a relatively algebraic coherent sheaf on* X'. *Then the natural map*

$$R^i f_*(F)^\wedge \to R^i \hat{f}_*(\hat{F})$$

is an isomorphism for all i, where ^ *denotes formal completion along* Y (resp. Y').

Proof. — The question is local on S, so for each $P \in S$ we choose a special compact neighborhood K of P such that X', X, Y, F all come from $X'_K, X_K, Y_K, F_K \in \mathcal{C}_K$. Copying the proof of (I.6.2), we use [EGA, III.4.1.5], in the category $\mathcal{C}_K$, and find that the inverse system $\{R^i f_*(F_{K,r})\}$ satisfies (ML), and that the kernel and cokernel of the map of inverse systems

$$R^i f_*(F_K) \otimes \mathcal{O}_{X_K}/I^r \to R^i f_*(F_{K,r})$$

are essentially zero.

Using the previous proposition does not give the analogous result for the analytic sheaves on any open neighborhood of K, but at least it does give the analogous result over the interior of K, which is still a neighborhood of P. Thus the rest of the proof is the same as that of (I.6.2).

Proposition **(2.5)** (Exact sequence of a proper birational morphism). — *Let* S *be an analytic space, let* $f: X' \to X$ *be a proper morphism of relative schemes over* S, *let* Y *be a closed relative subscheme of* X, *and let* $Y' = f^{-1}(Y)$. *Assume that f maps* $X' - Y'$ *isomorphically onto* $X - Y$. *Assume furthermore that there exist closed embeddings* $X \to Z$ *and* $X' \to Z'$ *of* X *and* X' *into complex manifolds* Z, Z', *which are relative schemes over* S, *and that there exists a proper morphism* $g: Z' \to Z$ *such that* $g|_{X'} = f$, *and g maps* $Z' - g^{-1}(Y)$ *isomorphically onto* $Z - Y$. *Then there is an exact sequence of analytic De Rham cohomology*

$$\ldots \to H^i(X) \to H^i(X') \oplus H^i(Y) \to H^i(Y') \to H^{i+1}(X) \to \ldots$$

where we calculate the De Rham cohomology using the embeddings into Z *and* Z': *thus* $H^i(X) = \mathbf{H}^i(\hat{Z}, \hat{\Omega}^{\bullet}_Z)$, *etc.*

Proof. — Using the fundamental theorem of a proper morphism just proved, and the analytic Mayer-Vietoris sequence (1.4), the proof proceeds exactly as in the algebraic case (II.4.4).

Proof of Theorem **(2.1)**. — Given the complex manifold X and the analytic subspace Y, it will be sufficient to show that for all $P \in Y$, and for all sufficiently small neighborhoods U of P, we have isomorphisms

$$H^i(U, \mathbf{C}_Y) \xrightarrow{\sim} \mathbf{H}^i(U, \hat{\Omega}^{\bullet}_X).$$

In fact, we will prove the following apparently more general result. Let S be an analytic space, and $P \in S$ a point. Let $Y \to X$ be a closed immersion of relative schemes over S, where X is a manifold. Then for all sufficiently small neighborhoods U of P, the natural maps

$$H^i(Y|_U, \mathbf{C}) \to \mathbf{H}^i(\hat{X}|_U, \hat{\Omega}^{\bullet}_X)$$

are isomorphisms. In particular, taking $S = X$ gives the theorem.

We prove this result by induction on dim Y. If Y is smooth, then there is a quasi-isomorphism $\hat{\Omega}^{\cdot}_X \to \Omega^{\cdot}_Y$ (see 1.3 above). Thus the result follows from the usual analytic Poincaré lemma. In particular, if $\dim Y = 0$, we may assume Y is reduced, hence smooth, so this starts the induction.

If Y is not smooth, we consider the scheme $Y_P \in \mathscr{C}_P$ over Spec $\mathcal{O}_{P,S}$ and a neighborhood U of P such that $Y^h_P \cong Y|_U$. We may also assume that there is a regular scheme $X_P \in \mathscr{C}_P$ such that $X^h_P \cong X|_U$. Let $W_P = \operatorname{Sing} Y_P$. We can apply resolution of singularities in the category $\mathscr{C}_P$, so we find a proper morphism $g : X'_P \to X_P$ of manifolds, such that g maps $X'_P - g^{-1}(W_P)$ isomorphically to $X_P - W_P$, and such that the proper transform Y'_P of Y_P is regular. Let $W'_P = g^{-1}(W_P) \cap Y'_P$.

Applying the functor h, this whole situation is transported to the category of relative schemes over a suitably diminished neighborhood U of P. Now we apply the exact sequence of a proper birational morphism just proved (with slightly different notation), the five-lemma, the induction hypothesis, and the corresponding topological exact sequence (1.6) to reduce to the smooth case which was already proved.

3. The local case.

Theorem (**3.1**). — *Let* S *be an analytic space, and let* P *be a point of* S. *Then there are natural isomorphisms*

$$H^i_P(S, \mathbf{C}) \cong H^i_{P,DR}(\operatorname{Spec} \hat{\mathcal{O}}_{P,S}).$$

Corollary (**3.2**). — *With the same hypotheses there are isomorphisms*

$$H^{BM}_i(S, \mathbf{C})_P \cong H^{DR}_i(\operatorname{Spec} \hat{\mathcal{O}}_{P,S})$$

where the group on the left is the stalk of the Borel-Moore homology sheaf of S.

Proof. — To prove the theorem, we will consider a slightly more general situation. Using the terminology and notations of the previous section, let X be a relative scheme, proper over S, and let $Y = f^{-1}(P)$. Let $X_P, Y_P \in \mathscr{C}_P$ be the corresponding schemes over Spec $\mathcal{O}_{P,S}$. Let ′ denote the base extension to Spec $\hat{\mathcal{O}}_{P,S}$. Furthermore, choose a special compact neighborhood K of P, such that X comes from a scheme $X_K \in \mathscr{C}_K$. Then we have a continuous map of topological spaces

$$\varphi : X|_K \to X_K.$$

Now we have natural maps of cohomology

$$\begin{array}{ccccc} H^i_{Y_K,DR}(X_K) & \xrightarrow{\alpha} & H^i_{Y_P,DR}(X_P) & \xrightarrow{\beta} & H^i_{Y'_P,DR}(X'_P) \\ \downarrow{\scriptstyle\gamma} & & & & \\ H^i_{Y,DR}(X|_K) & \xrightarrow{\delta} & H^i_{Y,DR}(X) & \xleftarrow{\varepsilon} & H^i_Y(X, \mathbf{C}). \end{array}$$

In this diagram, the top row consists of algebraic cohomology, and the bottom row of analytic cohomology. The maps α and β are excision maps. By the strong excision theorem (III, 3.1) which applies also in this case, β and $\beta.\alpha$ are isomorphisms. Hence α is also an isomorphism. The map γ comes from the continuous map φ above. The map δ is an isomorphism by the usual excision theorem, and ε is an isomorphism by the result (2.2) of the last section.

Using these isomorphisms, we obtain a natural map

$$\theta : H^i_{Y'_P}(X'_P) \to H^i_Y(X, \mathbf{C})$$

which we will show to be an isomorphism.

The proof proceeds along familiar lines, using the functorial properties of the cohomology theories on both sides. We use induction on the dimension of X_P. By the Mayer-Vietoris sequence, we may assume X_P is integral. Then we apply the resolution of singularities to X_P, and as before extend the resolution over a suitable neighborhood U of P. Then using the exact sequence of a proper birational morphism (with supports) we reduce to the case where X_P is a regular scheme. In this case, if $n = \dim X_P$, we have duality theorems

$$H^i_{Y_P}(X_P) \cong (H^{2n-i}(Y_P))'$$

and

$$H^i_Y(X, \mathbf{C}) \cong (H^{2n-i}(Y, \mathbf{C}))'.$$

Now the maps θ are compatible with duality; Y_P is a scheme of finite type over $\mathbf{C}$, and $Y \cong Y^h_P$, so the isomorphism follows from the global case (1.1).

We leave the proof of the corollary to the reader.

Remark. — In the case of an isolated singularity, P. Deligne proved by a different method (unpublished) that

$$H^i_P(S, \mathbf{C}) \cong H^i_{P, DR}(\operatorname{Spec} \mathcal{O}_{P,S}).$$

In fact, he proved more generally that if Y is an analytic subset of S such that $S - Y$ is smooth and $P \in Y$, then

$$(\mathcal{H}^i_Y(S, \mathbf{C}))_P \cong H^i_{Y, DR}(\operatorname{Spec} \mathcal{O}_{P,S}).$$

4. The relative Case.

Let $f : Y \to S$ be a morphism of schemes of finite type over $\mathbf{C}$. Then we wish to compare the sheaves of relative De Rham cohomology $R^i f_*(Y)$ on S with the sheaves of relative complex cohomology $R^i f^h_*(\mathbf{C})$ on S^h. One sees immediately that unless the sheaves $R^i f_*(Y)$ are coherent, there is not much hope for a reasonable comparison theorem. So we will show that under suitable conditions, when these sheaves are coherent, we have a comparison theorem.

In any case, we have natural maps

$$\alpha^i : R^i f_*(Y)^h \to R^i f^h_*(Y^h)$$

of the algebraic De Rham cohomology into the analogously defined analytic relative De Rham cohomology. On the other hand, the natural map of the complex numbers into the relative De Rham complex induces maps

$$\beta^i : R^i f^h_*(\mathbf{C}) \otimes_{\mathbf{C}} \mathcal{O}_{S^h} \to R^i f^h_*(Y^h).$$

Proposition **(4.1)**. — *Assume that f is proper. Let r be an integer such that* $R^i f_*(Y)$ *is coherent for $i=r$ and $i=r+1$. Then α^r is an isomorphism.*

Proof. — We will adapt the proof of (III, 5.2), with the continuous map $g : S^h \to S$ in place of the base extension there. The fact that g^* commutes with formation of cohomology of coherent sheaves is the relative GAGA theorem (quoted in Ch. I, § 6). Furthermore, (I.4.10) and its proof apply equally well to the functor g^* instead of the functor $\otimes E$ in the original statement. The rest of the proof of (III.5.2) carries over without change.

Proposition **(4.2)**. — *Assume that f is proper, and* S *smooth. Let r be an integer such that* $R^i f^h_*(Y^h)$ *is coherent for $i \leq r$. Then the sheaf* $R^i f_*(\mathbf{C})$ *is locally constant, and β^i is an isomorphism, for $i \leq r$.*

Proof. — We will use an analytic analogue of (III.5.4) suitably adapted. First note, as in the algebraic case, that $R^i f^h_*(Y^h)$ has an integrable connection. Now a coherent sheaf with integrable connection on a complex manifold is locally free, and the connection is locally trivial, so considering a neighborhood of any point $P \in S^h$, we may assume $R^i f^h_*(Y^h)$ is free with trivial connection for $i \leq r$.

Next we note that the Leray spectral sequence (III.4.1) holds equally well in the analytic case, so as in the proof of (III.5.4) we find an isomorphism

$$H^i_{DR}(Y^h_P) \to R^i f^h_*(Y^h)_P$$

where $Y_P = f^{-1}(P)$. On the other hand, one knows that the stalks of the sheaf $R^i f^h_*(\mathbf{C})$ give the cohomology of the fibre:

$$H^i(Y^h_P, \mathbf{C}) \simeq R^i f^h_*(\mathbf{C})_P.$$

Finally, the global comparison theorem tells us that

$$H^i_{DR}(Y^h_P) \simeq H^i(Y^h_P, \mathbf{C}).$$

We deduce that the natural map

$$R^i f^h_*(\mathbf{C})_P \to R^i f^h_*(Y^h)_P$$

is an isomorphism, whence the result of the proposition follows by tensoring with $\mathcal{O}_{S^h}$.

Corollary **(4.3)**. — *Assume that f is proper. Then there exists an open dense subset $U \subseteq S$ such that α^i and β^i are isomorphisms over* U, *for all i.*

Proof. — Apply (III.5.1).

Remark. — This Corollary was proved by Deligne in the case of a smooth morphism f, not necessarily proper [10, Thm. 6.13, p. 106].

BIBLIOGRAPHY

[1] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.*, **85** (1957), 181-207.

[2] M. F. Atiyah and W. V. D. Hodge, Integrals of the second kind on an algebraic variety, *Ann. of Math.*, **62** (1955), 56-91.

[3] W. Barth, Transplanting cohomology classes in complex projective space, *Amer. J. Math.*, **92** (1970), 951-967.

[4] P. Berthelot, Cohomologie p-cristalline des schémas, *C. R. Acad. Sci. Paris*, **269** (1969), 297-300; 357-360; 397-400.

[5] T. Bloom and M. Herrera, De Rham cohomology of an analytic space, *Inv. Math.*, **7** (1969), 275-296.

[6] A. Borel and J. Moore, Homology theory for locally compact spaces, *Mich. Math. J.*, **7** (1960), 137-159.

[7] A. Borel and J.-P. Serre, Le théorème de Riemann-Roch, *Bull. Soc. Math. France*, **86** (1958), 97-136.

[8] E. Brieskorn, Die Monodromie der isolierten Singularitäten von Hyperflächen, *Manuscripta Math.*, **2** (1970), 103-161.

[9] C. Chevalley, *Anneaux de Chow et applications*, Séminaire Chevalley, 1958.

[10] P. Deligne, Equations différentielles à points singuliers réguliers, *Springer Lecture Notes*, **163** (1970), 133 p.

[11] J. Frisch, Points de platitude d'un morphisme d'espaces analytiques complexes, *Inv. Math.*, **4** (1967), 118-138.

[12] R. Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1958.

[13] H. Grauert, Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen, *Publ. Math. I.H.E.S.*, **5** (1960), 5-64.

[14] A. Grothendieck, Sur quelques points d'algèbre homologique, *Tohoku Math. J.*, **9** (1957), 119-221.

[15] —, La théorie des classes de Chern, *Bull. Soc. Math. France*, **86** (1958), 137-154.

[16] —, On the De Rham cohomology of algebraic varieties, *Publ. Math. I.H.E.S.*, **29** (1966), 95-103.

[17] —, Local Cohomology (LC), *Springer Lecture Notes*, **41** (1966).

[18] —, Crystals and the De Rham cohomology of schemes (Notes by I. Coates and O. Jussila) in *Dix exposés sur la cohomologie des schémas*, North-Holland, 1968.

[19] —, Revêtements étales et groupe fondamental (SGA 1), *Springer Lecture Notes*, **224** (1971), 447 p.

[20] — and J. Dieudonné, Eléments de géométrie algébrique (EGA), *Publ. Math. I.H.E.S.*, **4**, **8**, **11**, **17**, **20**, **24**, **28**, **32** (1960-1967).

[21] R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, 1965.

[22] R. Hartshorne, Residues and Duality (RD), *Springer Lecture Notes*, **20** (1966).

[23] —, Cohomological dimension of algebraic varieties, *Annals of Math.*, **88** (1968), 403-450.

[24] —, Ample subvarieties of algebraic varieties, *Springer Lecture Notes*, **156** (1970).

[25] —, Cohomology of non-complete algebraic varieties, *Compositio Math.*, **23** (1971), 257-264.

[26] —, Cohomology with compact supports for coherent sheaves on an algebraic variety, *Math. Ann.*, **195** (1972), 199-207.

[27] —, Algebraic De Rham cohomology, *Manuscripta Math.*, **7** (1972), 125-140.

[28] —, Topological conditions for smoothing algebraic singularities, *Topology*, **13** (1974), 241-253.

[29] M. Herrera and D. Lieberman, Duality and the De Rham cohomology of infinitesimal neighborhoods, *Invent. Math.*, **13** (1971), 97-124.

[30] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, I, II, *Annals of Math.*, **79** (1964), 109-326.

[31] G. Hochschild, B. Kostant and A. Rosenberg, Differential forms on regular affine algebras, *Trans. Amer. Math. Soc.*, **102** (1962), 383-408.

[32] N. Katz, Nilpotent connections and the monodromy theorem: applications of a result of Turrittin, *Publ. Math. I.H.E.S.*, **39** (1970), 175-232.

[33] —, The regularity theorem in algebraic geometry, *Proc. Int. Cong. Math. in Nice*, 1970, vol. 1, 437-443.

[34] — and T. Oda, On the differentiation of De Rham cohomology classes with respect to parameters, *J. Math. Kyoto Univ.*, **8** (1968), 199-213.

[35] L. Kamp, Eine topologische Eigenschaft Steinscher Räume, *Nachr. Akad. Wiss. Göttingen*, **8** (1966).

[36] R. Kiehl, Note zu der Arbeit von J. Frisch « Points de platitude d'un morphisme d'espaces analytiques complexes », *Inv. Math.*, **4** (1967), 139-141.

[37] I. G. Macdonald, Duality over complete local rings, *Topology*, **1** (1962), 213-235.

[38] P. MONSKY, Formal cohomology II, *Annals of Math.*, **88** (1968), 218-238.
[39] —, Finiteness of De Rham cohomology, *Amer. J. Math.*, **94** (1972), 237-245.
[40] M. NAGATA, Imbedding of an abstract variety in a complete variety, *J. Math. Kyoto Univ.*, **2** (1962), 1-10.
[41] —, A generalization of the embedding problem, *J. Math. Kyoto Univ.*, **3** (1963), 89-102.
[42] R. NARASIMHAN, On the homology groups of Stein spaces, *Inv. Math.*, **2** (1967), 377-385.
[43] A. OGUS, *Local cohomological dimension of algebraic varieties*, Thesis, Harvard, 1972.
Added in proof : These results are published in two papers:
[43 *a*] A. OGUS, Local cohomological dimension of algebraic varieties, *Annals of Math.*, **98** (1973), 327-365.
[43 *b*] —, On the formal neighborhood of a subvariety of projective space (to appear in *Amer. J. Math.*).
[44] H.-J. REIFFEN, Das Lemma von Poincaré für holomorphe Differentialformen auf Komplexen Räumen, *Math. Zeit.*, **101** (1967), 269-284.
[45] J.-E. ROOS, Sur les foncteurs dérivés de $\varprojlim$. Applications. *C. R. Acad. Sc. Paris*, **252** (1961), 3702-3704.
[46] J.-P. SERRE, Faisceaux algébriques cohérents (FAC), *Annals of Math.*, **61** (1955), 197-278.
[47] —, Géométrie algébrique et géométrie analytique (GAGA), *Ann. Inst. Fourier*, **6** (1956), 1-42.
[48] E. H. SPANIER, *Algebraic Topology*, McGraw-Hill, 1966.
[49] D. LIEBERMAN, Generalizations of the De Rham complex with applications to duality theory and the cohomology of singular varieties, *Proc. Conf. of Complex Analysis*, Rice, 1972.

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