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ON THE STRUCTURE OF NON-EXCELLENT CURVE SINGULARITIES IN CHARACTERISTIC p

by BRUCE BENNETT

Notational and terminological conventions. — All rings considered are commutative with unit. Local rings are noetherian, unless otherwise specified. If R is a ring, Q(R) denotes is total quotient ring. If R is a ring and $\mathfrak{P}$ is an ideal of R, then, for x in R, $\nu_{\mathfrak{P}}(x)$ denotes the $\mathfrak{P}$-order of x, i.e. the highest power of $\mathfrak{P}$ containing x if this number exists, ∞ otherwise; $\mathrm{Bl}_{\mathfrak{P}}(\mathrm{R})$ denotes the blowing up of Spec(R) with center $\mathfrak{P}$.

0. INTRODUCTION

The object of this paper is to understand the phenomenon of a local integral domain $\mathfrak{O}$ of dimension 1 and characteristic p, whose completion has nilpotent elements. As is well known, this is equivalent to saying that the normalization of $\mathfrak{O}$ is not finite as $\mathfrak{O}$-module, or indeed that the singularity of $\mathfrak{O}$ cannot be resolved by finitely many quadratic transforms. Thus these rings cannot arise as the local rings of points on " standard " geometric objects, i.e. schemes of finite type over **Z** or over a complete local ring (in virtue of the famous theorems of Zariski, Nagata, and Grothendieck). How do they arise, and what is their structure?

We call such an $\mathfrak{O}$ as above a " non-excellent curve singularity ". Some authors may prefer the terminology " non-Japanese " or " non-pseudogeometric " here, since for local rings of dimension 1 the global aspects of excellence (in particular those relating to closedness of the singular loci) do not enter in. However the 1-dimensional local domains play an obvious elemental role in an inductive analysis of the relationship between any local ring and its completion.

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We will develop a structure theory for those local domains $\mathfrak{O}$ as above, for which also $\hat{\mathfrak{O}}_{\text{red}}$ is *regular*. This extra hypothesis on $\hat{\mathfrak{O}}$ is required morally by the observations that:

(i) The phenomenon of non-excellence which we seek to study is unaffected, in fact is *purified* by finitely many quadratic transforms, and

(ii) By finitely many such transforms we always arrive at an $\mathfrak{O}$ for which the hypothesis is satisfied (see § 1 for details). Here, at least, morality is rewarded: we find that such $\mathfrak{O}$ must have a discrete valuation subring R such that $\mathfrak{O}$ is a purely inseparable extension of R contained in $\hat{R}$, i.e. we have local homomorphisms

$$R \underset{i}{\hookrightarrow} \mathfrak{O} \underset{j}{\hookrightarrow} \hat{R}$$

with $j \circ i = 1_{\hat{R}}$ and $\mathfrak{O}$ purely inseparable over R. We call this a *presentation of* $\mathfrak{O}$ *over* R; its existence is proved in § 2, and it is the basic structural element of the theory.

Given a presentation $R \hookrightarrow \mathfrak{O} \hookrightarrow \hat{R}$, the structure of $\hat{\mathfrak{O}}$ can be completely described in terms of (*a*) the birational equivalence class of $\mathfrak{O}$ over R and (*b*) the " way " in which $\mathfrak{O}$ fails to be a finite R-module (for although the field of fractions of $\mathfrak{O}$ may be finite over that of R, $\mathfrak{O}$ need not be finitely generated as R-algebra). More precisely (say, for simplicity, in the case of finite fraction field extension) take a finite R-subalgebra S of $\mathfrak{O}$ such that

($\#$) $\mathfrak{O}$ and S have the same fraction field and $S \hookrightarrow \mathfrak{O}$ induces a *surjection* of completions.

Then we show that $\mathfrak{O}$ is obtained by an infinite sequence of birational operations on S, for which the kernel of $\hat{S} \rightarrow \hat{\mathfrak{O}} \rightarrow 0$ is a precise description, albeit in " coded " form (§ 3). The theorem of quasi-algebrization of § 6, which *a priori* is a technique for construction of rings with given completions, in effect accomplishes the breaking of this code. In combination with the uniqueness theorem of (6.3) it establishes an *isomorphism* between the set of all $\mathfrak{O}$ which satisfy ($\#$) above with respect to a given S, and the set $\mathrm{Hilb}_{\hat{S}/\hat{R}}(\hat{R})$, i.e. the set of all quotients of $\hat{S}$ which are flat over $\hat{R}$ (6.3.2). This may be viewed as a local description of a " classifying scheme " of say, all local R-algebras in a given birational class.

Thus, the results show that in characteristic p, all non-excellence of local rings of dimension 1 is due to inseparability in an extension $R \hookrightarrow \hat{R}$ for a suitable discrete valuation ring R. The interest of this seems enhanced by the fact that there exist examples of non-excellent local domains of dimension 1 over the complex numbers, e.g. the recent work of Ferrand and Raynaud [3]. These examples depend on certain differential operators, which however turn out to play a role analogous to that of the differential operator canonically attached to a presentation! This observation, together with some of the ideas of [3], suggest that a unified treatment may be possible from this point of view (keeping in mind that the operators arise transcendentally in characteristic 0, as contrasted with their algebraic origin in characteristic p). In any case their examples show that the algebraic approach of this paper cannot apply in characteristic 0, without what would appear at the moment to be very substantial modifications. These questions are treated in § 4.

In § 5 we indicate how to construct "geometrically" discrete valuation rings R with arbitrarily rich inseparability in $R \hookrightarrow \hat{R}$; the basic idea here is that of F. K. Schmidt. In combination with the results related to quasi-algebrization cited above, this construction implies that *any* finite flat $\hat{R}$-algebra C with a section and connected fibres over $\hat{R}$ is the completion of a local domain $\mathfrak{O}$, in such a way that the $\hat{R}$-structure of C is induced by an R-presentation structure of $\mathfrak{O}$ for suitable R (everything in char. p, of course).

In § 7 we give an example of a pathological $\mathfrak{O}$, whose fraction field is infinite over a maximal presentation.

I would like to mention that M. Nagata's beautiful and basic example in Appendix 3 of [2] provided me with many fundamental insights into this theory. It is my pleasure to thank H. Hironaka, with whom I have had numerous useful and encouraging conversations on this subject. I am also indebted to him for the proof of (2.1). I am grateful to R. Rasala for several helpful and pleasant discussions. Finally, I would thank the referee whose identity is unknown to me, but who, in observing a basic defect in an earlier manuscript of mine on this subject, played an indispensable role in the development of the theory.

1. PRELIMINARIES: THE EFFECT OF QUADRATIC TRANSFORMS

Let $\mathfrak{O}$ be a local domain of Krull dimension one, and of characteristic $p>0$. Let $\mathfrak{m}$ denote the maximal ideal of $\mathfrak{O}$. We want to study the "formal fibre" of $\mathfrak{O}$ i.e. the scheme-theoretic inverse image of the generic point by the natural morphism

$$\mathrm{Spec}(\hat{\mathfrak{O}}) \to \mathrm{Spec}(\mathfrak{O})$$

where $\hat{\mathfrak{O}}$ denotes the $\mathfrak{m}$-adic completion. Thus the formal fibre may be expressed as

$$\mathrm{Spec}(\hat{\mathfrak{O}} \otimes_{\mathfrak{O}} Q(\mathfrak{O}))$$

(where Q denotes field of quotients), or equivalently as

$$\coprod_{i=1}^{n} \mathrm{Spec}(\hat{\mathfrak{O}}_{\mathfrak{P}_i}) \tag{1.0}$$

where the $\mathfrak{P}_i$ are the minimal primes of $\hat{\mathfrak{O}}$. Thus we are reduced to study $\hat{\mathfrak{O}}_{\mathfrak{P}_i}$.

Since $\mathfrak{O}$ is of dimension 1, $\mathrm{Bl}_{\mathfrak{m}}(\mathfrak{O})$ has finitely many closed points, corresponding to the distinct points of $\mathrm{Proj}(\mathrm{Gr}_{\mathfrak{m}}(\mathfrak{O}))$. Thus $\mathrm{Bl}_{\mathfrak{m}}(\mathfrak{O}) = \mathrm{Spec}(B)$ is affine, where B is a semi-local $\mathfrak{O}$-algebra, finite over $\mathfrak{O}$ (since it is of finite type over $\mathfrak{O}$, and is contained in the normalization of $\mathfrak{O}$). Now since blowing up is compatible with flat base extension, we find that $\mathrm{Bl}_{\hat{\mathfrak{m}}}(\hat{\mathfrak{O}}) = \mathrm{Spec}(B \otimes_{\mathfrak{O}} \hat{\mathfrak{O}})$. Therefore $\mathrm{Bl}_{\mathfrak{m}}(\mathfrak{O})$ and $\mathrm{Bl}_{\hat{\mathfrak{m}}}(\hat{\mathfrak{O}})$ are topologically identical; if $\mathfrak{O}'$ and $\mathfrak{O}''$ are the local rings of corresponding (closed) points, then

$$\mathfrak{O}'' = \hat{\mathfrak{O}}' = \mathfrak{O}' \otimes_{\mathfrak{O}} \hat{\mathfrak{O}}.$$

(Note that the last equality holds even though $\mathfrak{O}'$ may not be finite over $\mathfrak{O}$.) These local rings are called the "quadratic transforms" of $\mathfrak{O}$ (or $\hat{\mathfrak{O}}$, as the case may be).

Let $\mathfrak{P}$ be a minimal prime of $\hat{\mathfrak{O}}$ and let $\mathfrak{O}'$ and $\hat{\mathfrak{O}}'$ be as above. We have the "cartesian" diagram

$$\text{(D)} \qquad \begin{array}{ccc} \mathfrak{O}' & \longrightarrow & \hat{\mathfrak{O}}' \\ \uparrow & & \uparrow \\ \mathfrak{O} & \longrightarrow & \hat{\mathfrak{O}} \end{array}$$

where the vertical arrows are quadratic transforms and the horizontal arrows are completions. Let $\mathfrak{P}'$ denote the strict transform of $\mathfrak{P}$ in $\hat{\mathfrak{O}}'$. We recall that if t is an element of $\hat{\mathfrak{O}}$ such that $\mathfrak{m}\hat{\mathfrak{O}}'=t\hat{\mathfrak{O}}$ then we may describe (letting $\hat{\mathfrak{m}}=\mathfrak{m}\hat{\mathfrak{O}}$):

$$\mathfrak{P}'=\{x/t^{\nu}\,|\,x\in\mathfrak{P} \quad \text{and} \quad \nu_{\hat{\mathfrak{m}}}(x)\geqslant\nu\}.$$

$\mathfrak{P}'$ has the property that $\hat{\mathfrak{O}}'/\mathfrak{P}'$ is a quadratic transform of $\hat{\mathfrak{O}}/\mathfrak{P}$ ([1], 0, § 3). We note:

(1.1) *If $\mathfrak{P}'$ is not the unit ideal then $\hat{\mathfrak{O}}\to\hat{\mathfrak{O}}'$ induces an isomorphism $\hat{\mathfrak{O}}_{\mathfrak{P}}\xrightarrow{\sim}\hat{\mathfrak{O}}'_{\mathfrak{P}'}$.*

In fact, $\mathfrak{P}'\neq\hat{\mathfrak{O}}'\Rightarrow t\notin\mathfrak{P}$. Therefore $\hat{\mathfrak{O}}_{\mathfrak{P}}=(\hat{\mathfrak{O}}_t)_{\mathfrak{P}}$ and $\hat{\mathfrak{O}}'_{\mathfrak{P}'}=(\hat{\mathfrak{O}}'_t)_{\mathfrak{P}'}$. But since $\hat{\mathfrak{O}}'\subset\hat{\mathfrak{O}}_t$ *(loc. cit.)*, and since obviously $\mathfrak{P}'\hat{\mathfrak{O}}'_t=\mathfrak{P}\hat{\mathfrak{O}}'_t$, we get the result.

Now take a quadratic sequence along $\mathfrak{P}$, i.e. a sequence

$$\hat{\mathfrak{O}}^{(0)}=\hat{\mathfrak{O}}\to\hat{\mathfrak{O}}^{(1)}=\hat{\mathfrak{O}}'\to\hat{\mathfrak{O}}^{(2)}\to\hat{\mathfrak{O}}^{(3)}\to\hat{\mathfrak{O}}^{(4)}\to\ldots,$$

of quadratic transforms such that, if $\mathfrak{P}^{(0)}=\mathfrak{P}$, and $\mathfrak{P}^{(i)}$ denotes the strict transform of $\mathfrak{P}^{(i-1)}$ in $\mathfrak{O}^{(i)}$, then $\mathfrak{P}^{(i)}\neq\mathfrak{O}^{(i)}$. This corresponds uniquely to a sequence

$$\mathfrak{O}^{(0)}=\mathfrak{O}\to\mathfrak{O}^{(1)}=\mathfrak{O}'\to\mathfrak{O}^{(2)}\to\mathfrak{O}^{(3)}\to\mathfrak{O}^{(4)}\to\ldots,$$

such that all the diagrams

$$\begin{array}{ccc} \mathfrak{O}^{(i+1)} & \longrightarrow & \hat{\mathfrak{O}}^{(i+1)} \\ \uparrow & & \uparrow \\ \mathfrak{O}^{(i)} & \longrightarrow & \hat{\mathfrak{O}}^{(i)} \end{array}$$

have the same properties as the diagram (D) above *(loc. cit.)*.

We now want to prove:

(1.2) *For i sufficiently large, $\mathfrak{P}^{(i)}$ is the unique minimal prime ideal of $\hat{\mathfrak{O}}^{(i)}$ and $\hat{\mathfrak{O}}^{(i)}/\mathfrak{P}^{(i)}$ is a regular local ring.*

This fact, in combination with (1.1) reduces our study of the formal fibre to the following situation:

(1.3) *$\mathfrak{O}$ is a local domain of dimension one such that $\hat{\mathfrak{O}}$ has a unique minimal prime $\mathfrak{P}$ and $\hat{\mathfrak{O}}/\mathfrak{P}$ is regular.*

The proof of (1.2) will follow from the considerations below.

Let $\mathfrak{O}$ be a reduced, complete local ring of dimension 1. If N denotes the normalization of $\mathfrak{O}$, then N may be written in the form $\prod_{i=1}^{n} R_i$, where the R_i are complete discrete valuation rings. On the other hand, we know Spec(N) may be realized as a succession of quadratic transformations beginning with Spec($\mathfrak{O}$). Hence Spec(N) and Spec($\mathfrak{O}$) are isomorphic outside the fibre above the closed point of Spec($\mathfrak{O}$). It is therefore clear that the R_i are in 1-1 correspondence with the minimal primes $\mathfrak{P}_i$ of $\mathfrak{O}$, so that R_i is the normalization of $\mathfrak{O}/\mathfrak{P}_i$. In particular

(1.4) *If* $\mathfrak{O}=\mathfrak{O}^{(0)}\rightarrow\mathfrak{O}^{(1)}\rightarrow\dots$ *is any sequence of quadratic transforms beginning with* $\mathfrak{O}$, *then there exists a* j *such that for all* $k\geqslant j$ $\mathfrak{O}^{(k)}$ *is regular*, i.e. the quadratic sequence separates the branches of Spec($\mathfrak{O}$) and resolves the singularity of each branch.

Now if $\mathfrak{O}$ is an arbitrary (not necessarily reduced) complete local ring of dimension 1, let $\mathfrak{J}$ be its nilradical. If $\mathfrak{O}'$ is a quadratic transform of $\mathfrak{O}$, let $\mathfrak{J}'$ denote the strict transform of $\mathfrak{J}$ in $\mathfrak{O}'$. *Then* $\mathfrak{J}'$ *is the nilradical of* $\mathfrak{O}'$. In fact, let $t\in\mathfrak{m}=\max(\mathfrak{O})$ such that $t\mathfrak{O}'=\mathfrak{m}\mathfrak{O}'$, and let f be an element of the nilradical of $\mathfrak{O}'$. Write $f=x/t^{\nu}$ with $x\in\mathfrak{m}^{\nu}$. Since $\mathfrak{O}'\subset\mathfrak{O}_t$, $f^n=0$ implies that $t^j x^n=0$ in $\mathfrak{O}$ for some j, and we may assume that $j=n$. Thus tx is in $\mathfrak{J}\cap\mathfrak{m}^{\nu+1}$, so that $tx/t^{\nu+1}=f$ is an element of $\mathfrak{J}'$. Thus $\mathrm{Nil}(\mathfrak{O}')\subset\mathfrak{J}'$, and since the other inclusion is obvious, we get the result.

Hence if $X=\mathrm{Spec}(\mathfrak{O})$, and prime (′) denotes blowing up with the closed point as center, $(X')_{\mathrm{red}}=(X_{\mathrm{red}})'$. Thus, applying (1.4) to $\mathfrak{O}_{\mathrm{red}}$, we obtain

(1.5) *If* $\mathfrak{O}=\mathfrak{O}^{(0)}\rightarrow\mathfrak{O}^{(1)}\rightarrow\dots$ *is any sequence of quadratic transforms, then there is a* j *such that for all* $k\geqslant j$, $\mathfrak{O}^{(k)}$ *is unibranch and* $\mathfrak{O}^{(k)}_{\mathrm{red}}$ *is regular.*

In particular let $\mathfrak{P}$ be any minimal prime of $\mathfrak{O}$, and let

$$\mathfrak{O}=\mathfrak{O}^{(0)}\rightarrow\mathfrak{O}^{(1)}\rightarrow\dots$$

be a quadratic sequence along $\mathfrak{P}$. (We can easily obtain such a sequence by choosing a quadratic sequence beginning with $\mathfrak{O}/\mathfrak{P}$ — which is necessarily unique by the above remarks — and taking the unique quadratic sequence beginning with $\mathfrak{O}$ to which it corresponds.) (1.2) now follows by applying (1.5) to this sequence, remembering that the $\mathfrak{O}$ in the discussion immediately preceding is actually $\hat{\mathfrak{O}}$ in our application.

2. THE EXISTENCE OF A PRESENTATION

(2.0) We begin with the hypotheses of (1.3): $\mathfrak{O}$ *is a local domain of dimension 1, char.* $p>0$, *and* $\hat{\mathfrak{O}}$ *has a unique minimal prime* $\mathfrak{P}$. *Moreover,* $\hat{\mathfrak{O}}/\mathfrak{P}$ *is regular.* Our goal in this section if to prove the following theorem:

Theorem 1. — With $\mathfrak{O}$ *as above there exists a regular local ring* R *of dimension 1 such that*

$$R\hookrightarrow\mathfrak{O}\hookrightarrow\hat{R}$$

where the inclusions are local homomorphisms, the composition is the canonical map $R \hookrightarrow \hat{R}$, *and* $\mathfrak{O}^q \subset R$ *for some* $q = p^e$.

Note that in view of § 1, this theorem has the following variant:

Theorem 1'. — *Let* $\mathfrak{O}$ *be a local domain of dimension 1, char.* $p > 0$ *and let*

$$\mathfrak{O} = \mathfrak{O}^{(0)} \to \mathfrak{O}^{(1)} \to \dots$$

be a sequence of quadratic transforms. Then there exists a discrete valuation ring R *such that for all j sufficiently large,*

$$R \hookrightarrow \mathfrak{O}^{(j)} \hookrightarrow \hat{R}$$

where the inclusions are local homomorphisms and the composition is the canonical map, and $(\mathfrak{O}^{(j)})^q \subset R$ *for some* $q = p^e$. (For Theorem 1', use Theorem 1 and (1.2) to get the result for some $\mathfrak{O}^{(j)}$. The same R then works for $k > j$ since $\hat{R}$ is normal and $\mathfrak{O}^{(k)}$ is contained in the normalization of $\mathfrak{O}^{(j)}$.)

To prove Theorem 1, we begin by showing that under the hypotheses of (1.3), $\mathfrak{O}$ contains a discrete valuation ring. This fact follows immediately from the lemma below, whose proof was suggested by Hironaka.

Lemma **(2.1)**. — *Suppose* $\mathfrak{O}$ *is a local ring of characteristic* $p > 0$, *and* $\hat{\mathfrak{O}}_{\text{red}}$ *is normal. Then* $\mathfrak{O}$ *contains a normal local subring* N *of the same dimension as* $\mathfrak{O}$ *with* $\mathfrak{O}^{q'} \subset N$ ($\mathfrak{O}^{q'}$ denotes the image of a suitably high iterate of the Frobenius endomorphism).

Proof. — Let $\mathfrak{P}$ denote the nilradical of $\hat{\mathfrak{O}}$. Then for a sufficiently high power q of the characteristic p, $\mathfrak{P}$ is the kernel of the Frobenius map $\hat{\mathfrak{O}} \to \hat{\mathfrak{O}}^q$. Hence

$$\hat{\mathfrak{O}}^q \simeq (\hat{\mathfrak{O}}/\mathfrak{P})^q = (\hat{\mathfrak{O}}_{\text{red}})^q.$$

Therefore, since $\hat{\mathfrak{O}}_{\text{red}} \to (\hat{\mathfrak{O}}_{\text{red}})^q$ is a ring isomorphism, $\hat{\mathfrak{O}}^q$ is normal.

Note that the Frobenius induces an injective local homomorphism $\mathfrak{O}^q \hookrightarrow \hat{\mathfrak{O}}^q$ which factors

$$\mathfrak{O}^q \hookrightarrow \widehat{\mathfrak{O}^q} \to \hat{\mathfrak{O}}^q$$

since $\hat{\mathfrak{O}}^q$ is complete. Now let N denote the normalization of $\mathfrak{O}^q$. Since $\hat{\mathfrak{O}}^q$ is normal we have

$$\begin{array}{ccccc} \mathfrak{O} & & \hookrightarrow & & \hat{\mathfrak{O}} \\ \cup & & & & \cup \\ \mathfrak{O}^q & \hookrightarrow & N & \hookrightarrow & \hat{\mathfrak{O}}^q. \end{array}$$

We claim that $N \subset \mathfrak{O}$. In fact, let a/b be an element of N, with a, b in $\mathfrak{O}^q$. Hence a/b is also in $\hat{\mathfrak{O}}^q$ and *a fortiori* in $\hat{\mathfrak{O}}$, i.e. b divides a in $\hat{\mathfrak{O}}$. But both a and b are also in $\mathfrak{O}$, so b divides a in $\mathfrak{O}$ by the faithful flatness of $\hat{\mathfrak{O}}$ over $\mathfrak{O}$. Q.E.D.

Of course in our situation where $\dim(\mathfrak{O}) = 1$, $\hat{\mathfrak{O}}_{\text{red}}$, $\hat{\mathfrak{O}}^q$ and N are discrete valuation rings, and we can obtain more precise information about the structure of N relative to that of $\mathfrak{O}$:

(2.2) *Let* K *denote the residue field of* $\mathfrak{O}$, *and let* x *be any element of* $\mathfrak{O}$ *with* $\nu_{\mathrm{m}}(x)=1$.

Then (i) $\hat{\mathfrak{O}}^q \xrightarrow{\sim} \mathrm{K}^q[[x^q]]$

and (ii) *The local inclusion* $\mathrm{N} \hookrightarrow \hat{\mathfrak{O}}^q$ *induces an isomorphism* $\hat{\mathrm{N}} \xrightarrow{\sim} \hat{\mathfrak{O}}^q$.

Proof. — Given x in $\mathfrak{O}$ with $\nu_{\mathrm{m}}(x)=1$, choose an isomorphism $\hat{\mathfrak{O}}_{\mathrm{red}} \xrightarrow{\sim} \mathrm{K}[[x]]$. Then, since $\hat{\mathfrak{O}}^q = (\hat{\mathfrak{O}}_{\mathrm{red}})^q$, we have:

$$\hat{\mathfrak{O}}^q \xrightarrow{\sim} \mathrm{K}^q[[x^q]],$$

which fits into a commutative diagram:

$$\begin{array}{ccc} \hat{\mathfrak{O}}_{\mathrm{red}} & \xrightarrow{\sim} & \mathrm{K}[[x]] \\ {\scriptstyle \mathrm{Frob}}\downarrow & & \downarrow{\scriptstyle \mathrm{Frob}} \\ \hat{\mathfrak{O}}^q & \xrightarrow{\sim} & \mathrm{K}^q[[x^q]] \\ \cap & & \cap \\ \hat{\mathfrak{O}}_{\mathrm{red}} & \xrightarrow{\sim} & \mathrm{K}[[x]]. \end{array}$$

For (ii), first note that since $\mathfrak{O}^q \hookrightarrow \mathrm{N} \hookrightarrow \hat{\mathfrak{O}}^q$ the residue field of N is K^q. Now since N is a discrete valuation ring and $\hat{\mathfrak{O}}^q$ is a domain, $\hat{\mathfrak{O}}^q$ is flat over N, so $\widehat{\hat{\mathfrak{O}}^q} = \hat{\mathfrak{O}}^q$ is flat over $\hat{\mathrm{N}}$, so $\hat{\mathrm{N}} \hookrightarrow \hat{\mathfrak{O}}^q$ is injective. Moreover, since x is in $\mathfrak{O}$, x^q is in $\mathfrak{O}^q \subset \mathrm{N} \subset \hat{\mathrm{N}}$, so $\hat{\mathrm{N}} \to \hat{\mathfrak{O}}^q$ is surjective in view of (i). Q.E.D.

(2.3) We now want to fatten N to obtain R as in Theorem 1. We first fix x in $\mathfrak{O}$ which becomes a regular parameter of $\hat{\mathfrak{O}}_{\mathrm{red}}$ as above. Let X be an indeterminate, and let

$$\mathrm{N}' = \mathrm{N}[\mathrm{X}]/(\mathrm{X}^q - x^q).$$

Then $\mathrm{X} \mapsto x$ defines a map

$$g : \mathrm{N}' \to \mathfrak{O}.$$

Let $\mathfrak{I} = \ker(g)$. Tensoring with $\hat{\mathrm{N}}$ over N, we obtain an exact sequence:

$$(0) \to \mathfrak{I} \otimes_{\mathrm{N}} \hat{\mathrm{N}} \to \mathrm{N}' \otimes_{\mathrm{N}} \hat{\mathrm{N}} \to \mathfrak{O} \otimes_{\mathrm{N}} \hat{\mathrm{N}}.$$

Now since $\hat{\mathrm{N}} \xrightarrow{\sim} \mathrm{K}^q[[x^q]]$, $\mathrm{N}' \otimes_{\mathrm{N}} \hat{\mathrm{N}} \xrightarrow{\sim} \mathrm{K}^q[[x]]$. On the other hand, since N' is finite over N, $\mathrm{N}' \otimes_{\mathrm{N}} \hat{\mathrm{N}} \xrightarrow{\sim} \hat{\mathrm{N}}'$. Hence N' is regular, and $\hat{\mathrm{N}}'$ is isomorphic to $\mathrm{K}^q[[x]]$. Consider now the composition

$$\mathrm{K}^q[[x]] = \mathrm{N}' \otimes_{\mathrm{N}} \hat{\mathrm{N}} = \hat{\mathrm{N}}' \xrightarrow{g^*} \mathfrak{O} \otimes_{\mathrm{N}} \hat{\mathrm{N}} \xrightarrow{\theta} \hat{\mathfrak{O}} \to \hat{\mathfrak{O}}_{\mathrm{red}} \xrightarrow{\sim} \mathrm{K}[[x]],$$

where θ is induced by the natural maps $\mathfrak{O} \to \hat{\mathfrak{O}}$ and $\hat{\mathrm{N}} \to \hat{\mathfrak{O}}$. Since this composition takes x to itself, it is injective, so also g^* is injective. Therefore $\mathfrak{I} \otimes_{\mathrm{N}} \hat{\mathrm{N}} = (0)$, so $\mathfrak{I} = (0)$

by faithful flatness. Thus: $N' = N[x]$ is a regular local subring of $\mathfrak{O}$, and the inclusion induces an injective map $\hat{N}' \hookrightarrow \hat{\mathfrak{O}}_{red}$ which fits into a commutative diagram

$$\begin{array}{ccccc} K^q[[x]] & \hookrightarrow & K[[x]] & & \\ \uparrow\wr & & \uparrow\wr & & \\ \hat{N}' & \hookrightarrow & \hat{\mathfrak{O}} & \rightarrow & \hat{\mathfrak{O}}_{red} \\ \uparrow & & \uparrow & & \\ N' & \hookrightarrow & \mathfrak{O} & & \end{array}$$

Now let $\{\bar{b}_a\}_{a\in A}$ be a p-base for K over K^p. Then the b_a are also a q-base for K over K^q, i.e. the set of all monomials of the form

$$\{\bar{b}_{a_1}^{n_1}\bar{b}_{a_2}^{n_2}\ldots\bar{b}_{a_j}^{n_j} \mid j\in\mathbf{Z}_+,\ 0\leq n_i\leq q-1\}$$

is a free base for K over K^q. In particular $K = K^q(\{\bar{b}_a\})$, and the irreducible equation of each $\bar{b}_a$ over K^q is $X^q - \bar{b}_a^q$.

Let $\{b_a\}$ be a set of representatives of the $\bar{b}_a$ in $\mathfrak{O}$, i.e. $\bar{b}_a \equiv b_a \pmod{\mathfrak{m}}$ for all a, and let $c^a = b_a^q$ in $\mathfrak{O}^q \subset N'$. Now define

$$R = N'[\{X_a\}_{a\in A}]/\{X_a^q - c_a\}$$

where the X_a are a system of indeterminates over N' indexed by A. We first note: R is regular, and $\hat{R} \xrightarrow{\sim} K[[x]]$, where the isomorphism is in the sense of N'-algebras. In fact, $R = \varinjlim R_S$ where the limit is taken over the inductive system of finite subsets S of A, and $R_S = N'[\{X_a\}_{a\in S}]/\{X_a^q - c_a\}$. Now each R_S is regular with parameter x:

$$\hat{R}_S \xrightarrow{\sim} R_S \otimes_{N'} \hat{N}' \underset{\hat{N}'}{\xrightarrow{\sim}} K^q(\{\bar{b}_a\}_{a\in S})[[x]].$$

Hence R is also regular. Namely, pick $y \in \max(R)$. Then y is in $\max(R_S)$ for some S, so x divides y in R_S and hence also in R. Moreover the residue field of

$$R = \varinjlim \mathrm{res}(R_S) = \varinjlim K^q(\{\bar{b}_a\}_{a\in S}) = K.$$

Hence $\hat{R} \underset{\hat{N}'}{\xrightarrow{\sim}} K[[x]]$ as asserted.

Now let $h: R \to \mathfrak{O}$ be the N'-algebra homomorphism defined by $X_a \mapsto b_a$. We claim that *h is injective.* To see this, simply observe that the above argument shows that the composition $\hat{R} \xrightarrow{\hat{h}} \hat{\mathfrak{O}} \to \hat{\mathfrak{O}}_{red}$ is an isomorphism, so that $\hat{h}$ is injective.

Hence we may view R *as a local subring of* $\mathfrak{O}$, *and the induced map* $\hat{R} \to \hat{\mathfrak{O}}_{red}$ *is an isomorphism*; *both are* $K[[x]]$. Now the composition

$$\mathfrak{O} \subset \hat{\mathfrak{O}} \to \hat{\mathfrak{O}}_{red}$$

is injective since $\mathfrak{O}$ is a domain. Hence, in view of the commutativity of the diagram

$$\begin{array}{ccccc} & & \overset{\approx}{\frown} & & \\ \hat{R} & \hookrightarrow & \hat{\mathfrak{O}} & \rightarrow & \hat{\mathfrak{O}}_{\text{red}} \\ \uparrow & & \uparrow & \nearrow & \\ R & \hookrightarrow & \mathfrak{O} & & \end{array}$$

we may view $R \subset \mathfrak{O} \subset \hat{R} \overset{\sim}{\rightarrow} K[[x]]$ in the sense of Theorem 1.

Definition **(2.3.1)**. — We will call the situation

$$R \hookrightarrow \mathfrak{O} \hookrightarrow \hat{R}$$

as above a *presentation of* $\mathfrak{O}$ *over* R.

Remark **(2.4)**. — If we want to give a theory only up to finitely many quadratic transforms, then we can make even stronger hypotheses on the presentation $R \hookrightarrow \mathfrak{O} \hookrightarrow \hat{R}$: If $\mathfrak{P}$ denotes the minimal prime of $\hat{\mathfrak{O}}$ (the kernel of the induced map $\hat{\mathfrak{O}} \rightarrow \hat{R}$), then we can assume that $\mathrm{Gr}_{\mathfrak{P}}(\hat{\mathfrak{O}})$ is *free* over $\hat{\mathfrak{O}}/\mathfrak{P} = \hat{R}$. (That this is achieved after finitely many quadratic transforms is an immeditae consequence of [1], chapter II (3.2).) In the terminology of Hironaka, this is expressed by saying that $\mathrm{Spec}(\hat{\mathfrak{O}})$ is *normally flat* along the subscheme defined by $\mathfrak{P}$, i.e. along the section (of $\mathrm{Spec}(\hat{\mathfrak{O}}) \rightarrow \mathrm{Spec}(\hat{R})$). If we think of $\mathrm{Spec}(\hat{\mathfrak{O}})$ as a family of (0-dimensional) singularities parametrized by $\mathrm{Spec}(\hat{R})$, it means that these singularities are *numerically equivalent*, i.e. they have the same Hilbert function.

We will not use this fact in the sequel, since our analysis has as its natural realm of application the class of those $\mathfrak{O}$ for which it is merely assumed that a presentation exists; their structure theory is not hampered by lack of such a normal flatness hypothesis. Thus we will not touch further on this point, except to suggest that any space which parametrizes the rings with presentation over R in a given birational equivalence class should be expected to have singularities at those points corresponding to those *finite* R-algebras which fail to satisfy the normal flatness hypothesis. The idea is that for generic S finite over R, the minimal number of generators of S as R-algebra should be no larger than the minimal number of generators of Q(S)/Q(R); however if S fails to satisfy the normal flatness hypothesis this need not be so.

3. BIRATIONAL STRUCTURE THEORY OVER A PRESENTATION

(3.0) We henceforth assume we are in the situation of a *presentation*:

(3.0.1)
$$R \underset{i}{\hookrightarrow} \mathfrak{O} \underset{j}{\hookrightarrow} \hat{R}$$

where of course $\mathfrak{O}$ is a local noetherian domain of dimension 1, R is a discrete valuation ring with $\mathfrak{O}$ purely inseparable and *flat* over R, and $j \circ i$ is the canonical homomorphism $R \rightarrow \hat{R}$. Moreover we assume $\hat{\mathfrak{O}}$ has a unique minimal prime ideal $\mathfrak{P}$, with $\hat{\mathfrak{O}}/\mathfrak{P}$ regular.

Upon completion, we obtain

$$\hat{R} \underset{\hat{i}}{\hookrightarrow} \hat{\mathfrak{O}} \underset{\hat{j}}{\rightarrow} \hat{R}$$

with $\hat{j} \circ \hat{i} = 1_{\hat{R}}$; then $\mathfrak{P} = \ker(\hat{j})$. We may also view $\mathfrak{P}$ as the inverse image of the generic point of $\mathrm{Spec}(\hat{R})$ by the morphism $\mathrm{Spec}(\hat{\mathfrak{O}}) \to \mathrm{Spec}(\hat{R})$ induced by $\hat{i}$, i.e.

$$\hat{\mathfrak{O}}_{\mathfrak{P}} = \hat{\mathfrak{O}} \otimes_{\hat{R}} Q(\hat{R}) \tag{3.0.2}$$

where as usual, Q denotes passage to the field of fractions. Consider for a moment the simplest case, when $\mathfrak{O}$ is *finite* over R so that $\hat{\mathfrak{O}} = \mathfrak{O} \otimes_R \hat{R}$. Combining this with (3.0.2) we obtain:

(3.0.3) When $\mathfrak{O}$ is a finite R-algebra

$$\hat{\mathfrak{O}}_{\mathfrak{P}} = Q(\mathfrak{O}) \otimes_{Q(R)} Q(\hat{R}).$$

Thus *in the case of finite* R-*algebras the formal fibre is a birational invariant.*

Our technique for analyzing the general case (when $\mathfrak{O}$ is not necessarily finite over R) is to approximate $\mathfrak{O}$ by a certain sequence $S_0 \subset S_1 \subset \ldots$ of birationally equivalent *finite* R-subalgebras of $\mathfrak{O}$. The fact that the formal fibres do not change in this sequence will enable us to get a good hold on the whole situation: we will be able to express $\hat{\mathfrak{O}}$ as a quotient of any of the $\hat{S}_i$ by an ideal which may be described precisely (3.4); this will also serve us in the *quasi-algebrization* procedure of § 6, as an essential part of the technique to construct $\mathfrak{O}$ with a given completion and presentation. The point is that the sequence (S_i) above may be defined in a canonical fashion, so that in the case when $[Q(\mathfrak{O}) : Q(R)] < \infty$ it characterizes $\mathfrak{O}$ uniquely (as well as $\hat{\mathfrak{O}}$); and in case $[Q(\mathfrak{O}) : Q(R)] = \infty$ (which may happen even when R is a *maximal presentation*, cf. § 4 and example of § 7) the technique of the sequence shows us at least how to construct R-subalgebras A of $\mathfrak{O}$ such that $[Q(A) : Q(R)] < \infty$ and $\hat{A} = \hat{\mathfrak{O}}$ (3.2).

(3.1) Beginning with (3.0.1), let $\mathfrak{m} = \max(\mathfrak{O})$, $\mathfrak{M} = \max(R)$. Choose a finite set of elements $f_1, \ldots, f_s$ in $\mathfrak{m}$ such that $(\mathfrak{M}, f_1, \ldots, f_s)\mathfrak{O} = \mathfrak{m}$. Let

$$S_0 = R[f_1, \ldots, f_s] \subset \mathfrak{O}.$$

It is clear that S_0 is a local R-subalgebra of $\mathfrak{O}$ with maximal ideal

$$\mathfrak{N}_0 = (\mathfrak{M}, f_1, \ldots, f_s) S_0,$$

$S_0 \hookrightarrow \mathfrak{O}$ induces an isomorphism of residue fields, and $\mathfrak{N}_0 \mathfrak{O} = \mathfrak{m}$. Moreover, since S_0 is integral over R, $\dim S_0 = 1$. We will need the following simple result:

Lemma **(3.1.1).** — *Let* $A \to B$ *be a local homomorphism which induces an isomorphism of residue fields. Let* $\mathfrak{M} = \max(A)$, $\mathfrak{N} = \max(B)$, *and suppose* $\mathfrak{M}B = \mathfrak{N}$. *Then, for every integer* $\nu > 0$, $A \to B/\mathfrak{N}^\nu$ *is surjective, and moreover* $\mathrm{Gr}_{\mathfrak{M}}(A) \to \mathrm{Gr}_{\mathfrak{N}}(B)$ *is surjective.*

Proof. — Since the residue fields are the same, if we let $\bar{A}$ denote the image of A in B, we have:

$$\begin{aligned} B &= \bar{A} + \mathfrak{N} = \bar{A} + \mathfrak{M}B \\ &= \bar{A} + \mathfrak{M}(\bar{A} + \mathfrak{M}B) = \bar{A} + \mathfrak{M}^2 B = \bar{A} + \mathfrak{N}^2 B \\ &= \vdots \\ &= \bar{A} + \mathfrak{N}^\nu B, \text{ etc.} \end{aligned}$$

Hence $\bar{A}/(\mathfrak{N}^\nu \cap \bar{A}) \xrightarrow{\sim} B/\mathfrak{N}^\nu B$ and $A \to \bar{A}/(\mathfrak{N}^\nu \cap \bar{A})$ is surjective.

For the second assertion, first note that since $A \to B/\mathfrak{N}^2$ is surjective, $\mathfrak{M}/\mathfrak{M}^2 \to \mathfrak{N}/\mathfrak{N}^2$ is surjective, so that if k denotes the common residue field:

$$\mathrm{Sym}_k(\mathfrak{M}/\mathfrak{M}^2) \to \mathrm{Sym}_k(\mathfrak{N}/\mathfrak{N}^2)$$

is surjective. But then, since in the commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}_k(\mathfrak{M}/\mathfrak{M}^2) & \longrightarrow & \mathrm{Sym}_k(\mathfrak{N}/\mathfrak{N}^2) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{\mathfrak{M}}(A) & \longrightarrow & \mathrm{Gr}_{\mathfrak{N}}(B) \end{array}$$

the vertical arrows are surjective, we get the result.

Note that as an immediate corollary to the lemma, we get:

(3.1.2) With the hypotheses of (3.1.1), $\hat{A} \to \hat{B}$ is surjective. In particular in our situation, *if* T *is any local subalgebra of* $\mathfrak{O}$ *containing* S_0, *then both* $\hat{T} \to \hat{\mathfrak{O}}$ *and*

$$\mathrm{Gr}_{\max(T)}(T) \to \mathrm{Gr}_{\mathfrak{m}}(\mathfrak{O})$$

are surjective.

Remark **(3.1.3)**. — Suppose $\mathfrak{O}$ is a local domain of characteristic 0, containing a discrete valuation ring R (with $R \hookrightarrow \mathfrak{O}$ a local homomorphism) such that R and $\mathfrak{O}$ have the same residue class field, and the *maximal ideal of* $\mathfrak{O}$ *is generated by elements* $f_1, \ldots, f_s$ *which are integral over* R. Then $\mathfrak{O}$ has dimension 1, and $\hat{\mathfrak{O}}$ is reduced (equivalently $\mathfrak{O}$ has finite normalization). Namely, let $S = R[f_1, \ldots, f_s] \subset \mathfrak{O}$. Then $\hat{S} \to \hat{\mathfrak{O}}$ is surjective by (3.1.1). Thus, since $\dim \mathfrak{O}$ is at least 1 by hypothesis, it must be precisely 1 ($\dim S = 1$ because S is integral over R). Moreover, the formal fibre of S is $Q(S) \otimes_{Q(R)} Q(\hat{R})$, since S is finite over R. But this is a direct sum of fields (because we are in char. 0). Thus by surjectivity, the same is true of the formal fibre of $\mathfrak{O}$.

(3.1.4) Returning to our situation (3.1), we observe that since S_0 is a finite purely inseparable extension of R, its formal fibre (3.0.3) is a *local* ring, so that S_0 has a *unique* minimal prime ideal $\mathfrak{P}_0$; of course, just as for $\mathfrak{O}$, $(\hat{S}_0)_{\mathrm{red}} \xrightarrow{\sim} \hat{R}$.

(3.2) We are going to use S_0 to construct a local R-subalgebra A of $\mathfrak{O}$ with the properties: $\hat{A} \xrightarrow{\sim} \hat{\mathfrak{O}}$ and $[Q(A):Q(R)]<\infty$. This will be accomplished by taking the normalization of S_0 and intersecting this with $\mathfrak{O}$. Moreover, by interpreting things in terms of the successive quadratic transforms of S_0, we will also express A as the limit of a sequence: $S_0 \subset S_1 \subset \dots$, which will prove to be an important invariant of the structure of $\mathfrak{O}$ relative to that of $\hat{\mathfrak{O}}$.

First observe that $\mathfrak{O}$ has a unique quadratic transform $\mathfrak{O}^{(1)}$, i.e. the exceptional fibre of $\mathrm{Bl}_{\mathfrak{m}}(\mathfrak{O}) \to \mathrm{Spec}(\mathfrak{O})$(*) has a unique closed point. In fact, this fibre is the same for $\mathfrak{O}$ as for $\hat{\mathfrak{O}}$, and $\hat{\mathfrak{O}}_{\mathrm{red}}$ is regular. Moreover, if x is a regular parameter of R, then it is also one for $\hat{\mathfrak{O}}_{\mathrm{red}} \xrightarrow{\sim} \hat{R}$. It follows that if $\hat{\mathfrak{O}}^{(1)}$ is the unique quadratic transform of $\hat{\mathfrak{O}}$, $\mathfrak{m}\hat{\mathfrak{O}}^{(1)}=(x)\hat{\mathfrak{O}}^{(1)}$, and hence also $\mathfrak{m}\mathfrak{O}^{(1)}=(x)\mathfrak{O}^{(1)}$. This is the same as saying that $\mathfrak{O}^{(1)}=\mathfrak{O}[\mathfrak{m}/x]$, i.e. the $\mathfrak{O}$-subalgebra of $\mathfrak{O}_x$ generated by all f/x, f in $\mathfrak{m}$. Now by (3.1.4) exactly the same argument applies to S_0, so that $S_0^{(1)}=S_0[N_0/x]$ is its unique quadratic transform. Clearly

$$S_0^{(1)}=S_0[N_0/x] \subset \mathfrak{O}[\mathfrak{m}/x]=\mathfrak{O}^{(1)}.$$

Applying the same argument inductively we obtain a diagram of quadratic sequences:

$$\begin{array}{ccccccccc} S_0 & \longrightarrow & S_0^{(1)} & \longrightarrow & \dots & \longrightarrow & S_0^{(i)} & \longrightarrow & \dots \\ \cap & & \cap & & & & \cap & & \\ \mathfrak{O} & \longrightarrow & \mathfrak{O}^{(1)} & \longrightarrow & \dots & \longrightarrow & \mathfrak{O}^{(i)} & \longrightarrow & \dots \end{array}$$

such that, if $\mathfrak{N}^{(i)}$ denotes $\max(S_0^{(i)})$, then for all i we have $\mathfrak{N}^{(i)}S_0^{(i)}=(x)S_0^{(i)}$ (and the analogous statement is of course true for the $\mathfrak{O}^{(i)}$'s).

Now let $S_i=S_0^{(i)} \cap \mathfrak{O}$, and let $\mathfrak{N}_i=\max(S_i)$. Let

(3.2.1) $$A=\bigcup_{i=0}^{\infty} S_i.$$

Note that if we let $S=\bigcup_{i=0}^{\infty} S_0^{(i)}$, we may also express

(3.2.2) $$A=S \cap \mathfrak{O},$$

and S is a discrete valuation ring with parameter x (the normalization of S_0).

Let us first check that $A \hookrightarrow \mathfrak{O}$ induces an isomorphism $\hat{A} \xrightarrow{\sim} \hat{\mathfrak{O}}$. Let $\mathfrak{N}=\max(A)$. Since $S_0 \hookrightarrow A \hookrightarrow \mathfrak{O}$, by lemma (3.1.2) all the maps

$$A \to \mathfrak{O}/\mathfrak{m}^{\nu}$$

are surjective, and hence for every ν

$$A/(\mathfrak{m}^{\nu} \cap A) \xrightarrow{\approx} \mathfrak{O}/\mathfrak{m}^{\nu}.$$

(*) $\mathrm{Bl}_{\mathfrak{m}}(\mathfrak{O})$ denotes the blowing up of $\mathrm{Spec}(\mathfrak{O})$ with center $\mathfrak{m}$.

Therefore it suffices to show that the topology on A defined by the ideals $\mathfrak{m}^\nu \cap \mathrm{A}$ is equivalent to the $\mathfrak{N}$-adic topology, i.e. that for every ν there exists a μ such that

$$(*) \qquad \mathfrak{m}^\mu \cap \mathrm{A} \subset \mathfrak{N}^\nu.$$

To see this, choose $j>0$ such that $\mathfrak{m}^j \subset (x)\mathfrak{O}$ (dim $\mathfrak{O}=1$, so $(x)\mathfrak{O}$ is $\mathfrak{m}$-primary). Then it is obvious that if $f \in \mathfrak{m}^{j\nu}$, f is divisible by x^ν. Thus if we can prove

$$(**) \qquad f \in \mathrm{A},\ x \mid f \text{ in } \mathfrak{O} \text{ implies } x \mid f \text{ in } \mathrm{A},$$

then $(*)$ follows, letting $\mu = j\nu$. But from (3.2.2) we see that if $f \in \mathrm{A}$, and $f = xg$ with g in $\mathfrak{O}$, then $g \in \mathrm{S}$ (since S is a discrete valuation ring with parameter x), so that also $g \in \mathrm{A}$. This completes the verification of the fact that $\hat{\mathrm{A}} \xrightarrow{\sim} \hat{\mathfrak{O}}$.

To show A is noetherian, we use a similar device: Let the integer j be as above. Then

$$\mathfrak{N}^j \subset \mathfrak{m}^j \cap \mathrm{A} \subset (x)\mathfrak{O} \cap \mathrm{A} \subset (x)\mathrm{A}$$

where the last inclusion is in virtue of $(**)$. Now since $\hat{\mathrm{A}} = \hat{\mathfrak{O}}$, $\mathrm{A}/\mathfrak{N}^j = \mathfrak{O}/\mathfrak{m}^j$, so that $\mathfrak{N} \equiv (x, f_1, \ldots, f_s)\mathrm{A} \pmod{\mathfrak{N}^j}$ (because $\mathfrak{m} = (x, f_1, \ldots, f_s)\mathfrak{O}$). But then since $\mathfrak{N}^j \subset (x)\mathrm{A}$, $\mathfrak{N} = (x, f_1, \ldots, f_s)\mathrm{A}$, i.e. the maximal ideal of A is finitely generated. On the other hand, $\mathfrak{N}$ is the only non-zero prime ideal of A (since for example A is integral over the discrete valuation ring R). Thus we can conclude by the following result of Cohen ([2], Chapter 1, Theorem (3.4)) : *A ring is noetherian if and only if every prime ideal has a finite basis.*

We finally note that Q(A) is finite over Q(R), simply because $\mathrm{Q}(\mathrm{A}) = \mathrm{Q}(\mathrm{S}_0)$.

(3.2.3) *Suppose, with the notation as above, that the* $f_1, \ldots, f_s$ *(of* (3.1)*) generate* $\mathrm{Q}(\mathfrak{O})$ *over* Q(R). *Then* $\mathrm{A} = \mathfrak{O}$. In fact, in this case $\mathrm{Q}(\mathrm{A}) = \mathrm{Q}(\mathrm{S}_0) = \mathrm{Q}(\mathfrak{O})$, and by the above results $\hat{\mathrm{A}} = \hat{\mathfrak{O}}$. Now take z in $\mathfrak{O}$, say $z = g/h$ with g and h in A. Thus h divides g in $\hat{\mathfrak{O}} = \hat{\mathrm{A}}$. But then h divides g in A (by faithful flatness), i.e. z is in A.

(3.3) The heart of the matter is now to interpret the structure of $\hat{\mathfrak{O}}$ in the case when $\mathfrak{O}$ is not necessarily finite over R (although $\mathrm{Q}(\mathfrak{O})/\mathrm{Q}(\mathrm{R})$ may be finite). The problem is to understand how the *ring theoretic structure* of $\mathfrak{O}/\mathrm{R}$ in this case modifies what would be expected from merely the *birational* data $\mathrm{Q}(\mathfrak{O})/\mathrm{Q}(\mathrm{R})$. When the latter is a finite extension, for example, by (3.0.3) the " birationally expected " formal fibre is just $\mathrm{Q}(\mathfrak{O}) \otimes_{\mathrm{Q}(\mathrm{R})} \mathrm{Q}(\hat{\mathrm{R}})$, but the actual formal fibre of $\mathfrak{O}$ will be a quotient of this by an ideal which expresses the *way* in which $\mathfrak{O}$ fails to be a finite R-algebra; of course, this is just the generic version of a similar statement about the relationship of $\mathfrak{O} \otimes_{\mathrm{R}} \hat{\mathrm{R}}$ and $\hat{\mathfrak{O}}$. For the rest of this section, we will retain the notations and hypotheses of (3.1) and (3.2).

We first observe that since for all i $\mathrm{S}_0 \subset \mathrm{S}_i \subset \mathfrak{O}$, the induced maps $\hat{\mathrm{S}}_i \to \hat{\mathrm{A}} = \hat{\mathfrak{O}}$ are surjective (3.1.3).

Lemma **(3.3.1).** — *Let* $\{S_i\}$ *be an inductive system of local rings whose limit is a local ring* A. *Let* $\mathfrak{N}_i = \max(S_i)$, *and* $\mathfrak{N} = \max(A)$. *Suppose that the induced maps of graded algebras* $\mathrm{Gr}_{\mathfrak{N}_i}(S_i) \xrightarrow{\rho_i} \mathrm{Gr}_{\mathfrak{N}}(A)$ *are all surjective (or equivalently that all the maps* $\hat{S}_i \xrightarrow{\gamma_i} \hat{A}$ *are surjective). Then*

(1) *Let* $G = \varinjlim \mathrm{Gr}_{\mathfrak{N}_i}(S_i)$. *Then the maps* ρ_i *induce an (obviously surjective) map* $\rho : G \to \mathrm{Gr}_{\mathfrak{N}}(A)$ *which is an isomorphism.*

(2) *Let* $L = \varinjlim \hat{S}_i$. *Then the maps* γ_i *induce an (obviously surjective) map* $\gamma : L \to \hat{A}$, *which in turn induces an isomorphism* $\mathrm{Gr}_{\mathfrak{L}}(L) \xrightarrow{\sim} \mathrm{Gr}_{\hat{\mathfrak{N}}}(\hat{A})$, *where* $\mathfrak{L} = \max(L)$.

Proof. — For (1), let z be a homogeneous element of G such that $\rho(z) = 0$, and let $z_i \in \mathrm{Gr}_{\mathfrak{N}_i}(S_i)$ (for a suitable i) represent z. Say $\nu = \deg(z_i) = \deg(z)$. Let $f \in S_i$ such that $\mathrm{In}_{\mathfrak{N}_i}(f) = z_i$. Since $\rho(z) = 0$, the image of z_i in $\mathrm{Gr}_{\mathfrak{N}}(A)$ by ρ_i is 0. This means that $\nu_{\mathfrak{N}}(f) > \nu = \nu_{\mathfrak{N}_i}(f)$. But then since $A = \varinjlim S_i$, for some $j > i$, $\nu_{\mathfrak{N}_j}(f) > \nu$, so that the image z_j of z_i in $\mathrm{Gr}_{\mathfrak{N}_j}(S_j)$ is 0. Hence also $z = 0$, which completes the proof. (Note that the inverse map $\rho^{-1} : \mathrm{Gr}_{\mathfrak{N}}(A) \to G$ may be obtained as follows: Let $w \in \mathrm{Gr}_{\mathfrak{N}}(A)$. Let $f \in A$ such that $w = \mathrm{In}_{\mathfrak{N}}(f)$, and choose an S_i such that $f \in \mathfrak{N}_i$. Then $\rho^{-1}(w) =$ image of $\mathrm{In}_{\mathfrak{N}_i}(f)$ in G.)

For (2), let $\mathfrak{L} = \max(L)$, and let α denote the map of graded algebras induced by γ, i.e.

$$\alpha : \mathrm{Gr}_{\mathfrak{L}}(L) \to \mathrm{Gr}_{\mathfrak{N}}(A).$$

We will prove α is injective:

Choose z in $\mathrm{Gr}_{\mathfrak{L}}(L)$, say $\deg(z) = \nu$, and $z = \mathrm{In}_{\mathfrak{L}}(f)$, $f \in L$. Choose a representative f_i of f in some $\hat{S}_i$, so that also $\nu_{\hat{\mathfrak{N}}_i}(f_i) = \nu$. In fact, write f in L as a sum of products $gx_1 \ldots x_\nu$ with all the $\nu_{\mathfrak{L}}(x_k) = 1$. Choose an i such that all the g's and x_k's are represented by elements $g_{(i)}$ and $x_{k(i)}$ of $\hat{S}_i$, and let f_i denote the corresponding sum of the products $g_{(i)} x_{1(i)} \ldots x_{\nu(i)}$. Then f_i represents f, and since $\hat{S}_i \to L$ is a local homomorphism (so that the $\nu_{\hat{\mathfrak{N}}_i}(x_{k(i)})$ are all $\geqslant 1$) we have $\nu_{\mathfrak{N}_i}(f_i) \geqslant \nu$, and hence is equal to ν.

Then $\alpha(z)$ is the image of f_i in $\hat{\mathfrak{N}}^\nu / \hat{\mathfrak{N}}^{\nu+1}$ (via $\hat{S}_i \to \hat{A}$). Now by (1) $\hat{\mathfrak{N}}^\nu / \hat{\mathfrak{N}}^{\nu+1} = \varinjlim \hat{\mathfrak{N}}_i^\nu / \hat{\mathfrak{N}}_i^{\nu+1}$. Hence if $\alpha(z) = 0$, we must have $\nu_{\hat{\mathfrak{N}}_j}(f_i) > \nu$ for some $j > i$, which is a contradiction since then $\nu_{\mathfrak{L}}(f) > \nu$. Q.E.D.

We remark that *the ring* L *need not be noetherian.*

Corollary **(3.3.2).** — *With notations and hypotheses of* (3.3.1):

$$\mathrm{Ker}\, \gamma = \bigcap_{\nu=0}^{\infty} \mathfrak{L}^\nu.$$

Proof. — Suppose $f \in \bigcap_{\nu=0}^{\infty} \mathfrak{L}^\nu$. Then $\gamma(f) \in \bigcap_{\nu=0}^{\infty} \mathfrak{N}^\nu$. But this ideal is (0), since $\hat{A}$ is noetherian. Thus $f \in \mathrm{Ker}\, \gamma$. Conversely, if $f \in \mathrm{Ker}\, \gamma$, then $\mathrm{In}_{\mathfrak{L}}(f) = 0$ since γ induces an isomorphism of graded algebras by (3.3.1), so that $f \in \bigcap_{\nu=0}^{\infty} \mathfrak{L}^\nu$.

We are now in a good position to analyze the structure of the surjective $\hat{R}$-algebra homomorphisms $\hat{S}_i \to \hat{A}$ in our situation. We first observe that since the S_i are flat and finite over R, and the maps $S_i \to S_j (j > i)$ are injective, the same is true after passing to completions, i.e.

(3.3.3) The $\hat{S}_i$ are flat and finite over $\hat{R}$, and the maps $\beta_{ij} : \hat{S}_i \to \hat{S}_j (j \geq i)$ are injective. Moreover, by (3.0.3), we get:

(3.3.4) The β_{ij} induce isomorphisms

$$\hat{S}_i \otimes_{\hat{R}} Q(\hat{R}) \overset{\approx}{\to} \hat{S}_j \otimes_{\hat{R}} Q(\hat{R}).$$

In fact we can express $\hat{S}_i \otimes_{\hat{R}} Q(\hat{R})$ as $S_i \otimes_R \hat{R} \otimes_{\hat{R}} Q(\hat{R}) = Q(S_i) \otimes_{Q(R)} Q(\hat{R})$. But the S_i's are birational, so that all these are just $F \otimes_K E$ where $F = Q(S_i)$ for any i, $K = Q(R)$, $E = Q(\hat{R})$. In other words, all the S_i have the same formal fibre. It follows that also

(3.3.5) $$L \otimes_{\hat{R}} Q(\hat{R}) = F \otimes_K E.$$

Namely, $L = \varinjlim \hat{S}_i$, so $L \otimes_{\hat{R}} Q(\hat{R}) = \varinjlim (\hat{S}_i \otimes_{\hat{R}} Q(\hat{R}))$.
The key technical result is now

(3.3.6) *In our situation,* $\ker(\gamma) = \bigcap_{\nu=0}^{\infty} (t)^{\nu} L$, *where t is any regular parameter of* R (*or* $\hat{R}$).

Proof. — Let $\tilde{\mathfrak{P}}$ denote the minimal prime ideal of L (if $\mathfrak{P}_i =$ the minimal prime of $\hat{S}_i$, $\tilde{\mathfrak{P}} = \bigcup_i \mathfrak{P}_i$). Then $L_{\tilde{\mathfrak{P}}} = L \otimes_{\hat{R}} Q(\hat{R}) = F \otimes_K E$ by (3.3.5). Thus although L may not be noetherian, $L_{\tilde{\mathfrak{P}}}$ is noetherian, so that $\tilde{\mathfrak{P}}^n L_{\tilde{\mathfrak{P}}} = (0)$ for some n. Now L is flat over $\hat{R}$, being the union of the flat $\hat{S}_i$'s. Hence if t is a regular parameter of R (or $\hat{R}$), $L \to L_t$ is injective. But clearly $L_{\tilde{\mathfrak{P}}} = L_t$. Hence $\tilde{\mathfrak{P}}^n = (0)$ in L. Now

$$\mathfrak{Q} = \max(L) = (t)L + \tilde{\mathfrak{P}}.$$

Hence for $\nu \gg 0$, $\mathfrak{Q}^{\nu} \subset (t)^{\nu - n} L$, so that $\bigcap_{\nu=0}^{\infty} \mathfrak{Q}^{\nu} = \bigcap_{\nu=0}^{\infty} (t)^{\nu} L$. Combining this with (3.3.2) we get the result.

(3.4) We now summarize the main results of this § 3:

Let $R \hookrightarrow \mathfrak{O} \hookrightarrow \hat{R}$ with the hypotheses of (3.0.3). Let t be a regular parameter of R, and let $f_1, \ldots, f_s$ be any elements of $\mathfrak{O}$ which along with t generate $\max(\mathfrak{O})$. Let $S_0 = R[f_1, \ldots, f_s] \subset \mathfrak{O}$, and let S_i denote the intersection with $\mathfrak{O}$ of the (unique) i^{th} iterated quadratic transform $S^{(i)}$ of S_0. If $A = \bigcup_{i=0}^{\infty} S_i$ then A is noetherian, $[Q(A) : Q(R)] < \infty$, $\hat{A} = \hat{\mathfrak{O}}$ (and $A = \mathfrak{O}$ if $Q(\mathfrak{O}) = Q(R)(f_1, \ldots, f_s)$). Moreover, let $\gamma_i : \hat{S}_i \to \hat{A} = \hat{\mathfrak{O}}$ be the map induced by the inclusion $S_i \subset \mathfrak{O}$. Then γ_i is surjective for all i. Let $\mathfrak{J}_i = \ker(\gamma_i)$, and let $\beta_{ij} : \hat{S}_i \to \hat{S}_j$ be the (injective) map of completions induced by $S_i \hookrightarrow S_j$. Then (by (3.3.6)):

$$\mathfrak{J}_i = \{f \in S_i \mid \text{given } \nu \text{ there is a } j > i \text{ such that } t^\nu \mid \beta_{ij}(f).\}$$

$$R[f_1, \ldots, f_s] = S_0 \subset S_1 \subset \ldots \subset S_i \subset S_j \subset \overset{\lim}{\overrightarrow{\ldots}} \subset A \subset \mathfrak{O} \quad \text{(over R)}$$

$$\begin{array}{ccccccccccc}
 & & & & 0 & & & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & \mathfrak{J}_i & & & & & & \\
 & & & & \downarrow & & & & & & \\
\hat{S}_0 & \hookrightarrow & \hat{S}_1 & \hookrightarrow \ldots & \hat{S}_i & \overset{\beta_{ij}}{\hookrightarrow} & \hat{S}_j & \hookrightarrow & \overset{\lim}{\overrightarrow{\ldots}} & \subset L & \text{(over } \hat{R}) \\
 & & & & \gamma_i \downarrow & & & \swarrow & & & \\
 & & 0 & \swarrow & \hat{A} = \hat{\mathfrak{O}} & & & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array}$$

(3.4.1) For the sequel, we need to note that all that is required for this analysis is the sequence $(S_i)_i$ of local R-algebra homomorphisms with the properties:

(i) The S_i are finite and flat over R and are all birationally equivalent.

(ii) For every j the map $S_j \to \varinjlim_i S_i$ induces a surjection of completions.

In other words, the hypothesis that there exists an $\mathfrak{O}$, given *a priori*, with $S_i = S^{(i)} \cap \mathfrak{O}$, plays no role. Thus if we are given any sequence (S_i) as above satisfying (i) and (ii), then we can define $\mathfrak{O} = \varinjlim S_i$, and the same conclusions hold: For every i $\hat{\mathfrak{O}} = \hat{S}_i / \mathfrak{J}_i$ (with notations as above).

4. d-THEORY AND MAXIMAL PRESENTATIONS

(4.1) We consider the general situation of a presentation (3.0.1) :

$$R \overset{i}{\hookrightarrow} \mathfrak{O} \overset{j}{\hookrightarrow} \hat{R} \qquad (j \circ i \text{ is the canonical map})$$

which yields upon completion the commutative diagram

$$\text{(D)} \qquad \begin{array}{ccccc}
\hat{R} & \overset{\hat{i}}{\hookrightarrow} & \hat{\mathfrak{O}} & \overset{\hat{j}}{\longrightarrow} & \hat{R} = \hat{\mathfrak{O}}_{\text{red}} \quad (\hat{j} \circ \hat{i} = 1_{\hat{R}}) \\
\uparrow & & \uparrow \alpha & & \| \\
R & \overset{i}{\hookrightarrow} & \mathfrak{O} & \overset{j}{\hookrightarrow} & \hat{R}
\end{array}$$

where the vertical arrows are the canonical inclusions. In particular we have *two* local homomorphisms α and β from $\mathfrak{O}$ to $\hat{\mathfrak{O}}$, where $\beta = \hat{i} \circ j$. Let $d = \alpha - \beta : \mathfrak{O} \to \hat{\mathfrak{O}}$.

Proposition **(4.1.1).** — (i) $\mathrm{Im}(d) \subset \mathrm{Ker}(\hat{j}) = \mathfrak{P}$.

(ii) $d(fg) = \alpha(f)d(g) + \beta(g)d(f)$.

(iii) $d(\mathrm{R}) = 0$.

(iv) *d is* R-*linear.*

(v) *Let* $\mathrm{R}' = \ker(d)$. *Then* R′ *is a discrete valuation ring and* $\mathrm{R} \subset \mathrm{R}'$ *induces an isomorphism of completions.*

Proof. — (i) follows from the commutativity of the diagram (D) above, remembering that $\hat{j} \circ \hat{i}$ is the identity of $\hat{\mathrm{R}}$.

(ii) is a simple computation based solely on the fact that d is the difference of the two ring homomorphisms α and β.

(iii) results also from the commutativity of (D), remembering that $j \circ i$ is the canonical inclusion.

(iv) follows immediately from (ii) and (iii).

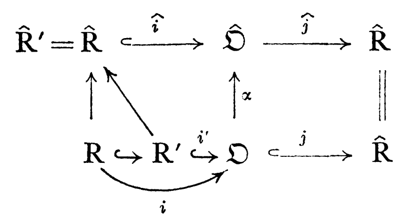
(v) We first note that (ii) and (iii) imply that R′ is an R-subalgebra of $\mathfrak{O}$. Moreover R′ is local: to see this, suppose g is in R′ and is also a *unit* in $\mathfrak{O}$. Then

$$0 = d(1) = d(g.g^{-1}) = \alpha(g)d(g^{-1}) + \beta(g^{-1})d(g) = \alpha(g)d(g^{-1}).$$

But then $d(g^{-1}) = 0$ (because $\hat{\mathfrak{O}}$ is flat over $\mathfrak{O}$), so g^{-1} is in R′.

Now let g be an element of $\max(\mathrm{R}') = \mathfrak{m} \cap \mathrm{R}'$, where $\mathfrak{m} = \max(\mathfrak{O})$. Then $\alpha(g) = \beta(g)$. Let t be a regular parameter of R. Now $t \mid j(g)$ in $\hat{\mathrm{R}}$, so $t \mid \beta(g)$ in $\mathfrak{O}$, hence also $t \mid \alpha(g)$ in $\mathfrak{O}$. Then by faithful flatness $t \mid g$ in $\mathfrak{O}$, say $g = tf$, f in $\mathfrak{O}$. We claim f is in R′. For this, note $0 = d(g) = d(tf) = td(f)$ (by (iv)); hence, since $\mathfrak{O}$ is flat over R, $d(f) = 0$. Thus we have shown: the maximal ideal of R′ is generated by t. This concludes the proof.

(4.2) Suppose now that $\mathrm{R} \hookrightarrow \mathfrak{O} \hookrightarrow \hat{\mathrm{R}}$ and $\mathrm{R}' \hookrightarrow \mathfrak{O} \hookrightarrow \hat{\mathrm{R}}' = \hat{\mathrm{R}}$ are two presentations, with $\mathrm{R} \subset \mathrm{R}'$. Then we get a commutative diagram



which is compatible with the identification of $\hat{\mathrm{R}}$ and $\hat{\mathrm{R}}'$. We conclude that the operator d' defined for R′ just as d was defined for R in (4.1) *coincides* with d. Thus the operator d is really an invariant of the *lattice of presentations of* $\mathfrak{O}$ *containing the given* R. By (v) of the proposition (4.1.1) we find that this lattice contains a *maximal* element, say R′, characterized simply as the kernel of d. Such an R′ is called a *maximal presentation* of $\mathfrak{O}$; its existence of course follows from the existence of a presentation (§ 2) as well as the above proposition. The question of whether there exists a *minimal* R which induces the given d

is interesting; I don't know the answer, however it is easy to see (as shown below) that every *maximal* presentation contains $\mathfrak{O}^q$ where q is some sufficiently high power of the characteristic p.

Remark **(4.2.1).** — Suppose we are given $R \hookrightarrow \mathfrak{O} \hookrightarrow \hat{R}$ with the usual hypotheses, except that we do not assume *a priori* any inseparability. Since the definition of d above does not depend on inseparability, it makes sense in this more general situation, and we can find a maximal $R' \supset R$ as above. But then $\mathfrak{O}$ is automatically purely inseparable over R'. In fact, choose a power q of p (= the characteristic) sufficiently large that $z^q = 0$ for all z in $\mathfrak{P}$ (the nilpotent prime ideal of $\mathfrak{O}$). Now if x is in $\mathfrak{O}$,

$$d(x^q) = (\alpha - \beta)(x^q) = d(x)^q = 0$$

(since $\operatorname{Im}(d) \subset \mathfrak{P}$ by (i) of (4.1.1)). Hence x^q is in R'.

(4.3) The operator d is closely related to the universal differential operator of $\mathfrak{O}/R$ of order $\leqslant \infty$. To see this, recall (3.3.6) that there is a canonical surjective homomorphism

$$L = \mathfrak{O} \otimes_R \hat{R} \xrightarrow{\gamma} \hat{\mathfrak{O}} \to 0$$

whose kernel is the " non-noetherian part " of L. Let

$$\nabla : \mathfrak{O} \to \mathfrak{O} \otimes_R \hat{R}$$

be the R-module homomorphism defined by $\nabla(x) = x \otimes 1 - 1 \otimes x$ (to make sense of $1 \otimes x$ we use the fact that $\mathfrak{O} \underset{j}{\hookrightarrow} \hat{R}$). Then $d = \gamma \circ \nabla$. In fact, we may write L in the form $(\mathfrak{O} \otimes_R \mathfrak{O}) \otimes_{\mathfrak{O}} \hat{R}$; via this identification, for any x in $\mathfrak{O}$,

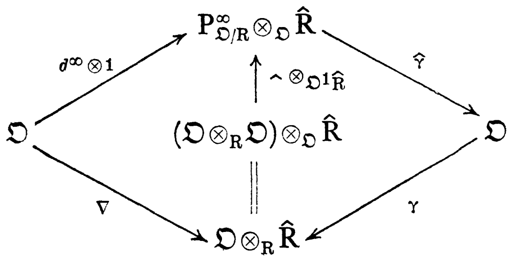
$$\gamma(x \otimes 1 \otimes 1) = \alpha(x), \quad \text{and} \quad \gamma(1 \otimes x \otimes 1) = \hat{i} \circ j(x) = \beta(x)$$

(with the notations of the diagram (D) above). In other words, we may view the map γ as induced by

$$\gamma_0 : \mathfrak{O} \otimes_R \mathfrak{O} \to \hat{\mathfrak{O}}$$

with $\gamma_0(x) = \alpha(x)\,\beta(x)$; γ is then obtained from γ_0 by viewing $\hat{\mathfrak{O}}$ as $\hat{R}$-module via $\hat{i}$, which is compatible with the structure of $\hat{\mathfrak{O}}$ as module over the $\mathfrak{O}$ in the right hand factor via α.

Let $\mathfrak{J}$ denote the diagonal ideal of $\mathfrak{O} \otimes_R \mathfrak{O}$; $\mathfrak{J}$ is generated by all $x \otimes 1 - 1 \otimes x$, x in $\mathfrak{O}$. Since $\gamma_0(\mathfrak{J}) \subset \mathfrak{P}$, and $\hat{\mathfrak{O}}$ is trivially $\mathfrak{P}$-adically complete, γ_0 factors naturally through the $\mathfrak{J}$-adic completion of $\mathfrak{O} \otimes_R \mathfrak{O}$, denoted $P^{\infty}_{\mathfrak{O}/R}$ ([5], (16.3) ff). Consequently we get a factorization $\hat{\gamma}$ of γ through $P^{\infty}_{\mathfrak{O}/R} \otimes_{\mathfrak{O}} \hat{R}$, and a commutative diagram



where $d^\infty(x)=x\otimes 1-1\otimes x\in P^\infty_{\mathfrak{O}/R}$ (" universal differential operator of order $\leqslant\infty$ " *(loc. cit.)*); we thus obtain the canonical expression of d as a differential operator:

$$d=\hat{\gamma}\circ(d^\infty\otimes 1).$$

We remark that the only reason for having to use $P^\infty_{\mathfrak{O}/R}$ here (rather than $P^N_{\mathfrak{O}/R}$, $N<\infty$) is the possibility that $Q(\mathfrak{O})$ is *infinite* over $Q(R)$, which can happen even if $R\hookrightarrow\mathfrak{O}$ is a *maximal presentation.* (We will give an example of this, but since it requires quasi-algebrization it is postponed until § 7.) For suppose $Q(\mathfrak{O})$ is finite over $Q(R)$, say generated by $x_1, \ldots, x_n$ in $\mathfrak{O}$. Then every u in $\mathfrak{O}$ satisfies $t^m u=F(x)$ for some integer m and some polynomial F in the x_i with coefficients in R, where t is a parameter of R (this follows from (3.2.3)). In particular,

$$t^m d^\infty(u)=d^\infty(F(x))=\sum_{\substack{\nu=(\nu_1,\ldots,\nu_n)\\ \nu_i>0}}\left(\frac{1}{\nu!}\partial^\nu F/\partial x^\nu\right)(d^\infty(x))^\nu$$

(the usual Taylor expansion; since we are in characteristic p we must be careful to interpret $\left(\frac{1}{\nu!}\partial^\nu F/\partial x^\nu\right)$ to mean that we first divide formally by $\nu!$ as though we were over $\mathbf{Z}$, and *then* reduce modulo p). Hence since $\hat{\mathfrak{J}}=P\mathfrak{J}^\infty_{\mathfrak{O}/R}$ is generated by the $d^\infty(u)$, u in $\mathfrak{O}$, we get:

$$(*)\qquad \hat{\mathfrak{J}}\subset\bigcup_m((d^\infty(x_1), \ldots, d^\infty(x_n))\,P^\infty_{\mathfrak{O}/R}:t^m).$$

Now there is a power q of p such that x_i^q is in R for all $i=1, \ldots, n$. Hence $d^\infty(x_i)^q$ $(=d^\infty(x_i^q))$ is 0 for all i. Hence, for $N>nq$, $(d^\infty(x_1), \ldots, d^\infty(x_n))\,P^\infty_{\mathfrak{O}/R}=(0)$. It follows from $(*)$ above that also $\hat{\mathfrak{J}}^N=(0)$, since $\mathfrak{O}\otimes_R\mathfrak{O}$ is flat over R (because $\mathfrak{O}$ is). Hence

$$P^\infty_{\mathfrak{O}/R}=P^N_{\mathfrak{O}/R}\underset{\text{def}}{=}(\mathfrak{O}\otimes_R\mathfrak{O})/\mathfrak{J}^{N+1}=\mathfrak{O}\otimes_R\mathfrak{O}.$$

Thus in this case we could equally well describe d as $\gamma\circ(d^N\otimes 1)$ where $d^N:\mathfrak{O}\to P^N_{\mathfrak{O}/R}$ is the universal differential operator of $\mathfrak{O}$ over R of order $\leqslant N$. Note that if $\mathfrak{O}$ is a finite R-module, the map γ is an isomorphism, so that we may regard $\hat{\mathfrak{O}}=P^N_{\mathfrak{O}/R}\otimes_{\mathfrak{O}}\hat{R}$ and then $d=d^n\otimes 1_{\hat{R}}$.

(4.4) We now want to study the relationship of the operator d with the normalization of $\mathfrak{O}$. The ideas here are inspired by the recent work of Ferrand and Raynaud [3]; in fact we include here a free presentation of a certain part of the contents of § 2 of that work which are relevant to our situation. They show that (regardless of characteristic and independent of questions of presentation) there is a differentia operator d' defined on the normalization of $\mathfrak{O}$, which determines $\mathfrak{O}$ completely. This operator depends on a choice of a *section* $\hat{\mathfrak{O}}_{\text{red}}\to\hat{\mathfrak{O}}$, which may of course be very " non-algebraic ". However if the section arises from a *presentation* (i.e. the map $\hat{i}$ of the diagram (D) at the beginning of § 4), we will show that the resulting d' induces our operator d, and so in particular the maximal presentation R corresponding to d is the kernel of d'.

In this regard it is interesting to note that frequently the normalization of a non-excellent $\mathfrak{O}$ is an excellent discrete valuation ring. This will be the case, for example, when in the situation of a presentation $R \hookrightarrow \mathfrak{O} \hookrightarrow \hat{R}$, $Q(\mathfrak{O})$ is the *inseparable closure* of $Q(R)$ in $Q(\hat{R})$.

We begin with the following hypotheses:

(4.4.1) *$\mathfrak{O}$ is a local domain of dimension 1,* $B =$ *the normalization of* $\mathfrak{O}$. *We will assume* B *is local (*i.e. *$\mathfrak{O}$ is unibranch) and $\hat{\mathfrak{O}}_{\mathrm{red}}$ is regular. Denote* $\mathfrak{m} = \max(\mathfrak{O})$ *and* $\mathfrak{n} = \max(B)$.

Since $\mathfrak{O} \to \hat{\mathfrak{O}}_{\mathrm{red}}$ is injective and $\hat{\mathfrak{O}}_{\mathrm{red}}$ is regular and hence normal, we may view $\mathfrak{O} \hookrightarrow B \hookrightarrow \hat{\mathfrak{O}}_{\mathrm{red}}$. From this we deduce:

(4.4.2) $\mathfrak{O} \hookrightarrow B$ induces an isomorphism of residue fields, and $\mathfrak{m}B = \mathfrak{n}$. In particular there is an element t of $\mathfrak{O}$ such that $(t)B = \mathfrak{n}$, and this t has the property: every iterated quadratic transform of $\mathfrak{O}$ is obtained by suitable divisions by t.

Let $\alpha : \mathfrak{O} \to \hat{\mathfrak{O}}$ be the canonical homomorphism, and $v : B \to B \otimes_{\mathfrak{O}} \hat{\mathfrak{O}}$ be $v(b) = b \otimes 1$. Then

(4.4.3) The diagram

$$\begin{array}{ccc} \mathfrak{O} & \longrightarrow & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle v} \\ \hat{\mathfrak{O}} & \longrightarrow & B \otimes_{\mathfrak{O}} \hat{\mathfrak{O}} \end{array}$$

is *cartesian*, i.e. it identifies $\mathfrak{O}$ with the ring-theoretic fibre product of $\hat{\mathfrak{O}}$ and B over $B \otimes_{\mathfrak{O}} \hat{\mathfrak{O}}$.

Proof. — It suffices to check that α and v induce an *isomorphism* $B/\mathfrak{O} \xrightarrow{\approx} (B \otimes_{\mathfrak{O}} \hat{\mathfrak{O}})/\hat{\mathfrak{O}}$. For this, first note that the term on the right is just $(B/\mathfrak{O}) \otimes_{\mathfrak{O}} \hat{\mathfrak{O}}$, and $B/\mathfrak{O}$ is a t-torsion $\mathfrak{O}$-module. Hence $B/\mathfrak{O}$ is the union of a family of finite length $\mathfrak{O}$-modules B_i, each of which is of course already complete. Hence

$$(B/\mathfrak{O}) \otimes_{\mathfrak{O}} \hat{\mathfrak{O}} = (\bigcup_i B_i) \otimes_{\mathfrak{O}} \hat{\mathfrak{O}} = \bigcup_i (B_i \otimes_{\mathfrak{O}} \hat{\mathfrak{O}}) = \bigcup_i B_i = B/\mathfrak{O}.$$

Now by the structure theorems of Cohen we can choose a subring $\hat{R}$ of $\hat{\mathfrak{O}}$ such that the composition $\hat{R} \hookrightarrow \hat{\mathfrak{O}} \to \hat{\mathfrak{O}}_{\mathrm{red}}$ is an isomorphism. (We remark that if $\mathfrak{O}$ is of characteristic p we can take the R of a presentation of $\mathfrak{O}$.) This gives a section σ of the natural projection $\hat{\mathfrak{O}} \to \hat{\mathfrak{O}}_{\mathrm{red}}$; use it to get a decomposition $\tau : \hat{\mathfrak{O}} \xrightarrow{\approx} \hat{R} \oplus \mathfrak{P}$, where $\mathfrak{P}$ is the nilpotent prime ideal of $\hat{\mathfrak{O}}$ ($\hat{R}$-module decomposition) (1). We also identify $\hat{B}$ with $\hat{R}$, in view of the fact that $\hat{\mathfrak{O}}_{\mathrm{red}} \xrightarrow{\approx} \hat{B}$ is obviously an isomorphism.

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(1) The ring operations are of course given by the multiplicative structure of $\mathfrak{P}$ together with its $\hat{R}$-module structure.

(4.4.4) Let $\mathfrak{P}'=\mathfrak{P}\otimes_{\mathfrak{O}}\mathrm{Q}(\mathfrak{O})$. Then there is an isomorphism

$$\theta : \mathrm{B}\otimes_{\mathfrak{O}}\hat{\mathfrak{O}}\overset{\approx}{\to}\hat{\mathrm{R}}\oplus\mathfrak{P}'$$

such that the diagram

$$\begin{array}{ccc} \hat{\mathfrak{O}} & \longrightarrow & \mathrm{B}\otimes_{\mathfrak{O}}\hat{\mathfrak{O}} \\ {\scriptstyle\tau}\downarrow\approx & & \approx\downarrow{\scriptstyle\theta} \\ \hat{\mathrm{R}}\oplus\mathfrak{P} & \longrightarrow & \hat{\mathrm{R}}\oplus\mathfrak{P}' \end{array}$$

is commutative, where the upper map is just $x\mapsto 1\otimes x$, and the lower one is induced by the natural map $\mathfrak{P}\to\mathfrak{P}'$ [1].

Proof. — We can write $\mathrm{B}=\varinjlim\mathfrak{O}_i$, where the $\mathfrak{O}_i$ are the successive quadratic transforms of $\mathfrak{O}$. Hence $\mathrm{B}\otimes_{\mathfrak{O}}\hat{\mathfrak{O}}=\varinjlim\hat{\mathfrak{O}}_i$, where $\hat{\mathfrak{O}}_i=\mathfrak{O}_i\otimes_{\mathfrak{O}}\hat{\mathfrak{O}}$ are the successive quadratic transforms of $\hat{\mathfrak{O}}$. Let $\mathfrak{P}_i$ denote the nilpotent ideal of $\hat{\mathfrak{O}}_i$ (so that $\mathfrak{P}=\mathfrak{P}_0$). Then since the discrete valuation subring $\hat{\mathrm{R}}$ of $\hat{\mathfrak{O}}$ is invariant under quadratic transform, we get compatible decompositions

$$\begin{array}{ccc} \hat{\mathfrak{O}}_{i+1} & \underset{\approx}{\overset{\tau_{i+1}}{\longrightarrow}} & \hat{\mathrm{R}}\oplus\mathfrak{P}_{i+1} \\ \uparrow & & \uparrow \\ \hat{\mathfrak{O}}_i & \underset{\approx}{\overset{\tau_i}{\longrightarrow}} & \hat{\mathrm{R}}\oplus\mathfrak{P}_i \end{array}$$

where the right-hand homomorphism is induced by $1_{\hat{\mathrm{R}}}$ and the natural map $\mathfrak{P}_i\to\mathfrak{P}_{i+1}$. Hence it suffices to show that $\varinjlim\mathfrak{P}_i=\mathfrak{P}'$. Let t be as in (4.4.2). Since $\mathrm{Q}(\mathfrak{O})=\mathfrak{O}_t$, if we denote $\mathfrak{P}_t=\mathfrak{P}\otimes_{\mathfrak{O}}\mathfrak{O}_t$, what we want is that

$$\varinjlim\mathfrak{P}_i=\mathfrak{P}_t.$$

Note that since $\mathfrak{O}$ is a domain and $\mathfrak{O}_{i+1}\subset(\mathfrak{O}_i)_t=\mathfrak{O}_t$, by applying $\otimes_{\mathfrak{O}}\hat{\mathfrak{O}}$ we find that $\hat{\mathfrak{O}}_i\hookrightarrow\hat{\mathfrak{O}}_{i+1}$ is injective and $\hat{\mathfrak{O}}_i\subset\hat{\mathfrak{O}}_t$ for all i. Now since $\mathfrak{P}_i$ and $\mathfrak{P}_{i+1}$ are the nilpotent ideals of $\hat{\mathfrak{O}}_i$ and $\hat{\mathfrak{O}}_{i+1}$, $\mathfrak{P}_{i+1}$ is the *strict transform* of $\mathfrak{P}_i$ in $\hat{\mathfrak{O}}_{i+1}$ (§ 1), and in particular $\mathfrak{P}_i\subset(t)\,\mathfrak{P}_{i+1}$. By iteration, we get

(*) $$\mathfrak{P}\subset(t^i)\,\mathfrak{P}_i.$$

On the other hand, the image of $\mathfrak{P}_i$ in $\hat{\mathfrak{O}}_t$ by the composition $\mathfrak{P}_i\subset\hat{\mathfrak{O}}_i\subset\hat{\mathfrak{O}}_t$ is clearly contained in the image of the inclusion $\mathfrak{P}_t\subset\hat{\mathfrak{O}}_t$ (obtained by applying $\otimes_{\mathfrak{O}}\mathfrak{O}_t$ to $\mathfrak{P}\subset\hat{\mathfrak{O}}$). Thus we have:

(**) $$\mathfrak{P}_i\subset\mathfrak{P}_t.$$

The result follows immediately from (*) and (**).

[1] Note $\mathfrak{P}\to\mathfrak{P}'$ is injective, since $\hat{\mathfrak{O}}$ is torsion-free over $\mathfrak{O}$.

In view of (4.4.3) and (4.4.4) we have a *cartesian* diagram

$$\text{(E)}\qquad \begin{array}{ccc} \mathfrak{O} & \hookrightarrow & B \\ {\scriptstyle \alpha}\downarrow & \square & \downarrow{\scriptstyle v=u+d'} \\ \hat{R}\oplus\mathfrak{P} & \hookrightarrow & \hat{R}\oplus\mathfrak{P}' \end{array}$$

Viewing $\hat{R}\overset{\approx}{\to}\hat{B}$, we may write $v=u+d'$, where $u : B\to\hat{B}=\hat{R}$ is the canonical map, and d' is a differential operator from B to $\mathfrak{P}'$ (the difference of two ring homomorphisms). It follows from (E) that

$$\textbf{(4.4.5)}\qquad \mathfrak{O}=\{x \text{ in } B \mid d'(x) \text{ is in } \mathfrak{P}\}.$$

(4.5) The above discussion is valid with no hypothesis on the characteristic of $\mathfrak{O}$, and it is clear that the differential operator d' depends only on the choice of the section $\sigma : \hat{\mathfrak{O}}_{\text{red}}\to\hat{\mathfrak{O}}$, i.e. on the choice of $\hat{R}$; of course, since this section may be chosen arbitrarily, it might have nothing to do with the arithmetic structure of $\mathfrak{O}$. However, suppose we start with a *presentation* $R\hookrightarrow\mathfrak{O}\hookrightarrow\hat{R}$, which we may as well assume to be *maximal*. We can use this to get a section σ, i.e. $\sigma=\hat{i}$ (of the diagram (D) of (4.1)), and also a differential operator $d : \mathfrak{O}\to\mathfrak{P}$ as in (4.1), canonically associated to the presentation. Let d' denote the operator $B\to\mathfrak{P}'$ arising from σ as in (E). Then an inspection of (E) reveals that *d is the restriction of d' to $\mathfrak{O}$*. Thus in characteristic p we can summarize as follows:

(4.5.1) *Let $\mathfrak{O}$ be a local (noetherian) domain, of dimension 1, char. p, unibranch, with $\hat{\mathfrak{O}}_{\text{red}}$ regular. Let* B *denote the normalization of $\mathfrak{O}$, $\mathfrak{P}$ the nilpotent prime ideal of $\hat{\mathfrak{O}}$, and $\mathfrak{P}'=\mathfrak{P}\otimes_{\mathfrak{O}}Q(\mathfrak{O})$. Then there is a maximal presentation $R\hookrightarrow\mathfrak{O}\hookrightarrow\hat{R}$ with the associated differential operator $d : \mathfrak{O}\to\mathfrak{P}$ of* (4.1), *and a differential operator $d' : B\to\mathfrak{P}'$, such that the diagrams*

$$\begin{array}{ccccc} R & \hookrightarrow & \mathfrak{O} & \hookrightarrow & B \\ \downarrow & \square & \downarrow{\scriptstyle d} & \square & \downarrow{\scriptstyle d'} \\ (0) & \hookrightarrow & \mathfrak{P} & \hookrightarrow & \mathfrak{P}' \end{array}$$

are cartesian.

Remark **(4.6)**. — The problem of finding a local domain $\mathfrak{O}$ *with a given completion* may be posed as a " converse " of the above results (neglecting questions of presentation) in the following way: Let $\hat{R}$ be a complete discrete valuation ring. Let C be a flat, augmented $\hat{R}$-algebra of finite type, of the form $\hat{R}\oplus\mathfrak{P}$ where $\mathfrak{P}$ is a nilpotent ideal (flat as $\hat{R}$-module). Let $\mathfrak{P}'=\mathfrak{P}\otimes_{\hat{R}}Q(\hat{R})$, and let $B'=\hat{R}\oplus\mathfrak{P}'$ (with its natural ring structure). Then we ask: does there exist a discrete valuation subring B of $\hat{R}$, with completion isomorphic to $\hat{R}$, and a homomorphism $v : B \to B'$ such that if $\mathfrak{D}$ denotes the fibre product in the cartesian diagram

$$\begin{array}{ccc} C \times_{B'} B = \mathfrak{D} & \xrightarrow{pr_1} & B \\ {\scriptstyle pr_2} \downarrow & & \downarrow {\scriptstyle v} \\ C & \hookrightarrow & B' \end{array}$$

(where the bottom map is the natural inclusion $\hat{R} \oplus \mathfrak{P} \hookrightarrow \hat{R} \oplus \mathfrak{P}'$), then $\mathfrak{D}$ *is a local ring with normalization* B *and completion* C (via pr_1 and pr_2)? This is the approach of Ferrand and Raynaud *(loc. cit.)*; they give an affirmative answer in the special case $\mathfrak{P}^2 = (0)$ in characteristic 0 and over certain fields of characteristic p. The technique involves the existence of the differential operator d' (actually a derivation in this case). One would hope that the same approach, suitably extended, would yield the general result (for arbitrary $\mathfrak{P}$) in characteristic 0. In characteristic p, however, the problem is solved by quasi-algebrization: the idea is to view the map $C \to C_{red} = \hat{R}$ as being induced by a *formal p-section* of affine space over a suitable discrete valuation ring R (§ 4 and 6); the procedure has the structure of a (purely inseparable) presentation built in.

5. SOME EXAMPLES OF FORMALLY IMPERFECT DISCRETE VALUATION RINGS; SCHMIDT RINGS

As we have seen, in characteristic p all non-excellent curve singularities arise from inseparability in an extension $R \hookrightarrow \hat{R}$, for some discrete valuation ring R; in this case we say that R is *formally imperfect*. We want to describe an easy method of constructing these R, beginning with any " geometric " discrete valuation ring R_0. In fact, the construction itself is of a geometric nature, and in particular it is unrelated to any question of " ground-field " structure. In a certain sense it generalizes the example of F. K. Schmidt (e.g. as reported by Zariski in [4]); hence the name *Schmidt ring* for those rings which arise in this manner. We will not consider here problems of classification of formally imperfect R; our purpose is only to indicate their relative abundance and in particular to insure that we have enough raw material for the quasi-algebrization of § 6. In contrast to the Schmidt rings, we will also recall a classic example of Nagata and a more recent one of Hironaka, in which the formal imperfectness depends on ground field structure in an essential way.

(5.1) Let R_0 be a discrete valuation ring of char. p such that $\hat{R}_0$ has infinite transcendence degree over R_0. This is not always true; in fact in the example of Nagata below the completion is even integral over the original ring. However it holds when R_0 is *geometric*, i.e. the local ring of a point (of codimension 1) on an algebraic scheme over a field k. (To see this we first note that $\mathrm{card}(Q(R_0)) = \mathrm{card}(k)$ if k is infinite, or $\aleph_0$

if k is finite, and the cardinality of the algebraic closure of $Q(R_0)$ is the same. But $\mathrm{card}(\hat{R}_0)$ is at least $\mathrm{card}(k)^{\aleph_0}$, which gives the result.)

Now, given $n>0$, choose elements $f_1, \ldots, f_n$ in $\hat{R}_0$ which are algebraically independent over R_0; let $e_1, \ldots, e_n$ be any positive integers, and let $g_i = f^{p^{e_i}}$ in $\hat{R}_0$, $i = 1, \ldots, n$. We view these $g_1, \ldots, g_n$ as defining a *formal section* σ of affine n-space over R_0 (which we call a "*formal p-section*" for obvious reasons):

$$\begin{array}{ccc} \mathrm{Spec}(R_0[X_1, \ldots, X_n]) = \mathbf{A}^n_{R_0} & \longleftarrow & \mathbf{A}^n_{\hat{R}_0} \\ \downarrow & \square & \downarrow \uparrow \sigma : X_i \mapsto g_i \\ \mathrm{Spec}(R_0) & \longleftarrow & \mathrm{Spec}(\hat{R}_0) \end{array}$$

We then define a discrete valuation ring R, called the *Schmidt ring* of (R_0, σ) in any of the following equivalent ways:

(i) Via the composition

$$R_0[X_1, \ldots, X_n] \longrightarrow \hat{R}_0[X_1, \ldots, X_n] \xrightarrow{X_i \mapsto g_i} \hat{R}_0$$

the formal section σ induces a discrete valuation of the function field of $\mathbf{A}^n_{R_0}$; let R be its valuation ring.

(ii) There is a unique infinite sequence

$$(*) \qquad \mathbf{A}^n_{\hat{R}_0} = \hat{Z}^{(0)} \xleftarrow{\pi_1} \hat{Z}^{(1)} \leftarrow \ldots \leftarrow \hat{Z}^{(j-1)} \xleftarrow{\pi_j} \hat{Z}^{(j)} \leftarrow \ldots$$

of iterated quadratic transforms with the following property: let $\hat{z}_0$ be the point $(g_1(0), \ldots, g_n(0))$ in the closed fibre of $\hat{Z}^{(0)}$. If $\hat{z}_{j-1} \in \hat{Z}^{(j-1)}$ is the center of π_j, then the strict transform of (the image of) σ in $\hat{Z}^{(j-1)}$ passes through $\hat{z}_{j-1}$. Let

$$\{\pi_j : Z^{(j)} \to Z^{(j-1)};\ Z^{(0)} = \mathbf{A}^n_{R_0}\}$$

be the unique sequence of quadratic transforms from which $(*)$ is deduced by the base extension $\mathrm{Spec}(\hat{R}_0) \to \mathrm{Spec}(R_0)$. Since the exceptional fibres in either sequence are identical, each point $\hat{z}_j$ in $\hat{Z}^{(j)}$ corresponds to a unique point z_j in $Z^{(j)}$ (so that the sequence π_j could equally well be described as that obtained by blowing up the successive points z_j). Then

$$R = \bigcup_{j=0}^{\infty} \mathcal{O}_{Z^{(j)}, z_j}.$$

(R = the local ring of the closed point on the " Zariski-Riemann space " of $\mathbf{A}^n_{R_0}$ determined by the sequence π_j.)

(iii) Write

$$g_i = \sum_{j=0}^{\infty} a_{ij} t^j$$

where the a_{ij} are units in R_0 and t is a regular parameter; then for every $m \geqslant 0$ let

$$g_{im} = \sum_{j \leq m} a_{ij} t^j.$$

Then R may be described as the R_0-subalgebra of $\hat{R}_0$ generated by all elements of the form

$$\frac{g_i - g_{im}}{t^{m+1}}$$

for $i = 1, \ldots, n$ and all $m \geqslant 0$.

It is clear that t is also a regular parameter of R, and that R_0 and R have the same residue field. Hence $R_0 \subset R$ induces an isomorphism of completions. It follows that R is formally imperfect; in fact we have f_i in $\hat{R}$, $f_i^{p^{e_i}}$ in R, but f_i is not in R (since $Q(R) = Q(R_0)(g_1, \ldots, g_n)$).

The following will be a convenient way of expressing the consequences of our construction of Schmidt rings:

(5.1.1) Let R_0 be a discrete valuation ring of char. p, such that $\hat{R}_0$ has infinite transcendence degree over R_0. Then for any integers $n, e_1, \ldots, e_n$ there exists a discrete valuation ring $R \supset R_0$ with $\hat{R}_0 \overset{\approx}{\leftarrow} \hat{R}$, and elements $f_1, \ldots, f_n$ in $\hat{R}$, such that, if we denote

$$S = R[f_1, \ldots, f_n] \subset \hat{R},$$

then S is R-isomorphic (via $X_i \mapsto f_i$) to

$$R[X_1, \ldots, X_n]/(X_i^{p^{e_i}} - g_i)_{1 \leqslant i \leqslant n}.$$

Moreover $Q(R) = Q(R_0)(g_1, \ldots, g_n)$, with the g_i algebraically independent over $Q(R_0)$.

(5.1.2) Note that with the terminology above, S is a finite R-algebra, so that

$$\hat{S} = S \otimes_R \hat{R} = R[Y_1, \ldots, Y_n]/(Y_i^{p^{e_i}})_{1 \leqslant i \leqslant n}$$

(letting $Y_i = X_i - f_i$ in $\hat{R}[X]$).

(5.2) We now give two examples, which, in contrast to the Schmidt rings, show how formal imperfectness can arise from specific properties of a ground field.

(1) Nagata (cf. [2], Appendix E 3.1 for details). — Let k be a field such that $[k : k^p] = \infty$, and let $R = k^p[[t]][k] \subset k[[t]]$. R may be described as the subring of $k[[t]]$ consisting of all those power series whose coefficients generate a *finite extension of* k^p. It is not hard to check that $\hat{R} = k[[t]]$, so that $\hat{R}^p \subset R$.

(2) Hironaka. — Let F denote the prime field, and let $u = \{u_i\}$, $i = 1, 2, \ldots$ be a countable system of algebraically independent elements over F. Let k denote the algebraic closure of $F(u)$. For every $n \geqslant 0$, let $F(u)_n$ denote the subfield

$$F(u)(u_1^{1/p^n}, u_2^{1/p^{n-1}}, \ldots, u_n^{1/p})$$

of k, and let k_n be the *separable closure* of $F(u)_n$. It is clear that $k_n \subset k_{n+1}$, and that $\bigcup_n k_n = k$. Then let $R \subset k[[t]]$ be the subring consisting of all those power series whose coefficients lie in *some* k_n (the n may be different for different power series in R). As in example (1), $\hat{R} = k[[t]]$. Now let (e_i) be a sequence of integers all of which are bounded by some integer N_0, and let $f = \sum_i u_i^{1/p^{e_i}} t^i$ in $\hat{R}$. Then f^N is in R for some $N \geq N_0$; but if infinitely many of the e_i are positive, f is not in R. In fact, the inseparable closure of R in $\hat{R}$ may be described as the ring of all power series whose coefficients generate an extension of $F(u)$ whose inseparable part is of bounded *height* over $F(u)$. Notice that in this example $\hat{R}$ still has infinite transcendence degree over R.

6. QUASI-ALGEBRIZATION

(6.0) Suppose C is the completion of a local domain $\mathfrak{O}$ of dimension 1 which comes with a presentation $R \hookrightarrow \mathfrak{O} \hookrightarrow \hat{R}$, as in (3.0.1). The "abstract" hypotheses satisfied by C are then

(6.0.1) $\hat{R}$ is a complete discrete valuation ring of char. p and C is a flat finite $\hat{R}$-algebra with nilpotent ideal $\mathfrak{P}$ such that $C/\mathfrak{P} \xrightarrow{\approx} \hat{R}$.

It is clear that (6.0.1) is equivalent to either of the following:

(6.0.2) $\hat{R}$ as in (6.0.1), and $C = \hat{R}[Y_1, \ldots, Y_n]/\mathfrak{J}$, flat over $\hat{R}$, and $\mathfrak{J}$ contains the ideal $(Y)^N$ for some N.

(6.0.3) $\hat{R}$ the same, $\operatorname{Spec}(C) \xrightarrow{\pi} \operatorname{Spec}(\hat{R})$ is flat of relative dimension 0, with connected fibres, and has a section σ.

A converse to the above is the following result:

Theorem **(6.0.4)**. — *Given* C *and* $\hat{R}$ *satisfying* (6.0.1), *there exists a local (noetherian) domain* $\mathfrak{O}$, *with a presentation* $R \underset{i}{\hookrightarrow} \mathfrak{O} \underset{j}{\hookrightarrow} \hat{R}$, *and an isomorphism* $\theta : \hat{\mathfrak{O}} \xrightarrow{\approx} C$ *such that the diagram*

$$\begin{array}{ccccc} \hat{R} & \xrightarrow{\pi} & C & \xrightarrow{\sigma} & \hat{R} \\ \| & & \theta \downarrow \wr & & \| \\ \hat{R} & \xrightarrow[\hat{i}]{} & \hat{\mathfrak{O}} & \xrightarrow[\hat{j}]{} & \hat{R} \end{array}$$

is commutative, where π *and* σ *correspond to the ones in* (6.0.3) *and* $\hat{i}, \hat{j}$ *arise from the presentation as in* (3.0.1).

Quasi-algebrization is a canonical procedure for constructing rings with a given completion; we have not attempted in this paper to describe the limits of its domain of application, but rather have restricted ourselves to giving a treatment in a setting appropriate to the situation at hand: purely inseparable R-subalgebras of $\hat{R}$, where R is a discrete valuation ring. In particular, we will get a proof of Theorem (6.0.4). For this, given the data (6.0.1) and a "*sufficiently*" *formally imperfect discrete valuation ring* R (whose completion is $\hat{R}$), we construct $\mathfrak{O}/R$ satisfying the conclusions of (6.0.4) by starting with a suitable *finite* R-subalgebra S of $\hat{R}$, purely inseparable over R, and then realize $\mathfrak{O}$ by an *infinite sequence of birational operations on* S (in such a way, however, that the result is noetherian), using the results of § 3 (especially (3.7.1)) as our guide. Thus, although the resulting $\mathfrak{O}$ is not a finite R-algebra, $Q(\mathfrak{O})$ is nevertheless a finite (purely inseparable) extension of $Q(R)$, so that $\operatorname{Spec}(\mathfrak{O})$ is a "quasi-algebraic" $\operatorname{Spec}(R)$-scheme. For the Theorem (6.0.4) the point is that we can *always* find R as above, for example in the form of a suitable *Schmidt ring* (§ 5).

(6.1) Preparation.

Suppose R is a discrete valuation ring with regular parameter t, and let $f_1, \ldots, f_n$ be elements of $\hat{R}$ which are purely inseparable over R, say $f_i^{p^{e_i}} = g_i$ in R. Let $S = R[f_1, \ldots, f_n] \subset \hat{R}$. Then via $X_i \mapsto f_i$ we have an isomorphism

(6.1.1) $$S \overset{\approx}{\to} R[X_1, \ldots, X_n]/\mathfrak{J}$$

where $\mathfrak{J}$ is an ideal of R[X] containing the ideal $\mathfrak{H}$ generated by the $X_i^{p^{e_i}} - g_i$, $i = 1, \ldots, n$. By subtracting a unit in R if necessary from each of the f_i, we may suppose that the f_i and g_i are non units (in $\hat{R}$ and R respectively). Now write, for each i

(6.1.2) $$f_i = \sum_{s=1}^{\infty} a_{is} t^s$$

where the a_{is} are *units in* R. Then of course

$$g_i = \sum_{s=1}^{\infty} a_{is}^{p^{e_i}} t^{sp^{e_i}}.$$

We will use the following terminology in the sequel: for every $\nu \geq 0$, and $i = 1, \ldots, n$ let

$$f_{i(\nu)} \underset{\text{def}}{=} \sum_{s=1}^{\nu} a_{is} t^s,$$

and $$g_i^{(\nu)} \underset{\text{def}}{=} (g_i - f_{i(\nu)}^{p^{e_i}}) / t^{\nu p^{e_i}} = (g_i - \sum_{s=1}^{\nu} a_{is}^{p^{e_i}} t^{sp^{e_i}}) / t^{\nu p^{e_i}}.$$

Note that the $g_i^{(\nu)}$ are in $(t^{p^{e_i}})R$.

(6.1.3) With notations and assumptions as above, the ν^{th} iterated quadratic transform $S^{(\nu)}$ of S is of the form

$$S^{(\nu)} = R[X_1^{(\nu)}, \ldots, X_n^{(\nu)}] / \mathfrak{J}^{(\nu)},$$

where

$$X_i^{(\nu)} = (X_i - \sum_{s=1}^{\nu} a_{is} t^s)/t^\nu = (X_i - f_{i(\nu)})/t^\nu$$

and $\mathfrak{J}^{(\nu)}$ contains the ideal $\mathfrak{H}^{(\nu)}$ generated by $(X_i^{(\nu)})^{p^{e_i}} - g_i^{(\nu)}$, $i = 1, \ldots, n$.

(We note that these successive quadratic transforms are unique, since S is unibranch; namely, as usual, the inseparability of S over R implies that $\hat{S} = S \otimes_R \hat{R}$ has a unique minimal prime ideal.)

Proof of (6.1.3). — Setting $S = S^{(0)}$, $X_i = X_i^{(0)}$, $g_i = g_i^{(0)}$, $\mathfrak{J} = \mathfrak{J}^{(0)}$, and $\mathfrak{H} = \mathfrak{H}^{(0)}$, the assertion is trivial for $\nu = 0$. Now assume it is true for $\nu \geqslant 0$. Since $\mathfrak{J}^{(\nu)}$ contains $\mathfrak{H}^{(\nu)}$, it is clear that the only maximal ideal of $R[X_1^{(\nu)}, \ldots, X_n^{(\nu)}]$ which contains $\mathfrak{J}^{(\nu)}$ is the one generated by t and the $X_i^{(\nu)}$; we denote this maximal ideal by $\mathfrak{M}^{(\nu)}$. Let $G = \mathrm{Gr}_{\mathfrak{M}^{(\nu)}}(R[X_1^{(\nu)}, \ldots, X_n^{(\nu)}])$. No power of $\mathrm{In}_{\mathfrak{M}^{(\nu)}}(t)$ (the $\mathfrak{M}^{(\nu)}$-initial form of t in G) is in $\mathrm{In}_{\mathfrak{M}^{(\nu)}}(\mathfrak{J}^{(\nu)})$ (the ideal of G generated by the initial forms of all elements in $\mathfrak{J}^{(\nu)}$). Otherwise, since this ideal contains the initial forms of elements of $\mathfrak{H}^{(\nu)}$, we would get:

$$\dim(S^{(\nu)}) = \dim(\mathrm{Gr}_{\mathfrak{M}^{(\nu)}}(S^{(\nu)})) = \dim(G/\mathrm{In}_{\mathfrak{M}^{(\nu)}}(\mathfrak{J}^{(\nu)})) = 0,$$

a contradiction, since $S^{(\nu)}$, being integral over R, has dimension 1. It follows from the elementary local theory of monoidal transforms (cf. [1], Chapter 0, § 3 for a summary) that

$$(*) \qquad S^{(\nu+1)} = R[X_1^{(\nu)}/t, \ldots, X_n^{(\nu)}/t]/\mathfrak{J}^{(\nu+1)}$$

where $R[X_1^{(\nu)}/t, \ldots, X_n^{(\nu)}/t]$ is the affine ring of the open piece of the blowing up of $\mathfrak{M}^{(\nu)}$ in $\mathrm{Spec}(R[X_1^{(\nu)}, \ldots, X_n^{(\nu)}])$ corresponding to those tangential directions where $t \neq 0$, and $\mathfrak{J}^{(\nu+1)}$ denotes the ideal of the strict transform of $\mathfrak{J}^{(\nu)}$ on this piece *(loc. cit.)*.

Note that by definition $X_i^{(\nu+1)} = (X_i^{(\nu)}/t) - a_{i,\nu+1}$. Hence we may use these as coordinates, and express $(*)$ equally well in the form

$$(**) \qquad S^{(\nu+1)} = R[X_1^{(\nu+1)}, \ldots, X_n^{(\nu+1)}]/\mathfrak{J}^{(\nu+1)}.$$

It remains to show that $\mathfrak{J}^{(\nu+1)}$ contains $\mathfrak{H}^{(\nu+1)}$. For this, since $\mathfrak{J}^{(\nu)} \supset \mathfrak{H}^{(\nu)}$, if $\mathfrak{H}^{(\nu)\prime}$ denotes the strict transform of $\mathfrak{H}^{(\nu)}$ in $R[X_1^{(\nu)}/t, \ldots, X_n^{(\nu)}/t]$, then of course $\mathfrak{J}^{(\nu+1)} \supset \mathfrak{H}^{(\nu)\prime}$. Hence it suffices to show that $\mathfrak{H}^{(\nu)\prime} \supset \mathfrak{H}^{(\nu+1)}$ (actually, it is even true that $\mathfrak{H}^{(\nu)\prime} = \mathfrak{H}^{(\nu+1)}$). To see this, we first note that each of the polynomials $X_i^{(\nu)p^{e_i}} - g_i^{(\nu)}$ which generate $\mathfrak{H}^{(\nu)}$ is of order p^{e_i} with respect to $\mathfrak{M}^{(\nu)}$; in fact, $g_i^{(\nu)}$ has order at least p^{e_i} in t. Hence $\mathfrak{H}^{(\nu)\prime}$ contains the elements $(X_i^{(\nu)p^{e_i}} - g_i^{(\nu)})/t^{p^{e_i}}$. But if we express this in terms of the coordinates $X_i^{(\nu+1)}$ (as in $(**)$), we get the right thing, namely:

$$\begin{aligned}(X_i^{(\nu)p^{e_i}} - g_i^{(\nu)})/t^{p^{e_i}} &= X_i^{(\nu+1)p^{e_i}} + a_{i,\nu+1}^{p^{e_i}} - (g_i^{(\nu)}/t^{p^{e_i}}) \\ &= X_i^{(\nu+1)p^{e_i}} - g_i^{(\nu+1)}.\end{aligned}$$

(The first equality is because $X_i^{(\nu+1)} = X_i^{(\nu)}/t - a_{i,\nu+1}$, and the second because

$$g_i^{(\nu+1)} = g_i^{(\nu)}/t^{p^{e_i}} - a_{i,\nu+1}). \qquad \text{Q.E.D.}$$

Corollary **(6.1.4)**. — *Let* R, $f_1, \ldots, f_n$ *and* S *be as above. Then for every* $\nu \geq 0$ *there exist elements* $f_{1(\nu)}, \ldots, f_{n(\nu)}$ *of* R *such that:*

(i) $\lim_{\nu \to \infty} f_{i(\nu)} = f_i$ *(in* $\hat{\mathrm{R}}$*) for all* i.

(ii) *Let* $\mathrm{Y}_{i(\nu)} = \mathrm{X}_i - f_{i(\nu)}$ *(in the sense of* (6.1.1) *and* (6.1.2)).

Then $\mathrm{Y}_{i(\nu)}$ *is divisible by* t^ν *in the (unique)* ν^{th} *iterated quadratic transform of* S.

Proof. — Let $f_{i(\nu)} = \sum_{s=1}^{\nu} a_{is} t^s$ as in (6.1.2). Then (i) is true by definition, and $\mathrm{Y}_{i(\nu)} = t^\nu \mathrm{X}_i^{(\nu)}$, for $\mathrm{X}_i^{(\nu)}$ as in (6.1.3). Thus the assertion (ii) is proved.

Remark. — It is helpful to think of the $\mathrm{Y}_{i(\nu)}$ as elements of S which approximate the differentials df_i in $\hat{\mathrm{S}}$, i.e. the generators of the nilpotent prime ideal of $\hat{\mathrm{S}}$ (d is the differential operator of § 4).

Quasi-algebrization **(6.2)**. — A quasi-algebrization requires two data:

(1) k is any field of characteristic p, $\hat{\mathrm{R}} = k[[t]]$ a formal power series ring, and C is a flat $\hat{\mathrm{R}}$-algebra of the form

$$\mathrm{C} = \hat{\mathrm{R}}[\mathrm{Y}_1, \ldots, \mathrm{Y}_n]/\mathfrak{L}$$

with $\mathfrak{L} \subset (\mathrm{Y})\,\mathrm{R}[\mathrm{Y}]$ and $(\mathrm{Y})^{\mathrm{N}} \subset \mathfrak{L}$ for some N (i.e. C satisfies the hypotheses (6.0.1)).

(2) R is a discrete valuation ring with completion $\hat{\mathrm{R}}$, and S is an R-subalgebra of $\hat{\mathrm{R}}$ of the form $\mathrm{R}[f_1, \ldots, f_n]$ where $f_i^{p^{e_i}} = g_i$ in R for some e_i (in particular S is a finite, purely inseparable R-algebra). We further suppose that via $\mathrm{X}_i \mapsto f_i$, S is R-isomorphic to $\mathrm{R}[\mathrm{X}_1, \ldots, \mathrm{X}_n]/\mathfrak{J}$, where $\mathfrak{J}$ of course contains the ideal $\mathfrak{H}$ generated by the $\mathrm{X}_i^{p^{e_i}} - g_i$, and moreover the following condition is satisfied:

(6.2.1) Identify $\hat{\mathrm{R}}[\mathrm{Y}]$ and $\hat{\mathrm{R}}[\mathrm{X}]$ by $\mathrm{Y}_i = \mathrm{X}_i - f_i$. Let $\hat{\mathfrak{J}}$ denote the ideal generated by $\mathfrak{J}$ in this ring. Then $\hat{\mathfrak{J}} \subset \mathfrak{L}$. (This enables us to view C as a quotient of $\hat{\mathrm{S}}$, which is crucial for the sequel.)

Remark **(6.2.2)**. — Given the datum (1), we can always find R, S as in (2). In fact, by the techniques of § 5 we can find a Schmidt ring R and elements $f_1, \ldots, f_n$ in the completion of R (which may be identified with $\hat{\mathrm{R}}$), so that if $\mathrm{S} = \mathrm{R}[f_1, \ldots, f_n]$, in the terminology of (2) $\mathfrak{J}$ is actually equal to the ideal $\mathfrak{H}$ in this case, with each $p^{e_i} \geq \mathrm{N}$. Hence $\mathfrak{J} = (\mathrm{Y}_i^{p^{e_i}}) \subset (\mathrm{Y})^{\mathrm{N}} \subset \mathfrak{L}$.

Given the data (1) and (2), we are now going to construct a local domain $\mathfrak{D}$ such that $\mathrm{R} \subset \mathrm{S} \subset \mathfrak{D} \subset \hat{\mathrm{R}}$, $\mathrm{Q}(\mathfrak{D}) = \mathrm{Q}(\mathrm{S})$ (so that we get a presentation $\mathrm{R} \hookrightarrow \mathfrak{D} \hookrightarrow \hat{\mathrm{R}}$), and the conclusions of the Theorem (6.0.4) are satisfied for R and $\mathfrak{D}$ with respect to C. This $\mathfrak{D}$ is called a *quasi-algebrization of* C *over* R *along* $(f_1, \ldots, f_n)$ [1]. Note that in view of the remark (6.2.2) we will then have *proved* (6.0.4). However since our interest lies not merely in the existence theorem, but also in the analysis of a *given* $\mathfrak{D}$, we want to

[1] In (6.3) we will show that there is a *unique* one of these.

reserve from the outset the right to start with a given R, S. We will see that in this case, quasi-algebrization determines $\mathfrak{O}$ uniquely (6.3).

We begin our quasi-algebrization: choose a set of generators $u_1, \ldots, u_r$ of the ideal $\mathfrak{L}$ of (1); each u_j is in the ideal generated by the Y_i, say

(6.2.3)
$$u_j = \sum_{\substack{\ell=(\ell_1, \ldots, \ell_n)\\ |\ell|>0}} c_{j\ell} Y^\ell, \quad j=1, \ldots, r$$

where Y^ℓ denotes the monomial $Y_1^{\ell_1} \ldots Y_n^{\ell_n}$, $|\ell| = \ell_1 + \ell_2 + \ldots + \ell_n$, the $c_{j\ell}$ are in $\hat{R}$, and the sum is finite for each j.

We are going to construct an infinite sequence

$$S = S_0 \subset S_1 \subset \ldots$$

of (finite) R-algebras, with S_ν contained in the ν^{th} iterated quadratic transform $S^{(\nu)}$ of S, with the following property: upon completion, in the resulting sequence

$$\hat{S} = \hat{S}_0 \subset \hat{S}_1 \subset \ldots$$

it is precisely the u_j which generate the ideal of $\hat{S}$ consisting of all those elements which become divisible by arbitrarily high powers of t (the regular parameter of R) in successive $\hat{S}_\nu$. Then if we let $\mathfrak{O} = \bigcup_\nu S_\nu$ we will find, essentially by (3.4), that $\hat{\mathfrak{O}}$ is a quotient of $\hat{S}$ by the ideal generated by the u_j, so that $\mathfrak{O} \overset{\approx}{\to} C$ as desired.

To do this, we first take elements $Y_{i(\nu)}$ of S as in (6.1.4) for $i=1, \ldots, n$, and all $\nu \geq 0$; we will use these to construct elements $u_{j(\nu)}$ of S for $j=1, \ldots, r$ and $\nu \geq 0$ as follows: for each $c_{j\ell}$ in the expression (6.2.3) for u_j take any sequence $c_{j\ell(\nu)}$ of elements of R which converges to $c_{j\ell}$ in $\hat{R}$, in such a way that $c_{j\ell} - c_{j\ell(\nu)}$ is divisible by t^ν in $\hat{R}$. Then define, for each j and ν,

(6.2.4)
$$u_{j(\nu)} = \sum_{\substack{\ell\\ |\ell|>0}} c_{j\ell(\nu)} Y_{1(\nu)}^{\ell_1} \ldots Y_{n(\nu)}^{\ell_n}.$$

Via the identification of (6.2.1), we will view the u_j as elements of $\hat{S}$, and the $u_{j(\nu)}$ as elements of S which approximate the u_j. Now let

(6.2.5)
$$S_\nu = S[(u_{1(\nu)}/t^\nu), \ldots, (u_{n(\nu)}/t^\nu)]$$

viewed in the following sense: since each $Y_{i(\nu)}$ is divisible by t^ν in the ν^{th} iterated quadratic transform $S^{(\nu)}$ of S (by (6.1.4)), and since $|\ell|>0$, S_ν is an S-subalgebra of $S^{(\nu)}$. Moreover, since $S \subset \hat{R}$, each $S^{(\nu)}$ is contained in $\hat{R}$, so that we may also regard $S_\nu \subset \hat{R}$. This could also be seen directly if we identify X_i with f_i in $\hat{R}$, and recall the definition of the $Y_{i(\nu)}$ in terms of these.

(6.2.6) To analyze this situation it will be convenient to introduce new variables: for each ν let $W_{1\nu}, \ldots, W_{r\nu}$ be independent variables over S, and let P_ν denote the polynomial ring $S[W_{1\nu}, \ldots, W_{r\nu}]$. For each ν we have a natural map $b_\nu : P_\nu \to S_\nu$

defined by $W_{j\nu} \mapsto (u_{j(\nu)}/t^\nu)$, $j=1, \ldots, r$. Let $\varphi_{j\nu}=t^\nu W_{j\nu}-u_{j(\nu)}$. Then the $\varphi_{j\nu}$ are in the kernel of b_ν. Note that the induced map $b_\nu \otimes 1_{Q(R)}$

$$P_\nu/(\{\varphi_{j\nu}\}_{1\leqslant j\leqslant r})\otimes_R Q(R) \overset{\approx}{\to} S_\nu \otimes_R Q(R)$$

is an *isomorphism*. Hence if $\mathfrak{K}_\nu$ denotes the kernel of b_ν, since S_ν is flat over R,

$$\textbf{(6.2.7)} \qquad \mathfrak{K}_\nu = \bigcup_m ((\{\varphi_{j\nu}\}_j)P_\nu : t^m)_{P_\nu}.$$

Now let h_ν denote the S-homomorphism $P_\nu \to P_{\nu+1}$ defined by

$$W_{j\nu} \mapsto tW_{j\nu+1} + (u_{j(\nu)}-u_{j(\nu+1)})/t^\nu, \quad \text{for} \quad j=1, \ldots, r.$$

To justify this, we need to show

(6.2.8) $u_{j(\nu)}-u_{j(\nu+1)}$ *is divisible by* t^ν *in* S.

Proof. — Remembering that $Y_{i(\nu)}=X_i-f_{i(\nu)}$ ((6.1) ff) we have

$$\begin{aligned} u_{j(\nu)}-u_{j(\nu+1)} &= \sum_{|\ell|>0} c_{j\ell(\nu)}(X_1-f_{1(\nu)})^{\ell_1}\ldots(X_n-f_{n(\nu)})^{\ell_n} \\ (*) \qquad & \quad - \sum_{|\ell|>0} c_{j\ell(\nu+1)}(X_1-f_{1(\nu+1)})^{\ell_1}\ldots(X_n-f_{n(\nu+1)})^{\ell_n} \\ &= \sum_{|\ell|>0} (c_{j\ell(\nu)}Z_1^{\ell_1}\ldots Z_n^{\ell_n} - c_{j\ell(\nu+1)}(Z_1-a_{1,\nu+1}t^{\nu+1})^{\ell_1}\ldots(Z_n-a_{n,\nu+1}t^{\nu+1})^{\ell_n}) \end{aligned}$$

where $Z_i=X_i-f_{i(\nu)}$, recalling that $f_{i(\nu)}=\sum_{s=1}^{\nu} a_{is}t^s$. Now

$$c_{j\ell(\nu+1)}(Z_1-a_{1,\nu+1}t^{\nu+1})^{\ell_1}\ldots(Z_n-a_{n,\nu+1}t^{\nu+1})^{\ell_n} = c_{j\ell(\nu+1)}Z_1^{\ell_1}\ldots Z_n^{\ell_n}+D_{j\ell}$$

with $D_{j\ell}$ divisible by $t^{\nu+1}$ in S. Thus from the equality (*) we find that

$$u_{j(\nu)}-u_{j(\nu+1)} = \sum_{|\ell|>0} (c_{j\ell(\nu)}-c_{j\ell(\nu+1)})Z_1^{\ell_1}\ldots Z_n^{\ell_n} - D_{j\ell}$$

which is divisible by t^ν in S, in virtue of the definition of the $c_{j\ell(\nu)}$. Note that the proof shows that $(u_{j(\nu)}-u_{j(\nu+1)})/t^\nu$ is in max(S). Q.E.D.

Hence the maps h_ν are well-defined, and it is obvious that for all ν the diagrams

$$\begin{array}{ccc} P_{\nu+1} & \xrightarrow{b_{\nu+1}} & S_{\nu+1} \\ {\scriptstyle h_\nu}\uparrow & & \cup \\ P_\nu & \xrightarrow{b_\nu} & S_\nu \end{array}$$

commute, where the homomorphism on the right is the natural inclusion. Moreover, observe that $h_\nu(\varphi_{j\nu})$ is by definition

$$t^\nu(tW_{j,\nu+1}+(u_{j(\nu)}-u_{j(\nu+1)})/t^\nu)-u_{j(\nu)} = t^{\nu+1}W_{j,\nu+1}-u_{j(\nu+1)} = \varphi_{j,\nu+1}.$$

Hence the h_ν induce commutative diagrams

$$\begin{array}{ccc} P_{\nu+1}/(\{\varphi_{j,\nu+1}\}_{1\leqslant j\leqslant r})P_{\nu+1} & \xrightarrow{b_{\nu+1}} & S_{\nu+1} \\ {\scriptstyle h_\nu}\uparrow & & \uparrow \\ P_\nu/(\{\varphi_{j\nu}\}_{1\leqslant j\leqslant r})P_\nu & \xrightarrow{b_\nu} & S_\nu \end{array}$$

(we preserve the notations h_ν and b_ν for the induced maps). Passing to completions, we get corresponding diagrams of exact sequences:

$$(6.2.9)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{K}_{\nu+1} & \longrightarrow & \overline{P}_{\nu+1} \underset{\text{def}}{=} \hat{P}_{\nu+1}/(\{\varphi_{j,\nu+1}\}_{1\leqslant j\leqslant r})\hat{P}_{\nu+1} & \xrightarrow{\hat{b}_{\nu+1}} & \hat{S}_{\nu+1} & \longrightarrow & 0 \\ & & \uparrow & & {\scriptstyle \hat{h}_\nu}\uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathfrak{K}_\nu & \longrightarrow & \overline{P}_\nu \underset{\text{def}}{=} \hat{P}_\nu/(\{\varphi_{j\nu}\}_{1\leqslant j\leqslant \nu})\hat{P}_\nu & \xrightarrow{\hat{b}_\nu} & \hat{S}_\nu & \longrightarrow & 0 \end{array}$$

(where we may interpret $\hat{P}_\nu$ to mean the completion of P_ν with respect to the ideal generated by max(S) and the $W_{j\nu}$, and $\mathfrak{K}_\nu$ is the t-torsion ideal of $\overline{P}_\nu$ (by (6.2.7))).

Let $h_{m\nu}: \overline{P}_\nu \to \overline{P}_m$ be the composition $\hat{h}_{m-1}\circ\ldots\circ\hat{h}_\nu$, with notations as in the diagram (6.2.9) above. Then let

$\mathfrak{T}_\nu = \{x\in\overline{P}_\nu \mid$ for every integer $M>0$, there is an m such that t^M divides $h_{m\nu}(x)$ in $\overline{P}_m\}$.

$\mathfrak{T}_\nu$ is clearly an ideal of $\overline{P}_\nu$. We first claim

(6.2.10) $W_{j\nu}+(u_j-u_{j(\nu)})/t^\nu$ *is in* $\mathfrak{T}_\nu$ *for* $j=1,\ldots,r$. (Note that *a priori* u_j is viewed as an element of $\hat{S}$, so it makes sense in $\overline{P}_\nu$ insofar as the latter is an $\hat{S}$-algebra, i.e. a quotient of $\hat{S}[[W_{1\nu},\ldots,W_{n\nu}]]$. Similarly $u_{j(\nu)}$ as an element on S also makes sense in $\overline{P}_\nu$, and in this way it is of course still true that $\varinjlim_\nu u_{j(\nu)} = u_j$.)

Proof of (6.2.10). — We first show that $h_{m\nu}(W_{j\nu}+(u_{j(m)}-u_{j(\nu)})/t^\nu)$ is divisible by $t^{m-\nu}$ in $\overline{P}_m$ (this makes sense because by iteration of (6.2.8), we see that $u_{j(m)}-u_{j(\nu)}$ is divisible by t^ν in S). In fact, we will show that

$$(*)\qquad h_{m\nu}(W_{j\nu}+(u_{j(m)}-u_{j(\nu)})/t^\nu) = t^{m-\nu}W_{jm}.$$

Namely, this is true for $m=\nu+1$ by definition of h_ν. By induction, suppose true for m. Then

$$\begin{aligned} & h_{m+1,\nu}(W_{j\nu}+(u_{j(m+1)}-u_{j(\nu)})/t^\nu) \\ &= h_{m+1,\nu}(W_{j\nu}+(u_{j(m+1)}-u_{j(m)}+u_{j(m)}-u_{j(\nu)})/t^\nu) \\ &= h_{m+1,m}(t^{m-\nu}W_{jm}+(t^{m-\nu}/t^m)(u_{j(m+1)}-u_{j(m)})) \\ &= t^{m-\nu}h_{m+1,m}(W_{jm}+(u_{j(m+1)}-u_{j(m)})/t^m) \\ &= t^{m-\nu}tW_{j,m+1} = t^{m+1-\nu}W_{j,m+1}, \end{aligned}$$

which completes the verification of $(*)$. Now, note that for any $m \geq \nu$ we can write

$$W_{j\nu} + (u_j - u_{j(\nu)})/t^\nu = W_{j\nu} + (u_{j(m)} - u_{j(\nu)})/t^\nu + (u_j - u_{j(m)})/t^\nu.$$

Hence by $(*)$, we get

(6.2.11) $$h_{m\nu}(W_{j\nu} + (u_j - u_{j(\nu)})/t^\nu) = t^{m-\nu} W_{jm} + (u_j - u_{jm})/t^\nu.$$

To conclude the proof of (6.2.10) it suffices to show that $u_j - u_{j(m)}$ is divisible by t^m in $\hat{S}$. For this, recall

$$u_j = \sum_\ell c_{j\ell} Y_1^{\ell_1} \dots Y_n^{\ell_n}$$

where $Y_i = X_i - f_i$, and f_i and the $c_{j\ell}$ are in $\hat{R}$; and

$$u_{j(m)} = \sum_\ell c_{j\ell(m)} Y_{1(m)}^{\ell_1} \dots Y_{n(m)}^{\ell_n}$$

with $Y_{i(m)} = X_i - f_{i(m)}$. Let v_{im} denote $f_i - f_{i(m)}$. Then $Y_{i(m)} = Y_i + v_{im}$, and we can write

$$u_j - u_{j(m)} = \sum_\ell c_{j\ell} Y_1^{\ell_1} \dots Y_n^{\ell_n} - c_{j\ell(m)} (Y_1 + v_{1m})^{\ell_1} \dots (Y_n + v_{nm})^{\ell_n}.$$

But then, since each v_{im} is divisible by t^m in R, as is $c_{j\ell} - c_{j\ell(m)}$, a computation analogous to that in the proof of (6.2.8) gives the result. Q.E.D.

We will write $\xi_{j\nu} = W_{j\nu} + (u_j - u_{j(\nu)})/t^\nu$. We have just shown then that $\xi_{j\nu}$ is in $\mathfrak{T}_\nu$ for all ν; more precisely, the formula (6.2.11) shows that $h_{m\nu}(\xi_{j\nu}) = t^{m-\nu}(\xi_{jm})$. Hence the $h_{m\nu}$ induce maps

$$\begin{array}{c} \overline{P}_m / (\{\xi_{jm}\}_{1 \leq j \leq r}) \overline{P}_m \\ {\scriptstyle \bar{h}_{m\nu}} \big\uparrow \\ \overline{P}_\nu / (\{\xi_{j\nu}\}_{1 \leq j \leq r}) \overline{P}_\nu \end{array}$$

(6.2.12) *For every* ν *we have a natural* R*-isomorphism*

$$\overline{P}_\nu / (\{\xi_{j\nu}\}_{1 \leq j \leq r}) \overline{P}_\nu \overset{\approx}{\to} C$$

(compatible with the $\bar{h}_{m\nu}$*).*

Proof. — In view of the definition of the $\overline{P}_\nu$ and the expression of $\hat{S}$ in the form $\hat{R}[X_1, \dots, X_n]/\hat{\mathfrak{J}}$ (see (6.2.1)), we can write

$$\overline{P}_\nu / (\{\xi_{j\nu}\}_{1 \leq j \leq r}) \overline{P}_\nu = \hat{R}[X_1, \dots, X_n, W_1, \dots, W_n] / (\{\xi_{j\nu}\}_j, \{\varphi_{j\nu}\}_j, \hat{\mathfrak{J}}).$$

Now since the $\xi_{j\nu}$ are 0, for each ν, $W_{j\nu} = -(u_j - u_{j(\nu)})/t^\nu$. Then, substituting in the expression $t^\nu W_{j\nu} - u_j$ for $\varphi_{j\nu}$, we find that $\varphi_{j\nu} = -u_j$. Thus

$$\overline{P}_\nu / (\{\xi_{j\nu}\}_{1 \leq j \leq r}) \overline{P}_\nu = \hat{R}[X_1, \dots, X_n] / (\{u_j\}_j, \hat{\mathfrak{J}}).$$

But now if we use the $Y_i = X_i - f_i$ as coordinates, and recall that by hypothesis (6.2.1) $\hat{\mathfrak{J}} \subset \mathfrak{L} = (u_1, \dots, u_r)$, we get the result. Q.E.D.

Thus all the maps $\bar{h}_{m\nu}$ are $\hat{\mathrm{R}}$-isomorphisms; in fact we have commutative diagrams

$$\begin{array}{ccc} \overline{\mathrm{P}}_m/(\{\xi_{jm}\}_j)\,\overline{\mathrm{P}}_m & \xrightarrow{\approx} & \mathrm{C} \\ {\scriptstyle \bar{h}_{m\nu}}\uparrow \wr & & \Vert \\ \overline{\mathrm{P}}_\nu/(\{\xi_{j\nu}\}_j)\,\overline{\mathrm{P}}_\nu & \xrightarrow{\approx} & \mathrm{C} \end{array}$$

It is now easy to see that

(6.2.13) *For each* ν, $\mathfrak{T}_\nu=(\{\xi_{j\nu}\}_j)\,\overline{\mathrm{P}}_\nu$ (*so that* $\overline{\mathrm{P}}_\nu/\mathfrak{T}_\nu=\mathrm{C}$). In fact, if x is in $\mathfrak{T}_\nu$, then *a fortiori*, if $\bar{x}$ denotes x $(\mathrm{mod}(\{\xi_{j\nu}\}_j)\,\overline{\mathrm{P}}_\nu)$, $\bar{h}_{m\nu}(\bar{x})$ becomes arbitrarily highly divisible by t as m gets large. But since all the $\bar{h}_{m\nu}$ are *isomorphisms* as we have just seen, this means that $x\equiv 0$ $(\mathrm{mod}(\{\xi_{j\nu}\}_j)\,\overline{\mathrm{P}}_\nu)$. Q.E.D.

Now let us return to our situation (6.2.9):

$$\begin{array}{ccccccccc} & & & & \vdots & & \vdots & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \overline{\mathfrak{R}}_{\nu+1} & \longrightarrow & \overline{\mathrm{P}}_{\nu+1} & \xrightarrow{\hat{b}_{\nu+1}} & \hat{\mathrm{S}}_{\nu+1} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle \hat{h}_\nu} & & \uparrow & & \\ 0 & \longrightarrow & \overline{\mathfrak{R}}_\nu & \longrightarrow & \overline{\mathrm{P}}_\nu & \xrightarrow{\hat{b}_\nu} & \hat{\mathrm{S}}_\nu & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & \vdots & & \vdots & & \end{array}$$

where $\overline{\mathfrak{R}}_\nu$ is the t-torsion ideal of $\overline{\mathrm{P}}_\nu$. Observe that since (by (6.2.13)) $\overline{\mathrm{P}}_\nu/\mathfrak{T}_\nu=\mathrm{C}$ is *flat* over $\hat{\mathrm{R}}$, $\overline{\mathfrak{R}}_\nu\subset\mathfrak{T}_\nu$. Let

$$\mathfrak{T}'_\nu=\left\{\begin{array}{l} x \text{ in } \hat{\mathrm{S}}_\nu \,|\, \text{for every integer } \mathrm{M}\geqslant 0 \text{ there is an } m \\ \text{such that } t^{\mathrm{M}} \text{ divides the image of } x \text{ in } \hat{\mathrm{S}}_m \end{array}\right\}$$

i.e. $\mathfrak{T}'_\nu$ is defined for $\hat{\mathrm{S}}_\nu$ just as $\mathfrak{T}_\nu$ is for $\overline{\mathrm{P}}_\nu$. It is clear that $\hat{b}_\nu(\mathfrak{T}_\nu)\subset\mathfrak{T}'_\nu$. We claim that in fact

(6.2.14) $$\hat{\mathrm{S}}_\nu/\mathfrak{T}'_\nu=\overline{\mathrm{P}}_\nu/\mathfrak{T}_\nu\ (=\mathrm{C}).$$

Proof. — Since $\overline{\mathfrak{R}}_\nu\subset\mathfrak{T}_\nu$, $\hat{\mathrm{S}}_\nu/\hat{b}_\nu(\mathfrak{T}_\nu)=\overline{\mathrm{P}}_\nu/\mathfrak{T}_\nu=\mathrm{C}$ for all ν. Now suppose x is an element of $\mathfrak{T}'_\nu$. *A fortiori*, the image $\bar{x}$ of x in $\hat{\mathrm{S}}_\nu/\hat{b}_\nu(\mathfrak{T}_\nu)$ becomes divisible by arbitrarily high powers of t in the successive $\hat{\mathrm{S}}_m/\hat{b}_m(\mathfrak{T}_m)$, $m\geqslant\nu$. But all these are *isomorphic* (to C), so that $\bar{x}$ must be 0, i.e. x is in $\hat{b}_\nu(\mathfrak{T}_\nu)$. Q.E.D.

Now let $\mathfrak{O}=\bigcup_\nu \mathrm{S}_\nu$ $(\subset\hat{\mathrm{R}})$. $\mathfrak{O}$ is of course an integral domain, and has Krull dimension 1 since it is integral, indeed purely inseparable, over R. Moreover since

$R \hookrightarrow \mathfrak{D} \hookrightarrow \hat{R}$, $R \to \mathfrak{D}$ induces an isomorphism of residue fields. Let $\mathfrak{m} = \max(\mathfrak{D})$. We claim that $\mathfrak{m} = \max(S)\mathfrak{D}$. For this, note that $\mathfrak{D}$ can be described as

$$\textbf{(6.2.15)} \qquad \mathfrak{D} = S[\{u_{j(\nu)}/t^\nu\},\ j = 1, \ldots, r,\ \nu = 1, 2, \ldots] \subset \hat{R}.$$

But we can write

$$u_{j(\nu)}/t^\nu = t(u_{j(\nu+1)}/t^{\nu+1}) + (u_{j(\nu)} - u_{j(\nu+1)})/t^\nu,$$

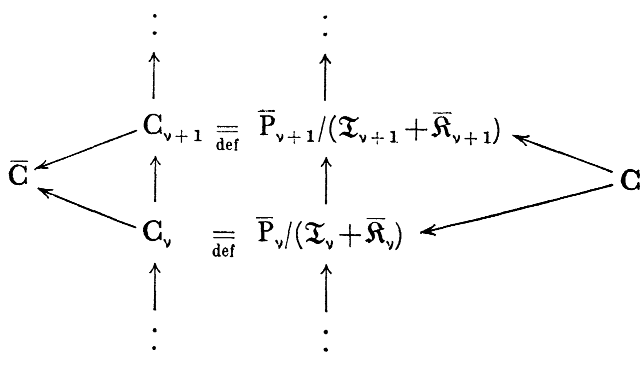
and the last summand is in $\max(S)$ (see the proof of (6.2.8)). This shows that $\mathfrak{D}$ is *noetherian* (since every prime ideal is finitely generated, by [2], Chap. I, Theorem (3.4)), and moreover that $\hat{S}_\nu \to \hat{\mathfrak{D}}$ is *surjective* for all ν. Hence we can apply (3.4.1) to deduce that $\hat{\mathfrak{D}} = \hat{S}_\nu/\mathfrak{T}'_\nu$ (for any ν) so that by (6.2.14) we have an $\hat{R}$-isomorphism $\mathfrak{D} \overset{\approx}{\to} C$ arising from the natural structure of R-presentation of $\mathfrak{D}$ (i.e. such that we have the commutative diagram of (6.0.4)). Thus $\mathfrak{D}$ is the desired quasi-algebrization of C.

Remark **(6.2.16).** — Suppose we had begun with an $\hat{R}$-algebra C as above which is *not necessarily flat over* $\hat{R}$, but which satifies all the other hypotheses of (1) at the beginning of (6.2). Let $\bar{C} = C$ (modulo its torsion ideal over $\hat{R}$) (so that $\bar{C}$ is flat over $\hat{R}$). Proceeding as above for C, we can construct the S_ν and $\mathfrak{D} = \bigcup_\nu S_\nu$. *Then we find that* $\hat{\mathfrak{D}} = \bar{C}$. To see this, first recall that the flatness of C was not used in the proof of the existence of quasi-algebrization until (6.2.14); at that point it was used in the form: $\bar{\mathfrak{K}}_\nu \subset \mathfrak{T}_\nu$ (with notations as above). For non-flat C, we replace (6.2.14) by the following argument (preserving all the notations and other assumptions of the proof above):

(6.2.14) *There exists a quotient ring* C′ *of* C, *between* C *and* $\bar{C}$ *(*i.e. *we have surjections* $C \to C'$ *and* $C' \to \bar{C}$*) such that for all* ν *sufficiently large*

$$C' = \bar{P}_\nu/(\mathfrak{T}_\nu + \bar{\mathfrak{K}}_\nu) = \hat{S}_\nu/\mathfrak{T}'_\nu.$$

Proof. — For each ν, we know that $\bar{P}_\nu/\mathfrak{T}_\nu = C$, so that, since $\bar{\mathfrak{K}}_\nu$ is the torsion ideal of $\bar{P}_\nu$ over $\hat{R}$, if we denote $\bar{P}_\nu/(\mathfrak{T}_\nu + \bar{\mathfrak{K}}_\nu)$ by C_ν, then $C \to \bar{C}$ factors through C_ν. Thus we have a commutative diagram



where the vertical maps are induced by the $\hat{h}_\nu$, and the maps emanating from C are all surjective, as are the ones terminating at $\overline{C}$. Now since each $C \to C_\nu$ is surjective, the maps $C_\nu \to C_{\nu+1}$ are also surjective, so by the noetherianness of C, for sufficiently large ν the $C_\nu \to C_{\nu+1}$ are isomorphisms, i.e. all the C_ν are equal to the same C′ with $C \to C' \to \overline{C}$ both surjective. On the other hand, for any ν we have

$$\overline{P}_\nu/(\mathfrak{T}_\nu + \overline{\mathfrak{K}}_\nu) = \hat{S}_\nu/\hat{b}_\nu(\mathfrak{T}_\nu).$$

Hence, for all ν sufficiently large, the $\hat{S}_\nu/\hat{b}_\nu(\mathfrak{T}_\nu)$ are all isomorphic (to C′). It then follows by definition of $\mathfrak{T}'_\nu$, and by virtue of the obvious inclusion $\hat{b}_\nu(\mathfrak{T}_\nu) \subset \mathfrak{T}'_\nu$, that in fact $\mathfrak{T}'_\nu = \hat{b}_\nu(\mathfrak{T}_\nu)$. Hence $\hat{S}_\nu/\mathfrak{T}'_\nu = C'$. Q.E.D.

Now it follows just as in the theorem that $\mathfrak{O}$ is noetherian and $\hat{\mathfrak{O}} = C'$. However $\mathfrak{O}$ is the limit of the S_ν, *which are flat over* R, so that also $\mathfrak{O}$ is flat over R. Hence $\hat{\mathfrak{O}}$ is flat over $\hat{R}$ (e.g. by Grothendieck's " local criterion " for flatness). Hence $C' = \overline{C}$, which gives the desired result.

Remark **(6.2.17)**. — The quasi-algebrization procedure appears to depend on the following choices:

(i) The choice of the *approximations* $f_{i(\nu)}$ of the f_i by elements of R.

(ii) The choice of the *generators* u_j of the ideal $\mathfrak{L}$, and the approximations of these by the elements $u_{j(\nu)}$ in S, i.e. the choice of the $c_{j\ell(\nu)}$.

However, we will see in (6.3) that the quasi-algebrization of C over R along $(f_1, \ldots, f_n)$ is unique, i.e. it is independent of any such choices.

We note the following consequences of Theorem (6.0.4):

(6.2.18) Let k be any field of characteristic p, and let $\overline{C}$ be any artinian local k-algebra with residue field k. Then $\overline{C}$ *can be deformed flatly over a discrete valuation ring* R *to a purely inseparable field extension* F *of* Q(R). In fact, let $\hat{R} = k[[t]]$, and let, for example, $C = \hat{R} \otimes_k \overline{C}$ (actually, any C over $\hat{R}$ as in (6.0.1) with $C \otimes_{\hat{R}} k = \overline{C}$ will do). Then, for a suitable discrete valuation ring R with completion isomorphic to $\hat{R}$, and elements $f_1, \ldots, f_n$ in $\hat{R}$, purely inseparable over R, we can form the quasi-algebrization $\mathfrak{O}$ of C over R along $(f_1, \ldots, f_n)$. The generic fibre of $\mathfrak{O}$ over R is then $F = Q(R)(f_1, \ldots, f_n)$ and the special fibre is $\mathfrak{O}/t\mathfrak{O} = \hat{\mathfrak{O}}/t\hat{\mathfrak{O}} = C/tC = \overline{C}$. Of course, $\mathfrak{O}$ is not necessarily a *finite type* R-algebra.

(6.2.19) Let E denote the field of Laurent series in one variable t over any field k of characteristic p. Let C be an artinian local E-algebra with residue field E. *Then* C *is the formal fibre of a local domain* $\mathfrak{O}$. Namely, in view of quasi-algebrization, this amounts to the following:

Lemma. — Any C *as above has flat reduction* $\tilde{C}$ mod(t), *such that* C *satisfies* (6.0.1).

Proof. — The hypotheses on C imply that we can write it in the form

$$C = E[Y_1, \ldots, Y_n]/\mathfrak{L},$$

where $\mathfrak{L}$ is an ideal such that $(Y)^N E[Y] \subset \mathfrak{L} \subset (Y)E[Y]$ for some N. Choose generators $u_1, \ldots, u_r$ for $\mathfrak{L}$ which are in R[Y], where $R = k[[t]]$ (so that $Q(R) = E$). Let $\mathfrak{L}_0$ be the ideal of R[Y] generated by the $u_1, \ldots, u_n$ *and* $(Y)^N$. Let $\widetilde{\mathfrak{L}} = \bigcup_\nu (\mathfrak{L}_0 : t^\nu)_{R[Y]}$. Then if we let $\widetilde{C} = R[Y]/\widetilde{\mathfrak{L}}$, $\widetilde{C}$ is torsion free over R, and is finite over R (since $(Y)^N \subset \widetilde{\mathfrak{L}}$). Hence $\widetilde{C}$ is flat over R. It is clear that $\widetilde{C} \otimes_R Q(R) = C$. Moreover, if we let $\mathfrak{P} = (Y)\widetilde{C}$, then $\widetilde{C}/\mathfrak{P} = R$, so (6.0.1) is satisfied. Q.E.D.

(6.2.20) At this point an example seems desirable. Let R be a discrete valuation ring of characteristic 5, and let $f \in \hat{R} - R$, with $f^5 = g \in R$. Let $S = R[f] \subset \hat{R}$. Then $\hat{S} = \hat{R}[Y]/(Y^5)$, where $Y = df = f \otimes 1 - 1 \otimes f$ (view S as $S \otimes_R \hat{R}$). Let $C = \hat{R}[Y]/(Y^n)$, where $1 \leqslant n \leqslant 5$. Then we can view C as a quotient of $\hat{S}$, and we can describe the ring $\mathfrak{O}$ which is the quasi-algebrization of C over R along f as follows: write $f = \sum_{i=1}^{\infty} a_i t^i$, with the a_i in R, as an element of $\hat{R}$. Then

$$\mathfrak{O} = S[\{1/t^\nu (f - \sum_{i=1}^{\nu} a_i t^i)^n\}_{\nu=1}^{\infty}],$$

viewed as an S-subalgebra of $\hat{S}$, the normalization of S. Note that when $n = 1$, $\mathfrak{O}$ is a discrete valuation ring — in fact in the terminology of § 5 it is the *Schmidt ring* over R corresponding to the formal p-section defined by f. For $n = 2, 3, 4$ $\mathfrak{O}$ is not regular; its maximal ideal is generated by f and t. When $n = 5$, $\mathfrak{O} = S$ because $f^5 \in R$, so $f^5 - (\sum_{i=1}^{\nu} a_i t^i)^5$ is divisible by t^ν in R and hence in S.

(6.3) *Uniqueness of quasi-algebrization, and some questions of classification.*

(6.3.1) *Let $\mathfrak{O}$ be a ring together with a presentation $R \hookrightarrow \mathfrak{O} \hookrightarrow \hat{R}$ satisfying the usual hypotheses* (3.0.1). *We will suppose in addition that* $Q(\mathfrak{O})$ *is finite over* $Q(R)$. *Choose $f_1, \ldots, f_n$ in $\mathfrak{m} = \max(\mathfrak{O})$ such that $\mathfrak{m} = (f_1, \ldots, f_n, t)\mathfrak{O}$ (where t is a regular parameter of* R*), and* $Q(\mathfrak{O}) = Q(R)(f_1, \ldots, f_n)$. *Then $\mathfrak{O}$ is the quasi-algebrization of $\hat{\mathfrak{O}}$ over* R *along $(f_1, \ldots, f_n)$, via the procedure which results from* any choices *as in* (i) *and* (ii) *of* (6.2.17).

Proof. — Before we consider the question of quasi-algebrization, we first analyze $\mathfrak{O}$ using the techniques of § 3: let $S = R[f_1 \ldots, f_n] \subset \mathfrak{O}$, and let $S^{(\mu)}$ denote the (unique) μ-th iterated quadratic transform of S. Write $S_\mu = S^{(\mu)} \cap \mathfrak{O}$. Then we know by (3.2.4) that $\mathfrak{O} = \bigcup_\mu S_\mu$. Moreover, by (3.4.1) $\hat{\mathfrak{O}}$ is naturally a quotient of $\hat{S}$ by the ideal $\mathfrak{T}$ consisting of all those elements whose images in successive terms of the sequence

$$\hat{S} = \hat{S}_0 \to \hat{S}_1 \to \ldots \to \hat{S}_\mu \to \ldots$$

become divisible by arbitrarily high powers of t (the regular parameter of R). Choose generators $u_1, \ldots, u_r$ of $\mathfrak{T}$ in $\hat{S}$. For each ν, let $\mu(\nu)$ be an integer such that $t^\nu | u_j$ in $\hat{S}_{\mu(\nu)}$ for each $j = 1, \ldots, r$. Identifying S with $R[X_1, \ldots, X_n]/\mathfrak{J}$ (as in (6.1.1)) and

$\hat{S}$ with $\hat{R}[Y_1, \dots, Y_n]/\hat{\mathfrak{J}}$ (as in (6.2.1)) with $Y_i = X_i - f_i$, we know that $\mathfrak{T} \subset (Y)S$, so that for each $j = 1, \dots, r$ we can write

$$u_j = \sum_{|\ell|>0} c_{j\ell} Y^\ell,$$

where the sum is finite for each j, and Y^ℓ denotes $Y_1^{\ell_1} \dots Y_n^{\ell_n}$ as usual, with $\ell = (\ell_1, \dots, \ell_n)$ and $|\ell| = \ell_1 + \dots + \ell_n$. Now for each $i = 1, \dots, n$ (resp. $j = 1, \dots, r$ and those ℓ for which $c_{j\ell} \neq 0$), choose a sequence of elements $f_{i(\nu)}$ (resp. $c_{j\ell(\nu)}$) in R such that $\lim_\nu f_{i(\nu)} = f_i$ and $t^\nu | (f_i - f_{i(\nu)})$ in $\hat{R}$ (resp. $\lim c_{j\ell(\nu)} = c_j$ and $t^\nu | c_{j\ell} - c_{j\ell(\nu)}$). Write

$$f_i^{(\nu)} = f_i - f_{i(\nu)}, \quad c_{j\ell}^{(\nu)} = c_{j\ell} - c_{j\ell(\nu)}.$$

Now for each $j = 1, \dots, r$ and each $\nu > 0$ (suppressing the indices i in conformity with the usual multi-index notation) we have :

$$u_j = \sum_\ell c_{j\ell} Y^\ell = \sum_\ell c_{j\ell}(X - f)^\ell = \sum_\ell (c_{j\ell(\nu)} + c_{j\ell}^{(\nu)})(X - f_{(\nu)} - f^{(\nu)})^\ell = A_{j\nu} + B_{j\nu},$$

where $$A_{j\nu} = \sum_\ell c_{j\ell(\nu)}(X - f_{(\nu)})^\ell \quad \text{and} \quad B_{j\nu} = u_j - A_{j\nu}.$$

Note that $A_{j\nu}$ is actually an element of S, so also an element of S_μ for all μ. Now in view of the definition of the $f_i^{(\nu)}$ and the $c_{j\ell}^{(\nu)}$, one checks easily that $t^\nu | B_{j\nu}$ in $\hat{S}$ (and *a fortiori* in $\hat{S}_\mu$ for any μ). On the other hand, we know $t^\nu | u_j$ in $\hat{S}_{\mu(\nu)}$. Hence $t^\nu | A_{j\nu}$ in $\hat{S}_{\mu(\nu)}$ for all j. Then, since $A_{j\nu}$ is in $S_{\mu(\nu)}$, also $t^\nu | A_{j\nu}$ in $S_{\mu(\nu)}$ by faithful flatness of the completion, i.e. $A_{j\nu}/t^\nu$ is in $S_{\mu(\nu)}$. *Hence* $A_{j\nu}/t^\nu$ *is in* $\mathfrak{O}$.

Now suppose we were to quasi-algebrize $\hat{\mathfrak{O}}$ over R along $(f_1, \dots, f_n)$, using the procedure that results from the choices of the generators u_j of the ideal $\mathfrak{L}$ of (6.2) (which corresponds to the ideal $\mathfrak{T}$ above in virtue of (6.2.1)), and the approximations $f_{i(\nu)}$ and $c_{j\ell(\nu)}$ in the terminology of (6.2). Then $A_{j\nu}$ is what was called $u_{j(\nu)}$ in (6.2), and hence if $\mathfrak{O}'$ denotes the quasi-algebrization, by (6.2.17) we get

$$\mathfrak{O}' = S[\{A_{j\nu}/t^\nu\}_\nu].$$

Thus $\mathfrak{O}' \subset \mathfrak{O}$. But then $\mathfrak{O}' \subset \mathfrak{O}$ is *birational* and induces an isomorphism of completions, so by the standard argument (see e.g. the proof of (3.2.4)) $\mathfrak{O}' = \mathfrak{O}$. This completes the proof of (6.3.1). Q.E.D.

The existence and uniqueness theorems have as an immediate consequence the following result, in the direction of classification:

(6.3.2) *Let* R *be a discrete valuation ring, and let* S *be a finite* R*-subalgebra of* $\hat{R}$, *purely inseparable over* R. *Let* $\tilde{S}$ *denote the normalization of* S, *and let* $|S|$ *be the class of all local (noetherian)* S*-subalgebras* $\mathfrak{O}$ *of* $\tilde{S}$ *such that* $S \hookrightarrow \mathfrak{O}$ *induces a surjection of completions (equivalently such that* $\max(S)\mathfrak{O} = \max(\mathfrak{O})$). *Then the assignment* $\mathfrak{O} \mapsto \hat{\mathfrak{O}}$ *is an* isomorphism of sets

$$|S| \xrightarrow[\approx]{\Phi} \mathrm{Hilb}_{\hat{S}/\hat{R}}(\hat{R})$$

(the latter denotes the set of subschemes of $\mathrm{Spec}(\hat{S})$ *flat over* $\hat{R}$*)*; Φ^{-1} *is given by quasi-algebrization over* R *along* $(f_1, \ldots, f_n)$, *where* $f_1, \ldots, f_n$ *are any set of elements of* S *which generate* max(S) *along with* max(R), *and which also generate* Q(S) *over* Q(R).

Proof. — Any element of $\mathrm{Hilb}_{\hat{S}/\hat{R}}(\hat{R})$ is of the form Spec(C), where C is of the form $\hat{S}/\mathfrak{a}$ for an ideal $\mathfrak{a}$. Let $\mathfrak{P}$ denote the nilpotent prime ideal of S. Then $\mathfrak{a} \subset \mathfrak{P}$, since C is flat over $\hat{R}$. Hence C satisfies (6.0.1). Therefore there exists a quasi-algebrization $\mathfrak{O}$ in $|S|$ as indicated, such that

(*) $S \subset \mathfrak{O}$ *induces a surjection of completions which identifies* $\hat{\mathfrak{O}}$ *with* C.

Moreover, by the uniqueness Theorem (6.3.1), this condition (*) uniquely determines $\mathfrak{O}$ in $|S|$. Thus the map $\mathrm{Hilb}_{\hat{S}/\hat{R}}(\hat{R}) \to |S|$ given by quasi-algebrization is well defined and is an inverse to Φ. Q.E.D.

Example **(6.3.3).** — To illustrate these ideas, take a discrete valuation ring R of characteristic 3, and an element $f \in \hat{R} - R$ with $f^3 = g$ in R. Let $S = R[f]$, so that $\hat{S} = \hat{R}[Y]/(Y^3)$ $(Y = df)$. Then $\mathrm{Hilb}_{\hat{S}/\hat{R}}(\hat{R})$ is the same as a set of certain ideals $\mathfrak{a} \subset (Y)\hat{S}$; in this case the Hilb has three components corresponding to the flat coverings of $\hat{R}$ of degree 1, 2, or 3 contained in $\mathrm{Spec}(\hat{S})$. There is a unique covering of degree 1, corresponding to the ideal $(Y)\hat{S}$. Via quasi-algebrization, this is associated to the discrete valuation ring $\tilde{S}$. The distinct ideals which give rise to coverings of $\hat{R}$ of degree 2 are of the form $(Y^2)\hat{S}$, or $(t^n Y + Y^2)\hat{S}$ for distinct $n > 0$. Via quasi-algebrization these are associated to the rings

$$\mathfrak{O}_\infty = S[\{1/t^\nu(f - \sum_{i=1}^{\nu} a_i t^i)^2\}_{\nu=1}^{\infty}]$$

and

$$\mathfrak{O}_n = S[\{1/t^\nu(t^n(f - \sum_{i=1}^{\nu} a_i t^i) + (f - \sum_{i=1}^{\nu} a_i t^i)^2)\}_{\nu=1}^{\infty}] \ (^1).$$

Note that this component is not " connected ". As for the coverings of degree 3, there is again a unique one corresponding to the ideal (0), which is associated by quasi-algebrization to the ring S itself. The situation is summarized in the diagram below:

$$|S = R[f]| \ \underset{\text{quasi-algebrization}}{\overset{\text{completion}}{\rightleftarrows}} \ \mathrm{Hilb}_{\hat{S}/\hat{R}}(\hat{R})$$

$\lvert S\rvert_1 = \{\tilde{S}\}$	deg. $1/\hat{R}$: $\{\hat{S}/(Y) \cong \hat{R}\}$
$\lvert S\rvert_2 = \{\mathfrak{O}_n\}_{n \in \mathbf{Z}_+ \cup \{\infty\}}$	deg. $2/\hat{R}$: $\{\hat{S}/(t^n Y + Y^2) \mid_{(t^\infty = 0)}^{n \in \mathbf{Z}_+ \cup \{\infty\}}\}$
$\lvert S\rvert_3 = \{S\}$	deg. $3/\hat{R}$: $\{\hat{S}\}$

Remark **(6.3.4).** — For the purpose of classification, say, of all those $\mathfrak{O}$ with a presentation over R and a given field of fractions, the classes $|S|$, parametrized by *all* those S finite over R, are not sufficiently precise. In fact, if $S \neq S'$, $|S| \cap |S'|$ is not empty

(1) As usual we take an expression $f = \sum_{i=1}^{\infty} a_i t^i$ for f in $\hat{R}$, a_i, $t \in R$.

in general. The simplest example of this is the fact that if S' is *any* one between S and $\widetilde{S}$, then $\widetilde{S}\in|S'|$. This does not really pose a problem, however, because $\widetilde{S}$ is isolated in any such $|S'|$; it corresponds to the unique point in the "deg. 1 over $\hat{R}$" part of $\mathrm{Hilb}_{\hat{S}'/\hat{R}}(\hat{R})$, i.e. $|S'|_1$. The analysis of the intersection of $|S|_n$ and $|S'|_n$ for $n>1$ is much more serious, and in view of the existence of the "local" description afforded by (6.3.2) is a crucial part of general birational classification; we will not treat this question here, except to say that the germs of many of the essential difficulties are present even in the simplest cases, e.g. the previous example (6.3.3).

7. AN EXAMPLE

Given a *maximal presentation* $R\hookrightarrow\mathfrak{O}\hookrightarrow\hat{R}$ (see § 4 for definitions) the question arises: *is* $Q(\mathfrak{O})$ *necessarily a finite extension of* $Q(R)$? (We know that $\mathfrak{O}$ is not necessarily a finitely generated R-algebra.) Since $\mathfrak{O}^q\subset R$ for some $q=p^e$, this question obviously has an affirmative answer whenever the following condition on R is satisfied:

(*) $Q(R)^{1/p}\cap Q(R)$ is a finite extension of $Q(R)$.

One would conjecture that (*) holds for example when R is a Schmidt ring over an excellent discrete valuation ring R_0 (§ 5). The example of this section shows, however, that if (*) does not hold for R, then the answer to our question above is negative in general: we will construct an $\mathfrak{O}$ with maximal presentation over R such that $Q(\mathfrak{O})$ is infinite over $Q(R)$. The point is that if (*) fails, there exist $\mathfrak{O}$ for which there will be too many *differential forms with coefficients in* R *which are not integrable over* $\mathfrak{O}$.

To begin, suppose we have an R for which

$$[(Q(R)^{1/p}\cap Q(R)) : Q(R)]=\infty.$$

For example, we can take R to be as in the examples of Nagata or Hironaka at the end of § 5, or the discrete valuation ring associated to a formal p-section of *infinite dimensional* affine space over an arbitrary discrete valuation ring. Let $\{f, g_i, \alpha_i\ (i=1, 2, \ldots)\}$ be elements of $R^{1/p}\cap\hat{R}$ which are p-independent over R (so that all monomials in the f, g_i and α_i of degree $<p$ in each factor are linearly independent over $Q(R)$). For each $n>0$ define $S_n=R[f, g_1, \ldots, g_n]\subset\hat{R}$ (so that S_n is a *finite* R-algebra) and let $\mathfrak{L}_n$ denote the ideal of $\hat{S}_n$ generated by $\{dg_i-\alpha_i df\}_{i=1}^n$. Then we can form the quasi-algebrization $\mathfrak{O}_n$ of $\hat{S}_n/\mathfrak{L}_n$ over R along $(f, g_1, \ldots, g_n)$: $\mathfrak{O}_n$ is contained in the normalization of S_n, and the inclusion $S_n\hookrightarrow\mathfrak{O}_n$ induces a surjective map of completions whose kernel is $\mathfrak{L}_n$ (6.2). Now if $i\leqslant n$, according to the quasi-algebrization procedure the element $dg_i-\alpha_i df$ occurs in the kernel of $\hat{S}_n\to\hat{\mathfrak{O}}_n$ because of the presence in $\mathfrak{O}_n$ of a certain infinite sequence of elements in the subfield $Q(R)(f, g_i)$; for a given i this sequence of elements is the same, regardless of n (provided of course that $n\geqslant i$). Hence $\mathfrak{O}_n\subset\mathfrak{O}_{n+1}$ for all n.

Now if we set $Y_i = dg_i$ and $X = df$ in $\hat{S}_n$, we may identify $\hat{S}_n$ with

$$\hat{R}[X, Y_1, \ldots, Y_n]/(X^p, Y_1^p, \ldots, Y_n^p).$$

Hence for all n,

$$\hat{\mathfrak{O}}_n = \hat{S}_n/(\{Y_i - \alpha_i X\}_{1 \leqslant i \leqslant n}) = \hat{R}[X]/(X^p).$$

Hence the inclusions $\mathfrak{O}_n \subset \mathfrak{O}_{n+1}$ induce isomorphisms of completions. Let

$$\mathfrak{O} = \bigcup_{n=1}^{\infty} \mathfrak{O}_n.$$

We first verify that $\mathfrak{O}$ is noetherian: it is easy to check that in general if

$$\ldots \to \mathfrak{O}_n \to \mathfrak{O}_{n+1} \to \ldots$$

is any inductive system of local rings (with $\mathfrak{m}_n = \max(\mathfrak{O}_n)$), and if $\mathfrak{O}$ is its limit (with $\mathfrak{m} = \max(\mathfrak{O})$), then the natural map

$$\varinjlim_n \operatorname{Gr}_{\mathfrak{m}_n}(\mathfrak{O}_n) \to \operatorname{Gr}_{\mathfrak{m}}(\mathfrak{O})$$

is surjective. However in our case all the $\operatorname{Gr}_{\mathfrak{m}_n}(\mathfrak{O}_n)$ are isomorphic, so that for all n, $\operatorname{Gr}_{\mathfrak{m}_n}(\mathfrak{O}_n) \to \operatorname{Gr}_{\mathfrak{m}}(\mathfrak{O})$ is surjective. Thus we can apply (3.3.2) to obtain: if $L = \bigcup_n \hat{\mathfrak{O}}_n$, with $\mathfrak{N} = \max(L)$, then $\hat{\mathfrak{O}} = L / \bigcap_{\nu=0}^{\infty} \mathfrak{N}^\nu$. However all the $\hat{\mathfrak{O}}_n$ are isomorphic and noetherian. Hence $\bigcap_{\nu=0}^{\infty} \mathfrak{N}^\nu = (0)$, and $\hat{\mathfrak{O}}_n \xrightarrow{\approx} \hat{\mathfrak{O}}$ is an *isomorphism* for all n. In particular $\hat{\mathfrak{O}}$ is noetherian; hence so is $\mathfrak{O}$ (since $\mathfrak{O}$ is a one-dimensional domain, we only have to check that its maximal ideal is finitely generated, by the theorem of Cohen cited at the end of (3.2.2)).

Now it is clear that $[Q(\mathfrak{O}) : Q(R)] = \infty$ by construction, and that $R \hookrightarrow \mathfrak{O} \hookrightarrow \hat{R}$. We want to show that this is a *maximal presentation*, i.e. if $x \in \mathfrak{O}$ and $dx = 0$ in $\hat{\mathfrak{O}}$ then $x \in R$ (where $d : \mathfrak{O} \to \hat{\mathfrak{O}}$ is the differential operator attached to the presentation; see § 4). To see this, take any element x of $\mathfrak{O}$, so in particular x is in $\mathfrak{O}_n$ for some n. Now $Q(\mathfrak{O}_n) = Q(S_n) = S_n \otimes_R Q(R)$. Hence, if t is a regular parameter of R, $t^m x$ is in S_n for some m, and since d is R-linear, $dx = 0$ if and only if $d(t^m x) = 0$. Moreover, since $x \in \mathfrak{O}_n \subset \hat{R}$, $t^m x$ is in R if and only if x is in R. Hence, replacing x by $t^m x$, we may assume x is actually in S_n. Then we can write

$$x = \sum_{\nu = (\nu_0, \ldots, \nu_n)} a_\nu f^{\nu_0} g_1^{\nu_1} \ldots g_n^{\nu_n}$$

with a_ν in R, and the sum is taken over those ν such that $0 \leqslant \nu_j < p$ for $j = 0, \ldots, n$. Then

$$dx = \sum_\mu \Big(\sum_\nu \big[{\nu \atop \mu}\big] a_\nu (f, g)^{\nu - \mu} \Big) (df, dg)^\mu \quad \text{in } \hat{\mathfrak{O}},$$

where $\mu = (\mu_0, \ldots, \mu_n)$ with each $\mu_j < p$ and $\mu_0 + \mu_1 + \ldots + \mu_n > 0$;

$$\big[{\nu \atop \mu}\big] = \prod_{j=0}^{n} \nu_j(\nu_j - 1) \ldots (\nu_j - \mu_j + 1); \quad \nu - \mu = (\nu_0 - \mu_0, \nu_1 - \mu_1, \ldots, \nu_n - \mu_n),$$

and $(f, g)^{\nu-\mu}$ (resp. $(df, dg)^{\mu}$) denotes the monomial in f and the g_i (resp. df and the dg_i) in which the factors appear to the power indicated by the multi-index $\nu-\mu$ (resp. μ) (*). But in $\hat{\mathfrak{O}}$, $dg_i = \alpha_i df$. Hence we may write

$$dx = \sum_{\mu}\Big(\sum_{\nu} \textstyle\binom{\nu}{\mu} a_\nu (f, g)^{\nu-\mu}\Big)(df, \alpha_i df)^{\mu}$$

with notational conventions as above, i.e.

$$dx = \sum_{\mu}\Big(\sum_{\nu} \textstyle\binom{\nu}{\mu} a_\nu (f, g)^{\nu-\mu}\Big)\, \alpha^{\mu}\, df^{|\mu|}$$

with $|\mu| = \mu_0 + \mu_1 + \ldots + \mu_n$ and $\alpha^{\mu} = \alpha_1^{\mu_1} \ldots \alpha_n^{\mu_n}$. Now, since $\hat{\mathfrak{O}} = \hat{\mathrm{R}}[\mathrm{X}]/(\mathrm{X}^p)$ with $\mathrm{X} = df$ (so that $\hat{\mathfrak{O}}$ is a free $\hat{\mathrm{R}}$-module on the 0^{th} through $p-1^{\text{st}}$ powers of df), $dx = 0$ implies that for each non-negative integer $\ell < p$,

$$\sum_{|\mu|=\ell}\Big(\sum_{\nu} \textstyle\binom{\nu}{\mu} a_\nu (f, g)^{\nu-\mu}\Big)\alpha^{\mu} = 0$$

in $\hat{\mathrm{R}}$. Since μ_0 plays no role in α^{μ}, it is possible that two distinct μ's give the same α^{μ}. However if we restrict our attention to those μ with $|\mu| = \ell$, μ is uniquely determined by $\mu_1, \ldots, \mu_n$. Hence the α^{μ} in the sum above are distinct monomials in the α_i and so they are linearly independent over $\mathrm{R}[f, g_1, \ldots, g_n]$. Thus we find that: for each μ with $|\mu| < p$,

$$\sum_{\nu \geqslant \mu} \textstyle\binom{\nu}{\mu} a_\nu (f, g)^{\nu-\mu} = 0$$

in $\hat{\mathrm{R}}$ (where $\nu \geqslant \mu$ means that $\nu_j \geqslant \mu_j$ for $j = 0, \ldots, n$). Now for fixed μ, the $(f, g)^{\nu-\mu}$ (with $\nu \geqslant \mu$) are distinct monomials in f and the g_i, so they are linearly independent over R. Hence for all μ such that $|\mu| < p$, and all $\nu \geqslant \mu$, $\binom{\nu}{\mu} a_\nu = 0$. Now each ν_j and μ_j is less than p, and hence, for $\nu \geqslant \mu$, $\binom{\nu}{\mu} \neq 0$. Thus we find: provided there is *some* μ with $|\mu| < p$ for which $\nu \geqslant \mu$, $a_\nu = 0$. And since the only ν for which $\nu \ngeqslant \mu$ for any μ is $\nu = (0, \ldots, 0)$ (we restricted μ to $|\mu| > 0$), we get: *all* the a_ν are 0 except possible for $\nu = (0, \ldots, 0)$. But this means precisely that x is in R. Q.E.D.

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(*) The symbol is defined to denote 0 if any of the components of the multi-index are negative.