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## An index theorem for systems of difference operators on a half space

# AN INDEX THEOREM FOR SYSTEMS
# OF DIFFERENCE OPERATORS ON A HALF SPACE

*by* David G. SCHAEFFER[1]

In our joint paper [3] we presented an index theorem for a certain class of operators analogous to Tœplitz operators. The index which appeared there was real-valued and it involved Breuer's theory [1] of Fredholm operators in a von Neumann algebra of type II — indeed, this result vas of interest precisely because it was the first example of an index theorem for the Breuer index. In the present paper we extend the theorem of [3] to the matrix case, and we apply this result to prove the equivalence of existence and uniqueness of solution for elliptic systems of difference equations, generalizing the result of [6] for a single equation.

Our main result is stated in § 1 and proved in § 2. These sections overlap somewhat with [2] and [3] — our slightly different point of view here allows us to give simpler proofs of the results from these papers which are needed. (See the discussion at the end of § 1.) In § 3 we state without proof the application of the index theorem to difference equations.

## 1. Statement of the main theorem.

For $j \in \mathbf{Z}^n$ a multi-integer let $T_j$ be the translation on $L^2(\mathbf{R}^n)$,

$$(1.1) \qquad T_j v(x) = v(x+j),$$

and let $M_\nu$ be the $\nu \times \nu$ matrix algebra. If $\varphi \in C(\mathbf{T}^n, M_\nu)$ is a continuous, matrix-valued function on the torus with Fourier series $\sum_j m_j e^{i\langle j, \cdot \rangle}$, then the formula $L_\varphi = \sum_j m_j \otimes T_j$ defines a bounded linear operator on $\mathbf{C}^\nu \otimes L^2(\mathbf{R}^n)$. In this paper we study the restriction of such operators to a half space $H = \{x \in \mathbf{R}^n : \langle x, N \rangle \geq 0\}$, where $N$ is a unit vector in $\mathbf{R}^n$. Let $E = \chi_{[0,\infty)}(\langle x, N \rangle)$ be the projection onto $L^2(H)$, and for $\varphi \in C(\mathbf{T}^n, M_\nu)$ let $W_\varphi = E L_\varphi E$.

Let $\mathcal{N}_0$ be the von Neumann algebra of operators on $L^2(\mathbf{R}^n)$ generated by multi-

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integer translations $T_j$ and multiplication by functions of $\langle x, \mathrm{N} \rangle$. Any operator $A \in \mathcal{N}_0$ may be expressed uniquely as a weakly convergent sum :

$$A = \sum_{j \in \mathbf{Z}^n} c_j(\langle x, \mathrm{N} \rangle) T_j$$

where $c_j \in \mathrm{L}^\infty(\mathbf{R}^1)$. Two such operators multiply according to the extension by linearity of the basic product formula

**(1.2)**           $c(\langle x, \mathrm{N} \rangle) T_j d(\langle x, \mathrm{N} \rangle) T_k = c(\langle x, \mathrm{N} \rangle) d(\langle x+j, \mathrm{N} \rangle) T_{j+k}.$

In § 1 of [6] we defined a faithful, normal trace $\tau : \mathcal{N}_0^+ \to [0, \infty]$ such that for non-negative operators

**(1.3)**           $\tau(\sum_j c_j(\langle x, \mathrm{N} \rangle) T_j) = \int_{-\infty}^{\infty} c_0(t)\, dt,$

where $c_0$ is the coefficient of the identity translation in the sum on the left in **(1.3)**. (In particular, for non-negative operators $c_0 \geq 0$ almost everywhere.) Let $\mathcal{N} = \mathrm{M}_\nu \otimes \mathcal{N}_0$; we shall also use $\tau$ for the natural trace on $\mathcal{N}$ defined by

**(1.4)**           $\tau(m \otimes A) = \mathrm{tr}(m)\tau(A).$

Of course $W_\varphi \in \mathcal{N}$ for every $\varphi \in \mathrm{C}(\mathbf{T}^n, \mathrm{M}_\nu)$.

If $P \in \mathcal{N}$ is a projection, we shall call $\tau(P)$ the (relative) dimension of the subspace range $P$. Let $\mathcal{K}$ be the uniformly closed, two sided ideal in $\mathcal{N}$ generated by operators whose range has finite relative dimension. Following Breuer [1] we shall say that $A \in \mathcal{N}$ is (generalized) Fredholm if $A$ is invertible modulo $\mathcal{K}$. It was shown by Breuer that if $A$ is Fredholm, then $\dim \ker A$ and $\dim \ker A^*$ are finite, so that one may define a real-valued index of $A$

**(1.5)**           $i(A) = \dim \ker A - \dim \ker A^*.$

This index has the same algebraic and invariance properties as the ordinary Fredholm index. In particular, even though the index is a real number, it is unchanged by a continuous deformation.

*Theorem 1.* — *If* $\varphi \in \mathrm{C}(\mathbf{T}^n, \mathrm{M}_\nu)$, $W_\varphi$ *is (generalized) Fredholm if and only if* $\det \varphi$ *is non-vanishing on* $\mathbf{T}^n$, *and in this case the index of* $W_\varphi$ *equals the mean winding number of* $\det \varphi$ *along the line* $\{t\mathrm{N}\}$,

$$\lim_{\mathrm{T} \to \infty} (2\mathrm{T})^{-1}(\arg \det \varphi(\mathrm{TN}) - \arg \det \varphi(-\mathrm{TN})).$$

The above theorem is the main result of this paper. The simplest example to which it applies occurs in the case of a pure shift $S_j = E T_j E$ for $\nu = 1$. Note that according to (1.2)

**(1.6)**           $S_j S_k - S_{j+k} = \begin{cases} \chi_\mathrm{J}(\langle x, \mathrm{N} \rangle) S_{j+k} & \text{if} \quad \langle j, \mathrm{N} \rangle < 0 \quad \text{and} \quad \langle k, \mathrm{N} \rangle > 0 \\ 0 & \text{otherwise} \end{cases}$

where $\chi_J$ is the characteristic function of the interval $J = [0, -\langle j, N\rangle]$. Thus $S_j$ is Fredholm, since by (1.6) $S_{-j}$ is an inverse of $S_j$ modulo $\mathcal{K}$. Of course $\ker S_j$ and $\ker S_j^*$ may be determined by inspection. For example, if $\langle j, N\rangle \geq 0$ then $\ker S_j^*$ is trivial and $\ker S_j$ equals the range of the projection $\chi_J(\langle x, N\rangle)$ where $J = [0, \langle j, N\rangle]$. Hence for $\langle j, N\rangle \geq 0$ :

$$i(S_j) = \tau(\chi_J(\langle x, N\rangle)) - 0 = \langle j, N\rangle,$$

which is equal to the mean winding number of $e^{i\langle j, \cdot\rangle}$ along $\{tN\}$. The case $\langle j, N\rangle \leq 0$ may be checked similarly.

Let $\mathcal{A}$ be the uniformly closed sub-algebra of $\mathcal{N}$ generated by the operators $\{W_\varphi : \varphi \in C(\mathbf{T}^n, M_v)\}$. In § 2 we introduce a symbol calculus for the algebra $\mathcal{A}$, a homomorphism $\sigma : \mathcal{A} \to C(\mathbf{T}^n, M_v)$ such that $\ker \sigma = \mathcal{A} \cap \mathcal{K}$ and $\sigma(W_\varphi) = \varphi$. Thus an operator $A \in \mathcal{A}$ is Fredholm if and only if $\sigma(A)$ is invertible in $C(\mathbf{T}^n, M_v)$; indeed $A - W_{\sigma(A)} \in \mathcal{K}$, so if $\sigma(A)$ is invertible the index of $A$ equals the mean winding number of $\sigma(A)$. Hence it is a trivial matter to extend theorem 1 to an index theorem for any operator in $\mathcal{A}$.

Let $\mathcal{G}$ be the group of invertible elements in $C(\mathbf{T}^n, M_v)$. Both the analytic and topological indices depend only on the symbol, and they are constant on any homotopy class of $\mathcal{G}$. Therefore to prove theorem 1 it would be sufficient to check the index formula on one representative from each homotopy class. In the scalar case $v = 1$, for any $\varphi \in \mathcal{G}$ we may write :

$$\varphi(\xi) = e^{i\langle k, \xi\rangle}\psi(\xi),$$

where $k \in \mathbf{Z}^n$ is a vector whose $\ell^{th}$ component if the winding number of $\varphi$ around the $\ell^{th}$ factor of $\mathbf{T}^n$ and $\psi$ is homotopic to a constant. Thus the components of $\mathcal{G}$ are classified by the winding numbers, and since $\sigma(S_k) = e^{i\langle k, \cdot\rangle}$, the computation above of the index of a pure shift suffices to prove theorem 1 when $v = 1$. In the matrix case, however, more invariants are required to classify the components of $\mathbf{T}$. For example note that $SU_2$ is homeomorphic with $S^3$; thus any symbol $\varphi \in C(\mathbf{T}^3, SU_2)$ of non-zero Brouwer degree cannot be homotopic to a constant, although the winding numbers of $\det \varphi$ certainly vanish, since $\det \varphi(\xi) \equiv 1$. In this paper we make no attempt to classify the components of $\mathcal{G}$; instead we prove theorem 1 by computing the analytic index for a wider class of operators.

The original result in this area, theorem (2.2) of [3], was an index theorem for a certain algebra of operators on $L^2(0, \infty)$. Our representation on a half space in $\mathbf{R}^n$ rather than a half line is suggested by the application to difference equations in § 3. Moreover on a half space the analytic index may be computed directly in the von Neumann algebra on $L^2(H)$ generated by $\mathcal{A}$, avoiding the slightly unnatural passage to a representation on $L^2(0, \infty) \otimes \ell^2(\mathbf{R}_d)$ that was required in [3]. In any event, the formulas of § 2 have obvious analogues which apply to the matrix generalization of the algebra considered in [3].

## 2. Proof of the main theorem.

In this paragraph we summarize certain facts about the trace class that will be needed below. (See chap. I, § 6 of Dixmier [4] for proofs.) Let

$$\mathcal{K}_1^+ = \{K \in \mathcal{N}^+ : \tau(K) < \infty\}$$

and let $\mathcal{K}_1$ be the linear span of $\mathcal{K}_1^+$. Then $\mathcal{K}_1$ is a two-sided ideal of $\mathcal{N}$, contained in $\mathcal{K}$, whose closure equals $\mathcal{K}$. The trace $\tau$ extends uniquely to a linear functional on $\mathcal{K}_1$ with the property that $\tau(AK) = \tau(KA)$ for $A \in \mathcal{N}$, $K \in \mathcal{K}_1$. If $K \in \mathcal{K}_1$ let $\|K\|_1 = \tau(|K|)$ where $|K| = (KK^*)^{1/2}$; then $\|\cdot\|_1$ is a norm on $\mathcal{K}_1$ such that $|\tau(K)| \leq \|K\|_1$ and moreover

$$(2.1) \qquad \|AK\|_1 \leq \|A\| \, \|K\|_1 \quad \text{and} \quad \|KA\|_1 \leq \|A\| \, \|K\|_1$$

for any $A \in \mathcal{N}$. Finally if $K = \sum_\ell K_\ell$ is a series that converges absolutely with respect to both the operator norm and the trace norm, then $K \in \mathcal{K}_1$.

*Lemma* (2.1). — *If* $\varphi, \psi \in C(\mathbf{T}^n, M_\nu)$ *then* $W_\varphi W_\psi - W_{\varphi\psi} \in \mathcal{K}$. *Indeed if*

$$\varphi = \sum_j m_j e^{i\langle j, \cdot \rangle} \quad \text{and} \quad \psi = \sum_j n_j e^{i\langle j, \cdot \rangle}$$

*are smooth, then* $W_\varphi W_\psi - W_{\varphi\psi} \in \mathcal{K}_1$ *and*

$$(2.2) \qquad \tau(W_\varphi W_\psi - W_{\varphi\psi}) = \sum_{\langle j, N \rangle < 0} \langle j, N \rangle \operatorname{tr}(m_j n_{-j})$$

*Proof.* — Suppose that $\varphi, \psi \in C^\infty(\mathbf{T}^n, M_\nu)$ have the indicated Fourier series; then

$$(2.3) \qquad W_\varphi W_\psi - W_{\varphi\psi} = \sum_{j,k} m_j n_k \otimes (S_j S_k - S_{j+k}).$$

Note from (1.6) that for any $m \in M_\nu$:

$$\|m \otimes (S_j S_k - S_{j+k})\|_1 \leq |\langle j, N \rangle| \operatorname{tr}(|m|)$$

Since $\varphi$ and $\psi$ are smooth, the Fourier coefficients of these functions are rapidly decreasing. Hence the series in (2.3) is absolutely convergent in both the operator and trace norms, so $W_\varphi W_\psi - W_{\varphi\psi} \in \mathcal{K}_1$. Moreover the trace of the sum in (2.3) may be evaluated term by term to yield (2.2). Finally it follows by limits that $W_\varphi W_\psi - W_{\varphi\psi} \in \mathcal{K}$ for general $\varphi, \psi \in C(\mathbf{T}^n, M_\nu)$. This completes the proof.

*Lemma* (2.2). — *If* $W_\varphi \in \mathcal{K}$ *then* $\varphi = 0$.

*Proof.* — We may assume without loss of generality in the proof of this lemma that $\nu = 1$, for the matrix operator $W_\varphi$ is generalized compact if and only if each of its entries is generalized compact. If $a \geq 0$ let $E_a$ be the projection $\chi_{[0,a)}(\langle x, N \rangle)$. Suppose that $\varphi \in C(\mathbf{T}^n)$ has Fourier series $\sum c_j e^{i\langle j, \cdot \rangle}$. We claim that

$$(2.4) \qquad \lim_{a \to \infty} a^{-1} \tau(S_{-j} W_\varphi E_a) = c_j.$$

It follows from (1.6) that $\tau(S_{-j}S_k E_a) = 0$ if $j \ne k$ and that

$$\tau(S_{-j}S_j E_a) = \begin{cases} a & \text{if } \langle j, N \rangle \le 0 \\ a - \langle j, N \rangle & \text{if } 0 < \langle j, N \rangle < a \\ 0 & \text{if } \langle j, N \rangle \ge a. \end{cases}$$

Thus for large $a$

$$a^{-1}\tau(S_{-j}S_k E_a) = \delta_{jk} + O(a^{-1}).$$

Therefore (2.4) holds if $\varphi$ is an exponential polynomial, and by limits it holds in the general case.

Suppose now that $W_\varphi \in \mathscr{K}$ for some $\varphi \in C(\mathbf{T}^n)$. Since $\mathscr{K}_1$ is dense in $\mathscr{K}$, for every $\varepsilon > 0$ there exists $K \in \mathscr{K}_1$ such that $\|W_\varphi - K\| \le \varepsilon$. Thus

$$|\tau(S_{-j}W_\varphi E_a)| \le |\tau(S_{-j}(W_\varphi - K)E_a)| + |\tau(S_{-j}K E_a)|$$
$$\le \varepsilon a + \|K\|_1$$

If $\varphi = \sum_j c_j e^{i\langle j, \cdot \rangle}$, then

$$|c_j| \le \limsup_{a \to \infty} a^{-1} |\tau(S_{-j}W_\varphi E_a)| \le \varepsilon.$$

Therefore every Fourier coefficient of $\varphi$ vanishes, so $\varphi$ itself vanishes. This completes the proof.

Let $\mathscr{A}$ be the uniformly closed sub-algebra of $\mathscr{N}$ generated by the operators $\{W_\varphi : \varphi \in C(\mathbf{T}^n, M_\nu)\}$, and let $\pi$ be the canonical projection $\pi : \mathscr{A} \to \mathscr{A}/(\mathscr{A} \cap \mathscr{K})$. By lemma (2.1) the formula $\rho(\varphi) = \pi(W_\varphi)$ defines a norm decreasing *-homomorphism of $C(\mathbf{T}^n, M_\nu)$ into $\mathscr{A}/(\mathscr{A} \cap \mathscr{K})$. (It is easily verified that $\rho(\varphi)^* = \rho(\varphi^*)$.) According to lemma (2.2) the kernel of $\rho$ is trivial. The range of $\rho$ contains a generating set for $\mathscr{A}/(\mathscr{A} \cap \mathscr{K})$; since the range of a homomorphism of two $C^*$-algebras is closed (corollary 1.8.3 of [5]), $\rho$ defines an isomorphism of $C(\mathbf{T}^n, M_\nu)$ onto $\mathscr{A}/(\mathscr{A} \cap \mathscr{K})$. Therefore if $\varphi \in C(\mathbf{T}^n, M_\nu)$, $W_\varphi$ is invertible mod $\mathscr{K}$ if and only if $\varphi$ is invertible in $C(\mathbf{T}^n, M_\nu)$; in other words, $W_\varphi$ is Fredholm if and only if $\det \varphi$ is non-vanishing on $\mathbf{T}^n$.

If $\varphi$ is an invertible element of $C(\mathbf{T}^n, M_\nu)$, then there exists $\psi \in C(\mathbf{T}^n, M_\nu)$ homotopic to $\varphi$, such that $\psi(\xi)$ belongs to the unitary group $U_\nu$ for each $\xi$. Indeed, for $0 \le t \le 1$, let :

$$\Phi(t, \xi) = (\varphi(\xi)\varphi^*(\xi))^{-t/2}\varphi(\xi);$$

then $\Phi(0, \xi) = \varphi(\xi)$ and $\Phi(1, \xi) \in U_\nu$. Of course both the analytic and topological indices are unchanged by this homotopy. Moreover any function in $C(\mathbf{T}^n, U_\nu)$ may be uniformly approximated by a smooth function. Therefore, to prove theorem 1 it suffices to check the index formula for $\{W_\varphi : \varphi \in C^\infty(\mathbf{T}^n, U_\nu)\}$.

Suppose that $W_\varphi$ is a Fredholm operator. Let $W_\varphi = HV$ be the polar decomposition of $W_\varphi$, where $H = (W_\varphi W_\varphi^*)^{1/2}$ and $V$ is a partial isometry. Then $V^*V = I - P$

and $VV^* = I - P'$, where $P$ and $P'$ are projections onto ker $W_\varphi$ and ker $W_\varphi^*$ respectively. Hence $[V, V^*] = P - P'$ belongs to $\mathcal{K}_1$ and $\tau([V, V^*]) = i(W_\varphi)$. However if

$$\varphi \in C^\infty(\mathbf{T}^n, U_\nu),$$

then by lemma (2.1) :

$$H^2 - I = W_\varphi W_\varphi^* - W_{\varphi\varphi^*} \in \mathcal{K}_1,$$

so that

$$W_\varphi - V = (H - I)V = (H + I)^{-1}(H^2 - I)V \in \mathcal{K}_1.$$

Hence $\tau([W_\varphi, W_\varphi^*]) = \tau([V, V^*])$, and we may use (2.2) to evaluate this trace. If $\varphi = \sum_j m_j e^{i\langle j, \cdot \rangle}$ one finds that :

$$\tau([W_\varphi, W_\varphi^*]) = \tau(W_\varphi W_\varphi^* - I) - \tau(W_\varphi^* W_\varphi - I) = \sum_j \langle j, N \rangle \operatorname{tr}(m_j^* m_j).$$

On the other hand, by a trivial computation with Fourier series one sees that if $\varphi = \sum_j m_j e^{i\langle j, \cdot \rangle}$ belongs to $C^\infty(\mathbf{T}^n, M_\nu)$, then

$$(2\pi)^{-n} \int_{\mathbf{T}^n} \operatorname{tr}(\varphi^*(\xi) \nabla_N \varphi(\xi)) \, d\xi = \sum_j \langle j, N \rangle \operatorname{tr}(m_j^* m_j),$$

where $\nabla_N = \frac{1}{i} \sum_{\ell=1}^n N_\ell \frac{\partial}{\partial \xi_\ell}$. Of course if $\varphi \in C^\infty(\mathbf{T}^n, U_\nu)$ then $\varphi^*(\xi) = \varphi^{-1}(\xi)$. Observe that

$$\operatorname{tr}(\varphi^{-1}(\xi) \nabla_N \varphi(\xi)) = \nabla_N(\log \det \varphi(\xi)).$$

Now we may write :

(2.5)                          $\log \det \varphi(\xi) = i\langle k, \xi \rangle + \psi(\xi),$

where $\psi \in C^\infty(\mathbf{T}^n)$ and the $\ell^{th}$ component of $k \in \mathbf{Z}^n$ is the winding number of det $\varphi$ around the $\ell^{th}$ factor of $\mathbf{T}^n$. Note that by (2.5), the mean winding number of det $\varphi$ along $\{tN\}$ equals $\langle k, N \rangle$. But

$$i(W_\varphi) = \tau([W_\varphi, W_\varphi^*]) = (2\pi)^{-n} \int_{\mathbf{T}^n} \nabla_N(\log \det \varphi(\xi)) d\xi = \langle k, N \rangle,$$

since $\int \nabla_N \psi \, d\xi$ vanishes by an integration by parts. This completes the proof of theorem 1.

## 3. Application to elliptic difference equations.

In this section we assume the reader is familiar with the result of [6]. Suppose

$$Q_{\alpha\beta}(D) = \sum_{j \in \mathbf{Z}^n} c_j^{\alpha\beta} T_j, \quad \alpha, \beta = 1, \ldots, \nu$$

is a system of difference operators on $\mathbf{R}^n$. Let $m_j$ be the $\nu \times \nu$ matrix with entries $c_j^{\alpha\beta}$, and let

$$Q(\xi) = \sum_j m_j e^{i\langle j, \xi \rangle}.$$

We whall call $Q_{\alpha\beta}(D)$ properly elliptic if $\det Q(\xi) \neq 0$ for $\xi \in \mathbf{R}^n$ and $\arg \det Q$ is periodic. Consider a boundary value problem for an elliptic system:

(3.1) $\qquad \overset{\nu}{\underset{\beta=1}{\Sigma}} Q_{\alpha\beta}(D) v_\beta(x) = 0 \qquad$ for $\quad \langle x, N \rangle \geq a_\alpha, \alpha = 1, \ldots, \nu$

(3.2) $\qquad \underset{\beta=1}{\Sigma} q_{\gamma\beta}(x, D) v_\beta(x) = g_\gamma(x) \quad$ for $\quad 0 \leq \langle x, N \rangle < b_\gamma, \gamma = 1, \ldots, M.$

Here the unknown functions $v_\beta$ belong to $L^2(H)$; we shall require that $c_j^{\alpha\beta} = 0$ for $\langle j, N \rangle < -a_\alpha$ so that in (3.1) no attempt to evaluate $v_\beta$ in $\mathbf{R}^n - H$ is made. As in [6] we assume the boundary operators $q_{\gamma\beta}(x, D)$ depend on $x$ only through $\langle x, N \rangle$. We regard the boundary data $g$ as an element of $\overset{M}{\underset{\gamma=1}{\oplus}} L^2([0, b_\gamma] \times \partial H)$. Finally we suppose that

$$\overset{\nu}{\underset{\alpha=1}{\Sigma}} a_\alpha = \overset{M}{\underset{\gamma=1}{\Sigma}} b_\gamma.$$

*Theorem 2.* — *Under the above hypotheses, the following two statements are equivalent.*

(i) *Equation* (3.1) *with homogeneous boundary conditions has the unique solution zero in* $\overset{\nu}{\underset{1}{\oplus}} L^2(H)$.

(ii) *For a dense set of boundary data* $g$ *there is at least one solution of* (3.1), (3.2) *in* $\overset{\nu}{\underset{1}{\oplus}} L^2(H)$.

We omit the proof of this theorem, as it is completely analogous to that of [6].

## ACKNOWLEDGEMENT

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