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An extension of Whitney’s spectral theorem


<http://www.numdam.org/item?id=PMIHES_1971__40__139_0>
AN EXTENSION OF WHITNEY'S SPECTRAL THEOREM
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1. Notations and results.

For \( y \in \mathbb{R}^p \) and \( Y \subseteq \mathbb{R}^n \), \(|y|\) denotes the euclidean norm of \( y \) and \( d(y,Y) \) the euclidean distance from \( y \) to \( Y \). If \( Y \) is empty, we write \( d(y,Y) = 1 \).

Let \( \Omega_p \) denote an open set in \( \mathbb{R}^p \) and \( \mathcal{E}(\Omega_p) \) the \( \mathbb{R} \)-algebra of all \( C^\infty \) real-valued functions in \( \Omega_p \). When \( y \in \Omega_p \), let \( \mathcal{F}_y^m \) denote the \( \mathbb{R} \)-algebra of Taylor expansions of order \( m \) at \( y \) of all elements in \( \mathcal{E}(\Omega_p) \); if \( m < \infty \), \( \mathcal{F}_y^m \) is isomorphic to the algebra \( \mathcal{F}_p/m_p^{m+1} \), where \( m_p \) denotes the maximal ideal of the formal power series ring \( \mathcal{F}_p = \mathbb{R}[[y_1, \ldots, y_p]] \); if \( m = +\infty \), \( \mathcal{F}_y^m \) (simply written \( \mathcal{F}_y \)) \(^{(1)}\) is isomorphic to \( \mathcal{F}_p \) (by the generalized Borel theorem).

Let \( T_y^m : \mathcal{E}(\Omega_p)^{\mathcal{F}_y^m} \rightarrow (\mathcal{F}_p)^{\mathcal{F}_y^m} \) denote the projection associating to each function \( G \) its Taylor expansion of order \( m \) at \( y \). If \( Y \) is a compact set in \( \Omega_p \), we write \( |G|_m^Y = \sup \{ |D^kG(y)| : y \in Y \} \). We provide \( \mathcal{E}(\Omega_p)^{\mathcal{F}_y^m} \) with its usual structure of a Fréchet space, defined by the family of all semi-norms \( G \mapsto |G|^Y_m \), where \( Y \) ranges over the set of compacts in \( \Omega_p \) and \( m \in \mathbb{N} \).

Let \( M \) be a submodule of \( \mathcal{E}(\Omega_p)^{\mathcal{F}_y^m} \) and let us write

\[
M = \{ G \in \mathcal{E}(\Omega_p)^{\mathcal{F}_y^m} : \forall y \in \Omega_p, \exists G' \in M \text{ so that } G - G' \text{ is flat at } y \} = \bigcap_{y \in \Omega_p} (T_y)^{-1}(T_yM).
\]

According to a standard result of Whitney (B. Malgrange \([1]\)), \( M \) is the closure \( \overline{M} \) of \( M \) in \( \mathcal{E}(\Omega_p)^{\mathcal{F}_y^m} \); we propose to extend this theorem.

Let \( \Phi \) denote a \( C^\infty \) function from an open set \( \Omega_n \) in \( \mathbb{R}^n \) to \( \mathbb{R}^p \). The mapping \( \Phi \) defines a homomorphism of \( \mathbb{R} \)-algebras \( \Phi^* : \mathcal{E}(\Omega_n) \rightarrow \mathcal{E}(\Omega_p) \). Let \( \Psi \) be a \( \Phi^* \)-homomorphism from \( \mathcal{E}(\Omega_p)^{\mathcal{F}_y^m} \) to \( \mathcal{E}(\Omega_n)^{\mathcal{F}_y^m} \), i.e. \( \Psi \) is a homomorphism of abelian groups and, \( \forall G \in \mathcal{E}(\Omega_p)^{\mathcal{F}_y^m} \) and \( \forall G \in \mathcal{E}(\Omega_n) \): \( \Psi(G) = \Phi^*(G) \Psi(G) \). For \( y \in \Omega_p \) and \( x \in \Phi^{-1}(y) \), the mapping \( \Psi \) induces an \( \mathbb{R} \)-linear mapping \( \Psi_y^m : (\mathcal{F}_y^m)^{\mathcal{F}_y^m} \rightarrow (\mathcal{F}_x^m)^{\mathcal{F}_y^m} \), so that \( T_y^m \circ \Psi = \Psi_y^m \circ T_y^m \).

For \( X \subset \Phi^{-1}(y) \), we note \( \Psi_X^m \) the \( \mathbb{R} \)-linear mapping \( (\mathcal{F}_y^m)^{\mathcal{F}_y^m} \rightarrow (\mathcal{F}_x^m)^{\mathcal{F}_y^m} \) \( x \in X \). Finally, let \( T_X^m \) be the mapping \( \mathcal{E}(\Omega_n)^{\mathcal{F}_y^m} \rightarrow (\mathcal{F}_x^m)^{\mathcal{F}_y^m} \).

We propose to determine the closure \( \overline{\Psi(M)} \) of \( \Psi(M) \) in \( \mathcal{E}(\Omega_n)^{\mathcal{F}_y^m} \). Therefore, let us write

\[
\overline{\Psi(M)} = \{ F \in \mathcal{E}(\Omega_n)^{\mathcal{F}_y^m} : \forall y \in \Omega_p, \exists G \in M \text{ such that } \Psi(G) - F \text{ is flat on } \Psi^{-1}(y) \} = \bigcap_{y \in \Omega_p} (T_y)^{-1}(\Psi_\Phi(y) \circ T_yM).
\]

\(^{(1)}\) We shall omit afterwards the index \( m \), if \( m = +\infty \), and shall write: \( T_y, \Psi_x, \ldots \) instead of \( T_y^m, \Psi_x^m, \ldots \).
We shall prove the following result:

**Theorem (1.1).** — Let us suppose that \( \Phi \) verifies the following condition:

\((H)\) For all compact sets \( X \subset \Omega_\alpha \) and \( Y \subset \Omega_\beta \), there exists a constant \( \alpha \geq 0 \) such that,

\[ \forall y \in Y : \Gamma(y) = \sup_{x \in X \setminus \Phi^{-1}(y)} (d(x, \Phi^{-1}(y))^\alpha / |\Phi(x) - y|) < \infty. \]

Then \( \overline{\Phi(M)} = \overline{\Phi(M)} \).

It is easy to find \( C^\infty \) mappings \( \Phi \) which do not satisfy this condition. Nevertheless, we shall prove the following result:

**Theorem (1.2).** — An analytic mapping \( \Phi \) verifies the condition \((H)\).

Both following paragraphs are devoted to the proofs of these theorems which are independent of each other. In the last paragraph, we give a refinement of the Theorem (1.2), when \( \Phi \) is a polynomial mapping.

### 2. Proof of theorem 1.2.

**Definition (2.1).** — Let \( \mathcal{I} \) be a finitely generated ideal of a subring of the ring of germs at \( x^0 \) in \( \mathbb{R}^n \) of continuous functions with real values. Let \( \varphi_1(x), \ldots, \varphi_r(x) \) denote real valued functions, continuous in a neighborhood of \( x^0 \) and such that their germs at \( x^0 \) generate \( \mathcal{I} \). Let \( V(\mathcal{I}) \) be the set of their zeros.

We say that \( \mathcal{I} \) verifies a \( \text{Lojasiewicz inequality of order } \alpha \geq 0 \) (or simply that \( \mathcal{I} \) verifies \( \mathcal{L}(\alpha) \)) if there exist a constant \( C > 0 \) and a neighborhood \( V \) of \( x^0 \) such that,

\[ \forall x \in V, \sum_{i=1}^s |\varphi_i(x)| \geq C \cdot d(x, V(\mathcal{I}))^\alpha. \]

Let \( \Omega_\beta \) be an open set in \( \mathbb{R}^\beta \), \( \Omega_\alpha \) an open set in \( \mathbb{R}^\alpha \), and \( x = (x_1, \ldots, x_\alpha) \) coordinate systems in \( \Omega_\alpha \) and \( \Omega_\beta \) respectively. Let \( \mathcal{O} \) be the sheaf of germs of analytic functions with real values on \( \Omega_\alpha \times \Omega_\beta \); \( \mathcal{I} \) a sheaf of ideals, analytic and coherent on \( \Omega_\alpha \times \Omega_\beta \). For \( (x^0, y^0) \in \Omega_\alpha \times \Omega_\beta \), we denote \( \mathcal{I}_{(x^0, y^0)} \) the stalk of \( \mathcal{I} \) at the point \( (x^0, y^0) \). Let \( \varphi_1, \ldots, \varphi_s \) be generators of the ideal \( \mathcal{I}_{(x^0, y^0)} \); we denote \( \mathcal{I}_{(x^0, y^0)} \) the ideal generated by \( \varphi_1(x, y^0), \ldots, \varphi_s(x, y^0) \) in the ring \( \mathcal{O}_{(x^0, y^0)} \) of germs at \( (x^0, y^0) \) in \( \Omega_\alpha \times \Omega_\beta \) of analytic functions with real values. Permuting \( x \) and \( y \), we define similarly the ideal \( \mathcal{I}_{(y^0, x^0)} \) of \( \mathcal{O}_{(y^0, x^0)} \). Finally, let \( V(\mathcal{I}) \) be the set of zeros of \( \mathcal{I} \).

Theorem (1.2) is an easy consequence of the following one (Lojasiewicz inequality with a parameter):

**Theorem (2.2).** — Let \( X \) be a compact set in \( \Omega_\alpha \), \( Y \) a compact set in \( \Omega_\beta \). There exists \( \alpha \geq 0 \) such that the ideal \( \mathcal{I}_{(x, y)} \) verifies \( \mathcal{L}(\alpha), \forall (x, y) \in X \times Y \).

Indeed, let us suppose this theorem is true, and let \( \Phi \) be an analytic mapping. Let \( \mathcal{I} \) denote the analytic and coherent sheaf generated on \( \Omega_\alpha \times \Omega_\beta \) by \( \Phi_1(x) - y_1, \ldots, \Phi_\beta(x) - y_\beta \). Let \( X, Y \) be compact sets in \( \Omega_\alpha, \Omega_\beta \) respectively. By (2.2) applied to \( \mathcal{I} \), \( \forall (x^0, y^0) \in X \times Y \), there exists a constant \( C_{(x^0, y^0)} > 0 \) such that for \( x \) in a neighborhood of
x': \left| \Phi(x) - y \right| \geq C_\omega(x', y), d(x, \Phi^{-1}(y))^\omega$. Hence, the set $X$ being compact, there exists a constant $C_\omega > 0$ such that, $\forall x \in X$:

$$\left| \Phi(x) - y \right| \geq C_\omega d(x, \Phi^{-1}(y))^\omega.$$ 

Clearly, condition (H) follows.

Proof of (2.2). — Obviously, condition $\mathcal{L}(x)$ is verified, with $\alpha = 0$, for $(x, y) \notin V(\mathcal{I})$. The set $X \times Y$ being compact, it suffices to find, for $(x^0, y^0) \in V(\mathcal{I})$, an $\alpha \geq 0$ such that $\mathcal{J}_{(x^0, y^0)}$ verifies $\mathcal{L}(x)$ for $(x, y)$ in a neighborhood of $(x^0, y^0)$. We shall suppose that $(x^0, y^0)$ is the origin of $\mathbb{R}^n \times \mathbb{R}^p$. Now, it is enough to prove the following result:

(2.3) There exists an $\alpha \geq 0$ such that $\mathcal{J}^n_{(x, y)}$ verifies $\mathcal{L}(x)$ for $(x, y) \in V(\mathcal{I})$ and $|y|$ small enough.

Indeed, let $\varphi_1(x, y), \ldots, \varphi_n(x, y)$ generate $\mathcal{I}$ in a neighborhood of $(0, 0)$, and let us consider the sheaf $\mathcal{J}$ generated on a neighborhood of the origin of $\mathbb{R}^n \times \mathbb{R}^p$ by $\varphi_1(x + z, y), \ldots, \varphi_n(x + z, y)$. By (2.3) applied to the sheaf $\mathcal{J}$ (with the parameter $(z, y)$ instead of $y$), there exists an $\alpha \geq 0$ such that $\mathcal{J}^n_{(z, y)} = \mathcal{J}^n_{(z, y)}$ verifies $\mathcal{L}(x)$ for $(z, y)$ in a neighborhood of the origin.

Proof of (2.3). — We proceed by induction on the height $k$ of the ideal $\mathcal{J}_{(0, 0)}$. There exist sheafs of ideals $\mathcal{P}^1, \ldots, \mathcal{P}^r$, analytic coherent on a neighborhood of the origin of $\mathbb{R}^n \times \mathbb{R}^p$, such that $\mathcal{P}^i_{(0, 0)}, \ldots, \mathcal{P}^i_{(0, 0)}$ are prime ideals of height $\geq k$, and an integer $\beta \geq 1$, such that:

$$\mathcal{J} \cap (\mathcal{P}^1 \cap \ldots \cap \mathcal{P}^\beta).$$

Clearly, if $\mathcal{P}^i_{(0, 0)}$ verifies $\mathcal{L}(x)$ for $y$ small enough, $\mathcal{J}^n_{(0, 0)}$ verifies $\mathcal{L}(\beta \sum_{i=1}^\beta a_i)$ for $y$ small enough. Hence, we may suppose that $\mathcal{J}^n_{(0, 0)}$ is prime and its height equals $k$.

Let $\varphi(y)$ be analytic in a neighborhood of the origin of $o \times \mathbb{R}^p$ and null in $V(\mathcal{I}) \cap (o \times \mathbb{R}^p)$ in a neighborhood of the origin. Let $\mathcal{J}$ be the analytic coherent sheaf on a neighborhood of the origin of $\mathbb{R}^n \times \mathbb{R}^p$, generated by $\mathcal{I}$ and $\varphi$: obviously, $\mathcal{J}^n_{(0, 0)} = \mathcal{J}^n_{(0, 0)}$ for $y$ small enough. If $\varphi \notin \mathcal{J}^n_{(0, 0)}$, we get $h(\mathcal{J}^n_{(0, 0)}) = k$ and hence the result is proved by the induction hypothesis. Therefore, we may suppose that $\varphi \in \mathcal{J}^n_{(0, 0)}$, i.e. $\mathcal{J}^n_{(0, 0)} = \mathcal{J}^n_{(0, 0)}$ and $\mathcal{J}^n_{(0, 0)}$ is the ideal of germs $\varphi(y)$ null in $V(\mathcal{I}) \cap (o \times \mathbb{R}^p)$.

Lemma (2.4). — With the preceding hypothesis, let $k = l$ be the height of the prime ideal $\mathcal{J}^n_{(0, 0)}$. After an eventual permutation on the coordinates $x_1, \ldots, x_n$, there exist $\varphi_{i_{1}}, \ldots, \varphi_{l} \in \mathcal{I}_{(0, 0)}$ such that $\xi_{i_{1}} = D(\varphi_{i_{1}}, \ldots, \varphi_{l}) \notin \mathcal{I}_{(0, 0)}$.

Proof. — We proceed by induction on the height $k$ of $\mathcal{J}^n_{(0, 0)}$. Let us suppose that $k = l$. There is a sequence $(0, y) \in V(\mathcal{I})$, $y \to 0$, such that for each $i$: $\mathcal{J}^n_{(0, 0)} = 0$ (otherwise $\mathcal{J}^n_{(0, 0)}$ would be generated by $\mathcal{J}^n_{(0, 0)}$). After an eventual linear change of coordinates on the variables $x_1, \ldots, x_n$, we know (following the analytic preparation theorem, Malgrange [1]) that there exists, for each $i$, a distinguished polynomial $\Psi_i = x_{i_{1}} + a_{i, 1}(x', y) . x_{i_{1}}^{i_{1} - 1} + \ldots + a_{i, i}(x', y) \in \mathcal{J}^n_{(0, 0)}$ (we write $x' = (x_2, \ldots, x_n)$ and the $a_{i, i}$, $141$
are analytic functions of \((x', y)\) in a neighborhood of \((o, y')\). Besides, we may suppose that 
\[ \frac{\partial^N Y_i}{\partial x_1^N} \notin \mathcal{I}(o, y'). \] 
(Indeed, there exists a smaller integer \( N \geq 0 \) such that 
\[ \frac{\partial^{N+1} Y_i}{\partial x_1^{N+1}} \notin \mathcal{I}(o, y'); \]
we have only to substitute \( \frac{\partial^N Y_i}{\partial x_1^N} \) for \( Y_i. \)) Hence, there exists \( \varphi_1 \in \mathcal{I}(o, 0) \) such that 
\[ \frac{\partial \varphi_1}{\partial x_1} \notin \mathcal{I}(o, 0). \]

Let \( \mathcal{O}' \) be the sheaf of germs of analytic functions with real values on 
\( \mathbb{R}^{n-1} \times \mathbb{R}^l = \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^l | x_1 = 0 \} \) and let us write \( \mathcal{I}' = \mathcal{I} \cap \mathcal{O}' \). There exists an integer \( i_0 \) such that \( \text{ht} \mathcal{I}_{(o, y')} = k \) for \( i \geq i_0 \); besides, \( \mathcal{O}_{(o, y')}|_{\mathcal{I}_{(o, y')}} \) is a finitely generated module over \( \mathcal{O}_{(o, y')}|_{\mathcal{I}_{(o, y')}} \) and hence their Krull dimensions are equal (by the Cohen-Seidenberg theorem, Malgrange [1]; th. (5.3), chap. III); therefore \( \text{ht} \mathcal{I}'_{(o, y')} = k - 1 \) for \( i \geq i_0 \). Since \( \mathcal{I}'_{(o, 0)} \) is prime, \( \text{ht} \mathcal{I}'_{(o, 0)} = \text{ht} \mathcal{I}'_{(o, y')} \) for \( i \) large enough, so that: \( \text{ht} \mathcal{I}'_{(o, 0)} = k - 1 \); finally, \( \mathcal{I}'_{(o, 0)} \supset \mathcal{I}'_{(o, 0)}. \) Applying the induction hypothesis to the sheaf \( \mathcal{I}' \) (after an eventual permutation on the variables \( x_2, \ldots, x_n \)) we see that there exist \( \varphi_2, \ldots, \varphi_l \in \mathcal{I}'_{(o, 0)} \) such that 
\[ \frac{D(\varphi_2, \ldots, \varphi_l)}{D(x_2, \ldots, x_l)} \notin \mathcal{I}'_{(o, 0)}. \]

\[ \text{D}(\varphi_1, \ldots, \varphi_l) \frac{\partial \varphi_1}{\partial x_1} \text{D}(x_2, \ldots, x_l) = \frac{D(\varphi_2, \ldots, \varphi_l)}{D(x_2, \ldots, x_l)} \notin \mathcal{I}'_{(o, 0)}. \]

Since \( \text{ht} \mathcal{I}'_{(o, 0)} = k - l \) and \( \mathcal{I}'_{(o, 0)} \) is prime, there exist \( \varphi_{l+1}, \ldots, \varphi_k \in \mathcal{I}'_{(o, 0)} \) such that, after an eventual permutation on the coordinates \( y_1, \ldots, y_k \):

\[ \xi_2 = \frac{D(\varphi_{l+1}, \ldots, \varphi_k)}{D(y_1, \ldots, y_{k-l})} \notin \mathcal{I}'_{(o, 0)}; \]
\[ \frac{D(\varphi_1, \ldots, \varphi_k)}{D(x_1, \ldots, x_l, y_1, \ldots, y_{k-l})} = \xi_1, \xi_2 \notin \mathcal{I}'_{(o, 0)}. \]

By the jacobian criterion for regular points, the localized ring \( (\mathcal{O}_{(o, 0)})_{\mathcal{I}(o, 0)} \) is regular of dimension \( k \) and its maximal ideal is generated by \( \varphi_1, \ldots, \varphi_k \). Hence there exists \( \xi_3 \in \mathcal{O}_{(o, 0)} \setminus \mathcal{I}_{(o, 0)} \) such that: \( \xi_3 \mathcal{I}_{(o, 0)} \subset \mathcal{I}_{(o, 0)} \).

Let \( \xi \) be analytic in a neighborhood of \( (o, 0) \in \mathbb{R}^{n+p} \) and inducing the germ \( \xi_1, \xi_2, \xi_3 \) at the origin. Let \( \mathcal{I} \) be the sheaf of ideals generated, on a neighborhood of the origin of \( \mathbb{R}^{n+p} \), by \( \mathcal{I} \) and \( \xi \). For \( (o, y) \in \mathcal{V}(\mathcal{I}), \ y \) small enough:

- There exists \( a \geq 0 \) such that \( \mathcal{I}_{(o, y)} = \mathcal{I}_{(o, y)} + \xi \mathcal{I}_{(o, y)} \) verifies \( \mathcal{L}(a) \) (because \( \text{ht} \mathcal{I}_{(o, 0)} > \text{ht} \mathcal{I}_{(o, 0)} \) and we apply the induction hypothesis).
- \( \xi \mathcal{I}_{(o, y)} \) is contained in the sub-ideal of \( \mathcal{I}_{(o, y)} \) generated by \( \varphi_1, \ldots, \varphi_l \).
- Finally, \( \xi \) belongs to the ideal generated in \( \mathcal{I}_{(o, 0)} \) by the jacobian \( \frac{D(\varphi_1, \ldots, \varphi_l)}{D(x_1, \ldots, x_l)} \).

So Theorem (2.3) is an immediate consequence of the following lemma (Tougeron and Merrien [2], prop. 3, chap. II):

**Lemma (2.5).** — Let \( \mathcal{I} \) be a finitely generated ideal of the ring \( \mathcal{O}_{n} \) of germs at the origin in \( \mathbb{R}^n \) of \( C^\infty \) functions with real values. Let \( \varphi_1, \ldots, \varphi_l \in \mathcal{I} \) and \( \xi \) belonging to the ideal generated
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in \(E_n\) by \(\varphi_1, \ldots, \varphi_\ell\) and all the jacobians \(D(\varphi_1, \ldots, \varphi_\ell)\), so that \(\mathcal{J} \subset (\varphi_1, \ldots, \varphi_\ell)\). Then if \(\mathcal{J} = \mathcal{I} + \xi, E_n\) verifies \(\mathcal{L}(\mathcal{J})\), the ideal \(\mathcal{I}\) verifies \(\mathcal{L}(\sup(2\alpha, \alpha + 1))\).

Remark (2.6). — Let \(\Phi = (\Phi_1, \ldots, \Phi_\ell)\) be a \(C^\infty\) mapping from \(\Omega_n\) to \(\Omega_p\). Let \(\mathcal{E}\) be the sheaf of \(C^\infty\) functions with real values on \(\Omega_n\) (or \(\Omega_p\)); let \(\mathcal{I}\) be the sheaf of ideals generated on \(\Omega_n\) by all the jacobians \(D(\Phi_1, \ldots, \Phi_\ell)\); the set \(\mathcal{V}(\mathcal{I})\) of zeros of \(\mathcal{I}\) is the set of singular points of the mapping \(\Phi\).

Let us consider the following condition:

\[ (H') \quad \forall \omega \in \mathcal{V}(\mathcal{I}), \quad \mathcal{E}_\omega \mathcal{I}_\omega \text{ is (by } \mathcal{E} \text{) a module of finite type over the ring } \mathcal{E}_\omega \text{ (we set } \gamma = \Phi(\omega)) \text{, i.e. by the Malgrange preparation theorem (Malgrange [1]):} \]

\[ (\mathcal{E}_\omega \mathcal{I}_\omega \otimes \mathcal{E}_\omega / \mathcal{m}_\mathcal{E} \to \mathcal{E}_\omega / \mathcal{I}_\omega + \mathcal{m}_\mathcal{E}) \]

is a real vector space of finite dimension \((m_\mathcal{E}: \text{maximal ideal of } \mathcal{E}_\omega)\).

The condition \((H')\) is a very strong one; nevertheless, it is a generic one, i.e. it is verified on an open dense subset of the space of \(C^\infty\) mappings from \(\Omega_n\) to \(\Omega_p\), this space being provided with the Whitney topology. Besides, \((H')\) implies \((H)\).

Indeed, let \(X\) and \(Y\) be compact sets in \(\Omega_n\) and \(\Omega_p\) respectively. By hypothesis, there exists an \(\alpha > 0\) such that, \(\forall \omega \in X \times Y\), the ideal generated by \(\Phi_1(\omega) - y_1, \ldots, \Phi_\ell(\omega) - y_\ell\) and all the jacobians \(D(\Phi_1, \ldots, \Phi_\ell)\) in \(\mathcal{E}_\omega / \mathcal{I}_\omega \) (ring of germs at \((\omega, y)\) in \(R^n \times \{\mathbf{y}\}\) of \(C^\infty\) functions with real values), verifies \(\mathcal{L}(\omega)\). By Lemma (2.5), the ideal generated by \(\Phi_1(\omega) - y_1, \ldots, \Phi_\ell(\omega) - y_\ell\) in \(\mathcal{E}_\omega / \mathcal{I}_\omega \) verifies \(\mathcal{L}(\omega')\), with an \(\omega'\) independent of the point \((\omega, y)\) in \(X \times Y\). Clearly, the condition \((H)\) follows.


With the notations of § 1, we must show that: \(\Psi(M) = \overline{\Psi(M)}\).

\(3.1\) We have: \(\overline{\Psi(M)} \subset \Psi(M)\).

Let \(F \in \Psi(M)\) and let \(\gamma \in \Omega_p\). A finite subset \(X_m\) of \(\Phi^{-1}(y)\) will be called \(m\)-essential \((m\) is a positive integer), if \(\ker \Psi_{\Phi^{-1}(y)} = \ker \Psi_{X_m}\); clearly, there always exist \(m\)-essential sets \(X_m\) such that \(\text{card } X_m \leq \text{card}(\mathcal{F}^m)\).

Let \(X\) be a finite subset of \(\Phi^{-1}(y)\) containing such an \(X_m\). By hypothesis, \(T_{\Phi^{-1}(y)} F\) is in the closure of the finite dimensional real space \(\Psi_{\Phi^{-1}(y)}(T_{\phi^{-1}(y)} M)\), and therefore belongs to it. So there exist \(G^m\) with \(G^m_{\Phi^{-1}(y)} T_{\phi^{-1}(y)} M\) such that: \(T_{\Phi^{-1}(y)} F = \Psi_{\Phi^{-1}(y)}(G^m)\) and \(T_{\phi^{-1}(y)} F = \Psi_{\phi^{-1}(y)}(G^m)\). Obviously, \(G^m - G^m_{\Phi^{-1}(y)} = \ker \Psi_{X_m} = \ker \Psi_{\Phi^{-1}(y)}\); thus: \(T_{\Phi^{-1}(y)} F = \Psi_{\Phi^{-1}(y)}(G^m)\), and \(X\) being arbitrary: \(T_{\Phi^{-1}(y)} F = \Psi_{\Phi^{-1}(y)}(G^m)\).

So, \(W = (\Psi_{\Phi^{-1}(y)} - (T_{\Phi^{-1}(y)} F) \cap T_{\phi^{-1}(y)} M\) is a finite dimensional and non empty affine space. The inverse limit \(W = \lim_{\downarrow} W_n\) is then non empty and contained in \(\lim_{\downarrow} T_{\phi^{-1}(y)} M = T_{\phi^{-1}(y)} M\); besides, \(T_{\Phi^{-1}(y)} F = \lim_{\downarrow} T_{\Phi^{-1}(y)} F_{\Phi^{-1}(y)}(W)\); hence, we have \((3.1)\).
(3.2) We have \( \mathcal{V}(M) \subseteq \mathcal{W}(M) \).

Let \( F \in \mathcal{V}(M) \) and let \( X' \) be a compact subset of \( \Omega \). Let \( X \) be a compact neighborhood of \( X' \) in \( \Omega \) and let us put \( Y = \Phi(X) \) and \( \Phi_0 = \Phi|_{X'} \). Finally let \( \varepsilon \) be a number \( > 0 \) and \( \mu \) be a positive integer. We have only to prove the following result:

(3.3) There exist \( g \in \mathcal{C}(\Omega_\varepsilon) \) with \( g = 1 \) in a neighborhood of \( Y \), and \( G \in M \), such that:

\[ |\Phi^*(g)F - \Psi(G)|^N < \varepsilon. \]

This easily results from two lemmas. We first give a definition:

Definition (3.4). — A subset \( K \) of \( Y \) is \((\alpha, m)\)-elementary if the following conditions are verified:

1) There exists a constant \( C > 0 \) such that, \( \forall x \in X \) and \( \forall y \in K \):

\[ |\Phi(x) - y| \geq C \cdot d(x, \Phi^{-1}(y))^\alpha. \]

2) The dimension of the real vector space \( \mathcal{V}_\alpha^{-m}(T^\alpha M) \) is constant, for \( y \in K \).

Lemma (3.5). — Let us suppose that \( \Phi \) verifies the condition \((H)\) and let \( Z \) be a compact and non empty subset of \( Y \). Then, there exists a closed set \( E(Z)^c \subseteq \mathcal{Z} \) such that each compact set in \( \mathcal{Z} - E(Z)^c \) is \((\alpha, m)\)-elementary (\( m \) is an arbitrary integer, but \( \alpha \) is the real number associated to \( X \) and \( Y \) by the condition \((H)\)).

Proof. — With the notations of (1.1), the function: \( Y \ni y \mapsto \Gamma(y) \) is lower semi-continuous (because, for a fixed \( x \), the mapping \( Y \ni y \mapsto d(x, \Phi^{-1}(y)) \) is lower semi-continuous). So there exists an open dense set \( Z_0 \) in \( \mathcal{Z} \), such that this function is bounded on each compact set in \( Z_0 \).

Let \( y^0 \in \mathcal{Z}_0 \); if \( x^0 \) belongs to the fiber \( \Phi_0^{-1}(y^0) \), we have:

\[ \lim_{y^0 \in \mathcal{Z}_0} \frac{\text{dim} \mathcal{V}_\alpha^{-m}(T^\alpha M)}{\text{dim} \mathcal{V}_\alpha^{-m}(T^\alpha M)} \geq \text{dim} \mathcal{V}_\alpha^{-m}(T^\alpha M). \]

(Indeed, by hypothesis, there exists a constant \( C > 0 \) such that, for each \( y \in \mathcal{Z}_0 \) in a neighborhood of \( y^0 \), we have \( |y^0 - y| \geq C \cdot d(x^0, \Phi^{-1}(y))^\alpha \).

Let \( X(y^0) = \{ x^1(y^0), \ldots, x^s(y^0) \} \) be an \( m \)-essential subset of the fiber \( \Phi_0^{-1}(y^0) \) for \( y^0 \in \mathcal{Z}_0 \).

We can associate to each \( y \in \mathcal{Z}_0 \) a subset \( X(y) = \{ x^1(y), \ldots, x^s(y) \} \) of \( \Phi_0^{-1}(y) \), so that \( \lim_{y^0 \to \mathcal{Z}_0} x^i(y) = x^i(y^0) \) for \( i = 1, \ldots, s \). Clearly, we have the following inequalities, for \( |y - y^0| \) small enough:

\[ \text{dim} \mathcal{V}_\alpha^{-m}(T^\alpha M) \geq \text{dim} \mathcal{V}_\alpha^{-m}(T^\alpha M) \geq \text{dim} \mathcal{V}_\alpha^{-m}(T^\alpha M) = \text{dim} \mathcal{V}_\alpha^{-m}(T^\alpha M). \]

So the function \( Z_0 \ni y \mapsto \text{dim} \mathcal{V}_\alpha^{-m}(T^\alpha M) \) is lower semi-continuous, bounded with integer values. Therefore, there exists an open and non empty subset \( Z_1 \) of \( Z_0 \) in which this function is constant. Then it suffices to put \( E(Z) = Z - Z_1 \).

Lemma (3.6). — Let \( K \) be a compact and \((\alpha, m)\)-elementary subset of \( Y \), and let us suppose that \( m \geq \mu \alpha \). Then we can find \( g \in \mathcal{C}(\Omega_\varepsilon) \) with \( g = 1 \) in a neighborhood of \( K \), and \( G \in M \), such that:

\[ |\Phi^*(g)F - \Psi(G)|^\mu < \varepsilon. \]
Proof. — The following proof takes inspiration from the proof of the spectral theorem (B. Malgrange [1], lemma (1.4), chap. II).

Let \( y_0 \in \mathbb{K} \). By hypothesis, there exists a neighborhood \( V_{y_0} \) of \( y_0 \) and \( G_1, \ldots, G_k \) in \( \mathcal{M} \) such that for \( y \in V_{y_0} \cap \mathbb{K} \), \( \Psi_{\mathcal{F}_y \cup \{y\}}(T_y^w G_1), \ldots, \Psi_{\mathcal{F}_y \cup \{y\}}(T_y^w G_k) \) is a basis of the real vector space \( \Psi_{\mathcal{F}_y \cup \{y\}}(T_y^w \mathcal{M}) \). Hence there exist continuous functions \( \lambda_1, \ldots, \lambda_k \) on \( V_{y_0} \cap \mathbb{K} \), such that:

\[
T_y^w F = \Psi_{\mathcal{F}_y \cup \{y\}}(\sum_{i=1}^{k} \lambda_i(y). T_y^w G_i)
\]

for all \( y \in V_{y_0} \cap \mathbb{K} \). Using a partition of unity, we can find \( G_1, \ldots, G_k \) in \( \mathcal{M} \), continuous functions \( \lambda_1, \ldots, \lambda_k \) on \( \mathbb{K} \), and a constant \( C \), such that, for all \( y \in \mathbb{K} \):

\[
T_y^w F = \Psi_{\mathcal{F}_y \cup \{y\}}(\sum_{i=1}^{t} \lambda_i(y). T_y^w G_i)
\]

and

\[
\sup_{1 \leq i \leq t, y \in \mathbb{K}} |\lambda_i(y)| \leq C.
\]

Let us put \( G_y = \sum_{i=1}^{t} \lambda_i(y) G_i \); clearly, \( F - \Psi(G_y) \) is \( m \)-flat on \( \Phi_0^{-1}(y) \). Let \( \omega \) be a modulus of continuity on the compact set \( \mathbb{X} \) for \( F, \Psi(G_1), \ldots, \Psi(G_k) \); there exists a constant \( C_1 > 0 \) such that \( C_1 \omega \) is a modulus of continuity on \( \mathbb{X} \) for all functions \( F - \Psi(G_y), y \in \mathbb{K} \).

Let \( x \in \mathbb{X} \) and \( a \in \Phi_0^{-1}(y) \) such that \( d(x, \Phi_0^{-1}(y)) = d(x, a) \). The function \( F - \Psi(G_y) \) being \( m \)-flat at \( a \), we have:

\[
|D^k F(x) - D^k \Psi(G_y)(x)| = |(R^n_m(F - \Psi(G_y)))^k(x)| \leq C_1 \cdot d(x, \Phi_0^{-1}(y))^{m-|k|} \cdot \omega(d(x, \Phi_0^{-1}(y))).
\]

Clearly, there exists a constant \( C_2 \) such that \( d(x, \Phi_0^{-1}(y)) \leq C_2 \cdot d(x, \Phi_0^{-1}(y)) \) for all \( x \in \mathbb{X} \) and \( y \in \mathbb{K} \). Hence, the compact \( \mathbb{K} \) being \((x, m)\)-elementary and \( m \geq \mu \), we see that there exist a constant \( C_3 \) and a modulus of continuity \( \omega' \) such that:

\[
|D^k F(x) - D^k \Psi(G_y)(x)| \leq C_3 \cdot |\Phi(x) - y|^{m-|k|} \cdot \omega'(|\Phi(x) - y|)
\]

for all \( n \)-integers \( k \) such that \( |k| \leq \mu \), all \( x \in \mathbb{X} \) and all \( y \in \mathbb{K} \).

Let \( d \) be a real number \( > 0 \). The open cubes of side \( 2d \), centered at the points \((j_i d, \ldots, j_p d)\) \((j_i, \ldots, j_p \) are integers\) constitute an open covering \( \mathcal{F} \) of \( \mathbb{R}^p \). Let \( g_i (i \in \mathcal{F}) \) be a partition of unity subordinate to \( \mathcal{F} \) such that, for \(|k| \leq \mu \),

\[
\sum_{i \in \mathcal{F}} |D^k g_i(y)| \leq \frac{C_3}{d^{\mu}} \quad \text{for all } y \in \mathbb{R}^p
\]

\((C_3 \) is a constant only depending on \( \mu \) and \( p \)). Let \( \mathcal{F}' \) be the finite family of those cubes \( L \in \mathcal{F} \) which meet \( \mathbb{K} \). For \( L \in \mathcal{F}' \), let \( j_L \) be a point in \( L \cap \mathbb{K} \). Let us put:

\[
g = \sum_{L \in \mathcal{F}'} g_L, \quad G = \sum_{L \in \mathcal{F}'} g_L \cdot G_L.
\]
Obviously, \( g = 1 \) in a neighborhood of \( K \) and:

\[
|\Phi^*(g)F - \Psi(G)|^X_{\mu} \leq \sum_{\xi \in Y^*} \sup_{|\xi| \leq \mu} |D^\xi(\Phi^*(g))(F - \Psi(G_{\mu}))| \leq C_{\epsilon} \cdot \omega'(d)
\]

and so, by Leibniz's formula and (3.6.1), (3.6.2):

\[
|\Phi^*(g)F - \Psi(G)|^X_{\mu} \leq C_{\epsilon} \cdot \omega'(d)
\]

where \( C_{\epsilon} \) is independent of \( d \). Hence if we choose \( d \) sufficiently small, the lemma follows.

*Proof of (3.3). —* First let us decompose the compact set \( Y \) with the help of Lemma (3.5). Let \( a \) be the real number associated to \( X \) and \( Y \) by the condition (H) and let \( m \) be an integer \( \geq \mu a \).

Let \( T \) be a well ordered set. We construct, by transfinite induction, a mapping \( T \mapsto Y_T \) with values in the set of compact subsets of \( Y \). If \( 1 \) denotes the first element of \( T \), we put \( Y_1 = Y \). Suppose the mapping is defined in the interval \([i, \tau_1[: \) we put \( Y_{\tau} = \cap_{\tau < \tau_1} Y_{\tau} \) if \( \tau_1 \) has no predecessor; on the other hand, if \( \tau_1 = \tau + 1 \), we put:

\[
Y_{\tau+1} = E(Y_{\tau}) \text{ if } Y_{\tau} = \emptyset \text{ and } Y_{\tau+1} = \emptyset \text{ if } Y_{\tau} = \emptyset.
\]

If the cardinal of \( T \) is sufficiently large, there exist some \( \tau \) such that \( Y_\tau = \emptyset \). Let \( \nu_1 \) be the smallest element \( \tau \) of \( T \) such that \( Y_\tau = \emptyset \): we have \( \nu_1 = \nu + 1 \) for a \( \nu \in T \) (otherwise, we should have \( \cap_{\nu \leq \nu_1} Y_{\nu} = \emptyset \), which is absurd, because the \( Y_\nu, \nu < \nu_1 \), are compact and non empty sets such that \( Y_{\nu+1} \subset Y_{\nu} \) for each \( \tau \)). Let us consider the following assertion:

\((H_\nu) \) *There exist \( g_\nu \in \mathcal{E}(\Omega_p) \) with \( g_\nu = 1 \) in a neighborhood \( V_\nu \) of \( Y_\nu \), and \( G_\nu \in M \), such that \( |\Phi^*(g_\nu)F - \Psi(G_\nu)|^X_{\mu} < \epsilon \).*

The set of all \( \tau \) such that \((H_\nu) \) is true is non empty: Indeed, by (3.6), it contains \( \nu \) (because \( Y_\nu \) is a compact and \( (x, m) \)-elementary set). Let \( \tau_1 \) be the smallest element of this set: we have to show that \( \tau_1 = 1 \).

Indeed, suppose that \( \tau_1 > 1 \). Necessarily, \( \tau_1 = \tau + 1 \) for an element \( \tau \in T \) (otherwise, we should have \( \cap_{\tau < \tau_1} Y_{\tau} = \emptyset \) and therefore \( Y_{\tau} \subset V_{\tau_1} \), hence \((H_\nu) \), for a \( \tau < \tau_1 \), which is absurd).

We have \( |\Phi^*(g_\nu)F - \Psi(G_\nu)|^X_{\mu} < \epsilon \), with \( g_\nu = 1 \) in an open neighborhood \( V_{\tau_1} \) of \( Y_{\tau_1} \). Let us put \( K = Y_{\tau_1} - V_\nu \): \( K \) is a compact and \( (x, m) \)-elementary subset of \( \Omega_p \).

By (3.6), applied to \( \Phi^*(1 - g_\nu)F \) instead of \( F \), there exist \( h \in \mathcal{E}(\Omega_p) \) with \( h = 1 \) in a neighborhood of \( K \), and \( G \in M \), such that:

\[
|\Phi^*(h(1 - g_\nu))F - \Psi(G)|^X_{\mu} < \epsilon - \epsilon'.
\]

Let us put \( g_\tau = g_\nu + h - h \cdot g_\nu \) and \( G_\tau = G + G_\nu \). Clearly, \( g_\tau \in \mathcal{E}(\Omega_p) \), \( g_\tau = 1 \) in a neighborhood of \( Y_{\tau} \), \( G_\tau \in M \), and \( |\Phi^*(g_\tau)F - \Psi(G_\tau)|^X_{\mu} < \epsilon \). Hence condition \((H_\nu) \) is fulfilled, which is absurd.

*Remark (3.7). —* I do not know if Theorem (1.1) is always true without the hypothesis (H): unfortunately, I have no counter-example.
4. A refinement of theorem 1.2 when \( \Phi \) is polynomial.

Let us recall the following definition: a set in \( \mathbb{R}^n \) is **semi-algebraic** if it is a finite union of subsets \( X_i \), each \( X_i \) being defined by a finite number of polynomial equalities or inequalities.

The image of a semi-algebraic set by a polynomial mapping \( \Phi : \mathbb{R}^n \rightarrow \mathbb{R}^p \) is semi-algebraic (this is a fundamental result of Seidenberg and Tarski, cf. [3]); if \( X \) and \( Y \) are semi-algebraic sets in \( \mathbb{R}^n \) and if \( X \subseteq Y \), the closure of \( X \) in \( Y \) and \( Y \setminus X \) are semi-algebraic. Finally, it is obvious that finite unions or finite intersections of semi-algebraic sets are semi-algebraic.

Let \( \Phi \) be a polynomial mapping from \( \Omega_\alpha = \mathbb{R}^n \) to \( \Omega_\beta = \mathbb{R}^p \) and let \( X \) and \( Y \) be compact and semi-algebraic sets in \( \mathbb{R}^n \) and \( \Phi(\mathbb{R}^n) \) respectively. The following theorem improves (1.2):

**Theorem (4.1).** There exists a closed and semi-algebraic set \( D(Y) \) in \( Y \), such that \( Y \setminus D(Y) \) is dense in \( Y \), and constants \( G > 0, \alpha > 0, \beta > 0 \) such that, for all \( x \in X \) and \( y \in Y \):

\[
|\Phi(x) - y| \geq C \cdot d(x, \Phi^{-1}(y))^\alpha \cdot d(y, D(Y))^{\beta}.
\]

*Proof.* By (1.2), there exists an \( \alpha \geq 0 \) (we suppose that \( \alpha \) is an integer, which is always possible) such that, for all \( y \in Y \):

\[
\Gamma(y) = \sup_{x \in X \setminus \Phi^{-1}(y)} (d(x, \Phi^{-1}(y))^\alpha) \cdot d(y, D(Y))^\beta < \infty.
\]

Let us put

\[
D(Y) = \{ y \in Y | \Gamma(y) \text{ is not bounded in every neighborhood of } y \}.
\]

Clearly, \( D(Y) \) is closed and \( Y \setminus D(Y) \) is dense in \( Y \) (because the mapping \( Y \ni y \mapsto \Gamma(y) \) is lower semi-continuous). Let us verify that \( D(Y) \) is semi-algebraic.

First, the set

\[
A_1 = \{(x, y, \tau) \in X \times Y \times \mathbb{R}^+ | \Phi(x) - y \geq \tau \cdot d(x, \Phi^{-1}(y))^\alpha \}
\]

is semi-algebraic. Indeed, \( A_1 \) is the image of the semi-algebraic set

\[
A_0 = \{(x', y) \in X \times \mathbb{R}^n \times Y \times \mathbb{R}^+ | \Phi(x') = y \text{ and } \Phi(x) - y \geq \tau \cdot |x - x'|^\alpha \}
\]

by the projection: \( X \times \mathbb{R}^n \times Y \times \mathbb{R}^+ \rightarrow X \times Y \times \mathbb{R}^+ \). Now the set

\[
A_2 = \{(y, \tau) \in Y \times \mathbb{R}^+ | \exists x \in X \text{ such that } \Phi(x) - y \leq \tau \cdot d(x, \Phi^{-1}(y))^\alpha \}
\]

is semi-algebraic, because it is the image of \( (X \times Y \times \mathbb{R}^+) \setminus A_1 \) by the projection: \( X \times Y \times \mathbb{R}^+ \rightarrow Y \times \mathbb{R}^+ \). Clearly, we have

\[
D(Y) \times \{0\} = \tilde{A}_2 \cap Y \times \{0\},
\]

and therefore \( D(Y) \) is semi-algebraic.

Let us prove inequality (4.1.1) (the proof is similar to that of Lemma 1 in [4]).

Let us put:

\[
B_1 = \{(y, \delta, \tau) \in Y \times \mathbb{R}^+ \times \mathbb{R}^+ | d(y, D(Y)) \geq \delta \}
\]

and

\[
B_2 = \{(y, \delta, \tau) \in B_1 | \forall x \in X, \Phi(x) - y \geq \tau d(x, \Phi^{-1}(y))^\alpha \}.
\]
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