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ON FINITE SIMPLE GROUPS OF CHARACTERISTIC 2 TYPE

by DANIEL GORENSTEIN (1)

1. Introduction. — A large portion of Thompson's fundamental classification of N-groups (nonsolvable finite groups all of whose local subgroups are solvable [5]) deals with the case that $2 \in \pi_4$; that is, $SCN_3(2)$ (2) is non-empty in the given group G and a Sylow 2-subgroup of G normalizes no nontrivial subgroups of G of odd order. The assumption that G is an N-group implies, in particular, that the 2-local subgroups of G are solvable and hence are 2-constrained by Theorem (6.3.3) of [4] and the definition of 2-constraint as given in Section (8.1) of [4]. These same two conditions: *2 is in* π_4 *and all 2-local subgroups are 2-constrained* appear to be satisfied by the simple groups of Lie type defined over fields of characteristic 2 (with certain obvious exceptions of low rank). In fact, we shall show in Section 5 that these two conditions hold in any group with $SCN_3(2)$ non empty in which the centralizer of every involution is 2-constrained and has no nontrivial normal subgroups of odd order. In those families of simple groups of Lie type over fields of characteristic 2 in which the centralizers of involutions are known, these latter conditions are satisfied and it is very likely that they hold in every such family. For this reason we shall say that an arbitrary group in which the italicized conditions hold is of *characteristic 2 type.*

However, these two properties by no means characterize the groups of Lie type over fields of characteristic 2 among even the known simple groups, for it can be shown, using the above-mentioned result of Section 5, that they are satisfied, for instance, by the groups $G_2(3)$, M_{22}, M_{23}, M_{24}, Suzuki's new simple group, and the large Leech-Conway group. On the other hand, at the present time at least, only a finite number of such examples are known. Hence it is reasonable to regard these two properties as somehow connected with simple groups whose definitions are related to fields of characteristic 2, and serves to explain our use of the term " characteristic 2 type ".

The enormous difficulties which Thompson overcame in classifying simple groups of characteristic 2 type under the additional assumption that all 2-local subgroup are solvable, together with the fact that some of the new " exceptional " simple groups are

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(2) Equivalently, a Sylow 2-subgroup of G contains a normal elementary abelian subgroup of order 8. In general, we follow the terminology and notation of [4].

of this type, clearly indicates that a complete classification of simple groups of characteristic 2 type may well be one of the major problems in the study of simple groups.

The purpose of this paper is primarily to call attention to this important family of simple groups and at the same time to establish two elementary general properties of such groups. The first of these generalizes a corresponding result of Thompson's in the N-group paper and the proof is modelled after his.

Theorem 1. — If G *is a simple group of characteristic* 2 *type, then* $O(H)=1$ *for every* 2*-local subgroup* H *of* G.

Our second result, which is perhaps surprising, is the following:

Theorem 2. — If G *is a simple group of characteristic* 2 *type, then the center of* $O_2(M)$ *is non cyclic for some maximal* 2*-local subgroup* M *of* G.

Theorem 2 is an easy consequence of Theorem 1 and the following characterization of the simple groups $L_2(2^n)$, $Sz(2^n)$, and $U_3(2^n)$, which in turn is a direct corollary of Bender's recent classification of groups which contain a strongly embedded subgroup ([1], [2]).

Theorem 3. — If G *is a simple group with only one conjugacy class of maximal* 2*-local subgroups, then* G *is isomorphic to* $L_2(2^n)$, $Sz(2^n)$, *or* $U_3(2^n)$ *for some* $n \geq 2$.

Although this last result is very special, it may have applicability in the study of simple groups and, in particular, those of characteristic 2 type, for in some situations it may enable one to obtain information concerning the subgroup structure of G related to *odd* primes. Indeed, assume that for some odd prime p, $SCN_3(p)$ is nonempty in G and every maximal 2-local subgroup of G contains an element of $A_i(p)$ for some i. (For the definition of the sets $A_i(p)$, see Section (8.6) of [4]. In particular, any p-subgroup of G which contains an elementary abelian subgroup of order p^3 lies in $A_3(p)$.) Suppose that in some particular case one is able to show that each element X of $A_i(p)$ is contained in a unique maximal local subgroup M_X of G and that, in addition, M_X is a 2-local subgroup of G. If also each maximal 2-local subgroup contains an element of $A_i(p)$, it will then follow directly that G possesses only one conjugacy class of maximal 2-local subgroups-namely, the subgroups M_X as X ranges over the elements of $A_i(p)$. Hence by Theorem 3, G must be isomorphic to $L_2(2^n)$, $Sz(2^n)$, or $U_3(2^n)$ for some $n \geq 2$. However, this is impossible as $SCN_3(p)$ is empty in each of these groups for all odd p. Thus under the given hypothesis on the maximal 2-local subgroups of G, we see that there must exist at least one element X of $A_i(p)$ which violates the above conclusions. Under appropriate circumstances, this fact should have consequences for the structure of the p-local subgroups of G and, in particular, for the centralizers of elements of order p in G.

2. Proof of Theorem 1. — We carry out the proof in a sequence of lemmas. First of all, the Thompson transitivity theorem (Theorem (8.5.4) of [4]) holds in G for the elements of $SCN_3(2)$ inasmuch as all the 2-local subgroups of G are 2-constrained by assumption. Since a Sylow 2-subgroup of G normalizes no non-identity subgroups of G of odd order, we obtain the following basic result as a corollary of the transitivity theorem:

Lemma 1. — *If* $A \in SCN_3(2)$, *then* A *normalizes no non-identity subgroups of* G *of odd order.*

Following the terminology of [1] and [5], if S is a Sylow 2-subgroup of G, $\mathscr{U}(S)$ denotes the set of normal abelian subgroups B of S of type (2, 2) subject to the condition that $B \subseteq Z(S)$ if Z(S) is noncyclic. Moreover, $\mathscr{U}(2)$ denotes the set of elements of $\mathscr{U}(S)$ as S ranges over the Sylow 2-subgroups of G. We note that, by Lemma (8.9) of [1], any element of $\mathscr{U}(S)$ is contained in some element of $SCN_3(S)$. Moreover, it follows at once from the definition of $\mathscr{U}(2)$ that if $B \in \mathscr{U}(2)$, then $B \in \mathscr{U}(S)$ for any Sylow 2-subgroup S of N(B) and S is a Sylow 2-subgroup of G.

We now prove

Lemma 2. — *If* $B \in \mathscr{U}(2)$, *then*:

(i) $B \subseteq O_2(C(b))$ *for each* b *in* $B^{\#}$;

(ii) B *centralizes every subgroup of* G *of odd order that it normalizes.*

Proof. — We first prove that (i) implies (ii). Indeed, if Q is a B-invariant subgroup of G of odd order, then $Q = \langle C_Q(b) \mid b \in B^{\#} \rangle$ by Theorem (6.2.4) of [4] as B is a noncyclic abelian group of order prime to that of Q. But $C_Q(b) \subseteq C(b)$ and $B \subseteq O_2(C(b))$ by (i), which we are assuming; so $[C_Q(b), B] \subseteq Q \cap O_2(C(b)) = 1$. Hence B centralizes $C_Q(b)$ for each b in $B^{\#}$ and so B centralizes Q, thus proving our assertion.

We turn now to (i). We let S be a Sylow 2-subgroup of G for which $B \in \mathscr{U}(S)$ and we let A be an element of $SCN_3(S)$ containing B. Now choose b in $B^{\#}$ and set $C = C(b)$. Then $A \subseteq C$ as A is abelian. Since A normalizes O(C), it follows from Lemma 1 that $O(C) = 1$. Since C is 2-constrained, we conclude that $C_C(O_2(C)) \subseteq O_2(C)$.

Now let R be a Sylow 2-subgroup of C containing $S \cap C$. By definition of B, $C_S(B)$ has index at most 2 in S and so $|R : (S \cap C)| \leq 2$. Moreover, if equality holds, then R is a Sylow 2-subgroup of G. Setting $P = O_2(C)$, we have that $P \subseteq R$ and that $C_C(P) \subseteq P$. Hence if $B \subseteq Z(R)$, then $B \subseteq C_R(P)$, whence $B \subseteq P$ and (i) holds. In particular, this will be the case if either $B \subseteq Z(S)$, in which case $R = S$, or if $R = S \cap C$ is of index 2 in S, in which case $R = C_S(B)$. Thus we can assume henceforth without loss that Z(S) is cyclic and that either $R = S$ or $R \supset S \cap C$.

Suppose first that $R = S$, in which case $\langle b \rangle = \Omega_1(Z(S))$. Setting $\overline{C} = C/\langle b \rangle$ and using bars for images in $\overline{C}$, we have that $\overline{B}$ is of order 2 and is normal in $\overline{S}$, so $\overline{B} \subseteq Z(\overline{S})$. Thus $\overline{B}$ centralizes $\overline{P}$ and consequently B stabilizes the chain: $P \supseteq \langle b \rangle \supset 1$. Since each member of this chain is normal in C, so also is its stabilizer K. But $K/C_C(P)$ is a 2-group by Theorem (5.3.2) of [4]. Since $C_C(P) \subseteq P$ is also a 2-group, K is thus a normal 2-subgroup of C, whence $K \subseteq O_2(C) = P$. However, as we have argued above, $B \subseteq K$ and so $B \subseteq P$ in this case as well.

Assume finally that $R \supset S \cap C$, in which case R is a Sylow 2-subgroup of G, $\langle b \rangle = \Omega_1(Z(R))$, and $b \notin Z(S)$. It will suffice to show that B is normal in R, for then $B \in \mathcal{U}(R)$ as R is a Sylow 2-subgroup of G and Z(R) is cyclic and hence the argument of the preceding paragraph can be repeated verbatim with R in place of S to yield the same conclusion $B \subseteq P$.

Since $b \notin Z(S)$, $S \cap C = C_S(B)$ in the present case. Thus $B \subseteq Z = \Omega_1(Z(S \cap C))$. Since $S \cap C$ is of index 2 in R, $S \cap C$ is normal in R and, as Z is characteristic in $S \cap C$, we see that Z is normal in R. Hence we are done if $B = Z$, so we may assume that $B \subset Z$, in which case Z is elementary abelian of order at least 8. We shall derive a contradiction in this case by arguing that Z(R) is noncyclic. Indeed, if $x \in R-(S \cap C)$, then $x^2 \in S \cap C$ and so x^2 centralizes Z. Thus x acts on Z, regarded as a vector space over the field with two elements, as a linear transformation of order 2. Since the dimension of Z is at least 3, x must therefore have at least two Jordan blocks and so $Z_0 = C_Z(x)$ has order at least 4. But Z_0 centralizes both $S \cap C$ and x and so centralizes $R = \langle S \cap C, x \rangle$. Thus $Z_0 \subseteq Z(R)$ and, as Z_0 is noncyclic, so also is Z(R). This completes the proof of (i) and the lemma.

Now let H be an arbitrary 2-local subgroup of G, so that $H = N(P)$ for some non-identity 2-subgroup P of G. Note that $O(H)P = O(H) \times P$ as O(H) and P are normal subgroups of H of coprime orders. In addition, we let S be any Sylow 2-subgroup of G containing P and fix this notation for the balance of the proof.

Lemma 3. — *If* O(H)P *is contained in a* 2-*local subgroup* K *of* G *for which* $O(K) = 1$, *then* $O(H) = 1$.

Proof. — Let K be such a 2-local subgroup of G and set $Q = O_2(K)$. Since K is 2-constrained, $C_K(Q) \subseteq Q$. In particular, $C_{O(H)}(Q) = 1$. Hence if we apply Theorem (5.3.4) of [4] to the action of $O(H) \times P$ on Q, we conclude that $C_{O(H)}(Q_0) = 1$, where $Q_0 = C_Q(P)$. On the other hand, $Q_0 \subseteq H = N(P)$, so $[O(H), Q_0] \subseteq O(H) \cap Q = 1$. Thus Q_0 centralizes O(H) and, as $C_{O(H)}(Q_0) = 1$, we obtain the desired conclusion $O(H) = 1$.

Lemma 4. — *If* H *contains an element* B *of* $\mathcal{U}(S)$, *then* $O(H) = 1$.

Proof. — Suppose B is an element of $\mathcal{U}(S)$ contained in H and set $K = N(B)$. It will suffice to prove that K contains O(H)P and that $O(K) = 1$, for then the desired conclusion will follow from the preceding lemma.

We have $S \subseteq K$ as B is normal in S. But S normalizes O(K) and so $O(K) = 1$ since $2 \in \pi_4$ by hypothesis. Furthermore, B normalizes O(H) and so B centralizes O(H) by Lemma 2. Thus $O(H) \subseteq K$. Since $P \subseteq S \subseteq K$, K satisfies all the required conditions and the lemma is proved.

Lemma 5. — *If* P *has order greater than* 2, *then* $O(H) = 1$.

Proof. — Assume $|P| > 2$ and let B be an arbitrary element of $\mathcal{U}(S)$. Since $C_S(B)$ has index at most 2 in S and $P \subseteq S$, it follows that $C_P(B) \neq 1$. However, $C_P(B)$ is normal in P as P normalizes B and therefore $Z = C_P(B) \cap Z(P) \neq 1$. Setting $K = N(Z)$, it follows that K is a 2-local subgroup of G containing both B and P. Since $Z \subseteq S$, we can apply Lemma 4 with K, Z, B in the roles of H, P, B respectively to conclude that $O(K)=1$. On the other hand, $Z \subseteq P$ and P centralizes $O(H)$, so also $O(H) \subseteq K$. Now Lemma 3 yields the desired conclusion $O(H)=1$.

Lemma 6. — *If* P *has order* 2, *then* $O(H)=1$.

Proof. — Assume $|P|=2$, so that $P=\langle x \rangle$, where x is an involution, and $H=C(x)$. Since G is simple, Lemma (5.38) of [5] implies that x centralizes some element B of $\mathscr{U}(2)$. (However, we do not know that $B \in \mathscr{U}(S)$.) Let R be a Sylow 2-subgroup of $C(B)$ containing x and let T be a Sylow 2-subgroup of $N(B)$ containing R. As we have noted earlier, T is a Sylow 2-subgroup of G and $B \in \mathscr{U}(T)$. But $B \subseteq C(x)=H$ and $P=\langle x \rangle \subseteq T$. We can therefore apply Lemma 4 with T in place of S to conclude that $O(H)=1$.

Lemmas 5 and 6 together imply that $O(H)=1$. Since H was an arbitrary 2-local subgroup of G, Theorem 1 is therefore proved.

3. Proof of Theorem 3. — We denote the set of maximal 2-local subgroups of G by $\mathscr{M}$ and we let S be a fixed Sylow 2-subgroup of G. We divide the proof into three short lemmas.

Lemma 1. — S *is contained in a unique element of* $\mathscr{M}$.

Proof. — First of all, $N(S)$ is clearly a 2-local subgroup of G and so is contained in some element M of $\mathscr{M}$. Suppose then that also $S \subseteq N$ with N in $\mathscr{M}$. Since the elements of $\mathscr{M}$ are all conjugate in G by assumption, we have $N=M^g$ for some g in G. Since S is a Sylow 2-subgroup of G, it follows that S and S^g are each Sylow 2-subgroups of M^g and so, by Sylow's theorem, they are conjugate by an element of M^g. Thus

$$(S^g)^{m^g}=S$$

for some m in M. Since $gm^g=mg$, this yields that $S^{mg}=S$, whence $mg \in N(S)$. But $m \in M$ and $N(S) \subseteq M$ by definition of M, so $g \in M$. Thus $N=M^g=M$ and therefore M is the unique element of $\mathscr{M}$ containing S.

Lemma 2. — *If* M *and* N *are distinct elements of* $\mathscr{M}$, *then* $M \cap N$ *has odd order.*

Proof. — Suppose this is false and choose M and N so that a Sylow 2-subgroup T of $M \cap N$ has maximal order. Since the elements of $\mathscr{M}$ are all conjugate in G and since S is contained in some element of $\mathscr{M}$ by Lemma 1, we can assume without loss that $S \subseteq M$. Replacing S by a suitable conjugate in M, we can also suppose that $T \subseteq S$. If $T=S$, then also $S \subseteq N$. But then $M=N$ as S is contained in a unique element of $\mathscr{M}$ by Lemma 1. However, $M \neq N$ by hypothesis. Thus $T \subset S$.

Now let L be an element of $\mathscr{M}$ containing $N(T)$. Then $N_S(T) \subseteq L$. But $B_S(T) \supset T$ as $S \supset T$, so $M \cap L \supseteq N_S(T) \supset T$. Our maximal choice of $M \cap N$ thus forces $M=L$. Similarly a Sylow 2-subgroup R of N containing T is a Sylow 2-subgroup of G and $T \subset R$. But now arguing with N and R as we have just done with M and S, we conclude that $N=L$. Therefore $M=N=L$, giving the same contradiction as in the preceding paragraph.

Lemma 3. — *If* $M \in \mathcal{M}$, *then* M *is strongly embedded in* G.

Proof. — As above, we can assume without loss that $S \subseteq M$. Then $N(S) \subseteq M$ by Lemma 1. Furthermore, if x is an involution of M, $C(x)$ is a 2-local subgroup of G, so $C(x) \subseteq N$ for some N in $\mathcal{M}$. But $x \in M \cap N$, so $M \cap N$ is of even order, whence $M = N$ by the preceding lemma. Thus $C(x) \subseteq M$ for any involution x of M. Finally not every involution of G is contained in M as G is simple and M is a local subgroup of G. We conclude therefore from the definition that M is strongly embedded in G.

Theorem 3 is now an immediate consequence of Bender's theorem ([1], [2]). Indeed, since G is simple and contains a strongly embedded subgroup, his theorem implies that G is isomorphic to $L_2(2^n)$, $Sz(2^n)$, or $U_3(2^n)$, $n \geq 2$.

4. Proof of Theorem 2. — Again we denote the set of maximal 2-local subgroups of G by $\mathcal{M}$ and let S be a Sylow 2-subgroup of G. We assume the theorem is false and first derive the following three lemmas:

Lemma 1. — *If* $M \in \mathcal{M}$ *and* $M \cap S$ *is a Sylow* 2*-subgroup of* M, *then* $M = N(Z(S))$.

Proof. — Since M is a 2-local subgroup of G, we have $O(M) = 1$ by Theorem 1. Hence if we set $P = O_2(M)$ and use the fact that M is 2-constrained, it follows that $C_M(P) \subseteq P$.

On the other hand, $M = N(T)$ for some nontrivial 2-subgroup T of G. Moreover, $T \subseteq S \cap M$ as T is normal in M and $S \cap M$ is a Sylow 2-subgroup of M by hypothesis. Thus $Z(S) \subseteq S \cap M$. But $P \subseteq S \cap M$ for the same reason that T is, so Z(S) centralizes P. Hence $Z(S) \subseteq P$ and therefore $Z(S) \subseteq Z(P)$. However, Z(P) is cyclic since we are assuming Theorem 2 to be false. Hence Z(S) is characteristic in Z(P) and consequently Z(S) is normal in M. Thus $M \subseteq N(Z(S))$ and now the maximality of M yields the desired conclusion $M = N(Z(S))$.

Since every subgroup of a cyclic group is cyclic, our argument also yields :

Lemma 2. — Z(S) *is cyclic.*

Lemma 3. — *If* $M \in \mathcal{M}$, *then* $M = N(Z(S))^g$ *for some* g *in* G.

Proof. — Let T be a Sylow 2-subgroup of G such that $M \cap T$ is a Sylow 2-subgroup of M. By Lemma 1, $M = N(Z(T))$. Since $T = S^g$ for some g in G by Sylow's theorem and since $N(Z(T)) = N(Z(S^g)) = N(Z(S))^g$, we conclude that $M = N(Z(S))^g$, as asserted.

But now the theorem follows directly from Theorem 3. Indeed, by Lemma 3, G has only one conjugacy class of maximal 2-local subgroups, so G is isomorphic to $L_2(2^n)$, $Sz(2^n)$, or $U_3(2^n)$ for some $n \geq 2$ by Theorem 3. However, as is well-known, the center of a Sylow 2-subgroup of each of these groups is elementary of order 2^n and so is non cyclic. On the other hand, S is a Sylow 2-subgroup of G and Z(S) is cyclic by Lemma 2. This contradiction establishes the theorem.

5. Sufficient conditions for characteristic 2 type. — The following result, which is valid for arbitrary primes, facilitates the task of showing in a particular group that its 2-local subgroups are all, in fact, 2-constrained.

Theorem 4. — *Suppose that one of the following two conditions holds in a group* G *for some prime* p:

a) *Every maximal* p*-local subgroup of* G *is* p*-constrained*; *or*

b) *The centralizer of every element of order* p *in* G *is* p*-constrained.*

Then every p*-local subgroup of* G *is* p*-constrained.*

Once again we divide the proof into a sequence of lemmas. Let H be an arbitrary p-local subgroup of G, let Q be a Sylow p-subgroup of $O_{p',p}(H)$ and set $K = QC(Q)$. We fix this notation.

Lemma 1. — *The following conditions hold*:

(i) $Q \neq 1$;

(ii) $K = QC_H(Q)$ *and* K *is a subgroup of* H;

(iii) $O_{p'}(H)K$ *is a normal subgroup of* H;

(iv) Q *is a Sylow* p*-subgroup of* $O_{p',p}(K)$.

Proof. — We have $H = N(T)$ for some non-identity p-subgroup T of G as H is a p-local subgroup of G. Since T is normal in H and Q is a Sylow p-subgroup of $O_{p',p}(H)$, clearly $T \subseteq Q$. In particular, $Q \neq 1$ and (i) holds. Furthermore, $K = QC(Q) \subseteq N(T) = H$, so K is a subgroup of H and hence $K = QC_H(Q)$, proving (ii). This last result implies that K is normal in $N_H(Q)$. But $H = O_{p'}(H)N_H(Q)$ by the Frattini argument, and therefore $O_{p'}(H)K$ is normal in H, so (iii) also holds.

Finally (iii) implies that $O_{p'}(K) \subseteq O_{p'}(H)$ and $O_{p',p}(K) \subseteq O_{p',p}(H)$. Thus every Sylow p-subgroup of $O_{p',p}(K)$ is contained in one of $O_{p',p}(H)$. But Q is normal in K by definition of K and so Q is contained in a Sylow p-subgroup R of $O_{p',p}(K)$, whence $R \subseteq S$ for some Sylow p-subgroup S of $O_{p',p}(H)$. However, $Q \subseteq S$ and Q is a Sylow p-subgroup of $O_{p',p}(H)$ by definition, forcing $Q = R = S$. Hence Q is a Sylow p-subgroup of $O_{p',p}(K)$, proving (iv).

Now define M to be a maximal p-local subgroup of G containing H if condition *a)* holds and to be the centralizer of an element y of $Z(Q)$ of order p if condition *b)* holds. Observe that $K \subseteq H$ by Lemma 1 and that $K = QC(Q) \subseteq C(y)$ as $y \in Z(Q)$. Therefore $K \subseteq M$ in either case. We let P be a Sylow p-subgroup of M containing Q and set $R = P \cap O_{p',p}(M)$, so that R is a Sylow p-subgroup of $O_{p',p}(M)$ and R is normal in P. In particular, Q normalizes R. We fix this notation as well.

Lemma 2. — *The following conditions hold*:

(i) $K \subseteq M$;

(ii) M *is* p*-constrained*;

(iii) $C_R(Q) \subseteq Q$.

Proof. — We have already noted that (i) holds. Furthermore, (ii) is an immediate consequence of the definition of M and the hypothesis of Theorem 4. As for (iii), set $R_0 = C_R(Q)$, so that $R_0 \subseteq K = QC(Q)$. But $R_0 \subseteq R \subseteq O_{p',p}(M)$ and hence $R_0 \subseteq O_{p',p}(M) \cap K$. However, $K \subseteq M$ by (i) and so clearly $O_{p',p}(M) \cap K \subseteq O_{p',p}(K)$.

Thus $R_0 \subseteq O_{p',p}(K)$. Since also $Q \subseteq O_{p',p}(K)$, we see that R_0Q is a p-subgroup of $O_{p',p}(K)$. On the other hand, Q is a Sylow p-subgroup of $O_{p',p}(K)$ by Lemma 1 and $R_0Q \supseteq Q$. Thus $R_0Q = Q$ and so $R_0 \subseteq Q$, proving (iii).

We need one additional result:

Lemma 3. — If K *has a normal p-complement, then* H *is p-constrained.*

Proof. — Suppose K has a normal p-complement. Since $O_{p'}(H)$ is a normal p'-subgroup of $O_{p'}(H)K$, it follows that $O_{p'}(H)K$ also has a normal p-complement. But $O_{p'}(H)K$ is normal in H by Lemma 1. Together, these two properties of $O_{p'}(H)K$ imply that $O_{p'}(H)K \subseteq O_{p',p}(H)$. Since $K = QC_H(Q)$ by Lemma 1, we conclude that $C_H(Q) \subseteq O_{p',p}(H)$ and so by definition H is p-constrained.

We can now easily establish the theorem. We need only prove that H is p-constrained inasmuch as H is an arbitrary p-local subgroup of G. We proceed by contradiction. In view of Lemma 3, K does not have a normal p-complement. Hence there exists a p'-element x in K with $x \notin O_{p'}(K)$. Since $K \subseteq M$ by Lemma 2, certainly $x \notin O_{p'}(M)$. Thus the image $\bar{x}$ of x in $\bar{M} = M/O_{p'}(M)$ is a non-trivial p'-element of $\bar{M}$. Let $\bar{Q}$, $\bar{R}$ be the respective images of Q, R in $\bar{M}$, so that $\bar{R} = O_p(\bar{M})$. But M is p-constrained by Lemma 2 and this implies that $C_{\bar{M}}(\bar{R}) \subseteq \bar{R}$. We conclude that $\bar{x}$ does not centralize $\bar{R}$.

We shall now contradict this conclusion. Indeed, we know that $\bar{x}$ centralizes $\bar{Q}$ as x is a p'-element of $K = QC(Q)$. But $C_R(Q) \subseteq Q$ by Lemma 2, so $C_{\bar{R}}(\bar{Q}) \subseteq \bar{Q}$. Thus $\bar{x}$ centralizes $C_{\bar{R}}(\bar{Q})$. Theorem (5.3.4) of [4] now yields that $\bar{x}$ centralizes $\bar{R}$, the desired contradiction. This completes the proof.

As a corollary of the theorem, we have the following sufficient conditions for a group to be of characteristic 2 type:

Theorem 5. — Let G *be a group with* $SCN_3(2)$ *non empty, which satisfies one of the following two conditions:*

a) *Every maximal* 2*-local subgroup of* G *is* 2*-constrained and has no non-trivial normal subgroups of odd order; or*

b) *The centralizer of every involution of* G *is* 2*-constrained and has no non-trivial normal subgroups of odd order.*

Then G *is of characteristic 2 type.*

Proof. — In view of Theorem 4, every 2-local subgroup of G is 2-constrained. Since $SCN_3(2)$ is non empty by assumption, to prove that G is of characteristic 2 type, we need only show that a Sylow 2-subgroup S of G normalizes no non-trivial subgroup of G of odd order.

Suppose this is false and let K be a non-trivial S-invariant subgroup of G of odd order. If B is an element of $\mathcal{U}(S)$, then B normalizes K and so $K = \langle C_K(b) \mid b \in B^\# \rangle$ by Theorem (6.2.4) of [4]. Hence $L = C_K(b) \neq 1$ for some b in $B^\#$. If condition *a)* holds, let M be a maximal 2-local subgroup of G containing $C(b)$; while if condition *b)* holds, set $M = C(b)$. In either case, L and $R = C_S(b)$ are contained in M.

By hypothesis, $O(M) = 1$ and M is 2-constrained. Therefore, if we set $Q = O_2(M)$, we have $C_M(Q) \subseteq Q$. In particular, $C_L(Q) = 1$. We shall now contradict this conclusion by showing that, in fact, L centralizes Q.

First of all, $[L, R \cap Q] \subseteq Q$ as Q is normal in M and $L \subseteq M$. On the other hand, $[L, R \cap Q] \subseteq K$ as $L \subseteq K$, $R \cap Q \subseteq S$, and K is S-invariant. Since Q and K are of coprime orders, it follows that $[L, R \cap Q] = 1$ and so L centralizes $R \cap Q$. But $R = C_S(b)$ is of index at most 2 in S and so is of index at most 2 in a Sylow 2-subgroup of M containing R. This implies that $R \cap Q$ is of index at most 2 in Q. Hence $R \cap Q$ is normal in Q and L acts on $Q/(R \cap Q)$, which is of order at most 2. Thus L centralizes $Q/(R \cap Q)$ and so L stabilizes the chain: $Q \supseteq R \cap Q \supseteq 1$. Theorem (5.3.2) of [4] now yields that L centralizes Q. This contradiction completes the proof of the theorem.

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