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ON A THEOREM OF BOCHNER

by P. L. FALB (1)

1. Introduction.

Let G be a locally compact abelian group and let H be a separable complex Hilbert space. A well-known theorem of Bochner ([1], [2]) states that a mapping ψ of G into **C** is positive definite and continuous if and only if there is a unique non-negative finite regular Borel measure μ_ψ on $\hat{G}$ (the dual group of G) such that $\psi(g)=\int_{\hat{G}}(\gamma, g)d\mu_\psi(\gamma)$ where (γ, g) denotes the action of the character γ on g. Here we shall extend this theorem to the context of maps of G into $\mathscr{L}(H, H)$ where $\mathscr{L}(H, H)$ is the space of bounded linear maps of H into itself. Combining this extension of Bochner's theorem with the transform theory on $L_1(G, \mathscr{L}(H, H))$ developed in [3], an inversion theorem and a Plancherel theorem for Hilbert-Schmidt class operators can be proved and applied to the solution of certain integral equations of convolution type arising in the study of the stability and control of systems described by parabolic partial differential equations [4]. We shall not, however, consider these matters here.

2. Bochner's Theorem.

We first recall that a mapping f of G into **C** is *positive definite* if, for any integer N, any $c_1, \ldots, c_N$ in **C** and any $g_1, \ldots, g_N$ in G, the inequality $\sum_{n=1}^{N}\sum_{m=1}^{N} c_n\bar{c}_m f(g_n-g_m)\geqq 0$ is satisfied. This leads to

Definition (**2.1**). — *A mapping ψ of* G *into* $\mathscr{L}(H, H)$ *is positive definite if the mappings ψ_h of* G *into* **C** *given by*

$$\psi_h(g)=\langle\psi(g)h, h\rangle \tag{2.2}$$

are positive definite for all h in H (2).

Lemma (**2.3**). — *Let ψ be a positive definite mapping of* G *into* $\mathscr{L}(H, H)$. *Then*
(i) $\psi(0)$ *is a positive element of* $\mathscr{L}(H, H)$, i.e., $\langle\psi(0)h, h\rangle\geqq 0$ *for all h in* H;
(ii) $\psi(-g)=\psi(g)^*$; *and,*
(iii) $||\psi(g)||\leqq 2\,||\psi(0)||$ *for all g in* G.

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(2) This is equivalent to the following condition: for any integer N, any $c_1, \ldots, c_N$ in **C**, and any $g_1, \ldots, g_N$ in G, $\sum_{n=1}^{N}\sum_{m=1}^{N} c_n\bar{c}_m\psi(g_n-g_m)$ is a positive element of $\mathscr{L}(H, H)$.

Proof. — Let h be any element of H. Since ψ_h is positive definite, we have $\langle\psi(0)h, h\rangle = \psi_h(0) = \sum_{n=1}^{1}\sum_{m=1}^{1} 1.\bar{1}\langle\psi(0-0)h, h\rangle \geqq 0$ and so, (i) is established. Now, note that for $N=2, g_1=g, g_2=0, c_1=1$ and $c_2=\lambda$,

$$\langle\psi(0)h, h\rangle+\bar{\lambda}\langle\psi(g)h, h\rangle+\lambda\langle\psi(-g)h, h\rangle+|\lambda|^2\langle\psi(0)h, h\rangle\geqq 0 \tag{2.4}$$

and hence, that $\bar{\lambda}\langle\psi(g)h, h\rangle+\lambda\langle\psi(-g)h, h\rangle$ is real for all λ in C. Letting $\lambda=1$ and $\lambda=i$, we find that $\langle\psi(-g)h, h\rangle=\overline{\langle\psi(g)h, h\rangle}=\langle\psi(g)^*h, h\rangle$. However,

$$\begin{aligned}\langle\psi(-g)h, k\rangle &= 1/4(\langle\psi(-g)(h+k), (h+k)\rangle-\langle\psi(-g)(h-k), (h-k)\rangle\\ &\quad +i\langle\psi(-g)(h+ik), (h+ik)\rangle-i\langle\psi(-g)(h-ik), (h-ik)\rangle)=\\ &1/4(\langle\psi(g)^*(h+k), (h+k)\rangle-\langle\psi(g)^*(h-k), (h-k)\rangle\\ &\quad +i\langle\psi(g)^*(h+ik), (h+ik)\rangle-i\langle\psi(g)^*(h-ik), (h-ik)\rangle)=\\ &\langle\psi(g)^*h, k\rangle\end{aligned}$$

for all h, k in H and so, (ii) is established. If $\psi_h(0)=0$, then setting $\lambda=-\psi_h(g)$ in (2.4), we deduce that $-2|\psi_h(g)|^2\geqq 0$ so that $\psi_h(g)=0=\psi_h(0)$. If, on the other hand, $\psi_h(0)\neq 0$, then, setting $\lambda=-\psi_h(g)/\psi_h(0)$ in (2.4), we deduce that $\psi_h(0)^2-|\psi_h(g)|^2\geqq 0$ so that $\psi_h(0)\geqq|\psi_h(g)|$ for all h in H and g in G. It follows that if $||h||=1$ and $||k||=1$, then $|\langle\psi(g)h, k\rangle|\leqq 2||\psi(0)||$. Since

$$||\psi(g)|| = \sup_{||h||=1,\, ||k||=1}\{|\langle\psi(g)h, k\rangle|\}, \tag{2.5}$$

(iii) is established.

Now let $\Sigma(\hat{G})$ denote the σ-field of Borel sets of $\hat{G}$. Any weakly countably additive set function μ mapping $\Sigma(\hat{G})$ into H shall be called a *vector measure* (cf. [5]). A vector measure μ is said to be *regular* if the set functions $T\circ\mu$ mapping $\Sigma(\hat{G})$ into **C** are regular complex valued measures for all T in H^*. We now have

Definition (**2.6**). — *A mapping* M *of* $\Sigma(\hat{G})$ *into* $\mathscr{L}(H, H)$ *is a positive regular measure if* (i) *the mapping* M_h *of* $\Sigma(\hat{G})$ *into* H *given by* $M_h(E)=M(E)h$ *for* $E\in\Sigma(\hat{G})$ *is a regular vector measure for all h in* H, *and*
(ii) M(E) *is a positive element of* $\mathscr{L}(H, H)$ *for every* E *in* $\Sigma(\hat{G})$.
M *is bounded if there is an* $A>0$ *such that* $||M(E)||\leqq A$ *for all* E *in* $\Sigma(\hat{G})$.

This is the notion of measure that we shall use in extending Bochner's theorem. We also require a notion of continuity for maps of G into $\mathscr{L}(H, H)$ which is "compatible" with definition (2.6). Now, let $\mathscr{F}$ be the weakest topology on $\mathscr{L}(H, H)$ for which all the functions $\Phi_{h,k}$ given by

$$\Phi_{h,k}(S)=\langle Sh, k\rangle \tag{2.7}$$

where $S\in\mathscr{L}(H, H)$ and h and k are in H, are continuous. The topology $\mathscr{F}$ is a locally convex topology on $\mathscr{L}(H, H)$ since it is generated by the family of seminorms $q_h(S)=|\langle Sh, h\rangle|$. Continuity of maps ψ of G into $\mathscr{L}(H, H)$ is understood to be with respect to the topology $\mathscr{F}$. Thus, a map ψ of G into $\mathscr{L}(H, H)$ is *continuous* if, given $\varepsilon>0$

and any h and k in H, there is a neighborhood $N_{\varepsilon,h,k}$ of o in G such that if $g-g'$ is in $N_{\varepsilon,h,k}$, then $|\langle(\psi(g)-\psi(g'))h, k\rangle|<\varepsilon$. We now have the following extension of Bochner's theorem:

Theorem **(2.8)**. — *A mapping* ψ *of* G *into* $\mathscr{L}(\mathrm{H}, \mathrm{H})$ *is positive definite and continuous if and only if there is a bounded positive regular measure* M_ψ *mapping* $\Sigma(\hat{\mathrm{G}})$ *into* $\mathscr{L}(\mathrm{H}, \mathrm{H})$ *sucy that*

$$\psi(g)h = \int_{\hat{\mathrm{G}}}(\gamma, g)d(\mathrm{M}_\psi(\gamma)h) \tag{2.9}$$

for all h *in* H *and* g *in* G.

Proof. — Suppose first that ψ is positive definite and continuous. Then the function $\psi_h(\cdot)$ is a positive definite and continuous map of G into **C**. By the standard Bochner theorem, there is a unique non-negative finite regular Borel measure μ_h on $\hat{\mathrm{G}}$ such that

$$\langle\psi(g)h, h\rangle = \int_{\hat{\mathrm{G}}}(\gamma, g)d\mu_h(\gamma) \tag{2.10}$$

for all g in G. If E is an element of $\Sigma(\hat{\mathrm{G}})$, then we let $m_{\mathrm{E}}(h)$ be the mapping of H into **R** given by

$$m_{\mathrm{E}}(h) = \mu_h(\mathrm{E}) \tag{2.11}$$

and we let $\mathrm{B}_{\mathrm{E}}(h, k)$ be the mapping of $\mathrm{H}\times\mathrm{H}$ into **C** given by

$$\mathrm{B}_{\mathrm{E}}(h, k) = \left(m_{\mathrm{E}}\left(\frac{h+k}{2}\right) - m_{\mathrm{E}}\left(\frac{h-k}{2}\right)\right) + i\left(m_{\mathrm{E}}\left(\frac{h+ik}{2}\right) - m_{\mathrm{E}}\left(\frac{h-ik}{2}\right)\right). \tag{2.12}$$

We note that

$$\langle\psi(g)h, k\rangle = \int_{\hat{\mathrm{G}}}(\gamma, g)d\mu_{h,k}(\gamma) \tag{2.13}$$

where $\mu_{h,k}$ is the map of $\Sigma(\hat{\mathrm{G}})$ into **C** given by

$$\mu_{h,k}(\mathrm{E}) = \mathrm{B}_{\mathrm{E}}(h, k) = (\mu_{\frac{h+k}{2}}(\mathrm{E}) - \mu_{\frac{h-k}{2}}(\mathrm{E})) + i(\mu_{\frac{h+ik}{2}}(\mathrm{E}) - \mu_{\frac{h-ik}{2}}(\mathrm{E})). \tag{2.14}$$

We claim that, for each fixed E in $\Sigma(\hat{\mathrm{G}})$, $\mathrm{B}_{\mathrm{E}}(h, k)$ is a bounded Hermitian bilinear functional on H. To verify this claim, we first observe that

$$\begin{aligned}&\langle\psi(g)(h_1+h_2+k), (h_1+h_2+k)\rangle - \langle\psi(g)(h_1+h_2-k), (h_1+h_2-k)\rangle \\ &\qquad = (\langle\psi(g)(h_1+k), (h_1+k)\rangle - \langle\psi(g)(h_1-k), (h_1-k)\rangle) \\ &\qquad\quad - (\langle\psi(g)(h_2+k), (h_2+k)\rangle - \langle\psi(g)(h_2-k), (h_2-k)\rangle)\end{aligned}$$

and

$$\begin{aligned}&i(\langle\psi(g)(h_1+h_2+ik), (h_1+h_2+ik)\rangle - \langle\psi(g)(h_1+h_2-ik), (h_1+h_2-ik)\rangle) \\ &\qquad = i((\langle\psi(g)(h_1+ik), (h_1+ik)\rangle - \langle\psi(g)(h_1-ik), (h_1-ik)\rangle) \\ &\qquad\quad + (\langle\psi(g)(h_2+ik), (h_2+ik)\rangle - \langle\psi(g)(h_2-ik), (h_2-ik)\rangle)).\end{aligned}$$

Since μ_h is unique for every h in H, it follows that $\mu_{h_1+h_2,k}(\mathrm{E}) = \mu_{h_1,k}(\mathrm{E}) + \mu_{h_2,k}(\mathrm{E})$ for all h_1, h_2, k in H. Similarly, $\mu_{ah_1,k}(\mathrm{E}) = a\mu_{h_1,k}(\mathrm{E})$ for all h_1, k in H and a in **C**. In other words, $B_E(h, k)$ is linear in h. Since $\langle\psi(g)(-h), (-h)\rangle = \langle\psi(g)h, h\rangle$ and $\langle\psi(g)(ih), (ih)\rangle = i.\bar{i}\langle\psi(g)h, h\rangle = \langle\psi(g)h, h\rangle$, we also have $B_E(h, k) = \overline{B_E(k, h)}$ so that $B_E(h, k)$ is Hermitian and conjugate linear in k. Thus, to show that $B_E(h, k)$ is bounded, it will be enough to show that $|B_E(h, h)| \leqq \|\psi(0)\| \|h\|^2$ for all h in H. Since $\langle\psi(g)(ah), (ah)\rangle = |a|^2\langle\psi(g)h, h\rangle$ for all a in $\mathbf{C}$ and μ_h is unique, $\mu_{ah} = |a|^2\mu_h$. It follows that $|B_E(h, h)| = |\mu_h(E)| \leqq \|\mu_h\|$ where

$$\|\mu_h\| = \mu_h(\hat{G}) = \int_{\hat{G}} d\mu_h(\gamma) = \int_{\hat{G}} (\gamma, 0)\, d\mu_h(\gamma) = \langle\psi(0)h, h\rangle$$

is the total variation of μ_h (note that μ_h is nonnegative and finite). But $\langle\psi(0)h, h\rangle \leqq \|\psi(0)\| \|h\|^2$ and so, $B_E(h, k)$ is bounded independently of E.

Since $B_E(h, k)$ is, for fixed E, a bounded Hermitian bilinear function on $H \times H$, there is a unique self-adjoint element $M_\psi(E)$ of $\mathscr{L}(H, H)$ such that $B_E(h, k) = \langle M_\psi(E)h, k\rangle$. Moreover, since $\|M_\psi(E)\| = \|B_E(\cdot, \cdot)\|$ and $B_E(h, k)$ is bounded independently of E, the mapping M_ψ is bounded. We shall show that the mapping M_ψ of $\Sigma(\hat{G})$ into $\mathscr{L}(H, H)$ is a positive regular measure such that (2.9) is satisfied.

Since $\langle M_\psi(E)h, h\rangle = B_E(h, h) = \mu_h(E) \geqq 0$, $M_\psi(E)$ is a positive element of $\mathscr{L}(H, H)$ for every E in $\Sigma(\hat{G})$. Now let h be a given element of H and consider the mapping $M_{\psi,h}$ of $\Sigma(\hat{G})$ into H given by $M_{\psi,h}(E) = M_\psi(E)h$. If k is any element of H^* $(= H)$ and $\{E_n\}$ is a sequence of disjoint sets in $\Sigma(\hat{G})$ with $F = \bigcup_{n=1}^{\infty} E_n$, then

$$k(M_{\psi,h}(F)) = \langle M_\psi(F)h, k\rangle = B_F(h, k) = (\mu_{\frac{h+k}{2}}(F) - \mu_{\frac{h-k}{2}}(F)) + i(\mu_{\frac{h+ik}{2}}(F) - \mu_{\frac{h-ik}{2}}(F)).$$

But μ_h is countably additive for all h in H. It follows that

$$\begin{aligned} (2.15) \qquad \langle M_\psi(F)h, k\rangle &= \sum_{n=1}^{\infty} ((\mu_{\frac{h+k}{2}}(E_n) - \mu_{\frac{h-k}{2}}(E_n)) + i(\mu_{\frac{h+ik}{2}}(E_n) - \mu_{\frac{h-ik}{2}}(E_n))) \\ &= \sum_{n=1}^{\infty} \mu_{h,k}(E_n) = \sum_{n=1}^{\infty} \langle M_\psi(E_n)h, k\rangle\text{[1]} \end{aligned}$$

or, in other words, that $M_{\psi,h}$ is a vector measure. Since

$$k(M_{\psi,h}(E)) = \langle M_\psi(E)h, k\rangle = \mu_{h,k}(E)$$

for all E in $\Sigma(\hat{G})$ and since the complex valued measure $\mu_{h,k}$ is regular, $M_{\psi,h}$ is a regular vector measure. So, all that remains for this part of the proof is to show that (2.9) is satisfied.

Suppose, for the moment, that (γ, g) is integrable with respect to $M_{\psi,h}(\gamma)$ so that $\int_{\hat{G}} (\gamma, g)\, dM_{\psi,h}(\gamma)$ exists as an element of H. Then, by the property of vector measures under linear transformations ([5], p. 324),

$$\begin{aligned} (2.16) \qquad \left\langle \int_{\hat{G}} (\gamma, g)\, dM_{\psi,h}(\gamma), k \right\rangle &= \int_{\hat{G}} (\gamma, g)\, d\langle M_{\psi,h}(\gamma), k\rangle \\ &= \int_{\hat{G}} (\gamma, g)\, d\mu_{h,k}(\gamma) = \langle\psi(g)h, k\rangle \end{aligned}$$

(1) The series can be rearranged as it is absolutely convergent, due to the finiteness of μ.

for all k in H. It follows immediately that (2.9) is satisfied. Now let us show that $f(\gamma)=(\gamma, g)$ is integrable. Since $\hat{\hat{G}}=G$, f is continuous on $\hat{G}$ (*a fortiori* a Borel function) and $|f(\gamma)|=1$ for all γ in $\hat{G}$. Thus, f is measurable with respect to the vector measure $M_{\psi,h}$. It follows that there is a sequence (f_n) of $M_{\psi,h}$-simple functions with $|f_n(\gamma)|\leqq 2$ $M_{\psi,h}$-almost everywhere, such that f_n converges to f $M_{\psi,h}$-almost everywhere. Since 2 is $M_{\psi,h}$-integrable, f is integrable with respect to $M_{\psi,h}$ by virtue of the dominated convergence theorem.

Now let us suppose that M_ψ is a bounded positive regular measure and that the mapping ψ of G is given by (2.9). Then, $\psi(g)$ maps H into H and $\psi(g)$ is linear since $M_\psi(\gamma)(ah_1+bh_2)=aM_\psi(\gamma)h_1+bM_\psi(\gamma)h_2$ as $M_\psi(\gamma)$ is in $\mathscr{L}(H, H)$ (note that the set consisting of γ alone is closed). Let $||M_{\psi,h}||(\cdot)$ denote the semi-variation of the vector measure $M_{\psi,h}$ [5]. Then

$$||M_{\psi,h}||(\hat{G})\leqq 4\sup_{E\subset\hat{G}}\{||M_{\psi,h}(E)||\}\leqq 4\sup_{E\subset\hat{G}}\{||M_\psi(E)||\}\,||h||\leqq 4A\,||h||$$

for some $A>0$ since M_ψ is bounded. Since $||\psi(g)h||\leqq 1\,.\,||M_{\psi,h}||(\hat{G})\leqq 4A\,||h||$, $\psi(g)$ is an element of $\mathscr{L}(H, H)$. If h is an element of H, then, by the property of vector measures under linear transformations,

$$\langle\psi(g)h, h\rangle=\int_{\hat{G}}(\gamma, g)\,d\langle M_\psi(\gamma)h, h\rangle \tag{2.17}$$

for all g in G. Letting $\mu_h(E)=\langle M_\psi(E)h, h\rangle$ and noting that $M_{\psi,h}$ is regular and that $M_\psi(E)$ is a positive element of $\mathscr{L}(H, H)$, we deduce that μ_h is a nonnegative regular Borel measure on $\hat{G}$. Since $|\mu_h(E)|\leqq||M_\psi(E)||\,.\,||h||^2$ and M_ψ is bounded, μ_h is finite. It then follows from Bochner's theorem and (2.17) that $\psi_h(\cdot)=\langle\psi(\cdot)h, h\rangle$ is continuous and positive definite. Thus, the mapping ψ of G into $\mathscr{L}(H, H)$ is continuous and positive definite. This completes the proof.

Theorem (2.8) suffers from the drawback that the mapping M_φ of $\Sigma(\hat{G})$ into $\mathscr{L}(H, H)$ need not be an $\mathscr{L}(H, H)$-valued measure so that the formula $\psi(g)=\int_{\hat{G}}(\gamma, g)\,dM_\psi(\gamma)$ need not make sense. In the next section, we prove two theorems relating to this drawback.

3. Two Theorems.

A mapping M of $\Sigma(\hat{G})$ into $\mathscr{L}(H, H)$ shall be called an *operator measure* if M is weakly countably additive. An operator measure M is said to be *regular* if the set functions $T\circ M$ mapping $\Sigma(\hat{G})$ into $\mathbf{C}$ are regular complex valued measures for all T in $\mathscr{L}(H, H)^*$. We then have

Definition **(3.1)**. — *A mapping* M *of* $\Sigma(\hat{G})$ *into* $\mathscr{L}(H, H)$ *is a positive strongly regular operator measure if*

(i) M *is a regular operator measure*;

(ii) $M(E)$ *is a positive element of* $\mathscr{L}(H, H)$ *for every* E *in* $\Sigma(\hat{G})$; *and,*
(iii) *given* $\varepsilon > 0$, *there is a compact set* C_ε *in* $\hat{G}$ *such that* $\|M\|(\hat{G}-C_\varepsilon)<\varepsilon$.

Theorem (**3.2**). — *Let* M_ψ *be a positive strongly regular operator measure and let*

$$(3.3) \qquad \psi(g)=\int_{\hat{G}}(\gamma, g)\,dM_\psi(\gamma).$$

Then $\psi(\cdot)$ *is positive definite and continuous with respect to the uniform topology on* $\mathscr{L}(H, H)$.

Proof. — The integrability of $(\gamma, g)=f(\gamma)$ with respect to M_ψ follows from the Lebesgue dominated convergence theorem [5] just as in the proof of Theorem (2.8). Thus, $\psi(g)\in\mathscr{L}(H, H)$.

Now, $M_\psi(\cdot)h$ is a regular vector measure for all h in H, for if k is an element of H^*, then the mapping $A\mapsto\langle Ah, k\rangle$, $A\in\mathscr{L}(H, H)$, is an element of $\mathscr{L}(H, H)^*$ which implies that $\langle M_\psi(\cdot)h, k\rangle$ is a regular complex valued measure and that $\langle M_\psi(E)h, k\rangle=\sum_i\langle M_\psi(E_i)h, k\rangle$ for $E=\bigcup_i E_i$ (disjoint). Since the mapping $A\mapsto Ah$, $A\in\mathscr{L}(H, H)$, is a bounded linear transformation of $\mathscr{L}(H, H)$ into H, we have $\psi(g)h=\int_{\hat{G}}(\gamma, g)\,dM_\psi(\gamma)h$ ([5], p. 324). Furthermore, if $k\in H^*$, then

$$\langle\psi(g)h, k\rangle=\int_{\hat{G}}(\gamma, g)\,d\langle M_\psi(\gamma)h, k\rangle.$$

Let $\mu_{\psi,h}(E)=\langle M_\psi(E)h, h\rangle$. Then $\mu_{\psi,h}$ is a non-negative regular Borel measure. Since $|\mu_{\psi,h}(E)|\leqq\|M_\psi(E)\|\,\|h\|^2$ and since operator valued measures are bounded, $|\mu_{\psi,h}(E)|\leqq\alpha\|h\|^2$ for some $\alpha>0$. In other words, $\mu_{\psi,h}$ is finite. It then follows from Bochner's theorem that $\langle\psi(g)h, h\rangle=\int_{\hat{G}}(\gamma, g)\,d\mu_{\psi,h}(\gamma)$ is a positive definite and continuous function for all h in H (1).

We now show that ψ is continuous with respect to the uniform topology. Let g be an element of G and let $\varepsilon>0$ be given. Then there is a compact set C in $\hat{G}$ such that $\|M_\psi\|(\hat{G}-C)<\varepsilon/4$ and we have

$$(3.4) \qquad \psi(g)-\psi(g')=\int_C((\gamma, g)-(\gamma, g'))\,dM_\psi(\gamma)+\int_{\hat{G}-C}((\gamma, g)-(\gamma, g'))\,dM_\psi(\gamma)$$

for all g' in G. Since $|(\gamma, g)-(\gamma, g')|\leqq 2$, it follows that

$$(3.5) \qquad \|\psi(g)-\psi(g')\|\leqq\sup_{\gamma\in C}(|1-(\gamma, g-g')|)\,\|M_\psi\|(\hat{G})+\varepsilon/2$$

for all g' in G. Letting $g-g'$ be in the neighborhood $N(C, \varepsilon/2\|M_\psi\|(\hat{G}))$ of 0 in G determined by C and $\varepsilon/2\|M_\psi\|(\hat{G})$, we immediately deduce that ψ is continuous. Thus, the theorem is established.

Now let $\mathscr{C}$ denote the ideal of compact operators in $\mathscr{L}(H, H)$. Then it is well-known [6] that $\mathscr{L}(H, H)^*=\mathscr{L}_1\oplus\mathscr{C}^\perp$ where $\mathscr{C}^\perp$ is the annihilator of $\mathscr{C}$ and $\mathscr{L}_1$ is

(1) Thus ψ is continuous with respect to the topology $\mathscr{F}$ even if M_ψ does not satisfy condition (iii) of definition (3.1).

the set of mappings S given by $S(\Phi)=\sum_{j=1}^{\infty}\langle\Phi l_j, k_j\rangle$ where Φ is in $\mathscr{L}(H, H)$, l_j, k_j are in H and $\sum_j \|l_j\|^2<\infty$, $\sum_j \|k_j\|^2<\infty$. We note that if S is in $\mathscr{L}_1$, then $S(\Phi)$ can also be written in the form $S(\Phi)=\sum_{r=1}^{\infty}\lambda_r\langle\Phi h_r, h_r\rangle$ with $|\lambda_r|=1$ and $\sum_{r=1}^{\infty}\|h_r\|^2<\infty$. We have

Lemma **(3.6)**. — *Let* M *be a mapping of* $\Sigma(\hat{G})$ *into* $\mathscr{C}$ (i.e. M(E) *is a compact operator for all* E *in* $\Sigma(\hat{G})$). *If* M_h *is a regular vector measure for all* h *in* H, *then* M *is a regular operator measure.*

Proof. — We first show that M is weakly countably additive. To do this, it is enough to show that $S\circ M$ is countably additive for all S in $\mathscr{L}_1$. If $E=\bigcup_i E_i$ (disjoint), then

$$SM(E)=\sum_{r=1}^{\infty}\lambda_r\langle M(E)h_r, h_r\rangle = \sum_{r=1}^{\infty}\lambda_r\Big(\sum_{i=1}^{\infty}\langle M(E_i)h_r, h_r\rangle\Big) \tag{3.7}$$

since M_{h_r} is a vector measure. But $|SM(E)|\leq\big(\sum_{r=1}^{\infty}\|h_r\|^2\big)\|M(E)\|<\infty$ and so, (3.7) can be rearranged to give

$$SM(E)=\sum_{i=1}^{\infty}\Big(\sum_{r=1}^{\infty}\lambda_r\langle M(E_i)h_r, h_r\rangle\Big)=\sum_{i=1}^{\infty}SM(E_i). \tag{3.8}$$

To show that M is regular, we again need only consider $S\circ M$ for S in $\mathscr{L}_1$. Since $|SM(E)|\leq\big(\sum_{r=1}^{\infty}\|h_r\|^2\big)\|M(E)\|$ and since the measure M is bounded [5], there is an $A>0$ such that $|SM(E)|\leq A\big(\sum_{r=1}^{\infty}\|h_r\|^2\big)$ for all E. Now let F be an element of $\Sigma(\hat{G})$ and let $\varepsilon>0$ be given. Then there is an N for which $\big(\sum_{r=N+1}^{\infty}\|h_r\|^2\big)A<\varepsilon/2$ and so, $\big|\sum_{r=N+1}^{\infty}\lambda_r\langle M(E)h_r, h_r\rangle\big|<\varepsilon/2$ for all E in $\Sigma(\hat{G})$. But $\langle M(\cdot)h_r, h_r\rangle$ is a regular measure. It follows that $\sum_{r=1}^{N}\lambda_r\langle M(\cdot)h_r, h_r\rangle$ is a finite regular complex measure. Thus, there is a compact set K and an open set U with $K\subset F\subset U$ such that if $E\subset U-K$, then $\big|\sum_{r=1}^{N}\lambda_r\langle M(E)h_r, h_r\rangle\big|<\varepsilon/2$ and hence,

$$|SM(E)|\leq\Big|\sum_{r=1}^{N}\lambda_r\langle M(E)h_r, h_r\rangle\Big|+\Big|\sum_{r=N+1}^{\infty}\lambda_r\langle M(E)h_r, h_r\rangle\Big|<\varepsilon.$$

In other words, M is regular and so the lemma is established.

Theorem **(3.9)**. — *If the mapping* ψ *of* G *into* $\mathscr{L}(H, H)$ *is positive definite and continuous and if the corresponding measure* M_ψ *of theorem* (2.8) *maps* $\Sigma(\hat{G})$ *into* $\mathscr{C}$, *then*

$$\psi(g)=\int_{\hat{G}}(\gamma, g)\,dM_\psi(\gamma) \tag{3.10}$$

for all g in G.

Proof. — By the lemma, M_ψ is a regular operator measure. Moreover, (γ, g) is integrable with respect to M_ψ so that $\int_{\hat{G}}(\gamma, g)dM_\psi(\gamma)$ exists. Since the mapping $\Phi \mapsto \Phi h$, $\Phi \in \mathscr{L}(H, H)$, is a bounded linear transformation of $\mathscr{L}(H, H)$ into H, we have

$$\left(\int_{\hat{G}}(\gamma, g)dM_\psi(\gamma)\right)h = \int_{\hat{G}}(\gamma, g)dM_\psi(\gamma)h = \psi(g)h \tag{3.11}$$

for all h in H which proves (3.10).

In view of theorem (3.9), it is of interest to determine direct conditions on ψ which insure that M_ψ maps $\Sigma(\hat{G})$ into $\mathscr{C}$. A typical condition is that ψ be "approximable" by finite dimensional maps. More precisely, we have

Definition **(3.12)**. — *Let* $\{e_1, \ldots\}$ *be an orthonormal basis of* H. *Let* H_n *be the span of* $\{e_1, \ldots, e_n\}$ *and let* P_n *be the projection of* H *onto* H_n. *An element* $\psi(\cdot)$ *of* $L_1(G, \mathscr{L}(H, H))$ *is approximable if*

$$\lim_{n \to \infty} \int_{\hat{G}} \|\hat{\psi}(\gamma) - \hat{\psi}_n(\gamma)\| dm(\gamma) = 0 \tag{3.13}$$

where $\psi_n(\cdot) = P_n \psi(\cdot) P_n$, $m(\gamma)$ *is Haar measure on* $\hat{G}$, *and the superscript* $\wedge$ *indicates the Fourier transform* [1].

Lemma **(3.14)**. — *If* ψ *is positive definite, continuous and approximable, then the corresponding measure* M_ψ *of Theorem* (2.8) *maps* $\Sigma(\hat{G})$ *into* $\mathscr{C}$.

Proof. — Let $M = M_\psi$ and $M_n = P_n M_\psi P_n$. Since P_n is a projection, the map ψ_n is positive definite and continuous. Moreover, it is clear that the measure M_{ψ_n} corresponding to ψ_n is simply M_n. Now, if $E \in \Sigma(\hat{G})$, then $M_n(E)$ has finite dimensional range and is therefore compact. Thus, we need only show that

$$\lim_{n \to \infty} \|M(E) - M_n(E)\| = 0 \tag{3.15}$$

for all E in $\Sigma(\hat{G})$. Since $\|M(E) - M_n(E)\| = \sup_{\|h\|=1, \|k\|=1} |\langle (M(E) - M_n(E))h, k\rangle|$ and

$$|\langle (M(E) - M_n(E))h, k\rangle| = |\mu_{h,k}(E) - \mu^{(n)}_{h,k}(E)| \leq |\mu_{\frac{h+k}{2}}(E) - \mu^{(n)}_{\frac{h+k}{2}}(E)|$$
$$+ |\mu_{\frac{h-k}{2}}(E)\mu^{(n)}_{\frac{h-k}{2}}(E)| + |\mu_{\frac{h+ik}{2}}(E)\mu^{(n)}_{\frac{h+ik}{2}}(E)| + |\mu_{\frac{h-ik}{2}}(E) - \mu^{(n)}_{\frac{h-ik}{2}}(E)|$$

where $\mu^{(n)}_h$ is the regular measure corresponding to ψ_n, we have

$$\|M(E) - M_n(E)\| \leq 4 \sup_{\|h\|=1} |\mu_h(E) - \mu^{(n)}_h(E)| \tag{3.16}$$

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[1] If $\Phi(\cdot)$ is in $L_1(G, \mathscr{L}(H, H))$, then the Fourier transform $\hat{\Phi}(\cdot)$ of $\Phi(\cdot)$ is the mapping of $\hat{G}$ into $\mathscr{L}(H, H)$ given by

$$\hat{\Phi}(\gamma) = \int_G \overline{(\gamma, g)}\Phi(g)dm(g)$$

where $m(g)$ is Haar measure on G. Properties of the Fourier transform are given in [3]. Note that $\hat{\Phi}(\cdot)$ may not be in $L_1(\hat{G}, \mathscr{L}(H, H))$ so that $\hat{\psi}(\cdot)$, $\hat{\psi}_n(\cdot)$ in $L_1(\hat{G}, \mathscr{L}(H, H))$ is a tacit assumption in (3.13).

and so, we need only show that

$$(3.17) \qquad \lim_{n\to\infty} \sup_{||h||=1} |\mu_h(E) - \mu_h^{(n)}(E)| = 0$$

for all E in $\Sigma(\hat{G})$. Now, ψ, ψ_n are positive definite, continuous and in $L_1(G, \mathscr{L}(H, H))$. It follows that $\langle\psi(\cdot)h, h\rangle$, $\langle\psi_n(\cdot)h, h\rangle$ are positive definite, continuous and in $L_1(G, C)$. Let $f_h(\gamma) = \langle\hat{\psi}(\gamma)h, h\rangle$ and $f_h^{(n)}(\gamma) = \langle\hat{\psi}_n(\gamma)h, h\rangle$. Then ([2], [3]) $f_h(\cdot), f_h^{(n)}(\cdot)$ are in $L_1(\hat{G}, C)$,

$$\langle\psi(g)h, h\rangle = \int_{\hat{G}} (\gamma, g) f_h(\gamma) dm(\gamma), \qquad \langle\psi_n(g)h, h\rangle = \int_{\hat{G}} (\gamma, g) f_h^{(n)}(\gamma) dm(\gamma)$$

and $f_h(\cdot), f_h^{(n)}(\cdot)$ define finite, nonnegative regular Borel measures μ_{f_h}, $\mu_{f_h}^{(n)}$ on $\hat{G}$ by

$$\mu_{f_h}(E) = \int_E f_h(\gamma) dm(\gamma), \qquad \mu_{f_h}^{(n)}(E) = \int_E f_h^{(n)}(\gamma) dm(\gamma).$$

By the uniqueness in Bochner's theorem, $\mu_{f_h} = \mu_h$ and $\mu_{f_h}^{(n)} = \mu_h^{(n)}$. It follows that

$$|\mu_h(E) - \mu_h^{(n)}(E)| = \left|\int_E (f_h(\gamma) - f_h^{(n)}(\gamma)) dm(\gamma)\right| \leq \int_{\hat{G}} ||\hat{\psi}(\gamma) - \hat{\psi}_n(\gamma)|| \, dm(\gamma) \, . \, ||h||^2$$

for all h in H [1]. Since ψ is approximable, (3.17) and with it the lemma, are established.

Corollary **(3.18)**. — *If* ψ *is positive definite, continuous and approximable, then* $\psi(g) = \int_{\hat{G}} (\gamma, g) dM_\psi$.

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[1] Thus, the notion of approximability can be weakened to the requirement

$$\lim_{n\to\infty} \sup_{||h||=1} \left|\int_{\hat{G}} (f_h(\gamma) - f_h^{(n)}(\gamma)) dm(\gamma)\right| = 0$$