PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

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Publications mathématiques de l'I.H.É.S., tome 35 (1968), p. 127-156
<http://www.numdam.org/item?id=PMIHES_1968__35__127_0>

STABILITY OF C^∞ MAPPINGS, III : FINITELY DETERMINED MAP-GERMS

by John N. MATHER [1]

INTRODUCTION

Given an equivalence relation $\sim$ on the set of C^∞ map-germs $f:(N, S) \dashrightarrow (P, y)$ (where N and P are manifolds, S is a finite subset of N, and y is a point of P), we say $f:(N, S) \dashrightarrow (P, y)$ is finitely determined if there exists an integer k such that any $g:(N, S) \dashrightarrow (P, y)$ which has the same k-jet as f satisfies $f \sim g$. The purpose of this paper is to obtain necessary and sufficient conditions for a C^∞ map-germ to be finitely determined with respect to two equivalence relations. The first is *contact equivalence*, which is defined in § 2. The second is *isomorphism*: we say that two map-germs f and g are isomorphic if there exist invertible C^∞ map-germs $h:(N, S) \dashrightarrow (N, S)$ and $h':(P, y) \dashrightarrow (P, y)$ such that $h' \circ g \circ h = f$.

The main result of this paper is theorem (3.5). In the case $\mathscr{S} = \mathscr{K}$, this gives the necessary and sufficient condition for f to be finitely determined with respect to the notion of contact equivalence, by (2.9). In the case $\mathscr{S} = \mathscr{A}$, this gives the necessary and sufficient condition for f to be finitely determined with respect to isomorphism. The cases $\mathscr{S} = \mathscr{K}$ and $\mathscr{S} = \mathscr{R}$ of (3.5) have been treated before by Tougeron ([4] and [5], Chap. II). The case $\mathscr{S} = \mathscr{C}$ is completely trivial. However in the cases $\mathscr{S} = \mathscr{A}$ or $\mathscr{L}$, theorem (3.5) is a new result in a non trivial sense.

From (3.5) and (3.6), we get:

Theorem. — f is finitely determined with respect to isomorphism if and only if there exists an integer k such that

$$(*) \qquad tf(B) + \omega f(A) + \mathfrak{m}_S^{l(k)}\theta(f) \supset \mathfrak{m}_S^k \theta(f),$$

where $l(k)$ is given by (3.6), (iv).

The problem of seeing whether $(*)$ holds for a given value of k is simply the question of seeing whether one sub-vector space of $\theta(f)/\mathfrak{m}_S^{l(k)}\theta(f)$ (which is finite dimensional) contains another, where each of these vector subspaces is the span of an explicitly given set of elements. This is about as close as we can get to a finite problem in this subject.

(1) Research supported by NSF postdoctoral fellowship.

The method of proving sufficiency in (3.5) is the following. We consider a map-germ f which satisfies the hypothesis of (3.5) and another map-germ g which has a high order of contact with f. We are to show that $g \sim f$ (for whichever equivalence relation is considered). We choose a one parameter family of map-germs g_t such that $g_0=f$ and $g_1=g$. We then try to show that $g_t \sim f$ for all t in the unit interval, or rather, we try to show a somewhat stronger one-parameter version of this result. Differentiating with respect to t, we get an equivalent problem (by (4.3)), which concerns modules over C_S in the case of contact equivalence and mixed homomorphisms ((1.12), definition 2) in the case of isomorphism. The problem concerning modules over C_S is solved by several applications of Nakayama's lemma; in contrast the problem concerning mixed homomorphisms requires a deep result of Malgrange.

In § 1, we develop the basic results about mixed homomorphisms which we will need in this paper. These are based on the Malgrange preparation theorem (our theorem (1.11), or, equivalently, theorem (4.1) of [2]). Our proof (starting with (1.8)) seems simpler than that of Malgrange; however, the least obvious step — the proof of (1.9) — comes from Malgrange.

In § 2, we define the notion of contact equivalence and develop some of its elementary properties. In § 3 the main result is stated and in §§ 4-8, it is proved. In § 9, we will consider briefly the analytic case.

We will refer to our papers Stability of Mappings, I, II ([3]) as I, II.

1. Modules over Rings of Differentiable Functions.

(1.1) We will use the following notation. If X and Y are topological spaces, $S \subset X$ and $T \subset Y$, then a *map-germ* $f : (X, S) \dashrightarrow (Y, T)$ will mean an equivalence class of continuous mappings $g : U \to Y$, where U is an open neighborhood of S in X and $g(S) \subset T$, with the relation of equivalence: $(g : U \to Y) \sim (h : V \to Y)$ if there exists a neighborhood W of S in $U \cap V$ such that $g|W = h|W$. Any member g of the equivalence class f will be called a *representative* of f. We will also say that f is the *germ at* S of g. A map-germ $f : (X, S) \dashrightarrow Y$ will mean a map-germ $f : (X, S) \dashrightarrow (Y, Y)$.

Many standard notions for mappings extend to map-germs in an obvious way. For example, if X and Y are (C^∞) manifolds, $f : (X, S) \dashrightarrow Y$ is said to be C^∞ if it has a representative which is C^∞. If $f : (X, S) \dashrightarrow Y$ and $g : (X, S) \dashrightarrow Z$ are map-germs, then $(f, g) : (X, S) \dashrightarrow Y \times Z$ is defined as the germ at S of $x \mapsto (\tilde{f}(x), \tilde{g}(x)) : U \to Y \times Z$, where $\tilde{f} : U \to Y$ and $\tilde{g} : U \to Z$ are representatives of f and g, respectively. If $f : (X_1, S_1) \dashrightarrow (Y_1, T_1)$ and $g : (X_2, S_2) \dashrightarrow (Y_2, T_2)$ are map-germs, we define

$$f \times g : (X_1 \times X_2, S_1 \times S_2) \dashrightarrow (Y_1 \times Y_2, T_1 \times T_2)$$

as the germ at $(S_1 \times S_2)$ of $(x_1, x_2) \mapsto (\tilde{f}(x_1), \tilde{g}(x_2)) : U_1 \times U_2 \to Y_1 \times Y_2$, where $\tilde{f} : U_1 \to Y_1$ and $\tilde{g} : U_2 \to Y_2$ are representatives of f and g, respectively.

If $f : (X, S) \dashrightarrow (Y, T)$ and $g : (Y, T) \dashrightarrow Z$ are map-germs, $g \circ f : (X, S) \dashrightarrow Z$ is defined in the obvious way. By the *germ at* S *of a subset of* X we will mean an equivalence class of subsets Z of X, where Z_1 and Z_2 are equivalent if there exists a neighborhood U of S in X such that $U \cap Z_1 = U \cap Z_2$. If $Z \subset X$, we will denote the germ of Z at S by (Z, S). We will write $W \subset (X, S)$ to indicate that W is the germ at S of a subset of X.

Suppose $f : (X, S) \dashrightarrow (Y, T)$ is a map-germ. If either $Z \subset Y$ or $Z \subset (Y, T)$, $f^{-1}Z \subset (X, S)$ is defined in the obvious way. If $f : (X, S) \dashrightarrow (Y, T)$ has a representative $\tilde{f} : U \to Y$ such that $\tilde{f}(U)$ is closed and $\tilde{f} : U \to \tilde{f}(U)$ is a homeomorphism, then for any $Z \subset X$ or $Z \subset (X, S)$ we may define $f(Z)$ as $(\tilde{f}(Z), T)$ or $(\tilde{f}(\tilde{Z}), T)$, where $\tilde{f}$ is any representative of f and (in the second case) $\tilde{Z}$ is any representative of Z. (In general, however, it is impossible to define $f(Z)$ in any sensible way.)

If $f : (X, S) \dashrightarrow (Y, T)$ is a map-germ, graph$(f) \subset (X \times Y, S \times T)$ is defined in the obvious way. If $X' \subset X$ and $S' \subset S$, then the restriction of f to (X', S') (denoted $f | (X', S')$) is defined as the germ at S' of $\tilde{f} | (X' \cap U)$, for any representative $\tilde{f} : U \to Y$ of f.

(1.2) Let N be a (C^∞) manifold and S a subset of N. We let $C_S = C(N)_S$ denote the set of C^∞ map-germs $(N, S) \dashrightarrow \mathbf{R}$. This set has a natural $\mathbf{R}$-algebra structure, induced by the $\mathbf{R}$-algebra structure on $\mathbf{R}$. Let $\mathfrak{m}_S = \mathfrak{m}(N)_S$ denote the ideal in C_S consisting of C^∞ map-germs $(N, S) \dashrightarrow (\mathbf{R}, 0)$.

Lemma **(1.3)**. — *Let* S *be a subset of a manifold* N. *If* $S = S_1 \cup \ldots \cup S_p$ *and* $\overline{S}_i \cap \overline{S}_j = \emptyset$ *for* $i \neq j$ (where $\overline{S}_i$ denotes the closure of S_i in N), *then*

$$C_S = C_{S_1} \times \ldots \times C_{S_p} \quad \text{(Cartesian product)}$$
$$\mathfrak{m}_S = \mathfrak{m}_{S_1} \times \ldots \times \mathfrak{m}_{S_p} \quad \text{(Cartesian product).}$$

The proof is trivial.

Lemma **(1.4)**. — *Let* $x \in N$. *Let* $x_1, \ldots, x_n$ *be a local system of coordinates for* N, *null at* x. *Let* K *be the subset of* N *defined by* $x_1 = \ldots = x_k = 0$. *Let* $\overline{x}_1, \ldots, \overline{x}_n \in C(N)_x$ *denote the germs of* $x_1, \ldots, x_n$. *Then the following are equivalent for* $u \in C(N)_x$:

a) $u \in \{\overline{x}_1, \ldots, \overline{x}_k\}^l C_x$.

b) *There exists a representative* $\tilde{u} : U \to \mathbf{R}$ *of* u *such that* $\tilde{u}$ *vanishes together with its derivatives of order* $< l$ *everywhere on* $K \cap U$.

Remarks. — By a local system of coordinates we will always mean a C^∞ system of coordinates.

The notation used in *a)* is the standard ring theoretic notation. In this notation, if A, B, and C are additive groups and a " product " $ab \in C$ is defined for each $a \in A$, $b \in B$, then for $S \subset A$ and $T \subset B$, the subset ST of C is defined as the set of all finite sums of elements of the form ab, where $a \in S$ and $b \in T$. If A is a ring and $S \subset A$, then S^k is SS...S (k times). In *a)*, $\{\overline{x}_1, \ldots, \overline{x}_k\}^l$ denotes (according to these conventions) the set of all homogeneous polynomials in $\overline{x}_1, \ldots, \overline{x}_k$ of degree l, with integer coefficients. Also $\{\overline{x}_1, \ldots, \overline{x}_k\}^l C(N)_x$ denotes the ideal in $C(N)_x$ generated by $\{\overline{x}_1, \ldots, \overline{x}_k\}^l$.

Proof. — Clearly *a)* ⇒ *b)*. We prove *b)* ⇒ *a)* by induction on l. Define $\widetilde{u}_i \in C_x$ $(i=1, \ldots, k)$ by

$$\widetilde{u}_i(x_1, \ldots, x_n) = \int_0^1 \widetilde{u}^{(i)}(0, \ldots, 0, tx_i, x_{i+1}, \ldots, x_n)\,dt,$$

where $\widetilde{u}^{(i)}$ denotes the first partial derivative of $\widetilde{u}$ with respect to the i^{th}-coordinate. Clearly, the germ u_i of $\widetilde{u}_i$ at x is C^∞. Then,

$$x_i\widetilde{u}_i(x_1, \ldots, x_n) = \widetilde{u}(0, \ldots, 0, x_i, x_{i+1}, \ldots, x_n) - \widetilde{u}(0, \ldots, 0, 0, x_{i+1}, \ldots, x_n),$$

so $u = \sum_{i=1}^{k} \overline{x}_i u_i$, assuming u vanishes everywhere on $K \cap U$. If $l=1$, we are done. If $l>1$, then the u_i vanish everywhere on $K \cap U$ together with their derivatives of order $<l-1$. Therefore $u_i \in \{\overline{x}_1, \ldots, \overline{x}_k\}^{l-1}C_x$, by induction. Then *a)* follows immediately.

Corollary 1. — *The ideal* $\mathfrak{m}_x$ *is generated by* $\overline{x}_1, \ldots, \overline{x}_n$.

Corollary 2. — *The ideal* $\mathfrak{m}_x^k$ *consists of exactly those germs vanishing at x together with their derivatives of order* $<k$.

Corollary 3. — $C_x/\mathfrak{m}_x^k \cong \mathbf{R}[[\overline{x}_1, \ldots, \overline{x}_n]]/\mathfrak{m}^k$, *where* $\mathfrak{m}$ *denotes the unique maximal ideal of the ring* $\mathbf{R}[[\overline{x}_1, \ldots, \overline{x}_n]]$ *of formal power series with coefficients in* $\mathbf{R}$.

The isomorphism in corollary 3 is given by the Taylor series expansion.

Nakayama's lemma (**1.5**). — *Let* R *be a commutative ring with identity,* $\alpha : E \to F$ *a homomorphism of* R*-modules, and* $\mathfrak{I}$ *an ideal in* R *such that* $1+z$ *is invertible for any* $z \in \mathfrak{I}$. *If* F *is finitely generated, then*

a) $\alpha(E) + \mathfrak{I}F = F$,

implies

b) $\alpha(E) = F$.

Remarks. — By R-module A, we shall mean a *unitary* R-module, i.e., one satisfying $1\,a = a$ for each $a \in A$.

For a proof, see for example II, § 6.

Corollary (**1.6**). — *Let* S *be a finite subset of a manifold* N. *Let* A *be a finitely generated* $C(N)_S$*-module and* B *a submodule of* A *such that for some integer* l,

a) $\dim_{\mathbf{R}} A/(\mathfrak{m}_S^{l+1}A + B) \leq l$;

then

b) $\mathfrak{m}_S^l A \subset B$.

Proof. — Let $A' = A/B$. Then *a)* is equivalent to

$$\dim_{\mathbf{R}} A'/\mathfrak{m}_S^{l+1}A' \leq l.$$

Hence there exists k, $0 \leq k \leq l$ such that

$$\mathfrak{m}_S^{k+1}A' = \mathfrak{m}_S^k A'.$$

It then follows from Nakayama's lemma, applied with $R = C_S$, $I = \mathfrak{m}_S$, $F = \mathfrak{m}_S^k A'$ and $E = 0$, that $\mathfrak{m}_S^k A' = 0$, which implies *b)*.

Corollary **(1.7).** — *Under the hypotheses of corollary* (1.6), *there exists a set of generators of* B *(over* C_S*) having not more than* $a\binom{n+l}{l}$ *elements, where a is the minimum number of elements in a set of generators of* A, *and n is the dimension of* N.

Proof. — First, consider the case when S consists of a single point x. Since $\mathfrak{m}_S^l A$ is finitely generated and $A/\mathfrak{m}_S^l A$ is finite dimensional, $\mathfrak{m}_S^l A + B$ is finitely generated. By (1.6), $B = \mathfrak{m}_S^l A + B$; hence B is finitely generated. Hence a set of generators of B modulo $\mathfrak{m}_S B$ is a set of generators of B, by Nakayama's lemma. But

$$\dim_{\mathbf{R}} B/\mathfrak{m}_S B \leq \dim_{\mathbf{R}} B/\mathfrak{m}_S^{l+1} A \leq \dim_{\mathbf{R}} A/\mathfrak{m}_S^{l+1} A \leq a\binom{n+l}{l},$$

where the first inequality follows from the fact that $\mathfrak{m}_S B \supseteq \mathfrak{m}_S^{l+1} A$ (by (1.6)), and the last from (1.4). (Note that if A is a free module, then the last inequality is an equality.)

The proof in general now follows by an application of (1.3). Let $S = \{x_1, \ldots, x_s\}$. Let $A_{x_i} = A \otimes C_{x_i}$, $B_{x_i} = B \otimes C_{x_i}$ (tensor product over C_S). Thus A_{x_i} is a C_{x_i}-module and B_{x_i} is a C_{x_i}-submodule of A_{x_i}. It is easily verified that (1.6 *a)*) is satisfied for C_{x_i}, A_{x_i}, B_{x_i} in place of C_S, A, B, respectively. Thus, we may conclude that B_{x_i} is generated as a C_{x_i}-module by $a\binom{n+l}{l}$ or fewer elements. From (1.3) it follows that $C_S = C_{x_1} \times \ldots \times C_{x_s}$ and $B = B_{x_1} \times \ldots \times B_{x_s}$, where for any $f = (f_1, \ldots, f_s) \in C_S$ (where $f_i \in C_{x_i}$) and any $b = (b_1, \ldots, b_s) \in B$ (where $b_i \in C_{x_i}$), $fb = (f_1 b_1, \ldots, f_s b_s)$. Thus, the fact that each B_{x_i} is generated by $a\binom{n+l}{l}$ or fewer elements as a C_{x_i}-module implies that B is generated by $a\binom{n+l}{l}$ or fewer elements as a C_S-module.

(1.8) Let P be a manifold. Let t and π denote the germs at $(y, 0)$ of the projections of $P \times \mathbf{R}$ on $\mathbf{R}$ and P, respectively. For any C^∞ map-germ $u = (u_1, \ldots, u_p) : (P, y) \to \mathbf{R}^p$, let $R_u = \sum_{i=1}^{p} (u_i \circ \pi) t^{p-i} \in C(P \times \mathbf{R})_{(y,0)}$ and let $\Gamma_u = t^p + R_u \in C(P \times \mathbf{R})_{(y,0)}$. From the division theorem of I, we have:

Lemma. — *Let* $g \in C(P \times \mathbf{R})_{(y,0)}$ *and let* $u : (P, y) \to \mathbf{R}^p$ *be a* C^∞ *map-germ. Then there exists* $q \in C(P \times \mathbf{R})_{(y,0)}$ *and a* C^∞ *map-germ* $h : (P, y) \to \mathbf{R}^p$ *such that*

$$(*) \qquad g = \Gamma_u q + R_h.$$

Proof. — Let $\tilde{g} : P \times \mathbf{R} \to \mathbf{R}$ and $\tilde{u} : P \to \mathbf{R}^p$ be C^∞ mappings whose germs at $(y, 0)$ and y are g and u, respectively. For each $z \in P$, define $\tilde{g}_z : \mathbf{R} \to \mathbf{R}$ by $\tilde{g}_z(b) = \tilde{g}(z, b)$. Let $\tilde{q} : P \times \mathbf{R} \to \mathbf{R}$ and $\tilde{h} : P \to \mathbf{R}^p$ be given by

$$\tilde{q}(z, b) = Q(\tilde{g}_z, b, \tilde{u}(z)), \qquad \tilde{h}(z) = H(\tilde{g}_z, \tilde{u}(z))$$

for all $z \in P$, $b \in \mathbf{R}$, where Q and H are as in the theorem in I, § 2. The proof of the corollary in I, § 10 shows that $\tilde{q}$ and $\tilde{h}$ are C^∞. From formula (1) in I, § 2, it follows that

$$\tilde{g}(z, b) = \Gamma(b, \tilde{u}(z)) \tilde{q}(z, b) + R(b, \tilde{h}(z)).$$

Thus the germs q of $\tilde{q}$ at $(y, 0)$ and h of $\tilde{h}$ at y are C^∞ and satisfy $(*)$.

(1.9) Let $f \in C(P \times \mathbf{R})_{(y,0)}$. If $f | (y \times \mathbf{R}, y \times 0)$ (considered as an element of $C(\mathbf{R})_0$) has the property that it vanishes at the origin together with its derivatives of order $<p$, but its derivative of order p does not vanish, we will say f is *regular of order* p.

Theorem. — If $f \in C(P \times \mathbf{R})_{(y,0)}$ *is regular of order* p, *then there exists an invertible* $q \in C(P \times \mathbf{R})_{(y,0)}$ *and a* C^∞ *map-germ* $u : (P, y) \dashrightarrow \mathbf{R}^p$ *such that*

$$(*) \qquad f = q\Gamma_u.$$

Proof. — According to (1.8), there exists $q_1 \in C^\infty(P \times \mathbf{R}^p \times \mathbf{R})_{(y,0,0)}$ and a C^∞ map-germ $h : (P \times \mathbf{R}^p, y \times 0) \dashrightarrow \mathbf{R}^p$ such that

$$\widetilde{f}(z, b) = (b^p + \sum_{i=1}^{p} a_i b^{p-i})\widetilde{q}_1(z, a, b) + \sum_{i=1}^{p} \widetilde{h}_i(z, a) b^{p-i},$$

for suitable representatives $\widetilde{f} : P_0 \times V \to \mathbf{R}$ of f, $\widetilde{q}_1 : P_0 \times U \times V \to \mathbf{R}$ of q_1, and $\widetilde{h} : P_0 \times U \to \mathbf{R}^p$, and all $z \in P_0 \subseteq P$, $a \in U \subseteq \mathbf{R}^p$, and $b \in V \subseteq \mathbf{R}$. Since f is regular of order p it follows that

$$\widetilde{q}_1(y, 0, 0) \neq 0, \qquad \widetilde{h}_i(y, 0) = 0$$

$$\frac{\partial \widetilde{h}_i}{\partial a_j}(y, 0) = 0, \qquad \text{if } i > j$$

$$\frac{\partial \widetilde{h}_i}{\partial a_i}(y, 0) \neq 0.$$

In particular $\det\left(\frac{\partial \widetilde{h}_i}{\partial a_j}(0)\right) \neq 0$. Thus by the implicit function theorem, there exists a C^∞ mapping $\widetilde{u} : P_1 \to \mathbf{R}^p$ (where P_1 is a neighborhood of y in P_0) such that $\widetilde{h}_i(z, \widetilde{u}(z)) = 0$ for $1 \leq i \leq p$ and $z \in P_1$. Set $\widetilde{q}(z, b) = \widetilde{q}_1(z, \widetilde{u}(z), b)$ and let q and u be the germs at $(y, 0)$ and y of $\widetilde{q}$ and $\widetilde{u}$, respectively. Then $(*)$ is satisfied. Since $\widetilde{q}_1(y, 0, 0) \neq 0$, q is invertible.

Theorem **(1.10)**. — *Let* $f, g \in C(P \times \mathbf{R})_{(y,0)}$ *and suppose* f *is regular of order* p. *Then there exists* $q \in C(P \times \mathbf{R})_{(y,0)}$ *and a* C^∞ *map-germ* $h : (P, y) \dashrightarrow \mathbf{R}^p$ *such that*

$$(*) \qquad g = fq + R_h.$$

Proof. — From (1.9) there exists an invertible $q_1 \in C(P \times \mathbf{R})_{(y,0)}$ and a C^∞ map-germ $u : (P, y) \dashrightarrow \mathbf{R}^p$ such that $f = q_1\Gamma_u$. From (1.8) there exists $q_2 \in C(P \times \mathbf{R})_{(y,0)}$ and a C^∞ map-germ h such that $g = \Gamma_u q_2 + R_h$. Set $q = q_2/q_1$. Then $(*)$ holds.

(1.11) If $f : (N, S) \dashrightarrow (P, T)$ is a C^∞ map-germ (where N and P are C^∞ manifolds), then we define $f^* : C(P)_T \to C(N)_S$ by $f^*(u) = u \circ f$. Clearly f^* is an $\mathbf{R}$-algebra homomorphism.

Malgrange " preparation " theorem. — Let $f : (N, S) \dashrightarrow (P, y)$ *be a* C^∞ *map-germ, where* S *is a finite subset of* N, *and* $y \in P$. *Let* A *be a finitely generated* C_S*-module. If* $A/f^*(\mathfrak{m}_y)A$ *is finitely generated as an* $\mathbf{R}$*-vector space, then* A *is finitely generated as a* C_y*-module, where the* C_y*-module structure on* A *is induced by* f^*.

Proof. — There are three steps.

Step 1. — Let $\pi : P\times\mathbf{R}\to P$ and $t : P\times\mathbf{R}\to\mathbf{R}$ denote the projections. We prove the theorem in the case $N=P\times\mathbf{R}$, $S=(y, 0)$, and $f=\pi$. Let $a_1, \ldots, a_q$ be a finite set of elements of A which generate A as a $C(N)_S$-module and whose images in $A/f^*(\mathfrak{m}_y)A$ span this $\mathbf{R}$-vector space. Then any $a\in A$ can be written in the form

$$a=\sum_{i=1}^{q} c_i a_i + \sum_{i=1}^{q} z_i a_i,$$

where $c_i\in\mathbf{R}$, $z_i\in f^*(\mathfrak{m}_y)C(N)_S$. In particular, there exist $c_{ij}\in\mathbf{R}$, $z_{ij}\in f^*(\mathfrak{m}_y)C(N)_S$ $(1\leq i,j\leq q)$ such that

$$ta_i=\sum_{j=1}^{q}(c_{ij}+z_{ij})a_j.$$

Let Δ be the determinant $|t\delta_{ij}-c_{ij}-z_{ij}|$. It follows from the above equation and Cramer's rule that $\Delta a_i=0$, $i=1, \ldots, q$. Expanding the determinant, we see that Δ is regular of order p, where $p\leq q$, since $\Delta|(y\times\mathbf{R}, y\times 0)$ is a monic polynomial in t of order q.

Since $\Delta.A=0$, A is a $(C(N)_S/\Delta.C(N)_S)$-module. By (1.10) and the fact that Δ is regular, it follows that $C(N)_S/\Delta.C(N)_S$ is finitely generated as a $C(P)_y$-module. Therefore the fact that A is finitely generated as a $(C(N)_S/\Delta.C(N)_S)$-module implies A is finitely generated as a $C(P)_y$-module.

Step 2. — We prove the theorem under the additional assumption that S is a single point, say x. Let $\varphi : (N, x)\dashrightarrow(\mathbf{R}^n, 0)$ be an invertible C^∞ map-germ. Factor f as follows:

$$(N, x)\xrightarrow{(f,\varphi)}(P\times\mathbf{R}^n, (y, 0))\xrightarrow{\pi_n}\ldots\xrightarrow{\pi_1}(P, y)$$

where

$$\pi_k : (P\times\mathbf{R}^k, (y, 0))\dashrightarrow(P\times\mathbf{R}^{k-1}, (y, 0))$$

is the germ of the projection

$$P\times\mathbf{R}^k\to P\times\mathbf{R}^{k-1} : (z, a_1, \ldots, a_k)\mapsto(z, a_1, \ldots, a_{k-1}).$$

For each k, $0\leq k\leq n$, we give A the $C(P\times\mathbf{R}^k)_{(y,0)}$-module structure induced by $(\pi_{k+1}\circ\ldots\circ\pi_n\circ(f, \varphi))^*$. In the case $k=0$, this is the $C(P)_y$-module structure we gave A in the statement of the theorem, since $f=\pi_1\circ\ldots\circ\pi_n\circ(f, \varphi)$. From the fact that $(f, \varphi)^*$ is surjective and the assumption that A is finitely generated as a $C(N)_x$-module, it follows that A is finitely generated as a $C(P\times\mathbf{R}^n)_{(y,0)}$-module.

Now we prove by decreasing induction on k that A is finitely generated as a $C(P\times\mathbf{R}^k)_{(y,0)}$-module for all k, $0\leq k\leq n$. We have just shown that this is true for the case $k=n$, so it suffices to carry out the inductive step. Assume that A is finitely generated as a $C(P\times\mathbf{R}^{k+1})_{(y,0)}$-module. Then

$$f^*(\mathfrak{m}_y)A=(\pi^*_{k+1}\circ\ldots\circ\pi^*_1)(\mathfrak{m}_y)A$$

(where on the left hand side A is considered as a $C(N)_x$-module and on the right hand side A is considered as a $C(P\times\mathbf{R}^{k+1})_{(y,0)}$-module). Thus, if $\mathfrak{n}$ denotes the maximal ideal of $C(P\times\mathbf{R}^k)_{(y,0)}$,

$$f^*(\mathfrak{m}_y)A\subset\pi_{k+1}^*(\mathfrak{n})A.$$

In particular $A/\pi_{k+1}^*(\mathfrak{n})A$ is finitely generated as an $\mathbf{R}$-vector space. In particular, the hypotheses of the theorem are satisfied for π_{k+1} in place of f. Thus, we may apply step 1 to see that A is finitely generated as a $C(P\times\mathbf{R}^k)_{(y,0)}$-module. This completes the inductive step, and also the proof, since the case $k=0$ is just the statement of the theorem.

Step 3. — The theorem in general follows from step 2 and (1.3).

(1.12) Let $\varphi: R\to S$ be a ring homomorphism.

Definition 1. — *Let* A *be an* R*-module and* B *an* S*-module. A mapping* $\alpha: A\to B$ *will be said to be a homomorphism* over φ *if* $\alpha(a_1+a_2)=\alpha(a_1)+\alpha(a_2)$ *and* $\alpha(ra_1)=\varphi(r)\alpha(a_1)$ *for all* $a_1, a_2\in A$ *and* $r\in R$.

Note that α is a homomorphism over φ if and only if α is a homomorphism of R-modules, where B is given the R-module structure induced by φ.

Definition 2. — *By* a mixed homomorphism over φ, *we will mean a quintuple* (α, β, A, B, C), *where* A *is an* R*-module,* B *and* C *are* S*-modules,* $\alpha: A\to C$ *is a homomorphism over* φ *and* $\beta: B\to C$ *is a homomorphism of* S*-modules. Such a mixed homomorphism will be said to be of* finite type *if* A *is finitely generated as an* R*-module and* C *is finitely generated as an* S*-module.*

Lemma. — *Let* $f:(N,S)\dashrightarrow(P,y)$ *be a* C^∞ *map-germ, where* S *is a finite subset of* N *and* $y\in P$. *Suppose* (α,β,A,B,C) *is a mixed homomorphism of finite type over* $f^*: C(P)_y\to C(N)_S$. *Then*

a) $$\alpha(A)+\beta(B)+f^*(\mathfrak{m}_y)C=C$$

implies

b) $$\alpha(A)+\beta(B)=C.$$

Proof. — Set $C'=C/\beta(B)$ and let $\rho: C\to C'$ be the projection. Then C' is finitely generated as a $C(N)_S$-module. From *a)*, we get

$$(\rho\circ\alpha)(A)+f^*(\mathfrak{m}_y)C'=C'.$$

If we consider C' as a $C(P)_y$-module, where the C_y-module structure is induced by f^*, then this equation reads

c) $$(\rho\circ\alpha)(A)+\mathfrak{m}_yC'=C'.$$

Since A is finitely generated as a C_y-module and $\rho\circ\alpha$ is a homomorphism of C_y-modules, it follows that $C'/\mathfrak{m}_yC'$ is finitely generated as a C_y-module, and hence is finite dimensional as an $\mathbf{R}$-vector space. But $C'/\mathfrak{m}_yC'=C'/f^*(\mathfrak{m}_y)C'$; hence, by Malgrange's preparation theorem (1.11), C' is finitely generated as a C_y-module. Then by Nakayama's lemma, *c)* implies $(\rho\circ\alpha)(A)=C'$, which trivially implies *b)*.

Theorem **(1.13)**. — *Let* $f:(N,S)\dashrightarrow(P,y)$ *be a* C^∞ *map-germ. Suppose* (α,β,A,B,C) *is a mixed homomorphism of finite type over* $f^*: C(P)_y\to C(N)_S$. *Let* $a=\dim_R A/\mathfrak{m}_y A$. *Then*

a) $$\alpha(A)+\beta(B)+(f^*(\mathfrak{m}_y)+\mathfrak{m}_S^{a+1})C=C$$

implies

b) $$\alpha(A)+\beta(B)=C.$$

Proof. — It is enough to show *a)* implies (1.12 *a)*). Set $B'=\beta(B)+f^*(\mathfrak{m}_y)C$. To show (1.12 *a)*), it is enough to show $\mathfrak{m}_S^{a+1}C\subset B'$. Note that $\alpha(\mathfrak{m}_y A)\subset f^*(\mathfrak{m}_y)C\subset B'$. Therefore *a)* implies

c) $$\dim_R C/(\mathfrak{m}_S^{a+1}C+B')\leq \dim \alpha(A)/\alpha(\mathfrak{m}_y A)\leq a.$$

Therefore $\mathfrak{m}_S^a C\subset B'$ by (1.6), so (1.12 *a)*) follows, which completes the proof.

Corollary **(1.14)**. — *Let* $f:(N,S)\dashrightarrow(P,y)$ *be a* C^∞ *map-germ. Suppose* (α,β,A,B,C) *is a mixed homomorphism of finite type over* f^*. *Let* $a=\dim_R A/\mathfrak{m}_y A$. *Let* C_0 *be a* $C(N)_S$-*submodule of* C *such that*

$$c=\dim_R C/C_0$$

is finite. Let $p=\dim P$. *Set* $l=a\binom{p+c}{c}$. *Then*

a) $$\alpha(A)+\beta(B)+\mathfrak{m}_S^{l+1}C_0\supseteq C_0$$

implies

b) $$\alpha(A)+\beta(B)\supseteq C_0.$$

Proof. — Set $C'=C_0+\beta(B)$, $A'=\alpha^{-1}(C')$, $\alpha'=\alpha|A'$. Then *a)* implies

c) $$\alpha'(A')+\beta(B)+\mathfrak{m}_S^{l+1}C'=C'.$$

Since $\dim_R A/A'\leq\dim_R C/C'\leq c$, it follows from (1.7) that A' is generated as a C_y-module by l or fewer elements. It follows that theorem (1.13) applies to the mixed homomorphism (α',β,A',B,C'), since *c)* implies (1.13 *a)*). Hence, we may conclude (1.13 *b)*) for this mixed homomorphism; that is

$$\alpha(A')+\beta(B)=C'.$$

But this clearly implies *b)*.

2. Contact equivalence.

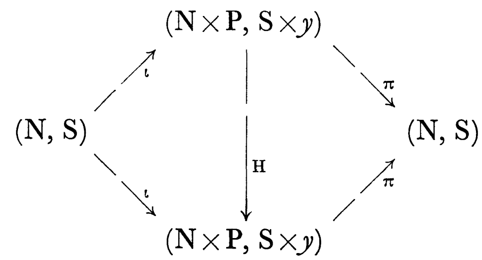
(2.1) Let N and P be manifolds, let S be a finite subset of N, and let $y\in P$. We let $\mathcal{F}$ denote the set of all C^∞ map-germs $f:(N,S)\dashrightarrow(P,y)$.

(2.2) We let $\mathcal{C}$ denote the group of invertible C^∞ map-germs

$$H:(N\times P,S\times y)\dashrightarrow(N\times P,S\times y)$$

such that the diagram below commutes.

diag. (2.2)



In this diagram, ι denotes the germ at S of the inclusion $N \to N \times P$, $n \mapsto (n, y)$ and π the germ of the projection $N \times P \to N$. We take composition as the group law.

We may describe $\mathscr{C}$ more simply (but more vaguely) as the group of germs of families of diffeomorphisms of P into itself which fix y, parameterized by N. Thus any H in $\mathscr{C}$ is of the form $H(n, p) = (n, H_1(n, p))$ where $H_1 : (N \times P, S \times y) \dashrightarrow (P, y)$ is a C^∞ map-germ and $H_1(n, y) = y$.

We define an action of $\mathscr{C}$ on $\mathscr{F}$ by the formula

a) $$(1, H.f) = H \circ (1, f), \quad H \in \mathscr{C}, \quad f \in \mathscr{F},$$

where 1 denotes the identity map-germ $(N, S) \dashrightarrow (N, S)$. (The bracket (g_1, g_2) of two map-germs with the same source was defined in (1.1).) Both sides of this equation are map-germs $(N, S) \dashrightarrow (N \times P, S \times y)$, so it makes sense. To see that $H.f$ is indeed defined by this equation, it is enough to see that $H \circ (1, f)$ can be written in the form $(1, g)$ for a suitable map-germ $g : (N, S) \dashrightarrow (P, y)$. But this is an immediate consequence of the fact that the right hand triangle in diag. (2.2) commutes.

Equation *a)* can also be written in the form

$$H.f(n) = H_1(n, f(n)),$$

where H_1 is as above.

Clearly,

b) $$\operatorname{graph}(H.f) = H(\operatorname{graph} f),$$

and this formula characterizes the action of $\mathscr{C}$ on $\mathscr{F}$.

(2.3) In the next proposition, we give three characterizations of the orbits of the action of $\mathscr{C}$ on $\mathscr{F}$.

If $f \in \mathscr{F}$, let $I(f) = f^*(\mathfrak{m}_y) C(N)_S$. Let $y_1, \ldots, y_p$ be a minimal set of generators of $\mathfrak{m}_y$ as an ideal in $C(P)_y$. (By (1.4), Corollary 2 and the implicit function theorem, $y_1, \ldots, y_p \in C(P)_y$ form a minimal set of generators of $\mathfrak{m}_y$ if and only if suitable representatives $\tilde{y}_1, \ldots, \tilde{y}_p$ form a local system of coordinates for P, null at y.) Then $I(f)$ is the ideal in $C(N)_S$ generated by $f^*(y_1), \ldots, f^*(y_p)$.

Proposition. — The following are equivalent for $f, g \in \mathscr{F}$.

(i) *f is in the same $\mathscr{C}$-orbit as g.*

(ii) $I(f) = I(g)$.

(iii) *There exists an invertible $p \times p$ matrix (u_{ij}) with entries in $C(N)_S$ such that*

$$f^*(y_i) = \sum_j u_{ij} g^*(y_j).$$

(iv) *There exists an invertible* C^∞ *map-germ* $H : (N\times P, S\times y) \dashrightarrow (N\times P, S\times y)$ *such that* $H|(N\times y, S\times y) =$ *identity and such that* $H(\text{graph} f) = \text{graph}\, g$.

Proof. — We will show that (i) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). In fact (i) $\Rightarrow$ (iv) is an immediate consequence of the fact that formula (2.2 *b*)) characterizes the action of $\mathscr{C}$ on $\mathscr{F}$ and the fact that $H|(N\times y, S\times y) = \text{identity}$, for any $H \in \mathscr{C}$ (since the left hand triangle in diag. (2.2) commutes).

To prove (iv) $\Rightarrow$ (ii), we introduce the following ideal in $C(N\times P)_{S\times y}$:

$$Z(f) = \{u : u|\text{graph} f = 0\}.$$

Let $\iota : (N, S) \dashrightarrow (N\times P, S\times y)$ denote the germ of the inclusion $n \mapsto (n, y)$. Then ι^* is a homomorphism of $C(N\times P)_{S\times y}$ onto $C(N)_S$. Furthermore $\iota^*(Z(f)) = I(f)$. For, by (1.4), $Z(f)$ is generated by $\{\pi_2^* y_i - \pi_1^* f^* y_i\}_{i=1,\dots,p}$, where $\pi_1 : (N\times P, S\times y) \dashrightarrow (N, S)$ and $\pi_2 : (N\times P, S\times y) \dashrightarrow (P, y)$ denote the projections. Since $\iota^*(\pi_2^* y_i - \pi_1^* f^* y_i) = -f^* y_i$, it follows $\iota^*(Z(f))$ is generated by $f^*y_1, \dots, f^*y_p$, and therefore that $\iota^*(Z(f)) = I(f)$, as asserted. Now let H be as in (iv). Since $H(\text{graph} f) = \text{graph}\, g$, $H^*(Z(g)) = Z(f)$. The assumption that $H|(N\times P, S\times y) = \text{identity}$ means that $H\circ\iota = \iota$. Thus

$$I(g) = \iota^*(Z(g)) = \iota^*(H^*(Z(g))) = \iota^*(Z(f)) = I(f).$$

Next, we prove (ii) $\Rightarrow$ (iii). For this, we need the following.

Lemma. — *Let A and B be $p\times p$ matrices with entries in* $\mathbf{R}$. *Then there exists a $p\times p$ matrix C with entries in* $\mathbf{R}$ *such that $C(I-AB)+B$ is invertible.*

Proof. — Let $\alpha, \beta : \mathbf{R}^p \to \mathbf{R}^p$ be the linear transformations corresponding to A and B, resp. Choose a basis $e_1, \dots, e_p$ of $\mathbf{R}^p$ such that $\beta e_i = 0$, $i \geq r+1$, where r is the rank of β. Choose $e'_{r+1}, \dots, e'_p$ in $\mathbf{R}^p$ such that $\beta e_1, \dots, \beta e_r, e'_{r+1}, \dots, e'_p$ is a basis of $\mathbf{R}^p$. Let $\gamma : \mathbf{R}^p \to \mathbf{R}^p$ be the linear transformation defined by $\gamma e_i = 0$, $1 \leq i \leq r$ and $\gamma e_i = e'_i$, $r+1 \leq i \leq p$. Then

$$\begin{aligned}(\gamma(1-\alpha\beta)+\beta)(e_i) &= \beta e_i + \sum_{j=r+1}^{p} C_{ij} e'_i, &&\text{if } 1 \leq i \leq r\\ &= e'_i, &&\text{if } r+1 \leq i \leq p,\end{aligned}$$

so $\gamma(1-\alpha\beta)+\beta$ is invertible. This proves the lemma, where we take for C the matrix corresponding to γ.

Assuming (ii), we see that there exist $p\times p$ matrices (w_{ij}) and (v_{ij}) with entries in $C(N)_S$ such that

a) $$f^*(y_i) = \sum w_{ij} g^*(y_j), \qquad g^*(y_i) = \sum v_{ij} f^*(y_j).$$

For each $x \in S$, let $V(x)$ and $W(x)$ be the matrices $(v_{ij}(x))$ and $(w_{ij}(x))$ respectively. Since the entries of $V(x)$ and $W(x)$ are in $\mathbf{R}$, we may apply the lemma, with $A = V(x)$ and $B = W(x)$. Thus, there exists a matrix $C(x) = (c_{ij}(x))$ (with entries in $\mathbf{R}$) such that $C(x)(I - V(x)W(x)) + W(x)$ is invertible. Let $C = (c_{ij})$ be the matrix with entries in $C(N)_S$ such that c_{ij} is the constant $c_{ij}(x)$ in a neighborhood of x, for each $x \in S$. Let V, W be the matrices (v_{ij}), (w_{ij}), respectively. Set $U = C(I - VW) + W$ and let u_i

be the $(i,j)^{\text{th}}$-entry of U. From $a)$, it follows that $f^*(y_i)=\Sigma u_{ij}g^*(y_j)$, and from the fact that $U(x)=C(x)(I-V(x)W(x))+W(x)$ is invertible for each $x\in S$, it follows that U is invertible; hence (iii) is proved.

Now, we show (iii) $\Rightarrow$ (i). Let (u_{ij}) be as in (iii). Let

$$\pi_1 : (N\times P, S\times y) \dashrightarrow (N, S) \quad \text{and} \quad \pi_2 : (N\times P, S\times y) \dashrightarrow (P, y)$$

be the projections. Define $H : (N\times P, S\times y) \dashrightarrow (N\times P, S\times y)$ by

$$\pi_1\circ H=\pi_1; \quad y_i\circ\pi_2\circ H=\sum_j (u_{ij}\circ\pi_1)(y_j\circ\pi_2).$$

Clearly the right hand triangle of diagram (2.2) commutes. The left hand triangle also commutes, since the right hand triangle commutes and H maps the set-germ $(N\times y, S\times y)=\{y_1\circ\pi_2=\ldots=y_p\circ\pi_2=0\}$ into itself. Hence $H\in\mathscr{C}$. Now

$$\begin{aligned} y_i\circ\pi_2\circ H\circ(1, g) &= \sum_j (u_{ij}\circ\pi_1)(y_j\circ\pi_2)\circ(1, g) \\ &= \sum_j u_{ij}g^*(y_j)=f^*(y_i). \end{aligned}$$

Hence $(1,f)=H\circ(1, g)$, so by the definition of the action of $\mathscr{C}$ on $\mathscr{F}$ (formula (2.2 a))), it follows that $f=H.g$, so f is in the same $\mathscr{C}$-orbit as g, as asserted.

(2.4) The theory of $\mathscr{C}$-orbits reduces trivially to the case where S is a point. For, consider the general case (when S is a finite set in N). For each $x\in S$, let $\mathscr{F}_x$, $\mathscr{C}_x$, and the action of $\mathscr{C}_x$ on $\mathscr{F}_x$ be defined in the same way as $\mathscr{F}$, $\mathscr{C}$, and the action of $\mathscr{C}$ on $\mathscr{F}$, except with x in place of S. Note that if $f\in\mathscr{F}$, then $f|(N, x)\in\mathscr{F}_x$.

Proposition. — Let $f, g\in\mathscr{F}$. Then f and g are in the same $\mathscr{C}$-orbit if and only if $f|(N, x)$ and $g|(N, x)$ are in the same $\mathscr{C}_x$-orbit for each $x\in S$.

The proof is trivial.

Thus, if we were only interested in $\mathscr{C}$-orbits, there would be no reason to consider any case other than that in which S is a single point. However, we will be interested in other groups, for which the theory does not reduce trivially to the case where S is a single point, and since we wish to retain the same language throughout, we state all our results in the general setting.

Definition **(2.5)**. — *Let $f_i : (N_i, S_i) \dashrightarrow (P_i, y_i)$ $(i=1, 2)$ be C^∞ map-germs, where for each i, S_i is a finite subset of N_i and $y_i\in P_i$. We say f_1 and f_2 are* contact equivalent *if and only if there exists an invertible C^∞ map-germ $H : (N_1\times P_1, S_1\times y_1) \dashrightarrow (N_2\times P_2, S_2\times y_2)$ such that*

a) $$H(N_1\times y_1, S_1\times y_1)=(N_2\times y_2, S_2\times y_2)$$

b) $$H(\text{graph} f_1)=\text{graph} f_2.$$

Note that f_1 being contact equivalent to f_2 implies $\dim N_1=\dim N_2$ and $\dim P_1=\dim P_2$.

(2.6) We let $\mathscr{K}$ denote the group of invertible C^∞ map-germs

$$H : (N\times P, S\times y) \dashrightarrow (N\times P, S\times y)$$

such that there exists a map-germ $h : (N, S) \dashrightarrow (N, S)$ which makes the diagram below commutative

diag. (**2.6**)

$$\begin{array}{ccccc} (N, S) & \overset{\iota}{\dashrightarrow} & (N\times P, S\times y) & \overset{\pi}{\dashrightarrow} & (N, S) \\ \downarrow h & & \downarrow H & & \downarrow h \\ (N, S) & \overset{\iota}{\dashrightarrow} & (N\times P, S\times y) & \overset{\pi}{\dashrightarrow} & (N, S) \end{array}$$

Here, ι and π are as in (2.2). Clearly h is uniquely determined by H, and h is necessarily C^∞ and invertible. We take composition as group law.

We define an action of $\mathscr{K}$ on $\mathscr{F}$ by the formula,

a) $$(\mathbf{1}, H\circ f) = H\circ(\mathbf{1}, f)\circ h^{-1}, \quad H\in\mathscr{K}, \quad f\in\mathscr{F},$$

where $\mathbf{1}$ denotes the identity map-germ $(N, S) \dashrightarrow (N, S)$ and h is the map-germ which makes diag. (2.6) commutative. We leave it to the reader to check that this is a valid definition and defines a left action.

It is easily seen that

b) $$H(\mathrm{graph} f) = \mathrm{graph}(H.f)$$

and that this formula characterizes the action of $\mathscr{K}$ on $\mathscr{F}$.

(**2.7**) We let $\mathscr{R}$ denote the group of invertible C^∞ map-germs $h : (N, S) \dashrightarrow (N, S)$, with composition as group law. We define an action of $\mathscr{R}$ on $\mathscr{F}$ by the formula

$$h.f = f\circ h^{-1}, \quad f\in\mathscr{F}, \quad h\in\mathscr{R}.$$

(**2.8**) Clearly $\mathscr{C}$ is a subgroup of $\mathscr{K}$. For each $h\in\mathscr{R}$, we identify h with $(h, \mathbf{1})\in\mathscr{K}$, where $\mathbf{1}$ denotes the identity map-germ $(P, y) \dashrightarrow (P, y)$. This makes $\mathscr{R}$ a subgroup of $\mathscr{K}$. The action of $\mathscr{R}$ on $\mathscr{F}$ is the restriction of the action of $\mathscr{K}$ on $\mathscr{F}$. $\mathscr{K}$ is the semi-direct product of $\mathscr{R}$ and $\mathscr{C}$ in the sense that $\mathscr{C}$ is a normal subgroup of $\mathscr{K}$ and each element of $\mathscr{K}$ can be represented in one and only one way as a product rc, where $r\in\mathscr{R}$ and $c\in\mathscr{C}$.

Proposition (**2.9**). — *Let $f, g\in\mathscr{F}$. Then f is in the same $\mathscr{K}$-orbit as g if and only if f is contact equivalent to g.*

Proof. — This follows from the equivalence of (i) and (iv) in proposition (2.3). For, f is in the same $\mathscr{K}$-orbit as g if and only if there exists $h\in\mathscr{R}$ such that $f\circ h$ is in the same $\mathscr{C}$-orbit as g, and f is contact equivalent to g if and only if there exists $h\in\mathscr{R}$ such that $f\circ h$ and g satisfies the relation in (iv).

(**2.10**) The theory of $\mathscr{K}$-orbits reduces trivially to the case where S is a point. We state this result in terms of contact equivalence:

Proposition. — *Let $f, g\in\mathscr{F}$. Then f is contact equivalent to g if and only if there exists a permutation π of S such that for each $x\in S$, $f|(N, x)$ is contact equivalent to $g|(N, \pi x)$.*

3. Finitely Determined Germs of Mappings.

(3.1) We let $\mathscr{F}$ be as in § 2. If $f, g \in \mathscr{F}$, we say f and g have the same k-jet at S if for each $x \in S$ the derivatives of f and of g at x of order $\leq k$ are the same (with respect to one and therefore every pair of coordinate systems about x and y). By the k-jet (at S) of $f \in \mathscr{F}$, we mean the equivalence class of all $g \in \mathscr{F}$ having the same k-jet at S as f. We will write $f^{(k)}$ for the k-jet of f, if $f \in \mathscr{F}$.

Definition. — Let $f \in \mathscr{F}$ and let $\mathscr{S}$ be a group which acts on $\mathscr{F}$. We will say f is k-determined relative to $\mathscr{S}$ (abbreviated k-det. rel. to $\mathscr{S}$) *if for any $g \in \mathscr{F}$ such that $g^{(k)} = f^{(k)}$, the $\mathscr{S}$-orbit of f contains g. We will say f is finitely determined relative to $\mathscr{S}$* (abbreviated f.d. rel. $\mathscr{S}$) *if f is k-determined for some positive integer k.*

(3.2) We will state necessary and sufficient conditions for f to be f.d. rel. $\mathscr{S}$ for five groups which act on $\mathscr{F}$. We defined three of these groups in the previous section: $\mathscr{R}$, $\mathscr{C}$ and $\mathscr{K}$. The other two are: $\mathscr{L} = \{$invertible C^∞ map-germs $h' : (P, y) \dashrightarrow (P, y)\}$ whose group law is composition, and whose action on $\mathscr{F}$ is given by

$$h'.f = h' \circ f, \quad h' \in \mathscr{L},\ f \in \mathscr{F};$$

and $\mathscr{A} = \mathscr{R} \times \mathscr{L}$, whose action on $\mathscr{F}$ is the direct product of the actions of $\mathscr{R}$ and $\mathscr{L}$ on $\mathscr{F}$, that is

$$(h, h').f = h' \circ f \circ h^{-1}, \quad h \in \mathscr{R},\ h' \in \mathscr{L},\ f \in \mathscr{F}.$$

For any $h' \in \mathscr{L}$, we will identify h' with $(\mathbf{1}_{(N, S)}, h') \in \mathscr{A}$. This makes $\mathscr{L}$ into a subgroup of $\mathscr{A}$. The action of $\mathscr{L}$ on $\mathscr{F}$ is the restriction of the action of $\mathscr{A}$.

Since $\mathscr{K}$ is the semi-direct product of $\mathscr{R}$ and $\mathscr{C}$, $\mathscr{A}$ is identified in an evident fashion with a subgroup of $\mathscr{K}$. The action of $\mathscr{A}$ on $\mathscr{F}$ is the restriction of the action of $\mathscr{K}$ on $\mathscr{F}$.

(3.3) Let $f : (U, \Sigma) \dashrightarrow (V, \Pi)$ be a C^∞ map-germ, where U and V are manifolds, $\Sigma \subset U$, and $\Pi \subset V$. By a *vector field along f*, we will mean a map-germ $\zeta : (U, \Sigma) \dashrightarrow TV$ such that $\pi \circ \zeta = f$, where $\pi : TV \to V$ denotes the projection. We let $\theta(f)$ denote the $C(U)_\Sigma$-module consisting of all C^∞ vector fields along f. Another way of describing $\theta(f)$ is as the $C(U)_\Sigma$-module consisting of all germs at Σ of sections of the bundle $\tilde{f}^* TV$, for any representative $\tilde{f} : U_0 \to V$ of f.

We will set

$$A = \theta(\mathbf{1}_{(P, y)}), \quad B = \theta(\mathbf{1}_{(N, S)})$$

where $\mathbf{1}_{(P,y)}$ and $\mathbf{1}_{(N, S)}$ denote the identity map-germs $(P, y) \dashrightarrow (P, y)$ and $(N, S) \dashrightarrow (N, S)$ respectively. For example, A is the $C(P)_y$-module consisting of all germs at y of C^∞ vector fields on P.

(3.4) If $f \in \mathscr{F}$, we let $Tf : (TN, \pi_N^{-1} S) \dashrightarrow (TP, \pi_P^{-1} y)$ denote the tangent map-germ of f, i.e. the germ at $\pi_N^{-1} S$ of the tangent mapping of any representative of f. (Here $\pi_N : TN \to N$ and $\pi_p : TP \to P$ denote the projections.) The diagram below commutes

$$\begin{array}{ccc} (TN, \pi_N^{-1}S) & \xrightarrow{Tf} & (TP, \pi_p^{-1}y) \\ \downarrow{\scriptstyle \pi_N} & & \downarrow{\scriptstyle \pi_P} \\ (N, S) & \xrightarrow{f} & (P, y) \end{array}$$

Thus, for any $\xi \in B$, $Tf\circ\xi\in\theta(f)$, and for any $\eta\in A$, $\eta\circ f\in\theta(f)$. We define

$$tf : B\to\theta(f), \qquad \omega f : A\to\theta(f)$$

by $tf(\xi)=Tf\circ\xi$, $\omega f(\eta)=\eta\circ f$. Then tf is a homomorphism of $C(N)_S$-modules and ωf is a homomorphism over $f^* : C(P)_y\to C(N)_S$. In short, $(\omega f, tf, A, B, \theta(f))$ is a mixed homomorphism over f^* (in the sense of (1.12) definition 2).

(3.5) For $f\in\mathscr{F}$, we set:

$$d(f, \mathscr{K})=\dim_{\mathbf{R}}\theta(f)/(tf(B)+f^*(\mathfrak{m}_y)\theta(f))$$
$$d(f, \mathscr{A})=\dim_{\mathbf{R}}\theta(f)/(tf(B)+\omega f(A))$$
$$d(f, \mathscr{R})=\dim_{\mathbf{R}}\theta(f)/tf(B)$$
$$d(f, \mathscr{L})=\dim_{\mathbf{R}}\theta(f)/\omega f(A)$$
$$d(f, \mathscr{C})=\dim_{\mathbf{R}}\theta(f)/f^*(\mathfrak{m}_y)\theta(f).$$

Theorem. — If $\mathscr{S}$ is any of the groups $\mathscr{K}$, $\mathscr{A}$, $\mathscr{R}$, $\mathscr{L}$, or $\mathscr{C}$, then the necessary and sufficient condition that $f\in\mathscr{F}$ be f.d. rel. $\mathscr{S}$ is that $d(f, \mathscr{S})<\infty$.

We will prove this theorem only in the cases $\mathscr{S}=\mathscr{K}$ and $\mathscr{S}=\mathscr{A}$. The proofs in the other cases are similar, and the results are much less interesting. The proofs in the two cases which interest us will occupy §§ 4-8.

We conclude this section by stating a result related to and an extension of the above theorem.

Proposition **(3.6)**. — a) *$d(f, \mathscr{K})<\infty$ if and only if there exists a positive integer k such that:*

(i) $tf(B)+f^*(\mathfrak{m}_y)\theta(f)\supseteq\mathfrak{m}_S^k\theta(f)$.

b) *$d(f, \mathscr{A})<\infty$ if and only if there exists a positive integer k such that*

(ii) $tf(B)+\omega f(A)\supseteq\mathfrak{m}_S^k\theta(f)$.

The analogous results hold for the other groups $\mathscr{S}$.

In both cases " if " is trivial because the displayed formula implies

$$d(f, \mathscr{S})\leq\dim_{\mathbf{R}}\theta(f)/\mathfrak{m}_S^k\theta(f),$$

which is finite.

" Only if " follows from

Lemma. — a) *If*

(iii) $tf(B)+(f^*(\mathfrak{m}_y)+\mathfrak{m}_S^{k+1})\theta(f)\supseteq\mathfrak{m}_S^k\theta(f)$

then (i) *holds.*

b) *Let $n=\dim N$, $p=\dim P$, $|S|=$ number of elements of S. For any positive integer k, let $l=l(k)$ be given by*

(iv) $b=|S|\binom{n+k-1}{n}$, $a=p\binom{p+b}{p}$, $l=k+a+1$.

Then

(v) $tf(B)+\omega f(A)+\mathfrak{m}_S^{l(k)}\theta(f)\supseteq \mathfrak{m}_S^k\theta(f)$

implies (ii).

Proof. — *a)* Apply Nakayama's lemma with $R=C(N)_S$, $E=tf(B)+f^*(\mathfrak{m}_y)\theta(f)$ $F=tf(B)+(f^*(\mathfrak{m}_y)+\mathfrak{m}_S^k)\theta(f)$, α the inclusion mapping, and $\mathfrak{I}=\mathfrak{m}_S$. The hypothesis (iii) implies $E+\mathfrak{I}F=F$ and the conclusion of Nakayama's lemma, $E=F$, implies (i).

b) Apply (1.14) with the following substitutions:

$$\begin{array}{cccccc} \alpha & \beta & A & B & C & C_0 \\ \omega f & tf & A & B & \theta(f) & \mathfrak{m}_S^k\theta(f). \end{array}$$

Since $\dim_{\mathbf{R}}(C/C_0)\leq b$, we deduce *b)* from (1.14).

Proof of " only if " in the proposition.

We will consider only *b)*, the proof of *a)* being similar. Let

$$d_r=\dim_{\mathbf{R}}\theta(f)/(tf(B)+\omega f(A)+\mathfrak{m}_S^r\theta(f)).$$

Then $d_1\leq d_2\leq\ldots\leq d_r\leq\ldots$, and the assumption that $d(f,\mathscr{A})<\infty$ implies that there exists k such that $d_l=d_k$ for all $l\geq k$. In particular, $d_{l(k)}=d_k$ implies that (v) holds. Hence (ii) holds, which completes the proof.

(3.7) Let $\mathscr{K}_r$ denote the subgroup of $\mathscr{K}$ consisting of all $H\in\mathscr{K}$ whose r-jet at $S\times y$ is equal to the r-jet at $S\times y$ of the identity. We set $\mathscr{A}_r=\mathscr{A}\cap\mathscr{K}_r$, $\mathscr{L}_r=\mathscr{L}\cap\mathscr{K}_r$, $\mathscr{R}_r=\mathscr{R}\cap\mathscr{K}_r$, and $\mathscr{C}_r=\mathscr{C}\cap\mathscr{K}_r$, using the fact that each of the groups $\mathscr{A}$, $\mathscr{L}$, $\mathscr{R}$, $\mathscr{C}$ is identified with (or *is*) a subgroup of $\mathscr{K}$. We may describe $\mathscr{L}_r$ more directly as the set of all $h\in\mathscr{L}$ whose r-jet at y is the r-jet of the identity, $\mathscr{R}_r$ as the set of all $h\in\mathscr{R}$ whose r-jet at S is the r-jet of the identity, and $\mathscr{A}_r$ as $\mathscr{R}_r\times\mathscr{L}_r$.

Addendum to **(3.5)**. — For any non-negative integers r and d, and for $\mathscr{S}$ equal to any one of the groups $\mathscr{K}$, $\mathscr{A}$, $\mathscr{R}$, $\mathscr{L}$, or $\mathscr{C}$, there exists a non-negative integer $l=l(r,d;\mathscr{S})$ with the following property. Let $f\in\mathscr{F}$ be such that $d\geq d(f,\mathscr{S})$. Let g have the same l-jet at S as f. Then there exists $H\in\mathscr{S}_r$ such that $g=H.f$.

Clearly this implies sufficiency in (3.5). The proof will be carried out in §§ 4, 5 and 6. The proof of necessity in (3.5) will be carried out in §§ 7 and 8.

4. Beginning of the proof of (3.7).

(4.1) Let U and V be manifolds, and suppose $\Sigma\subset U\times\mathbf{R}$. A map-germ $g:(U\times\mathbf{R},\Sigma)\dashrightarrow V\times\mathbf{R}$ will be said to be *level preserving* if the diagram

$$\begin{array}{ccc} (U\times\mathbf{R},\Sigma) & \xrightarrow{g} & V\times\mathbf{R} \\ & \searrow^{\pi_2} \quad \swarrow^{\pi_2} & \\ & \mathbf{R} & \end{array}$$

commutes, where π_2 denotes the projection on the second factor. In other words, to say that g is level preserving is to say that g has the form $g(u, t)=(g_1(u, t), t)$, where $g_1 : (U\times\mathbf{R}, \Sigma) \dashrightarrow V$ is a C^∞ map-germ. If g is level preserving and C^∞, we define $\partial g/\partial t : (U\times\mathbf{R}, \Sigma) \dashrightarrow \pi_1^*TV$ (where $\pi_1 : V\times\mathbf{R}\to V$ denotes the projection on the first factor), as follows. Let $\widetilde{g} : W\to V\times\mathbf{R}$ be a C^∞ representative of g (where W is a neighborhood of Σ in $U\times\mathbf{R}$). For $u\in U$, $a\in\mathbf{R}$ such that $(u, a)\in W$, let $(\partial\widetilde{g}/\partial t)(u, a)\in TV_u=\pi_1^*TV_{(u,a)}$ denote the tangent at $t=0$ to the curve $t\mapsto(\pi_1\circ\widetilde{g})(u, a+t)$. Then $\partial\widetilde{g}/\partial t$ is a mapping of W into π_1^*TV. Let $\partial g/\partial t$ denote the germ of $\partial\widetilde{g}/\partial t$ at Σ.

If $a\in\mathbf{R}$ and g is as above, we define $g_a : (U, \Sigma_a) \dashrightarrow V$ as follows. We let $\iota_a : U\to U\times\mathbf{R}$ be given by $\iota_a(u)=(u, a)$. We set $\Sigma_a=\iota_a^{-1}(\Sigma)$ and $g_a=\pi_1\circ g\circ\iota_a$, where ι_a is the germ of ι_a at Σ_a.

Lemma (**4.2**). — *Let* U *be a manifold,* Σ *a finite subset of* U, *a a real number, and* ξ *a germ at* $\Sigma\times a$ *of a* C^∞ *section of the vector bundle* π_1^*TU. *Here* $\pi_1 : U\times\mathbf{R}\to U$ *denotes the projection, so that* π_1^*TU *is a vector bundle over* $U\times\mathbf{R}$. *Then there exists an invertible* C^∞ *level preserving map-germ* $H : (U\times\mathbf{R}, \Sigma\times a) \dashrightarrow (U\times\mathbf{R}, \Sigma\times a)$ *such that* $H_a=\mathbf{1} : (U, \Sigma) \dashrightarrow (U, \Sigma)$ *and* $(\partial H/\partial t)\circ H^{-1}=\xi$.

This is simply a form of the fundamental existence theorem for ordinary differential equations. Our proof consists of showing how this form follows from a more familiar form.

Proof. — Let $\widetilde{\xi} : V\times J\to\pi_1^*TU$ be a representative of ξ, where V is an open neighborhood of Σ in U and J is an open interval containing a. We may assume that $\widetilde{\xi}$ is C^∞ and that it is a section of π_1^*TU over $V\times J$. By Lang [1], IV, § 1, theorem 1, there exists a neighborhood V_0 of Σ in V, an open subinterval J_0 of J containing a, and a unique C^∞ " local flow " $\alpha : V_0\times J_0\to V$ for $\widetilde{\xi}$. The statement that α is a local flow means $\alpha(x, a)=x$ for all $x\in V_0$ and $\partial\alpha(x, t)/\partial t=\widetilde{\xi}(\alpha(x, t), t)$ for all $x\in V_0$ and $t\in J_0$. Define $\widetilde{H} : V_0\times J_0 \to V\times J_0$ by $\widetilde{H}(x, t)=(\alpha(x, t), t)$. Let H denote the germ of $\widetilde{H}$ at $\Sigma\times a$. Clearly H is a C^∞ level preserving map-germ and $H_a=\mathbf{1} : (U, \Sigma) \dashrightarrow (U, \Sigma)$. Hence by the inverse function theorem, H is invertible. Finally $(\partial H/\partial t)\circ H^{-1}=\xi$ is an immediate consequence of $\partial\alpha(x, t)/\partial t=\widetilde{\xi}(\alpha(x, t), t)$.

It also follows from the uniqueness theorem for ordinary differential equations that ξ uniquely determines H. We will call H the *integral* of ξ.

(**4.3**) For any manifold U, any $\Sigma\subset U$, and any $a\in\mathbf{R}$, we let

$$\pi_1^U : (U\times\mathbf{R}, \Sigma\times a) \dashrightarrow (U, \Sigma)$$

denote the projection on the first factor.

Now suppose Σ is finite. Let V be a second manifold, Π a finite subset of V, and $F : (U\times\mathbf{R}, \Sigma\times a) \dashrightarrow (V\times\mathbf{R}, \Pi\times a)$ a C^∞ level preserving map-germ. We define $t_1F : \theta(\pi_1^U) \to \theta(\pi_1^V\circ F)$ and $\omega_1F : \theta(\pi_1^V) \to \theta(\pi_1^V\circ F)$, (where $\theta(f)$ is as defined in (3.3)), as follows. For $\eta\in\theta(\pi_1^V)$, set $\omega_1F(\eta)=\eta\circ F$. For $\xi\in\theta(\pi_1^U)$, take C^∞ representatives

$$\widetilde{\xi} : W\times J\to TU \text{ of } \xi \quad\text{and}\quad \widetilde{F} : W\times J\to V\times J \text{ of } F,$$

where W is an open neighborhood of Σ in U and J is an open interval in $\mathbf{R}$ containing Σ. Then we define $t_1 F(\xi)$ to be the germ at $\Sigma\times a$ of $(w, t)\mapsto T\widetilde{F}_t\circ\widetilde{\xi}_t(w)$. (Compare the definition of tf and ωf in (3.4).)

Lemma. — Suppose $\xi\in\theta(\pi_1^U)$ *and* $\eta\in\theta(\pi_1^V)$ *are such that*

$$\frac{\partial F}{\partial t}=t_1F(\xi)+\omega_1F(\eta).$$

Let

$$H:(U\times\mathbf{R},\Sigma\times a)\to(U\times\mathbf{R},\Sigma\times a)$$

be the integral of $-\xi$ *and*

$$H':(V\times\mathbf{R},\Pi\times a)\to(V\times\mathbf{R},\Pi\times a)$$

the integral of η. *Then*

$$\pi_1^V\circ H'^{-1}\circ F\circ H=F_a\circ\pi_1^U.$$

Proof. — It is enough to show

$$\frac{\partial}{\partial t}(H'^{-1}\circ F\circ H)=0. \qquad (*)$$

Take representatives $\widetilde{H}'$ of H', $\widetilde{F}$ of F, and $\widetilde{H}$ of H. Then

$$\begin{aligned}&\frac{\partial}{\partial t}(\widetilde{H}'^{-1}\circ\widetilde{F}\circ\widetilde{H})\\&=\frac{\partial\widetilde{H}'^{-1}}{\partial t}\circ\widetilde{F}\circ\widetilde{H}+T\widetilde{H}_t'^{-1}\circ\frac{\partial\widetilde{F}}{\partial t}\circ\widetilde{H}+T\widetilde{H}'^{-1}\circ T\widetilde{F}_t\circ\frac{\partial\widetilde{H}}{\partial t}\\&=T\widetilde{H}_t'^{-1}\circ\left(-\frac{\partial\widetilde{H}'}{\partial t}\circ\widetilde{H}'^{-1}\circ\widetilde{F}+\frac{\partial\widetilde{F}}{\partial t}+T\widetilde{F}_t\circ\frac{\partial\widetilde{H}}{\partial t}\circ\widetilde{H}^{-1}\right)\circ\widetilde{H}.\end{aligned}$$

The germ at $\Sigma\times a$ of the quantity inside the parentheses is

$$-\omega_1F(\eta)+\partial F/\partial t-t_1F(\xi)=0.$$

Hence (*) holds, which proves the lemma.

(4.4) Let $f, g\in\mathscr{F}$ (where $\mathscr{F}$ is as in § 3). Let $y_1,\dots,y_p$ be a system of coordinates for P, null at y. Let t denote the projection $(N\times\mathbf{R}, S\times\mathbf{R})\to\mathbf{R}$ and also the projection $(P\times\mathbf{R}, y\times\mathbf{R})\to\mathbf{R}$. We let $G:(N\times\mathbf{R}, S\times\mathbf{R})\to(P\times\mathbf{R}, y\times\mathbf{R})$ be given by

$$y_i\circ G=(1-t)(y_i\circ f)+t(y_i\circ g), \qquad t\circ G=t,$$

so that G is a level preserving C^∞ map-germ with $G_0=f$ and $G_1=g$. For each $a\in\mathbf{R}$, we let $G^a:(N\times\mathbf{R}, S\times a)\to(P\times\mathbf{R}, y\times a)$ denote the restriction of G. Note that $G_a:(N, S)\to(P, y)$ is the restriction of G^a to $(N\times a, S\times a)$.

We identify any $u \in C(N)_S$ with $(\pi_1^N)^*(u) \in C(N \times \mathbf{R})_{S \times a}$ and any $v \in C(P)_y$ with $(\pi_1^P)^*(v) \in C(P \times \mathbf{R})_{y \times a}$. With these identifications, we have:

$$\mathfrak{m}_S \subset \mathfrak{m}_S C(N \times \mathbf{R})_{S \times a} \subset \mathfrak{m}_{S \times a}.$$

These are all proper inclusions.

We identify any $\xi \in B$ with $\xi \circ \pi_1^N \in \theta(\pi_1^N)$ and any $\eta \in A$ with $\eta \circ \pi_1^P \in \theta(\pi_1^P)$, so that $B \subset \theta(\pi_1^N)$ and $A \subset \theta(\pi_1^P)$.

For each differential operator $\partial/\partial y_i$ there is an associated unique vector field on P defined in a neighborhood of y. We will denote the germ at y of this vector field by $\partial/\partial y_i$, so that $\partial/\partial y_1, \ldots, \partial/\partial y_p$ form a free basis of A as a C_y-module. Then $(\partial/\partial y_1)\circ f, \ldots, (\partial/\partial y_p)\circ f$ form a free basis of $\theta(f)$ as a C_S-module and

$$(\partial/\partial y_1)\circ\pi_1^P\circ G^a, \ \ldots, \ (\partial/\partial y_p)\circ\pi_1^P\circ G^a$$

form a free basis of $\theta(\pi_1^P \circ G^a)$ as a $C_{S\times a}$-module. Using the identification we have just made of C_S with a subring of $C_{S\times a}$ we identify $\theta(f)$ with a subset of $\theta(\pi_1^P\circ G^a)$ by setting

$$\sum_i u_i(\partial/\partial y_i)\circ f = \sum_i u_i(\partial/\partial y_i)\circ\pi_1^P\circ G^a, \qquad u_i \in C_S.$$

With these identifications, we have:

Lemma. — Suppose g has the same l-jet at S *as f.*

a) *If* $\xi \in B$, *then* $tf(\xi) - t_1G^a(\xi) \in \mathfrak{m}_S^{l-1}\theta(\pi_1^P\circ G^a)$.

b) *If* $v \in C_y$ *then* $f^*(v) - G^{a*}(v) \in \mathfrak{m}_S^l C_{S\times a}$.

c) *If* $\eta \in A$ *then* $\omega f(\eta) - \omega_1 G^a(\eta) \in \mathfrak{m}_S^l \theta(\pi_1^P\circ G^a)$.

Proof. — a) Let $x_1, \ldots, x_n$ be a local system of coordinates for N, defined in a neighborhood of S. Then

$$tf(\partial/\partial x_i) = \sum_j \frac{\partial(y_j\circ f)}{\partial x_i}\left(\frac{\partial}{\partial y_j}\circ f\right)$$

$$t_1G^a(\partial/\partial x_i) = \sum_j \left(\frac{\partial(y_j\circ f)}{\partial x_i} + t\frac{\partial(y_j\circ g) - \partial(y_j\circ f)}{\partial x_i}\right)\left(\frac{\partial}{\partial y_j}\circ\pi_1^P\circ G^a\right).$$

It follows that

$$t_1G^a(\partial/\partial x_i) - tf(\partial/\partial x_i) \in \mathfrak{m}_S^{l-1}\theta(\pi_1^P\circ G^a)$$

since f and g have the same l-jet.

Then *a)* follows, since tf is a C_S-module homomorphism of B into $\theta(f)$ and t_1G is a $C_{S\times a}$-module homomorphism of $\theta(\pi_1^N)$ into $\theta(\pi_1^P\circ G^a)$.

b) We can write v as the sum of a polynomial in $y_1, \ldots, y_p$ and an element of $\mathfrak{m}_S^l$. Clearly f^* and G^{a*} map $\mathfrak{m}_y^l$ into $\mathfrak{m}_S^l C_{S\times a}$. It follows that we may suppose that v is a polynomial in $y_1, \ldots, y_p$. In fact, we may suppose that v is a monomial in $y_1, \ldots, y_p$. If degree $v = 0$, then v is a constant and $f^*(v) - G^{a*}(v) = 0$.

For the case when v is a monomial and degree$(v) > 0$, we give the proof by induction on degree(v). If degree$(v) = 1$, then $v = y_i$ for some i, and

$$f^*(y_i) - G^{a*}(y_i) = t(y_i\circ f - y_i\circ g) \in \mathfrak{m}_S^l C_{S\times a},$$

since f and g have the same l-jet. If degree$(v)>1$, we can write $v=y_i w$, where w is a suitable monomial; then

$$f^*(v)-G^{a*}(v)=(f^*(y_i)-G^{a*}(y_i))f^*(w)+G^{a*}(y_i)(f^*(w)-G^{a*}(w))\in \mathfrak{m}_S^l C_{S\times a},$$

by induction.

c) This follows immediately from *b)*, together with the observations that ωf is a homomorphism over f^* and $\omega_1 G^a$ is a homomorphism over G^{a*}.

5. Proof of (3.7) in the case $\mathscr{S}=\mathscr{K}$.

(5.1) We will prove (3.7) in the case $\mathscr{S}=\mathscr{K}$ with $l(r,d;\mathscr{K})=d+r+1$, if $r>0$ and $l(0,d;\mathscr{K})=d+2$. Let $f\in\mathscr{F}$ be such that $d=d(f,\mathscr{K})$ is finite. Let $g\in\mathscr{F}$ have the same l-jet at S as f, where $l=l(r,d;\mathscr{K})$. We will show that g is in the same $\mathscr{K}_r$-orbit as f.

We shall continue to use the notations and terminology introduced in §§ 3 and 4. In particular, we will assume that $y_1, \ldots, y_p$ and t are as in (4.4), and that G and G^a are as defined there. Moreover, we will continue to use the identifications that we introduced in (4.4).

(5.2) It follows from corollary (1.6), that

a) $$\mathfrak{m}_S^d\theta(f)\subseteq f^*(\mathfrak{m}_y)\theta(f)+tf(B),$$

since the right hand side is a sub-C_S-module of $\theta(f)$ having codimension d (by the definition of $d=d(f,\mathscr{K})$).

Fix $a\in\mathbf{R}$ and let $\Psi=\theta(\pi_1^P\circ G^a)$. From *a)* and the approximation lemma of (4.4), it follows that

b) $$\mathfrak{m}_S^d\Psi\subseteq G^{a*}(\mathfrak{m}_y)\Psi+t_1G^a(\theta(\pi_1^N))+\mathfrak{m}_S^{d+1}\Psi.$$

To see this, we observe that since the right hand side is a $C(N\times\mathbf{R})_{S\times a}$-module, and since $\mathfrak{m}_S^d\theta(f)$ generates $\mathfrak{m}_S^d\Psi$ as a $C_{S\times a}$-module, it is enough to show that $\mathfrak{m}_S^d\theta(f)$ is in the right hand side. Let $\zeta\in\mathfrak{m}_S^d\theta(f)$. By *a)*, we may write ζ in the form

$$\zeta=\sum_i f^*(v_i)\zeta_i+tf(\xi),$$

where $\zeta_i\in\theta(f)$, $v_i\in\mathfrak{m}_y$, and $\xi\in B$. From lemma (4.4), it follows that

$$\zeta-(\sum_i G^{a*}(v_i)\zeta_i+t_1G^a(\xi))\in\mathfrak{m}_S^{l-1}\Psi.$$

Since $l-1\geq d+1$, it follows that ζ is in the right hand side of *b)*. This proves *b)*.

From *b)* and Nakayama's lemma, it follows that

c) $$\mathfrak{m}_S^d\Psi\subseteq G^{a*}(\mathfrak{m}_y)\Psi+t_1G^a(\theta(\pi_1^N)).$$

To see this, apply Nakayama's lemma with $R=C_{S\times a}$, $E=$ right hand side of *c)*, $F=E+\mathfrak{m}_S^d\Psi$, $\alpha=$ the inclusion mapping, and $\mathfrak{I}=\mathfrak{m}_S C_{S\times a}$. From *b)*, it follows that $F=E+\mathfrak{I}F$, so by Nakayama's lemma, $F=E$, which implies *c)*.

Multiplying both sides of *c)* by $\mathfrak{m}_S^{r+1}$, we obtain

d) $$\mathfrak{m}_S^{d+r+1}\Psi \subseteq G^{a*}(\mathfrak{m}_y)\mathfrak{m}_S^{r+1}\Psi + t_1 G^a(\mathfrak{m}_S^{r+1}\theta(\pi_1^N)).$$

(5.3) From the fact that g and f have the same l-jet, it follows that $\partial G^a/\partial t \in \mathfrak{m}_S^l \Psi$. From (5.2 *d)*), it follows that there exist $\xi \in \mathfrak{m}_S^{r+1}\theta(\pi_1^N)$ and $u_{ij} \in \mathfrak{m}_S^r C_{S\times a}$ such that

a) $$\frac{\partial G^a}{\partial t} = \sum_{i,j=1}^{p} G^{a*}(y_j)u_{ij}\left(\frac{\partial}{\partial y_i}\circ \pi_1^P \circ G^a\right) + t_1 G^a(\xi).$$

Let

$$\pi_1^{N\times P} : (N\times P\times \mathbf{R}, S\times y\times a) \to (N\times P, S\times y)$$

denote the projection. Let $\eta \in \theta(\pi_1^{N\times P})$ be given by

$$\eta = \sum_{i,j=1}^{p} y_j u_{ij}\frac{\partial}{\partial y_i} - \xi.$$

We are going to apply lemma (4.3) with

$$(U, \Sigma) = (N, S), \qquad (V, \Pi) = (N\times P, S\times y),$$

and

$$F = (\pi_1^N, G^a) : (N\times \mathbf{R}, S\times a) \to (N\times P\times \mathbf{R}, S\times y\times a).$$

From *a)* and the definition of η it follows that

b) $$\frac{\partial F}{\partial t} = \omega_1 F(\eta) + t_1 F(\xi).$$

Let H be the integral of $-\xi$ and H′ the integral of η. From the definition of η it follows that the diagram below commutes

$$\begin{array}{ccccc} (N\times\mathbf{R}, S\times a) & \xrightarrow{\iota} & (N\times P\times\mathbf{R}, S\times y\times a) & \xrightarrow{\pi} & (N\times\mathbf{R}, S\times a) \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H'} & & \downarrow{\scriptstyle H} \\ (N\times\mathbf{R}, S\times a) & \xrightarrow{\iota} & (N\times P\times\mathbf{R}, S\times y\times a) & \xrightarrow{\pi} & (N\times\mathbf{R}, S\times a) \end{array}$$

Here ι denotes the germ of the inclusion $(n, t) \mapsto (n, y, t)$ and π the germ of the projection, $(n, p, t) \mapsto (n, t)$.

Let

$$\widetilde{H} : (N\times\mathbf{R}, S\times J) \to (N\times\mathbf{R}, S\times J)$$

and $$\widetilde{H}' : (N\times P\times\mathbf{R}, S\times y\times J) \to (N\times P\times\mathbf{R}, S\times y\times J)$$

be C^∞ map-germs which restrict to H and H′, respectively, where J is an open interval in **R** containing a. Then, for $b \in J$,

$$\widetilde{H}_b : (N, S) \to (N, S) \quad \text{and} \quad \widetilde{H}'_b : (N\times P, S\times y) \to (N\times P, S\times y)$$

are defined, as in (4.1). From the commutativity of the above diagram, it follows that (by taking J sufficiently small), we may suppose that $\widetilde{H}'_b \in \mathscr{K}$ for all $b \in J$ and that the diagram in (2.6) commutes, where $\widetilde{H}_b$ is substituted for h and $\widetilde{H}'_b$ is substituted for H.

Furthermore, from *b)* and lemma (4.3) it follows that $\pi_1^V \circ H'^{-1} \circ F \circ H = F_a \circ \pi_1^U$, which implies that if we take J sufficiently small then

$$\widetilde{H}_b'^{-1} \circ (\mathbf{1}, G_b) \circ \widetilde{H}_b = (\mathbf{1}, G_a), \qquad b \in J.$$

According to the definition of the action of $\mathscr{K}$ on $\mathscr{F}$, this means

$$G_b = \widetilde{H}'_b . G_a.$$

Since $\xi \in \mathfrak{m}_S^{r+1}\theta(\pi_1^N)$, it follows from the definition of η that $\eta \in \mathfrak{m}_{S\times y}^{r+1}\theta(\pi_1^{N\times P})$. It follows that (by taking J sufficiently small) we may suppose that $\widetilde{H}'_b$ has the same r-jet at $S \times y$ as the identity, for all $b \in J$. Thus, $\widetilde{H}'_b \in \mathscr{K}_r$. Hence G_b is in the same $\mathscr{K}_r$ orbit as G_a for all $b \in \mathbf{R}$ sufficiently close to a. Since $a \in \mathbf{R}$ is arbitrary, this shows that all G_a are in the same $\mathscr{K}_r$ orbit. In particular $f = G_0$ and $g = G_1$ are in the same $\mathscr{K}_r$ orbit, which is what was to be proved.

6. Proof of (3.7) in the case $\mathscr{S} = \mathscr{A}$.

(6.1) We begin by defining $l = l(d, r; \mathscr{A})$. The value of l that we will take depends on $n = \dim N$, $p = \dim P$, and $|S| =$ the number of elements in S, as well as on d and r; however, these are the only things it depends on.

For each non-negative integer k, and each $f \in \mathscr{F}$, we set

$$c_k(f) = \dim_{\mathbf{R}} \theta(f) / (\omega f(\mathfrak{m}_y^{r+1}A) + tf(\mathfrak{m}_S^{r+1}B) + \mathfrak{m}_S^k \theta(f)).$$

Lemma 1. — For each non-negative integer k, there is an integer $q(k) > k$ such that $c_{q(k)}(f) = c_k(f)$ implies

$$\omega f(\mathfrak{m}_y^{r+1}A) + tf(\mathfrak{m}_S^{r+1}B) \supset \mathfrak{m}_S^k \theta(f).$$

Remark. — For $k < r+1$, this lemma is vacuous, since for $k \leq r+1$,

$$c_k(f) = \dim_{\mathbf{R}} \theta(f) / \mathfrak{m}_S^k \theta(f),$$

which implies that for $k < r+1$, $c_{k+1}(f) > c_k(f)$.

Proof. — The hypothesis $c_q(f) = c_k(f)$ means

$$\omega f(\mathfrak{m}_y^{r+1}A) + tf(\mathfrak{m}_S^{r+1}B) + \mathfrak{m}_S^q \theta(f) \supseteq \mathfrak{m}_S^k \theta(f).$$

Thus the lemma follows from corollary (1.14) applied to the mixed homomorphism

$$(\omega f, tf, \mathfrak{m}_y^{r+1}A, \mathfrak{m}_S^{r+1}B, \theta(f)),$$

where $C_0 = \mathfrak{m}_S^k \theta(f)$.

Let d be an integer, which will be fixed throughout the rest of this section. We set

$$d' = d + \dim_{\mathbf{R}}(A/\mathfrak{m}_y^{r+1}A) + \dim_{\mathbf{R}}(B/\mathfrak{m}_S^{r+1}B).$$

Under the assumption that $d \geq d(f, \mathscr{A})$ (where $d(f, \mathscr{A})$ is defined as in (3.5)) we have

a) $$d' \geq \dim_{\mathbf{R}} \theta(f) / (\omega f(\mathfrak{m}_y^{r+1}\mathrm{A}) + tf(\mathfrak{m}_{\mathrm{S}}^{r+1}\mathrm{B}).$$

Let q be the function from the non-negative integers into themselves given by lemma 1. Define $\overline{q}(s)$ inductively for all non-negative integers s by

$$\overline{q}(0) = q(0), \quad \overline{q}(s+1) = q(\overline{q}(s)).$$

Set

b) $$k = \overline{q}(d'-1)$$

c) $$l = \min(q(k), k+\alpha+1)$$

where α is given by the following formulas

d) $$\begin{cases} b = p\binom{p+r}{p} + p|\mathrm{S}|n\binom{n+k-1}{n} \\ \alpha = p\binom{p+b}{b} \end{cases}$$

In this section, we will show that if $f \in \mathscr{F}$ and $d(f, \mathscr{A}) \leq d$, then f is l-determined relative to $\mathscr{A}_r$, thus proving (3.7) in the case $\mathscr{S} = \mathscr{A}$. For the proof the following lemma will be useful.

Lemma 2. — Let $f \in \mathscr{F}$ *be such that* $d(f, \mathscr{A}) \leq d$. *If* k *is given by* b), *then*

$$\omega f(\mathfrak{m}_y^{r+1}\mathrm{A}) + tf(\mathfrak{m}_{\mathrm{S}}^{r+1}\mathrm{B}) \supset \mathfrak{m}_{\mathrm{S}}^k \theta(f).$$

Proof. — Since $\overline{q}(0) > 0$, $c_{\overline{q}(0)} > 0$ (compare the remark following lemma 1). On the other hand, $c_{\overline{q}(d')} \leq d'$ by *a)* and the definition of c_k. Thus, for some s, $0 \leq s < d'$, $c_{\overline{q}(s)} = c_{\overline{q}(s+1)}$, since the sequence $c_{\overline{q}(0)}, c_{\overline{q}(1)}, \ldots$, is never decreasing. Since $\overline{q}(s+1) = q(\overline{q}(s))$, it then follows from lemma 1 that

$$\omega f(\mathfrak{m}_y^{r+1}\mathrm{A}) + tf(\mathfrak{m}_{\mathrm{S}}^{r+1}\mathrm{B}) \supset \mathfrak{m}_{\mathrm{S}}^{\overline{q}(s)} \theta(f).$$

Since $s \leq d'-1$, it follows that $\overline{q}(s) \leq k$ (where k is given by *b)*). Thus, lemma 2 follows.

(6.2) Throughout the remainder of this section, we let f and g be as in the statement of addendum (3.7). In other words, we suppose $d \geq d(f, \mathscr{A})$ and that g has the same l-jet at S as f, where l is given by (6.1 *c)*). We shall continue to use the notations and terminology introduced in §§ 3 and 4. In particular, we will assume that $y_1, \ldots, y_p$ and t are as in (4.4), and that G and G^a are as defined there. Moreover, we will continue to use the identifications that we introduced in (4.4).

Fix $a \in \mathbf{R}$ and let $\Psi^a = \theta(\pi_1^{\mathrm{P}} \circ \mathrm{G}^a)$. Set $\mathrm{F} = \mathrm{G}^a$. Since F is a level preserving map-germ from $(\mathrm{N} \times \mathbf{R}, \mathrm{S} \times a)$ to $(\mathrm{P} \times \mathbf{R}, y \times a)$, it follows that $t_1\mathrm{F} : \theta(\pi_1^{\mathrm{N}}) \to \Psi^a$ and $\omega_1\mathrm{F} : \theta(\pi_1^{\mathrm{P}}) \to \Psi^a$ are defined as in (4.3).

Let A_0 be the $\mathrm{C(P)}_y$-submodule of A given by

$$\mathrm{A}_0 = (\omega f)^{-1}(tf(\mathfrak{m}_{\mathrm{S}}^{r+1}\mathrm{B}) + \mathfrak{m}_{\mathrm{S}}^k \theta(f)) \cap \mathfrak{m}_y^{r+1}\mathrm{A}.$$

Let $\overline{A}_0$ be the $C(P\times\mathbf{R})_{y\times a}$-submodule of $\theta(\pi_1^P)$ generated by A_0. (Here we use the identification, introduced in (4.4), of A with a subset of $\theta(\pi_1^P)$.) Using (6.1), lemma 2 and the approximation lemma of (4.4), we show that

$$a) \qquad \omega_1 F(\overline{A}_0)+t_1F(\mathfrak{m}_S^{r+1}\theta(\pi_1^N))+\mathfrak{m}_{S\times a}^{l-k}\mathfrak{m}_S^k\Psi=t_1F(\mathfrak{m}_S^{r+1}\theta(\pi_1^N))+\mathfrak{m}_S^k\Psi.$$

First, we show that the l.h.s. is contained in the r.h.s. It is enough to show that $\omega_1F(\overline{A}_0)$ is contained in the r.h.s. However, since A_0 generates $\overline{A}_0$ as a $C(P\times\mathbf{R})_{y\times a}$-module and the r.h.s. is a $C(P\times\mathbf{R})_{y\times a}$-module, it is enough to show that $\omega_1F(A_0)$ is contained in the r.h.s. Consider $\eta\in A_0$. By definition of A_0, there exist $\xi\in\mathfrak{m}_S^{r+1}B$ and $\zeta\in\mathfrak{m}_S^k\theta(f)$ such that $\omega f(\eta)=tf(\xi)+\zeta$. From lemma (4.4) and the assumption that g has the same l-jet as f, it follows that $\omega_1F(\eta)-\omega f(\eta)$ and $t_1F(\xi)-tf(\xi)$ are both in $\mathfrak{m}_S^k\Psi$. Hence

$$\omega_1F(\eta)=(\omega_1F(\eta)-\omega f(\eta))+(tf(\xi)-t_1F(\xi))+t_1F(\xi)+\zeta$$

is in the r.h.s.

Second, we show that the r.h.s. is contained in the l.h.s. It is enough to show that $\mathfrak{m}_S^k\Psi$ is contained in the l.h.s. Let $\zeta\in\mathfrak{m}_S^k\Psi$. Then ζ can be written in the form

$$\zeta'+\sum_{i=0}^{l-k-1}t^i\zeta_i \quad \text{where} \quad \zeta'\in\mathfrak{m}_{S\times a}^{l-k}\mathfrak{m}_S^k\Psi$$

and $\zeta_i\in\mathfrak{m}_S^k\theta(f)$ for $0\leq i\leq l-k-1$. (Here we use the identification, introduced in (4.4), of $\theta(f)$ with a subset of Ψ). Since ζ' in contained in the l.h.s. and the l.h.s. is closed under multiplication by t, it is enough to show $\mathfrak{m}_S^k\theta(f)$ is contained in the left hand side. Let $\zeta\in\mathfrak{m}_S^k\theta(f)$. By (6.1), lemma 2, there exist $\xi\in\mathfrak{m}_S^{r+1}B$ and $\eta\in\mathfrak{m}_y^{r+1}A$ such that $\zeta=tf(\xi)+\omega f(\eta)$. By lemma (4.4),

$$\zeta-t_1F(\xi)-\omega_1F(\eta)=(tf(\xi)-t_1F(\xi))+(\omega f(\eta)-\omega_1F(\eta))\in\mathfrak{m}_S^l\Psi,$$

which shows that ζ is contained in the l.h.s. and thus completes the proof of *a)*.

From *a)* and theorem (1.13), it follows that

$$b) \qquad \omega_1F(\overline{A}_0)+t_1F(\mathfrak{m}_S^{r+1}\theta(\pi_1^N))=t_1F(\mathfrak{m}_S^{r+1}\theta(\pi_1^N))+\mathfrak{m}_S^k\Psi.$$

To show this, we apply theorem (1.13) to the mixed homomorphism

$$(*) \qquad (\omega_1F,\ t_1F,\ \overline{A}_0,\ \mathfrak{m}_S^{r+1}\theta(\pi_1^N),\ t_1F(\mathfrak{m}_S^{r+1}\theta(\pi_1^N))+\mathfrak{m}_S^k\Psi).$$

(This is a mixed homomorphism over $F^*: C(P\times\mathbf{R})_{y\times a}\to C(N\times\mathbf{R})_{S\times a}$).

Let α be given as in (6.1 *d)*). Then $\overline{A}_0$ is generated as a $C(P\times\mathbf{R})_{y\times a}$-module by α or fewer elements. To see this, it is enough to see that A_0 is generated as a $C(P)_y$-module by α or fewer elements, since A_0 generates $\overline{A}_0$ as a $C(P\times\mathbf{R})_{y\times a}$-module. But by definition of A_0,

$$\dim_{\mathbf{R}}\mathfrak{m}_y^{r+1}A/A_0\leq\dim_{\mathbf{R}}\theta(f)/\mathfrak{m}_S^k\theta(f)=p|S|\binom{n+k-1}{n}.$$

This implies $\dim_{\mathbf{R}}A/A_0\leq b$, where b is as in (6.1 *d)*), so by (1.7) A_0 is generated by α or fewer elements.

From (6.1 *c)*), it follows that $l-k\geq\alpha+1$; thus the hypotheses of theorem (1.13) for the mixed homomorphism (*) follow. Thus the conclusion of theorem (1.13) holds for (*), i.e. *b)* holds.

From the definition of A_0, it follows that $A_0\subseteq\mathfrak{m}_y^{r+1}A$. Hence $\overline{A}_0\subseteq\mathfrak{m}_y^{r+1}\theta(\pi_1^P)$. Hence *b)* implies

c) $$\mathfrak{m}_S^k\Psi\subseteq\omega_1F(\mathfrak{m}_y^{r+1}\theta(\pi_1^P))+t_1F(\mathfrak{m}_S^{r+1}\theta(\pi_1^N)).$$

(6.3) From the fact that g has the same l-jet at S as f it follows that $\partial F/\partial t\in\mathfrak{m}_S^l\Psi$. Thus, according to (6.2 *c)*) and the fact that $l\geq k$, there exist $\xi\in\mathfrak{m}_S^{r+1}\theta(\pi_1^N)$ and $\eta\in\mathfrak{m}_y^{r+1}\theta(\pi_1^N)$ such that

$$\partial F/\partial t=\omega_1F(\eta)+t_1F(\xi).$$

Thus lemma (4.3) applies. Let H be the integral of $-\xi$ and H′ the integral of η. Let J be an open interval in **R** containing a, and let

$$\widetilde{H}:(N\times\mathbf{R},S\times J)\to(N\times\mathbf{R},S\times J)$$
$$\widetilde{H}':(P\times\mathbf{R},y\times J)\to(P\times\mathbf{R},y\times J)$$

be C∞ map-germs which restrict to H and H′ respectively. By lemma (4.3), $\pi_1^P\circ H'^{-1}\circ F\circ H=F_a\circ\pi_1^N$. This implies that if we take J sufficiently small, then

$$\widetilde{H}_b'^{-1}\circ G_b\circ\widetilde{H}_b=G_a,\quad \text{for all } b\in J.$$

From $\xi\in\mathfrak{m}_S^{r+1}\theta(\pi_1^N)$ it follows that for all b in J sufficiently close to a, $\widetilde{H}_b$ has the same r-jet at S as the identity. Likewise $\widetilde{H}_b'$ has the same r-jet at y as the identity. In other words $(\widetilde{H}_b,\widetilde{H}_b')\in\mathscr{A}_r$. Hence the above formula shows that G_b is in the same $\mathscr{A}_r$-orbit as G_a for all b sufficiently near a.

Since $a\in\mathbf{R}$ was chosen arbitrarily, this implies that all G_b ($b\in\mathbf{R}$) are in the same $\mathscr{A}_r$-orbit. Since $G_0=f$ and $G_1=g$, this shows that f and g are in the same $\mathscr{A}_r$-orbit, which is what was to be proved.

7. The tangent space of an orbit of any one of the groups $\mathscr{R}_k^l$, $\mathscr{L}_k^l$, $\mathscr{A}_k^l$, $\mathscr{K}_k^l$, $\mathscr{C}_k^l$.

(7.1) In (3.7), we introduced the subgroup $\mathscr{S}_l$ of $\mathscr{S}$, where $\mathscr{S}$ was any one of the groups $\mathscr{R}$, $\mathscr{L}$, $\mathscr{A}$, $\mathscr{K}$, or $\mathscr{C}$, and l was any positive integer. Clearly $\mathscr{S}_l$ is a normal subgroup of $\mathscr{S}$. We set $\mathscr{S}_k^l=\mathscr{S}_k/\mathscr{S}_l$ for $k\leq l$. In effect, $\mathscr{S}_k^l$ is the set of all l-jets of members of $\mathscr{S}$, whose k-jets are equal to the identity. We set $\mathscr{S}^l=\mathscr{S}_0^l$. We have an obvious identification: $\mathscr{S}_k^l\subseteq\mathscr{S}^l$. If $g\in\mathscr{S}$, we let $g^{(k)}\in\mathscr{S}^k$ denote its image under the canonical projection $\mathscr{S}\to\mathscr{S}^k$. We let J^k denote the set of k-jets at S of members of $\mathscr{F}$. We will continue with the notation introduced in (3.1): if $f\in\mathscr{F}$ then $f^{(k)}\in J^k$ denotes the k-jet at S of f.

It is easily verified that the action of $\mathscr{S}$ on $\mathscr{F}$ induces an action of $\mathscr{S}^l$ on J^l uniquely defined by $g^{(l)}f^{(l)}=(gf)^{(l)}$ for each $g\in\mathscr{S}$ and $f\in\mathscr{F}$.

(7.2) Suppose given a local system of coordinates for N about each of the points of S and a local system of coordinates for P about y. In terms of these coordinate systems, each $z \in J^l$ has a well defined Taylor series expansion of order l. The mapping which assigns to each $z \in J^l$ the $N = ps\binom{n+l}{n}$ coefficients of the Taylor series expansion of z is a bijection of J^l onto $\mathbf{R}^N$. Thus we may consider J^l as a C^∞ manifold with the set of these coefficients as a global system of coordinates. If we change the given systems of coordinates (about the points of S and about y), we get a new system of coordinates for J^l. Each of the components of the new system of coordinates for J^l is a rational function in terms of the old system of coordinates; it follows that the C^∞ manifold structure on J^l is independent of the initial choice of coordinates in N and P.

In the same way, we may provide each of the groups $\mathscr{S}^l$ with the structure of a C^∞ manifold. With this structure $\mathscr{S}^l$ is a Lie group and the action of $\mathscr{S}^l$ on J^l is of class C^∞. In addition $\mathscr{S}^l_k$ is a sub-Lie group of $\mathscr{S}^l$. It follows that the orbits of $\mathscr{S}^l_k$ in J^l are submanifolds.

(7.3) In this section, we will give a formula for the tangent space at z (where $z \in J^l$) to the orbit of $\mathscr{S}^l_k$ containing z, where $\mathscr{S} = \mathscr{R}, \mathscr{L}, \mathscr{A}, \mathscr{C}$, or $\mathscr{K}$. This formula is a step in the proof of "necessity" in theorem (3.5) and will be used again in later papers in this series.

Let $f \in \mathscr{F}$ and set $z = f^{(l)}$. We define the *projection* π^l of $\mathfrak{m}_S\theta(f)$ on $T_z J^l$ (where T_z means "the tangent space at z of") as follows. Let $\zeta \in \mathfrak{m}_S\theta(f)$. Let $F : (N \times \mathbf{R}, S \times I_0) \dashrightarrow (P \times \mathbf{R}, y \times I_0)$ be a level preserving map-germ, where I_0 is an open interval in $\mathbf{R}$ containing 0. Suppose that $\partial F/\partial t|_{t=0} = \zeta$. (Here the map-germ $\partial F/\partial t : (N \times \mathbf{R}, S \times I_0) \dashrightarrow TP$ is defined as in (4.1), and $\partial F/\partial t|_{t=0}$ denotes the restriction of $\partial F/\partial t$ to $(N \times 0, S \times 0)$, considered as a map-germ $(N, S) \dashrightarrow TP$.) For each $t \in I_0$, set $z_t = F^{(l)}_t \in J^l$. Then $t \mapsto z_t$ is a C^∞ mapping of I_0 into J^l. We define $\pi^l(\zeta)$ to be dz_t/dt.

One verifies easily that $\pi^l(\zeta)$ is independent of the choice of F, that π^l is $\mathbf{R}$-linear and onto, and that

$$\text{kernel } \pi^l = \mathfrak{m}_S^{l+1}\theta(f).$$

Proposition **(7.4)**. — *Let* $f \in \mathscr{F}$, $z = f^{(l)}$, *and* $U = \mathscr{S}^l_k z$. (In other words, U is the orbit of $\mathscr{S}^l_k$ through z). *Then* $T_z U$ *is equal to*

a) $\pi^l(tf(\mathfrak{m}_S^{k+1}B) + \mathfrak{m}_S^k f^*(\mathfrak{m}_y)\theta(f))$, *in the case* $\mathscr{S} = \mathscr{K}$,

b) $\pi^l(tf(\mathfrak{m}_S^{k+1}B) + \omega f(\mathfrak{m}_y^{k+1}A))$, *in the case* $\mathscr{S} = \mathscr{A}$,

c) $\pi^l(tf(\mathfrak{m}_S^{k+1}B))$, *in the case* $\mathscr{S} = \mathscr{R}$,

d) $\pi^l(\omega f(\mathfrak{m}_y^{k+1}A))$, *in the case* $\mathscr{S} = \mathscr{L}$,

e) $\pi^l(\mathfrak{m}_S^k f^*(\mathfrak{m}_y)\theta(f))$, *in the case* $\mathscr{S} = \mathscr{C}$.

Proof. — Given any Lie group G acting on a manifold M, and any $x \in M$, we have $T_x(Gx) = T\alpha_x(T_1 G)$, where $\alpha_x : G \to M$ denotes the mapping $g \mapsto gx$, and 1 is the identity element of G. We will apply this fact in proving each of *a)-e)*.

First, consider *c)*. There exists a "canonical" projection $\pi^l_{\mathscr{R}} : \mathfrak{m}_S B \to T_1 \mathscr{R}^l$ (given by the construction which was used to define the projection $\pi^l : \mathfrak{m}_S\theta(f) \to T_z J^l$), such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{m}_S B & \xrightarrow{tf} & \mathfrak{m}_S\theta(f) \\ \Big\downarrow{\scriptstyle \pi^l_{\mathscr{R}}} & & \Big\downarrow{\scriptstyle \pi^l} \\ T_1\mathscr{R}^l & \xrightarrow{T\alpha_z} & T_z J^l \end{array}$$

Since π^l is onto, and

$$\mathfrak{m}_S^{k+1}B = (\pi^l_{\mathscr{R}})^{-1}(T_1\mathscr{R}^l_k),$$

c) follows from the remark in the previous paragraph.

The proof of *d)* is similar. Now *b)* follows immediately from *c)* and *d)*.

Next, we prove *e)*. Let π_2 denote the germ at $S\times y$ of the projection $N\times P \to P$. Then $\theta(\pi_2)$ denotes (according to the notation introduced in (3.3)) the $C(N\times P)_{S\times y}$-module of C^∞ map-germs $\eta : (N\times P, S\times y) \to TP$ such that $\pi_P\circ\eta = \pi_2$, where $\pi_P : TP \to P$ denotes the projection. For any $u \in C(P)_y$ identify u with $\pi_2^*(u) \in C(N\times P)_{S\times y}$. We construct a projection $\pi^l : \mathfrak{m}_y\theta(\pi_2) \to T_1\mathscr{C}^l$, as follows. One verifies easily that for any $\eta \in \mathfrak{m}_y\theta(\pi_2)$ there exists a family $\{H_t : -\varepsilon < t < \varepsilon\}$ of members of $\mathscr{C}$ with the following properties. First, for suitable representatives $\widetilde{H}_t$ of H_t, $\widetilde{H}_t(x, x')$ is C^∞ in the variables t, x and x' simultaneously, where t varies in $(-\varepsilon, \varepsilon)$, x in a neighborhood of S in N, and x' in a neighborhood of y in P. Second, $H_0 = \mathbf{1}$. Third, $\partial H_t/\partial t|_{t=0} = \eta$. We set $z_t = H_t^{(l)} \in \mathscr{C}^l$ and $\pi^l_{\mathscr{C}}(\eta) = \partial z_t/\partial t|_{t=0}$. It is easily verified that $\pi^l_{\mathscr{C}}(\eta)$ is independent of the choice of $\{H_t\}$, and that $\pi^l_{\mathscr{C}}$ is **R**-linear and onto.

Define $\kappa : \theta(\pi_2) \to \theta(f)$ by $\kappa(\eta) = \eta\circ(\mathbf{1}, f)$. Then the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{m}_y\theta(\pi_2) & \xrightarrow{\kappa} & \mathfrak{m}_S\theta(f) \\ \Big\downarrow{\scriptstyle \pi^l_{\mathscr{C}}} & & \Big\downarrow{\scriptstyle \pi^l} \\ T_1\mathscr{C}^l & \xrightarrow{T\alpha_z} & T_z J^l \end{array}$$

To prove that this diagram commutes, we first observe that if $\eta \in \theta(\pi_2)$ and $\{H_t\}$ is as in the previous paragraph, then $\kappa(\eta) = \partial(H_t f)/\partial t|_{t=0}$. Then it follows that both $T\alpha_z\circ\pi^l_{\mathscr{C}}(\eta)$ and $\pi^l\circ\kappa(\eta)$ are equal to $\partial(H_t f)^{(l)}/\partial t|_{t=0}$, for $\eta \in \mathfrak{m}_y\theta(\pi_2)$.

From the commutativity of the above diagram, the fact that π^l is onto, and the formula

$$(\pi^l_{\mathscr{C}})^{-1}(T_1\mathscr{C}^l_k) = \mathfrak{m}^k_{S\times y}\mathfrak{m}_y\theta(\pi_2),$$

it follows that to prove *e)*, it is enough to show:

$$\kappa(\mathfrak{m}^k_{S\times y}\mathfrak{m}_y\theta(\pi_2)) = f^*(\mathfrak{m}_y)\mathfrak{m}^k_S\theta(f).$$

But this follows immediately from the easily verified facts that $\kappa(\theta(\pi_2))=\theta(f)$, that κ is a homomorphism over $(1,f)^*$, that

$$(1,f)^*(\mathfrak{m}_y C(N\times P)_{S\times y})=f^*(\mathfrak{m}_y)C(N)_S,$$

and that $(1,f)^*(\mathfrak{m}^k_{S\times y})=\mathfrak{m}^k_S$. This completes the proof of *e)*.

Finally *a)* follows from *c)*, *e)*, the fact that $\mathscr{K}$ is the semi-direct product of $\mathscr{R}$ and $\mathscr{C}$ and the fact that for $h\in\mathscr{R}$, $H\in\mathscr{C}$ and $f\in\mathscr{F}$, $(hH)f=h(Hf)$. For, it follows that $\mathscr{K}^l_k$ is the semi-direct product of $\mathscr{R}^l_k$ and $\mathscr{C}^l_k$ and that its action on J^l is of the form $(hH)z=h(Hz)$ for $h\in\mathscr{R}^l_k$, $H\in\mathscr{C}^l_k$ and $z\in J^l$. Thus $T_1\mathscr{K}^l_k=T_1\mathscr{R}^l_k\oplus T_1\mathscr{C}^l_k$ and $T_1\alpha_z(\xi\oplus\eta)=T_1\alpha_z(\xi)+T_1\alpha_z(\eta)$ for all $\xi\in T_1\mathscr{R}^l_k$ and all $\eta\in T_1\mathscr{L}^l_k$.

8. Proof of necessity in (3.5).

(8.1) We will give the proof only in the case $\mathscr{S}=\mathscr{A}$, since the same method works in the other cases.

Suppose $f\in\mathscr{F}$ is r det. rel. $\mathscr{A}$. Consider $l>r$, let $z=f^{(l)}\in J^l$, and let $E=\pi^{-1}(\pi(z))\subset J^l$, where $\pi: J^l\to J^r$ denotes the projection. The assumption that z is r determined relative to $\mathscr{A}$ implies that $E\subset U$, where U denotes the orbit of z in J^l under the action of $\mathscr{A}^l$. It is easily seen that

$$T_z E=\pi^l(\mathfrak{m}^{r+1}_S\theta(f)),$$

where $\pi^l: \mathfrak{m}_S\theta(f)\to T_zJ^l$ is the " projection " defined in (7.3). It follows from this formula, (7.4 *b)*), the fact that $T_zE\subset T_zU$ and the fact that kernel $(\pi^l)=\mathfrak{m}^{l+1}_S\theta(f)$, that

$$\mathfrak{m}^{r+1}_S\theta(f)\subset tf(\mathfrak{m}_S B)+\omega f(\mathfrak{m}_y A)+\mathfrak{m}^{l+1}_S\theta(f).$$

Now (1.14) shows that if l is taken sufficiently large, then the above inclusion implies

$$\mathfrak{m}^{r+1}_S\theta(f)\subset tf(\mathfrak{m}_S B)+\omega f(\mathfrak{m}_y A).$$

Then $d(f,\mathscr{A})<\infty$ follows immediately; in fact we obtain $d(f,\mathscr{A})\leq\dim\theta(f)/\mathfrak{m}^{r+1}_S\theta(f)$.

(8.2) The argument given in the previous section proves somewhat more than we have stated; in fact we have:

Theorem. — For each integer r there exists an integer $l>r$ (depending also on dim N, dim P, *and* $|S|$, *of course) such that if $f\in\mathscr{F}$ has the following property*:

— *for any $g\in\mathscr{F}$ such that $j^r(g)=j^r(f)$ there exists $(h,h')\in\mathscr{A}$ such that*

$$j^l(h'gh^{-1})=j^l(f);$$

then f is r determined relative to $\mathscr{A}$.

Proof. — The property that we assume f to have is equivalent to the statement that $E\subset U$, in the notation of (8.1). Our reasoning in (8.1) shows that $E\subset U$ implies $d(f,\mathscr{A})\leq\dim\theta(f)/\mathfrak{m}^{r+1}_S\theta(f)$ (provided l is sufficiently large). Hence f is f.d. rel. $\mathscr{A}$, say l' det. From (3.7) it follows that we can choose l' independently of f. If $l'\leq l$, we are done; if not we can replace l by l'.

9. Analytic case.

(9.1) Instead of considering the category of C^∞ mappings, we could have considered the category of real analytic manifolds and real analytic mappings, or the category of complex analytic manifolds and complex analytic (i.e. holomorphic) mappings. If we replace " C^∞ " throughout by " real analytic " or " complex analytic ", the above theory goes through word for word with the exception of the proof of lemma (1.8). However, in either of these cases lemma (1.8) is the classical Weierstrass preparation theorem in the form of Rückert. In the complex case it follows from the Lagrange interpolation formula in the following form. If γ is a simple closed curve in $\mathbf{C}$, D is a closed disk which γ bounds, g is a holomorphic function defined in a neighborhood of D, and $z=(z_1, \ldots, z_p)\in\mathbf{C}^p$ is such that $\Gamma_p(t, z)$ never vanishes for $t\in\gamma$ (where $\Gamma_i(t, z)=t^i+\sum_{j=1}^{i} z_j t^{i-j}$) then

$$g(t)=\Gamma_p(t, z)q(t, z)+\sum_{i=1}^{p} h_i(z)t^{p-i}$$

for t in the interior of D, where

$$q(t, z)=\int_\gamma \frac{g(s)ds}{\Gamma_p(s, z)(s-t)}$$

$$h_i(z)=\int_\gamma \frac{g(s)\Gamma_{i-1}(s, z)}{\Gamma_p(s, z)}ds$$

(Compare I, § 1.) In the real case, lemma (1.8) is a consequence of lemma (1.8) in the complex case.

(9.2) From the fact that the above theory works in all three cases (C^∞, real analytic, complex analytic) we obtain that for a real analytic map-germ, the notion of finite determinancy (for any one of the groups $\mathscr{S}$) is independent of whether we consider g as a real analytic, C^∞, or complex analytic map-germ. To be explicit, we consider a real analytic map-germ $f:(\mathbf{R}^n, S)\to(\mathbf{R}^p, y)$ and we will work with the equivalence relation defined by the group $\mathscr{A}$. Let $\tilde{f}:(\mathbf{C}^n, S)\to(\mathbf{C}^p, y)$ be the unique complex analytic map-germ extending f.

Theorem. — The following conditions are equivalent:

a) *There exists an integer k such that if $g:(\mathbf{R}^n, S)\to(\mathbf{R}^p, y)$ is a real analytic map-germ with the same k-jet as f then there exist invertible real analytic map-germs $h:(\mathbf{R}^n, S)\to(\mathbf{R}^n, S)$ and $h':(\mathbf{R}^p, y)\to(\mathbf{R}^p, y)$ such that $g=h'fh$.*

b) *The same, except " C^∞ " replaces " real analytic " throughout.*

c) *There exists an integer k such that if $g:(\mathbf{C}^n, S)\to(\mathbf{C}^p, y)$ is a complex analytic map-germ with the same k-jet as $\tilde{f}$ then there exist invertible complex analytic map-germs $h:(\mathbf{C}^n, S)\to(\mathbf{C}^n, S)$ and $h':(\mathbf{C}^p, y)\to(\mathbf{C}^p, y)$ such that $g=h'\tilde{f}h$.*

Proof. — It follows from (3.5) and (3.6) that f is f.d. rel. $\mathscr{A}$ (i.e. *b)* is satisfied) if and only if there exists an integer k such that

$$(*) \qquad tf(\mathrm{B}) + \omega f(\mathrm{A}) + \mathfrak{m}_{\mathrm{S}}^{l(k)}\theta(f) \supseteq \mathfrak{m}_{\mathrm{S}}^{k}\theta(f)$$

(compare (3.6 (v)), where $l(k)$ is given by (3.6 (iv)). Similarly *a)* is satisfied if and only if there exists an integer k such that $(*)$ is satisfied where $\theta(f)$ is replaced by the module of *real analytic* vector fields along f, A is replaced by ...

But since $(*)$ can be checked by looking at terms of finite order (precisely: order $< l(k)$) it makes no differences whether we look at real analytic or C^∞ things. Thus *a)* $\Leftrightarrow$ *b)*. The proof that *b)* $\Leftrightarrow$ *c)* is similar.

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Manuscrit reçu le 2 septembre 1968.

1969. — Imprimerie des Presses Universitaires de France. — Vendôme (France)
ÉDIT. N° 30 493 IMPRIMÉ EN FRANCE IMP. N° 21 116