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INVARIANT EIGENDISTRIBUTIONS ON A SEMISIMPLE LIE ALGEBRA

by HARISH-CHANDRA

§ 1. INTRODUCTION

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbf{R}$ and T an invariant distribution on $\mathfrak{g}$ which is an eigendistribution of all invariant differential operators on $\mathfrak{g}$ with constant coefficients. Then the first result of this paper (Theorem 1) asserts that T is a locally summable function F which is analytic on the regular set $\mathfrak{g}'$ of $\mathfrak{g}$ (cf. Lemma 1 of [3(*g*)]). The second result (Theorem 5) can be stated as follows. Let D be an invariant analytic differential operator on $\mathfrak{g}$ such that $Df=0$ for every invariant C^∞ function f on $\mathfrak{g}$. Then $DS=0$ for any invariant distribution S on $\mathfrak{g}$ (cf. [3(*g*), Lemma 3]). This will be needed in the next paper of this series, in order to lift the first result, from $\mathfrak{g}$ to the corresponding group G (see [3(*g*), Theorem 1]), by means of the exponential mapping.

Proof of Theorem 1 proceeds by induction on dim $\mathfrak{g}$. In § 2 we show that there exists an analytic function F on $\mathfrak{g}'$ such that $T=F$ on $\mathfrak{g}'$. Moreover we verify that F is locally summable on $\mathfrak{g}$ and therefore it defines a distribution T_F on $\mathfrak{g}$. Thus it remains to prove that $\theta=T-T_F$ is actually zero. The results of § 3 enable us to reduce this to the verification of the fact that no semisimple element H of $\mathfrak{g}$ lies in Supp θ. If $H\neq 0$, this follows easily from [3 (*i*), Theorem 2] and the induction hypothesis. Hence we conclude (see Corollary 1 of Lemma 8) that Supp $\theta\subset\mathcal{N}$ where $\mathcal{N}$ is the set of all nilpotent elements of $\mathfrak{g}$. Let ω be the Killing form of $\mathfrak{g}$. Then $\partial(\omega)T=cT$ $(c\in\mathbf{C})$. Since $T=\theta+T_F$, we get

$$(\partial(\omega)-c)\theta=J$$

where $J=-(\partial(\omega)-c)T_F$. By making use of [3 (*j*), Theorem 4], one proves that $J=0$ and therefore it follows from [3 (*h*), Theorem 5] that $\theta=0$.

In § 8 we study the function F in greater detail. Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\pi^{\mathfrak{a}}$ the product of all the positive roots of $(\mathfrak{g}, \mathfrak{a})$. Define $g_{\mathfrak{a}}(H)=\pi^{\mathfrak{a}}(H)F(H)$ for $H\in\mathfrak{a}'=\mathfrak{a}\cap\mathfrak{g}'$. Then we show that $\partial(\pi^{\mathfrak{a}})g_{\mathfrak{a}}$ can be extended to a continuous function $h_{\mathfrak{a}}$ on $\mathfrak{a}$ and if $\mathfrak{b}$ is another Cartan subalgebra of $\mathfrak{g}$, then $h_{\mathfrak{a}}=h_{\mathfrak{b}}$ on $\mathfrak{a}\cap\mathfrak{b}$ (Theorem 3). These results will be used in subsequent papers for a detailed study of the irreducible characters of a semisimple Lie group. In § 10 we apply Theorem 3 to give a new and simpler proof of the main result of [3 (*e*)].

The rest of this paper is devoted to the proof of the second result mentioned at the beginning. It depends, in an essential way, on Theorem 1 and the theory of Fourier transforms for distributions. However, since the given distribution S is not assumed to be tempered, one has to construct a method of reducing the problem to the tempered case. This is done by means of Lemma 29. The last seven sections (§§ 15-21) are devoted to the proof of this lemma.

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§ 2. BEHAVIOUR OF T ON THE REGULAR SET

We use the terminology of [3 (*h*)] and [3 (*i*)]. Let $\mathfrak{g}$ be a reductive Lie algebra over **R** and $\mathfrak{g}'$ the set of all regular elements of $\mathfrak{g}$. Let $I(\mathfrak{g}_c)$ denote the subalgebra of all invariants in $S(\mathfrak{g}_c)$ (see [3 (*i*), § 9]). Fix a Euclidean measure dX on $\mathfrak{g}$.

Lemma 1. — Let T *be a distribution on an open subset* Ω *of* $\mathfrak{g}$. *Assume that:*

1) T *is locally invariant;*

2) *There exists an ideal* $\mathfrak{U}$ *in* $I(\mathfrak{g}_c)$ *such that* $\dim(I(\mathfrak{g}_c)/\mathfrak{U})<\infty$ *and* $\partial(u)T=0$ *for* $u\in\mathfrak{U}$. *Then there exists an analytic function* F *on* $\Omega'=\Omega\cap\mathfrak{g}'$ *such that*

$$T(f)=\int fF\,dX$$

for all $f\in C_c^\infty(\Omega')$.

Fix a point $H_0\in\Omega'$. It is obviously enough to show that T coincides with an analytic function around H_0. Fix a connected Lie group G with Lie algebra $\mathfrak{g}$ and let $\mathfrak{h}$ and A be the centralizers of H_0 in $\mathfrak{g}$ and G respectively. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and A is the corresponding Cartan subgroup of G [3(*j*), Lemma 8]. Let $x\to x^*$ denote the natural projection of G on $G^*=G/A$. As usual we define $x^*H=xH$ ($x\in G, H\in\mathfrak{h}$). Then if $n=\dim\mathfrak{g}$ and $\mathfrak{h}'=\mathfrak{h}\cap\mathfrak{g}'$, the mapping $\varphi:(x^*,H)\to x^*H$ has rank n everywhere on $G^*\times\mathfrak{h}'$ [3(*i*), Lemma 15]. Therefore we can select open connected neighborhoods G_0 and $\mathfrak{h}_0$ of 1 and H_0 in G and $\mathfrak{h}'$ respectively such that $\Omega_0=G_0\mathfrak{h}_0\subset\Omega$ and φ is univalent on $G_0^*\times\mathfrak{h}_0$. Then Ω_0 is open in $\mathfrak{g}$ and φ defines an analytic diffeomorphism φ_0 of $G_0^*\times\mathfrak{h}_0$ onto Ω_0.

Fix a Euclidean measure dH on $\mathfrak{h}$ and let σ_T denote the distribution on $\mathfrak{h}_0$ which corresponds to T under Lemma 17 of [3(*i*)]. As usual let π denote the product of all the positive roots of $(\mathfrak{g},\mathfrak{h})$. Then if $\sigma=\pi\sigma_T$, we conclude from Theorem 2 of [3(*i*)] that $\partial(u_\mathfrak{h})\sigma=0$ for $u\in\mathfrak{U}$. Let $\mathfrak{U}_\mathfrak{h}$ denote the image of $\mathfrak{U}$ under the homomorphism $p\to p_\mathfrak{h}$ of $I(\mathfrak{g}_c)$ into $S(\mathfrak{h}_c)$. Put $\mathfrak{V}=S(\mathfrak{h}_c)\mathfrak{U}_\mathfrak{h}$. Since $\dim(I(\mathfrak{g}_c)/\mathfrak{U})<\infty$, it follows from Lemma 19 of [3(*i*)] that $\dim(S(\mathfrak{h}_c)/\mathfrak{V})<\infty$. Moreover it is obvious that $\partial(v)\sigma=0$ for $v\in\mathfrak{V}$. Therefore from the corollary of [3(*b*), Lemma 27], we get the following result.

Lemma 2. — We can choose linear functions λ_i *and polynomial functions* p_i *on* $\mathfrak{h}_c$ $(1\leq i\leq r)$ *such that*

$$\sigma(\beta)=\int\beta g\,dH \qquad (\beta\in C_c^\infty(\mathfrak{h}_0))$$

where

$$g(H)=\sum_{1\leq i\leq r} p_i(H)e^{\lambda_i(H)} \qquad (H\in\mathfrak{h}_c).$$

This shows that

$$T(f_\alpha)=\int \beta_\alpha \pi^{-1} g\,dH \qquad (\alpha\in C_c^\infty(G_0\times\mathfrak{h}_0))$$

in the notation of [3(*i*), Lemma 17].

Since φ_0 is an analytic diffeomorphism, we can now define an analytic function F on Ω_0 as follows:

$$F(x^*H)=g(H)\pi(H)^{-1} \qquad (x^*\in G_0^*,\ H\in\mathfrak{h}_0).$$

Then if $\alpha\in C_c^\infty(G_0\times\mathfrak{h}_0)$, we have

$$\int f_\alpha F\,dX=\int\alpha(x:H)F(xH)\,dx\,dH=\int\beta_\alpha\pi^{-1}g\,dH=T(f_\alpha).$$

Since the mapping $\alpha\to f_\alpha$ of $C_c^\infty(G_0\times\mathfrak{h}_0)$ into $C_c^\infty(\Omega_0)$ is surjective [3(*h*), Theorem 1], this implies that $T=F$ on Ω_0 and so Lemma 1 is proved.

Lemma 3. — The function F *of Lemma 1 is locally summable on* Ω.

Let $l=\operatorname{rank}\mathfrak{g}$ and t an indeterminate. We denote by $\eta(X)$ $(X\in\mathfrak{g}_c)$ the coefficient of t^l in $\det(t-\operatorname{ad}X)$. Then we know (see [3(*j*), Corollary 2 of Lemma 30]) that $|\eta|^{-1/2}$ is locally summable on $\mathfrak{g}$. Since the singular set of $\mathfrak{g}$ is of measure zero, it would be enough to show that there exists a neighborhood V (in Ω) of any given point $X_0\in\Omega$, such that $|\eta|^{1/2}|F|$ is bounded on $V\cap\Omega'$.

Fix X_0 in Ω and a positive-definite quadratic form Q on $\mathfrak{g}$. For $\varepsilon>0$, let Ω_ε be the set of all $X\in\mathfrak{g}$ such that $Q(X-X_0)<\varepsilon^2$. Then $\Omega_\varepsilon\subset\Omega$ if ε is sufficiently small. Put

$$p(X)=(Q(X-X_0)-\varepsilon^2)\eta(X) \qquad (X\in\mathfrak{g}).$$

Then p is a polynomial function on $\mathfrak{g}$. Let $\mathfrak{g}''$ be the set of all points $X\in\mathfrak{g}$ where $p(X)\neq 0$. By a theorem of Whitney [4, Theorem 4, p. 547] $\mathfrak{g}''$ has only a finite number of connected components. It is obvious that any connected component of $\Omega'_\varepsilon=\Omega_\varepsilon\cap\mathfrak{g}'$ is also a connected component of $\mathfrak{g}''$. Hence Ω'_ε has only a finite number of connected components (1). So it would be enough to show that $|\eta|^{1/2}|F|$ remains bounded on a connected component Ω^0 of Ω'_ε.

We now fix an element $H_0\in\Omega^0$ and use the notation of the proof of Lemma 1. In particular φ is the mapping $(x^*,H)\to x^*H$ of $G^*\times\mathfrak{h}'$ into $\mathfrak{g}'$. Let U denote the connected component of $(1^*,H_0)$ in $\varphi^{-1}(\Omega^0)$. We claim that $\varphi(U)=\Omega^0$. Since φ is everywhere regular, $\varphi(U)$ is open in Ω^0. Therefore since Ω^0 is connected, it would be enough to show that $\varphi(U)$ is closed in Ω^0. So let (x_k^*,H_k) $(k\geq 1)$ be a sequence in U such that $X_k=x_k^*H_k$ converges to some point $X\in\Omega^0$. Then $\eta(H_k)=\eta(x_k^*H_k)\to\eta(X)\neq 0$.

(1) This proof was pointed out to me by A. Borel.

Moreover $X_k \in \Omega^0 \subset \Omega_\varepsilon$. Since Ω_ε is a bounded set in $\mathfrak{g}$, we can conclude from Lemma 23 of [3(j)] that H_k remains bounded. Hence by selecting a subsequence, we can arrange that H_k converges to some $H' \in \mathfrak{h}$. But then $\eta(H') = \eta(X) \neq 0$ and therefore $H' \in \mathfrak{h}'$. Hence [3(j), Lemma 8] A is the centralizer of H' in G and therefore [3(j), Lemma 7] x_k^* remains within a compact subset of G^*. So again by selecting a subsequence we can assume that $x_k^* \to x^*$ for some $x^* \in G^*$. Then $(x_k^*, H_k) \to (x^*, H')$ in $G^* \times \mathfrak{h}'$. Since $X_k \to X$, it follows that $x^*H' = X \in \Omega^0$ and therefore $(x^*, H') \in \varphi^{-1}(\Omega^0)$. But U, being a connected component of $\varphi^{-1}(\Omega^0)$, is closed in $\varphi^{-1}(\Omega^0)$. Hence $(x^*, H') \in U$ and $X = x^*H' \in \varphi(U)$. This proves that $\varphi(U)$ is closed in Ω^0 and therefore $\varphi(U) = \Omega^0$.

Now choose G_0, $\mathfrak{h}_0$ as in the proof of Lemma 1. We may assume that $G_0^* \times \mathfrak{h}_0 \subset U$. Moreover we recall (see Lemma 2) that g is defined and analytic on $\mathfrak{h}$. Consider the function $v : (x^*, H) \to F(x^*H) - \pi(H)^{-1} g(H)$ on U. It is obviously analytic and it vanishes identically on $G_0^* \times \mathfrak{h}_0$. Therefore, since U is connected, $v = 0$. This shows that

$$|\eta(x^*H)|^{1/2} |F(x^*H)| = |g(H)|$$

for $(x^*, H) \in U$. However $\varphi(U) = \Omega^0$ is contained in the bounded set Ω_ε. Therefore if V is the projection of U on $\mathfrak{h}$, it follows from [3(j), Lemma 23] that V is bounded. Hence g is bounded on V and therefore $|\eta|^{1/2}|F|$ is bounded on $\varphi(U) = \Omega^0$. This proves Lemma 3.

Corollary. — *Let* $p \in I(\mathfrak{g}_c)$. *Then* $\partial(p)F$ *is also locally summable on* Ω.

Since F is analytic and $T = F$ on Ω', it is clear that $\partial(p)T = \partial(p)F$ on Ω'. However the distribution $\partial(p)T$ obviously also satisfies all the conditions of Lemma 1. Therefore our assertion follows by applying Lemma 3 to $(\partial(p)T, \partial(p)F)$ in place of (T, F).

Φ being a locally summable function on Ω, define the distribution T_Φ on Ω by

$$T_\Phi(f) = \int f \Phi \, dX \qquad (f \in C_c^\infty(\Omega)).$$

We intend to show (under some mild extra conditions) that $T = T_F$.

Let Ω_a be the set of all points $X \in \Omega$ such that T coincides around X with an analytic function. Clearly Ω_a is open and there exists an analytic function F_a on Ω_a such that $T = F_a$ on Ω_a. Moreover $\Omega_a \supset \Omega'$ from Lemma 1 and therefore $F_a = F$ on Ω'. But then, since the singular set of $\mathfrak{g}$ has measure zero, it is obvious that $T_{F_a} = T_F$. Hence we shall write F instead of F_a.

We say that an element $H \in \mathfrak{g}$ is of compact type if 1) ad H is semisimple and 2) the derived algebra of the centralizer $\mathfrak{z}$ of H in $\mathfrak{g}$ is compact. (It follows from 1) that $\mathfrak{z}$ is reductive in $\mathfrak{g}$ and therefore $[\mathfrak{z}, \mathfrak{z}]$ is semisimple.)

Lemma 4. — *Every element of* Ω *of compact type lies in* Ω_a.

Fix an element H_0 in Ω of compact type and let $\mathfrak{z}$ denote the centralizer of H_0 in $\mathfrak{g}$. Then it is clear that $\mathfrak{z}$ satisfies the conditions of [3(i), § 2]. Define ζ and $\mathfrak{z}'$ as in [3(i), § 2]. Let Ξ be the analytic subgroup of G corresponding to $\mathfrak{z}$ and $x \to x^*$ the natural mapping of G on $G^* = G/\Xi$. Since $\mathfrak{z}$ is reductive, Ξ is unimodular and therefore there exists an invariant measure dx^* on G^*. Select open neighborhoods G_0 and $\mathfrak{z}_0$

of 1 and H_0 in G and $\mathfrak{z}'$ respectively such that $\mathfrak{z}_0^{G_0}\subset\Omega$ and G_0 is connected. Let G_0^* denote the image of G_0 in G^*. Then if G_0 and $\mathfrak{z}_0$ are sufficiently small, the following conditions hold (see [3(*e*), pp. 654-655]).

1) There exists an analytic mapping ψ of G_0^* into G such that $(\psi(x^*))^*=x^*$ $(x^*\in G_0^*)$ and ψ is regular on G_0^*.

2) The mapping $\varphi:(x^*, Z)\rightarrow\psi(x^*)Z$ of $G_0^*\times\mathfrak{z}_0$ into Ω is univalent.

Put $\Omega_0=\varphi(G_0^*\times\mathfrak{z}_0)$. Then Ω_0 is open in Ω and φ is an analytic diffeomorphism of $G_0^*\times\mathfrak{z}_0$ onto Ω_0. Moreover since $\mathfrak{z}_1=[\mathfrak{z},\mathfrak{z}]$ is compact, $\mathfrak{z}_{00}=\bigcap_{\xi\in\Xi}\mathfrak{z}_0^{\xi}$ is open. Hence by replacing $\mathfrak{z}_0$ by $\mathfrak{z}_{00}$, we can assume that $\mathfrak{z}_0^{\Xi}=\mathfrak{z}_0$.

Let σ_T be the distribution on $\mathfrak{z}_0$ which corresponds to T under Lemma 17 of [3(*i*)]. Since $|\zeta|^{1/2}$ is an analytic function on $\mathfrak{z}_0$, $\sigma=|\zeta|^{1/2}\sigma_T$ is also a distribution on $\mathfrak{z}_0$. Moreover since $\zeta^2>0$ on $\mathfrak{z}_0$ it follows from Theorem 2 of [3(*i*)] that $\partial(u_{\mathfrak{z}})\sigma=0$ for $u\in\mathfrak{U}$. Let $\mathfrak{U}_{\mathfrak{z}}$ denote the image of $\mathfrak{U}$ in $I(\mathfrak{z}_c)$ under the mapping $p\rightarrow p_{\mathfrak{z}}$ of $I(\mathfrak{g}_c)$ into $I(\mathfrak{z}_c)$. Put $\mathfrak{V}=I(\mathfrak{z}_c)\mathfrak{U}_{\mathfrak{z}}$. Then $\partial(v)\sigma=0$ for $v\in\mathfrak{V}$ and it follows from Lemma 19 of [3(*i*)] that $\dim(I(\mathfrak{z}_c)/\mathfrak{V})<\infty$.

Let $\mathfrak{c}_{\mathfrak{z}}$ be the center and $\mathfrak{z}_1$ the derived algebra of $\mathfrak{z}$. We identify $\mathfrak{z}_1$ with its dual under the Killing form ω_1 of $\mathfrak{z}_1$. Select a base $H_1,\ldots,H_r$ for $\mathfrak{c}_{\mathfrak{z}}$ over **R** and put

$$\omega=H_1^2+\ldots+H_r^2-\omega_1.$$

Then $\omega\in I(\mathfrak{z}_c)$ and since $\mathfrak{z}_1$ is compact, $\square=\partial(\omega)$ is an elliptic differential operator on $\mathfrak{z}$. Let $N=\dim(I(\mathfrak{z}_c)/\mathfrak{V})$. Then we can choose complex numbers $c_1,\ldots,c_N$ such that

$$\omega^N+c_1\omega^{N-1}+\ldots+c_N\in\mathfrak{V}.$$

Hence

$$(\square^N+c_1\square^{N-1}+\ldots+c_N)\sigma=0.$$

This shows that σ satisfies an elliptic differential equation with constant coefficients. Therefore there exists an analytic function g on $\mathfrak{z}_0$ such that

$$\sigma(\beta)=\int\beta g dZ \qquad (\beta\in C_c^\infty(\mathfrak{z}_0)).$$

Since ζ is invariant under Ξ, it follows from [3(*i*), Lemma 17] that g is locally invariant (with respect to $\mathfrak{z}$). Therefore since Ξ is connected and $\mathfrak{z}_0^{\Xi}=\mathfrak{z}_0$, it follows that g is invariant under Ξ.

Now consider the analytic function F_0 on Ω_0 defined by

$$F_0(\varphi(x^*, Z))=|\zeta(Z)|^{-1/2}g(Z) \qquad (x^*\in G_0^*, Z\in\mathfrak{z}_0)$$

Then if $\alpha\in C_c^\infty(G_0\times\mathfrak{z}_0)$, we have (see [3(*i*), § 7])

$$\int f_\alpha F_0 dX=\int\alpha(x:Z)F_0(xZ)dxdZ.$$

However if $x\in G_0$, it is clear that $x=\psi(x^*)\xi$ where $\xi\in\Xi$. Therefore

$$F_0(xZ)=F_0(\varphi(x^*,\xi Z))=|\zeta(Z)|^{-1/2}g(Z) \qquad (Z\in\mathfrak{z}_0)$$

since ζ and g are invariant under Ξ. This shows that

$$\int f_\alpha F_0 dX = \int \alpha(x : Z)|\zeta(Z)|^{-1/2} g(Z) dx dZ$$
$$= \int \beta_\alpha |\zeta|^{-1/2} g dZ = \sigma_T(\beta_\alpha) = T(f_\alpha)$$

from Lemma 17 of [3(*i*)]. Hence $T = F_0$ on Ω_0 and this proves that $H_0 \in \Omega_a$.

§ 3. SOME PROPERTIES OF COMPLETELY INVARIANT SETS

We keep to the above notation. An element $H \in \mathfrak{g}$ is called semisimple if ad H is semisimple. Moreover $X \in \mathfrak{g}$ is called nilpotent if $X \in \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ and ad X is nilpotent. It is obvious that if X is both semisimple and nilpotent then $X = 0$.

Lemma 5. — Any element $Y \in \mathfrak{g}$ *can be written uniquely in the form* $Y = H + X$ *where* H *is a semisimple and* X *a nilpotent element of* $\mathfrak{g}$ *and* $[H, X] = 0$.

Let $\mathfrak{c}$ be the center of $\mathfrak{g}$. Since $\mathfrak{g} = \mathfrak{c} + \mathfrak{g}_1$, the lemma follows from well-known facts about semisimple Lie algebras (see Bourbaki [2, p. 79]). H and X respectively are called the semisimple and the nilpotent components of Y.

Lemma 6. — Let $\mathfrak{z}$ *be a subalgebra of* $\mathfrak{g}$ *which is reductive in* $\mathfrak{g}$. *An element* Z *of* $\mathfrak{z}$ *is semisimple (or nilpotent) in* $\mathfrak{z}$, *if and only if the same holds in* $\mathfrak{g}$.

Let $\mathfrak{c}_\mathfrak{z}$ be the center of $\mathfrak{z}$. Since $\mathfrak{z}$ is reductive in $\mathfrak{g}$, every element of $\mathfrak{c}_\mathfrak{z}$ is semisimple in $\mathfrak{g}$. The lemma follows easily from this (see [2, p. 79]).

Corollary. — Let $Z \in \mathfrak{z}$. *Then the semisimple component of* Z *in* $\mathfrak{z}$ *is the same as in* $\mathfrak{g}$. *Similarly for the nilpotent component.*

This is obvious from Lemma 5.

Lemma 7. — Let U_1 *be a neighborhood of zero in* $\mathfrak{g}_1$ *and* X *a nilpotent element of* $\mathfrak{g}$. *Then we can choose* $x \in G$ *such that* $xX \in U_1$.

We may assume that $X \neq 0$. Then by the Jacobson-Morosow theorem [3(*h*), Lemma 24], we can choose $H \in \mathfrak{g}_1$ sucht hat $[H, X] = 2X$. Put $a_t = \exp(-tH) \in G$ $(t \in \mathbf{R})$. Then $a_t X = e^{-2t} X$ and therefore $a_t X \in U_1$ if t is positive and sufficiently large.

Corollary. — Let H *denote the semisimple component of an element* $Z \in \mathfrak{g}$. *Then* (1) $H \in Cl(Z^G)$.

Let X be the nilpotent component of Z so that $Z = H + X$. Consider the centralizer $\mathfrak{z}$ of H in $\mathfrak{g}$. Then $\mathfrak{z}$ is reductive in $\mathfrak{g}$ and $X \in \mathfrak{z}$. Hence X is nilpotent in $\mathfrak{z}$ (Lemma 6). Let Ξ be the analytic subgroup of G corresponding to $\mathfrak{z}$. Then by Lemma 7, applied to $\mathfrak{z}$, we have

$$H \in H + Cl(X^\Xi) = Cl(Z^\Xi) \subset Cl(Z^G).$$

Let Ω be a subset of $\mathfrak{g}$. We say that Ω is completely invariant if it has the following property: C being any compact subset of Ω, $Cl(C^G) \subset \Omega$.

(1) ClS denotes the closure of S.

Lemma 8. — *Let* Ω *be a completely invariant subset of* $\mathfrak{g}$ *and* Z *an element in* Ω. *Then if* H *is the semisimple component of* Z, $H\in\Omega$.

This is obvious from the corollary of Lemma 7.

Let $\mathcal{N}$ be the set of all nilpotent elements of $\mathfrak{g}$.

Corollary 1. — *Let* S *be the set of all semisimple elements of* Ω *and* Φ *an invariant subset of* Ω *which is closed in* Ω. *Then* $\Phi\cap S=\emptyset$ *implies that* $\Phi=\emptyset$. *Similarly* $\Phi\cap S\subset\{0\}$ *implies that* $\Phi\subset\Omega\cap\mathcal{N}$.

For suppose $Z\in\Phi$. Then if H is the semisimple component of Z, $H\in \mathrm{Cl}(Z^G)\subset\Omega$. Since Φ is invariant and closed in Ω, it follows that $H\in\Phi\cap S$. The two statements of the corollary are now obvious.

Corollary 2. — *Let* Ω_0 *be an open and invariant subset of* Ω. *Assume that* $S\subset\Omega_0$. *Then* $\Omega_0=\Omega$.

This follows from Corollary 1 by taking Φ to be the complement of Ω_0 in Ω.

Let $\mathfrak{c}$ be the center of $\mathfrak{g}$. Fix an open and completely invariant subset Ω of $\mathfrak{g}$ and a point $X_0=C_0+Z_0$ ($C_0\in\mathfrak{c}$, $Z_0\in\mathfrak{g}_1$) in Ω. Select a relatively compact and open neighborhood $\mathfrak{c}_0$ of C_0 in $\mathfrak{c}$ such that $\mathrm{Cl}(\mathfrak{c}_0)+Z_0\subset\Omega$.

Lemma 9. — *Let* Ω_1 *be the set of all* $Z\in\mathfrak{g}_1$ *such that* $Z+\mathrm{Cl}\,\mathfrak{c}_0\subset\Omega$. *Then* Ω_1 *is an open and completely invariant neighborhood of* Z_0 *in* $\mathfrak{g}_1$.

It is obvious that Ω_1 is an open neighborhood of Z_0 in $\mathfrak{g}_1$. Fix a compact set Q in Ω_1. Then $\mathrm{Cl}\,\mathfrak{c}_0+Q$ is a compact subset of Ω and therefore

$$\mathrm{Cl}(\mathrm{Cl}\,\mathfrak{c}_0+Q)^G=\mathrm{Cl}\,\mathfrak{c}_0+\mathrm{Cl}(Q^G)\subset\Omega,$$

since Ω is completely invariant. This shows that $\mathrm{Cl}(Q^G)\subset\Omega_1$ and therefore Ω_1 is also completely invariant.

Lemma 10. — *The following three conditions on* Ω *are equivalent:*

1) $\Omega\cap\mathcal{N}\neq\emptyset$;

2) $0\in\Omega$;

3) $\mathcal{N}\subset\Omega$.

Let $X\in\Omega\cap\mathcal{N}$. By Lemma 8, $0\in\Omega$. Hence 1) implies 2). Now assume $0\in\Omega$. Then if $X\in\mathcal{N}$, it follows from Lemma 7 that $X^x\in\Omega$ for some $x\in G$. Since Ω is invariant, this means that $X\in\Omega$. Therefore 2) implies 3). It is obvious that 3) implies 1).

§ 4. THE MAIN PART OF THE PROOF OF THEOREM 1

We shall now begin the proof of the following theorem (cf. [3(*g*), Lemma 1]).

Theorem 1. — *Let* $\mathfrak{g}$ *be a reductive Lie algebra over* $\mathbf{R}$, Ω *an open and completely invariant subset of* $\mathfrak{g}$ *and* T *a distribution on* Ω. *Assume that:*

1) T *is invariant;*

2) *There exists an ideal* $\mathfrak{U}$ *in* $I(\mathfrak{g}_c)$ *such that* $\dim(I(\mathfrak{g}_c)/\mathfrak{U})<\infty$ *and* $\partial(u)T=0$ *for* $u\in\mathfrak{U}$.

Then T *is a locally summable function on* Ω *which is analytic on* $\Omega'=\Omega\cap\mathfrak{g}'$.

We use induction on dim $\mathfrak{g}$. Let F be the analytic function on Ω' corresponding to Lemma 1. Then by Lemma 3, F is locally summable on Ω and we have to show that $T=T_F$.

Let $\mathfrak{c}$ be the center and $\mathfrak{g}_1$ the derived algebra of $\mathfrak{g}$. First assume that $\mathfrak{c}\neq\{0\}$. Fix a point $X_0=C_0+Z_0$ ($C_0\in\mathfrak{c}$, $Z_0\in\mathfrak{g}_1$) in Ω. We have to prove that $T=T_F$ around X_0. Select an open and relatively compact neighborhood $\mathfrak{c}_0$ of C_0 in $\mathfrak{c}$ such that $(\mathrm{Cl}\,\mathfrak{c}_0)+Z_0\subset\Omega$. Let Ω_1 be the set of all elements $Z\in\mathfrak{g}_1$ such that $\mathrm{Cl}\,\mathfrak{c}_0+Z\subset\Omega$. Then by Lemma 9, Ω_1 is also completely invariant.

Fix Euclidean measures dC and dZ on $\mathfrak{c}$ and $\mathfrak{g}_1$ respectively such that $dX=dCdZ$ for $X=C+Z$ ($C\in\mathfrak{c}$, $Z\in\mathfrak{g}_1$) and, for any $\alpha\in C_c^\infty(\mathfrak{c}_0)$, consider the distribution θ_α on Ω_1 given by

$$\theta_\alpha(\beta)=T(\alpha\times\beta) \qquad (\beta\in C_c^\infty(\Omega_1)).$$

Then if G_1 is the analytic subgroup of G corresponding to $\mathfrak{g}_1$, it is clear that θ_α is invariant under G_1. Moreover $I(\mathfrak{g}_c)=S(\mathfrak{c}_c)I(\mathfrak{g}_{1c})$ since $\mathfrak{g}=\mathfrak{c}+\mathfrak{g}_1$. Put $\mathfrak{U}_1=\mathfrak{U}\cap I(\mathfrak{g}_{1c})$. Then it is obvious that

$$\dim(I(\mathfrak{g}_{1c})/\mathfrak{U}_1)\leq\dim(I(\mathfrak{g}_c)/\mathfrak{U})<\infty$$

and $\partial(u)\theta_\alpha=0$ for $u\in\mathfrak{U}_1$. Therefore, since $\dim\mathfrak{g}_1<\dim\mathfrak{g}$, it follows by the induction hypothesis that θ_α coincides on Ω_1 with a locally summable function g_α. Put $\Omega_1'=\Omega_1\cap\mathfrak{g}_1'$ where $\mathfrak{g}_1'$ is the set of those elements of $\mathfrak{g}_1$ which are regular in $\mathfrak{g}_1$. Since $\mathfrak{g}'=\mathfrak{c}+\mathfrak{g}_1'$, it is clear that $\mathfrak{c}_0+\Omega_1'\subset\Omega'$. Moreover since $T=T_F$ on Ω', it follows that

$$\theta_\alpha(\beta)=T(\alpha\times\beta)=T_F(\alpha\times\beta)=\int\alpha(C)\beta(Z)F(C+Z)dCdZ$$

for $\beta\in C_c^\infty(\Omega_1')$. Since g_α is analytic on Ω_1' (by the induction hypothesis), it is clear from the above relation that

$$g_\alpha(Z)=\int\alpha(C)F(C+Z)dC \qquad (Z\in\Omega_1').$$

But since g_α and F are locally summable on Ω_1 and Ω respectively, we can now conclude that

$$T(\alpha\times\beta)=\theta_\alpha(\beta)=\int\beta(Z)\alpha(C)F(C+Z)dCdZ=T_F(\alpha\times\beta)$$

for $\beta\in C_c^\infty(\Omega_1)$. This proves (see [3(*h*), Lemma 3]) that $T=T_F$ on $\mathfrak{c}_0+\Omega_1$.

So now we can assume that $\mathfrak{c}=\{0\}$ and therefore $\mathfrak{g}$ is semisimple. Fix a semisimple element $H_0\neq 0$ in Ω. We shall first prove that $T=T_F$ around H_0. Let $\mathfrak{z}$ be the centralizer of H_0 in $\mathfrak{g}$ and Ξ the analytic subgroup of G corresponding to $\mathfrak{z}$. Define ζ and $\mathfrak{z}'$ as in [3(*i*), § 2]. Then $\zeta(H_0)\neq 0$. Let $\Omega_\mathfrak{z}$ be the set of all $Z\in\mathfrak{z}\cap\Omega$ such that $|\zeta(Z)|>|\zeta(H_0)|/2$. Then $\Omega_\mathfrak{z}$ is an open neighborhood of H_0 in $\mathfrak{z}'$. Moreover since ζ is invariant under Ξ, it follows easily that $\Omega_\mathfrak{z}$ is completely invariant in $\mathfrak{z}$. Let σ_T be the distribution on $\Omega_\mathfrak{z}$ corresponding to T under [3(*i*), Lemma 17] with $G_0=G$ and $\mathfrak{z}_0=\Omega_\mathfrak{z}$. Then by Corollary 1 of [3(*i*), Lemma 17], σ_T is invariant under Ξ. Now $\zeta^2>0$ on $\Omega_\mathfrak{z}$. Hence $\sigma=|\zeta|^{1/2}\sigma_T$ is also an invariant distribution on $\Omega_\mathfrak{z}$ and it follows from Theorem 2 of [3(*i*)] that $\partial(u_{\mathfrak{z}})\sigma = 0$ for $u \in \mathfrak{U}$. Let $\mathfrak{U}_{\mathfrak{z}}$ denote the image of $\mathfrak{U}$ under the homomorphism $p \rightarrow p_{\mathfrak{z}}$ of $I(\mathfrak{g}_c)$ into $I(\mathfrak{z}_c)$. Then if $\mathfrak{V} = I(\mathfrak{z}_c)\mathfrak{U}_{\mathfrak{z}}$, it is clear from Lemma 19 of [3(*i*)] that $\dim(I(\mathfrak{z}_c)/\mathfrak{V}) < \infty$. On the other hand $\dim \mathfrak{z} < \dim \mathfrak{g}$ since $\mathfrak{g}$ is semisimple and $H_0 \neq 0$. Therefore the induction hypothesis is applicable to $(\sigma, \Omega_{\mathfrak{z}}, \mathfrak{V})$ in place of $(T, \Omega, \mathfrak{U})$. Let $\Omega'_{\mathfrak{z}}$ be the set of all points in $\Omega_{\mathfrak{z}}$ which are regular in $\mathfrak{z}$. Then σ coincides with a locally summable function g on $\Omega_{\mathfrak{z}}$ which is analytic on $\Omega'_{\mathfrak{z}}$. This shows that

$$T(f_\alpha) = \sigma_T(\beta_\alpha) = \int \beta_\alpha |\zeta|^{-1/2} g dZ \qquad (\alpha \in C_c^\infty(G \times \Omega_{\mathfrak{z}}))$$

in the notation of [3(*i*), Lemma 17]. On the other hand since $\Omega_{\mathfrak{z}} \subset \mathfrak{z}'$, it is clear that $\Omega'_{\mathfrak{z}} \subset \Omega'$. Moreover $T = T_F$ on Ω'. Therefore

$$T(f_\alpha) = T_F(f_\alpha) = \int \alpha(x : Z) F(xZ) dx dZ$$

for $\alpha \in C_c^\infty(G \times \Omega'_{\mathfrak{z}})$. However T is invariant and therefore the same holds for F. Hence

$$T(f_\alpha) = \int \beta_\alpha(Z) F(Z) dZ.$$

This proves that $g(Z) = |\zeta(Z)|^{1/2} F(Z)$ for $Z \in \Omega'_{\mathfrak{z}}$. Now fix $\alpha \in C_c^\infty(G \times \Omega_{\mathfrak{z}})$. Then

$$\begin{aligned} T(f_\alpha) &= \int \beta_\alpha |\zeta|^{-1/2} g dZ = \int_{\Omega'_{\mathfrak{z}}} \beta_\alpha |\zeta|^{-1/2} g dZ \\ &= \int_{G \times \Omega'_{\mathfrak{z}}} \alpha(x : Z) F(xZ) dx dZ = T_F(f_\alpha) \end{aligned}$$

from Corollary 2 of [3(*h*), Theorem 1]. This proves that $T = T_F$ around H_0.

Put $\theta = T - T_F$. Then θ is an invariant distribution on Ω.

Lemma 11. — Let $\mathcal{N}$ be the set of all nilpotent elements of $\mathfrak{g}$. *Then*

$$\operatorname{Supp} \theta \subset \mathcal{N} \cap \Omega.$$

It follows from the above proof that no semisimple element of Ω, other than zero, can lie in Supp θ. Therefore our assertion follows immediately by taking $\Phi = \operatorname{Supp} \theta$ in Corollary 1 of Lemma 8.

As usual we identify $\mathfrak{g}_c$ with its dual under the Killing form ω of $\mathfrak{g}$.

Lemma 12. — Assume that there exists a complex number c and an integer $r \geq 0$ such that $(\partial(\omega) - c)^r T = 0$. Then $T = T_F$.

We shall prove this by induction on r. If $r = 0$ then $T = 0$ and our statement is true. So assume that $r \geq 1$. Put $T_0 = (\partial(\omega) - c)T$. Then T_0 satisfies all the conditions of Theorem 1 and $(\partial(\omega) - c)^{r-1} T_0 = 0$. Moreover since $T = F$ on Ω' and F is analytic on Ω', it is obvious that $T_0 = (\partial(\omega) - c)F$ on Ω'. Therefore it follows by the induction hypothesis that $T_0 = T_{F_0}$ where $F_0 = (\partial(\omega) - c)F$ (see also the corollary of Lemma 3). Hence

$$(\partial(\omega) - c)(\theta + T_F) = T_{F_0}$$

and therefore

$$(\partial(\omega) - c)\theta = T_{\partial(\omega)F} - \partial(\omega) T_F.$$

Lemma 13. — $T_{\partial(\omega)F}\partial - (\omega)T_F = 0$.

Assuming this for a moment, we shall complete the proof of Lemma 12. For then we have $(\partial(\omega)-c)\theta = 0$ and therefore we conclude from [3(*h*), Theorem 5] that $\theta = 0$. Hence $T = T_F$.

The proof of Lemma 13 is based on Theorem 4 of [3(*j*)] and requires some preparation. Select a system of generators [1] $(p_1, \ldots, p_m)$ for the algebra $I(\mathfrak{g}_c)$ over $\mathbf{C}$.

Lemma 14. — *Fix* $X_0 \in \mathfrak{g}$ *and for any* $\varepsilon > 0$, *let* $U_{X_0}(\varepsilon)$ *denote the set of all* $X \in \mathfrak{g}$ *such that* $|p_i(X) - p_i(X_0)| < \varepsilon$ $(1 \leq i \leq m)$. *Then* $U_{X_0}(\varepsilon)$ *is open and completely invariant.*

$U_{X_0}(\varepsilon)$ is obviously open. Let C be a compact subset of $U_{X_0}(\varepsilon)$. Then it is clear that we can choose a $(0 < a < \varepsilon)$ such that

$$\sup_{X \in C} |p_i(X) - p_i(X_0)| \leq a \qquad (1 \leq i \leq m).$$

Since p_i is invariant, it is obvious that $|p_i(Y) - p_i(X_0)| \leq a$ for any $Y \in \mathrm{Cl}(C^G)$ and therefore $U_{X_0}(\varepsilon)$ is completely invariant.

Now put $J_0 = T_{\partial(\omega)F} - \partial(\omega)T_F$ and fix $X_0 \in \Omega$. We have to prove that $J_0 = 0$ around X_0. Define $\Omega(\varepsilon) = \Omega \cap U_{X_0}(\varepsilon)$ for $\varepsilon > 0$. Then $\Omega(\varepsilon)$ is an open and completely invariant neighborhood of X_0. We shall now use the notation of [3(*j*), Theorem 4]. Put $\Phi_i(\varepsilon) = \mathfrak{h}_i \cap \Omega(\varepsilon)$ and $\Phi_i = \bigcap_{\varepsilon>0} \Phi_i(\varepsilon)$ $(1 \leq i \leq r)$. If $H \in \Phi_i$, it is clear that $p(H) = p(X_0)$ for $p \in I(\mathfrak{g}_c)$. Hence it follows from Chevalley's theorem [3(*c*), Lemma 9] that Φ_i is a finite set. For each $H \in \Phi_i$, choose two open convex neighborhoods U_H, V_H of H in $\mathfrak{h}_i$ such that $\mathrm{Cl}U_H \subset V_H \subset \Phi_i(1)$ and $V_H \cap V_{H'} = \emptyset$ for $H \neq H'$ $(H, H' \in \Phi_i)$. Then $\mathrm{Cl}V_H$ is compact (see the proof of Lemma 23 of [3(*j*)]). Put

$$U_i = \bigcup_{H \in \Phi_i} U_H, \qquad V_i = \bigcup_{H \in \Phi_i} V_H$$

and select $\alpha_H \in C_c^\infty(V_H)$ such that $\alpha_H = 1$ on U_H $(H \in \Phi_i)$. Define

$$\alpha_i = \sum_{H \in \Phi_i} \alpha_H.$$

Let F_i denote the restriction of F on $\Omega' \cap \mathfrak{h}_i = \Omega \cap \mathfrak{h}_i'$. Fix i and let P_c be the set of all complex positive roots of $\mathfrak{h}_i$. Let Q be a connected component of $\mathfrak{h}_i'(S)$ and Q_1 the set consisting of all regular and semiregular points of Q. If β is a root of $(\mathfrak{g}, \mathfrak{h}_i)$ which vanishes at some point H_0 in Q_1, then it is clear that β is compact and therefore H_0 is of compact type in $\mathfrak{g}$. Obviously Q_1 is open in $\mathfrak{h}_i$. Therefore by Lemma 4, F_i can be extended to an analytic function on $Q_1 \cap \Omega$ which we again denote by F_i.

Now fix $H \in \Phi_i$ and consider $Q_1 \cap V_H$. Then $Q_1 \cap V_H$ is connected (see the corollary of [3(*j*), Lemma 19]). Also $V_H \subset \Omega$ and therefore F_i is analytic on the connected set $Q_1 \cap V_H$. Hence by Lemma 2, there exists an analytic function h_H on $\mathfrak{h}_i$ such that $\pi_i F_i = h_H$ on $Q_1 \cap V_H$. Then $\alpha_H \pi_i F_i = \alpha_H h_H$ on $Q_1 \cap \Omega$ and therefore

$$\alpha_i \pi_i F_i = \sum_{H \in \Phi_i} \alpha_H h_H$$

[1] Since $\mathfrak{g}$ is semisimple, it follows from the theory of invariants that $I(\mathfrak{g}_c)$ is finitely generated.

on $Q_1 \cap \Omega$. Put $g_i' = \alpha_i \pi_i F_i$. Then the above result shows that g_i' is of class C^∞ on $Cl(Q_1) = Cl(Q)$.

Choose $\varepsilon > 0$ so small that $\Phi_i(\varepsilon) \subset U_i$ $(1 \leq i \leq r)$. Then from Corollary 1 of [3(*j*), Lemma 30] we can choose numbers c_i $(1 \leq i \leq r)$ such that

$$\int fu dX = \sum_{1 \leq i \leq r} c_i \int_{\mathfrak{h}_i} \psi_{f,i} \varepsilon_{R,i} \pi_i u_i d_i H \qquad (f \in C_c^\infty(\Omega))$$

for any invariant and locally summable function u on Ω. (Here u_i is the restriction of u on $\Omega \cap \mathfrak{h}_i$.) Now suppose $f \in C_c^\infty(\Omega(\varepsilon))$. Since $\Omega(\varepsilon)$ is completely invariant, it follows from [3(*j*), Lemma 22] that

$$\text{Supp } \psi_{f,i} \subset \Omega(\varepsilon) \cap \mathfrak{h}_i = \Phi_i(\varepsilon) \subset U_i.$$

Hence

$$\int fu dX = \sum_{1 \leq i \leq r} c_i \int \psi_{f,i} \varepsilon_{R,i} \alpha_i \pi_i u_i d_i H.$$

Now take $u = F$. Then $c_i \varepsilon_{R,i} \alpha_i \pi_i u_i = c_i \varepsilon_{R,i} g_i' = g_i$ (say). On the other hand, by the corollary of Lemma 3, we can also take $u = \partial(\omega) F$. Then it follows from [3(*c*), Lemma 3] that $u_i = \pi_i^{-1} \partial(\omega_i)(\pi_i F_i)$ on $\mathfrak{h}_i' \Omega$ and therefore

$$c_i \varepsilon_{R,i} \alpha_i \pi_i u_i = \partial(\omega_i) g_i$$

on $U_i \cap \mathfrak{h}_i'$. Therefore

$$\begin{aligned} J_0(f) &= \int (F \partial(\omega) f - \partial(\omega) F . f) dX \\ &= \sum_{1 \leq i \leq r} \int_{\mathfrak{h}_i} (\partial(\omega_i) \psi_{f,i} \cdot g_i - \psi_{f,i} \cdot \partial(\omega_i) g_i) d_i H \end{aligned}$$

from [3(*d*), Theorem 3]. Now define J as in [3(*j*), Theorem 4], corresponding to the above functions g_i $(1 \leq i \leq r)$. Then the above result shows that $J = J_0$ on $\Omega(\varepsilon)$. Since g_i is obviously of class C^∞ on the closure of each connected component of $\mathfrak{h}_i'(S)$, Theorem 4 of [3(*j*)] is applicable. Fix an open and relatively compact neighborhood V of X_0 in $\Omega(\varepsilon)$. Then since $\Omega(\varepsilon)$ is completely invariant, $Cl(V^G) \subset \Omega(\varepsilon)$. Let $\mathcal{S}$ denote the set of all semiregular elements of $\Omega(\varepsilon)$ of noncompact type. Then in order to prove that $J_0 = 0$ on V, it is enough, from [3(*j*), Theorem 4], to verify that $\text{Supp } J_0 \cap \mathcal{S} = \emptyset$. However $J_0 = (\partial(\omega) - c)\theta$ and so it follows from Lemma 11 that

$$\text{Supp } J_0 \subset \text{Supp } \theta \subset \mathcal{N} \cap \Omega.$$

Since zero is the only semisimple element in $\mathcal{N}$, it is clear that $\mathcal{N} \cap \mathcal{S} \subset \{0\}$. Therefore we may assume that $\mathcal{S}$ contains zero. But then it follows from [3(*j*), § 4] that $\mathfrak{g}$ is isomorphic to the three dimensional noncompact semisimple algebra $\mathfrak{l}$ of [3(*j*), § 2]. We shall consider this case in detail in the next section.

§ 5. SOME COMPUTATIONS ON $\mathfrak{l}$

So we now assume that $\mathfrak{g}=\mathfrak{l}$ and $0\in\Omega$. Then we have to show that $J_0=0$ around zero. Hence we take $X_0=0$ (see § 4). Then it follows from Lemma 10 that $\mathcal{N}\subset\Omega(\varepsilon)$. Now $\mathcal{N}$ is also the singular set of $\mathfrak{g}$ in the present case. Therefore $\operatorname{Supp} J\subset\mathcal{N}$. However $J=J_0$ on $\Omega(\varepsilon)$ and so it is obvious that $J=J_0$ on Ω.

Lemma 15. — We can choose complex numbers a, a^+ and a^- such that

$$J(f)=af(0)+a^+c^+(f)+a^-c^-(f) \qquad (f\in C_c^\infty(\mathfrak{g}))$$

in the notation of [3(*j*), Lemma 34].

This is obvious from [3(*j*), Lemmas 2, 3 and 26].

Corollary. — $\omega J=0$.

Since $\omega=0$ on $\mathcal{N}$, this is an immediate consequence of Lemma 15.

We have seen in § 4 that $\operatorname{Supp} J_0\subset\mathcal{N}$. Fix an element $X\neq 0$ in $\mathcal{N}$. We shall first prove that $J_0=0$ around X. By the Jacobson-Morosow theorem [3(*h*), Lemma 24], we can select H, Y in $\mathfrak{g}$ such that

$$[H, X]=2X, \qquad [H, Y]=-2Y, \qquad [X, Y]=H.$$

Then $\mathfrak{z}_X=\mathbf{R}X$ is the centralizer of X in $\mathfrak{g}$ and $\mathfrak{g}_X=[X, \mathfrak{g}]=\mathbf{R}H+\mathbf{R}X$. Take $U=\mathbf{R}Y$ and $V=\mathbf{R}H+\mathbf{R}Y$ so that $\mathfrak{g}=U+\mathfrak{g}_X=\mathfrak{z}_X+V$. We now use the notation of [3(*h*), § 7]. Then $4\omega=2^{-1}H^2+2XY$, $\omega(X+tY)=8t$ and

$$\Gamma_{X+tY}(Y^2\otimes 1)=H^2-2Y, \qquad \Gamma_{X+tY}(H\otimes Y)=2(XY-Y-tY^2)$$

for $t\in\mathbf{R}$. Hence

$$\Gamma_{X+tY}\left(\frac{1}{2}Y^2\otimes 1+H\otimes Y+1\otimes(3Y+2tY^2)\right)=4\omega.$$

This means that

$$4\Delta(\partial(\omega))=3D+2tD^2$$

on U′ in the notation of [3(*h*), § 8]. (Here $D=d/dt$.) On the other hand $(\partial(\omega)-c)\theta=J_0$ and θ and J_0 are both invariant distributions. Hence

$$(\Delta-c)\sigma_\theta=\sigma_{J_0}$$

in the notation of [3(*h*), Theorem 3] where $\Delta=\Delta(\partial(\omega))$.

Since $\operatorname{Supp}\theta\subset\mathcal{N}$, we can regard σ_θ as a distribution on an open neighborhood U_0 of the origin in $\mathbf{R}$ and assume that $\operatorname{Supp}\sigma_\theta\subset\{0\}$ (see [3(*h*), Lemma 23]). If $\sigma_\theta=0$, it follows from [3(*h*), Theorem 2] that $\theta=0$ around X and therefore the same holds for J_0. Hence we may assume that $0\in\operatorname{Supp}\sigma_\theta$. Then (see [3(*h*), Lemma 20])

$$\sigma_\theta=\sum_{0\leq k\leq m} a_k D^k\delta$$

where δ denotes the Dirac distribution $\beta \to \beta(0)$ $(\beta \in C_c^\infty(U_0))$ and a_k are complex numbers $(a_m \neq 0)$. Now $\omega J = \omega J_0 = 0$ on Ω. Since $\omega(X+tY) = 8t$, it follows that $t\sigma_{J_0} = 0$ on U_0. Hence

$$\sum_{0 \leq k \leq m} a_k t(3D + 2tD^2 - 4c)D^k \delta = 0.$$

But it is easy to verify that

$$tD^k\delta = -kD^{k-1}\delta,$$
$$t^2 D^k \delta = k(k-1)D^{k-2}\delta \qquad (k \geq 0)$$

where $D^\nu\delta$ should be interpreted to mean zero if $\nu < 0$. Therefore

$$\sum_{0 \leq k \leq m} a_k\{(k+1)(2k+1)D^k\delta + 4ckD^{k-1}\delta\} = 0.$$

But since the distributions $D^k\delta$ $(k \geq 0)$ on U_0 are linearly independent, we conclude that $(m+1)(2m+1)a_m = 0$. However this is impossible since $m \geq 0$ and $a_m \neq 0$. This contradiction shows that $\sigma_0 = 0$ and therefore $\theta = J_0 = 0$ around X. This proves that $\mathrm{Supp}\,\theta \subset \{0\}$ and $\mathrm{Supp}\,J_0 \subset \{0\}$.

Now $J = J_0$ on Ω. Hence it follows (see [3(*e*), p. 685]) that $a^+ = a^- = 0$ in Lemma 15 and therefore $J_0 = J = a\delta_0$ on Ω. Here δ_0 is the Dirac distribution $f \to f(0)$ $(f \in C_c^\infty(\mathfrak{g}))$ on $\mathfrak{g}$. But since $\mathrm{Supp}\,\theta \subset \{0\}$, we conclude from [3(*h*), Lemma 20] that $\theta = \partial(p)\delta_0$ where $p \in S(\mathfrak{g}_c)$. On the other hand $(\partial(\omega) - c)\theta = J_0 = a\delta_0$ on Ω. Therefore $(\omega - c)p = a$ again from [3(*h*), Lemma 20]. Since ω is homogeneous of degree 2, this is possible only if $p = a = 0$. Therefore $\theta = J_0 = 0$ and so Lemma 13 is now proved.

§ 6. COMPLETION OF THE PROOF OF THEOREM 1

It remains to complete the proof of Theorem 1 in case $\mathfrak{g}$ is semisimple. Let $\mathfrak{T}$ be the vector space of all distributions on Ω of the form $\partial(p)T$ $(p \in I(\mathfrak{g}_c))$. Then it is clear that

$$\dim \mathfrak{T} \leq \dim(I(\mathfrak{g}_c)/\mathfrak{U}) < \infty$$

and every element of $\mathfrak{T}$ satisfies all the conditions of Theorem 1. The mapping $S \to \partial(\omega)S$ $(S \in \mathfrak{T})$ is obviously an endomorphism of $\mathfrak{T}$. Hence we can choose a base T_j $(1 \leq j \leq N)$ for $\mathfrak{T}$ over $\mathbf{C}$ with the following property. There exist complex numbers c_j and integers $r_j \geq 0$ such that

$$(\partial(\omega) - c_j)^{r_j} T_j = 0 \qquad (1 \leq j \leq N).$$

Then Lemma 12 is applicable to T_j. Let F_j be the analytic function on Ω' such that $T_j = F_j$ on Ω' (Lemma 1). Then F_j is locally summable on Ω (Lemma 3) and $T_j = T_{F_j}$ (Lemma 12). Since (T_j) $(1 \leq j \leq N)$ is a base for $\mathfrak{T}$, $T = \sum_j a_j T_j$ for some $a_j \in \mathbf{C}$. Then if $F = \sum_j a_j F_j$, it is obvious that $T = T_F$. This proves Theorem 1.

§ 7. SOME CONSEQUENCES OF THEOREM 1

We shall now derive some consequences of Theorem 1. Define $\mathfrak{J}(\mathfrak{g}_c)$ as in [3(*i*), § 4]. We keep to the notation of Theorem 1.

Lemma 16. — Fix $D\in\mathfrak{J}(\mathfrak{g}_c)$. *Then the distribution* DT *also satisfies the conditions of Theorem 1. Hence* DF *is locally summable on* Ω *and* $DT=T_{DF}$.

Corollary. — Let D^* *denote, as usual, the adjoint of* D. *Then*

$$\int f\, DF\, dX = \int D^*f . F\, dX \qquad (f\in C_c^\infty(\Omega)).$$

This is merely a restatement of the relation $DT=T_{DF}$.

Since the distribution DT is obviously invariant, it is enough to verify that the dimension of the space of all distributions of the form $\partial(p)(DT)$ $(p\in I(\mathfrak{g}_c))$ is finite. This requires some preparation.

Let us now use the notation of [3(*i*), § 3]. For any $p\in S(E)$, let r_p and d_p denote the endomorphisms $D\rightarrow D\circ\partial(p)$ and (1) $D\rightarrow\{\partial(p), D\}$ $(D\in\mathfrak{D}(E))$ respectively of $\mathfrak{D}(E)$.

Lemma 17. — Fix $p\in S(E)$. *Then for every* $D\in\mathfrak{D}(E)$ *we can choose an integer* $N\geq 0$ *such that* $d_p^N D=0$.

Let A be the set of all $p\in S(E)$ for which the lemma holds. We claim that A is a subalgebra of S(E). Observe that d_p, r_p, d_q, r_q $(p, q\in S(E))$ all commute with each other and

$$d_{pq}=d_p d_q + r_p d_q + d_p r_q.$$

Now fix p, q in A and $D\in\mathfrak{D}(E)$ and choose an integer $N\geq 0$ such that $d_p^N D=d_q^N D=0$. Then it is obvious that $(d_p+d_q)^{2N}D=0$ and

$$d_{pq}^{3N}D=(d_p d_q + r_p d_q + d_p r_q)^{3N}D=0.$$

This shows that $p+q$ and pq are both in A and therefore A is a subalgebra. On the other hand if $p\in P(E)$, $q\in S(E)$ and $X\in E$, it is obvious that

$$d_X^N(p\partial(q))=(d_X^N p)\partial(q)=0$$

if $N>d^0p$. This shows that $E\subset A$ and therefore $A=S(E)$.

We now return to the proof of Lemma 16. Let $\mathfrak{T}$ denote the space of all distributions of the form $\partial(p)T$ $(p\in I(\mathfrak{g}_c))$. Then $\dim\mathfrak{T}<\infty$. Since the algebra $I(\mathfrak{g}_c)$ is abelian, we can choose a base $T_1, \ldots, T_m$ for $\mathfrak{T}$ over **C** and homomorphisms $\chi_1, \ldots, \chi_m$ of $I(\mathfrak{g}_c)$ into **C** such that

$$(\partial(p)-\chi_i(p))^m T_i=0 \qquad (1\leq i\leq m).$$

Since T is a linear combination of T_i, it would be enough to prove Lemma 16 under the additional assumption that

$$(\partial(p)-\chi(p))^m T=0 \qquad (p\in I(\mathfrak{g}_c))$$

(1) As usual $\{D_1, D_2\}=D_1\circ D_2 - D_2\circ D_1$ for two differential operators D_1, D_2.

for some integer $m \geq 0$ and some homomorphism χ of $I(\mathfrak{g}_c)$ into $\mathbf{C}$. Now fix $p \in I(\mathfrak{g}_c)$ and choose $N \geq 0$ so large that $d_p^N D = 0$ (in the notation of Lemma 17 with $E = \mathfrak{g}_c$). Then

$$(\partial(p) - \chi(p))^{N+m} \circ D = (d_p + r_p - \chi(p))^{N+m} D$$
$$= \sum_{0 \leq k \leq N+m} C_k^{N+m} (r_p - \chi(p))^{N+m-k} d_p^k D,$$

where C_k^{N+m} stands for the usual binomial coefficient. Now consider

$$((r_p - \chi(p))^{N+m-k} d_p^k D) T = (d_p^k D)((\partial(p) - \chi(p))^{N+m-k} T).$$

If $k \geq N$, $d_p^k D = 0$ and if $k \leq N$, $(\partial(p) - \chi(p))^{N+m-k} T = 0$. Hence

$$(\partial(p) - \chi(p))^{N+m} (DT) = 0.$$

Choose $p_1, \ldots, p_l$ in $I(\mathfrak{g}_c)$ such that $I(\mathfrak{g}_c) = \mathbf{C}[p_1, \ldots, p_l]$. Then we can choose an integer $M \geq 0$ such that

$$(\partial(p_i) - \chi(p_i))^M DT = 0 \qquad (1 \leq i \leq l).$$

But this implies that the space of all distributions of the form $\partial(p)DT$ ($p \in I(\mathfrak{g}_c)$) has dimension at most M^l. This proves that Theorem 1 is applicable to DT.

It is obvious that $DT = DF$ on Ω'. Hence by applying Theorem 1 to DT we conclude that DF is locally summable on Ω and $DT = T_{DF}$.

§ 8. FURTHER PROPERTIES OF F

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let us use the notation of [3(*j*), § 4]. Define the analytic function g on $\mathfrak{h}' \cap \Omega$ by

$$g(H) = \pi(H) F(H) \qquad (H \in \mathfrak{h}' \cap \Omega).$$

Theorem 2. — *g can be extended to an analytic function on* $\mathfrak{h}'(R) \cap \Omega$.

Fix an element $H_0 \in \mathfrak{h}'(R) \cap \Omega$. It is enough to show that there exists an analytic function g_1 on an open neighborhood U of H_0 in $\mathfrak{h}'(R) \cap \Omega$ such that $g_1 = g$ on $U \cap \mathfrak{h}'$. First assume that H_0 is semiregular. Let β be the unique positive root of $\mathfrak{h}$ which vanishes at H_0. Then clearly β is imaginary. If β is compact, the required result follows immediately from Lemma 4. Hence we may assume that β is singular. Define $\mathfrak{a}$ and $\mathfrak{b}$ as in [3(*j*), § 7] corresponding to H_0. Then it follows from [3(*j*), Lemmas 12 and 13] that we may assume that $\mathfrak{h} = \mathfrak{b}$.

Let $\mathfrak{z}$ be the centralizer of H_0 in $\mathfrak{g}$. Define ζ and $\mathfrak{z}'$ as usual [3(*i*), §§ 2, 7]. We now use the notation of [3(*j*), § 7]. Let $\mathfrak{z}_0$ be the set of all $Z \in \mathfrak{z} \cap \Omega$ such that $|\zeta(Z)| > |\zeta(H_0)|/2$. Then $\mathfrak{z}_0$ is an open neighborhood of H_0 in $\mathfrak{z}'$ which is completely invariant (with respect to $\mathfrak{z}$). Now σ is the center of $\mathfrak{z}$. Fix an open and convex neighborhood σ_0 of H_0 in σ such that $\mathrm{Cl}\,\sigma_0$ is compact and contained in $\mathfrak{z}_0$. Let $\Omega_{\mathfrak{l}}$ denote the set of all $Z \in \mathfrak{l}$ such that $\mathrm{Cl}\,\sigma_0 + Z \subset \mathfrak{z}_0$. Then by Lemma 9, $\Omega_{\mathfrak{l}}$ is a completely invariant and open neighborhood of zero in $\mathfrak{l}$.

Now apply Lemma 17 of [3(*i*)] with $G_0 = G$ and put $T_{\mathfrak{z}} = |\zeta|^{1/2}\sigma_T$. Then it follows from Theorem 2 of [3(*i*)] that $\partial(u_{\mathfrak{z}})T_{\mathfrak{z}} = 0$ $(u \in \mathfrak{U})$. Put $\mathfrak{V} = I(\mathfrak{z}_c)\mathfrak{U}_{\mathfrak{z}}$ where $\mathfrak{U}_{\mathfrak{z}}$ is the image of $\mathfrak{U}$ in $I(\mathfrak{z}_c)$ under the mapping $p \to p_{\mathfrak{z}}$ $(p \in I(\mathfrak{g}_c))$. Then by [3(*i*), Lemma 19], $\dim (I(\mathfrak{z}_c)/\mathfrak{V}) < \infty$. Let $\omega_{\mathfrak{l}}$ denote the Killing form of $\mathfrak{l}$. Then, if we identify $\mathfrak{l}$ with its dual under $\omega_{\mathfrak{l}}$, we have $\omega_{\mathfrak{l}} \in I(\mathfrak{l}_c) \subset I(\mathfrak{z}_c)$. Hence we can choose complex numbers $c_1, \ldots, c_r$ such that

$$\sum_{0 \leq k \leq r} c_k \omega^{r-k} \in \mathfrak{V}$$

where $c_0 = 1$. This proves that

$$\sum_{0 \leq k \leq r} c_k \partial(\omega_{\mathfrak{l}})^{r-k} T_{\mathfrak{z}} = 0.$$

Now fix $\gamma \in C_c^\infty(\sigma_0)$ and let τ_γ denote the distribution

$$\tau_\gamma : f \to T_{\mathfrak{z}}(\gamma \times f) \qquad (f \in C_c^\infty(\Omega_{\mathfrak{l}}))$$

on $\Omega_{\mathfrak{l}}$. Obviously τ_γ is invariant under L and it is clear from the above relation that

$$\sum_{0 \leq k \leq r} c_k \partial(\omega_{\mathfrak{l}})^{r-k} \tau_\gamma = 0.$$

Since $I(\mathfrak{l}_c) = \mathbf{C}[\omega_{\mathfrak{l}}]$, Theorem 1 and Lemma 16 are both applicable to $(\mathfrak{l}, \Omega_{\mathfrak{l}}, \tau_\gamma)$ in place of $(\mathfrak{g}, \Omega, T)$. Let $\Omega'_{\mathfrak{l}}$ be the set of those points of $\Omega_{\mathfrak{l}}$ which are regular in $\mathfrak{l}$. Fix a Euclidean measure $d\mathfrak{l}$ on $\mathfrak{l}$. Then we can choose an analytic function φ_γ on $\Omega'_{\mathfrak{l}}$ which is locally summable on $\Omega_{\mathfrak{l}}$ and such that

$$\tau_\gamma(f) = \int f\varphi_\gamma d\mathfrak{l} \qquad (f \in C_c^\infty(\Omega_{\mathfrak{l}})).$$

Hence it follows from Lemma 16 that

$$\int \{ \partial(\omega_{\mathfrak{l}})^k f . \varphi_\gamma - f . \partial(\omega_{\mathfrak{l}})^k \varphi_\gamma \} d\mathfrak{l} = 0$$

for $k \geq 0$ and $f \in C_c^\infty(\Omega_{\mathfrak{l}})$. For any $\varepsilon > 0$, let $\Omega_{\mathfrak{l}}(\varepsilon)$ denote the set of all $Z \in \mathfrak{l}$ with $|\omega_{\mathfrak{l}}(Z)| < 8\varepsilon^2$. If ε is sufficiently small, it is obvious that tH' and $t(X'-Y')$ both lie in $\Omega_{\mathfrak{l}}$ whenever $|t| \leq \varepsilon$ $(t \in \mathbf{R})$. Since $\Omega_{\mathfrak{l}}$ is completely invariant under L, we can conclude (see [3(*e*), p. 681]) that $\mathrm{Cl}\,\Omega_{\mathfrak{l}}(\varepsilon) \subset \Omega_{\mathfrak{l}}$. It follows from Lemma 2 that there exist three analytic functions g_γ, g_γ^+, g_γ^- on $\mathbf{R}$ such that

$$g_\gamma(t) = t\varphi_\gamma(tH') \qquad (0 < t \leq \varepsilon)$$

$$g_\gamma^+(\theta) = \theta\varphi_\gamma(\theta(X'-Y')) \qquad (0 < \theta \leq \varepsilon)$$

$$g_\gamma^-(\theta) = \theta\varphi_\gamma(\theta(X'-Y')) \qquad (-\varepsilon \leq \theta < 0).$$

Now define the distributions T_k $(k \geq 0)$ on $\mathfrak{l}$ as in Corollary 1 of [3(*j*), Lemma 35] with (g, g^+, g^-) replaced by $(g_\gamma, g_\gamma^+, g_\gamma^-)$. Then it follows from [3(*e*), Lemma 16] and [3(*c*), Theorem 1] that

$$T_k(f) = c\int \{ \partial(\omega_{\mathfrak{l}})^k f . \varphi_\gamma - f\partial(\omega_{\mathfrak{l}})^k \varphi_\gamma \} d\mathfrak{l} = 0$$

for $f \in C_c^\infty(\Omega_I(\varepsilon/2))$. (Here c is a positive constant.) Therefore we conclude from the corollaries of [3(*j*), Lemma 35] that $g_Y^+ = g_Y^-$ and

$$(-1)^k (d^{2k+1} g_Y / dt^{2k+1})_0 = (d^{2k+1} g_Y^+ / d\theta^{2k+1})_0 \qquad (k \geq 0)$$

where the subscript 0 denotes the value at zero.

On the other hand let $F_{\mathfrak{z}}$ denote the restriction of F to $\mathfrak{z}_0$. Then by Corollary 2 of [3(*h*), Theorem 1], $F_{\mathfrak{z}}$ is locally summable on $\mathfrak{z}_0$ and since F is obviously invariant under G, we have

$$T(f_\alpha) = \int f_\alpha F dX = \int \beta_\alpha F_{\mathfrak{z}} dZ \qquad (\alpha \in C_c^\infty(\mathfrak{z}_0))$$

in the notation of [3(*i*), Lemma 17]. This proves that $\sigma_T = F_{\mathfrak{z}}$ and therefore $T_{\mathfrak{z}} = |\zeta|^{1/2} F_{\mathfrak{z}}$. Now $\mathfrak{a} = \sigma + \mathbf{R}H'$ and $\mathfrak{b} = \mathfrak{h} = \sigma + \mathbf{R}(X' - Y')$. Let τ and λ be the unique positive roots of $\mathfrak{a}$ and $\mathfrak{b}$ respectively which vanish at H_0. We may assume that $\tau(H') = 2$, $\lambda(X' - Y') = -2(-1)^{1/2}$ and the positive roots of $\mathfrak{a}$ go into positive roots of $\mathfrak{b}$ under the automorphism ν of [3(*j*), § 7]. Put $\pi_\tau^{\mathfrak{a}} = \tau^{-1}\pi^{\mathfrak{a}}$, $\pi_\lambda^{\mathfrak{b}} = \lambda^{-1}\pi^{\mathfrak{b}}$. Then it is clear that

$$|\zeta(H)|^{1/2} = |\pi_\tau^{\mathfrak{a}}(H)| \qquad (H \in \mathfrak{a}),$$

$$|\zeta(H)|^{1/2} = |\pi_\lambda^{\mathfrak{b}}(H)| \qquad (H \in \mathfrak{b}).$$

Let I denote the open interval $(-\varepsilon, \varepsilon)$ in $\mathbf{R}$. Put $\mathfrak{a}(\varepsilon) = \sigma_0 + IH'$ and $\mathfrak{b}(\varepsilon) = \sigma_0 + I(X' - Y')$. Then $\mathfrak{a}(\varepsilon)$ and $\mathfrak{b}(\varepsilon)$ are both connected sets. Since $\mathfrak{a}(\varepsilon) \subset \mathfrak{z}_0$, it is obvious that no positive root of $(\mathfrak{g}, \mathfrak{a})$ other than τ can vanish anywhere on $\mathfrak{a}(\varepsilon)$. Hence $|\pi_\tau^{\mathfrak{a}}(H)| / \pi_\tau^{\mathfrak{a}}(H)$ is a continuous function on $\mathfrak{a}(\varepsilon)$. But since its fourth power is 1 (see [3(*j*), Lemma 9]), it must be a constant. Put $c = |\pi_\tau^{\mathfrak{a}}(H_0)| / \pi_\tau^{\mathfrak{a}}(H_0)$. Since $\pi_\lambda^{\mathfrak{b}} = (\pi_\tau^{\mathfrak{a}})^\nu$ and H_0 remains fixed under ν, it is clear that

$$c = |\pi_\lambda^{\mathfrak{b}}(H_0)| / \pi_\lambda^{\mathfrak{b}}(H_0).$$

Hence we conclude by a similar argument that

$$|\pi_\lambda^{\mathfrak{b}}(H)| = c\pi_\lambda^{\mathfrak{b}}(H) \qquad (H \in \mathfrak{b}(\varepsilon)).$$

This shows that

$$t|\zeta(H + tH')|^{1/2} = 2^{-1} c \pi^{\mathfrak{a}}(H + tH') \qquad (|t| < \varepsilon)$$

$$\theta|\zeta(H + \theta(X' - Y'))|^{1/2} = 2^{-1}(-1)^{1/2} c \pi^{\mathfrak{b}}(H + \theta(X' - Y')) \qquad (|\theta| < \varepsilon)$$

for $H \in \sigma_0$. Now put

$$g^{\mathfrak{a}}(H) = \pi^{\mathfrak{a}}(H) F(H) \qquad (H \in \mathfrak{a}' \cap \Omega),$$

$$g^{\mathfrak{b}}(H) = \pi^{\mathfrak{b}}(H) F(H) \qquad (H \in \mathfrak{b}' \cap \Omega).$$

and fix a Euclidean measure $d\sigma$ on σ such that $d\sigma dI$ is equal to the Euclidean measure dZ on $\mathfrak{z}$ used above. Since $T_{\mathfrak{z}} = |\zeta|^{1/2} F_{\mathfrak{z}}$, it is obvious that

$$\varphi_Y(Y) = \int \gamma(H) |\zeta(H + Y)|^{1/2} F(H + Y) d\sigma \qquad (Y \in \Omega_I').$$

Hence

$$g_\gamma(t) = 2^{-1}c\int g^{\mathfrak{a}}(\mathrm{H}+t\mathrm{H}')\gamma(\mathrm{H})d\sigma \qquad (0<t<\varepsilon),$$

$$g_\gamma^+(\theta) = 2^{-1}(-1)^{1/2}c\int g^{\mathfrak{b}}(\mathrm{H}+\theta(\mathrm{X}'-\mathrm{Y}'))\gamma(\mathrm{H})d\sigma \qquad (0<\theta<\varepsilon),$$

$$g_\gamma^-(\theta) = 2^{-1}(-1)^{1/2}c\int g^{\mathfrak{b}}(\mathrm{H}+\theta(\mathrm{X}'-\mathrm{Y}'))\gamma(\mathrm{H})d\sigma \qquad (-\varepsilon<\theta<0).$$

On the other hand if J is the open interval $(0, \varepsilon)$ in $\mathbf{R}$, it is clear that $\sigma_0 \pm \mathrm{J}\mathrm{H}'$ are connected sets contained in $\mathfrak{a}' \cap \Omega$. Let $\mathfrak{a}^\pm$ denote the connected component of $\mathfrak{a}'\cap\Omega$ containing $\sigma_0\pm\mathrm{J}\mathrm{H}'$. Similarly let $\mathfrak{b}^\pm$ be the connected component of $\mathfrak{b}'\cap\Omega$ containing $\sigma_0\pm\mathrm{J}(\mathrm{X}'-\mathrm{Y}')$. Then by Lemmas 1 and 2, there exist analytic functions $g_\pm^{\mathfrak{a}}$ and $g_\pm^{\mathfrak{b}}$ on $\mathfrak{a}$ and $\mathfrak{b}$ respectively such that $g^{\mathfrak{a}} = g_+^{\mathfrak{a}}$ on $\mathfrak{a}^+$, $g^{\mathfrak{a}} = g_-^{\mathfrak{a}}$ on $\mathfrak{a}^-$, $g^{\mathfrak{b}} = g_+^{\mathfrak{b}}$ on $\mathfrak{b}^+$ and $g^{\mathfrak{b}} = g_-^{\mathfrak{b}}$ on $\mathfrak{b}^-$. It is then obvious that

$$g_\gamma(t) = 2^{-1}c\int g_+^{\mathfrak{a}}(\mathrm{H}+t\mathrm{H}')\gamma(\mathrm{H})d\sigma \qquad (t\in\mathbf{R}),$$

$$g_\gamma^\pm(\theta) = 2^{-1}(-1)^{1/2}c\int g_\pm^{\mathfrak{b}}(\mathrm{H}+\theta(\mathrm{X}'-\mathrm{Y}'))\gamma(\mathrm{H})d\sigma \qquad (\theta\in\mathbf{R}).$$

On the other hand we have seen above that $g_\gamma^+ = g_\gamma^-$ for every $\gamma\in \mathrm{C}_c^\infty(\sigma_0)$. Therefore it is clear that $g_+^{\mathfrak{b}} = g_-^{\mathfrak{b}}$. This shows that $g = g^{\mathfrak{b}} = g_+^{\mathfrak{b}}$ on $\mathfrak{b}(\varepsilon)\cap\mathfrak{b}'$. Since $\mathfrak{b}(\varepsilon)$ is a neighborhood of H_0 in $\mathfrak{b}$, our assertion is proved in this case.

Moreover since

$$(d^{2k+1}g_\gamma/dt^{2k+1})_0 = (-1)^k(d^{2k+1}g_\gamma^+/d\theta^{2k+1})_0 \qquad (k\geq 0)$$

and $\nu(\mathrm{H}') = (-1)^{1/2}(\mathrm{X}'-\mathrm{Y}')$, we find in the same way that

$$g_+^{\mathfrak{a}}(\mathrm{H};\,\partial(\mathrm{H}')^{2k+1}) = g_+^{\mathfrak{b}}(\mathrm{H};\,\partial(\nu(\mathrm{H}'))^{2k+1})$$

for $\mathrm{H}\in\sigma$.

We now use the notation of [3(j), § 8].

Lemma 18. — Let s_τ *be the Weyl reflexion in* $\mathfrak{a}$ *corresponding to* τ. *Then* $(g^{\mathfrak{a}})^{s_\tau} = -g^{\mathfrak{a}}$. *If* D *is an element in* $\mathfrak{D}(\mathfrak{a}_c)$ *such that* $\mathrm{D}^{s_\tau} = -\mathrm{D}$, *then* $\mathrm{D}g^{\mathfrak{a}}$ *can be extended to a continuous function on* $\mathfrak{a}(\varepsilon)$ *and* (1)

$$g^{\mathfrak{a}}(\mathrm{H};\mathrm{D}) = g^{\mathfrak{b}}(\mathrm{H};\mathrm{D}^\nu)$$

for $\mathrm{H}\in\sigma_0$.

Since τ is real we know from [3(j), Lemma 6] that $s_\tau\in\mathrm{W}_{\mathrm{G}}^{\mathfrak{a}}$. Therefore since F is invariant under G, it is obvious that $(g^{\mathfrak{a}})^{s_\tau} = -g^{\mathfrak{a}}$ and hence $(\mathrm{D}g^{\mathfrak{a}})^{s_\tau} = \mathrm{D}g^{\mathfrak{a}}$. This implies that $(\mathrm{D}g_+^{\mathfrak{a}})^{s_\tau} = \mathrm{D}g_-^{\mathfrak{a}}$ and therefore $\mathrm{D}g_+^{\mathfrak{a}} = \mathrm{D}g_-^{\mathfrak{a}}$ on σ. It is now clear that $\mathrm{D}g^{\mathfrak{a}}$ can be extended to a continuous function on $\mathfrak{a}(\varepsilon)$. So it remains to show that

$$g_+^{\mathfrak{a}}(\mathrm{H};\mathrm{D}) = g^{\mathfrak{b}}(\mathrm{H};\mathrm{D}^\nu)$$

for $\mathrm{H}\in\sigma_0$. Since $\mathfrak{D}(\mathfrak{a}_c) = \mathfrak{D}(\sigma_c)\mathfrak{D}(\mathbf{C}\mathrm{H}')$ and since s_τ leaves σ pointwise fixed, it is sufficient to consider the case when $\mathrm{D} = \Delta\circ\tau^i\partial(\mathrm{H}')^j$. Here $\Delta\in\mathfrak{D}(\sigma_c)$ and $i+j$ is odd.

(1) $g^{\mathfrak{a}}(\mathrm{H};\mathrm{D})$ denotes, as usual, the value of the continuous function $\mathrm{D}g^{\mathfrak{a}}$ at H. Similarly in other cases.

Now Δ and τ commute. Therefore, if $i \geq 1$, our assertion is obvious from the fact that τ and λ are both zero on σ. So we may assume that $i = 0$ so that j is odd. It is enough to verify that

$$g_+^{\mathfrak{a}}(H;\, \partial(H')^j) = g^{\mathfrak{b}}(H;\, \partial(\nu(H'))^j) \qquad (H \in \sigma_0)$$

since the required relation would then follow by applying the differential operator Δ to this equation. However $g^{\mathfrak{b}} = g_+^{\mathfrak{b}}$ on $\mathfrak{b}(\varepsilon)$ and so this follows from the result proved above.

Now we return to the proof of Theorem 2. Fix a point $H_0 \in \mathfrak{h}'(R) \cap \Omega$ and an open convex neighborhood U of H_0 in $\mathfrak{h}'(R) \cap \Omega$. Let U_1 be the set consisting of all regular and semiregular elements of U. Then U_1 is open, and if β is a root of $(\mathfrak{g}, \mathfrak{h})$ which vanishes at some point of U_1, it is clear that β is imaginary. Hence it follows from the above proof that there exists an analytic function g_1 on U_1 such that $g_1 = g$ on $U_1 \cap \mathfrak{h}'$. Now fix a connected component U_2 of $U_1 \cap \mathfrak{h}' = U \cap \mathfrak{h}'$. Then by Lemma 2 there exists an analytic function g_2 on $\mathfrak{h}$ such that $g = g_2$ on U_2. Since U_1 is connected (see the corollary of [3(*j*), Lemma 19]), we conclude that $g_1 = g_2$ and therefore $g = g_2$ on $U \cap \mathfrak{h}'$. Since g_2 is analytic on $\mathfrak{h}$, we have shown that g can be extended to an analytic function on U. Thus Theorem 2 is proved.

We denote the extended analytic function on $\mathfrak{h}'(R) \cap \Omega$ again by g.

Lemma 19 (1). — *Let* H_0 *be a point in* $\mathfrak{h} \cap \Omega$ *and* D *an element in* $\mathfrak{D}(\mathfrak{h}_c)$ *such that* $D^{s_\alpha} = -D$ *for every real root* α *of* $(\mathfrak{g}, \mathfrak{h})$ *which vanishes at* H_0. *Then* Dg *can be extended to a continuous function around* H_0.

Fix an open, convex and relatively compact neighborhood U of H_0 in $\Omega \cap \mathfrak{h}$. By taking it sufficiently small we can arrange that no real root α of $(\mathfrak{g}, \mathfrak{h})$ vanishes anywhere on U unless $\alpha(H_0) = 0$. Let U_0 be the set consisting of all regular and semiregular elements of U. Then, as before, U_0 is open and connected and it follows from Theorem 2 and Lemma 18 that there exists a continuous function g_0 on U_0 such that $Dg = g_0$ on $U_0 \cap \mathfrak{h}'(R)$. The set $U \cap \mathfrak{h}'$ has only a finite number of connected components, say $U_1, \ldots, U_r$. By Lemma 2 we can choose an analytic function g_i on $\mathfrak{h}$ such that $g = g_i$ on U_i $(1 \leq i \leq r)$. This shows that Dg is of class C^∞ on $\mathrm{Cl}\,U_i$ (see [3(*j*), § 14]). But $\mathrm{Cl}\,U_i = \mathrm{Cl}(U_i \cap U_0)$ and $Dg = g_0$ on $U_i \cap U_0$. Therefore g_0 is also of class C^∞ on $\mathrm{Cl}\,U_i$ $(1 \leq i \leq r)$. Fix a Euclidean norm on $\mathfrak{h}$ and put

$$\nu(g_0) = \sup |g_0(H_1;\, \partial(H_2))|$$

where H_1, H_2 vary in $U \cap \mathfrak{h}'$ and $\mathfrak{h}$ respectively under the sole restriction that $\|H_2\| \leq 1$. Then it is obvious from what we have said above that $\nu(g_0) < \infty$. Moreover (see [3(*j*), § 10])

$$|g_0(H_1) - g_0(H_2)| \leq \nu(g_0)\, \|H_1 - H_2\|$$

for any two points H_1, H_2 in $U \cap \mathfrak{h}'$. Obviously this means that Dg can be extended to a continuous function on U.

(1) Cf. [3(*j*), Theorem 1].

Corollary. — *Let* D *be an element of* $\mathfrak{D}(\mathfrak{h}_c)$ *such that* $D^{s_\alpha}=-D$ *for every real root* α *of* $\mathfrak{h}$. *Then* Dg *can be extended to a continuous function on* $\mathfrak{h}\cap\Omega$.

This is obvious from the above lemma. We denote the extended function again by Dg. Moreover $g(H; D)$ $(H\in\mathfrak{h}\cap\Omega)$ will stand for the value of Dg at H.

Put $\varpi=\prod_{\alpha>0} H_\alpha$ where α runs over all positive roots of $(\mathfrak{g}, \mathfrak{h})$. Then $\varpi\in S(\mathfrak{h}_c)$ and $\varpi^{s_\alpha}=-\varpi$ for every root α. Hence $\partial(\varpi)g$ is a continuous function on $\mathfrak{h}\cap\Omega$. Since the differential operator $\partial(\varpi)\circ\pi$ is obviously independent of the choice of positive roots of $\mathfrak{h}$, it is clear that the function $\partial(\varpi)g$ also does not depend on this choice. Corresponding to any Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, we define $\varpi^{\mathfrak{a}}$, $g^{\mathfrak{a}}$ and $\partial(\varpi^{\mathfrak{a}})g^{\mathfrak{a}}$ in an analogous way.

Theorem 3. — *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two Cartan subalgebras of* $\mathfrak{g}$. *Then*

$$\partial(\varpi^{\mathfrak{a}})g^{\mathfrak{a}}=\partial(\varpi^{\mathfrak{b}})g^{\mathfrak{b}}$$

on $\mathfrak{a}\cap\mathfrak{b}\cap\Omega$.

Before giving the proof we derive a consequence of this theorem. Let g_T denote the function g of Theorem 2 corresponding to the distribution T. For any $D\in\mathfrak{J}(\mathfrak{g}_c)$, DT also fulfills the conditions of Theorem 1 (Lemma 16). Hence we can consider the corresponding function g_{DT}. It follows from [3(*i*), Theorem 1] that $g_{DT}=\delta_{\mathfrak{g}/\mathfrak{h}}(D)g_T$. Therefore

$$\partial(\varpi)g_{DT}=(\partial(\varpi)\circ\delta_{\mathfrak{g}/\mathfrak{h}}(D))g$$

can also be extended to a continuous function on $\mathfrak{h}\cap\Omega$.

Corollary. — $(\partial(\varpi^{\mathfrak{a}})\circ\delta_{\mathfrak{g}/\mathfrak{a}}(D))g^{\mathfrak{a}}=(\partial(\varpi^{\mathfrak{b}})\circ\delta_{\mathfrak{g}/\mathfrak{b}}(D))g^{\mathfrak{b}}$ *on* $\mathfrak{a}\cap\mathfrak{b}\cap\Omega$ *for any* $D\in\mathfrak{J}(\mathfrak{g}_c)$.

This follows by applying Theorem 3 to DT instead of T.

We shall prove Theorem 3 by induction on dim $\mathfrak{g}$. Fix a point $H_0\in\mathfrak{a}\cap\mathfrak{b}\cap\Omega$. We have to show that $g^{\mathfrak{a}}(H_0; \partial(\varpi^{\mathfrak{a}}))=g^{\mathfrak{b}}(H_0; \partial(\varpi^{\mathfrak{b}}))$. Let $\mathfrak{c}$ be the center and $\mathfrak{g}_1$ the derived algebra of $\mathfrak{g}$ and first suppose that $\mathfrak{c}\neq\{0\}$. Let $H_0=C_0+H_1$ where $C_0\in\mathfrak{c}$ and $H_1\in\mathfrak{g}_1$. Then it is clear that $H_1\in\mathfrak{a}_1\cap\mathfrak{b}_1$ where $\mathfrak{h}_1=\mathfrak{h}\cap\mathfrak{g}_1$ ($\mathfrak{h}=\mathfrak{a}$ or $\mathfrak{b}$). Choose an open and relatively compact neighborhood $\mathfrak{c}_0$ of C_0 in $\mathfrak{c}$ and let Ω_1 be the set of all $Z\in\mathfrak{g}_1$ such that $\mathrm{Cl}\,\mathfrak{c}_0+Z\subset\Omega$. Then (Lemma 9) Ω_1 is an open and completely invariant neighborhood of H_1 in $\mathfrak{g}_1$, if c_0 is sufficiently small. Fix $\alpha\in C_c^\infty(\mathfrak{c}_0)$ and consider the distribution

$$\tau_\alpha: f\to T(\alpha\times f) \qquad (f\in C_c^\infty(\Omega_1))$$

on Ω_1. Put $\mathfrak{U}_1=\mathfrak{U}\cap I(\mathfrak{g}_{1c})$. Then it is clear that

$$\dim(I(\mathfrak{g}_{1c})/\mathfrak{U}_1)\le\dim(I(\mathfrak{g}_c)/\mathfrak{U})<\infty$$

and $\partial(u_1)\tau_\alpha=0$ for $u_1\in\mathfrak{U}_1$. Hence Theorem 1 also holds if we replace $(\mathfrak{g}, \Omega, T)$ by $(\mathfrak{g}_1, \Omega_1, \tau_\alpha)$. Since $\dim\mathfrak{g}_1<\dim\mathfrak{g}$, Theorem 3 applies to τ_α by the induction hypothesis. Put

$$g_\alpha^{\mathfrak{h}}(H)=\int\alpha(C)g^{\mathfrak{h}}(C+H)dC \qquad (H\in\mathfrak{h}'\cap\Omega_1)$$

where dC is a Euclidean measure on $\mathfrak{c}$. Then we conclude that

$$g_\alpha^{\mathfrak{a}}(H; \partial(\varpi^{\mathfrak{a}})) = g_\alpha^{\mathfrak{b}}(H; \partial(\varpi^{\mathfrak{b}}))$$

for $H \in \mathfrak{a} \cap \mathfrak{b} \cap \Omega_1$. Since this is true for every $\alpha \in C_c^\infty(\mathfrak{c}_0)$, it is clear that $\partial(\varpi^{\mathfrak{a}})g^{\mathfrak{a}}$ and $\partial(\varpi^{\mathfrak{b}})g^{\mathfrak{b}}$ coincide around H_0 on $\mathfrak{a} \cap \mathfrak{b} \cap \Omega$.

So now we can assume that $\mathfrak{c} = \{0\}$ and therefore $\mathfrak{g}$ is semisimple. Then we identify $\mathfrak{g}$ and $\mathfrak{h}$ with their respective duals by means of the Killing form (see [3(*i*), § 6]) so that $\varpi^{\mathfrak{h}} = \pi^{\mathfrak{h}}$ ($\mathfrak{h} = \mathfrak{a}$ or $\mathfrak{b}$). First assume that $H_0 \neq 0$ and let $\mathfrak{z}$ be the centralizer of H_0 in $\mathfrak{g}$. Then $\dim \mathfrak{z} < \dim \mathfrak{g}$ and we can identify $\mathfrak{z}$ with its dual by means of the restriction (to $\mathfrak{z}$) of the Killing form of $\mathfrak{g}$. Define ζ and $\mathfrak{z}'$ as in [3(*i*), § 2] and put $\Omega_{\mathfrak{z}} = \mathfrak{z}' \cap \Omega$. Then $\Omega_{\mathfrak{z}}$ is an open neighborhood of H_0 in $\mathfrak{z}$ which is completely invariant (with respect to $\mathfrak{z}$). Take $G_0 = G$ and $\mathfrak{z}_0 = \Omega_{\mathfrak{z}}$ in Lemma 17 of [3(*i*)] and let σ_T denote the corresponding distribution on $\Omega_{\mathfrak{z}}$. Then

$$\sigma_T(\beta_\alpha) = T(f_\alpha) = \int f_\alpha F\,dX \qquad (\alpha \in C_c^\infty(G \times \Omega_{\mathfrak{z}})).$$

But since F is invariant under G, we conclude from Corollary 2 of [3(*h*), Theorem 1] that the function $F_{\mathfrak{z}} : Z \to F(Z)$ $(Z \in \Omega_{\mathfrak{z}})$ is locally summable on $\Omega_{\mathfrak{z}}$ and

$$\int f_\alpha F\,dX = \int \beta_\alpha F_{\mathfrak{z}}\,dZ.$$

This shows that $\sigma_T = F_{\mathfrak{z}}$.

Let $\mathfrak{h} = \mathfrak{a}$ or $\mathfrak{b}$. Then $\mathfrak{h} \subset \mathfrak{z}$. Define $\mathfrak{q}$ as in [3(*i*), § 2]. $P^{\mathfrak{h}}$ being the set of all positive roots of $(\mathfrak{g}, \mathfrak{h})$, let $P_{\mathfrak{z}}^{\mathfrak{h}}$ and $P_{\mathfrak{q}}^{\mathfrak{h}}$ denote the subsets of those $\alpha \in P^{\mathfrak{h}}$ for which X_α lies in $\mathfrak{z}_c$ and $\mathfrak{q}_c$ respectively. Let $\pi_{\mathfrak{z}}^{\mathfrak{h}}$ and $\pi_{\mathfrak{q}}^{\mathfrak{h}}$ be the products of all roots in $P_{\mathfrak{z}}^{\mathfrak{h}}$ and $P_{\mathfrak{q}}^{\mathfrak{h}}$ respectively. Then $\pi^{\mathfrak{h}} = \pi_{\mathfrak{z}}^{\mathfrak{h}} \pi_{\mathfrak{q}}^{\mathfrak{h}}$ and it is clear that $(\pi_{\mathfrak{q}}^{\mathfrak{h}})^{s_\alpha} = \pi_{\mathfrak{q}}^{\mathfrak{h}}$ for all $\alpha \in P_{\mathfrak{z}}^{\mathfrak{h}}$. Hence, by Chevalley's theorem [3(*c*), Lemma 9], there exists an invariant polynomial function p on $\mathfrak{z}$ such that $p(H) = \pi_{\mathfrak{q}}^{\mathfrak{a}}(H)$ for $H \in \mathfrak{a}$. But

$$\zeta(H) = (-1)^q (\pi_{\mathfrak{q}}^{\mathfrak{a}}(H))^2 \qquad (H \in \mathfrak{a})$$

where $q = 2^{-1} \dim \mathfrak{q}$ is the number of roots in $P_{\mathfrak{q}}^{\mathfrak{a}}$. Therefore $\zeta = (-1)^q p^2$ again by Chevalley's theorem. Let $p_{\mathfrak{h}}$ denote the restriction of p to $\mathfrak{h}$. Then since ζ coincides with $(-1)^q (\pi_{\mathfrak{q}}^{\mathfrak{b}})^2$ on $\mathfrak{b}$, it is clear that $p_{\mathfrak{b}} = \varepsilon \pi_{\mathfrak{q}}^{\mathfrak{b}}$ where $\varepsilon = \pm 1$.

Now put $T_{\mathfrak{z}} = p\sigma_T$. Then it follows from Theorem 2 of [3(*i*)] (see also § 4) that Theorem 1 still holds if we replace $(\mathfrak{g}, \Omega, T)$ by $(\mathfrak{z}, \Omega_{\mathfrak{z}}, T_{\mathfrak{z}})$. Put

$$g_{\mathfrak{z}}^{\mathfrak{h}}(H) = \pi_{\mathfrak{z}}^{\mathfrak{h}}(H) p(H) F(H) \qquad (H \in \mathfrak{h}' \cap \Omega_{\mathfrak{z}}).$$

Since $\dim \mathfrak{z} < \dim \mathfrak{g}$, both Theorem 3 and its corollary are applicable to $T_{\mathfrak{z}}$. Moreover $\delta_{\mathfrak{z}/\mathfrak{h}}(\partial(p)) = \partial(p_{\mathfrak{h}})$ [3(*c*), Theorem 1] and so we conclude that

$$\partial(\pi_{\mathfrak{z}}^{\mathfrak{a}} p_{\mathfrak{a}}) g_{\mathfrak{z}}^{\mathfrak{a}} = \partial(\pi_{\mathfrak{z}}^{\mathfrak{b}} p_{\mathfrak{b}}) g_{\mathfrak{z}}^{\mathfrak{b}}$$

on $\Omega_{\mathfrak{z}} \cap \mathfrak{a} \cap \mathfrak{b}$. However $\pi_{\mathfrak{z}}^{\mathfrak{a}} p_{\mathfrak{a}} = \pi^{\mathfrak{a}}$ and $\pi_{\mathfrak{z}}^{\mathfrak{b}} p_{\mathfrak{b}} = \varepsilon \pi^{\mathfrak{b}}$. Therefore $g_{\mathfrak{z}}^{\mathfrak{a}} = g^{\mathfrak{a}}$ and $g_{\mathfrak{z}}^{\mathfrak{b}} = \varepsilon g^{\mathfrak{b}}$ on $\Omega_{\mathfrak{z}} \cap \mathfrak{a}'$ and $\Omega_{\mathfrak{z}} \cap \mathfrak{b}'$ respectively. So it follows that $\partial(\pi^{\mathfrak{a}}) g^{\mathfrak{a}} = \partial(\pi^{\mathfrak{b}}) g^{\mathfrak{b}}$ on $\Omega_{\mathfrak{z}} \cap \mathfrak{a} \cap \mathfrak{b}$. Since $H_0 \in \Omega_{\mathfrak{z}} \cap \mathfrak{a} \cap \mathfrak{b}$, our assertion is proved in this case.

So it remains to consider the case when $H_0 = 0$. Hence we may assume that $0 \in \Omega$. For any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, put $c(\mathfrak{h}) = g^{\mathfrak{h}}(0; \partial(\pi^{\mathfrak{h}}))$.

Lemma 20. — *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two Cartan subalgebras of* $\mathfrak{g}$. *Then, if* $\mathfrak{a} \cap \mathfrak{b} \neq \{0\}$, $c(\mathfrak{a}) = c(\mathfrak{b})$.

Since Ω is an open neighborhood of zero in $\mathfrak{g}$, we can choose $H \neq 0$ in $\mathfrak{a} \cap \mathfrak{b}$ such that $tH \in \mathfrak{a} \cap \mathfrak{b} \cap \Omega$ for $0 \leq t \leq 1$. Then in view of what we have proved above, it is clear that

$$g^{\mathfrak{a}}(tH; \partial(\pi^{\mathfrak{a}})) = g^{\mathfrak{b}}(tH; \partial(\pi^{\mathfrak{b}})) \qquad (0 < t \leq 1).$$

Making t tend to zero we get $c(\mathfrak{a}) = c(\mathfrak{b})$.

Lemma 21. — *Let* $\mathfrak{h}$ *be a Cartan subalgebra of* $\mathfrak{g}$. *Then* $c(\mathfrak{h}) = c(\mathfrak{h}^x)$ *for any* x *in* G.

Let $\mathfrak{a} = \mathfrak{h}^x$. Without loss of generality we may assume that $\pi^{\mathfrak{a}} = (\pi^{\mathfrak{h}})^x$. Then it is clear that

$$g^{\mathfrak{a}}(H^x) = g^{\mathfrak{h}}(H) \qquad (H \in \Omega \cap \mathfrak{h}')$$

and therefore

$$g^{\mathfrak{a}}(H^x; \partial(\pi^{\mathfrak{a}})) = g^{\mathfrak{h}}(H; \partial(\pi^{\mathfrak{h}}))$$

for $H \in \Omega \cap \mathfrak{h}'$. We obtain the required result by making H tend to zero.

Fix a Cartan involution θ of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ which is stable under θ, we put

$$l_+(\mathfrak{h}) = \dim(\mathfrak{h} \cap \mathfrak{p}), \qquad l_-(\mathfrak{h}) = \dim(\mathfrak{h} \cap \mathfrak{k}).$$

Then $l_+(\mathfrak{h}) + l_-(\mathfrak{h}) = \dim \mathfrak{h} = l$ where $l = \operatorname{rank} \mathfrak{g}$. Let $l_+ = \sup_{\mathfrak{h}} l_+(\mathfrak{h})$, $l_- = \sup_{\mathfrak{h}} l_-(\mathfrak{h})$ where $\mathfrak{h}$ runs over all Cartan subalgebras stable under θ. Fix two Cartan subalgebras $\mathfrak{h}_+$, $\mathfrak{h}_-$, both stable under θ, such that $l_+ = l_+(\mathfrak{h}_+)$, $l_- = l_-(\mathfrak{h}_-)$.

Lemma 22. — *Let* $\mathfrak{h}$ *be a Cartan subalgebra of* $\mathfrak{g}$ *which is stable under* θ. *Then* $c(\mathfrak{h}) = c(\mathfrak{h}_+)$ *if* $l_+(\mathfrak{h}) > 0$ *and* $c(\mathfrak{h}) = c(\mathfrak{h}_-)$ *if* $l_-(\mathfrak{h}) > 0$.

Let K be the analytic subgroup of G corresponding to $\mathfrak{k}$. Then $\mathfrak{h}_+ \cap \mathfrak{p}$ and $\mathfrak{h}_- \cap \mathfrak{k}$ are maximal abelian subspaces of $\mathfrak{p}$ and $\mathfrak{k}$ respectively. Since any two maximal abelian subspaces of $\mathfrak{p}$ (or $\mathfrak{k}$) are conjugate under K, it is clear that we can choose $k_1, k_2 \in K$ such that

$$(\mathfrak{h} \cap \mathfrak{p})^{k_1} \subset \mathfrak{h}_+ \cap \mathfrak{p}, \qquad (\mathfrak{h} \cap \mathfrak{k})^{k_2} \subset \mathfrak{h}_- \cap \mathfrak{k}.$$

Then

$$\dim(\mathfrak{h}^{k_1} \cap \mathfrak{h}_+) \geq \dim(\mathfrak{h} \cap \mathfrak{p})^{k_1} = l_+(\mathfrak{h})$$

and similarly

$$\dim(\mathfrak{h}^{k_2} \cap \mathfrak{h}_-) \geq l_-(\mathfrak{h}).$$

Hence our assertion follows from Lemmas 20 and 21.

Lemma 23. — $c(\mathfrak{h}_+) = c(\mathfrak{h}_-)$.

We may obviously assume that $l \geq 1$. If $l_-(\mathfrak{h}_+) + l_+(\mathfrak{h}_-) \geq 1$, our statement follows from Lemma 22. Hence we may assume that $\mathfrak{h}_+ \subset \mathfrak{p}$ and $\mathfrak{h}_- \subset \mathfrak{k}$. Then $\mathfrak{h}_+$ is not fundamental in $\mathfrak{g}$ and so there exists a positive real root α of $(\mathfrak{g}, \mathfrak{h}_+)$ (see [3(*d*), Lemma 33]). We assume, as we may (see [3(*d*), Lemma 46]), that $\theta(X_\alpha)=-X_{-\alpha}$ and X_α, $X_{-\alpha}$ are in $\mathfrak{g}$. Take $H'=a^{-2}H_\alpha$, $X'=a^{-1}X_\alpha$, $Y'=a^{-1}X_{-\alpha}$ where $a=(\alpha(H_\alpha)/2)^{1/2}$. Define the automorphism ν of $\mathfrak{g}_c$ as in [3(*j*), § 7]. Then $\theta(X')=-Y'$ and $\mathfrak{b}=\nu((\mathfrak{h}_+)_c)\cap\mathfrak{g}=\sigma_\alpha+\mathbf{R}(X'-Y')$ is a Cartan subalgebra of $\mathfrak{g}$ which is stable under θ. Here σ_α is the hyperplane consisting of all points $H\in\mathfrak{h}_+$ where $\alpha(H)=0$. Since $\mathfrak{b}\cap\mathfrak{p}=\sigma_\alpha$ and $\mathfrak{b}\cap\mathfrak{k}=\mathbf{R}(X'-Y')$, it is obvious that $l_+(\mathfrak{b})=l-1$ and $l_-(\mathfrak{b})=1$. Hence, if $l\geq 2$, it follows from Lemma 22 that $c(\mathfrak{h}_+)=c(\mathfrak{b})=c(\mathfrak{h}_-)$. On the other hand if $l=1$, zero is a semiregular element of $\mathfrak{g}$ and our assertion follows immediately from Lemmas 18 and 21.

We shall now finish the proof of Theorem 3. Choose x, y in G such that $\mathfrak{a}^x$ and $\mathfrak{b}^y$ are stable under θ (see [3(*b*), p. 100]). Then it is clear from Lemmas 21, 22 and 23 that $c(\mathfrak{a})=c(\mathfrak{b})$. The proof of Theorem 3 is now complete.

§ 9. THE DIFFERENTIAL OPERATOR $\nabla_\mathfrak{g}$ AND THE FUNCTION $\nabla_\mathfrak{g}F$

Lemma 24. — There exists a unique differential operator $\nabla_\mathfrak{g}$ on $\mathfrak{g}'$ with the following two properties:

1) *$\nabla_\mathfrak{g}$ is invariant under* G.

2) *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then*

$$f(H;\nabla_\mathfrak{g})=f(H;\partial(\varpi^\mathfrak{h})\circ\pi^\mathfrak{h})$$

for $f\in C^\infty(\mathfrak{g})$ and $H\in\mathfrak{h}'$.
Moreover $\nabla_\mathfrak{g}$ is analytic.

Since two distinct Cartan subalgebras cannot have a regular element in common, the uniqueness is obvious. The existence is proved as follows. Fix a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ and define $\mathfrak{g}_\mathfrak{a}=(\mathfrak{a}')^G$. Then $\mathfrak{g}_\mathfrak{a}$ is an open subset of $\mathfrak{g}$. Let A be the Cartan subgroup of G corresponding to $\mathfrak{a}$ and $x\to x^*$ the natural projection of G on $G^*=G/A$. Then the mapping $\varphi:(x^*,H)\to x^*H$ of $G^*\times\mathfrak{a}'$ onto $\mathfrak{g}_\mathfrak{a}$ (in the notation of § 2) is everywhere regular. Define $W_G=\widetilde{A}/A$ where $\widetilde{A}$ is the normalizer of $\mathfrak{a}$ in G. Then W_G operates on G^* and $\mathfrak{a}$ as follows. Fix $s\in W_G$ and choose $y\in\widetilde{A}$ lying in the coset s. Then

$$sH=H^y,\qquad x^*s=(xy)^*$$

for $H\in\mathfrak{a}$ and $x\in G$. It is clear that the complete inverse image under φ of a point $x^*H\in\mathfrak{g}_\mathfrak{a}$ $(x^*\in G^*, H\in\mathfrak{a}')$ consists of the elements $(x^*s, s^{-1}H)$ $(s\in W_G)$, which are all distinct. Since φ is locally an analytic diffeomorphism and since $\partial(\varpi^\mathfrak{a})\circ\pi^\mathfrak{a}$ is obviously invariant under W_G, it is clear that there exists an analytic differential operator ∇ on $\mathfrak{g}_\mathfrak{a}$ such that

$$f(x^*H;\nabla)=f(x^*:H;\partial(\varpi^\mathfrak{a})\circ\pi^\mathfrak{a})\qquad (x^*\in G^*, H\in\mathfrak{a}')$$

for $f\in C^\infty(\mathfrak{g}_\mathfrak{a})$. Here $f(x^*:H)=f(x^*H)$ as usual. It is easy to verify that ∇ satisfies the two conditions of the lemma on $\mathfrak{g}_\mathfrak{a}$.

Now select a maximal set $\mathfrak{h}_1, \ldots, \mathfrak{h}_r$ of Cartan subalgebras of $\mathfrak{g}$, no two of which are conjugate under G. Put $\mathfrak{g}_i = (\mathfrak{h}_i')^G$ and define a differential operator ∇_i on $\mathfrak{g}_i$ as above corresponding to $\mathfrak{a} = \mathfrak{h}_i$. Since $\mathfrak{g}'$ is the disjoint union of the open sets $\mathfrak{g}_1, \ldots, \mathfrak{g}_r$, we can define $\nabla_\mathfrak{g}$ by setting $\nabla_\mathfrak{g} = \nabla_i$ on $\mathfrak{g}_i$ $(1 \leq i \leq r)$.

Lemma 25. — For any $D \in \mathfrak{I}(\mathfrak{g}_c)$, $(\nabla_\mathfrak{g} \circ D)F$ *can be extended to a continuous function on* Ω.

We shall use induction on dim $\mathfrak{g}$. In view of Lemma 16, it is enough to consider the case when $D = 1$. Define $\mathfrak{c}$ and $\mathfrak{g}_1$ as in § 4 and first assume that $\mathfrak{c} \neq \{0\}$. Fix a point $X_0 = C_0 + Z_0$ $(C_0 \in \mathfrak{c}, Z_0 \in \mathfrak{g}_1)$ in Ω. We have to show that $\nabla_\mathfrak{g} F$ can be extended to a continuous function around X_0. Select an open, connected and relatively compact neighborhood $\mathfrak{c}_0$ of C_0 in $\mathfrak{c}$ such that $(\mathrm{Cl}\,\mathfrak{c}_0) + Z_0 \subset \Omega$. Define Ω_1 to be the set of all $Z \in \mathfrak{g}_1$ such that $\mathrm{Cl}\,\mathfrak{c}_0 + Z \subset \Omega$. Then, by Lemma 9, Ω_1 is also open and completely invariant in $\mathfrak{g}_1$. Since $S(\mathfrak{c}_c) \subset I(\mathfrak{g}_c)$, it is clear that

$$\dim(S(\mathfrak{c}_c)/\mathfrak{U} \cap S(\mathfrak{c}_c)) \leq \dim(I(\mathfrak{g}_c)/\mathfrak{U}) < \infty.$$

Let E be the space of all analytic functions χ on $\mathfrak{c}_0$ such that $\partial(u)\chi = 0$ for $u \in \mathfrak{U} \cap S(\mathfrak{c}_c)$. Then (see the proof of Lemma 13 of [3(*c*)]) $\dim E < \infty$. Let χ_j $(1 \leq j \leq N)$ be a base for E over **C**. Fix $Z \in \Omega_1' = \Omega_1 \cap \mathfrak{g}'$. Then it is obvious that $F(Z + C; \partial(u)) = 0$ for $u \in \mathfrak{U} \cap S(\mathfrak{c}_c)$ and $C \in \mathfrak{c}_0$. Therefore

$$F(C+Z) = \sum_{1 \leq j \leq N} \chi_j(C) F_j(Z) \qquad (C \in \mathfrak{c}_0)$$

where $F_j(Z) \in \mathbf{C}$. Since F is analytic on Ω', it is obvious that F_j $(1 \leq j \leq N)$ are analytic functions on Ω_1'.

Fix Euclidean measures dC and dZ on $\mathfrak{c}$ and $\mathfrak{g}_1$ respectively such that $dX = dC\,dZ$ for $X = C + Z$ $(C \in \mathfrak{c}, Z \in \mathfrak{g}_1)$ and for any $\alpha \in C_c^\infty(\mathfrak{c}_0)$ define the distribution θ_α on Ω_1 by

$$\theta_\alpha(\beta) = T(\alpha \times \beta) \qquad (\beta \in C_c^\infty(\Omega_1)).$$

Then, as we have seen in § 4, the induction hypothesis is applicable to $(\mathfrak{g}_1, \theta_\alpha, \Omega_1)$ in place of $(\mathfrak{g}, T, \Omega)$. Put

$$F_\alpha(Z) = \sum_{1 \leq j \leq N} F_j(Z) \int \chi_j(C)\alpha(C)\,dC \qquad (Z \in \Omega_1').$$

Then $\nabla_{\mathfrak{g}_1} F_\alpha$ can be extended to a continuous function on Ω_1. Since this is true for every $\alpha \in C_c^\infty(\mathfrak{c}_0)$, the same holds for $\nabla_{\mathfrak{g}_1} F_j$, $1 \leq j \leq N$ (see [3(*e*), Lemma 20]). But it is obvious that

$$F(C+Z; \nabla_\mathfrak{g}) = \sum_{1 \leq j \leq N} \chi_j(C) F_j(Z; \nabla_{\mathfrak{g}_1})$$

for $C \in \mathfrak{c}_0$ and $Z \in \Omega_1'$. Hence $\nabla_\mathfrak{g} F$ extends to a continuous function on $\mathfrak{c}_0 + \Omega_1$, which proves our assertion.

So now we may assume that $\mathfrak{g}$ is semisimple. Let Ω^0 be the set of all points $X_0 \in \Omega$ such that $\nabla_\mathfrak{g} F$ can be extended to a continuous function around X_0. Clearly Ω^0 is an open and invariant subset of Ω. Therefore, in view of Corollary 2 of Lemma 8, it would be enough to show that every semisimple element of Ω lies in Ω^0.

Fix a semisimple element $H_0 \in \Omega$. First assume that $H_0 \neq 0$. Let $\mathfrak{z}$ be the centralizer of H_0 in $\mathfrak{g}$. Define $\mathfrak{q}$ and ζ as in [3(*i*), § 2]. Then as we have seen during the proof of Theorem 3, there exists an invariant polynomial function p on $\mathfrak{z}$ such that $\zeta = (-1)^q p^2$ where $q = (\dim \mathfrak{q})/2$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{z}$. We identify $\mathfrak{g}$, $\mathfrak{z}$, $\mathfrak{h}$ with their respective duals by means of the Killing form of $\mathfrak{g}$. Define $\pi_\mathfrak{z}$ and $\pi_\mathfrak{q}$ as in the proof of Theorem 3. Since $\zeta = (-1)^q \pi_\mathfrak{q}^2$ on $\mathfrak{h}$, it follows from [3(*c*), Theorem 1] that

$$\delta'_{\mathfrak{z}/\mathfrak{h}}(\partial(p)\circ p) = \pi_\mathfrak{z}^{-1}\partial(\pi_\mathfrak{q})\circ\pi.$$

Hence if $H \in \mathfrak{h} \cap \Omega'$, we get

$$\begin{aligned} F(H; \nabla_\mathfrak{g}) &= F(H; \partial(\pi)\circ\pi) \\ &= F(H; \partial(\pi_\mathfrak{z})\circ\pi_\mathfrak{z}\circ\delta'_{\mathfrak{z}/\mathfrak{h}}(\partial(p)\circ p)). \end{aligned}$$

On the other hand define $\Omega_\mathfrak{z}$, σ_T and $F_\mathfrak{z}$ as before (see the proof of Theorem 3) and put $T_\mathfrak{z} = p\sigma_T$. Then by [3(*i*), Theorem 2 and Lemma 19], the induction hypothesis is applicable to $(\mathfrak{z}, \Omega_\mathfrak{z}, T_\mathfrak{z})$ in place of $(\mathfrak{g}, \Omega, T)$. On the other hand we have seen during the proof of Theorem 3 that $\sigma_T = F_\mathfrak{z}$. Therefore $T_\mathfrak{z} = pF_\mathfrak{z}$ and so by the induction hypothesis $(\nabla_\mathfrak{z}\circ\partial(p))(pF_\mathfrak{z})$ extends to a continuous function $g_\mathfrak{z}$ on $\Omega_\mathfrak{z}$.

Let Ξ denote the analytic subgroup of G corresponding to $\mathfrak{z}$ and $x \to x^*$ the natural projection of G on $G^* = G/\Xi$. Select open connected neighborhoods G_0 and $\mathfrak{z}_0$ of 1 and H_0 in G and $\Omega_\mathfrak{z}$ respectively and let G_0^* denote the image of G_0 in G^*. Then if G_0 and $\mathfrak{z}_0$ are sufficiently small, we can define ψ, φ and Ω_0 as in the proof of Lemma 4. Define a function g on Ω_0 as follows:

$$g(\varphi(x^*, Z)) = g_\mathfrak{z}(Z) \qquad (x^* \in G_0^*, Z \in \mathfrak{z}_0).$$

Since φ is an analytic diffeomorphism of $G_0^* \times \mathfrak{z}_0$ with Ω_0, g is obviously continuous. Fix $X \in \Omega_0 \cap \mathfrak{g}'$. We claim that $g(X) = F(X; \nabla_\mathfrak{g})$. Let $X = \varphi(x^*, H)$ $(x^* \in G_0^*, H \in \mathfrak{z}_0)$. Then it is clear that $g(X) = g(H)$. Similarly, since $\nabla_\mathfrak{g} F$ is invariant under G, it follows that $F(X; \nabla_\mathfrak{g}) = F(H; \nabla_\mathfrak{g})$. Hence it would be enough to show that $g(H) = F(H; \nabla_\mathfrak{g})$. Obviously H is regular in both $\mathfrak{g}$ and $\mathfrak{z}$. Let $\mathfrak{h}$ be the centralizer of H in $\mathfrak{z}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{z}$ and $H \in \mathfrak{h} \cap \Omega'$. Therefore, as we have seen above,

$$F(H; \nabla_\mathfrak{g}) = F(H; \partial(\pi_\mathfrak{z})\circ\pi_\mathfrak{z}\circ\delta'_{\mathfrak{z}/\mathfrak{h}}(\partial(p)\circ p)).$$

Put $F'_\mathfrak{z} = (\partial(p)\circ p)F_\mathfrak{z}$. Since $F_\mathfrak{z}$ is invariant under Ξ and $\partial(p)\circ p \in \mathfrak{I}(\mathfrak{z}_c)$, it follows from [3(*i*), Lemma 14] that

$$F'_\mathfrak{z}(H') = F_\mathfrak{z}(H'; \delta'_{\mathfrak{z}/\mathfrak{h}}(\partial(p)\circ p)) \qquad (H' \in \mathfrak{h}' \cap \Omega_\mathfrak{z}),$$

and therefore

$$\begin{aligned} g_\mathfrak{z}(H) = F'_\mathfrak{z}(H; \nabla_\mathfrak{z}) &= F_\mathfrak{z}(H; \partial(\pi_\mathfrak{z})\circ\pi_\mathfrak{z}\circ\delta'_{\mathfrak{z}/\mathfrak{h}}(\partial(p)\circ p)) \\ &= F(H; \nabla_\mathfrak{g}) \end{aligned}$$

from the definition of $\nabla_\mathfrak{z}$. This proves that $\nabla_\mathfrak{g} F = g$ on $\Omega_0 \cap \mathfrak{g}'$ and therefore $H_0 \in \Omega^0$.

So in order to complete the proof of Lemma 25, we may assume that $0 \in \Omega$. Then, by Lemma 10, $\mathcal{N} \subset \Omega$ and it follows from Corollary 1 of Lemma 8 that $\nabla_\mathfrak{g} F$ can be extended to a continuous function g on [1] $\Omega \cap {}^c\mathcal{N}$. Hence it would be sufficient to prove the following result.

Lemma 26. — There exists a number c with following property. If (X_k) $(k \geq 1)$ *is a sequence in* Ω' *which converges to some element* X *in* $\mathcal{N}$, *then* $g(X_k) \rightarrow c$.

Define $\mathfrak{h}_i$ and $\mathfrak{g}_i$ $(1 \leq i \leq r)$ as in the proof of Lemma 24 and $c(\mathfrak{h}_i)$ as in § 8. Then $c(\mathfrak{h}_1) = \ldots = c(\mathfrak{h}_r) = c$ (say) from the results of § 8. Since $\mathfrak{g}'$ is the union of $\mathfrak{g}_1, \ldots, \mathfrak{g}_r$ we can select, for each k, an index i_k and elements $x_k \in G$, $H_k \in \mathfrak{h}'_{i_k}$ such that $X_k = x_k H_k$. Since $X_k \rightarrow X$, it is clear (see the proof of [3(*j*), Lemma 23]) that $H_k \rightarrow 0$. Hence it follows from the definition of $\nabla_\mathfrak{g}$ and c that

$$g(H_k) = F(H_k; \nabla_\mathfrak{g}) \rightarrow c.$$

But since $g = \nabla_\mathfrak{g} F$ is invariant under G, $g(X_k) = g(H_k)$ and therefore $g(X_k) \rightarrow c$.

§ 10. A DIGRESSION

We shall now apply Theorem 3 to give a new proof of the main result of [3(*e*)]. We keep to the notation of Theorem 1.

Lemma 27. — Assume that F *is locally constant on* Ω'. *Then* T *is locally constant on* Ω.

Given any point $H_0 \in \Omega$, we have to show that T coincides with a constant around H_0. In view of Corollary 2 of Lemma 8, it would be sufficient to consider the case when H_0 is semisimple. However we first prove the following lemma.

Lemma 28. — There exists a number $a > 0$ *such that*

$$\partial(\varpi^\mathfrak{h})\pi^\mathfrak{h} = a$$

for every Cartan subalgebra $\mathfrak{h}$ *of* $\mathfrak{g}$.

Take $T = 1$ and $\Omega = \mathfrak{g}$ in Theorem 1. Then $\partial(\varpi^\mathfrak{h}) g^\mathfrak{h} = \partial(\varpi^\mathfrak{h})\pi^\mathfrak{h}$ in the notation of Theorem 3. But $\partial(\varpi^\mathfrak{h})\pi^\mathfrak{h}$ is obviously a constant which we denote by $a(\mathfrak{h})$. Since zero belongs to every Cartan subalgebra $\mathfrak{h}$, it follows from Theorem 3 that $a(\mathfrak{h})$ is actually independent of $\mathfrak{h}$. Hence we may denote it by a. On the other hand we know (see [3(*c*), p. 110]) that $a(\mathfrak{h}) > 0$. This proves the lemma [2].

Let us now return to the proof of Lemma 27. Since F is locally constant on Ω', it follows from Lemma 28 that $\nabla_\mathfrak{g} F = aF$. Therefore we conclude from Lemma 25 that F can be extended to a continuous function on Ω. Since $T = T_F$, this proves that T is locally constant on Ω.

Now we know that the distribution T′ of [3(*d*), Lemma 30] is locally constant on $\mathfrak{g}'$ (see [3(*d*), p. 235]). Hence from Lemma 27, it is a constant. This gives a new proof of Lemma 17 of [3(*j*)].

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(1) ${}^cS$ denotes the complement of any set S.

(2) It is obviously possible to give a direct proof of Lemma 28.

§ 11. PROOF OF THEOREM 4

In order to prove that the irreducible unitary characters of G are actually functions [3(*g*), Theorem 1], we have to develop a method of lifting our results from $\mathfrak{g}$ to G. The remainder of this paper is devoted to this task.

We use the notation of [3(*i*), Theorem 1].

Theorem 4. — Let Ω *be a completely invariant open set in* $\mathfrak{g}$ *and* T *an invariant distribution on* Ω. *Let* D *be a differential operator in* $\mathfrak{I}(\mathfrak{g}_c)$ *such that* $Dp=0$ *for all* $p\in J(\mathfrak{g}_c)$. *Then* $DT=0$.

We proceed by induction on dim $\mathfrak{g}$. Let $\mathfrak{c}$ be the center and $\mathfrak{g}_1$ the derived algebra of $\mathfrak{g}$ and first assume that $\mathfrak{c}\neq\{0\}$. Then $\mathfrak{I}(\mathfrak{g}_c)=\mathfrak{D}(\mathfrak{c}_c)\mathfrak{I}(\mathfrak{g}_{1c})$ (see [3(*i*), § 3]). Hence $D=\sum_{1\leq i\leq r}\xi_i D_i$ where $\xi_i\in\mathfrak{D}(\mathfrak{c}_c)$, $D_i\in\mathfrak{I}(\mathfrak{g}_{1c})$ and $\xi_1, \ldots, \xi_r$ are linearly independent over **C**.

Fix a point $X_0\in\Omega$ and let $X_0=C_0+Z_0$ $(C_0\in\mathfrak{c},\ Z_0\in\mathfrak{g}_1)$. Define $\mathfrak{c}_0$ and Ω_1 as in the proof of Lemma 25. Since $J(\mathfrak{g}_c)=P(\mathfrak{c}_c)J(\mathfrak{g}_{1c})$, we conclude that

$$\sum_i(\xi_i q)(D_i p_1)=0$$

for all $q\in P(\mathfrak{c}_c)$ and $p_1\in J(\mathfrak{g}_{1c})$. Fix $p_1\in J(\mathfrak{g}_{1c})$. Then it follows from the above result that

$$\sum_i(D_i p_1)\xi_i=0$$

in $\mathfrak{D}(\mathfrak{g}_c)$. Therefore we can conclude (see [3(*i*), § 3]) that $D_i p_1=0$ $(1\leq i\leq r)$.

Now fix $\alpha\in C_c^\infty(\mathfrak{c}_0)$. Then if $\beta\in C_c^\infty(\Omega_1)$, we have

$$(DT)(\alpha\times\beta)=\sum_i T(\xi_i^*\alpha\times D_i^*\beta)=\sum_i(D_iT_i)(\beta)$$

where $T_i(\beta)=T(\xi_i^*\alpha\times\beta)$. Since $\dim\mathfrak{g}_1<\dim\mathfrak{g}$, Theorem 4 holds for (Ω_1, T_i, D_i) in place of (Ω, T, D) by the induction hypothesis. Hence $D_iT_i=0$. In view of [3(*h*), Lemma 3] this shows that $DT=0$ on $\mathfrak{c}_0+\Omega_1$ and therefore $X_0\notin\operatorname{Supp} DT$. Since X_0 was an arbitrary point in Ω, this proves that $DT=0$.

Hence we may now assume that $\mathfrak{c}=\{0\}$ and therefore $\mathfrak{g}$ is semisimple. Let $H_0\neq 0$ be a semisimple element in Ω. We intend to show that $H_0\notin\operatorname{Supp} DT$. Let $\mathfrak{z}$ be the centralizer of H_0 in $\mathfrak{g}$. Define ζ and $\mathfrak{z}'$ as usual (see [3(*i*), § 2]) and let $\Omega_\mathfrak{z}$ be the set of all $Z\in\Omega\cap\mathfrak{z}$ such that $|\zeta(Z)|>|\zeta(H_0)|/2$. Then $\Omega_\mathfrak{z}$ is open and completely invariant in $\mathfrak{z}$. Take $G_0=G$ and $\mathfrak{z}_0=\Omega_\mathfrak{z}$ in [3(*i*), Lemma 17] and let σ_T be the corresponding distribution on $\Omega_\mathfrak{z}$. Let Ξ be the analytic subgroup of G corresponding to $\mathfrak{z}$. Then σ_T is invariant under Ξ (see Corollary 1 of [3(*i*), Lemma 17]). Now it follows from [3(*i*), Lemma 10 and Corollary 2 of Lemma 2] that $D_1=\zeta^m\delta'_{\mathfrak{g}/\mathfrak{z}}(D)\in\mathfrak{I}(\mathfrak{z}_c)$, if *m* is a sufficiently large positive integer. Moreover

$$\sigma_{DT}=\delta'_{\mathfrak{g}/\mathfrak{z}}(D)\sigma_T$$

by Corollary 2 of [3(*i*), Lemma 17]. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{z}$. Then

$$\delta'_{\mathfrak{z}/\mathfrak{h}}(D_1)=\zeta_{\mathfrak{h}}^{m}\delta'_{\mathfrak{g}/\mathfrak{h}}(D)$$

from [3(*i*), Lemma 11] where $\zeta_{\mathfrak{h}}$ is the restriction of ζ on $\mathfrak{h}$. Moreover we know from [3(*i*), Theorem 1] that $\delta'_{\mathfrak{g}/\mathfrak{h}}(D)=0$. Therefore, by applying [3(*i*), Theorem 1] to $\mathfrak{z}_c$ instead of $\mathfrak{g}_c$, we conclude that $D_1 p_1=0$ for all $p_1\in J(\mathfrak{z}_c)$. Therefore, since $\dim\mathfrak{z}<\dim\mathfrak{g}$, we conclude from the induction hypothesis that $D_1\sigma_T=\zeta^m\sigma_{DT}=0$. Since ζ is nowhere zero on $\Omega_{\mathfrak{z}}$, this implies that $\sigma_{DT}=0$ and therefore $DT=0$ around H_0.

In view of the above result, it follows from Corollary 1 of Lemma 8 that $\operatorname{Supp} DT\subset\Omega\cap\mathscr{N}$. Hence, in order to complete the proof of Theorem 4, we may assume that $\Omega\cap\mathscr{N}\neq\emptyset$. But then $\mathscr{N}\subset\Omega$ from Lemma 10.

Lemma 29. — We can select a function $f\in C^\infty(\mathfrak{g})$ *such that:*

1) f *is invariant under* G;

2) $\operatorname{Supp} f\subset\Omega$;

3) $f=1$ *on some neighborhood of zero in* $\mathfrak{g}$;

4) *the distribution* fT *on* $\mathfrak{g}$ *is tempered.*

Notice that since $\operatorname{Supp} f\subset\Omega$, the distribution $fT: g\to T(fg)$ $(g\in C_c^\infty(\mathfrak{g}))$ is well defined on $\mathfrak{g}$. The proof of this lemma is rather long and therefore, in order not to interrupt our main line of argument, we shall postpone it until later (see § 19).

We have seen above that $\operatorname{Supp} DT\subset\mathscr{N}$. Choose an open neighborhood V of zero in Ω such that $f=1$ on V. Fix a point $X\in\mathscr{N}$. Then, by Lemma 7, we can choose $y\in G$ such that $y^{-1}X\in V$. Now T and fT are both invariant distributions which obviously coincide on V. Hence they also coincide on V^y. Therefore, in order to show that $DT=0$ around X, it would be sufficient to verify that $D(fT)=0$ around X. This means that in order to complete the proof of Theorem 4, it is enough to prove that $D(fT)=0$. Therefore, replacing T by fT, we may now assume that T is an invariant and tempered distribution on $\mathfrak{g}$. Moreover we know from the above proof that $\operatorname{Supp} DT\subset\mathscr{N}$.

Define the space $\mathscr{C}(\mathfrak{g})$ as in [3(*c*), p. 91] and for any $f\in\mathscr{C}(\mathfrak{g})$ define its Fourier transform $\hat{f}$ by

$$\hat{f}(Y)=\int_{\mathfrak{g}}\exp((-1)^{1/2}B(Y,X))f(X)\,dX \qquad (Y\in\mathfrak{g})$$

where dX is a fixed Euclidean measure on $\mathfrak{g}$ and $B(Y,X)=\operatorname{tr}(\operatorname{ad}Y\operatorname{ad}X)$ $(X,Y\in\mathfrak{g}_c)$ as usual. If σ is any tempered distribution on $\mathfrak{g}$, its Fourier transform $\hat{\sigma}$ is also a tempered distribution on $\mathfrak{g}$ given by $\hat{\sigma}(f)=\sigma(\hat{f})$ $(f\in\mathscr{C}(\mathfrak{g}))$. Since $f\to\hat{f}$ is a topological mapping of $\mathscr{C}(\mathfrak{g})$ onto itself, $\hat{\sigma}=0$ implies that $\sigma=0$.

As usual we identify $\mathfrak{g}_c$ with its dual under B and use the notation of [3(*i*), Lemma 12]. Then the mapping $\alpha:\Delta\to(\hat{\Delta})^*$ $(\Delta\in\mathfrak{D}(\mathfrak{g}_c))$ is an anti-automorphism of $\mathfrak{D}(\mathfrak{g}_c)$ and therefore α^2 is an automorphism. However it is easy to check that α^2 leaves $\mathfrak{g}_c+\partial(\mathfrak{g}_c)$ fixed pointwise and therefore it must be the identity. The relation

$\alpha^2\Delta=\Delta\ (\Delta\in\mathfrak{D}(\mathfrak{g}_c))$ implies that $\Delta^*=(\alpha\Delta)\hat{}$. On the other hand $(\alpha\Delta)^*=\hat{\Delta}$ from the definition of α. Therefore

$$(\Delta\sigma)\hat{}\,(f)=\sigma(\Delta^*\hat{f})=\sigma(((\alpha\Delta)f)\hat{}\,)$$
$$=\hat{\sigma}((\alpha\Delta)f)=(\hat{\Delta}\hat{\sigma})(f) \qquad (f\in\mathscr{C}(\mathfrak{g})),$$

for any tempered distribution σ. This proves that $(\Delta\sigma)\hat{}=\hat{\Delta}\hat{\sigma}$. Similarly, since B is invariant under G, $(f^x)\hat{}=(\hat{f})^x$ for $f\in\mathscr{C}(\mathfrak{g})$ and $x\in G$. Therefore $(\sigma^x)\hat{}=(\hat{\sigma})^x$.

Thus $\hat{T}$ is an invariant distribution on $\mathfrak{g}$ and $(DT)\hat{}=\hat{D}\hat{T}$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then it follows from [3(*i*), Theorem 1] and our hypothesis on D, that $\delta'_{\mathfrak{g}/\mathfrak{h}}(D)=0$. Therefore we conclude from [3(*i*), Lemma 13] that $\hat{D}\in\mathfrak{J}(\mathfrak{g}_c)$ and $\hat{D}p=0$ for every $p\in J(\mathfrak{g}_c)=I(\mathfrak{g}_c)$. So the above proof is also applicable to $(\hat{D},\hat{T})$ instead of (D, T). Hence $\operatorname{Supp}\hat{D}\hat{T}\subset\mathcal{N}$.

Now put $\sigma=DT$ and fix an element $p\in I(\mathfrak{g}_c)$ such that p vanishes at zero. Then it follows from Lemma 7 that $p=0$ on $\mathcal{N}$. Hence (see [3(*h*), Lemma 21]) we can choose an integer $m\geq 0$ such that $p^m\sigma=0$ around zero. Then, if we take $\Omega=\mathfrak{g}$ and $\Phi=\operatorname{Supp}(p^m\sigma)$ in Corollary 1 of Lemma 8, we can conclude that $p^m\sigma=0$. Choose a finite number of homogeneous elements $p_1,\dots,p_l$ of positive degrees in $I(\mathfrak{g}_c)$ such that $I(\mathfrak{g}_c)=\mathbf{C}[p_1,\dots,p_l]$. Fix an integer $m\geq 0$ such that $p_i^m\sigma=0\ (1\leq i\leq l)$. Then, if $\mathfrak{B}$ is the ideal in $I(\mathfrak{g}_c)$ generated by $p_1^m,\dots,p_l^m$, it is obvious that $\dim(I(\mathfrak{g}_c)/\mathfrak{B})\leq m^l$ and $v\sigma=0$ for $v\in\mathfrak{B}$. On the other hand, by [3(*i*), Lemmas 12 and 13], $\Delta\to\hat{\Delta}\ (\Delta\in\mathfrak{D}(\mathfrak{g}_c))$ is an automorphism of $\mathfrak{D}(\mathfrak{g}_c)$ of order 4 which maps $I(\mathfrak{g}_c)$ onto $\partial(I(\mathfrak{g}_c))$. Therefore $\hat{\mathfrak{B}}$ is an ideal in $\partial(I(\mathfrak{g}_c))$ and $\hat{v}\hat{\sigma}=(v\sigma)\hat{}=0$ for $v\in\mathfrak{B}$. Hence we conclude from Theorem 1, applied to $\hat{\sigma}$ instead of T, that $\hat{\sigma}$ is a locally summable function on $\mathfrak{g}$. But $\hat{\sigma}=\hat{D}\hat{T}$ and therefore, as we have seen above, $\operatorname{Supp}\hat{\sigma}\subset\mathcal{N}$. Since $\mathcal{N}$ is of measure zero in $\mathfrak{g}$, it follows that $\hat{\sigma}=0$ and therefore $DT=\sigma=0$. This proves Theorem 4.

§ 12. ANALYTIC DIFFERENTIAL OPERATORS

For applications we have to generalize Theorem 4 to the case when D is an analytic differential operator on Ω. For this we need some preparation.

Let E be a vector space over $\mathbf{R}$ of finite dimension, Ω a non-empty open subset of E and $\mathfrak{D}_\infty(\Omega:E)$ the algebra of all differential operators on Ω. Then any such operator D can be written in the form

$$D=\sum_{1\leq i\leq r} f_i\partial(p_i)$$

where $f_i\in C^\infty(\Omega)$ and $p_i\in S(E)$. For any $X\in\Omega$, D_X denotes, as usual the local expression of D at X (see [3(*c*), p. 90]) so that

$$D_X=\sum_i f_i(X)\,\partial(p_i).$$

Let $\mathscr{A}(\Omega)$ be the algebra of all analytic functions on Ω. Then $\mathscr{A}(\Omega)$ is a subalgebra of $C^\infty(\Omega)$. We denote by $\mathfrak{D}_a(\Omega : E)$ the subalgebra of $\mathfrak{D}_\infty(\Omega : E)$ generated by $\mathscr{A}(\Omega) \cup \partial(S(E))$. If Ω is empty, we define $\mathfrak{D}_\infty(\Omega : E) = \mathfrak{D}_a(\Omega : E) = \{0\}$.

Let Ω_1 be an open subset of Ω. Then we get a homomorphism

$$j : \mathfrak{D}_\infty(\Omega : E) \to \mathfrak{D}_\infty(\Omega_1 : E)$$

as follows. If $D \in \mathfrak{D}_\infty(\Omega : E)$, then $j(D)$ is the restriction of D on Ω_1. It is clear that j maps $\mathfrak{D}_a(\Omega : E)$ into $\mathfrak{D}_a(\Omega_1 : E)$. We say that an element $D \in \mathfrak{D}_\infty(\Omega : E)$ is analytic on Ω_1 if $j(D) \in \mathfrak{D}_a(\Omega : E)$. In particular D is analytic if $D \in \mathfrak{D}_a(\Omega : E)$.

§ 13. EXTENSION OF SOME RESULTS TO ANALYTIC DIFFERENTIAL OPERATORS

Now let $\mathfrak{g}$ be a reductive Lie algebra over **R** and Ω a non-empty open set in $\mathfrak{g}$. If Ω is invariant, G operates on $\mathfrak{D}_\infty(\Omega : \mathfrak{g})$ (see [3(*h*), § 5]). We denote by $\mathfrak{I}_\infty(\Omega : \mathfrak{g})$ the subalgebra consisting of all invariant elements and put $\mathfrak{I}_a(\Omega : \mathfrak{g}) = \mathfrak{I}_\infty(\Omega : \mathfrak{g}) \cap \mathfrak{D}_a(\Omega : \mathfrak{g})$.

Fix $\mathfrak{z}$ and define ζ and $\mathfrak{z}'$ as in [3(*i*), §§ 2, 7]. Put $\Omega_\mathfrak{z} = \Omega \cap \mathfrak{z}'$ and for any $D \in \mathfrak{D}_\infty(\Omega : \mathfrak{g})$ define an element $\Delta(D) \in \mathfrak{D}_\infty(\Omega_\mathfrak{z} : \mathfrak{z})$ as follows. Fix $Z \in \Omega_\mathfrak{z}$ and choose $p_Z \in S(\mathfrak{g}_c)$ such that $D_Z = \partial(p_Z)$. Then, corresponding to Corollary 1 of [3(*i*), Lemma 2], $\alpha_Z(p_Z) \in S(\mathfrak{z}_c)$. It follows from Corollary 2 of [3(*i*), Lemma 2] that there exists a unique element $\nabla \in \mathfrak{D}_\infty(\Omega_\mathfrak{z} : \mathfrak{z})$ such that $\nabla_Z = \partial(\alpha_Z(p_Z))$ for $Z \in \Omega_\mathfrak{z}$. We define $\Delta(D) = \nabla$. (In case $\Omega_\mathfrak{z}$ is empty, $\Delta(D) = 0$ by definition.)

Let $\delta'_{\mathfrak{g}/\mathfrak{z}}$ denote the mapping $D \to \Delta(D)$ of $\mathfrak{D}_\infty(\Omega : \mathfrak{g})$ into $\mathfrak{D}_\infty(\Omega_\mathfrak{z} : \mathfrak{z})$ (cf. [3(*i*), § 4]).

Lemma 30. — $\delta'_{\mathfrak{g}/\mathfrak{z}}$ *maps* $\mathfrak{D}_a(\Omega : \mathfrak{g})$ *into* $\mathfrak{D}_a(\Omega_\mathfrak{z} : \mathfrak{z})$. *Moreover, if* Ω *is invariant,* $\delta'_{\mathfrak{g}/\mathfrak{z}}$ *maps* $\mathfrak{I}_\infty(\Omega : \mathfrak{g})$ *into* $\mathfrak{I}_\infty(\Omega_\mathfrak{z} : \mathfrak{z})$.

The first statement is obvious from Corollary 2 of [3(*i*), Lemma 2]. Moreover, if Ω is invariant, then $\Omega_\mathfrak{z}$ is invariant in $\mathfrak{z}$ and the second assertion follows from [3(*i*), Lemma 3].

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{z}$.

Lemma 31. — $\delta'_{\mathfrak{g}/\mathfrak{h}}(D) = \delta'_{\mathfrak{z}/\mathfrak{h}}(\delta'_{\mathfrak{g}/\mathfrak{z}}(D))$ *for* $D \in \mathfrak{D}_\infty(\Omega : \mathfrak{g})$.

The proof of this is the same as that of [3(*i*), Lemma 11].

Lemma 32. — *Let f be a locally invariant* C^∞ *function on an open subset* Ω_0 *of* Ω. *Then*

$$f(Z; D) = f(Z; \delta'_{\mathfrak{g}/\mathfrak{z}}(D))$$

for $Z \in \Omega_0 \cap \mathfrak{z}'$ *and* $D \in \mathfrak{D}_\infty(\Omega : \mathfrak{g})$.

This is proved in the same way as Lemma 14 of [3(*i*)].

Lemma 33. — *Let* D *be a differential operator on an open subset* Ω *of* $\mathfrak{g}$. *Then the following two conditions on* D *are equivalent.*

1) *For every Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$, $\delta'_{\mathfrak{g}/\mathfrak{h}}(D) = 0$.

2) *If* Ω_0 *is an open subset of* Ω *and f a locally invariant* C^∞ *function on* Ω_0, *then* $Df = 0$.

Assume 1) holds and let f be a locally invariant C^∞ function on Ω_0. Since $\Omega'_0 = \Omega_0 \cap \mathfrak{g}'$ is dense in Ω_0, it is enough to verify that $Df=0$ on Ω'_0. Fix $H_0 \in \Omega'_0$ and let $\mathfrak{h}$ be the centralizer of H_0 in $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and since f is locally invariant, it follows from Lemma 32 that $f(H_0; D) = f(H_0; \delta'_{\mathfrak{g}/\mathfrak{h}}(D)) = 0$. This proves that $Df=0$ on Ω'_0.

Conversely assume that 2) holds. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a point $H_0 \in \Omega \cap \mathfrak{h}'$. Let A be the Cartan subgroup of G corresponding to $\mathfrak{h}$. We now use the notation of the proof of Lemma 1. Then φ defines an analytic diffeormorphism of $G_0^* \times \mathfrak{h}_0$ with Ω_0. Fix $\beta \in C^\infty(\mathfrak{h}_0)$ and define $f \in C^\infty(\Omega_0)$ by the relation $f(x^*H) = \beta(H)$ $(x^* \in G_0^*, H \in \mathfrak{h}_0)$. Then it is obvious that f is locally invariant and therefore

$$0 = f(H; D) = f(H; \delta'_{\mathfrak{g}/\mathfrak{h}}(D)) = \beta(H; \delta'_{\mathfrak{g}/\mathfrak{h}}(D)) \qquad (H \in \mathfrak{h}_0)$$

Since β was an arbitrary element of $C^\infty(\mathfrak{h}_0)$, this implies that $\delta'_{\mathfrak{g}/\mathfrak{h}}(D) = 0$ on $\mathfrak{h}_0$. Hence in particular $(\delta'_{\mathfrak{g}/\mathfrak{h}}(D))_{H_0} = 0$. This shows that 2) implies 1).

Corollary. — Assume Ω is invariant. Then either one of the two conditions above is equivalent to the following.

3) *For every invariant function f in* $C^\infty(\Omega)$, $Df=0$.

Obviously 2) implies 3). Now assume 3) holds. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a point $H_0 \in \Omega \cap \mathfrak{h}'$. Let $\mathfrak{h}_0$ be an open neighborhood of H_0 in $\mathfrak{h}' \cap \Omega$. We assume that $\mathfrak{h}_0$ is relatively compact in $\mathfrak{h}'$ and $s\mathfrak{h}_0 \cap \mathfrak{h}_0 = \emptyset$ for $s \neq 1$ in W_G (see § 9 for the definition of W_G). Fix $\beta_0 \in C_c^\infty(\mathfrak{h}_0)$ and put

$$\beta(H) = \sum_{s \in W_G} \beta_0(sH) \qquad (H \in \mathfrak{h}).$$

Then $\beta^s = \beta$ $(s \in W_G)$ and the mapping $\varphi : G^* \times \mathfrak{h}' \to \mathfrak{g}$ is everywhere regular. Put $\mathfrak{g}_{\mathfrak{h}} = \varphi(G^* \times \mathfrak{h}') = (\mathfrak{h}')^G$. The group W_G operates on $G^* \times \mathfrak{h}'$ on the right as follows:

$$(x^*, H)s = (x^*s, s^{-1}H) \qquad (s \in W_G)$$

in the notation of the proof of Lemma 24. Since no point of $G^* \times \mathfrak{h}'$ is left fixed by s if $s \neq 1$, it follows that φ defines an analytic diffeomorphism of the quotient manifold $(G^* \times \mathfrak{h}')/W_G$ with $\mathfrak{g}_{\mathfrak{h}}$. Now define a function F on $G^* \times \mathfrak{h}'$ by

$$F(x^* : H) = \beta(H) \qquad (x^* \in G^*, H \in \mathfrak{h}').$$

Then $F(x^*s : s^{-1}H) = \beta(s^{-1}H) = \beta(H) = F(x^* : H)$ and therefore F defines a C^∞ function f on $\mathfrak{g}_{\mathfrak{h}}$. Since $\mathfrak{h}_0$ is relatively compact in $\mathfrak{h}'$ it follows from [3(*j*), Lemma 7] that $\mathrm{Cl}(\mathfrak{h}_0{}^G) \subset \mathfrak{g}_{\mathfrak{h}}$ and therefore we can extend f to a C^∞ function on $\mathfrak{g}$ by defining it to be zero outside $\mathfrak{g}_{\mathfrak{h}}$. Then it is clear that f is invariant and therefore $Df=0$ on Ω by 3). But

$$f(H; D) = f(H; \delta'_{\mathfrak{g}/\mathfrak{h}}(D)) = \beta_0(H; \delta'_{\mathfrak{g}/\mathfrak{h}}(D)) \qquad (H \in \mathfrak{h}_0)$$

because $f = \beta = \beta_0$ on $\mathfrak{h}_0$. Since β_0 was arbitrary in $C_c^\infty(\mathfrak{h}_0)$, this shows that $(\delta'_{\mathfrak{g}/\mathfrak{h}}(D))_{H_0} = 0$. Therefore 3) implies 1) and the corollary is proved.

§ 14. PROOF OF THEOREM 5

We shall now prove the following generalization of Theorem 4.

Theorem 5. — Let Ω *be a completely invariant open set in* $\mathfrak{g}$ *and* T *an invariant distribution on* Ω. *Let* D *be an analytic and invariant differential operator on* Ω *such that* $Df=0$ *for every invariant* C^∞ *function* f *on* Ω. *Then* $DT=0$.

We again use induction on dim $\mathfrak{g}$. Define $\mathfrak{c}$ and $\mathfrak{g}_1$ as in § 4 and fix a semisimple element $H_0\in\Omega$ such that $H_0\notin\mathfrak{c}$. We shall prove that $DT=0$ around H_0. Let $\mathfrak{z}$ denote the centralizer of H_0 in $\mathfrak{g}$ and define ζ and $\mathfrak{z}'$ as in [3(*i*), § 2]. Then $\Omega_{\mathfrak{z}}=\Omega\cap\mathfrak{z}'$ is an open and completely invariant set in $\mathfrak{z}$. Let σ_T and σ_{DT} be the distributions on $\Omega_{\mathfrak{z}}$ corresponding to T and DT respectively under [3(*i*), Lemma 17] with $G_0=G$ and $\mathfrak{z}_0=\Omega_{\mathfrak{z}}$. Then it would be enough to show that $\sigma_{DT}=0$. However it is easy to prove (cf. Corollary 2 of [3(*i*), Lemma 17]) that $\sigma_{DT}=\Delta\sigma_T$ where $\Delta=\delta'_{\mathfrak{g}/\mathfrak{z}}(D)$. Now σ_T is an invariant distribution on $\Omega_{\mathfrak{z}}$ (see Corollary 1 of [3(*i*), Lemma 17]) and $\Delta\in\mathfrak{J}_a(\Omega_{\mathfrak{z}}:\mathfrak{z})$ by Lemma 30. Therefore since $\dim\mathfrak{z}<\dim\mathfrak{g}$, it follows by induction hypothesis that $\Delta\sigma_T=0$ (see Lemma 31 and the corollary of Lemma 33).

Now fix $C_0\in\mathfrak{c}\cap\Omega$. We claim that $T=0$ around C_0. Applying the translation by $-C_0$ to the whole problem, we are reduced to the case when $C_0=0$. Let

$$D=\sum_{1\leq i\leq r} a_i\,\partial(p_i)$$

where $p_1,\ldots,p_r$ are linearly independent homogeneous elements in $S(\mathfrak{g}_c)$ and $a_1,\ldots,a_r$ are analytic functions on Ω. Let V be the subspace of $S(\mathfrak{g}_c)$ spanned by p_i^x $(1\leq i\leq r,\ x\in G)$. Then obviously $\dim V<\infty$ and we may, without loss of generality, assume that $(p_1,\ldots,p_r)$ is a base for V. Then

$$p_i^x=\sum_j c_{ji}(x)p_j \qquad (x\in G)$$

where c_{ji} are analytic functions on G. Since $D=D^x$, it follows that $D_{xX}=(D_X)^x$ and therefore

$$\sum_i a_i(xX)\,\partial(p_i)=\sum_i a_i(X)\,\partial(p_i^x) \qquad (X\in\Omega).$$

This shows that

$$a_i(xX)=\sum_j c_{ij}(x)a_j(X) \qquad (x\in G,\ X\in\Omega).$$

For any integer m, let $\mathfrak{D}_m$ denote the subspace of $\mathfrak{D}(\mathfrak{g}_c)$ spanned by elements of the form $p\partial(q)$ where p and q are homogeneous elements in $P(\mathfrak{g}_c)$ and $S(\mathfrak{g}_c)$ respectively and $\deg p-\deg q=m$. Then if $\Delta\in\mathfrak{D}_m$ and Q is a homogeneous polynomial function on $\mathfrak{g}$, it is clear that ΔQ is homogeneous and

$$\deg(\Delta Q)=\deg Q+m.$$

Choose an open and convex neighborhood Ω_0 of zero in Ω such that each a_i $(1 \leq i \leq r)$ can be expanded in a power series around zero, which converges absolutely on Ω_0. Then

$$a_i(X) = \sum_{\nu \geq 0} q_{\nu i}(X) \qquad (X \in \Omega_0)$$

where $q_{\nu i}$ is a homogeneous polynomial function on $\mathfrak{g}$ of degree ν. It is obvious from our result above that

$$q_{\nu i}(xX) = \sum_{1 \leq j \leq r} c_{ij}(x) q_{\nu j}(X) \qquad (x \in G,\ X \in \Omega_0)$$

and therefore

$$_\nu D = \sum_i q_{\nu i} \partial(p_i)$$

lies in $\mathfrak{J}(\mathfrak{g}_c)$. On the other hand it is clear that $\mathfrak{D}(\mathfrak{g}_c)$ is the direct sum of $\mathfrak{D}_m$ for all m $(-\infty < m < \infty)$ and each $\mathfrak{D}_m$ is stable under G. Let $_\nu D_m$ denote the component of $_\nu D$ in $\mathfrak{D}_m$ is this direct sum. Then it is clear that $_\nu D_m \in \mathfrak{J}(\mathfrak{g}_c)$. Moreover $_\nu D_m \neq 0$ implies that $\nu = m + \deg p_i$ for some i. Hence if $m_0 = \sup_i \deg p_i$, it follows that $_\nu D_m = 0$ for $\nu > m + m_0$. Put

$$D_m = \sum_{\nu \geq 0} {_\nu D_m}.$$

Then $D_m \in \mathfrak{J}(\mathfrak{g}_c) \cap \mathfrak{D}_m$. Moreover if p is a homogeneous element in $J(\mathfrak{g}_c)$, then by hypothesis

$$0 = Dp = \sum_{m \geq -m_0} D_m p$$

on Ω_0. Since $D_m p$ is homogeneous of degree $m + \deg p$, it is clear that $D_m p = 0$. Therefore $D_m T = 0$ by Theorem 4.

Now fix $f \in C_c^\infty(\Omega_0)$. It is clear that for any $p \in S(\mathfrak{g}_c)$, the series

$$\sum_{\nu \geq 0} |\partial(p) q_{\nu i}| \qquad (1 \leq i \leq r)$$

converge uniformly on any compact subset of Ω_0. Hence it follows without difficulty that the series

$$\sum_{m \geq -m_0} D_m^* f$$

converges in $C_c^\infty(\Omega_0)$ to $D^* f$. (Here the star denotes adjoint, as usual.) This implies that the series

$$\sum_{m \geq -m_0} T(D_m^* f)$$

converges to $T(D^* f)$. But $T(D_m^* f) = 0$ since $D_m T = 0$. Therefore $T(D^* f) = 0$. This means that $DT = 0$ on Ω_0.

The above proof shows that Supp DT contains no semisimple element of Ω.

Hence it follows from Corollary 1 of Lemma 8 that $DT=0$. This completes the proof of Theorem 5.

Remark. — I do not know whether Theorem 5 continues to hold when the condition of analyticity of D is dropped.

§ 15. SOME PREPARATION FOR THE PROOF OF LEMMA 29

Let G be a connected semisimple Lie group with a faithful finite-dimensional representation, $\mathfrak{g}$ its Lie algebra over **R**, θ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. We introduce an order in the space of all (real) linear functions on $\mathfrak{a}$ and denote by Σ the set of all positive roots of $(\mathfrak{g}, \mathfrak{a})$ (see [3(*f*), p. 244]). Let $\mathfrak{a}^+$ be the set of all points $H\in\mathfrak{a}$ where $\alpha(H)\geq 0$ for every $\alpha\in\Sigma$. Put $A=\exp\mathfrak{a}$ and $A^+=\exp(\mathfrak{a}^+)$ in G. The exponential mapping from $\mathfrak{a}$ to A is bijective. We denote its inverse by log. Introduce a partial order in A as follows. Given two elements h_1, h_2 in A, we write $h_1 \succ h_2$ if $h_1h_2^{-1}\in A^+$. Let $l=\dim\mathfrak{a}$. Then we can choose a simple system of roots $\alpha_1, \ldots, \alpha_l$ in Σ (see [3(*d*), Lemma 1]).

Lemma 34. — Fix some norm ν on the finite-dimensional space $\mathfrak{g}$. Then for any number $a\geq 0$, we can choose two numbers b, c $(b\geq a, c\geq 1)$ with the following property. Suppose $X\in\mathfrak{g}$, $h\in A^+$ and $\nu(X)\leq a$. Then there exist elements $X_0\in\mathfrak{g}$ and $h_0\in A^+$ such that

1) $X^h=X_0^{h_0}$, $\nu(X_0)\leq b$, $1\prec h_0\prec h$;

2) $\max_{1\leq i\leq l} \exp\alpha_i(\log h_0)\leq c(1+\nu(X^h))^l$.

(In case $l=0$, $\max_i \exp\alpha_i(\log h_0)$ should be taken to mean 1.) We shall give a proof of this lemma in § 20.

As usual put $B(X, Y)=\mathrm{tr}(\mathrm{ad}\, X \mathrm{ad}\, Y)$ $(X, Y\in\mathfrak{g})$. Then the quadratic form

$$||X||^2=-B(X, \theta(X)) \qquad (X\in\mathfrak{g})$$

is positive-definite and defines the structure of a real Hilbert space on $\mathfrak{g}$. For any $a>0$, let ω_a denote the set of all $X\in\mathfrak{g}$ with $||X||<a$ and put $\Omega_a=(\omega_a)^G$.

Lemma 35. — Suppose $a>b>0$. Then $\mathrm{Cl}\,\Omega_b\subset\Omega_a$.

We shall give a proof of this in § 21.

Corollary. — Ω_a is an open and completely invariant subset of $\mathfrak{g}$.

It is obvious that Ω_a is open and invariant. Fix $X_0\in\Omega_a$. Then $X_0=Y_0^{x_0}$ where $||Y_0||<a$ and $x_0\in G$. Choose b such that $||Y_0||<b<a$. Then $X_0\in\Omega_b$ and $\mathrm{Cl}(\Omega_b)\subset\Omega_a$ by Lemma 35. This shows that every point of Ω_a has an open invariant neighborhood whose closure (in $\mathfrak{g}$) is contained in Ω_a. From this it is clear that Ω_a is completely invariant.

For any linear transformation T in $\mathfrak{g}$, let T^* denote its adjoint (in the sense of Hilbert-space theory). Put

$$||x||^2=\mathrm{tr}(\mathrm{Ad}(x)^*\mathrm{Ad}(x)) \qquad (x\in G).$$

Choose a base $(H_1, \ldots, H_l)$ for $\mathfrak{a}$ over $\mathbf{R}$ dual to $(\alpha_1, \ldots, \alpha_l)$ so that

$$\alpha_i(H_j) = \delta_{ij} \qquad (1 \leq i, j \leq l)$$

and define $m(\alpha) = \sum_i \alpha(H_i)$ for $\alpha \in \Sigma$. Since $H_i \in \mathfrak{a}^+$, it is clear that $m(\alpha)$ is a positive integer.

Lemma 36. — *Given $a > 0$, we can choose numbers b, c $(b \geq a, c \geq 1)$ such that the following condition holds. For any $X \in \Omega_a$, we can select $x \in G$ such that*

$$1) \quad \|X^{x^{-1}}\| \leq b, \qquad 2) \quad \|x\| \leq c(1 + \|X\|)^m$$

where $m = l \max_{\alpha \in \Sigma} m(\alpha)$.

(If $l = 0$ then $m = 0$ by definition.) Choose b_0, c_0 such that Lemma 34 holds for (b_0, c_0) instead of (b, c) with the norm $\nu(Z) = \|Z\|$ $(Z \in \mathfrak{g})$. Let K be the analytic subgroup of G corresponding to $\mathfrak{k}$. Then K is compact, $G = KA^+K$ and $\mathrm{Ad}(k)$ is unitary for $k \in K$. Hence if $x = k_1 h k_2$ $(k_1, k_2 \in K, h \in A^+)$, it follows that $\|x\| = \|h\|$. However $\mathrm{Ad}(h)$ is self-adjoint ([1]) and its eigenvalues are 1 and $e^{\pm \alpha(\log h)}$ $(\alpha \in \Sigma)$. Since $\alpha(\log h) \geq 0$, it is clear that

$$\|h\| \leq n^{1/2} \max_{\alpha \in \Sigma} e^{\alpha(\log h)}$$

where $n = \dim \mathfrak{g}$. But $\alpha = \sum_{1 \leq i \leq} m_i \alpha_i$ where $m_i = \alpha(H_i)$ are integers ≥ 0. Hence

$$\alpha(\log h) \leq m(\alpha) \max_i \alpha_i(\log h) \leq m_0 \max_i \alpha_i(\log h)$$

where $m_0 = \max_{\alpha \in \Sigma} m(\alpha)$. Therefore

$$\|h\| \leq n^{1/2} \left(\max_i e^{\alpha_i(\log h)} \right)^{m_0}$$

(This holds also if $l = 0$. We define $m_0 = 0$ in that case.)

Now fix $X \in \Omega_a$ and choose $Y \in \mathfrak{g}$, $y \in G$ such that $\|Y\| < a$ and $X = Y^y$. Let $y = k_1 h k_2$ $(k_1, k_2 \in K, h \in A^+)$. Replacing (Y, y) by $(Y^{k_2}, k_1 h)$, we can assume that $y = k_1 h$. Select $Y_0 \in \mathfrak{g}$ and $h_0 \in A^+$ such that $Y^h = Y_0^{h_0}$, $\|Y_0\| \leq b_0$, $1 \prec h_0 \prec h$ and

$$\max_i e^{\alpha_i(\log h_0)} \leq c_0(1 + \|Y^h\|)^l.$$

This is possible from the definition of b_0, c_0. Then

$$\|h_0\| \leq n^{1/2} c_0^{m_0} (1 + \|Y^h\|)^m.$$

Now put $x = k_1 h_0$. Then $X = Y^y = Y_0^x$ and therefore $\|X^{x^{-1}}\| \leq b_0$. Moreover

$$\|x\| = \|h_0\| \leq n^{1/2} c_0^{m_0} (1 + \|X\|)^m.$$

Therefore we can take $b = b_0$ and $c = n^{1/2} c_0^{m_0}$ in the lemma.

([1]) The facts stated here are all well known. They can be found in [3(*a*)].

§ 16. SOME INEQUALITIES

For $t \geq 1$, let $G(t)$ denote the set of all $x \in G$ with $\|x\| \leq t$. Then $G(t)$ is obviously compact.

Lemma 37. — *Let* μ *denote the Haar measure of* G. *Then there exists a number* $c>0$ *and an integer* $M \geq 0$ *such that*

$$\mu(G(t)) \leq ct^M$$

for $t \geq 1$.

The statement is trivial if G is compact. Hence we may assume that $l \geq 1$. Put $A^+(t) = A^+ \cap G(t)$. Then it is clear that $G(t) = KA^+(t)K$ and therefore (see [3(*a*), Lemma 22])

$$\mu(G(t)) = \int_{A^+(t)} D(h)\, dh$$

where dh is the (suitably normalized) Haar measure on A,

$$D(h) = \prod_{\alpha \in \Sigma} (e^{\alpha(\log h)} - e^{-\alpha(\log h)})^{m_\alpha} \qquad (h \in A)$$

and m_α is the multiplicity of α (m_α is the dimension of the space $\mathfrak{g}_\alpha$ consisting of all $X \in \mathfrak{g}$ such that $[H, X] = \alpha(H)X$ for all H in $\mathfrak{a}$.) Put $2\rho = \sum_{\alpha \in \Sigma} m_\alpha \alpha$. Then it is obvious that

$$D(h) \leq e^{2\rho(\log h)} \qquad (h \in A^+).$$

Now $2\rho = \sum_{1 \leq i \leq l} m_i \alpha_i$ where m_i are positive integers. Put $\tau_i = \alpha_i(\log h)$. Then $dh = c_1 d\tau_1 \dots d\tau_l$ where c_1 is a positive constant and

$$e^{2\rho(\log h)} = \exp(m_1\tau_1 + \dots + m_l\tau_l).$$

Now if $h \in A^+(t)$, we have

$$1 \leq e^{\alpha_i(\log h)} \leq \| h \| \leq t$$

and therefore $0 \leq \tau_i \leq \log t$. Hence

$$\mu(G(t)) \leq \int_{A^+(t)} e^{2\rho(\log h)}\, dh \leq ct^M$$

where $c = c_1/(m_1 m_2 \dots m_l)$ and $M = m_1 + \dots + m_l$.

Lemma 38. — *There exists a compact neighborhood* U *of* 1 *in* G *and two constants* $a_1, c_1 > 0$ *with the following property. For any* $t \geq 1$, *we can choose a finite set of points* x_i $(1 \leq i \leq N(t))$ *in* G *such that*

1) $G(t) \subset \bigcup_i x_i U$;
2) $\|x_i\| \leq a_1 t$;
3) $N(t) \leq c_1 t^M$.

By a theorem of Borel [1, Theorem C], there exists a discrete subgroup Γ of G such that $\Gamma \backslash G$ is compact. Choose a compact neighborhood U of 1 in G such that $U = U^{-1}$ and $G = \Gamma U$. Put

$$\Gamma(t) = \Gamma \cap (G(t)U).$$

Select a compact neighborhood $V=V^{-1}$ of 1 in U such that $V^2 \cap \Gamma=\{1\}$ $(V^2=VV)$. Then the union

$$\bigcup_{\gamma \in \Gamma(t)} \gamma V=\Gamma(t)V$$

is disjoint and

$$\Gamma(t)V \subset G(t)UV \subset G(t)U^2.$$

Choose $a_1 \geq 1$ so large that $U^2 \subset G(a_1)$. Then

$$\Gamma(t)V \subset G(t)G(a_1) \subset G(ta_1)$$

since $\|xy\| \leq \|x\|.\|y\|$ $(x, y \in G)$. Hence

$$\mu(\Gamma(t)V) \leq \mu(G(ta_1)) \leq c{a_1}^M t^M$$

from Lemma 37. But since the above union was disjoint,

$$\mu(\Gamma(t)V)=N(t)\mu(V)$$

where $N(t)$ is the number of elements in $\Gamma(t)$. Hence $N(t) \leq c_1 t^M$ where $c_1=ca_1^M/\mu(V)$. Let x_i $(1 \leq i \leq N(t))$ be all the elements of $\Gamma(t)$. Since $\Gamma(t) \subset G(t)U \subset G(ta_1)$, it follows that $\|x_i\| \leq a_1 t$. Finally since $G=\Gamma U$, it is obvious that

$$G(t) \subset \Gamma(t)U=\bigcup_i x_i U.$$

§ 17. PROOF OF LEMMA 39

Fix a number $a>0$ and let $\Omega=\Omega_a$ in the notation of Lemma 35. For $0 \leq s<t$, let $\Omega(s, t)$ denote the set of all $X \in \Omega$ with $s \leq \|X\|<t$. Also put $\Omega(t)=\Omega(0, t)$.

Lemma 39. — Let T *be an invariant distribution on* $\mathfrak{g}$. *Then there exist elements* $p_1, \ldots, p_r$ *in* $S(\mathfrak{g}_c)$ *and an integer* $\nu \geq 0$ *such that*

$$|T(f)| \leq (1+t)^\nu \sum_{1 \leq i \leq r} \sup|\partial(p_i)f|$$

for all $f \in C_c^\infty(\Omega(t))$ *and* $t>0$.

This requires some preparation. As before let ω_t $(t>0)$ be the set of all points $X \in \mathfrak{g}$ with $\|X\|<t$.

Lemma 40. — Define b, c and m as in Lemma 36 and for any $t \geq 0$ *let* G_t *denote the set of all* $x \in G$ *with* $\|x\| \leq c(1+t)^m$. *Then* $\omega_b^{G_t} \supset \Omega(t)$.

This is obvious from Lemma 36.

Define U and M as in Lemma 38.

Lemma 41. — There exist two numbers $c_1, c_2>0$ *with the following property. For any* $t>0$, *we can choose a finite set* F_t *of points in* G *such that:*

1) $G_t \subset F_t U$;
2) $\|x\| \leq c_1(1+t)^m$ *for* $x \in F_t$;
3) $[F_t] \leq c_2(1+t)^{mM}$.

Here $[F_t]$ *denotes the number of elements in* F_t.

This follows immediately from Lemma 38 if we note that $G_t = G(t')$ where $t' = c(1+t)^m$.

Now choose $b_1 > b$ such that $\mathrm{Cl}(\omega_b)^U \subset \omega_{b_1}$ and fix $\alpha \in C_c^\infty(\omega_{b_1})$ such that $0 \leq \alpha \leq 1$ and $\alpha = 1$ on $\omega_b{}^U$. For any $t > 0$, put

$$\varphi_t = \sum_{x \in F_t} \alpha^x.$$

Since $\Omega(t) \subset \omega_b^{G_t} \subset (\omega_b^U)^{F_t}$ and $\alpha = 1$ on ω_b^U, it is clear that $\varphi_t \geq 1$ on $\Omega(t)$. Put $\alpha_x = \alpha^x/\varphi_t$ on $\Omega(t)$ $(x \in F_t)$.

Lemma 42. — Given $p \in S(\mathfrak{g}_c)$, *we can choose a number* $c(p) \geq 0$ *and an integer* $m(p) \geq 0$ *such that*

$$\sup |\partial(p)\alpha^x| \leq c(p)(1+t)^{m(p)}$$

for $x \in F_t$ *and* $t > 0$.

Let V be the subspace of $S(\mathfrak{g}_c)$ spanned by p^y $(y \in G)$ and let $p_1, \ldots, p_r$ be a base for V. Then

$$p^y = \sum_{1 \leq i \leq r} a_i(y) p_i$$

where a_i are analytic functions on G. We can choose $c' \geq 0$ and an integer $\nu \geq 0$ such that (see [1] [3(*d*), p. 203])

$$|a_i(y)| \leq c' \|y\|^\nu \qquad (y \in G,\ 1 \leq i \leq r).$$

Then

$$\partial(p)\alpha^x = (\partial(p^{x^{-1}})\alpha)^x = \sum_i a_i(x^{-1})(\partial(p_i)\alpha)^x.$$

Hence

$$\sup |\partial(p)\alpha^x| \leq c_0 \|x^{-1}\|^\nu$$

where $c_0 = c' \sum_i \sup |\partial(p_i)\alpha|$. If $x = k_1 h k_2$ $(k_1, k_2 \in K, h \in A)$, it is obvious that $\|x\| = \|h\|$. Moreover θ is a unitary transformation of $\mathfrak{g}$ and therefore since $\theta \mathrm{Ad}(h)\theta^{-1} = \mathrm{Ad}(h^{-1})$, it is clear that $\|h\| = \|h^{-1}\| = \|x^{-1}\|$. This shows that $\|x^{-1}\| = \|x\|$ and therefore

$$\sup |\partial(p)\alpha^x| \leq c_0 \|x\|^\nu \leq c_0 c_1{}^\nu (1+t)^{m\nu}$$

since $\|x\| \leq c_1(1+t)^m$ for $x \in F_t$. So we can take $c(p) = c_0 c_1{}^\nu$ and $m(p) = m\nu$.

Corollary 1. — $\sup |\partial(p)\varphi_t| \leq c(p) c_2 (1+t)^{m(p)+mM}$ $(t > 0)$.

This is obvious since $[F_t] \leq c_2(1+t)^{mM}$.

Corollary 2. — Given $p \in S(\mathfrak{g}_c)$, *we can choose* $c'(p) \geq 0$ *and an integer* $\mu(p) \geq 0$ *such that*

$$\sup_{\Omega(t)} |\partial(p)\alpha_x| \leq c'(p)(1+t)^{\mu(p)}$$

for $x \in F_t$ *and* $t > 0$.

Since $\alpha_x = \alpha^x/\varphi_t$ and $\varphi_t \geq 1$ on $\Omega(t)$, this is an immediate consequence of Lemma 42 and Corollary 1 above.

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[1] The proof of Lemma 6 of [3(*d*)] is clearly independent of the assumption that rank $\mathfrak{g}$ = rank $\mathfrak{k}$ which was made at the beginning of § 3 of [3(*d*)].

Now we come to the proof of Lemma 39. Put $f_x = \alpha_x f$ $(x \in F_t)$. Since $\sum_{x \in F_t} \alpha_x = 1$ on $\Omega(t)$, it is obvious that

$$f = \sum_{x \in F_t} f_x$$

and therefore

$$T(f) = \sum_{x \in F_t} T(f_x).$$

But $T(f_x) = T((f_x)^{x^{-1}})$, since T is invariant. Moreover

$$\operatorname{Supp} f_x \subset \operatorname{Supp} f \cap \operatorname{Supp} \alpha_x \subset \operatorname{Supp} \alpha^x.$$

Hence

$$\operatorname{Supp} (f_x)^{x^{-1}} \subset \operatorname{Supp} \alpha \subset \omega_{b_1}.$$

Since ω_{b_1} is relatively compact in $\mathfrak{g}$, we can select $p_1, \ldots, p_r \in S(\mathfrak{g}_c)$ such that

$$|T(g)| \leq \sum_{1 \leq i \leq r} \sup |\partial(p_i) g| \qquad (g \in C_c^\infty(\omega_{b_1})).$$

Therefore

$$|T(f_x)| = |T((f_x)^{x^{-1}})| \leq \sum_i \sup |\partial(p_i) f_x^{x^{-1}}|.$$

But

$$\sup |\partial(p_i) f_x^{x^{-1}}| = \sup |\partial(p_i^x) f_x|.$$

Let V be the subspace of $S(\mathfrak{g}_c)$ spanned by p_i^y $(y \in G, 1 \leq i \leq r)$ and let q_j $(1 \leq j \leq s)$ be a base for V. Then

$$p_i^y = \sum_{1 \leq j \leq s} a_{ij}(y) q_j \qquad (y \in G)$$

where a_{ij} are analytic functions on G. Moreover we can choose $c_0 \geq 0$ and an integer $\nu \geq 0$ such that $|a_{ij}(y)| \leq c_0 \|y\|^\nu$ for $y \in G$ (see [3(d), p. 302]). Then

$$\sum_{1 \leq i \leq r} \sup |\partial(p_i) f_x^{x^{-1}}| \leq c_0 \|x\|^\nu r \sum_{1 \leq j \leq s} \sup |\partial(q_j) f_x|.$$

We can obviously select q_{kj}, q'_{kj} in $S(\mathfrak{g}_c)$ $(1 \leq k \leq u, 1 \leq j \leq s)$ such that

$$\partial(q_j)(\beta\gamma) = \sum_{1 \leq k \leq u} \partial(q_{kj})\beta \cdot \partial(q_{kj}')\gamma \qquad (1 \leq j \leq s)$$

for any two C^∞ functions β and γ on $\mathfrak{g}$. Then since $f_x = \alpha_x f$, we get

$$\sum_j \sup |\partial(q_j) f_x| \leq \sum_{k,j} \sup_{\Omega(t)} |\partial(q_{kj})\alpha_x| \, |\partial(q'_{kj}) f|.$$

Therefore

$$|T(f_x)| \leq c_0 r \|x\|^\nu \sum_{1 \leq k \leq u} \sum_{1 \leq j \leq s} \sup_{\Omega(t)} |\partial(q_{kj})\alpha_x| \cdot \sup |\partial(q'_{kj}) f|.$$

Now $\|x\| \leq c_1(1+t)^m$ for $x \in F_t$ (Lemma 41). Therefore we get the following result from Corollary 2 of Lemma 42. There exists a number $c_3 \geq 0$ and an integer $m_3 \geq 0$ such that

$$|T(f_x)| \leq c_3(1+t)^{m_3} \sum_{k,j} \sup |\partial(q_{kj}') f|$$

for $x \in F_t$, $f \in C_c^\infty(\Omega(t))$ and $t > 0$. Since $f = \sum_{x \in F_t} f_x$ and $[F_t] \leq c_2(1+t)^{mM}$ (Lemma 41), we conclude that

$$|T(f)| \leq c_4(1+t)^{m_4} \sum_{k,j} \sup |\partial(q_k')f|$$

where $c_4 = c_2 c_3$ and $m_4 = m_3 + mM$. Obviously this implies the statement of Lemma 39.

§ 18. PROOF OF LEMMA 43

For any $a > 0$ define Ω_a as in Lemma 35.

Lemma 43. — *Let* T *be an invariant distribution on* $\mathfrak{g}$ *and fix a number* $a > 0$. *Then we can choose* $p_1, \ldots, p_r$ *in* $S(\mathfrak{g}_c)$ *and an integer* $d \geq 0$ *such that*

$$|T(f)| \leq \sum_{1 \leq i \leq r} \sup (1 + \|X\|)^d |f(X; \partial(p_i))|$$

for all $f \in C_c^\infty(\Omega_a)$.

We need some preliminary work. Fix a function $\alpha \in C_c^\infty(\mathbf{R})$ such that 1) $\alpha(-t) = \alpha(t)$, 2) $0 \leq \alpha \leq 1$, 3) $\alpha(t) = 1$ if $|t| \leq 1/2$ and $\alpha(t) = 0$ if $|t| \geq 3/4$ ($t \in \mathbf{R}$). Put $\alpha_k(t) = \alpha(t-k)$ for any integer k and let

$$\beta = \sum_{-\infty < k < \infty} \alpha_k.$$

Fix $t_0 \in \mathbf{R}$ and select an integer k_0 such that $|t_0 - k_0| \leq 1/2$. Then $\alpha_{k_0}(t_0) = 1$ and therefore $\beta(t_0) \geq 1$. Moreover $t_0 \notin \operatorname{Supp} \alpha_k$ unless $|t_0 - k| \leq 3/4$. Since the closed interval of length $3/2$ with t_0 at its center, can contain at most two integral points, it is clear that

$$|(d^m\beta/dt^m)_{t=t_0}| \leq 2 \sup_t |(d^m\alpha/dt^m)|$$

for any integer $m \geq 0$. Therefore $1 \leq \beta \leq 2$ everywhere and

$$\sup_t |(d^m\beta/dt^m)| \leq 2 \sup_t |(d^m\alpha/dt^m)| < \infty.$$

Put $\gamma_k = \alpha_k/\beta$. Then it is clear that

$$\sup_t |(d^m\gamma_k/dt^m)|$$

is finite and independent of k. We denote it by c_m.

Since $0 \notin \operatorname{Supp} \alpha_k$ if $k \neq 0$, it is clear that $\beta = 1$ around the origin. Hence $B(s) = \beta(|s|^{1/2})$ ($s \in \mathbf{R}$) is a C^∞ function on $\mathbf{R}$ and

$$\sup_s |(d^mB/ds^m)| = \sup_t |(d/2tdt)^m\beta|.$$

Since $\beta = 1$ around zero, it is clear that

$$\sup_t |t^{-p}(d^q\beta/dt^q)| < \infty$$

for two integers p and q ($p\geq 0$, $q\geq 1$). Hence it follows that

$$\sup_s |(d^m B/ds^m)| < \infty.$$

Similarly if $A_k(s)=\alpha_k(|s|^{1/2})$ ($s\in\mathbf{R}$), one sees that A_k is a function of class C^∞. Moreover if $k\geq 0$, it follows in the same way that

$$\sup_s |(d^m A_k/ds^m)| = \sup_{t\geq 0} |(d/2tdt)^m \alpha_k|$$
$$\leq \sup_{t\geq 0} |(d/2(t+k)dt)^m \alpha| \leq c'_m$$

where c'_m is a positive number independent of k.

Now put $g(X)=\beta(\|X\|)=B(\|X\|^2)$, $h_k(X)=\alpha_k(\|X\|)=A_k(\|X\|^2)$ for $X\in\mathfrak{g}$ and $k\geq 0$. Since $Q: X\rightarrow\|X\|^2$ is a quadratic form on $\mathfrak{g}$, it is obvious that g and h_k are C^∞ functions on $\mathfrak{g}$.

Lemma 44. — Let p be an element in $S(\mathfrak{g}_c)$ of degree $\leq d$. Then we can choose a number $c_p\geq 0$ such that

$$|g(X;\partial(p))|\leq c_p(1+\|X\|)^d, \qquad |h_k(X;\partial(p))|\leq c_p(1+\|X\|)^d$$

for $X\in\mathfrak{g}$ and $k\geq 0$.

One proves by an easy induction on d that there exist polynomial functions q_j ($0\leq j\leq d$) on $\mathfrak{g}$ of degrees $\leq d$ such that

$$g(X;\partial(p))=\sum_{0\leq j\leq d} q_j(X)(d^jB/ds^j)_{s=\|X\|^2},$$
$$h_k(X;\partial(p))=\sum_{0\leq j\leq d} q_j(X)(d^jA_k/ds^j)_{s=\|X\|^2}$$

for $X\in\mathfrak{g}$ and $k\geq 0$. Our assertion now follows immediately from the facts proved above.

Put $g_k=h_k/g$ ($k\geq 0$). Since $g\geq 1$, g_k is also of class C^∞.

Corollary. — We can choose $c'_p\geq 0$ such that

$$|g_k(X;\partial(p))|\leq c'_p(1+\|X\|)^d$$

for $X\in\mathfrak{g}$ and $k\geq 0$.

This is obvious from Lemma 44 if we take into account the fact that $g\geq 1$.

We now come to the proof of Lemma 43. Since $\alpha_k(t)=0$ for $k<0$ and $t\geq 0$, it follows that $\sum_{k\geq 0} g_k=1$. Fix $f\in C_c^\infty(\Omega_a)$ and put $f_k=g_kf$. Then $\sum_{k\geq 0} f_k=f$. It is clear that if $X\in\operatorname{Supp} g_k$, then $|\|X\|-k|\leq 3/4$. Therefore $f_k=0$ if k is large. Hence

$$T(f)=\sum_{k\geq 0} T(f_k).$$

Define $\Omega(s,t)$ and $\Omega(t)$ $(0 \leq s < t)$ for $\Omega = \Omega_a$ as in the beginning of § 17. Then $\mathrm{Supp} f_k \subset \Omega(k+1)$. Therefore, by Lemma 39, we can choose $p_1, \ldots, p_r$ in $S(\mathfrak{g}_c)$ and an integer $\nu \geq 0$ such that

$$|T(f_k)| \leq (2+k)^\nu \sum_{1 \leq i \leq r} \sup |\partial(p_i) f_k|$$

for all $f \in C_c^\infty(\Omega_a)$ and all $k \geq 0$. Moreover since $\|X\| \geq k-1$ if $X \in \mathrm{Supp} f_k$, it follows that

$$(2+k)^{\nu+2} \sup |\partial(p_i) f_k| \leq \sup (3 + \|X\|)^{\nu+2} |f_k(X; \partial(p_i))|.$$

Choose q_{ij}, q'_{ij} in $S(\mathfrak{g}_c)$ $(1 \leq i \leq r,\ 1 \leq j \leq s)$ such that

$$\partial(p_i)(F_1 F_2) = \sum_j \partial(q_{ij}) F_1 \cdot \partial(q'_{ij}) F_2 \qquad (1 \leq i \leq r)$$

for any two C^∞ functions F_1, F_2 on $\mathfrak{g}$. Then

$$|f_k(X; \partial(p_i))| \leq \sum_j |g_k(X; \partial(q_{ij}))| \, |f(X; \partial(q'_{ij}))|$$

since $f_k = g_k f$. Therefore there exist, from the corollary of Lemma 44, an integer $d_0 \geq 0$ and a number $c \geq 0$ such that

$$|f_k(X; \partial(p_i))| \leq c(1 + \|X\|)^{d_0} \sum_j |f(X; \partial(q'_{ij}))|$$

for all $f \in C_c^\infty(\Omega_a)$, $k \geq 0$, $X \in \mathfrak{g}$ and $1 \leq i \leq r$. Hence

$$(2+k)^{\nu+2} \sup |\partial(p_i) f_k| \leq 3^{\nu+2} c \sum_j \sup (1 + \|X\|)^d |\partial(q'_{ij}) f|$$

where $d = d_0 + \nu + 2$. Put

$$c_0 = 3^{\nu+2} c \sum_{k \geq 0} (k+2)^{-2} < \infty.$$

Then it follows that

$$|T(f)| \leq \sum_{k \geq 0} |T(f_k)| \leq c_0 \sum_{i,j} \sup (1 + \|X\|)^d |\partial(q'_{ij}) f|$$

for $f \in C_c^\infty(\Omega_a)$. This completes the proof of Lemma 43.

§ 19. COMPLETION OF THE PROOF OF LEMMA 29

As usual we identify $\mathfrak{g}_c$ with its dual under the Killing form. Call an element $p \in S(\mathfrak{g}_c)$ real if $p(X)$ is real for $X \in \mathfrak{g}$. Then we can select $p_1, \ldots, p_r$ in $I(\mathfrak{g}_c)$ such that 1) p_i is real and homogeneous of degree ≥ 1 and 2) $I(\mathfrak{g}_c) = \mathbf{C}[p_1, \ldots, p_r]$. Put

$$q(X) = \sum_{1 \leq i \leq r} p_i(X)^2 \qquad (X \in \mathfrak{g}).$$

Lemma 45. — *We can choose a number $\delta > 0$ such that $q(X) < \delta$ $(X \in \mathfrak{g})$ implies that $X \in \Omega_a$.*

Suppose this is false. Then we can choose a sequence $X_k \in \mathfrak{g}$ $(k \geq 1)$ such that $q(X_k) \to 0$ and $X_k \notin \Omega_a$. Let Y_k and Z_k respectively be the semisimple and nilpotent components of X_k (see § 3). Then $Y_k \in \mathrm{Cl}(X_k^G)$ from the corollary of Lemma 7. Therefore $q(Y_k)=q(X_k)$. Since Ω_a is open and invariant and $X_k \notin \Omega_a$, it is clear that $Y_k \notin \Omega_a$. Therefore $q(Y_k)=q(X_k)\to 0$ and $Y_k \notin \Omega_a$.

Let $\mathfrak{h}_1, \ldots, \mathfrak{h}_m$ be a maximal set of Cartan subalgebras of $\mathfrak{g}$, no two of which are conjugate under G. Y_k, being semisimple, lies in some Cartan subalgebra of $\mathfrak{g}$ which must be conjugate to $\mathfrak{h}_j$ for some j. Hence we can choose $x_k \in G$ and an index j_k such that $Y_k^{x_k} \in \mathfrak{h}_{j_k}$. By choosing a subsequence we may assume that $H_k = Y_k^{x_k} \in \mathfrak{h}$ $(k \geq 1)$ where $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{g}$. Then $q(H_k)=q(Y_k)\to 0$ and therefore it is obvious that $p(H_k)\to 0$ for any $p \in I(\mathfrak{g}_c)$ which is homogeneous of degree ≥ 1. Now define q_j $(1 \leq j \leq n)$ in $I(\mathfrak{g}_c)$ by

$$\det(t - \mathrm{ad}\, X) = t^n + \sum_{1 \leq j \leq n} q_j(X) t^{n-j} \qquad (X \in \mathfrak{g}),$$

where t is an indeterminate. Then q_j is homogeneous of positive degree and therefore $q_j(H_k)\to 0$. However

$$\det(t - \mathrm{ad}\, H) = t^l \prod_{\alpha>0} (t - \alpha(H)^2) \qquad (H \in \mathfrak{h})$$

where $l = \dim \mathfrak{h}$ and α runs over all positive roots of $(\mathfrak{g}, \mathfrak{h})$. Therefore $\alpha(H_k)\to 0$ for every root α and hence $H_k \to 0$. But then $\|H_k\| < a$ if k is large and therefore $Y_k = x_k^{-1} H_k \in \Omega_a$, giving a contradiction with our earlier result. This proves Lemma 45.

Corollary 1. — There exists a C^∞ *function* g *on* $\mathfrak{g}$ *such that:*

1) g *is invariant and* $\mathrm{Supp}\, g \subset \Omega_a$;

2) $g = 1$ *around zero*;

3) *for any* $p \in S(\mathfrak{g}_c)$, *we can choose* $c_p, m_p \geq 0$ *such that*

$$|g(X; \partial(p))| \leq c_p (1 + \|X\|)^{m_p} \qquad (X \in \mathfrak{g}).$$

Select a C^∞ function F on $\mathbf{R}$ such that 1) $F(t)=F(-t)$, 2) $F(t)=1$ if $|t| \leq \delta/3$ and $F(t)=0$ if $|t| \geq \delta/2$ $(t \in \mathbf{R})$. Put

$$g(X) = F(q(X)) \qquad (X \in \mathfrak{g}).$$

If $X \in \mathrm{Supp}\, g$, it is clear that $q(X) \leq \delta/2$ and therefore $X \in \Omega_a$. Moreover $g(X)=1$ if $q(X) \leq \delta/3$. Fix $p \neq 0$ in $S(\mathfrak{g}_c)$ and let $d = d^0 p$. Then it is clear that

$$g(X; \partial(p)) = \sum_{0 \leq j \leq d} (d^j F / dt^j)_{t=q(X)} p_j(X) \qquad (X \in \mathfrak{g})$$

where p_j $(0 \leq j \leq d)$ are suitable elements in $S(\mathfrak{g}_c)$. Hence g obviously satisfies condition 3).

Corollary 2. — Let T *be an invariant distribution on* Ω_a. *Then* $g^2 T$ *is a tempered distribution on* $\mathfrak{g}$.

Put $T_k = g^k T$ $(k=1, 2)$. Then T_k is an invariant distribution on $\mathfrak{g}$. We now apply Lemma 43 to T_1. So we can choose an integer $d \geq 0$ and elements $p_i \in S(\mathfrak{g}_c)$ $(1 \leq i \leq r)$ such that

$$|T_1(f)| \leq \sum_{1 \leq i \leq r} \sup (1 + \|X\|)^d |f(X; \partial(p_i))|$$

for $f\in C_c^\infty(\Omega_a)$. Therefore if $f\in C_c^\infty(\mathfrak{g})$, we have

$$|T_2(f)|=|T_1(f_1)|\leq\sum_i \sup(1+||X||)^d|f_1(X;\partial(p_i))|$$

where $f_1=gf$. Now select $p_{ij}, q_{ij}\in S(\mathfrak{g}_c)$ $(1\leq i\leq r, 1\leq j\leq s)$ in such a way that

$$\partial(p_i)(\varphi_1\varphi_2)=\sum_j \partial(p_{ij})\varphi_1 . \partial(q_{ij})\varphi_2 \qquad (1\leq i\leq r)$$

for any two C^∞ functions φ_1, φ_2 on $\mathfrak{g}$. Then

$$\partial(p_i)f_1=\sum_j \partial(p_{ij})g . \partial(q_{ij})f.$$

Therefore, by condition 3) of Corollary 1 above, it is obvious that there exist $c, m\geq 0$ such that

$$|T_2(f)|\leq c\sum_{i,j} \sup(1+||X||)^{d+m}|f(X;\partial(q_{ij}))|$$

for $f\in C_c^\infty(\mathfrak{g})$. This proves that T_2 is tempered.

We can now complete the proof of Lemma 29. Since Ω is an open neighborhood of zero, we can choose $a>0$ such that $X\in\Omega$ whenever $||X||<a$ $(X\in\mathfrak{g})$. Therefore $\Omega_a\subset\Omega$. Now take $f=g^2$ where g is defined as in Corollary 1 of Lemma 45. Then it follows from Corollary 2 above that fT is a tempered distribution on $\mathfrak{g}$. This proves Lemma 29.

§ 20. PROOF OF LEMMA 34

We shall now begin the proof of Lemma 34. Since any two norms on $\mathfrak{g}$ are equivalent, it is enough to consider the case when $\nu(X)=||X||$ $(X\in\mathfrak{g})$. The case $l=0$ being trivial, we assume $l\geq 1$ and use induction. For any (real-valued) linear function λ on $\mathfrak{a}$, let $\mathfrak{g}_\lambda$ denote the space of all $X\in\mathfrak{g}$ such that $[H, X]=\lambda(H)X$ for all $H\in\mathfrak{a}$. We denote by E_λ the orthogonal projection of $\mathfrak{g}$ on $\mathfrak{g}_\lambda$. Then $\mathfrak{g}_\lambda=\{0\}$ unless $\lambda=0$ or $\pm\alpha$ for some $\alpha\in\Sigma$. Since ad H is self-adjoint for $H\in\mathfrak{a}$ (see [3(*h*), Lemma 27]), the spaces $\mathfrak{g}_\lambda$ and $\mathfrak{g}_\mu$ $(\lambda\neq\mu)$ are mutually orthogonal. Therefore if

$$E_+=\sum_{\alpha\in\Sigma}E_\alpha, \qquad E_-=\sum_{\alpha\in\Sigma}E_{-\alpha},$$

it is clear that $E_++E_0+E_-=1$.

Let S denote the set $\{1, 2, \ldots, l\}$ and for any subset Q of S, let Σ_Q denote the set of all $\alpha\in\Sigma$ which are linear combinations of α_i $(i\in Q)$. Define $\mathfrak{n}_Q=\sum_{\alpha\in\Sigma_Q}\mathfrak{g}_\alpha$ and let $\mathfrak{g}_Q$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{n}_Q+\theta(\mathfrak{n}_Q)$. Then $\theta(\mathfrak{g}_Q)=\mathfrak{g}_Q$ and therefore $\mathfrak{g}_Q=\mathfrak{k}_Q+\mathfrak{p}_Q$ where $\mathfrak{k}_Q=\mathfrak{k}\cap\mathfrak{g}_Q$, $\mathfrak{p}_Q=\mathfrak{p}\cap\mathfrak{g}_Q$.

Lemma 46. — $\mathfrak{g}_Q$ *is semisimple.*

Let $\langle X, Y\rangle=-B(X, \theta(Y))$ $(X, Y\in\mathfrak{g})$ denote the scalar product in the Hilbert space $\mathfrak{g}$ and, for any linear function λ on $\mathfrak{a}$, let H_λ denote the element in $\mathfrak{a}$ such that

$\langle H, H_\lambda \rangle = \lambda(H)$ for all $H \in \mathfrak{a}$. We know (see [3(*d*), Lemma 3]) that if $X \in \mathfrak{g}_\lambda$ and $\|X\| = 1$, then $[\theta(X), X] = H_\lambda$.

First we claim that $\mathfrak{g}_Q$ is reductive in $\mathfrak{g}$. Let U be any subspace of $\mathfrak{g}$ such that $[\mathfrak{g}_Q, U] \subset U$. Since $\mathfrak{g}_Q = \theta(\mathfrak{g}_Q)$, ad $\mathfrak{g}_Q$ is a self-adjoint family of transformations in $\mathfrak{g}$ [3(*h*), Lemma 27]. Hence if V is the orthogonal complement of U in $\mathfrak{g}$, V is stable under ad $\mathfrak{g}_Q$. This proves our assertion. Therefore $\mathfrak{g}'_Q = [\mathfrak{g}_Q, \mathfrak{g}_Q]$ is semisimple. Now fix $\alpha \in \Sigma_Q$ and $X \in \mathfrak{g}_\alpha$ with $\|X\| = 1$. Then $[\theta(X), X] = H_\alpha \in \mathfrak{g}_Q$ and therefore $[H_\alpha, X] = \alpha(H_\alpha) X \in \mathfrak{g}'_Q$. Since $\alpha(H_\alpha) = \|H_\alpha\|^2 > 0$, this proves that $\mathfrak{g}_\alpha \subset \mathfrak{g}'_Q$. However $\mathfrak{g}'_Q$ is obviously stable under θ and so we conclude that $\mathfrak{n}_Q + \theta(\mathfrak{n}_Q) \subset \mathfrak{g}'_Q$. But, in view of the definition of $\mathfrak{g}_Q$, this implies that $\mathfrak{g}'_Q = \mathfrak{g}_Q$. This proves that $\mathfrak{g}_Q$ is semisimple.

Let F_Q denote the orthogonal projection of $\mathfrak{g}$ on $\mathfrak{g}_Q$. We have seen above that $H_\alpha \in \mathfrak{a} \cap \mathfrak{g}_Q$ for $\alpha \in \Sigma_Q$. Put $\mathfrak{a}_Q = \sum_{i \in Q} \mathbf{R} H_{\alpha_i}$ and let $\mathfrak{b}_Q$ denote the orthogonal complement of $\mathfrak{a}_Q$ in $\mathfrak{a}$. Then $\mathfrak{a}_Q = \sum_{\alpha \in \Sigma_Q} \mathbf{R} H_\alpha \subset \mathfrak{g}_Q$.

Lemma 47. — $\mathfrak{a}_Q = \mathfrak{a} \cap \mathfrak{g}_Q$. *Moreover* F_Q *commutes with* θ *and* E_λ *for any linear function* λ *on* $\mathfrak{a}$.

Let $H \in \mathfrak{b}_Q$. Then $\alpha_i(H) = \langle H_{\alpha_i}, H \rangle = 0$ $(i \in Q)$ and therefore $\alpha(H) = 0$ for $\alpha \in \Sigma_Q$. Hence H commutes with $\mathfrak{n}_Q + \theta(\mathfrak{n}_Q)$ and therefore also with $\mathfrak{g}_Q$. Since $\mathfrak{g}_Q$ is semisimple, it follows that $\mathfrak{g}_Q \cap \mathfrak{b}_Q = \{0\}$. Therefore since $\mathfrak{a}_Q \subset \mathfrak{g}_Q$, it is obvious that $\mathfrak{a} \cap \mathfrak{g}_Q = \mathfrak{a}_Q$.

Let $\mathfrak{m}_Q$ be the set of all $X \in \mathfrak{g}_Q$ such that $[H, X] \in \mathfrak{g}_Q$ for all $H \in \mathfrak{a}$. Then $\mathfrak{m}_Q$ is a subalgebra of $\mathfrak{g}_Q$ which contains $\mathfrak{n}_Q + \theta(\mathfrak{n}_Q)$. Hence $\mathfrak{m}_Q = \mathfrak{g}_Q$. Therefore $\mathfrak{g}_Q$ is stable under ad H $(H \in \mathfrak{a})$ and this implies that $E_\lambda \mathfrak{g}_Q \subset \mathfrak{g}_Q$ for any linear function λ on $\mathfrak{a}$. This shows that F_Q commutes with E_λ. Similarly since $\mathfrak{g}_Q$ is stable under θ, F_Q commutes with θ.

Corollary. — $\mathfrak{a}_Q$ *is maximal abelian in* $\mathfrak{p}_Q$ *and* $\mathfrak{a}_Q = F_Q \mathfrak{a}$.

Since $\mathfrak{g}_0 + \mathfrak{n}_Q + \theta(\mathfrak{n}_Q)$ is a subalgebra of $\mathfrak{g}$, it must contain $\mathfrak{g}_Q$. Therefore

$$X = E_0 X + \sum_{\alpha \in \Sigma_Q} E_\alpha X + \sum_{\alpha \in \Sigma_Q} E_{-\alpha} X \qquad (X \in \mathfrak{g}_Q).$$

Now suppose $X \in \mathfrak{p}_Q$ and it commutes with $\mathfrak{a}_Q$. Then

$$0 = [H, X] = \sum_{\alpha \in \Sigma_Q} \alpha(H) E_\alpha X - \sum_{\alpha \in \Sigma_Q} \alpha(H) E_{-\alpha} X \qquad (H \in \mathfrak{a}_Q)$$

and therefore $\alpha(H) E_{\pm\alpha} X = 0$ for $H \in \mathfrak{a}_Q$ and $\alpha \in \Sigma_Q$. But $H_\alpha \in \mathfrak{a}_Q$ for $\alpha \in \Sigma_Q$ and $\alpha(H_\alpha) = \|H_\alpha\|^2 > 0$. Hence $E_{\pm\alpha} X = 0$ $(\alpha \in \Sigma_Q)$ and therefore $X = E_0 X \in \mathfrak{g}_0$. This means that $X \in \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$ since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$. But then $X \in \mathfrak{a} \cap \mathfrak{g}_Q = \mathfrak{a}_Q$. This proves that $\mathfrak{a}_Q$ is maximal abelian in $\mathfrak{p}_Q$.

Since F_Q commutes with θ and E_0 and $\mathfrak{a} \subset \mathfrak{p}$, it is clear that $F_Q \mathfrak{a} \subset \mathfrak{p}_Q \cap \mathfrak{g}_0$. But since $\mathfrak{a}_Q$ is maximal abelian in $\mathfrak{p}_Q$, $\mathfrak{p}_Q \cap \mathfrak{g}_0 = \mathfrak{a}_Q$. This proves that $F_Q \mathfrak{a} = \mathfrak{a}_Q$.

Let l_Q denote the number of elements in Q. Then dim $\mathfrak{a}_Q = l_Q$. Let G_Q and A_Q be the analytic subgroups of G corresponding to $\mathfrak{g}_Q$ and $\mathfrak{a}_Q$ respectively. If $Q \neq S$, Lemma 34 holds for $(\mathfrak{g}_Q, \mathfrak{a}_Q)$ instead of $(\mathfrak{g}, \mathfrak{a})$ by the induction hypothesis. Let A_Q^+ be the set of all $h \in A_Q$ such that $\alpha_i(\log h) \geq 0$ $(i \in Q)$. Then we obviously have the following result.

Lemma 48. — Assume that $Q \neq S$. *Then there exist numbers* $b_Q, c_Q \geq 1$ *with the following properties. Suppose* $X \in \mathfrak{g}_Q$, $||X|| \leq 1$ *and* $h \in A_Q^+$. *Then we can choose* $X_0 \in \mathfrak{g}_Q$, $h_0 \in A_Q^+$ *such that:*

1) $X^h = X_0^{h_0}$, $||X_0|| \leq b_Q$, $0 \leq \alpha_i(\log h_0) \leq \alpha_i(\log h)$ $\quad (i \in Q)$,
2) $\max_{i \in Q} \exp(\alpha_i(\log h_0)) \leq c_Q(1 + ||X_0^{h_0}||)^{l_Q}$.

Let $A^+(Q)$ be the set of all $h \in A^+$ such that $\alpha_j(\log h) = 0$ $(j \notin Q)$. For any $h \in A$, define

$$h_Q = \exp(\sum_{i \in Q} \alpha_i(\log h) H_i).$$

Then $\alpha_i(\log h) = \alpha_i(\log h_Q)$ $(i \in Q)$ and therefore $\log h - \log h_Q$ commutes with $\mathfrak{g}_Q$ so that $X^h = X^{h_Q}$ $(X \in \mathfrak{g}_Q)$. Moreover if $h \in A^+$, it is clear that $1 \prec h_Q \prec h$ and $h_Q \in A^+(Q)$.

Corollary. — Suppose $X \in \mathfrak{g}_Q$, $||X|| \leq 1$ *and* $h \in A^+(Q)$. *Then we can choose* $X_0 \in \mathfrak{g}_Q$ *and* $h_0 \in A^+(Q)$ *such that*

1) $X^h = X_0^{h_0}$, $||X_0|| \leq b_Q$, $1 \prec h_0 \prec h$,
2) $\max_{1 \leq i \leq l} \exp \alpha_i(\log h_0) \leq c_Q(1 + ||X_0^{h_0}||)^{l_Q}$.

Put $h' = \exp(\sum_{i \in Q} \alpha_i(\log h) F_Q H_i)$. Then $h' \in A_Q^+$ and $(h')_Q = h$ from the corollary of Lemma 47. Hence $X^{h'} = X^h$. Choose $h_0' \in A_Q^+$ and $X_0 \in \mathfrak{g}_Q$ such that the conditions of Lemma 48 hold for (X, h', X_0, h_0') in place of (X, h, X_0, h_0). Then if we put $h_0 = (h_0')_Q$ all the conditions of the corollary are fulfilled.

For any $i \in S$ and $Z \in \mathfrak{g}$, define

$$\mu(i : Z) = \max_{\substack{\alpha \in \Sigma \\ \alpha(H_i) \neq 0}} ||E_\alpha Z||$$

and let $Q(Z)$ be the set of all $i \in S$ for which $\mu(i : Z) \geq 1$. Moreover for any subset Q of S, let Σ_Q' denote the complement of Σ_Q in Σ.

Lemma 49. — Let Z *be an element of* $\mathfrak{g}$. *Then* $||E_\alpha Z|| < 1$ *for every* $\alpha \in (\Sigma_{Q(Z)})'$.

Suppose $||E_\alpha Z|| \geq 1$ for some $\alpha \in \Sigma$. We have to show that $\alpha \in \Sigma_{Q(Z)}$. Fix $i \in S$ such that $\alpha(H_i) \neq 0$. Then

$$\mu(i : Z) \geq ||E_\alpha Z|| \geq 1$$

and therefore $i \in Q(Z)$. Since this holds for every i for which $\alpha(H_i) \neq 0$, it is clear that $\alpha \in \Sigma_{Q(Z)}$.

Put $F_Q' = 1 - F_Q$ for any subset Q of S. Fix $X \in \mathfrak{g}$ and $h \in A^+$ and assume that $||X|| \leq 1$. Put $Q_0 = Q(X^h)$ and let s denote the number of elements in Σ.

Lemma 50. — $||F_{Q_0}' X^a|| \leq 1 + s^{1/2}$ *for any* $a \in A$ *such that* $1 \prec a \prec h$.

Let λ be a linear function on $\mathfrak{a}$ such that $\mathfrak{g}_\lambda \neq \{0\}$. Then

$$E_\lambda X^a = e^{\lambda(\log a)} E_\lambda X.$$

Now $a \succ 1$ and therefore $\lambda(\log a) \leq 0$ if $\lambda \leq 0$. Therefore since F'_{Q_0} commutes with E_0 and E_-, it is obvious that

$$||(E_0+E_-)F'_{Q_0}X^a|| \leq ||(E_0+E_-)X^a|| \leqslant ||X|| \leq 1.$$

On the other hand $\alpha(\log a) \leq \alpha(\log h)$ $(\alpha \in \Sigma)$ since $a \prec h$. Therefore

$$||E_+ F'_{Q_0} X^a||^2 = \sum_{\alpha \in \Sigma'_{Q_0}} ||E_\alpha X^a||^2 \leq \sum_{\alpha \in \Sigma'_{Q_0}} ||E_\alpha X^h||^2 \leq s$$

from Lemma 49. Since

$$F'_{Q_0}X^a = (E_+ + E_0 + E_-)F'_{Q_0}X^a,$$

our assertion is now obvious.

Lemma 51. — For any (¹) $Q<S$, *select* b_Q *and* c_Q *corresponding to Lemma 48 and define*

$$b_0 = 1 + s^{1/2} + \max_{Q<S} b_Q, \qquad c_0 = \max_{Q<S} c_Q.$$

Let $X \in \mathfrak{g}$, $h \in A^+$ *and suppose that* $||X|| \leq 1$ *and* $Q(X^h) \neq S$. *Then we can choose* $X_0 \in \mathfrak{g}$ *and* $h_0 \in A^+$ *such that*

1) $X^h = X_0^{h_0}$, $1 \prec h_0 \prec h$, $||X_0|| \leq b_0$;
2) $\max_{1 \leq i \leq l} \exp \alpha_i(\log h_0) \leq c_0(1 + ||X_0^{h_0}||)^{l-1}$.

Put $Q = Q(X^h)$. Then $X^h = F_Q X^h + F'_Q X^h$. But since F_Q commutes with $\mathrm{Ad}(h)$ (Lemma 47), we have

$$F_Q X^h = (F_Q X)^h = X_Q^{h_Q}$$

where $X_Q = F_Q X$. Since $Q<S$, we can apply the corollary of Lemma 48 to (X_Q, h_Q). Hence we can choose $X_1 \in \mathfrak{g}_Q$ and $h_0 \in A^+(Q)$ such that:

1) $X_Q^{h_Q} = X_1^{h_0}$, $||X_1|| \leq b_Q$, $1 \prec h_0 \prec h_Q$;
2) $\max_{1 \leq i \leq l} \exp \alpha_i(\log h_0) \leq c_Q(1 + ||X_1^{h_0}||)^{l_Q}$.

Then

$$X^h = X_1^{h_0} + F'_Q X^h = (X_1 + F'_Q X^{h_2})^{h_0}$$

where $h_2 = h h_0^{-1}$. Since $1 \prec h_0 \prec h_Q \prec h$, it follows that $1 \prec h_2 \prec h$. Put

$$X_0 = X_1 + F'_Q X^{h_2}.$$

Then

$$||X_0|| \leq ||X_1|| + ||F'_Q X^{h_2}|| \leq b_Q + 1 + s^{1/2} \leq b_0$$

from Lemma 50. Moreover

$$X_1^{h_0} = X_Q^{h_Q} = F_Q X^h.$$

Therefore

$$||X_1^{h_0}|| \leq ||X^h|| = ||X_0^{h_0}||.$$

Hence

$$\max_{1 \leq i \leq l} \exp \alpha_i(\log h_0) \leq c_Q(1 + ||X_1^{h_0}||)^{l_Q} \leq c_0(1 + ||X_0^{h_0}||)^{l-1}$$

and so the lemma is proved.

(¹) $Q<S$ means that Q is a subset of S and $Q \neq S$.

Put $c=2^l c_0$ and $b=b_0$. Then in order to prove Lemma 34, itis obviously enough to verify the following result.

Lemma 52. — *Let* $X\in\mathfrak{g}$ *and* $h\in A^+$ *and suppose* $\|X\|\leq 1$. *Then we can choose* $X_0\in\mathfrak{g}$ *and* $h_0\in A^+$ *such that:*

1) $X^h=X_0^{h_0}$, $\|X_0\|\leq b$, $1\prec h_0\prec h$;

2) $\max_{1\leq i\leq l} \exp \alpha_i(\log h_0)\leq c(1+\|X_0^{h_0}\|)^l$.

If $Q(X^h)<S$, our statement follows immediately from Lemma 51. So we may assume that

$$\mu(i:X^h)\geq 1 \qquad (1\leq i\leq l).$$

Let Ω be the set of all $a\in A^+$ such that 1) $1\prec a\prec h$ and 2) $\mu(i:X^a)\geq 1/2$ $(1\leq i\leq l)$. Obviously Ω is a compact set containing h. Put

$$f(a)=\sum_{1\leq i\leq l}\mu(i:X^a) \qquad (a\in\Omega).$$

Then f is a continuous function on Ω which must take its minimum at some point $a_0\in\Omega$. First suppose $a_0=1$. Then $1\in\Omega$ and therefore

$$\mu(i:X)\geq 1/2 \qquad (1\leq i\leq l).$$

Now fix $i\in S$ and choose $\alpha\in\Sigma$ such that $\alpha(H_i)\neq 0$ and $\|E_\alpha X\|\geq 1/2$. Then

$$\|E_+X^h\|\geq e^{\alpha(\log h)}\|E_\alpha X\|\geq 2^{-1}e^{\alpha_i(\log h)}.$$

Therefore

$$\max_i e^{\alpha_i(\log h)}\leq 2\|E_+X^h\|\leq 2\|X^h\|.$$

Since $b\geq 1$ and $c=2^l c_0\geq 2$, we can take $X_0=X$ and $h_0=h$ in this case.

So now assume that $a_0\neq 1$. Then we claim that $\mu(i:X^{a_0})=1/2$ for some i. For otherwise suppose $\mu(i:X^{a_0})>1/2$ for every i. Choose j such that $\alpha_j(\log a_0)\neq 0$. Put $a_\varepsilon=a_0(\exp(-\varepsilon H_j))$ where ε is a small positive number. If ε is sufficiently small, it is clear that $a_\varepsilon\in\Omega$. Hence $f(a_\varepsilon)\geq f(a_0)$. On the other hand since

$$\|E_\alpha X^{a_\varepsilon}\|=e^{-\varepsilon\alpha(H_j)}\|E_\alpha X^{a_0}\| \qquad (\alpha\in\Sigma),$$

it is clear that

$$\mu(i:X^{a_\varepsilon})\leq\mu(i:X^{a_0})$$

for every i. Moreover $e^{-\varepsilon\alpha(H_j)}<1$ if $\alpha(H_j)\neq 0 (\alpha\in\Sigma)$ and therefore since $\mu(j:X^{a_0})\geq 1/2$, it is obvious that

$$\mu(j:X^{a_\varepsilon})<\mu(j:X^{a_0}).$$

But this implies that $f(a_\varepsilon)<f(a_0)$ and so we get a contradiction. Hence $\mu(i:X^{a_0})=1/2$ for some i and therefore $Q(X^{a_0})<S$. But then by Lemma 51 we can choose $X_0\in\mathfrak{g}$ and $a_1\in A^+$ such that $X^{a_0}=X_0^{a_1}$, $\|X_0\|\leq b_0$, $1\prec a_1\prec a_0$ and

$$\max_{1\leq i\leq l}\exp\alpha_i(\log a_1)\leq c_0(1+\|X_0^{a_1}\|)^{l-1}.$$

Now put $h_0 = ha_0^{-1}a_1$. Then

$$X^h = (X^{a_0})^{ha_0^{-1}} = (X_0^{a_1})^{ha_0^{-1}} = X_0^{h_0}$$

and therefore

$$\|X^h\| \geq \|E_\alpha X^{a_0}\| \exp \alpha(\log(ha_0^{-1})) \qquad (\alpha \in \Sigma).$$

Fix $i \in S$. Then since $\mu(i : X^{a_0}) \geq 1/2$, we can select $\alpha \in \Sigma$ such that $\alpha(H_i) \neq 0$ and $\|E_\alpha X^{a_0}\| \geq 1/2$. Therefore since $1 \prec a_0 \prec h$, we have

$$\|X^h\| \geq 2^{-1} \exp \alpha_i(\log(ha_0^{-1})).$$

On the other hand

$$e^{\alpha_i(\log a_1)} \leq c_0(1 + \|X_0^{a_1}\|)^{l-1} = c_0(1 + \|X^{a_0}\|)^{l-1}.$$

Therefore since $h_0 = ha_0^{-1}a_1$, we get

$$e^{\alpha_i(\log h_0)} \leq 2c_0 \|X^h\| (1 + \|X^{a_0}\|)^{l-1}.$$

But since $1 \prec a_0 \prec h$, we have (see the proof of Lemma 50)

$$\|E_+ X^{a_0}\| \leq \|E_+ X^h\| \leq \|X^h\|$$

and

$$\|(E_0 + E_-)X^{a_0}\| \leq \|X\| \leq 1.$$

Therefore

$$\|X^{a_0}\| \leq 1 + \|X^h\|$$

and hence

$$e^{\alpha_i(\log h_0)} \leq 2c_0 \|X^h\| (2 + \|X^h\|)^{l-1} \leq c(1 + \|X^h\|)^l.$$

Since $\|X_0\| \leq b_0 = b$, Lemma 51 (and therefore also Lemma 34) is proved.

§ 21. PROOF OF LEMMA 35

We have still to prove Lemma 35. Fix $a > b > 0$ and let x_i and X_i $(i \geq 1)$ be two sequences in G and $\mathfrak{g}$ respectively such that $\|X_i\| < b$ and $x_i X_i$ converges to some $Y \in \mathfrak{g}$. We have to prove that $Y \in \Omega_a$. Let $x_i = k_i h_i k_i'$ $(k_i, k_i' \in K; h_i \in A^+)$. Replacing (x_i, X_i) by $(k_i h_i, k_i' X_i)$ we may assume that $x_i = k_i h_i$. Moreover by selecting a subsequence we can arrange that $k_i \to k$ and $X_i \to X$ $(k \in K, X \in \mathfrak{g})$. Then by replacing (x_i, X_i, Y) by $(k^{-1}x_i, X_i, k^{-1}Y)$, we are reduced to the case when $k = 1$. Now

$$X_i^{x_i} - X_i^{h_i} = (1 - \mathrm{Ad}(k_i^{-1}))X_i^{x_i}.$$

Since $X_i^{x_i} \to Y$ and $k_i \to 1$, it is clear that $\|X_i^{x_i} - X_i^{h_i}\| \to 0$. Hence $X_i^{h_i} \to Y$.

By selecting a subsequence we can obviously arrange that the following condition holds. There exists a subset Q of S such that $\alpha_j(\log h_i) \to t_j$ $(t_j \in \mathbf{R})$ for $j \in Q$ and $\alpha_j(\log h_i) \to +\infty$ for $j \notin Q$ $(1 \leq j \leq l)$ as $i \to \infty$. Then it is clear that

$$E_{-\alpha} X_i^{h_i} = e^{-\alpha(\log h_i)} E_{-\alpha} X_i \to 0$$

for $\alpha \in \Sigma_Q'$. Put

$$E = E_0 + \sum_{\alpha \in \Sigma_Q} (E_\alpha + E_{-\alpha})$$

and

$$h = \exp(\sum_{j \in Q} t_j H_j).$$

Then it is clear that

$$EX_i^{h_i} \to EX^h.$$

On the other hand if

$$E'_+ = \sum_{\alpha \in \Sigma'_Q} E_\alpha,$$

we have

$$1 = E + E'_+ + \sum_{\alpha \in \Sigma'_Q} E_{-\alpha}.$$

Therefore since $E_{-\alpha} X_i^{h_i} \to 0$ $(\alpha \in \Sigma'_Q)$, we conclude that

$$(E + E'_+) X_i^{h_i} \to Y.$$

Therefore $Y = EY + E'_+ Y$ and $EY = EX^h$. Now select $H \in \mathfrak{a}$ such that $\alpha_j(H) = 0$ for $j \in Q$ and $\alpha_j(H) > 0$ for $j \notin Q$ $(1 \leq j \leq l)$. Then $\alpha(H) > 0$ for $\alpha \in \Sigma'_Q$ and therefore

$$\mathrm{Ad}(\exp(-tH)) E'_+ Y \to 0$$

as $t \to +\infty$. Put $y_t = (h \exp tH)^{-1}$. Then

$$Y^{y_t} = EX + (E'_+ Y)^{y_t} \to EX$$

as $t \to +\infty$. Since $||EX|| \leq ||X|| \leq b$, it follows that $||Y^{y_t}|| < a$ if t is sufficiently large and positive. Therefore $Y \in \Omega_a$ and this proves Lemma 35.

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