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CARLEMAN ESTIMATES  
FOR THE LAPLACE-BELTRAMI EQUATION  
ON COMPLEX MANIFOLDS

by ALDO ANDREOTTI and EDOARDO VESENTINI <sup>(1)</sup>

Let  $P(x, D)$  be a differential operator defined in an open set  $\Omega$  of  $\mathbf{R}^n$ , with  $C^\infty$  coefficients. Let  $u$  be a  $C^\infty$  function such that  $P(x, D)u$  has compact support in  $\Omega$ . Assume that for any such function  $u$  we have an inequality of the type

$$\int e^{\tau\Phi} |u|^2 dx \leq C \int e^{\tau\Phi} |P(x, D)u|^2 dx$$

for any  $\tau > \tau_0$  and where  $\Phi$  is a positive  $C^\infty$  function on  $\Omega$ . Then it follows that, on the support of  $u$ ,  $\Phi$  does not exceed the maximum of  $\Phi$  on the support of  $P(x, D)u$ . An inequality of the above type is an inequality of Carleman's type [8]; its essential feature is in the presence of the exponential weight factor which permits to give information on the support of  $u$  in terms of the support of  $P(x, D)u$ . This remark which we learned from a paper of L. Hörmander [13] is at the origin of the present paper.

In the first part we establish a general criterion for the vanishing of cohomology with compact support on a complex manifold  $X$ , the coefficients being chosen in a locally free sheaf,  $\mathcal{F}$ , i.e. in the sheaf of germs of holomorphic sections of a holomorphic vector bundle  $E$  on  $X$ .

This is done by the study of the Laplace-Beltrami operator and by use of an inequality of Carleman's type. It turns out that the role of the exponential factor is nothing else than the choice of a metric in the fibres of  $E$ . The possibility of a large freedom of choice in this metric replaces the parameter  $\tau$  of Carleman's inequality.

In the second part of the paper we show how the general theory gives the vanishing theorems for  $q$ -complete spaces established elsewhere by other methods [2]. Here the presence, on the manifold, of a  $C^\infty$  positive function  $\Phi$  whose Levi form has a given signature, gives the desired freedom in the choice of the metric on the fibres of  $E$ . The Carleman inequality is established by using a generalized form of an inequality given by K. Kodaira [14] using a method of Bochner [23].

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The last part of the paper gives, we hope, a sufficiently detailed indication of how to apply the previous results to establish the finiteness theorems for  $q$ -pseudoconvex and  $q$ -pseudoconcave manifolds [2]. Since we deal with cohomology with compact support, we are able to avoid the use of the approximation theorem which was, on the contrary, the essential point in establishing the results for the cohomology with closed supports in [2]. Moreover, for  $q$ -pseudoconcave manifolds we gain additional information (by Serre's duality [20]), namely that the groups  $H^{n-q}(X, \Omega^r(E))$  have a topology of a (separated) Fréchet space.

The case of a general complex space (in the sense of Serre) and of the cohomology on it with values in any coherent sheaf is not treated here. We believe that the methods developed in [14] will be sufficient for the reader to see how to extend the above result to cover this more general case (the starting point being always the case of a locally free sheaf on a manifold).

We are indebted to B. Malgrange for many valuable suggestions and, in particular, for the idea of reducing the theorem of finiteness to a classical theorem of finiteness of L. Schwartz [17]. E. Calabi gave us the idea of the proof of Lemma 18. M. K. V. Murthy and B. V. Singbal of Tata Institute of Fundamental Research helped us in learning the theory of topological vector spaces and the works of L. Hörmander and A. Grothendieck.

To all of these we wish to express our sincere thanks. The results of this paper have been announced in [5].

### § 1. Preliminaries

1. a) Let  $X$  be a complex manifold and let  $E \rightarrow X$  be a holomorphic vector bundle over  $X$  with fibre  $\mathbf{C}^m$ . Let  $\mathcal{U} = (U_i)_{i \in I}$  be a coordinate covering of  $X$  such that on each  $U_i$ ,  $E|_{U_i}$  is isomorphic to the trivial bundle. If  $\Phi_i : U_i \times \mathbf{C}^m \rightarrow E$  are these trivialisations of  $E$ , we denote by

$$e_{ij} : U_i \cap U_j \rightarrow \mathrm{GL}(m, \mathbf{C})$$

the holomorphic cocycle defined by the conditions:

$$\Phi_j^{-1} \circ \Phi_i(z, \xi_i) = (z, e_{ij} \xi_j)$$

where  $\xi_i$  denote the fibre coordinates over  $U_i$ .

The dual bundle  $E^* \rightarrow X$  of the given bundle  $E$  is thus defined on the same covering  $\mathcal{U}$  by the cocycle  ${}^t e_{ij}^{-1}$ .

In particular the tangent bundle  $\Theta$  will be defined in terms of a choice of local coordinates  $(z_i^1, \dots, z_i^n)$  on  $U_i$  by the cocycle  $J_{ij} = \partial(z_i)/\partial(z_j)$ , and the dual bundle  $\Theta^*$  by the cocycle  ${}^t J_{ij}^{-1}$ .

b) A  $\mathbf{C}^\infty$  form of type  $(p, q)$  with values in the bundle  $E$  is a  $\mathbf{C}^\infty$  section of the bundle  $E \otimes \Theta^{*p} \otimes \overline{\Theta^{*q}}$  where  $\Theta^{*p}$  stands for  $\bigwedge^p \Theta^*$  and where the bar over  $\Theta^{*q}$  denotes the complex conjugate of  $\Theta^{*q}$ . Locally on  $U_i$  such a form is given by a column vector

$\varphi_i = (\varphi_i^1, \dots, \varphi_i^m)$  whose components are  $C^\infty$  forms of type  $(p, q)$  on  $U_i$ . In  $U_i \cap U_j$  we will have

$$\varphi_i = e_{ij} \varphi_j.$$

Let  $\bar{\partial}$  denote the exterior differentiation with respect to the complex conjugates of the local holomorphic coordinates. If  $\mathcal{A}^{pq}(E)$  denotes the sheaf of germs of  $C^\infty$  forms of type  $(p, q)$  with values in  $E$ , then  $\bar{\partial}$  defines a sheaf homomorphism

$$\bar{\partial} : \mathcal{A}^{pq}(E) \rightarrow \mathcal{A}^{p, q+1}(E)$$

because  $E$  is a holomorphic vector bundle.

If  $\Omega^p(E)$  is the sheaf of germs of holomorphic sections of  $E \otimes \Theta^{*p}$  we get, by the Dolbeault theorem, an exact sequence

$$0 \rightarrow \Omega^p(E) \rightarrow \mathcal{A}^{p0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p1} \xrightarrow{\bar{\partial}} \dots$$

and therefore the isomorphism

$$H_\Phi^q(X, \Omega^p(E)) \approx H_\Phi^{pq}(X, E)$$

where  $\Phi$  is the family of closed or the family of compact subsets of  $X$  and where  $H_\Phi^{pq}(X, E)$  denotes the homology of the complex  $\bigoplus_{q=0}^\infty \Gamma_\Phi(X, \mathcal{A}^{pq}(E))$ .

In the sequel the space  $\Gamma_\Phi(X, \mathcal{A}^{pq}(E))$  will be denoted by  $C^{pq}(X, E)$  if  $\Phi$  is the family of closed sets, and by  $\mathcal{D}^{pq}(X, E)$  if  $\Phi$  is the family of compact sets.

**2. a)** We introduce on the fibres of  $E$  a hermitian metric. This will be given by a hermitian scalar product  $h(v, w)$ ,  $v, w \in \pi^{-1}(z)$ , which depends differentiably on the base point  $z$ .

Locally on  $U_i$ , if  $\xi_i, \eta_i$  are the fibre-coordinates of  $v$  and  $w$ ,  $h(v, w)$  will be expressed in the form

$$h(v, w) = {}^t \bar{\eta}_i h_i \xi_i$$

where  $h_i$  will be a positive definite hermitian matrix, whose elements depend differentiably (i.e.  $C^\infty$ ) on  $z \in U_i$ .

For this local representation, the consistency conditions are given in  $U_i \cap U_j$  by

$$(1) \quad h_i = {}^t \bar{e}_{ij} h_j e_{ij}.$$

Consider in  $U_i$  the matrix of (1, 0) forms

$$l_i = h_i^{-1} \partial h_i$$

where  $\partial$  is the exterior differentiation with respect to holomorphic coordinates.

From (1) we deduce that

$$e_{ij}^{-1} \partial e_{ij} = l_j - e_{ij}^{-1} l_i e_{ij}$$

and this means that  $\{l_i\}$  are the local components of a  $\partial$ -connection in the bundle  $E$ .

The obstruction for this connection to be holomorphic is given by the curvature form

$$(2) \quad s_i = \bar{\partial} l_i$$

for which the consistency conditions are now on  $U_i \cap U_j$ :

$$(3) \quad s_i = e_{ij} s_j e_{ji}$$

In particular, if  $E$  is a line bundle ( $m = 1$ ) then the curvature form  $s = \bar{\partial} \log h_i$  is a global  $(1, 1)$  form on the base.

*b)* The datum of a  $\partial$ -connection in the holomorphic bundle  $E$  enables us to consider for any  $C^\infty$  section of  $E$  the absolute differentiation with respect to local holomorphic coordinates. If  $t = \{t_i\}$  are the local components of a section of  $E$ ,

$$t_i = e_{ij} t_j \quad \text{in } U_i \cap U_j,$$

and if  $l_i$  are the local components of the  $\partial$ -connection, then the absolute differential  $\nabla t$  of  $t$  has the local components

$$(\nabla t)_{U_i} = \partial t_i + l_i t_i,$$

with the consistency conditions

$$(\nabla t)_{U_i} = e_{ij} (\nabla t)_{U_j}.$$

The absolute differentiation is therefore a linear map

$$\nabla : \Gamma(X, \mathcal{A}(E)) \rightarrow \Gamma(X, \mathcal{A}(E \otimes \Theta^*)),$$

$\mathcal{A}$  denoting the sheaf of differentiable sections.

If  $\{h_i\}$  are the local components of a hermitian metric on the fibres of  $E$  then  $\{l_i = h_i^{-1} \partial h_i\}$  are the local components of a  $\partial$ -connection on  $E$ . Analogously in the antiholomorphic bundle  $\bar{E}$  the forms  $\{\bar{l}_i = \bar{h}_i^{-1} \bar{\partial} \bar{h}_i\}$  are the local components of a  $\bar{\partial}$ -connection.

On the dual bundle  $E^*$  it is natural to assume  $\{h_i^{-1}\}$  as metric on the fibres and correspondingly  $\{-\partial h_i^{-1}\}$  as  $\partial$ -connection. We have the corresponding formulae for the bundle  $\bar{E}^*$  passing to the complex conjugate forms.

Given any tensor product of holomorphic and antiholomorphic bundles with corresponding  $\partial$  and  $\bar{\partial}$ -connections, the absolute differentials  $\nabla$  and  $\bar{\nabla}$  for the sections of that tensor product are then defined in a natural way.

We remark that the choice of a "metric" connection has the advantage that the absolute differentials of the metric tensor  $\{h_i\}$  are zero:

$$\nabla h = 0, \quad \bar{\nabla} h = 0.$$

**3.** *a)* In particular, a hermitian metric on the fibres of the holomorphic tangent bundle  $\Theta$  will be the datum on  $X$  of a hermitian metric

$$ds^2 = 2 \sum g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$$

The corresponding  $\partial$ -connection will be given by

$$\omega_\alpha^\beta = \sum_{\bar{\gamma}} g^{\beta\bar{\gamma}} \partial g_{\bar{\gamma}\alpha} = \sum_{\rho} C_{\alpha\rho}^\beta dz^\rho.$$

The curvature form is given by

$$\bar{\partial}\omega_\alpha^\beta = L_{\alpha\sigma\rho}^\beta \bar{d}z^\sigma \wedge dz^\rho$$

where

$$L_{\alpha\sigma\rho}^\beta = \frac{\partial C_{\alpha\rho}^\beta}{\partial z^\sigma}$$

In this case one can consider the torsion tensor

$$S_{\alpha\rho}^\beta = \frac{1}{2} \{ C_{\alpha\rho}^\beta - C_{\rho\alpha}^\beta \}$$

whose vanishing represents the necessary and sufficient condition for the hermitian metric to be a Kähler metric.

b) It is more convenient to operate in the case of the tangent bundle with a symmetric connection in which the metric tensor has absolute differential zero. This is the corresponding riemannian connection whose components are

$$\begin{aligned} \Gamma_{\beta\gamma}^\alpha &= \frac{1}{2} g^{\alpha\bar{\rho}} \left\{ \frac{\partial g_{\bar{\rho}\beta}}{\partial z^\gamma} + \frac{\partial g_{\bar{\rho}\gamma}}{\partial z^\beta} \right\}, \\ \Gamma_{\bar{\gamma}\bar{\beta}}^\alpha &= \Gamma_{\beta\bar{\gamma}}^\alpha = \frac{1}{2} g^{\alpha\bar{\rho}} \left\{ \frac{\partial g_{\bar{\rho}\beta}}{\partial \bar{z}^\gamma} + \frac{\partial g_{\bar{\rho}\gamma}}{\partial \bar{z}^\beta} \right\}, \\ \Gamma_{\beta\bar{\gamma}}^\alpha &= 0. \end{aligned}$$

The local forms of this connection are thus given by

$$\begin{aligned} \Omega_\beta^\alpha &= \sum_{\gamma} \Gamma_{\beta\gamma}^\alpha dz^\gamma + \sum_{\bar{\gamma}} \Gamma_{\beta\bar{\gamma}}^\alpha d\bar{z}^\gamma; \quad \Omega_{\bar{\beta}}^\alpha = \overline{\Omega_\beta^\alpha} \\ \Omega_{\bar{\beta}}^\alpha &= \sum_{\gamma} \Gamma_{\gamma\bar{\beta}}^\alpha dz^\gamma; \quad \Omega_{\beta}^\alpha = \overline{\Omega_{\bar{\beta}}^\alpha}. \end{aligned}$$

Let  $\Omega$  denote the matrix of 1-forms  $\Omega_\star^\star$ ; then the curvature form is given by

$$d\Omega + \Omega \wedge \Omega$$

whose components are denoted by

$$R_{ikl}^j dz^k \wedge dz^l \quad (i, j, k, l = 1, \dots, n, \bar{1}, \dots, \bar{n}).$$

If the metric is a Kähler metric, then

$$(4) \quad C_{\alpha\rho}^\beta = \Gamma_{\alpha\rho}^\beta, \quad L_{\alpha\sigma\rho}^\beta = R_{\alpha\sigma\rho}^\beta.$$

c) If  $\{t^a\}$  ( $a = 1, \dots, m$ ) is a section of  $E$ , and if we take covariant derivatives, we see that the covariant derivatives  $\nabla_\mu \nabla_\nu t^a$  and  $\nabla_\nu \nabla_\mu t^a$  are related by the "Ricci identity"

$$(\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu) t^a = s_{b\nu\mu}^a t^b.$$

4. a) Let  $X$  be a complex manifold of complex dimension  $n$ ,  $E$  a holomorphic vector bundle on  $X$ ,  $h = \{h_i\}$  a hermitian metric on the fibres of  $E$  and let  $ds^2 = 2\sum g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$  be a hermitian metric on  $X$ .

On the space  $C(X, E) = \oplus C^{pq}(X, E)$  we can define a number of local <sup>(1)</sup> operators:  
 α) the operator

$$\bar{\partial} : C^{pq}(X, E) \rightarrow C^{p, q+1}(X, E)$$

defined before, with the property  $\bar{\partial}\bar{\partial} = 0$ ;

β) the isomorphism

$$* : C^{pq}(X, E) \rightarrow C^{n-q, n-p}(X, E)$$

locally defined, with the evident block indices notation, by

$$(5) \quad * \varphi = c \det(g_{\alpha\bar{\beta}}) \sum \text{sgn}(MA) \text{sgn}(GB) \varphi^{\bar{G}M} dz^A \wedge d\bar{z}^B$$

the constant  $c$  being so chosen that

$$**\varphi = (-1)^{p+q} \varphi$$

(see e.g. [22]).

The datum of a hermitian metric on the fibres of  $E$  defines an “anti-isomorphism” of  $E$  onto the dual bundle  $E^*$ . If  $\xi_i$  is the fibre-coordinate over  $U_i$  on  $E$ , it is given by  $\xi_i \rightarrow \bar{h}_i \bar{\xi}_i$ .

This anti-isomorphism extends to an anti-isomorphism

$$\# : C^{pq}(X, E) \rightarrow C^{qp}(X, E^*),$$

which is defined locally by

$$(\# \varphi)_i = \bar{h}_i \bar{\varphi}_i,$$

and which commutes with the operator  $*$ .

Using  $*$  and  $\#$  we obtain:

γ) the operator

$$\theta : C^{pq}(X, E) \rightarrow C^{p, q-1}(X, E),$$

defined by

$$\theta = - \#^{-1} * \bar{\partial} * \# ;$$

we have  $\theta\theta = 0$ .

Using  $\bar{\partial}$  and  $\theta$  we define the Laplace-Beltrami operator

$$\square = \bar{\partial}\theta + \theta\bar{\partial} : C^{pq}(X, E) \rightarrow C^{pq}(X, E).$$

The operators  $\theta$  and  $\square$  depend on the hermitian metric on  $X$  and on the metric along the fibres of  $E$ . To emphasize this fact, we may write occasionally  $\theta_E$  and  $\square_E$  for these operators.

It follows from the very definition of  $\theta$ , that for any  $\varphi \in C^{pq}(X, E)$ :

$$(6) \quad \bar{\partial} \# * = (-1)^{p+q} * \# \theta_E, \quad \theta_{E^*} \# * \varphi = (-1)^{p+q+1} * \# \bar{\partial} \varphi.$$

<sup>(1)</sup> We call an operator  $A$  on  $C(X, E)$  local if, for any  $\varphi \in C(X, E)$ ,  $\text{support of } A\varphi \subset \text{support of } \varphi$ .

Hence

$$(7) \quad \square_{E^*} * \# = * \# \square_E.$$

$$b) \text{ Let } \omega = \sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$$

be the exterior form associated with the hermitian metric on  $X$ . Let

$$L : C^{pq}(X, E) \rightarrow C^{p+1, q+1}(X, E)$$

be the linear mapping locally defined by

$$(L\varphi)_i^a = \omega \wedge \varphi_i^a.$$

We shall consider also the linear mapping

$$\Lambda = (-1)^{p+q} * L * : C^{pq}(X, E) \rightarrow C^{p-1, q-1}(X, E).$$

Let  $e(s_i)\varphi_i$  be the local vector form locally defined by

$$(e(s_i)\varphi_i)^a = \sqrt{-1} s_{ib}^a \wedge \varphi_i^b.$$

It follows from (3) that

$$e(s_i)\varphi_i = e_{ij}(e(s_j)\varphi_j) \quad \text{on } U_i \cap U_j.$$

This allows us to define a linear mapping

$$e(s) : C^{pq}(X, E) \rightarrow C^{p+1, q+1}(X, E).$$

Let

$$i(s) : C^{pq}(X, E) \rightarrow C^{p-1, q-1}(X, E)$$

be the linear mapping defined by

$$i(s) = (-1)^{p+q} * e(s) *.$$

c) Given  $\varphi, \psi \in C^{pq}(X, E)$  we can construct the global scalar  $(n, n)$ -form

$${}^t\varphi \wedge * \# \psi.$$

If  $dX$  is the volume element in the considered metric on  $X$  we will denote this form by  $A(\varphi, \psi)dX$ . One has

$$A(\varphi, \psi) = \overline{A(\psi, \varphi)}, \quad A(\varphi, \varphi) \geq 0$$

Moreover  $A(\varphi, \varphi) = 0$  if and only if  $\varphi = 0$ . We shall call  $A(\varphi, \varphi)^{1/2}$  the *length* of the form  $\varphi$ .

In the space  $L^{pq}(X, E) = \{\varphi \in C^{pq}(X, E) \mid \int_X A(\varphi, \varphi) dX < \infty\}$

the scalar product

$$(\varphi, \psi) = \int_X A(\varphi, \psi) dX$$

is defined and gives  $L^{pq}(X, E)$  the structure of a complex prehilbert space. One verifies immediately that for  $\varphi \in C^{pq}(X, E)$  we have

$$\Lambda_{E^*}(* \# \varphi, * \# \varphi) = \Lambda_E(\varphi, \varphi).$$

If  $\varphi, \psi$  are forms of suitable degree in  $C^{pq}(X, E)$  one has the formulae:

$$\begin{aligned} A(L\varphi, \psi) &= A(\varphi, \Lambda\psi), \\ A(e(s)\varphi, \psi) &= A(\varphi, i(s)\psi). \end{aligned}$$

If  $\varphi \in C^{pq}(X, E)$ ,  $\psi \in C^{p, q+1}(X, E)$  one has

$${}^t\bar{\partial}\varphi \wedge * \neq \psi - {}^t\varphi \wedge * \neq \theta\varphi = d({}^t\varphi \wedge * \neq \varphi).$$

Thus, by Stokes' theorem we have that, if  $\text{Supp } \varphi \cap \text{Supp } \psi$  is compact, then

$$(8) \quad (\bar{\partial}\varphi, \psi) = (\varphi, \theta\psi).$$

If  $\varphi_1, \varphi_2 \in C^{pq}(X, E)$  and if  $\text{Supp } \varphi_1 \cap \text{Supp } \varphi_2$  is compact, then

$$(\square\varphi_1, \varphi_2) = (\varphi_1, \square\varphi_2) = (\bar{\partial}\varphi_1, \bar{\partial}\varphi_2) + (\theta\varphi_1, \theta\varphi_2).$$

We will be concerned with forms with (locally) Lipschitz coefficients. We observe that for such a form  $\varphi$ ,  $\bar{\partial}\varphi$  and  $\theta\varphi$  are defined almost everywhere. Since Stokes' theorem holds for Lipschitz forms, (8) remains valid also in this case.

d) If  $E', E''$  are two holomorphic vector bundles on  $X$  of rank  $m', m''$  respectively and if  $\{h'_i\}, \{h''_i\}$  are hermitian metrics on the fibres of  $E', E''$  then  $\{h'_i \otimes h''_i\}$  is a hermitian metric on the fibres of  $E' \otimes E''$ . The corresponding connections and curvature forms are then represented locally by

$$\begin{aligned} l'_i \otimes I_{m''} + I_{m'} \otimes l''_i \\ s'_i \otimes I_{m''} + I_{m'} \otimes s''_i \end{aligned}$$

where  $l'_i, l''_i, s'_i, s''_i$  are the connections and curvature forms corresponding to  $h'_i, h''_i$  respectively and where  $I_r$  denotes the identity matrix of rank  $r$ .

## § 2. W-ellipticity of vector bundles

5. *The spaces  $W^{pq}(X, E)$ .* — a) In the space  $\mathcal{D}^{pq}(X, E)$  we introduce the hermitian sesquilinear non-degenerate positive form

$$a(\varphi, \psi) = (\varphi, \psi) + (\bar{\partial}\varphi, \bar{\partial}\psi) + (\theta\varphi, \theta\psi).$$

We denote by

$\mathcal{L}^{pq}(X, E)$  the completion of  $\mathcal{D}^{pq}(X, E)$  with respect to the norm  $\|\varphi\| = (\varphi, \varphi)^{1/2}$ ;  
 $W^{pq}(X, E)$  the completion of  $\mathcal{D}^{pq}(X, E)$  with respect to the norm  $N(\varphi) = a(\varphi, \varphi)^{1/2}$ .

The canonical map  $W^{pq}(X, E) \rightarrow \mathcal{L}^{pq}(X, E)$  is an injective map, as it follows from a remark of K. O. Friedrichs (cf. e.g. [10]). The elements of  $W^{pq}(X, E)$  are those elements  $\varphi \in \mathcal{L}^{pq}(X, E)$  which admit simultaneously  $\bar{\partial}$  and  $\theta$  in the generalized sense of Friedrichs; i.e. there exists a Cauchy sequence  $(\varphi_n) \subset \mathcal{D}^{pq}(X, E)$ , converging to  $\varphi$  in  $\mathcal{L}^{pq}(X, E)$ , such that the sequences  $(\bar{\partial}\varphi_n)$  and  $(\theta\varphi_n)$  are also Cauchy sequences in  $\mathcal{L}^{p, q+1}(X, E)$  and  $\mathcal{L}^{p, q-1}(X, E)$  respectively.

The extension of the operators  $\bar{\partial}$  and  $\theta$  to  $W^{pq}(X, E)$  will be denoted by the same letters.

Consider now the dual bundle  $E^*$  endowed with the metric  $\{h_i^{-1}\}$ .

We obtain from (6) and (7) the following

*Proposition 1.* — *The anti-isomorphism  $* \neq$  defines an isometry of  $\mathcal{L}^{pq}(X, E)$  onto  $\mathcal{L}^{n-p, n-q}(X, E^*)$  which maps  $W^{pq}(X, E)$  isometrically onto  $W^{n-p, n-q}(X, E^*)$ .*

In  $W^{pq}(X, E)$  we introduce the Dirichlet sesquilinear hermitian form

$$d(\varphi, \psi) = (\bar{\partial}\varphi, \bar{\partial}\psi) + (\theta\varphi, \theta\psi).$$

Then  $d(\varphi, \varphi)^{1/2}$  is a seminorm on  $W^{pq}(X, E)$ .

*Definition.* — *We say that the vector bundle  $E$  is  $W$ -elliptic in the degree  $(p, q)$  (or briefly  $W^{pq}$ -elliptic) if there exists*

- a hermitian metric on  $X$ ,*
- a hermitian metric on the fibres of  $E$ ,*
- a constant  $c > 0$ ,*

*such that for every  $\varphi \in \mathcal{D}^{pq}(X, E)$  we have the inequality:*

$$(9) \quad (\varphi, \varphi) \leq cd(\varphi, \varphi).$$

If  $E$  is  $W^{pq}$  elliptic, then the Dirichlet seminorm  $d(\varphi, \varphi)^{1/2}$  is a norm on  $W^{pq}$  and defines on it the same topology as the natural norm  $N(\varphi)$ .

Conversely, if the Dirichlet seminorm  $d(\varphi, \varphi)^{1/2}$  defines in  $W^{pq}(X, E)$  the same topology as the natural norm  $N(\varphi)$ , then  $E$  is  $W^{p, q}$ -elliptic [21].

Since, by (6) and (7)

$$(10) \quad A_{E^*}(\bar{\partial} * \neq \varphi, \bar{\partial} * \neq \varphi) + A_{E^*}(\theta_{E^*} * \neq \varphi, \theta_{E^*} * \neq \varphi) = A_E(\bar{\partial}\varphi, \bar{\partial}\varphi) + A_E(\theta_E\varphi, \theta_E\varphi)$$

for all  $\varphi \in C^{pq}(X, E)$ , then the anti-isomorphism  $* \neq$  transforms the Dirichlet seminorm in  $W^{pq}(X, E)$  onto the Dirichlet seminorm in  $W^{n-p, n-q}(X, E^*)$ . This proves the following

*Lemma 2.* — *If  $E$  is  $W^{pq}$ -elliptic (with respect to a metric  $\{h_i\}$ ), then  $E^*$  is  $W^{n-p, n-q}$  elliptic (with respect to the metric  $\{h_i^{-1}\}$ ).*

*b)* From the Riesz representation theorem one obtains the following

*Theorem 1.* — *If the vector bundle  $E$  is  $W^{pq}$ -elliptic, then, for any  $\alpha \in \mathcal{L}^{pq}(X, E)$ , the equation*

$$\square x = \alpha$$

*has one and only one weak solution  $x \in W^{pq}(X, E)$*

*(i.e. for any  $u \in W^{pq}(X, E)$  one has*

$$(\bar{\partial}x, \bar{\partial}u) + (\theta x, \theta u) = (\alpha, u).$$

Moreover, since  $\square$  represents a strongly elliptic system, it follows from the regularization theorem (see e.g. [16]) that if  $\alpha \in \mathcal{L}^{pq}(X, E) \cap C^{pq}(X, E)$  then

$$x \in W^{pq}(X, E) \cap C^{pq}(X, E),$$

and one has

$$\square x = \alpha$$

in the classical sense.

**6.** *The case of a complete metric on X.* — a) Let  $o$  be a point of  $X$  and let  $d(p, o)$  be the geodesic distance from  $o$  to  $p \in X$  in the fixed hermitian metric on  $X$ . Let

$$B(c) = \{x \in X \mid d(x, o) < c\};$$

one has the following useful

*Proposition 3.* — *There exists a constant  $A > 0$  such that if  $0 < r < R$  and if  $B(R)$  is relatively compact in  $X$ , then, for any  $\sigma > 0$  and any  $\varphi \in C^{pq}(X, E)$ , one has the inequality (which will be referred to as Stampacchia's inequality):*

$$(11) \quad \|\bar{\partial}\varphi\|_{B(r)}^2 + \|\theta\varphi\|_{B(r)}^2 \leq \sigma \|\square\varphi\|_{B(R)}^2 + \left(\frac{1}{\sigma} + \frac{A}{(R-r)^2}\right) \|\varphi\|_{B(R)}^2.$$

The proof of this proposition has been given in [3] in the case of a line-bundle. Although the same proof holds, with some slight changes, in the general case [21], we reproduce it here for the sake of completeness.

We start with a lemma, which has been established in [3] (see also [21]).

$\alpha$ ) *The distance  $\rho(x) = d(o, x)$  is a locally Lipschitz function. At points where the derivatives exist, we have in terms of local real coordinates,*

$$\sum g^{ij} \frac{\partial \rho}{\partial x^i} \frac{\partial \rho}{\partial x^j} \leq 2n \quad (n = \dim_{\mathbb{C}} X)$$

$\beta$ ) A straightforward calculation yields the following:

*There exists a constant  $c_0 > 0$  (which depends only on the dimension of  $X$ ) such that, at any point  $x \in X$ , for any scalar form  $u$  and for any form  $v$  with values in  $E$ , one has*

$$A(u \wedge v, u \wedge v) \leq c_0 |u|^2 A(v, v),$$

where  $|u|$  denotes the length of the scalar form  $u = \sum_{i_1 < \dots < i_p} u_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$  expressed by

$$|u|^2 = \sum u_{i_1 \dots i_p} \overline{u^{i_1 \dots i_p}}.$$

$\gamma$ ) We choose a  $C^\infty$  function  $\mu(t)$  on  $\mathbf{R}$ , with the following properties

$$\begin{aligned} 0 &\leq \mu(t) \leq 1 \\ \mu(t) &= \begin{cases} 1 & \text{for } t \leq 1 \\ 0 & \text{for } t \geq 2, \end{cases} \end{aligned}$$

and we set

$$M = \text{Sup} \left| \frac{d\mu}{dt} \right|.$$

We consider the function  $w(x) = \mu \left( \frac{\rho(x) + R - 2r}{R - r} \right)$

for any choice of  $R > r > 0$ . It is a real locally Lipschitz function, and satisfies the following conditions:

$$\begin{aligned} 0 &\leq w(x) \leq 1 \\ w(x) &= \begin{cases} 1 & \text{for } x \in \overline{B(r)} \\ 0 & \text{for } x \in X - B(R), \end{cases} \\ \left| \frac{dw}{d\varphi} \right| &\leq \frac{M}{R-r}. \end{aligned}$$

It follows from the first condition that, for every form  $\varphi \in C^{pq}(X, E)$  and at any point  $x \in X$ ,

$$(12) \quad A(w\varphi, w\varphi) \leq A(\varphi, \varphi).$$

From the third condition and from  $\alpha$ ) we see that, where the derivatives  $\frac{\partial \rho}{\partial x^i}$  exist,

$$|dw|^2 = \sum g^{ij} \frac{\partial w}{\partial x^i} \frac{\partial w}{\partial x^j} \leq 2n \frac{M^2}{(R-r)^2}.$$

We get from this and from  $\beta$ ) that for every  $\varphi \in C^{pq}(X, E)$ , we have almost everywhere in  $X$ :

$$(13) \quad A(\bar{\partial}w \wedge \varphi, \bar{\partial}w \wedge \varphi) \leq \frac{2nc_0 M^2}{(R-r)^2} A(\varphi, \varphi),$$

$$(14) \quad A(\partial w \wedge * \varphi, \partial w \wedge * \varphi) \leq \frac{2nc_0 M^2}{(R-r)^2} A(\varphi, \varphi).$$

$\delta$ ) If  $\alpha$  is any locally Lipschitz  $(p, q)$ -form with values in  $E$  and with support contained in  $B(R)$ , then

$$(15) \quad (\bar{\partial}\varphi, \bar{\partial}\alpha)_{B(R)} + (\theta\varphi, \theta\alpha)_{B(R)} = (\square\varphi, \alpha)_{B(R)}.$$

Letting  $\alpha = w^2\varphi$ , we have almost everywhere

$$\bar{\partial}\alpha = w^2\bar{\partial}\varphi + 2w\bar{\partial}w \wedge \varphi, \quad \theta\alpha = w^2\theta\varphi - *(2w\partial w \wedge * \varphi).$$

Substituting in (15) we have

$$(16) \quad \begin{aligned} \|\bar{\partial}\varphi\|_{B(R)}^2 + \|\theta\varphi\|_{B(R)}^2 &\leq \\ &|(\square\varphi, w^2\varphi)_{B(R)}| + |(\bar{\partial}\varphi, 2w\bar{\partial}w \wedge \varphi)_{B(R)}| + |(\theta\varphi, *(2w\partial w \wedge * \varphi))_{B(R)}|. \end{aligned}$$

On the other hand, the Schwarz inequality gives the following

$$|(\square\varphi, w^2\varphi)_{B(R)}| \leq \frac{1}{2} \left\{ \frac{1}{\sigma} \|\varphi\|_{B(R)}^2 + \sigma \|\square\varphi\|_{B(R)}^2 \right\} \quad \text{for every } \sigma > 0,$$

$$|(\bar{\partial}\varphi, 2w\bar{\partial}w \wedge \varphi)_{B(R)}| \leq \frac{1}{2} \left\{ \|\bar{\partial}\varphi\|_{B(R)}^2 + 4 \|\bar{\partial}w \wedge \varphi\|_{B(R)}^2 \right\},$$

$$|(\theta\varphi, *(2w\partial w \wedge * \varphi))_{B(R)}| \leq \frac{1}{2} \left\{ \|\theta\varphi\|_{B(R)}^2 + 4 \|\partial w \wedge * \varphi\|_{B(R)}^2 \right\}.$$

Substituting in (16) we have

$$\|w\bar{\partial}\varphi\|_{B(R)}^2 + \|\theta\varphi\|_{B(R)}^2 \leq \sigma \|\square\varphi\|_{B(R)}^2 + \frac{1}{\sigma} \|w\varphi\|_{B(R)}^2 + 4\|\bar{\partial}w\wedge\varphi\|_{B(R)}^2 + 4\|\partial w\wedge*\varphi\|_{B(R)}.$$

It follows from (12), (13), (14) that

$$\|w\varphi\|_{B(R)}^2 \leq \|\varphi\|_{B(R)}^2, \quad (17)$$

$$\|\bar{\partial}w\wedge\varphi\|_{B(R)}^2 \leq \frac{2nc_0M^2}{(R-r)^2} \|\varphi\|_{B(R)}^2,$$

$$\|\partial w\wedge*\varphi\|_{B(R)}^2 \leq \frac{2nc_0M^2}{(R-r)^2} \|\varphi\|_{B(R)}^2. \quad (18)$$

Thus, since  $w \geq 0$  on  $B(R)$ ,  $w = 1$  on  $B(r)$ , we obtain (11) with  $A = 16nc_0M^2$ .  
Q.E.D.

b) Let  $\varphi \in \mathcal{L}_{\frac{p,q}{2}}^{p,q}(X, E)$  be a form which admits a  $\bar{\partial}\varphi \in \mathcal{L}^{p,q+1}(X, E)$  and a  $\theta\varphi \in \mathcal{L}^{p,q-1}(X, E)$  generalized in the sense of distributions, i.e. such that

$$\begin{aligned} (\varphi, \theta u) &= (\bar{\partial}\varphi, u) & \text{for all } u \in \mathcal{D}^{p,q+1}(X, E), \\ (\varphi, \bar{\partial}v) &= (\theta\varphi, v) & \text{for all } v \in \mathcal{D}^{p,q-1}(X, E). \end{aligned}$$

*Lemma 4.* — *If the hermitian metric on  $X$  is complete, then  $\varphi \in W^{p,q}(X, E)$ , and  $\bar{\partial}\varphi, \theta\varphi$  are the  $\bar{\partial}$  and  $\theta$  of  $\varphi$  in the strong sense.*

*Proof.* — As in Proposition 3 we consider the open balls  $B(R)$  and  $B(r)$  of radii  $R = 2\nu, r = \nu$  ( $\nu = 1, 2, \dots$ ), and we construct the function

$$w_\nu = \mu \left( \frac{\rho(x) + R - 2r}{R - r} \right) = \mu \left( \frac{\rho(x)}{\nu} \right).$$

Let  $\varphi_\nu$  be the form  $\varphi_\nu = w_\nu\varphi$ ;

the support of  $\varphi_\nu$  belongs to  $B(2\nu)$ .

Since on  $B(\nu)$ ,  $w_\nu = 1$ , we have

$$\|\varphi - \varphi_\nu\| = \|(1 - w_\nu)\varphi\| \leq \|\varphi\|_{X - B(\nu)}.$$

Therefore  $\lim_{\nu \rightarrow \infty} \|\varphi - \varphi_\nu\| = 0$ .

On the other hand we have

$$\bar{\partial}\varphi_\nu = w_\nu\bar{\partial}\varphi + \bar{\partial}w_\nu\wedge\varphi, \quad \theta\varphi_\nu = w_\nu\theta\varphi - *(\partial w_\nu\wedge*\varphi)$$

in the sense of distributions.

Hence, by (17) and (18)

$$\|\bar{\partial}\varphi - \bar{\partial}\varphi_\nu\| \leq \|(1 - w_\nu)\bar{\partial}\varphi\| + \frac{c}{\nu} \|\varphi\|_{B(2\nu)} \leq \|\bar{\partial}\varphi\|_{X - B(\nu)} + \frac{c}{\nu} \|\varphi\|_{B(2\nu)} \rightarrow 0,$$

$$\|\theta\varphi - \theta\varphi_\nu\| \leq \|(1 - w_\nu)\theta\varphi\| + \frac{c}{\nu} \|\varphi\|_{B(2\nu)} \leq \|\theta\varphi\|_{X - B(\nu)} + \frac{c}{\nu} \|\varphi\|_{B(2\nu)} \rightarrow 0$$

with  $c = M\sqrt{2nc_0}$ .

Let  $(U_i)$  be a locally finite open coordinate covering of  $X$ , such that only a finite number of  $U_i$  has a non-void intersection with  $\text{Supp } \varphi_\nu$ . We denote by  $V_i$  open sets  $V_i \subset \subset U_i$  such that  $X = \bigcup V_i$ .

By an elementary convolution argument we can construct, for every  $V_i$  and every  $\nu$ , a sequence  $(\varphi_{\nu,\mu}^i)$  of forms  $\varphi_{\nu,\mu}^i \in C^{p,q}(\overline{V}_i, E)$ , such that

$$\lim_{\mu \rightarrow +\infty} \|\varphi_{\nu|V_i} - \varphi_{\nu,\mu}^i\|_{V_i} = 0, \quad \lim_{\mu \rightarrow +\infty} \|\bar{\partial}\varphi_{\nu|V_i} - \bar{\partial}\varphi_{\nu,\mu}^i\|_{V_i} = 0, \quad \lim_{\mu \rightarrow +\infty} \|\theta\varphi_{\nu|V_i} - \theta\varphi_{\nu,\mu}^i\|_{V_i} = 0.$$

If  $U_i \cap \text{Supp } \varphi_\nu = \emptyset$ , we can assume  $\varphi_{\nu,\mu}^i = 0$ .

Let  $(\pi_i)$  be a  $C^\infty$  partition of unity associated with the covering  $(V_i)$  ( $\text{Supp } \pi_i \subset V_i$ ).

Let 
$$\varphi_{\nu,\mu} = \sum_i \pi_i \varphi_{\nu,\mu}^i.$$

Since only a finite number of  $U_i$  are not disjoint from  $\text{Supp } \varphi_\nu$ , then

$$\varphi_{\nu,\mu} \in \mathcal{D}^{p,q}(X, E).$$

Furthermore, it is easily checked that

$$\lim_{\mu \rightarrow +\infty} \|\varphi_\nu - \varphi_{\nu,\mu}\| = 0, \quad \lim_{\mu \rightarrow +\infty} \|\bar{\partial}\varphi_\nu - \bar{\partial}\varphi_{\nu,\mu}\| = 0, \quad \lim_{\mu \rightarrow +\infty} \|\theta\varphi_\nu - \theta\varphi_{\nu,\mu}\| = 0,$$

i.e. 
$$\lim_{\mu \rightarrow +\infty} N(\varphi_\nu - \varphi_{\nu,\mu}) = 0.$$

Let  $(\varepsilon_\nu)_{\nu \in \mathbb{N}}$  be a sequence of positive numbers  $\varepsilon_\nu$  converging to zero.

For every  $\nu$  we can find an index  $\mu(\nu)$  such that for any  $\mu \geq \mu(\nu)$  we have

$$N(\varphi_\nu - \varphi_{\nu,\mu}) < \varepsilon_\nu.$$

We have  $N(\varphi - \varphi_{\nu,\mu(\nu)}) \leq N(\varphi - \varphi_\nu) + N(\varphi_\nu - \varphi_{\nu,\mu(\nu)}) < N(\varphi - \varphi_\nu) + \varepsilon_\nu \rightarrow 0$ .

Hence the sequence  $(\varphi_{\nu,\mu(\nu)}) \subset \mathcal{D}^{p,q}(X, E)$  converges to  $\varphi$  in the norm  $N$ , i.e.  $(\varphi_{\nu,\mu(\nu)}) \rightarrow \varphi$ ,  $(\bar{\partial}\varphi_{\nu,\mu(\nu)}) \rightarrow \bar{\partial}\varphi$ ,  $(\theta\varphi_{\nu,\mu(\nu)}) \rightarrow \theta\varphi$  in the norm  $\|\cdot\|$ . Q.E.D.

*Proposition 5.* — *If the hermitian metric on  $X$  is complete,  $W^{p,q}(X, E)$  can be identified with the space of forms  $\varphi \in \mathcal{L}^{p,q}(X, E)$  which admit a  $\bar{\partial}\varphi \in \mathcal{L}^{p,q+1}(X, E)$  and a  $\theta\varphi \in \mathcal{L}^{p,q-1}(X, E)$  in the sense of distributions.*

Under the assumption that the metric on  $X$  is complete, setting in (11)  $R = 2r$  and letting  $r \rightarrow +\infty$  we obtain the inequality

$$(19) \quad \|\bar{\partial}\varphi\|^2 + \|\theta\varphi\|^2 \leq \sigma \|\square\varphi\|^2 + \frac{1}{\sigma} \|\varphi\|^2 \quad \text{for every } \sigma > 0.$$

The following statement is a consequence of (19) and of Proposition 5.

*Corollary 6.* — *If the metric on  $X$  is complete, any form  $\varphi \in C^{p,q}(X, E)$  such that  $\|\varphi\| < +\infty$ ,  $\|\square\varphi\| < +\infty$  can be identified with an element of  $W^{p,q}(X, E)$ .*

If, in particular,  $\square\varphi = 0$ , letting in (19)  $\sigma \rightarrow +\infty$ , we obtain

*Proposition 7.* — *Let the metric on  $X$  be complete. If  $\varphi \in \mathcal{L}^{p,q}(X, E) \cap C^{p,q}(X, E)$  is such that  $\square\varphi = 0$ , then  $\bar{\partial}\varphi = 0$ ,  $\theta\varphi = 0$ .*

c) We now make one more assumption, namely that  $E$  is  $W^{p,q}$ -elliptic with respect to a complete metric on  $X$  (and to a suitable hermitian metric on the fibres of  $E$ ).

We then have the following

*Proposition 8.* — *Under the above assumptions, if*

$$\varphi \in \mathcal{L}^{pq}(\mathbf{X}, \mathbf{E}) \cap \mathbf{C}^{pq}(\mathbf{X}, \mathbf{E}) \quad \text{and} \quad \square \varphi = 0$$

then also  $\bar{\partial} \varphi = 0$ .

*Proof.* — By the  $W^{pq}$ -ellipticity condition there exists an  $x \in W^{pq}(\mathbf{X}, \mathbf{E}) \cap \mathbf{C}^{pq}(\mathbf{X}, \mathbf{E})$  such that

$$\varphi = \square x.$$

By the previous proposition  $\bar{\partial} \varphi = 0 = \theta \varphi$ . Hence  $\square \bar{\partial} x = 0 = \square \theta x$ . Since  $\bar{\partial} x$  and  $\theta x$  are square integrable, again by the proposition, one has  $\theta \bar{\partial} x = 0 = \bar{\partial} \theta x$ . Hence  $\square x = 0$ .

**7. Vanishing theorem (weak form).**

*Theorem 21.* — *If the vector bundle  $\mathbf{E}$  is  $W^{pq}$ -elliptic with respect to a complete metric on  $\mathbf{X}$ , then*

$$\text{if } \varphi \in \mathcal{L}^{pq}(\mathbf{X}, \mathbf{E}) \cap \mathbf{C}^{pq}(\mathbf{X}, \mathbf{E}) \quad \text{and} \quad \bar{\partial} \varphi = 0,$$

there exists a  $\psi \in \mathcal{L}^{p, q-1}(\mathbf{X}, \mathbf{E}) \cap \mathbf{C}^{p, q-1}(\mathbf{X}, \mathbf{E})$  such that

$$\varphi = \bar{\partial} \psi.$$

*Proof.* — By the  $W^{pq}$ -ellipticity condition there exists an  $x \in W^{pq}(\mathbf{X}, \mathbf{E}) \cap \mathbf{C}^{pq}(\mathbf{X}, \mathbf{E})$  such that

$$\varphi = \square x.$$

Since  $\bar{\partial} \varphi = 0$  one has  $\square \bar{\partial} x = 0$ . By proposition 7 of the previous section it follows that  $\theta \bar{\partial} x = 0$ . Therefore

$$\varphi = \bar{\partial} \theta x = \bar{\partial} \psi,$$

where  $\psi = \theta x \in \mathcal{L}^{p, q-1}(\mathbf{X}, \mathbf{E}) \cap \mathbf{C}^{p, q-1}(\mathbf{X}, \mathbf{E})$ .

*Corollary 9.* — *Under the above hypothesis, the natural map*

$$H_k^q(\mathbf{X}, \Omega^p(\mathbf{E})) \rightarrow H^q(\mathbf{X}, \Omega^p(\mathbf{E}))$$

is the zero homomorphism.

### § 3. Green's operator and Carleman's inequality

**8. Green's operator.** — *a)* Let  $\mathbf{E}$  be a holomorphic vector bundle on  $\mathbf{X}$  which is  $W^{pq}$ -elliptic with respect to a given choice of hermitian metrics on the fibres of  $\mathbf{E}$  and on the base  $\mathbf{X}$ .

For any  $f \in \mathcal{L}^{pq}(\mathbf{X}, \mathbf{E})$  there exists one and only one element  $x \in W^{pq}(\mathbf{X}, \mathbf{E})$  such that

$$f = \square x$$

in the generalized sense. This means that for any  $u \in W^{pq}(\mathbf{X}, \mathbf{E})$  we have

$$(f, u) = (\bar{\partial} x, \bar{\partial} u) + (\theta x, \theta u).$$

We thus define this unique solution  $x$  as the image of an operator  $G : \mathcal{L}^{pq}(\mathbf{X}, \mathbf{E}) \rightarrow W^{pq}(\mathbf{X}, \mathbf{E})$ :

$$x = Gf.$$

b) From the inequality of the  $W^{pq}$ -ellipticity one obtains

$$\begin{aligned} \|x\|^2 &\leq c\{\|\bar{\partial}x\|^2 + \|\theta x\|^2\} \\ &= c|(f, x)| \\ &\leq c\|f\|\|x\|. \end{aligned}$$

We then have

$$(20) \quad \|Gf\| \leq c\|f\|.$$

**9. Carleman's inequality.** — a) Let  $E$  be a holomorphic vector bundle on  $X$ ; let  $h = \{h_i\}$  be a hermitian metric on the fibres of  $E$  and let  $ds^2$  be a complete hermitian metric on  $X$ .

We now make the following

*Assumption:* There exists a  $C^\infty$  function  $\Phi : X \rightarrow \mathbf{R}$  with the following properties:

(i)  $\Phi \geq 0$ ;

(ii) for any non decreasing  $C^\infty$  convex function  $\lambda(t)$ ,  $0 \leq t < +\infty$ , the vector bundle  $E$  is  $W^{pq}$ -elliptic with respect to the metric  $e^{\lambda(\Phi)}h$  on the fibres and the complete metric  $ds^2$  on  $X$  (the constant  $c$  of  $W$ -ellipticity which appears in (9) being independent of  $\lambda$ :  $(\varphi, \varphi)_\lambda \leq c\{(\bar{\partial}\varphi, \bar{\partial}\varphi)_\lambda + (\theta_\lambda\varphi, \theta_\lambda\varphi)_\lambda\}$ <sup>(1)</sup>). Let  $f \in \mathcal{L}_\lambda^{pq}(X, E) \cap C^{pq}(X, E)$ ,  $\bar{\partial}f = 0$ ; then, by theorem 2, there exists a form  $\psi_\lambda \in \mathcal{L}_\lambda^{p, q-1}(X, E) \cap C^{p, q-1}(X, E)$  such that

$$f = \bar{\partial}\psi_\lambda.$$

And indeed it is enough to take for  $\psi$  the form

$$\psi_\lambda = \theta_\lambda G_\lambda f$$

Since the operators  $\theta$  and  $G$  depend on the metric considered on the fibres of  $E$  we have put the subscript  $\lambda$  to indicate dependence on the choice of the function  $\lambda(t)$ .

Similarly we will denote by  $A_\lambda(\varphi, \psi)$  the pointwise scalar product of the two forms  $\varphi, \psi$  of the same degree with respect to the metric  $e^{\lambda(\Phi)}h$  on the fibres of  $E$ . We have

$$A_\lambda(\varphi, \psi) = e^{\lambda(\Phi)}A(\varphi, \psi),$$

where  $A(, )$  stands for  $A_0(, )$ .

From (19) we get for  $x_\lambda = G_\lambda f$  and any  $\sigma > 0$

$$(\bar{\partial}x_\lambda, \bar{\partial}x_\lambda)_\lambda + (\theta_\lambda x_\lambda, \theta_\lambda x_\lambda)_\lambda \leq \frac{1}{\sigma}(f, f)_\lambda + \sigma(x_\lambda, x_\lambda)_\lambda.$$

Since  $x_\lambda \in W_\lambda^{pq}(X, E)$  we obtain from the above assumption

$$(x_\lambda, x_\lambda)_\lambda \leq c\{(\bar{\partial}x_\lambda, \bar{\partial}x_\lambda)_\lambda + (\theta_\lambda x_\lambda, \theta_\lambda x_\lambda)_\lambda\}$$

with  $c$  independent of  $\lambda$ .

<sup>(1)</sup> The index  $\lambda$  denotes the dependence of the symbol on the function  $\lambda$ .

Taking  $\sigma = \frac{1}{2c}$  we thus obtain

$$(\bar{\partial}x_\lambda, \bar{\partial}x_\lambda)_\lambda + (\theta_\lambda x_\lambda, \theta_\lambda x_\lambda)_\lambda \leq 4c(f, f)_\lambda.$$

In particular we have proved the following

*Lemma 10.* — *If assumptions (i) and (ii) are satisfied for the vector bundle E, then, for any  $C^\infty$  form*

$$f \in \bigcap_\lambda \mathcal{L}_\lambda^{pq}(X, E) \quad \text{such that} \quad \bar{\partial}f = 0,$$

*we have the inequality*

$$(21) \quad \int_X e^{\lambda(\Phi)} A(\psi_\lambda, \psi_\lambda) dx \leq 4c \int_X e^{\lambda(\Phi)} A(f, f) dx,$$

*where*

$$\psi_\lambda = \theta_\lambda G_\lambda f.$$

The inequality (21) will be called the *Carleman inequality* for the operator  $\bar{\partial}$  in degree  $(p, q)$  [8], [13].

*Remark.* — Inequality (21) is in particular valid for any  $f \in \mathcal{D}^{pq}(X, E)$  with  $\bar{\partial}f = 0$ .

b) Let us now choose  $f \in \mathcal{D}^{pq}(X, E)$  with  $\bar{\partial}f = 0$ , and  $\varepsilon > 0$ . Let  $c_0 = \sup_{\text{supp}(f)} \Phi$  and select a  $C^\infty$  function  $\lambda(t)$  for  $0 \leq t < +\infty$  with the following properties:

- (i)  $\lambda(t) \geq 0, \quad \lambda'(t) \geq 0, \quad \lambda''(t) \geq 0$
- (ii) 
$$\lambda(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq c_0 \\ t - \left(c_0 + \frac{\varepsilon}{2}\right) & \text{for } t \geq c_0 + \varepsilon. \end{cases}$$

Let  $\lambda_\nu = \nu\lambda, \nu = 1, 2, \dots$  Construct the forms  $\psi_\nu = \theta_{\lambda_\nu} G_{\lambda_\nu} f$ . Then the Carleman inequality gives

$$\int_X e^{\nu\lambda(\Phi)} A(\psi_\nu, \psi_\nu) dx \leq 4c \int_X e^{\nu\lambda(\Phi)} A(f, f) dx.$$

Since  $e^{\nu\lambda(\Phi)} = 1$  on  $\text{supp}(f)$ , we obtain the inequality,

$$\int_{\Phi \geq c_0 + \varepsilon} e^{\nu\lambda(\Phi)} A(\psi_\nu, \psi_\nu) \leq 4c \int_{\Phi \leq c_0} A(f, f) dx.$$

Hence

$$\int_{\Phi \geq c_0 + \varepsilon} A(\psi_\nu, \psi_\nu) \leq 4ce^{-\nu\varepsilon/2} \int_{\Phi \leq c_0} A(f, f) dx$$

and letting  $\nu \rightarrow +\infty$  we see that

$$\int_{\Phi \geq c_0 + \varepsilon} A(\psi_\nu, \psi_\nu) dx \rightarrow 0.$$

Moreover since  $e^{\nu\lambda(\Phi)} \geq 1$  we have

$$\int_X A(\psi_\nu, \psi_\nu) dx \leq 4c \int_X A(f, f) dx.$$

Therefore the elements  $\psi_\nu$  all lie in a ball of fixed radius in  $\mathcal{L}^{pq}(X, E)$ . We can extract a weakly convergent subsequence  $\psi_{\nu_i} \rightarrow \psi \in \mathcal{L}^{pq}(X, E)$ . This means that for any  $u \in \mathcal{L}^{pq}(X, E)$  (and in particular for any  $u \in \mathcal{D}^{pq}(X, E)$ ) we have

$$\lim_{\nu_i \rightarrow \infty} \int_X A(\psi_{\nu_i}, u) dx = \int_X A(\psi, u) dx.$$

If  $\text{supp}(u) \cap \{\Phi \leq c_0 + \varepsilon\} = \emptyset$  we have also

$$\int_X A(\psi, u) dx = 0$$

This means that, as a distribution,  $\psi$  has support contained in the region  $\{\Phi \leq c_0 + \varepsilon\}$ . Finally since  $\bar{\partial}\psi = f$  we have in the sense of distributions <sup>(1)</sup> that

$$\bar{\partial}\psi = f$$

with  $\text{supp } \psi \subset \{x \in X \mid \Phi(x) \leq c_0 + \varepsilon\}$ .

We have therefore proved the following

*Proposition 11.* — Let  $E$  satisfy the assumption stated at the beginning of this number, that gives Carleman's inequality in degree  $(p, q)$ .

Let  $f \in \mathcal{D}^{p,q}(X, E)$  with  $\bar{\partial}f = 0$ . Then, given any  $\varepsilon > 0$ , there exists an element  $\psi \in \mathcal{L}^{p, q-1}(X, E)$  which satisfies the following conditions

- (i)  $\bar{\partial}\psi = f$  in the sense of distributions;
- (ii)  $\text{supp}(\psi) \subset \{x \in X \mid \Phi(x) \leq \sup_{\text{supp}(f)} \Phi + \varepsilon\}$ .

**10.** *Regularization of the solution.* — We first prove the following

*Lemma 12.* — Let  $\psi$  be a form of type  $(p, q-1)$  with values in  $E$  and distribution coefficients.

We assume that  $f = \bar{\partial}\psi \in \mathcal{C}^{p,q}(X, E)$ .

Let  $C = \text{supp } \psi$  and let  $D$  be any open neighborhood of  $C$ . Then there exists an  $\eta \in \mathcal{C}^{p, q-1}(X, E)$  such that  $\text{supp } \eta \subset D$ ,  $\bar{\partial}\eta = f$ .

*Proof.* — We choose a covering of  $X$  with coordinate balls  $\mathcal{U} = (V_i)_{i \in I}$  with the following property that

$$\text{if } V_i \cap C \neq \emptyset \quad \text{then} \quad V_i \subset D.$$

This is possible, taking for instance, a covering of  $X - D$  by balls not meeting  $C$ , a covering of  $C$  by balls not meeting  $X - D$  and a covering of  $D - C$  by balls contained in  $D - C$ .

We will denote by  $A^r$  the sheaf of germs of  $C^\infty$  forms of type  $(p, r)$  with values in  $E$  and by  $A_c^r$  the subsheaf of germs of those forms of  $A^r$  which are  $\bar{\partial}$ -closed. Analogously by  $K^r, K_c^r$  will be denoted the analogous sheafs of germs of distribution forms.

We note that  $A_c^0 \simeq \Omega^p(E) \simeq K_c^0$ .

$\alpha)$  It follows from  $\bar{\partial}f = 0$  that we can find  $\varphi^{q-1} \in C^0(\mathcal{U}, A^{q-1})$  such that

$$f^q|_{V_i} = \bar{\partial}\varphi^{q-1}(i);$$

<sup>(1)</sup> For any  $u \in \mathcal{D}^{p,q}(X, E)$  we have

$$\bar{\partial}\psi[u] = (\partial u, \psi) = \lim_{v_i \rightarrow +\infty} (\partial u, \psi_{v_i}) = (u, f) = f[u].$$

we have  $\delta\varphi^{q-1} \in Z^1(\mathcal{U}, A_c^{q-1})$  and hence we can find  $\varphi^{q-2} \in C^1(\mathcal{U}, A^{q-2})$  such that

$$\delta\varphi^{q-1} = \bar{\partial}\varphi^{q-2}.$$

In this way we proceed till we find

$$\delta\varphi^0 \in Z^q(\mathcal{U}, A^0).$$

We note that the supports of  $\varphi^{q-1}, \varphi^{q-2}, \dots$ , can be chosen to be contained in  $D$ .

$\beta$ ) From  $\bar{\partial}(\psi^{q-1} - \varphi^{q-1}) = 0$  we see that

$$\psi^{q-1} - \varphi^{q-1} \in C^0(\mathcal{U}, K_c^{q-1});$$

hence we can find  $\psi^{q-2} \in C^0(\mathcal{U}, K^{q-2})$  such that

$$\psi^{q-1} - \varphi^{q-1} = \bar{\partial}\psi^{q-2}.$$

We will have, since  $\psi$  is global,

$$\delta\varphi^{q-1} = \delta\bar{\partial}\psi^{q-2}.$$

We proceed remarking that  $\bar{\partial}(\varphi^{q-2} - \delta\psi^{q-2}) = 0$  and make an analogous argument. Continuing in this way we find an element  $\psi^0 \in C^{q-2}(\mathcal{U}, K^0)$  such that

$$\delta\psi^0 - \varphi^0 \in C^{q-1}(\mathcal{U}, K_c^0).$$

We remark that the supports of  $\psi^{q-1}, \dots, \psi^0$  can be chosen to be contained in  $D$ . Moreover the element

$$h^0 = \delta\psi^0 - \varphi^0$$

is a holomorphic co-chain.

$\gamma$ ) We then have  $\delta(\varphi^0 + h^0) = 0$ ,

and since  $A^0$  is a fine sheaf we can find  $l^0 \in C^{q-2}(\mathcal{U}, A^0)$  such that

$$\varphi^0 + h^0 = \delta l^0.$$

We then have, since  $h^0$  is holomorphic,  $\bar{\partial}\varphi^0 = \bar{\partial}\delta l^0$ , hence

$$\delta(\varphi^1 - \bar{\partial}l^0) = 0,$$

and we can find  $l^1 \in C^{q-3}(\mathcal{U}, A^1)$  such that

$$\varphi^1 - \bar{\partial}l^0 = \delta l^1.$$

We continue in this way till we find  $l^{q-2} \in C^0(\mathcal{U}, A^{q-2})$  such that

$$\delta(\varphi^{q-1} - \bar{\partial}l^{q-2}) = 0.$$

This means that  $\eta^{q-1} = \varphi^{q-1} - \bar{\partial}l^{q-2}$  is a global  $(p, q)$ -form on  $X$ .

We remark that by construction the elements  $l^0, l^1, \dots, l^{q-1}$  can be chosen to have support in  $D$ . Therefore the form  $\eta^{q-1}$  is a  $C^\infty$  form with support in  $D$ , and we have

$$\bar{\partial}\eta^{q-1} = f.$$

*Corollary 13.* — Let  $E$  satisfy the assumptions of proposition 11. Then for any  $f \in \mathcal{D}^{p,q}(X, E)$  with  $\bar{\partial}f = 0$  and for any  $\varepsilon > 0$  we can find an  $\eta \in C^{p, q-1}(X, E)$  such that

- (i)  $\bar{\partial}\eta = f$   
(ii)  $\text{supp } \eta \subset \{x \in X \mid \Phi(x) \leq \sup_{\text{supp } (f)} \Phi + \varepsilon\}.$

**II.** *Cohomology with compact supports.* — a) We have the following

*Theorem 3.* — Suppose that  $E$  satisfies the assumptions of proposition 11. Suppose, furthermore, that the function  $\Phi$  satisfies the condition

- (iii) for any  $c \in \mathbf{R}$  the sets  $B_c = \{x \in X \mid \Phi(x) < c\}$  are relatively compact in  $X$ .

Then we have  $H_k^q(X, \Omega^p(E)) = 0$ .

Moreover, given any  $C^\infty$  form  $f \in \mathcal{D}^{p,q}(X, E)$  with  $\bar{\partial}f = 0$  and any  $\varepsilon > 0$ , we can find a  $C^\infty$  form  $\eta \in \mathcal{D}^{p, q-1}(X, E)$  such that

- $\alpha$ )  $\bar{\partial}\eta = f,$   
 $\beta$ )  $\text{supp } \eta \subset \{x \in X \mid \Phi(x) \leq \sup_{\text{supp } (f)} \Phi + \varepsilon\}.$

The proof of this theorem is a straightforward consequence of the above corollary 13.

b) We want now to prove that, under the above assumptions, the image of

$$\bar{\partial} : \mathcal{D}^{p,q}(X, E) \rightarrow \mathcal{D}^{p, q+1}(X, E)$$

is a closed subspace of  $\mathcal{D}^{p, q+1}(X, E)$ .

The following remark will be useful.

*Remark.* — Let  $(K_\nu)$  be a sequence of compact subsets, with  $K_\nu \subset \overset{\circ}{K}_{\nu+1}$  and  $\bigcup_\nu K_\nu = X$ . Let  $\mathcal{D}^{p,q}(K_\nu, E)$  be the space of  $C^\infty$  forms of type  $(p, q)$  with values in  $E$  and support in  $K_\nu$ . This space, with the topology of uniform convergence of the forms and of all their derivatives, is a Fréchet space [9].

We have a natural injection

$$\alpha_\nu : \mathcal{D}^{p,q}(K_\nu, E) \rightarrow \mathcal{D}^{p,q}(K_{\nu+1}, E)$$

The image of  $\alpha_\nu$  is closed in  $\mathcal{D}^{p,q}(K_{\nu+1}, E)$  and the induced topology on  $\alpha_\nu(\mathcal{D}^{p,q}(K_\nu, E))$  coincides with the natural topology of  $\mathcal{D}^{p,q}(K_\nu, E)$ . This shows that the space  $\mathcal{D}^{p,q}(X, E)$  is a strict inductive limit of Fréchet spaces.

*Theorem 4.* — Under the same assumptions as in Theorem 3,  $\bar{\partial}\mathcal{D}^{p,q}(X, E)$  is closed in  $\mathcal{D}^{p, q+1}(X, E)$ . In particular the group  $H_k^{q+1}(X, \Omega^p(E))$  has a structure of a separated topological vector space.

*Proof.* — In view of the above remark it will be sufficient to show that  $\bar{\partial}\mathcal{D}^{p,q}(X, E)$  is sequentially closed [15, p. 228] in  $\mathcal{D}^{p, q+1}(X, E)$ . Let  $(\varphi_\nu)$  be a sequence in  $\bar{\partial}\mathcal{D}^{p,q}(X, E)$

converging to an element  $\varphi \in \mathcal{D}^{p,q+1}(X, E)$ . We want to prove that  $\varphi \in \bar{\partial} \mathcal{D}^{p,q}(X, E)$ . Now any Cauchy sequence in a topological vector space is a bounded set. By the structure of  $\mathcal{D}^{p,q+1}(X, E)$  as strict inductive limit, any bounded set must be contained in some  $\mathcal{D}^{p,q+1}(K_\nu, E)$  [11, p. 257]. Thus the forms  $\varphi_\nu$  and  $\varphi$  have all their supports in a fixed compact set  $K_s$ .

Let  $\varphi_\nu = \bar{\partial} \eta_\nu$  with  $\eta_\nu \in \mathcal{D}^{p,q}(X, E)$ .

Because of the assumption of  $W_\lambda^{p,q}$ -ellipticity we can find  $\mu_\nu^\lambda \in W_\lambda^{p,q}(X, E) \cap C^{p,q}(X, E)$  such that

$$\eta_\nu = \square_\lambda \mu_\nu^\lambda = \bar{\partial} \theta_\lambda \mu_\nu^\lambda + \theta_\lambda \bar{\partial} \mu_\nu^\lambda.$$

Setting

$$\psi_\nu^\lambda = \theta_\lambda \bar{\partial} \mu_\nu^\lambda$$

we thus have

$$\varphi_\nu = \bar{\partial} \psi_\nu^\lambda.$$

Now applying Proposition 3 to the form  $x_\lambda = \bar{\partial} \mu_\nu^\lambda$  we get

$$(\psi_\nu^\lambda, \psi_\nu^\lambda)_{\lambda, B(r)} \leq \sigma(\varphi_\nu^\lambda, \varphi_\nu^\lambda)_{\lambda, B(R)} + \left( \frac{1}{\sigma} + \frac{A}{(R-r)^2} \right) (\bar{\partial} \mu_\nu^\lambda, \bar{\partial} \mu_\nu^\lambda)_{\lambda, B(R)},$$

and letting  $R = 2r$  and  $r \rightarrow +\infty$  we see that

$$\psi_\nu^\lambda \in \mathcal{L}_\lambda^{p,q}(X, E).$$

Moreover

$$\theta_\lambda \psi_\nu^\lambda = 0, \quad \bar{\partial} \psi_\nu^\lambda = \varphi_\nu.$$

Hence by proposition 5 we have the inequality

$$(\psi_\nu^\lambda, \psi_\nu^\lambda)_\lambda \leq c(\varphi_\nu, \varphi_\nu)_\lambda.$$

$\beta$ ) Now  $\lim(\varphi_\nu, \varphi_\nu)_\lambda = (\varphi, \varphi)_\lambda$ .

If  $\varphi = 0$  there is nothing to prove. Otherwise we can select an index  $\nu_0$  such that for  $\nu > \nu_0$  we have

$$(\psi_\nu^\lambda, \psi_\nu^\lambda)_\lambda \leq 2c(\varphi, \varphi)_\lambda.$$

From the sequence  $(\psi_\nu)$  we can extract a subsequence, that we denote again by  $(\psi_\nu)$ , which converges weakly to an element  $\psi \in \mathcal{L}^{p,q}(X, E)$  having compact support.

Since  $\bar{\partial} \psi_\nu = \varphi_\nu$ ,

we have in the sense of distributions that

$$\bar{\partial} \psi = \varphi \quad (1).$$

By lemma 12 we can then find a  $\eta \in \mathcal{D}^{p,q}(X, E)$  such that

$$\varphi = \bar{\partial} \eta.$$

(1) In fact for any  $u \in \mathcal{D}^{p,q+1}(X, E)$  we have

$$\begin{aligned} \bar{\partial} \psi[u] &= (\theta u, \psi) = \lim(\theta u, \psi_\nu) = \lim(u, \bar{\partial} \psi_\nu) \\ &= \lim(u, \varphi_\nu) = (u, \varphi) = \varphi[u]. \end{aligned}$$

§ 4. Criteria for W-ellipticity

12. *Local expression of the Laplace-Beltrami operator.* — This and the following section are not essential for the comprehension of the rest of this paper and may be omitted.

a) Any form of type  $(p, q)$  with values in  $E$  can be considered as a form of type  $(0, q)$  with values in  $E \otimes \Theta^{*p}$ . We have thus an isomorphism

$$C^{pq}(X, E) \cong C^{0q}(X, E \otimes \Theta^{*p}).$$

If  $\varphi \in C^{pq}(X, E)$  is given locally by the forms

$$\varphi = \{ \sum \varphi_{A\bar{B}}^a dz^A d\bar{z}^{\bar{B}} \} \quad (1 \leq a \leq m = \text{rank of } E),$$

then its image  $\tilde{\varphi} \in C^{0q}(X, E \otimes \Theta^{*p})$  is given locally by the forms

$$\tilde{\varphi} = \{ \sum \varphi_{A\bar{B}}^a d\bar{z}^{\bar{B}} \}.$$

By a direct computation one establishes the following formulae <sup>(1)</sup>:

$$(22) \quad \bar{\partial} \tilde{\varphi} = (-1)^p \bar{\partial} \tilde{\varphi},$$

$$(23) \quad \theta_E \tilde{\varphi} = (-1)^p \theta_{E \otimes \Theta^{*p}} \tilde{\varphi},$$

$$(24) \quad \square_E \tilde{\varphi} = \square_{E \otimes \Theta^{*p}} \tilde{\varphi}.$$

Moreover one has

$$(25) \quad A_E(\varphi, \varphi) = A_{E \otimes \Theta^{*p}}(\tilde{\varphi}, \tilde{\varphi}).$$

b) By the above remark we can restrict our considerations to forms of type  $(0, q)$  only.

Let  $\varphi \in C^{0q}(X, E)$  be given locally by

$$\varphi = \{ \sum \varphi_{\bar{\beta}_1 \dots \bar{\beta}_q}^a d\bar{z}^{\bar{\beta}_1} \wedge \dots \wedge d\bar{z}^{\bar{\beta}_q} \}.$$

If  $h$  is the hermitian metric on the fibres of  $E$  we use on  $E$  the connection given by  $h^{-1}\partial h$ . On  $\bar{\Theta}^*$  we use instead the riemannian connection given by the hermitian metric on  $X$  (cf. n° 3b). We will use greek indices in the range  $1, \dots, n = \dim_{\mathbb{C}} X$  and latin indices in the range  $1, \dots, n, \bar{1}, \dots, \bar{n}$ . Since the riemannian connection is symmetric one obtains the following formulae:

$$(26) \quad (\bar{\partial} \varphi)^a = \sum_{\beta_1 < \dots < \beta_{q+1}} \{ \sum (-1)^{r-1} \nabla_{\bar{\beta}_r} \varphi_{\bar{\beta}_1 \dots \hat{\beta}_r \dots \bar{\beta}_{q+1}} \} d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_{q+1}},$$

$$(27) \quad (\theta \varphi)^a = \sum_{\beta_1 < \dots < \beta_{q-1}} \{ \sum \nabla_r \varphi_{\bar{\beta}_1 \dots \bar{\beta}_{q-1}}^{ar} \} d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_{q-1}}.$$

<sup>(1)</sup> Only the formula (23) needs verification. Formula (22) is obvious, and formula (24) follows from (22) and (23).

Let  $B = (\beta_1, \dots, \beta_q)$ ,  $B'_i = (\beta_1, \dots, \hat{\beta}_i, \dots, \beta_q)$ ,  $B''_{ij} = (\beta_1, \dots, \hat{\beta}_i, \dots, \hat{\beta}_j, \dots, \beta_q)$ .  
From (26) and (27) we obtain

$$(\square \varphi)_B^a = -\sum g^{\alpha\bar{\beta}} \nabla_\alpha \nabla_{\bar{\beta}} \varphi_B^a + \sum g^{\alpha\bar{\beta}} \Gamma_{\alpha\bar{\beta}}^{\bar{\lambda}} \nabla_{\bar{\lambda}} \varphi_B^a + \sum_{i=1}^q \sum (-1)^{i-1} (\nabla_r \nabla_{\bar{\beta}_i} - \nabla_{\bar{\beta}_i} \nabla_r) \varphi_{B_i}^{ar}.$$

Using Ricci's identity for the last summand we finally get

$$(28) \quad (\square \varphi)_B^a = -\sum g^{\alpha\bar{\beta}} \nabla_\alpha \nabla_{\bar{\beta}} \varphi_B^a + \sum g^{\alpha\bar{\beta}} \Gamma_{\alpha\bar{\beta}}^{\bar{\lambda}} \nabla_{\bar{\lambda}} \varphi_B^a + (\mathcal{K} \varphi)_B^a,$$

where  $\mathcal{K} \varphi \in C^{0q}(X, E)$  has the following expression:

$$(29) \quad (\mathcal{K} \varphi)_B^a = \sum_{i=1}^q (-1)^i \{ \sum s_{b\bar{\beta}_i \beta}^a \varphi_{B_i}^{b\bar{\beta}} + \sum R_{\beta\bar{\beta}_i}^{\alpha\bar{\beta}} \varphi_{B_i}^{\alpha\bar{\beta}} + \sum (-1)^j R_{\bar{\beta}_j \beta_i \beta}^{\bar{\gamma}} \varphi_{B_{ij}}^{\alpha\bar{\beta}} \},$$

and  $R_{\beta\bar{\gamma}} = \sum_r R_{\beta\bar{\gamma}r}^r$  is the Ricci tensor.

The endomorphism  $\mathcal{K} : C^{0q}(X, E) \rightarrow C^{0q}(X, E)$  is hermitian, i.e.  $A(\mathcal{K} \varphi, \varphi)$  is real. For  $q=0$ ,  $\mathcal{K} = 0$ .

c) We now use formulae (28) and (29) and the remarks made in a) to obtain the corresponding formulae for any form  $\varphi \in C^{pq}(X, E)$ . Following the above procedure we must use on  $E \otimes \Theta^{*p} \otimes \bar{\Theta}^{*q}$  the connection which is obtained from the metric connections  $h^{-1} \partial h$  on  $E$ ,  $g^{*-1} \partial g^*$  on  $\Theta^*$  (where  $g^* = {}^t g^{-1}$  is the metric on the fibres of  $\Theta^*$ ), and the riemannian connection on  $\bar{\Theta}^*$ .

From (24) and (28) we then obtain

$$(30) \quad (\square \varphi)_{A\bar{B}}^a = -\sum g^{\alpha\bar{\beta}} \nabla_\alpha \nabla_{\bar{\beta}} \varphi_{A\bar{B}}^a + \sum g^{\alpha\bar{\beta}} \Gamma_{\alpha\bar{\beta}}^{\bar{\lambda}} \nabla_{\bar{\lambda}} \varphi_{A\bar{B}}^a + (\mathcal{K} \varphi)_{A\bar{B}}^a$$

where  $\mathcal{K} \varphi \in C^{pq}(X, E)$  is defined by

$$\widetilde{\mathcal{K}} \varphi = \mathcal{K} \widetilde{\varphi}.$$

If we compute the curvature form of the metric connection of  $E \otimes \Theta^{*p}$  in terms of the curvature forms

$$s = \{ \sum s_{b\bar{\beta}\alpha}^a d\bar{z}^{\bar{\beta}} dz^\alpha \}, \quad L = \{ \sum L_{\beta\bar{\gamma}\delta}^\alpha d\bar{z}^{\bar{\gamma}} dz^\delta \}$$

of  $E$  and  $\Theta^*$  respectively we obtain for  $\mathcal{K} \varphi$  the following expression:

$$(31) \quad (\mathcal{K} \varphi)_{A\bar{B}}^a = \sum_{i=1}^q (-1)^i \{ \sum s_{b\bar{\beta}_i \beta}^a \varphi_{A\bar{B}_i}^{b\bar{\beta}} + \sum_{k=1}^p (-1)^k L_{\alpha_k \bar{\beta}_i \beta}^\alpha \varphi_{A\bar{B}_i}^{\alpha \beta} + \sum R_{\beta\bar{\beta}_i}^{\alpha \beta} \varphi_{A\bar{B}_i}^{\alpha \beta} + \sum (-1)^j R_{\bar{\beta}_j \beta_i \beta}^{\bar{\gamma}} \varphi_{A\bar{B}_{ij}}^{\alpha \beta} \}$$

with the usual conventions and  $A'_h = (\alpha_1, \dots, \hat{\alpha}_h, \dots, \alpha_p)$ .

**13.** The operator  $*^{-1} \square * - \square$ . For any  $\varphi \in C^{0q}(X, E)$  one has the following formulae

$$\begin{aligned} (\nabla_\alpha * \varphi)_{\alpha_1 \dots \alpha_{n-q} \bar{1} \dots \bar{n}} &= (* \nabla_\alpha \varphi)_{\alpha_1 \dots \alpha_{n-q} \bar{1} \dots \bar{n}} - \sum_{i=1}^{n-q} \sum S_{\alpha_i \alpha}^\lambda (* \varphi)_{\alpha_1 \dots (\lambda)_i \dots \alpha_{n-q} \bar{1} \dots \bar{n}}, \\ (\nabla_{\bar{\beta}} * \varphi)_{\alpha_1 \dots \alpha_{n-q} \bar{1} \dots \bar{n}} &= (* \nabla_{\bar{\beta}} \varphi)_{\alpha_1 \dots \alpha_{n-q} \bar{1} \dots \bar{n}} + \sum_{i=1}^{n-q} \sum \Gamma_{\alpha_i \bar{\beta}}^\lambda (* \varphi)_{\alpha_1 \dots (\lambda)_i \dots \alpha_{n-q} \bar{1} \dots \bar{n}}. \end{aligned}$$

If the hermitian metric on  $X$  is Kähler then  $S_{\alpha_i \alpha}^\lambda = 0$  and  $\Gamma_{\alpha_i \bar{\beta}}^\lambda = 0$ ; thus for any  $\varphi \in C^{pq}(X, E)$  one has

$$\nabla_\alpha * \varphi = * \nabla_\alpha \varphi, \quad \nabla_{\bar{\beta}} * \varphi = * \nabla_{\bar{\beta}} \varphi,$$

so that using the local expression of the Laplace Beltrami operator one obtains that

$$*^{-1} \square * - \square = *^{-1} \mathcal{H} * - \mathcal{H},$$

where  $\mathcal{H} \varphi$  simplifies into the following expression first given by Kodaira [14]:

$$(\mathcal{H} \varphi)_{A\bar{B}}^a = \sum_{i=1}^q (-1)^i \{ \sum s_{b\bar{\beta}_i \beta}^a \varphi_{A \bar{B}_i}^{b \beta} + \sum R_{\beta \bar{\beta}_i} \varphi_{A \bar{B}_i}^{a \beta} + \sum_{j=1}^q (-1)^j R_{\alpha_j \bar{\beta}_i \beta}^\alpha \varphi_{A_j \bar{B}_i}^{a \beta} \}.$$

Moreover computing  $*^{-1} \mathcal{H} * - \mathcal{H}$  one obtains the following expression

$$\{ (*^{-1} \mathcal{H} * - \mathcal{H}) \varphi \}_{A\bar{B}}^a = \sum_{i=1}^q (-1)^{i-1} \sum s_{b\bar{\beta}_i \beta}^a \varphi_{A \bar{B}_i}^{b \beta} + \sum_{j=1}^p (-1)^{j-1} s_{b\bar{\alpha}_j \alpha}^a \varphi_{A_j \bar{B}}^{b \bar{\alpha}} - \sum s_{b\bar{\beta}}^a \varphi_{A\bar{B}}^{b \bar{\beta}}$$

Using the operators  $\Lambda$  and  $e(s)$  defined in § 1 we then have

$$*^{-1} \square * - \square = (\Lambda e(s) - e(s) \Lambda),$$

a formula which was first obtained in [7].

We remark that  $*^{-1} \square * = \neq^{-1} \square_{\mathbb{R}} * \neq$  according to formula (7).

b) If the hermitian metric is not Kähler, then one has a more complicated expression of the form

$$(32) \quad (*^{-1} \square * - \square) \varphi = (*^{-1} \mathcal{H} * - \mathcal{H}) \varphi + F_1 \varphi + F_2 \nabla \varphi + F_3 \bar{\nabla} \varphi,$$

where  $F_i$  are linear combinations of the components of  $\varphi$ ,  $\nabla \varphi$ ,  $\bar{\nabla} \varphi$  with coefficients involving (linearly) the torsion tensor, the tensor  $\Gamma_{\alpha \bar{\beta}}^\lambda$  and its covariant derivatives.

From the formulae we have given it will be clear as to the connection between the vanishing theorem we give here, which is a generalisation of the vanishing theorems of Kodaira and Nakano.

**14. A basic identity.** — a) We first remark that for any scalar 1-form on  $X$

$$\psi = \sum \psi_\alpha dz^\alpha + \sum \psi_{\bar{\beta}} d\bar{z}^\beta$$

one has the following identity  $d* \psi = (\sum \nabla_r \psi^r) dx$ ,

when we take on the tangent bundle the riemannian connection. The expression  $\sum \nabla_r \psi^r$  is called the divergence of the vector  $\psi^r$ .

b) Let  $\varphi \in C^{pq}(X, E)$  and let  $q > 0$ . We construct from  $\varphi$  the following tangent vectors:

$$\xi = \{ \xi^\beta = \sum h_{\bar{\alpha}\alpha} (\nabla_{\bar{\gamma}} \varphi_{\alpha \bar{B}'}^{a \beta}) \overline{\varphi^{b \gamma B'}}, \xi^{\bar{\beta}} = 0 \},$$

$$\eta = \{ \eta^\gamma = 0, \eta^{\bar{\gamma}} = \sum h_{\bar{\alpha}\alpha} (\nabla_\gamma \varphi_{\alpha \bar{B}'}^{a \bar{\gamma}}) \overline{\varphi^{b \gamma B'}} \}.$$

Computing the divergence of  $\xi$  and using the Ricci identity we obtain

$$\operatorname{div} \xi = \sum h_{\bar{b}a} (\nabla_{\bar{\gamma}} \nabla_r \varphi^{a\bar{r}\bar{b}'} + \frac{1}{q} (\mathcal{K} \varphi)^a_{\bar{\gamma}\bar{b}'} \overline{\varphi^{b\bar{\gamma}B'}} + (\nabla_{\bar{\gamma}} \varphi^{a\bar{b}\bar{b}'})(\nabla_{\bar{\beta}} \overline{\varphi^{b\bar{\gamma}B'}}).$$

Analogously we have

$$\operatorname{div} \eta = \sum h_{\bar{b}a} (\nabla_{\bar{\gamma}} \nabla_r \varphi^{a\bar{r}\bar{b}'} \overline{\varphi^{b\bar{\gamma}B'}} + (\nabla_r \varphi^{a\bar{r}\bar{b}'})(\nabla_s \overline{\varphi^{bsB'}}).$$

Therefore

$$\operatorname{div} \xi - \operatorname{div} \eta = \sum h_{\bar{b}a} \{ (\nabla_{\bar{\gamma}} \varphi^{a\bar{b}\bar{b}'})(\nabla_{\bar{\beta}} \overline{\varphi^{b\bar{\gamma}B'}}) - (\nabla_r \varphi^{a\bar{r}\bar{b}'})(\nabla_s \overline{\varphi^{bsB'}}) + \frac{1}{q} (\mathcal{K} \varphi)^a_{\bar{\gamma}\bar{b}'} \overline{\varphi^{b\bar{\gamma}B'}} \}.$$

By formula (26) we have

$$\sum h_{\bar{b}a} (\nabla_{\bar{\gamma}} \varphi^{a\bar{b}\bar{b}'})(\nabla_{\bar{\beta}} \overline{\varphi^{b\bar{\gamma}B'}}) = (q-1)! (A(\bar{\nabla}\varphi, \bar{\nabla}\varphi) - A(\bar{\partial}\varphi, \bar{\partial}\varphi)),$$

while by formula (27) we have

$$\sum h_{\bar{b}a} (\nabla_r \varphi^{a\bar{r}\bar{b}'})(\nabla_s \overline{\varphi^{bsB'}}) = (q-1)! A(\theta\varphi, \theta\varphi).$$

We obtain therefore the following identity:

$$(33) \quad \frac{1}{(q-1)!} (\operatorname{div} \xi - \operatorname{div} \eta) = A(\bar{\nabla}\varphi, \bar{\nabla}\varphi) + A(\mathcal{K} \varphi, \varphi) - A(\bar{\partial}\varphi, \bar{\partial}\varphi) - A(\theta\varphi, \theta\varphi).$$

**15.** a) We now suppose that the metric on  $X$  is *complete*. Let  $o \in X$  and let  $\rho(x) = d(o, x)$  be the geodesic distance of  $x$  from  $o$ . We set as before  $B(r) = \{x \in X \mid \rho(x) < r\}$ .

With the same notations as in n° 6  $\gamma$ ), we now consider the following expression

$$F = \frac{1}{(q-1)!} (\operatorname{div} w^2 \xi - \operatorname{div} w^2 \eta).$$

Since  $\int_X F dx = 0$ , we deduce from (33) the following equality:

$$\begin{aligned} \frac{2}{(q-1)!} \int (\sum w \frac{\partial w}{\partial z^{\bar{\beta}}} \xi^{\bar{\beta}}) dX - \frac{2}{(q-1)!} \int (\sum w \frac{\partial w}{\partial z^{\bar{\gamma}}} \eta^{\bar{\gamma}}) dX + (w \bar{\nabla}\varphi, w \bar{\nabla}\varphi) + \\ + (\mathcal{K} w\varphi, w\varphi) - (w \bar{\partial}\varphi, w \bar{\partial}\varphi) - (w\theta\varphi, w\theta\varphi) = 0. \end{aligned}$$

Now we remark that one has

$$\begin{aligned} |2 \int (\sum w \frac{\partial w}{\partial z^{\bar{\beta}}} \xi^{\bar{\beta}}) dX| &= |2 \int w \sum \frac{\partial w}{\partial z^{\bar{\beta}}} h_{\bar{b}a} (\nabla_{\bar{\gamma}} \varphi^{a\bar{b}\bar{b}'} \overline{\varphi^{b\bar{\gamma}B'}}) dX| \\ &\leq \frac{2c'}{R-r} \|w \bar{\nabla}\varphi\| \|\varphi\|_{B(R)} \\ &\leq \frac{c'}{R-r} \{ \|w \bar{\nabla}\varphi\|^2 + \|\varphi\|_{B(R)}^2 \}, \end{aligned}$$

$c'$  being an absolute constant. Analogously one has

$$|2 \int (\sum w \frac{\partial w}{\partial z^{\bar{\gamma}}} \eta^{\bar{\gamma}}) dX| \leq \frac{c'}{R-r} \{ \|w\theta\varphi\|^2 + \|\varphi\|_{B(R)}^2 \}.$$

Hence, using the fact that  $(\mathcal{H}w\varphi, w\varphi)$  is real we obtain the inequality:

$$\begin{aligned} \|\bar{w}\bar{\nabla}\varphi\|_{B(R)}^2 \left(1 - \frac{c'}{(q-1)!(R-r)}\right) + (\mathcal{H}w\varphi, w\varphi) &\leq \\ &\leq \|\bar{\partial}\varphi\|_{B(R)}^2 + \|\theta\varphi\|_{B(R)}^2 + \frac{c'}{(q-1)!(R-r)} \{2\|\varphi\|_{B(R)}^2 + \|\theta\varphi\|_{B(R)}^2\}. \end{aligned}$$

Letting  $R-r \rightarrow \infty$  and  $r \rightarrow \infty$  we obtain the following:

*Lemma 14.* — *If the hermitian metric on X is complete, for any form  $\varphi \in C^{0q}(X, E)$ , with  $q > 0$ , such that  $\varphi \in \mathcal{L}^{0q}(X, E)$ ,  $\bar{\partial}\varphi \in \mathcal{L}^{0, q+1}(X, E)$ ,  $\theta\varphi \in \mathcal{L}^{0, q-1}(X, E)$ , one has the following inequality:*

$$\|\bar{\nabla}\varphi\|^2 + \limsup_{(R-r) \rightarrow +\infty} (\mathcal{H}w\varphi, w\varphi) \leq \|\bar{\partial}\varphi\|^2 + \|\theta\varphi\|^2 \quad (1).$$

b) Let us consider now any form  $\varphi \in C^{pq}(X, E)$  ( $q > 0$ ). Applying Lemma 14 to the form  $\tilde{\varphi} \in C^{0q}(X, E \otimes \Theta^{*p})$  and taking into account (22), (23), (25) and (31) we get the following

*Proposition 15.* — *If the hermitian metric on X is a complete metric, then, for any form  $\varphi \in C^{pq}(X, E)$  ( $q > 0$ ) such that  $\|\varphi\| < +\infty$ ,  $\|\bar{\partial}\varphi\| < +\infty$ ,  $\|\theta\varphi\| < +\infty$ , the following inequality holds*

$$\|\bar{\nabla}\varphi\|^2 + \limsup_{R-r \rightarrow +\infty} (\mathcal{H}w\varphi, w\varphi)_{B(R)} \leq \|\bar{\partial}\varphi\|^2 + \|\theta\varphi\|^2.$$

Suppose, in particular, that at any point  $x \in X$  and for every  $\varphi \in C^{pq}(X, E)$ ,  $A(\mathcal{H}\varphi, \varphi) \geq 0$ .

Then, under the hypothesis of Proposition 15, it follows that

$$\limsup_{r \rightarrow +\infty} (\mathcal{H}w\varphi, w\varphi)_{B(2r)}$$

and  $\|\bar{\nabla}\varphi\|$  are bounded. Since, moreover

$$0 \leq (\mathcal{H}\varphi, \varphi)_{B(r)} \leq (\mathcal{H}w\varphi, w\varphi)_{B(2r)},$$

we have

$$(\mathcal{H}\varphi, \varphi) = \lim_{r \rightarrow +\infty} (\mathcal{H}\varphi, \varphi)_{B(r)} < +\infty.$$

We have therefore the following

*Corollary 16.* — *If the hermitian metric on X is a complete metric and if at each point  $x \in X$  and for any  $\varphi \in C^{pq}(X, E)$  ( $q > 0$ )*

$$A(\mathcal{H}\varphi, \varphi) \geq 0,$$

(1) The proof of this lemma is independent of prop. 5. One could instead obtain directly from (33) that for any  $\varphi \in \mathcal{L}^{0q}(X, E)$  ( $q > 0$ ) one has

$$\|\bar{\nabla}\varphi\|^2 + (\mathcal{H}\varphi, \varphi) = \|\bar{\partial}\varphi\|^2 + \|\theta\varphi\|^2,$$

and then, using proposition 5, deduce lemma 14 by a « closure » argument.

then, for any  $\varphi \in C^{pq}(X, E)$  ( $q > 0$ ) such that

$$\varphi \in \mathcal{L}^{pq}(X, E), \quad \bar{\partial}\varphi \in \mathcal{L}^{p, q+1}(X, E), \quad \theta\varphi \in \mathcal{L}^{p, q-1}(X, E),$$

$\|\bar{\nabla}\varphi\|$  and  $(\mathcal{K}\varphi, \varphi)$  are finite, and moreover

$$\|\bar{\nabla}\varphi\|^2 + (\mathcal{K}\varphi, \varphi) \leq \|\bar{\partial}\varphi\|^2 + \|\theta\varphi\|^2.$$

*Corollary 17.* — If the hermitian metric on  $X$  is a complete metric and if there exists a positive constant  $k$  such that at each point  $x \in X$  and for every  $\varphi \in C^{pq}(X, E)$  ( $q > 0$ )

$$A(\mathcal{K}\varphi, \varphi) \geq kA(\varphi, \varphi)$$

then  $E$  is  $W^{pq}$ -elliptic. In fact for any form  $\varphi \in C^{pq}(X, E) \cap W^{pq}(X, E)$  the following inequality holds:

$$\|\bar{\nabla}\varphi\|^2 + k\|\varphi\|^2 \leq \|\bar{\partial}\varphi\|^2 + \|\theta\varphi\|^2.$$

### § 5. Vanishing theorem for $q$ -complete manifolds

**16.** *A lemma on hermitian forms.* — a) On the complex manifold  $X$  ( $\dim_{\mathbb{C}} X = n$ ) we consider two hermitian forms given locally on a covering  $\mathcal{U} = \{U_i\}$  of  $X$  by

$$\sigma = \sum g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta = \{ {}^t d\bar{z}_i G_i dz_i \}, \quad {}^t \bar{G}_i = G_i$$

$$\eta = \sum h_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta = \{ {}^t d\bar{z}_i H_i dz_i \}, \quad {}^t \bar{H}_i = H_i$$

We will assume that

$\sigma$  is positive definite so that it defines a hermitian metric on  $X$ ,

$\eta$  has at each point  $x \in X$  at least  $p$  positive eigenvalues.

If  $J_{ij}$  are the transition functions  $\partial(z_i)/\partial(z_j)$  of the tangent bundle, then on  $U_i \cap U_j$   $G_i = {}^t \bar{J}_{ji} G_j J_{ji}$  and  $H_i = {}^t \bar{J}_{ji} H_j J_{ji}$  so that the characteristic polynomial  $\det(H_i G_i^{-1} - \lambda I)$  is a  $C^\infty$  function on  $X$ . The eigenvalues of  $HG^{-1}$  at each point  $x$  are real, let them be

$$\varepsilon_1(x) \geq \dots \geq \varepsilon_n(x),$$

so that each  $\varepsilon_\alpha(x)$  is a real continuous function on  $X$ . Because of the assumptions we will have at each point  $x \in X$

$$\varepsilon_1(x) \geq \dots \geq \varepsilon_p(x) > 0.$$

Let  $c_1 > 0$ ,  $c_2 > 0$  be two positive constants and let

$$l_H(x) = c_1 \varepsilon_p(x) + c_2 \inf(0, \varepsilon_n(x)).$$

*Lemma 18.* — Given the form  $\eta$  we can find a complete hermitian metric on  $X$  such that

$$l_H(x) > 0 \quad \forall x \in X.$$

*Proof.* —  $\alpha$ ) Consider the function

$$f(\lambda, t) = \frac{1}{\lambda} (e^{\lambda t} - 1)$$

for  $\lambda, t$  real. This function has the following properties:

- (i)  $f(\lambda, t) = t + \frac{\lambda t^2}{2!} + \frac{\lambda^2 t^3}{3!} + \dots$  is an entire function;
- (ii)  $\frac{\partial f(\lambda, t)}{\partial t} = e^{\lambda t} > 0$ ;
- (iii) If  $\lambda \geq 0$  then  $f(\lambda, t) \geq t$ .

$\beta$ ) We choose a hermitian metric  $\sigma$  on  $X$  which we may as well suppose to be a complete metric <sup>(1)</sup>. We set on  $U_i$

$$\hat{G}_i^{-1} = G_i^{-1} \left\{ I + \frac{\lambda(x) (H_i G_i^{-1})}{2!} + \frac{\lambda(x)^2 (H_i G_i^{-1})^2}{3!} + \dots \right\},$$

where  $\lambda = \lambda(x)$  is a non negative  $C^\infty$  function on  $X$ . Then  $\hat{G}_i$  is a positive definite hermitian metric; moreover on  $U_i \cap U_j$

$$\hat{G}_i = {}^t \bar{J}_{ji} \hat{G}_j J_{ji}.$$

Therefore  $\hat{G}$  defines a hermitian metric on  $X$ .

Now 
$$H \hat{G}^{-1} = f(\lambda, H G^{-1}).$$

Hence the eigenvalues  $\hat{\varepsilon}_1(x), \dots, \hat{\varepsilon}_n(x)$  of  $H \hat{G}^{-1}$  are given by

$$\hat{\varepsilon}_\alpha(x) = f(\lambda(x), \varepsilon_\alpha(x)).$$

From  $\varepsilon_\alpha(x) \geq \varepsilon_{\alpha+1}(x)$  it follows (by (ii)) that

$$\hat{\varepsilon}_\alpha(x) \geq \hat{\varepsilon}_{\alpha+1}(x).$$

Also by (iii) we have

$$\hat{\varepsilon}_\alpha(x) \geq \varepsilon_\alpha(x).$$

In particular

$$\hat{\varepsilon}_1(x) \geq \dots \geq \hat{\varepsilon}_p(x) > 0.$$

$\gamma$ ) Let  $o \in X$  and let  $d(o, x)$  be the distance of  $o$  from  $x$  in the metric  $\sigma$ . Let

$$B_\nu = \{x \in X \mid d(o, x) < \nu\}, \quad \nu = 1, 2, \dots$$

Then, since  $\sigma$  is complete,  $\bar{B}_\nu$  is compact.

Let 
$$b_\nu = \inf_{x \in \bar{B}_\nu} \varepsilon_p(x).$$

Then 
$$b_\nu \geq b_{\nu+1}, \quad b_\nu > 0 \quad \text{for each } \nu.$$

We select a  $C^\infty$  function  $b(x)$  on  $X$  such that

$$\begin{aligned} b(x) &> 0 & \forall x \in X, \\ b(x) &< b_\nu & \text{for } x \in B_\nu - B_{\nu-1}. \end{aligned}$$

<sup>(1)</sup> One may use the following remark: given a riemannian metric  $ds^2$  on a manifold  $X$  we can find a  $C^\infty$  function  $F(x) > 0$  such that  $F(x)ds^2$  is a complete metric. In fact if  $(K_\nu)$  is a sequence of compact sets on  $X$  such that  $K_\nu \subset \overset{\circ}{K}_{\nu+1}$ ,  $\bigcup K_\nu = X$ , we set  $c_\nu = \text{dist}(\partial K_{\nu+1}, K_\nu)$ . Then  $c_\nu > 0$  and for any  $F$  such that  $F^2(x) > \nu/c_\nu$  for  $x \in K_{\nu+1} - K_\nu$ ,  $Fds^2$  becomes a complete metric.

Hence on all  $X$  we will have  $b(x) < b_1$ ,  
 $b(x) \leq \varepsilon_p(x)$ .

Finally we select a  $C^\infty$  function  $\rho(x)$  on  $X$  such that

$$\rho(x) \geq d(o, x)$$

and we set

$$\lambda(x) = \frac{2ke^{\rho(x)}}{b^2(x)}$$

where  $k > \sqrt{\frac{c_2}{c_1}} b_1$ .

$\delta$ ) We have then for every  $x \in X$ , since  $\varepsilon_p(x) > 0$ ,

$$\begin{aligned} \hat{\varepsilon}_p(x) = f(\lambda(x), \varepsilon_p(x)) &> \frac{\lambda(x) \varepsilon_p^2(x)}{2} = ke^{\rho(x)} \frac{\varepsilon_p^2(x)}{b^2(x)} \\ &\geq ke^{\rho(x)} \geq k, \end{aligned}$$

$$\begin{aligned} \hat{\varepsilon}_n(x) = f(\lambda(x), \varepsilon_n(x)) &= \frac{1}{\lambda(x)} (e^{\lambda(x)\varepsilon_n(x)} - 1) \\ &\geq -\frac{1}{\lambda(x)} = -\frac{b^2(x)}{2ke^{\rho(x)}} > -\frac{b_1^2}{k}. \end{aligned}$$

Thus  $c_1 \hat{\varepsilon}_p(x) + c_2 \inf(o, \hat{\varepsilon}_n(x)) > c_1 k - c_2 \frac{b_1^2}{k} = \frac{1}{k} (c_1 k^2 - c_2 b_1^2) > 0$ .

Then for the metric  $\hat{\sigma}$  defined by  $\hat{G}$  we will have  $l_H(x) > 0$ . Now if we multiply  $\hat{\sigma}$  by a convenient  $C^\infty$  positive function  $F(x)$ , the condition  $l_H(x) > 0$  is preserved, while  $F(x)\hat{\sigma}$  can be made into a complete metric.

$b$ ) Let  $\Phi : X \rightarrow \mathbf{R}$  be a  $C^\infty$  function on  $X$  which is strongly  $q$ -pseudoconvex. This means that the Levi form of  $\Phi$

$$\mathcal{L}(\Phi) = \sum \frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha d\bar{z}^\beta$$

has at least  $n - q + 1$  positive eigenvalues <sup>(1)</sup>.

Given a hermitian metric  $ds^2$  on  $X$  we can evaluate the eigenvalues of  $\mathcal{L}(\Phi)$  with respect to  $ds^2$ . Let  $\varepsilon_1(x) \geq \dots \geq \varepsilon_n(x)$  be these eigenvalues. By assumption  $\varepsilon_1(x) \geq \dots \geq \varepsilon_{n-q+1}(x) > 0$  at each point of  $X$ . We set  $p = n - q + 1$ .

*Lemma 19.* — *If  $\Phi$  is strongly  $q$ -pseudoconvex, for any scalar form  $u = \sum u_{A\bar{B}} dz^A d\bar{z}^{\bar{B}}$  of type  $(a, b)$  with  $b \geq q$  the following inequality holds at any point  $x \in X$ :*

$$\sum \frac{\partial^2 \Phi}{\partial z^\beta \partial \bar{z}^\gamma} u_{A\bar{B}} \overline{u^{\bar{A}\gamma B'}} \geq \{ \varepsilon_p(x) + n \inf(o, \varepsilon_n(x)) \} \sum_{\substack{\alpha_1 < \dots < \alpha_a \\ \beta_1 < \dots < \beta_b}} u_{A\bar{B}} \overline{u^{\bar{A}\bar{B}}}$$

<sup>(1)</sup> (Added in proof) We prefer now to call strongly  $q$ -pseudoconvex a function whose Levi form has at least  $n - q$  positive eigenvalues.

*Proof.* — At any point  $x \in X$  we have

$$\sum \frac{\partial^2 \Phi}{\partial z^\beta \partial \bar{z}^\gamma} u_{A\bar{B}}^{\beta\gamma} \overline{u^{\alpha\gamma B'}} = \sum_{\beta=1}^n \varepsilon_\beta(x) u_{A\bar{B}}^{\beta\beta} \overline{u^{\alpha\beta B'}}.$$

If  $b \geq q$  then  $b + n - q + 1 \geq n + 1$ , thus any block of  $b$  indices taken in  $(\bar{1}, \dots, \bar{n})$  must contain one of the indices  $\bar{1}, \dots, \overline{n - q + 1}$ , i.e. one of the indices of the positive eigenvalues  $\varepsilon_1(x), \dots, \varepsilon_{n-q+1}(x)$ . It follows that

$$\sum_{\beta=1}^{n-q+1} \sum u_{A\bar{B}}^{\beta\beta} \overline{u^{\alpha\beta B'}} \geq \sum_{\substack{\alpha_1 < \dots < \alpha_a \\ \beta_1 < \dots < \beta_b}} u_{A\bar{B}}^{\alpha\beta} \overline{u^{\alpha\beta B'}}$$

From this the assertion of the lemma follows.

We will now apply lemma 18 to the form  $\eta = \mathcal{L}(\Phi)$  taking as  $l_{\mathcal{L}(\Phi)}$  the expression

$$l_{\mathcal{L}(\Phi)} = \varepsilon_{n-q+1}(x) + n \inf(0, \varepsilon_n(x))$$

( $c_1 = 1, c_2 = n, p = n - q + 1$ ). It thus follows that there exists on  $X$  a complete hermitian metric  $ds^2$  such that at any point  $x \in X$  we have

$$(34) \quad l_{\mathcal{L}(\Phi)}(x) > 0.$$

We will keep this hermitian metric  $ds^2$  fixed throughout the remainder of this section.

c) A complex manifold  $X$  is called  $q$ -complete if there exists on  $X$  a  $C^\infty$  strongly  $q$ -pseudoconvex function  $\Phi : X \rightarrow \mathbf{R}$  such that the sets

$$B_c = \{x \in X \mid \Phi(x) < c\}$$

are relatively compact in  $X$ .

Adding, if necessary, a constant to  $\Phi$ , in view of the last condition we may assume that  $\Phi \geq 0$ .

Let  $\mu = \mu(t)$  be a  $C^\infty$  function on  $0 \leq t < \infty$  which is increasing and convex i.e.  $\mu'(t) > 0, \mu''(t) \geq 0$ . Consider the function  $\mu(\Phi)$ : we have

$$(35) \quad \begin{aligned} \mathcal{L}(\mu(\Phi)) &= \mu'(\Phi) \mathcal{L}(\Phi) + \mu''(\Phi) |\partial\Phi|^2 \\ &\geq \mu'(\Phi) \mathcal{L}(\Phi). \end{aligned}$$

It follows then that, for any such choice of  $\mu$ , the function  $\mu(\Phi)$  is again strongly  $q$ -pseudoconvex, and again the sets  $\{\mu(\Phi) < \text{const.}\}$  are relatively compact.

From (35) it follows also that <sup>(1)</sup>

$$(36) \quad l_{\mathcal{L}(\mu(\Phi))} \geq \mu'(\Phi) l_{\mathcal{L}(\Phi)}.$$

*Lemma 20.* — Let  $X$  be a  $q$ -complete manifold with respect to the strongly  $q$ -pseudoconvex positive function  $\Phi : X \rightarrow \mathbf{R}$ . Let  $g : X \rightarrow \mathbf{R}$  be any continuous function on  $X$ . We can find a

<sup>(1)</sup> Cf. R. COURANT and D. HILBERT, *Methods of Mathematical Physics, Interscience, N.Y., 1953, vol. I, p. 33.*

sequence  $(a_\nu)_{\nu \in \mathbf{N}}$  of real numbers such that for any function  $\mu = \mu(t)$  defined on  $0 \leq t < \infty$  and satisfying the conditions

$$\mu'(t) > 0, \quad \mu''(t) \geq 0, \quad \mu'(t) > a_\nu \quad \text{for} \quad \nu \leq t < \nu + 1 \quad (\nu = 0, 1, \dots),$$

we have

$$I_{\mathcal{L}(\mu(\Phi))}(x) \geq g(x).$$

*Proof.* — Let  $a_\nu > 0$  be chosen so that

$$a_\nu I_{\mathcal{L}(\Phi)}(x) \geq g(x) \quad \text{for} \quad \nu \leq \Phi(x) < \nu + 1.$$

This is possible since the sets  $\{\nu \leq \Phi(x) < \nu + 1\}$  are relatively compact. Then the lemma follows from (36).

**17.** *W-ellipticity conditions.* — a) Let  $E$  be a holomorphic vector bundle on the complex manifold  $X$  and let  $h$  be a hermitian metric on the fibres of  $E$ . Let  $s = \{\sum s_{\bar{\nu}\mu}^a d\bar{z}^\nu dz^\mu\}$  be the curvature form of  $h$ . If  $f = f(x)$  is a real valued  $C^\infty$  function on  $X$ , then  $e^{f(\cdot)} > 0$  and therefore  $e^f h$  defines a new hermitian metric on the fibres of  $E$ . Its curvature form is represented locally by (cf. 4 d)

$$\{\delta_b^a \bar{\partial} \partial f + s_b^a\}.$$

Let us now assume that  $X$  is  $q$ -complete with respect to the positive strongly  $q$ -pseudoconvex function  $\Phi$ . Let  $\mu = \mu(t)$  be an increasing convex function on  $0 \leq t < \infty$ , and let us consider the hermitian metric  $e^{-\mu(\Phi)} h$  on the fibres of  $E$ .

We consider on  $X$  the hermitian metric  $ds^2$  of the previous section. Accordingly, if we use, on the fibres of  $E$ , the metric  $e^{-\mu(\Phi)} h$  or  $h$  we will affect the symbols which depend on that choice with the index  $-\mu$ . In particular, we will have for any  $\varphi \in C^{rs}(X, E)$

$$A_{-\mu}(\varphi, \varphi) = e^{-\mu(\Phi)} A(\varphi, \varphi),$$

$$\mathcal{K}_{-\mu}(\varphi) = \mathcal{K}(\varphi) - \left\{ \sum_{i=1}^s (-1)^i \sum \frac{\partial^2 \mu(\Phi)}{\partial z^\beta \partial \bar{z}^{\beta_i}} \varphi_{A \bar{B}_i}^{a\beta} \right\} dz^A \wedge \overline{dz^B}$$

*Lemma 21.* — Let  $X$  be  $q$ -complete. There exists a sequence of positive real constants  $a_\nu$  such that for any increasing convex function  $\mu(t) \geq 0$  satisfying

$$\mu'(t) \geq a_\nu \quad \text{for} \quad \nu \leq t < \nu + 1 \quad (\nu = 0, 1, \dots),$$

we will have the following inequality

$$A_{-\mu}(\mathcal{K}_{-\mu}(\varphi), \varphi) \geq A_{-\mu}(\varphi, \varphi)$$

for any  $\varphi \in C^{rs}(X, E)$  with  $s \geq q$ .

*Proof.* — First we can find a continuous function  $f: X \rightarrow \mathbf{R}$  such that  $A(\mathcal{K}\varphi, \varphi) \geq f(x)A(\varphi, \varphi)$ . Then we can write

$$A_{-\mu}(\mathcal{K}_{-\mu}\varphi, \varphi) \geq e^{-\mu(\Phi)} \sum h_{\bar{b}a} \frac{\partial^2 \mu(\Phi)}{\partial z^\beta \partial \bar{z}^\gamma} \varphi_{A \bar{B}'}^{a\beta} \overline{\varphi^{b\bar{A}\gamma B'}} + f(x)A_{-\mu}(\varphi, \varphi).$$

We now choose fibre coordinates at  $x$  such that  $h_{\bar{b}a} = \delta_{ba}$ . It then follows from lemma 19 that we will have the inequality

$$A_{-\mu}(\mathcal{K}_{-\mu}\varphi, \varphi) \geq (l_{\mathcal{D}(\mu)} + f(x))A_{-\mu}(\varphi, \varphi).$$

Applying lemma 20 to the function  $g(x) = 1 - f(x)$  we obtain the statement of this lemma.

b) We now fix a function  $\mu_0 = \mu_0(t)$  for  $0 \leq t < \infty$  satisfying the conditions of lemma 21 and we replace the metric  $h$  with  $e^{-\mu_0(t)}h$ . If  $\lambda = \lambda(t) \geq 0$  is any  $C^\infty$  function on  $0 \leq t < \infty$  which is non decreasing and convex ( $\lambda'(t) \geq 0, \lambda''(t) \geq 0$ ) then  $\mu = \mu_0 + \lambda$  satisfies again the conditions of lemma 21.

We can then state the following

*Proposition 22.* — Let  $X$  be a  $q$ -complete manifold and  $E$  a holomorphic vector bundle on  $X$ . We can select a complete hermitian metric  $ds^2$  on  $X$  and a hermitian metric  $h$  on the fibres of  $E$  such that for any non decreasing convex function  $\lambda = \lambda(t) \geq 0$  on  $0 \leq t < \infty$ , we have with respect to the metrics  $ds^2$  and  $e^{-\lambda(t)}h$  that

$$A_{-\lambda}(\mathcal{K}_{-\lambda}\varphi, \varphi) \geq A_{-\lambda}(\varphi, \varphi)$$

for any  $\varphi \in C^{rs}(X, E)$  with  $s \geq q$ .

c) We apply the previous proposition to the vector bundle  $E^*$  and to the form  $* \neq \varphi \in C^{n-r, n-s}(X, E^*)$ . If  $n-s \geq q$  i.e. if  $s \leq n-q$  we then will have

$$A_{E^*, -\lambda}(\mathcal{K}_{E^*, -\lambda}(* \neq \varphi), * \neq \varphi) \geq A_{E^*, -\lambda}(* \neq \varphi, * \neq \varphi),$$

i.e. 
$$A_{E, \lambda}(\varphi, \varphi) \leq A_{E^*, -\lambda}(\mathcal{K}_{E^*, -\lambda}(* \neq \varphi), * \neq \varphi).$$

Moreover we remark that

$$A_{E^*, -\lambda}(\bar{\partial} * \neq \varphi, \bar{\partial} * \neq \varphi) = A_{E, \lambda}(\theta\varphi, \theta\varphi),$$

$$A_{E^*, -\lambda}(\theta * \neq \varphi, \theta * \neq \varphi) = A_{E, \lambda}(\bar{\partial}\varphi, \bar{\partial}\varphi).$$

Therefore by applying corollary 17 to  $E^*$  and  $* \neq \varphi$  we obtain the following

*Proposition 23.* — Let  $X$  be  $q$ -complete and let  $E$  be a holomorphic vector bundle on  $X$ . We can select a complete hermitian metric  $ds^2$  on  $X$  and a hermitian metric  $h$  on the fibres of  $E$  such that for any non decreasing  $C^\infty$  convex function  $\lambda = \lambda(t)$  on  $0 \leq t < \infty$ , we have with respect to  $ds^2$  and  $e^{\lambda(t)}h$  the inequality

$$\|\varphi\|_\lambda^2 \leq c\{\|\bar{\partial}\varphi\|_\lambda^2 + \|\theta_\lambda\varphi\|_\lambda^2\}, \quad c = \text{absolute constant}$$

for any  $\varphi \in \mathcal{D}^{r, s}(X, E)$ , provided  $s \leq n - q$ .

This proposition enables us to apply the results of § 3 and we thus obtain the following

*Theorem 5.* — If the complex manifold  $X$ , of complex dimension  $n$ , is  $q$ -complete, then for any holomorphic vector bundle  $E$  on  $X$  we have

- (i)  $H_k^s(X, \Omega^r(E)) = 0$  for  $s \leq n - q$  and any  $r$ ,
- (ii)  $H_k^{n-q+1}(X, \Omega^r(E))$  is a separated topological vector space for any  $r$ .

d) We end this section with some remarks about the "a priori" estimates one can derive, with the method used here, in the case of complex manifolds with boundary.

Let  $Y \subset\subset X$  be an open relatively compact subset of the complex manifold  $X$  with smooth (i.e.  $C^\infty$ ) boundary  $\partial Y$ .

Let  $A^{rs}(Y, E)$  be the image of the restriction map

$$C^{rs}(X, E) \rightarrow C^{rs}(Y, E).$$

Let  $t : A^{rs}(Y, E) \rightarrow C^{rs}(\partial Y, E|_{\partial Y})$

be the natural map induced by the natural imbedding of  $\partial Y$  in  $X$ .

We set  $B^{rs}(Y, E) = \{\varphi \in A^{rs}(Y, E) \mid \widetilde{t} \varphi = 0\}$ .

One verifies that if  $\varphi \in A^{rs}(Y, E)$ ,  $\psi \in B^{r, s+1}(Y, E)$ , then

$$(\bar{\partial}\varphi, \psi)_Y = (\varphi, \theta\psi)_Y.$$

Let  $f$  be a real  $C^\infty$  function on  $X$  such that

$$\begin{aligned} Y &= \{x \in X \mid f(x) < 0\}, \\ df &\neq 0 \quad \text{on} \quad \partial Y. \end{aligned}$$

The condition for  $\varphi \in A^{rs}(Y, E)$  to belong to  $B^{rs}(Y, E)$  is given by

$$*\varphi \wedge \partial f = 0 \quad \text{on} \quad \partial Y,$$

i.e. in a neighborhood of every point  $z_0 \in \partial Y$  we will have

$$\sum_{\beta} \varphi_{A\bar{B}'}^{\alpha} \frac{\partial f}{\partial z^{\beta}} = f \eta_{A\bar{B}'}^{\alpha} \quad (\eta_{A\bar{B}'}^{\alpha} \in C^\infty).$$

Given any  $\varphi \in A^{rs}(Y, E)$ , we can consider  $\widetilde{\varphi} \in A^{rs}(Y, E \otimes \Theta^{*r})$  and we can construct the tangent vectors  $\xi$  and  $\eta$  as in n° 14 b).

If  $dS$  is the area element of  $\partial Y$  we get from identity (33) and from Stokes' theorem

$$\frac{1}{(s-1)!} \int_{\partial Y} (\xi - \eta)_n dS = \|\bar{\nabla}\varphi\|_Y^2 + (\kappa\varphi, \varphi)_Y - \|\bar{\partial}\varphi\|_Y^2 - \|\theta\varphi\|_Y^2,$$

where for any vector  $\lambda = (\lambda^\alpha, \lambda^{\bar{\beta}})$  we set

$$\lambda_n = \left( \sum g^{\mu\bar{\nu}} \frac{\partial f}{\partial z^\mu} \frac{\partial f}{\partial z^{\bar{\nu}}} \right)^{-\frac{1}{2}} \sum \left( \lambda^\alpha \frac{\partial f}{\partial z^\alpha} + \lambda^{\bar{\beta}} \frac{\partial f}{\partial z^{\bar{\beta}}} \right) \quad \text{on} \quad \partial Y.$$

Now if  $\varphi \in B^{rs}(Y, E)$  then one obtains from the boundary conditions that

$$\begin{aligned} \eta_n &= 0, \\ \xi_n &= - \left( \sum g^{\mu\bar{\nu}} \frac{\partial f}{\partial z^\mu} \frac{\partial f}{\partial z^{\bar{\nu}}} \right)^{-\frac{1}{2}} \sum h_{\bar{b}a} \nabla_{\bar{Y}} \frac{\partial f}{\partial z^{\bar{\beta}}} \varphi_{A\bar{B}'}^{\alpha} \overline{\varphi^{b\bar{A}YB'}} \quad \text{on} \quad \partial Y. \end{aligned}$$

We set

$$|\partial f| = \left( \sum g^{\mu\bar{\nu}} \frac{\partial f}{\partial z^\mu} \overline{\frac{\partial f}{\partial z^\nu}} \right)^{\frac{1}{2}}, \quad \mathcal{L}(f)\{\varphi, \varphi\} = \sum h_{\bar{b}a} \frac{\partial^2 f}{\partial z^\beta \partial \bar{z}^\gamma} \varphi_A^\alpha \beta_{\bar{B}'} \overline{\varphi^{\bar{A}\gamma B'}},$$

$$\tilde{\mathcal{L}}(f)\{\varphi, \varphi\} = \sum h_{\bar{b}a} \nabla_{\bar{\gamma}} \frac{\partial f}{\partial z^\beta} \varphi_A^\alpha \beta_{\bar{B}'} \overline{\varphi^{\bar{A}\gamma B'}}.$$

Hence we obtain the relation

$$\|\bar{\nabla}\varphi\|_Y^2 + (\kappa\varphi, \varphi) + \frac{1}{(s-1)!} \int_{\partial Y} \frac{1}{|\partial f|} \tilde{\mathcal{L}}(f)\{\varphi, \varphi\} dS = \|\bar{\partial}\varphi\|_Y^2 + \|\theta\varphi\|_Y^2.$$

Assume that  $\mathcal{L}(f)$  has  $n-q+1$  positive eigenvalues at each point of  $\partial Y$ . We can choose a hermitian metric on  $X$  such that

$$l_{\mathcal{L}(f)} \geq c_0$$

with  $c_0 > 0$ , at each point of  $\partial Y$  (if  $q=1$  any hermitian metric will satisfy this condition). This same relation will hold in a neighborhood  $U$  of  $\partial Y$  in  $X$ . It will still hold if we multiply the hermitian metric on  $X$  by a  $C^\infty$  positive function. By a suitable choice of this function, we can find a positive constant  $c_1$  such that for every  $\varphi \in B^{rs}(Y, E)$  with  $s \geq q$  we have

$$\frac{1}{(s-1)!} \int_{\partial Y} \frac{1}{|\partial f|} \tilde{\mathcal{L}}(f)\{\varphi, \varphi\} dS \geq c_1 \int_{\partial Y} A(\varphi, \varphi) dS \quad (1).$$

Replacing the metric  $h$  on the fibres of  $E$  by  $e^{\tau}h$  we can also find a  $\tau_0 \geq 0$  and a positive constant  $c_2$  ( $\tau_0 = 0$  if the metric on  $X$  is euclidean) such that, if  $\tau \geq \tau_0$ , we have for any  $\varphi \in B^{rs}(Y, E)$  with  $s \geq q$  and  $\text{supp } \varphi \subset U$

$$A_\tau(\kappa_\tau \varphi, \varphi) \geq c_2 \tau A_\tau(\varphi, \varphi).$$

Hence for  $\tau \geq \tau_0$ ,  $\varphi \in B^{rs}(Y, E)$ ,  $\text{supp } \varphi \subset U$ ,  $s \geq q$ , one obtains

$$\|\bar{\nabla}\varphi\|_{\tau, Y}^2 + c_2 \tau \|\varphi\|_{\tau, Y}^2 + c_1 \int_{\partial Y} A(\varphi, \varphi) dS \leq \|\bar{\partial}\varphi\|_{\tau, Y}^2 + \|\theta_\tau \varphi\|_{\tau, Y}^2.$$

We can incorporate  $e^{\tau_0}h$  into  $h$ , so that we may assume that the above inequality holds for  $\tau \geq 0$ .

(1) Let  $\mu = \mu(t)$  be a positive  $C^\infty$  function on  $\mathbf{R}$  such that  $\mu(0) = 1$ . We replace the hermitian metric  $ds^2$  on  $X$  by  $\mu(f)ds^2$ , and we denote by  $\tilde{\mathcal{L}}_\mu(f)\{\varphi, \varphi\}$  the hermitian form  $\tilde{\mathcal{L}}(f)\{\varphi, \varphi\}$  calculated with respect to this metric. We have on  $\partial Y$

$$\tilde{\mathcal{L}}_\mu(f)\{\varphi, \varphi\} = \mathcal{L}(f)\{\varphi, \varphi\} + \frac{d \log \mu(t)}{dt} \Big|_{t=0} |\partial f|^2 A(\varphi, \varphi) - 2\text{Re} \left( \sum \Gamma_{\beta\bar{\gamma}}^\alpha \frac{\partial f}{\partial z^\alpha} h_{\bar{b}a} \varphi_A^\alpha \beta_{\bar{B}'} \overline{\varphi^{\bar{A}\gamma B'}} \right).$$

Hence, by a suitable choice of  $\frac{d \log \mu(t)}{dt} \Big|_{t=0}$  we can make the hermitian form  $\tilde{\mathcal{L}}(f)\{\varphi, \varphi\}$  positive definite on  $\partial Y$ . We point out that this can be done regardless of the number of positive eigenvalues of the Levi form  $\mathcal{L}(f)$  on  $\partial Y$ .

Let  $V$  be a neighborhood of a boundary point and let us select an orthonormal basis  $\omega^1, \dots, \omega^n$  of  $(1, 0)$  forms on  $V$ , with  $\omega^n = g \cdot \partial f$ , proportional to  $\partial f$ . If

$$\varphi = \sum_{\substack{\alpha_1 < \dots < \alpha_r \\ \beta_1 < \dots < \beta_s}} \varphi_{\alpha_1 \dots \alpha_r \bar{\beta}_1 \dots \bar{\beta}_s} \omega^{\alpha_1} \wedge \dots \wedge \omega^{\alpha_r} \wedge \overline{\omega^{\beta_1}} \wedge \dots \wedge \overline{\omega^{\beta_s}}$$

is the local representation of  $\varphi$  in  $V$ , then the condition for  $\varphi$  to belong to  $B^{rs}(Y, E)$  can be expressed in  $V$  by saying that  $\varphi_{\alpha_1 \dots \alpha_r \bar{\beta}_1 \dots \bar{\beta}_s} = 0$  on  $\partial Y$  for  $\beta_1 < \dots < \beta_s = n$ . Letting  $\theta = \theta_0$  we get

$$\begin{aligned} (\theta_\tau \varphi)^a &= (\theta \varphi)^a - \tau * (\partial f \wedge * \varphi)^a = \\ &= (\theta \varphi)^a + (-1)^{r+s} \tau \sum_{\substack{\alpha_1 < \dots < \alpha_r \\ \beta_1 < \dots < \beta_{s-1}}} \varphi_{\alpha_1 \dots \alpha_r \bar{\beta}_1 \dots \bar{\beta}_{s-1} \bar{n}} \omega^{\alpha_1} \wedge \dots \wedge \omega^{\alpha_r} \wedge \overline{\omega^{\beta_1}} \wedge \dots \wedge \overline{\omega^{\beta_{s-1}}} \end{aligned}$$

Thus  $\|\theta_\tau \varphi\|_{\tau, Y} \leq \|\theta \varphi\|_{\tau, Y} + \tau \|\varphi\|_{\tau, Y}^{(n)}$ .

where  $\|\varphi\|_{\tau, Y}^{(n)} = \left( \frac{1}{r!(s-1)!} \int_Y e^{\tau f} \sum h_{\bar{b}a} \varphi_{\alpha_1 \dots \alpha_r \bar{\beta}_1 \dots \bar{\beta}_{s-1} \bar{n}} \overline{\varphi^{b\bar{\alpha}_1 \dots \bar{\alpha}_r \beta_1 \dots \beta_{s-1} n}} dX \right)^{\frac{1}{2}}$ .

Hence there exists a positive constant  $c_3 > 0$  such that, for any  $\varphi \in B^{rs}(Y, E)$  with  $s \geq q$ ,

$$\|\bar{\nabla} \varphi\|_{\tau, Y}^2 + c_2 \tau \|\varphi\|_{\tau, Y}^2 + c_1 \int_{\partial Y} A(\varphi, \varphi) dS \leq c_3 \{ \|\bar{\partial} \varphi\|_{\tau, Y}^2 + \|\theta \varphi\|_{\tau, Y}^2 + \tau^2 \|\varphi\|_{\tau, Y}^{(n)2} \}.$$

This inequality is to be compared with a similar one given by J. J. Kohn for  $q = 1$  (Regularity at the boundary of the  $\bar{\partial}$ -Neumann problem, *Proc. Nat. Acad. Sci., U.S.A.*, 40 (1963), 206-213; see also J. J. Kohn, Harmonic integrals on strongly pseudoconvex manifolds, II, *Ann. of Math.* (to appear)). The spaces  $B^{rs}(Y, E)$  were first introduced in general by H. Grauert in a lecture at a seminar in Bonn, Summer 1961. This inequality can be considered as a generalization of estimates given for the first time by C. B. Morrey for forms of type  $(0, 1)$  on a strongly pseudoconvex manifold with boundary (C. B. Morrey, The analytic embedding of abstract real analytic manifolds, *Ann. of Math.*, 68 (1958), 159-201).

### § 6. Applications : finiteness theorems

**18.** *Preliminaries on topological vector spaces.* — a) Let  $F$  be a locally convex topological vector space on which we will make the following assumptions.

(i) the topology of  $F$  is metrizable. This means that there exists a sequence  $(V_n)_{n \in \mathbf{N}}$  of disked open neighborhoods of the origin which is a fundamental sequence of neighborhoods and such that  $\bigcap_{n \in \mathbf{N}} V_n = 0$ . It is not restrictive to assume that  $V_n = \overset{\circ}{V}_n$ .

The second condition says that the topology of the space is Hausdorff. If

$$p_n(x) = \inf \{ \lambda \in \mathbf{R} \mid \lambda > 0, \lambda V_n \ni x \}$$

then  $p_n(x)$  is a continuous seminorm on  $F$  ([6] Chap. II, p. 94). We have

$$V_n = \{x \in F \mid p_n(x) < 1\}.$$

The topology of  $F$  can be defined by the sequence of seminorms  $(p_n)_{n \in \mathbf{N}}$ .

Since it is not restrictive to assume

$$V_n \supset V_{n+1} \quad \forall n \in \mathbf{N},$$

we have

$$p_n(x) \leq p_{n+1}(x) \quad \forall x \in F, \quad \forall n \in \mathbf{N}.$$

As a distance defining the topology of  $F$  we can assume the expression

$$d(x, y) = \sum_0^{\infty} \frac{1}{2^r} \frac{p_r(x-y)}{1 + p_r(x-y)}.$$

(ii) The space  $F$  is complete. Thus  $F$  is a Fréchet space. Given any  $\varepsilon > 0$  the set

$$B_n(a, \varepsilon) = \{x \in F \mid p_n(x-a) < \varepsilon\}$$

will be called a  $n$ -ball of radius  $\varepsilon$  and center  $a$ . We will make the following assumption:

(iii) Given  $\varepsilon > 0$  and  $n \geq 1$  we can cover the unit  $n$ -ball  $V_n = \{x \in F \mid p_n(x) < 1\}$  with a finite number of  $(n-1)$ -balls of radius  $\leq \varepsilon$ .

A space satisfying the conditions (i), (ii), (iii) will be called a space of Fréchet-Schwartz.

b) We want to prove the following

*Proposition 24.* —  $\alpha$ ) A space  $F$  of Fréchet-Schwartz is a Montel space (i.e. every bounded set of  $F$  is relatively compact);

$\beta$ ) if  $N$  is any closed subspace of  $F$  then  $N$  is again a space of Fréchet-Schwartz (with respect to its natural topology);

$\gamma$ ) if  $N$  is a closed subspace of  $F$  then  $F/N$  is again a space of Fréchet-Schwartz (with respect to its natural topology).

*Proof.* —  $\alpha$ ) We have to prove that for any  $\varepsilon > 0$  we can find a finite set of balls of radius  $\leq \varepsilon$  for the distance  $d$  which covers a given bounded set  $B$ .

Choose  $r_0 > 0$  such that  $\sum_{r_0+1}^{\infty} \frac{1}{2^r} < \varepsilon/3$ . Then if  $p_{r_0}(x-y) < \varepsilon/3$  we will have  $d(x, y) < \varepsilon$ .

In fact since  $p_r(x) \leq p_{r+1}(x)$  we have

$$d(x, y) \leq \frac{\varepsilon}{3} \sum_0^{r_0} \frac{1}{2^r} + \frac{\varepsilon}{3} < \varepsilon.$$

Since  $B$  is bounded there exists a  $\lambda_0 > 0$  such that

$$p_{r_0+1}(b) < \lambda_0 \quad \forall b \in B.$$

We can then, by assumption (iii), cover  $B$  by a finite set of  $r_0$ -balls of radius  $< \varepsilon/3$ . This proves our assertion.

$\beta$ ) is a direct consequence of the definitions.

$\gamma$ ) For any  $x \in F$  we denote by  $\dot{x}$  its image in  $F/N$ . Let

$$\dot{p}_r(y) = \inf \{ p_r(x) \mid \forall x \in F \text{ with } \dot{x} = y \}$$

Then  $\{\dot{p}_r\}$  is a system of seminorms on  $F/N$  defining on it the quotient topology. We know that this is the topology of a Fréchet space. The unit  $n$ -ball  $\{\dot{p}_n < 1\}$  in  $F/N$  is the image of the unit  $n$ -ball of  $F$  by the natural projection  $F \rightarrow F/N$ . From this the conclusion follows.

$c$ ) Given the seminorm  $p_n$ , the set  $\{x \in F \mid p_n(x) = 0\}$  is a closed subspace  $N_n$  of  $F$ . Consider then the space  $F/N_n$  and on it the norm

$$\|y\|_n = \text{value of } p_n(x) \text{ for all } x \in F \text{ with } \dot{x} = y$$

$\dot{x}$  being the natural image of  $x \in F$  in  $F/N_n$ .

Let  $F_n$  denote then the completion of  $F/N_n$  under the norm  $\|\cdot\|_n$ . There is a natural continuous map

$$\beta_n : F \rightarrow F_n$$

whose image is dense in  $F_n$ ; this associates to every  $x \in F$  its image  $\dot{x}$  in  $F/N_n$  as a point of  $F_n$ . We have also a sequence of natural maps

$$\alpha_{n+1} : F_{n+1} \rightarrow F_n$$

which associates to every Cauchy sequence  $\{\dot{x}_v\} \in F/N_{n+1}$  for  $\|\cdot\|_{n+1}$  the same sequence as a Cauchy sequence in the norm  $\|\cdot\|_n$ . This map is linear and continuous, and indeed we have for every  $x \in F$

$$\|\beta_{n+1}(x)\|_{n+1} \geq \|\beta_n(x)\|_n.$$

Hence for every  $y \in F_{n+1}$

$$\|\alpha_{n+1}(y)\|_n \leq \|y\|_{n+1}.$$

We have, in fact,

$$\alpha_{n+1} \circ \beta_{n+1} = \beta_n.$$

Thus the image of  $\alpha_{n+1}$  is dense in  $F_n$ .

*The maps  $\alpha_{n+1}$  are compact maps.*

*Proof.* — Given  $\varepsilon > 0$  and the set  $\{x \in F \mid p_{n+1}(x) < 1\}$  we can find a finite number of  $x_i \in F$  ( $1 \leq i \leq k$ ) such that

$$\{x \in F \mid p_{n+1}(x) < 1\} \subset \bigcup_{i=1}^k \{x \in F \mid p_n(x - x_i) < \varepsilon/2\}.$$

Therefore we must have

$$\alpha_{n+1} \{x \in F_{n+1} \mid \|x\|_{n+1} < 1\} \subset \bigcup_{i=1}^k \{x \in F_n \mid \|x - x_i\|_n < \varepsilon\}.$$

This proves that the image of the unit ball of  $F_{n+1}$  under  $\alpha_{n+1}$  is a relatively compact subset of  $F_n$ .

As a consequence, the maps  $\beta_n$  are also compact maps.

Consider now the space

$$\varprojlim F_n = \left\{ (x_n) \in \prod_{n=0}^{\infty} F_n \mid \alpha_{n+1}(x_{n+1}) = x_n \text{ for } n = 0, 1, \dots \right\}.$$

The topology being that induced by the product topology on  $\prod_0^{\infty} F_n$ .

The space  $\prod_{n=0}^{\infty} F_n$  is a Fréchet space and  $\varprojlim F_n$ , as a closed subspace of it, is again a Fréchet space.

We have a natural map  $\gamma : F \rightarrow \varprojlim F_n$

which associates to each  $x \in F$  the sequence  $(\beta_n(x)) \in \varprojlim F_n$ . The mapping  $\gamma$  is linear and continuous since every map  $\beta_n$  is a continuous map.

Moreover  $\gamma$  is injective since  $\beta_n(x) = 0$  implies  $p_n(x) = 0$ , and hence  $x = 0$  if  $\beta_n(x) = 0$  for all  $n$ .

Finally  $\gamma$  is surjective.

*Proof.* — Let  $(x_n) \in \varprojlim F_n$ .

We select  $y_0 \in F$  such that  $\|\beta_0(y_0) - x_0\|_0 \leq \frac{1}{2^2}$ .

We select  $y_1 \in F$  such that  $\|\beta_1(y_1) - x_1\|_1 \leq \frac{1}{2^3}$ ,

$$\|\alpha_1 \beta_1(y_1) - x_0\|_0 \leq \frac{1}{2^2};$$

then  $p_0(y_1 - y_0) \leq \frac{1}{2}$ .

We select  $y_2 \in F$  such that  $\|\beta_2(y_2) - x_2\|_2 \leq \frac{1}{2^4}$ ,

$$\|\alpha_2 \beta_2(y_2) - x_1\|_1 \leq \frac{1}{2^3}.$$

Then  $p_1(y_2 - y_1) \leq \frac{1}{2^2}$ .

In this way we construct a sequence  $(y_n) \subset F$  with the properties

$$p_n(y_{n+1} - y_n) \leq \frac{1}{2^{n+1}},$$

$$\|\beta_n(y_n) - x_n\|_n \leq \frac{1}{2^{n+2}}.$$

Now

$$\begin{aligned} d(y_{n+1}, y_n) &= \sum \frac{1}{2^r} \frac{p_r(y_{n+1} - y_n)}{1 + p_r(y_{n+1} - y_n)} \\ &\leq \sum_0^n \frac{1}{2^r} \frac{p_r(y_{n+1} - y_n)}{1 + p_r(y_{n+1} - y_n)} + \sum_{n+1}^{\infty} \frac{1}{2^r} \\ &\leq \frac{1}{2^{n+1}} \left\{ \sum_0^n \frac{1}{2^r} \right\} + \frac{1}{2^{n+1}} \sum_0^{\infty} \frac{1}{2^r} \\ &\leq \frac{1}{2^{n-1}}. \end{aligned}$$

Therefore the sequence  $(y_n)$  is a Cauchy sequence and converges to an element  $y \in F$ .

We want to show that  $\beta_n(y) = x_n$ . We have, for  $v > n$ ,

$$\begin{aligned} \|\beta_n(y) - x_n\|_n &\leq \|\beta_n(y) - \beta_n(y_v)\|_n + \|\beta_n(y_v) - x_n\|_n \\ &\leq p_n(y - y_v) + \|\beta_v(y_v) - x_v\|_v. \end{aligned}$$

For  $v \rightarrow +\infty$ ,  $p_n(y - y_v) \rightarrow 0$  and  $\|\beta_v(y_v) - x_v\|_v \rightarrow 0$ . This proves our assertion.

Using the theorem of Banach we conclude with the following

*Proposition.* — For any Fréchet space  $F$  we can find a sequence  $(F_n)$  of Banach spaces and continuous maps  $\alpha_{n+1} : F_{n+1} \rightarrow F_n$  with dense images, such that  $F \simeq \varprojlim F_n$  (in the topological sense).

If moreover  $F$  is a space of Fréchet-Schwartz then the maps  $\alpha_{n+1}$  are compact maps.

d) Let  $F$  be a space of Fréchet-Schwartz and let  $F'$  be the strong dual of  $F$ . We have the following proposition ([18], p. 404).

*Proposition 25.* — The strong dual  $F'$  of a space of Fréchet-Schwartz is the inductive limit of a sequence  $(F'_n)$  of Banach spaces. For each  $n$ ,  $F'_n$  is a subspace of  $F'_{n+1}$ , the injection map being compact.

*Proof.* —  $\alpha$ ) Let  $F = \varprojlim F_n$ . Let  $F'_n$  be the Banach space strong dual of  $F_n$ . The map  $\alpha_{n+1} : F_{n+1} \rightarrow F_n$  gives by transposition a compact injective map

$$\alpha'_n : F'_n \rightarrow F'_{n+1}.$$

Let  $G = \bigcup F'_n = \varinjlim F'_n$ . A fundamental system of neighborhoods of the origin in  $G$  is constituted by those convex disked sets  $V$  of  $G$  such that  $V \cap F'_n$  is a neighborhood of the origin in  $F'_n$ .

$\beta$ ) There is a natural algebraic isomorphism  $G \rightarrow F'$ .

In fact, if  $\alpha \in G$ , for  $n$  large enough  $\alpha \in F'_n$  and thus  $\alpha \circ \beta_n \in F'$ . The element thus defined  $\alpha' \in F'$  is independent of the choice of  $n$ .

If  $\alpha' = 0$  then  $\alpha = 0$  since  $\beta_n F$  is dense in  $F_n$ . Finally if  $\alpha' \in F'$  for some  $n$  we must have

$$|\alpha'(x)| \leq p_n(x).$$

Therefore  $\alpha'$  defines an element  $\alpha \in F'_n$  such that  $\alpha' = \alpha \circ \beta_n$ .

$\gamma$ ) We have to show that the isomorphism  $G \rightarrow F'$  is a topological isomorphism

Let  $B$  be a bounded set in  $F$ , then

$$B^0 = \{\alpha \in F' \mid \sup \langle \alpha, B \rangle < 1\}$$

is a neighborhood of the origin in  $F'$  for the strong topology. When  $B$  describes the system of bounded sets in  $F$ ,  $B^0$  describes a fundamental system of neighborhoods for the strong topology of  $F'$ . Consider the set

$$B^0 \cap F'_n = \{\alpha_n \in F'_n \mid \sup \langle \alpha_n, \beta_n(B) \rangle < 1\}.$$

Since  $\beta_n(B)$  as a subset of  $F_n$  is bounded,  $B^0 \cap F'_n$  is a neighborhood of the origin in  $F'_n$ .

Therefore  $B^0$  is a neighborhood of the origin in  $G'$ .

Let now  $A$  be a disked set in  $G$  which is a neighborhood of the origin in  $G$ . Consider the set

$$A^0 = \{x \in F \mid \sup \langle x, A \rangle < 1\}.$$

The set  $A^0$  is a bounded set in  $F$ . It is enough to show that  $A^0$  is weakly bounded. If  $\alpha \in F'$  we can find a  $\lambda > 0$  such that  $\lambda \in A$ . Thus for  $x \in A^0$  we have

$$|\alpha(x)| < \frac{1}{\lambda},$$

and we have  $A \subset (A^0)^0$ .

This proves that the topologies of  $F'$  and  $G$  coincide. Finally we remark that  $F'$  as a dual of Fréchet space is complete with respect to the strong topology ([15], p. 266). It then follows that every bounded set  $B \subset F$  is contained as a bounded set in some space  $F'_n$  of the sequence of definition ([11], p. 270).

*e)* We consider the class  $\mathcal{C}$  of spaces which are a product of a space of Fréchet-Schwartz and of the strong dual of a space of Fréchet-Schwartz.

Every element  $E$  of  $\mathcal{C}$  has the following properties

(i)  $E$  is a complete Montel space of type  $\mathcal{LF}$  ([11], p. 248).

(ii) If  $(E_n)_{n \in \mathbb{N}}$  is a sequence of definition of  $E$ , then every bounded set  $B \subset E$  is contained as a bounded set in some  $E_n$ .

(iii) There is a sequence of definition  $(E_n)$  of  $E$  such that, if  $K$  is a compact set of  $E$ , then, for some  $n$ ,  $K \subset E_n$  and is compact for the natural topology of  $E_n$ .

The properties (i) and (ii) follow from the remarks made in *d)*. To prove (iii) we proceed as follows. Let  $E = F' \times G$  where  $G$  is a Fréchet-Schwartz space and  $F'$  the dual of a Fréchet-Schwartz space. Let  $K_1, K_2$  be the projections of  $K$  on  $F', G$  respectively. Then  $K_1$  and  $K_2$  are compact. Let  $\hat{K} = K_1 \times K_2$ . It is enough to prove the statement for  $\hat{K}$ . It is thus not restrictive to assume  $G = \{0\}$  and  $E = F'$ . Now, with the notations used before,  $\hat{K} \subset F'_n$  for some  $n$ , and is bounded in  $F'_n$ , hence  $\hat{K} \subset F'_{n+1}$ . But  $\hat{K}$  is closed in  $F'_{n+1}$  for the topology induced by  $F'$ , hence closed also for the topology of  $F'_{n+1}$ . Moreover, the injection  $F'_n \rightarrow F'_{n+1}$  being compact, it follows that  $\hat{K}$ , as a subset of  $F'_{n+1}$ , is relatively compact, hence compact.

Let  $E$  and  $F$  be elements of  $\mathcal{C}$  and  $u : E \rightarrow F$  a surjective continuous linear map. Then we have

(iv)  $u$  is a homomorphism.

(v) every convex compact subset of  $F$  is the image by  $u$  of a convex compact set of  $E$ .

*Proof.* — The first assertion follows from ([11], p. 269). To prove the second assertion we remark that, if  $(F_n)$  is a sequence of definition of  $F$  then  $K \subset F_n$  for some  $n$  and is compact in  $F_n$ . Let  $(E_m)$  be a sequence of definition of  $E$  then, for some  $m$  we must have  $F_n \subset u(E_m)$ , for the injection map is continuous for the topology of Fréchet space of  $F_n$  and for the topology of Fréchet space of  $u(E_m) \simeq E_m / (\text{Ker } u|_{E_m})$ . It follows that  $K$ , as a subset of  $u(E_m)$ , is compact and it is thus the image of a compact set  $\hat{K} \subset E_m$  by the mapping  $u$ .

Now  $\hat{K}$  is a compact set in  $E_m$  and *a fortiori* compact for the topology induced on  $E_m$  by  $E$  which is weaker than the topology of  $E_m$ .

If  $\Gamma(K)$  and  $\Gamma(\hat{K})$  are the closed disked envelopes of  $K$  and  $\hat{K}$  these are also compact, since  $E$  and  $F$  are complete and we have  $u(\Gamma(\hat{K})) = \Gamma(K)$ .

By a theorem of L. Schwartz [17] we then conclude with the following

*Proposition 26.* — *Let  $u, v$  be two continuous linear maps of  $E$  into  $F$ ,  $E$  and  $F$  being elements of  $\mathcal{C}$ .*

*If  $u$  is surjective and  $v$  is compact, then  $u + v$  has closed image of finite codimension.*

**19.** a) Let  $X$  be a complex manifold of pure complex dimension  $n$ . Let  $\Omega$  be a relatively compact open subset of  $X$ .

Let  $E$  be a holomorphic vector bundle on  $X$  and let  $\mathcal{U} = (U_i)_{i \in \mathbf{N}}$  be a locally finite covering of  $X$  with the following properties:

- (i) for each  $i \in \mathbf{N}$  there exists a coordinate patch  $V_i \supset \supset U_i$ ;
- (ii) on  $V_i$ ,  $E|_{V_i}$  is a trivial bundle.

Let  $\Theta$  be the holomorphic tangent bundle; by condition (i)  $\Theta|_{V_i}$  is also a trivial bundle.

Let  $D_i^p$  a symbol of derivation of order  $|p|$  with respect to the local  $z_i$  and  $\bar{z}_i$  coordinates in  $U_i$ .

If we introduce a hermitian metric on  $X$  and a hermitian metric on the fibres of  $E$  we can also consider the symbols  $\nabla^p$  of covariant derivation of order  $|p|$  with respect to the local coordinates  $z_i$  and  $\bar{z}_i$ . Given a form  $\varphi \in C^{r,s}(X, E)$ ,  $\varphi$  can be represented by a system  $(\varphi_i)_{i \in \mathbf{N}}$  of  $C^\infty$  forms of type  $(r, s)$  on the sets  $U_i$  satisfying the consistency condition  $\varphi_i = e_{ij} \varphi_j$  in  $U_i \cap U_j$ . For any compact set  $K \subset X$  we can consider the seminorms

$$p_K^k(\varphi) = \sup_{i, V_i \cap K \neq \emptyset} \sup_{x \in U_i \cap K} \sum_{|r| \leq k} |D^r \varphi(x)|,$$

$$\pi_K^k(\varphi) = \sup_{x \in K} \left\{ \sum_{|r| \leq k} A(\nabla^r \varphi, \nabla^r \varphi)_x \right\}^{1/2}.$$

There exists a constant  $C(K) > 0$  such that for any  $\varphi \in C^{r,s}(X, E)$  we have

$$(37) \quad C(K)^{-1} p_K^k(\varphi) \leq \pi_K^k(\varphi) \leq C(K) p_K^k(\varphi).$$

b) We will consider the following topological vector spaces.  $\mathcal{E}^{r,s}$  = the vector space  $C^{r,s}(X, E)$  with the topology defined by the family of seminorms  $\pi_K^k$ .

Let  $(K_r)_{r \in \mathbf{N}}$  be an increasing sequence of compact sets in  $X$  such that  $K_r \subset \overset{\circ}{K}_{r+1}$ ,  $X = \bigcup_r K_r$ ; then, setting  $\pi_r = \pi_{K_r}^r$ , the family  $(\pi_r)_{r \in \mathbf{N}}$  defines on  $\mathcal{E}^{r,s}$  the same topology.

From inequality (37) it follows then that  $\mathcal{E}^{r,s}$  is a Fréchet space; as a distance defining its topology we can take

$$d(\varphi, \psi) = \sum_0^\infty \frac{1}{2^r} \frac{\pi_r(\varphi - \psi)}{1 + \pi_r(\varphi - \psi)}.$$

We remark that the seminorms  $\pi_r$  verify the following inequality:

$$\pi_r(\varphi) \leq \pi_{r+1}(\varphi)$$

for any  $r \in \mathbf{N}$ .

*Lemma 27.* — Let  $B_{r+1} = \{\varphi \in \mathcal{E}^{r,s} \mid \pi_{r+1}(\varphi) < 1\}$ .

Given  $\varepsilon > 0$  we can find a finite number of points  $\eta_i \in B_{r+1}$  such that

$$B_{r+1} \subset \bigcup_i \{\varphi \in \mathcal{E}^{r,s} \mid \pi_r(\varphi - \eta_i) < \varepsilon\}.$$

*Proof.* — If the contention of the lemma is not true, given  $\varphi_1 \in B_{r+1}$  we can find  $\varphi_2 \in B_{r+1}$  such that  $\pi_r(\varphi_1 - \varphi_2) \geq \varepsilon$ . Also, we can find  $\varphi_3 \in B_{r+1}$  such that  $\pi_r(\varphi_1 - \varphi_3) \geq \varepsilon$ ,  $\pi_r(\varphi_2 - \varphi_3) \geq \varepsilon$ . By this procedure we find a sequence  $(\varphi_\nu) \subset B_{r+1}$  such that for  $\nu \neq \mu$ ,  $\pi_r(\varphi_\nu - \varphi_\mu) \geq \varepsilon$ .

Now by inequality (37) we see that the functions  $\varphi_{\nu_i}$  on  $U_i \cap K_r$  are uniformly bounded with all their derivatives up to the order  $r+1$ . By Ascoli's theorem we can thus select a subsequence  $(\varphi_{\nu_h})$  which is a Cauchy sequence in the seminorm  $\pi_r$ . This is a contradiction. In conclusion the space  $\mathcal{E}^{r,s}$  is a space of the class  $\mathcal{C}$ , and in fact a space of Fréchet-Schwartz.

b) Let  $\mathcal{E}''^{r,s}$  be the strong dual of  $\mathcal{E}^{r,s}$ . This space is again in  $\mathcal{C}$  and can be identified with the space of distributions with compact support and of type  $(r, s)$  if, for  $T \in \mathcal{E}''^{r,s}$ , we define the value of  $T$  on  $\varphi \in \mathcal{E}^{r,s}$  by

$$T[\varphi] = (\varphi, T).$$

Then the operator

$$\bar{\partial} : \mathcal{E}''^{r,s} \rightarrow \mathcal{E}''^{r,s+1}$$

is given by

$$\bar{\partial}T[\varphi] = (\theta\varphi, T).$$

This is a continuous linear map. Hence the space

$$Z''^{r,s} = \{T \in \mathcal{E}''^{r,s} \mid \bar{\partial}T = 0\}$$

is a closed subspace of  $\mathcal{E}''^{r,s}$  and therefore is in the category  $\mathcal{C}$ .

c) Finally we consider the space  $C_0^{r,s}(\bar{\Omega}, E)$  of  $C^\infty(r, s)$ -forms with support contained in  $\bar{\Omega}$  with the topology defined by the seminorms  $\pi_{\bar{\Omega}}^k$ . It is easy to verify that this space is also a space of the class  $\mathcal{C}$  and in fact a space of Fréchet-Schwartz. We put

$$Z_{\bar{\Omega}}^{r,s} = \{\varphi \in C_0^{r,s}(\bar{\Omega}, E) \mid \bar{\partial}\varphi = 0\}.$$

This is a closed subspace of  $C_0^{r,s}(\bar{\Omega}, E)$ , hence again a space of Fréchet-Schwartz.

There is a natural map  $i : Z_{\bar{\Omega}}^{r,s} \rightarrow Z''^{r,s}$

which associates to every form  $\tau \in Z_{\bar{\Omega}}^{r,s}$  the distribution  $T_\tau$  defined by

$$T_\tau[\varphi] = \int_{\bar{\Omega}} A(\varphi, \tau) dX.$$

*Lemma 28.* — The inclusion map  $i : Z_{\bar{\Omega}}^{r,s} \rightarrow Z''^{r,s}$  is a compact map.

*Proof.* — Let  $B = \{\tau \in Z_{\bar{\Omega}}^{r,s} \mid \pi_{\bar{\Omega}}^0(\tau) < 1\}$ .

This is a neighborhood of the origin in  $Z_{\Omega}^{r,s}$ . For any  $\varphi \in \mathcal{E}^{r,s}$  and  $\tau \in B$  we have

$$\begin{aligned} \langle i(\tau), \varphi \rangle &= (\varphi, \tau)_{\Omega} && \text{and therefore} \\ \langle i(\tau), \varphi \rangle &\leq (\varphi, \varphi)_{\Omega}. \end{aligned}$$

This shows that  $i(B)$  is bounded in  $Z^{r,s}$ , hence relatively compact (Note that  $(Z^{r,s})' = \mathcal{E}^{r,s} / (Z^{r,s})^0$ ).

d) From proposition 26 we then obtain the following

*Theorem 6.* — Consider the linear map

$$w : \mathcal{E}^{r,s-1} \oplus Z_{\Omega}^{r,s} \rightarrow Z^{r,s}$$

defined by  $w(e' \oplus z) = \bar{\partial} e' + i(z)$ .

If  $w$  is surjective, then  $\dim_{\mathbb{C}} H_k^r(X, \Omega^s(E)) < \infty$ .

**20.** *Finiteness theorem for  $q$ -pseudoconvex spaces.*

a) We first prove the following

*Lemma 29.* — Let  $X$  be a complex manifold

Let  $\Phi : X \rightarrow \mathbb{R}$  be a  $C^\infty$  strongly  $q$ -pseudoconvex function on  $X$  such that the sets  $\{\Phi < \text{const.}\}$  are relatively compact in  $X$ .

$$Y = \{x \in X \mid \Phi(x) < \sup_Y \Phi\}$$

Then the natural map:

$$H_k^{n-q+1}(Y, \Omega^r(E)) \rightarrow H_k^{n-q+1}(X, \Omega^r(E))$$

is injective.

*Proof.* — Let  $\varphi \in \mathcal{D}^{r,n-q+1}(Y, E)$  with  $\bar{\partial} \varphi = 0$ . Let us assume that there exists a  $\eta \in \mathcal{D}^{r,n-q}(X, E)$  such that  $\varphi = \bar{\partial} \eta$  on  $X$ .

We want to show that there exists a  $\rho \in \mathcal{D}^{r,n-q}(Y, E)$  such that

$$\varphi = \bar{\partial} \rho.$$

With the same notations as in § 3 n. 9, and using Proposition 23, we can find a  $C^\infty$   $x_\lambda \in W_\lambda^{r,n-q}(X, E)$  such that

$$\eta = \bar{\partial} \theta_\lambda x_\lambda + \theta_\lambda \bar{\partial} x_\lambda;$$

thus

$$\varphi = \bar{\partial} \theta_\lambda \bar{\partial} x_\lambda.$$

Let  $\psi_\lambda = \theta_\lambda \bar{\partial} x_\lambda$ . Then  $\bar{\partial} \psi_\lambda = \varphi$ ,  $\theta_\lambda \psi_\lambda = 0$  so that, by proposition 5 we have

$$(\psi_\lambda, \psi_\lambda)_\lambda \leq c(\varphi, \varphi)_\lambda$$

with a positive constant  $c$  independent of  $\lambda$ .

This is an inequality of Carleman type; therefore we can find a  $\rho \in C^{r,n-q}(X, E)$  with

$$\bar{\partial} \rho = \varphi;$$

$$\text{supp } \rho \subset \{x \in X \mid \Phi(x) \leq \sup_{\text{supp } \varphi} \Phi + \varepsilon\}.$$

Then  $\rho \in \mathcal{D}^{r,n-q}(Y, E)$  if  $\varepsilon$  is sufficiently small. This proves the lemma.

b) Lemma 30. — Let  $X$  be a complex manifold. Let  $p, \varphi$  be  $C^\infty$  functions on  $X$  with the following properties;

- (i)  $p$  is strongly pseudoconvex;
- (ii)  $\varphi$  is strongly  $q$ -pseudoconvex;
- (iii) the set  $\Omega = \{x \in X \mid \sup(p, \varphi) < 0\}$

is relatively compact in  $X$ .

Then we can find a sequence of open sets  $A_\nu \subset \subset \Omega$  for  $\nu \in \mathbf{N}$  such that

- (i)  $A_\nu \subset A_{\nu+1}$  for  $\nu \in \mathbf{N}$ ;
- (ii)  $\Omega = \bigcup_\nu A_\nu$ ;
- (iii) each  $A_\nu$  is a  $q$ -complete manifold.

Proof. — Let  $a = \min_{\bar{\Omega}} p, \quad b = \min_{\bar{\Omega}} \varphi$  and let

$$P = \frac{p + |a|}{|a|}, \quad \Phi = \frac{\varphi + |b|}{|b|}.$$

Then  $\Omega = \{x \in X \mid \sup(P, \Phi) < 1\}$ .

Let  $\psi_\nu = P^\nu + \Phi^\nu$ .

We set  $A_\nu = \left\{ x \in \Omega \mid \psi_\nu < 1 - \frac{1}{\nu} \right\}$ .

Then the sequence  $A_\nu, \nu = 1, 2, \dots$ , has the required properties.

In fact  $\psi_\nu$  is  $C^\infty$  and strongly  $q$ -pseudoconvex. Moreover for  $x_0 \in A_\nu$  we have

$$\sup(P(x_0), \Phi(x_0)) = \sqrt[\nu]{1 - \frac{1}{\nu}},$$

and for  $x_1 \in \Omega$  and  $\nu$  sufficiently large

$$P(x_1)^\nu + \Phi(x_1)^\nu < 1 - \frac{1}{\nu}.$$

c) Using these lemmas and the arguments of § 21 of [2] one obtains the following

Proposition 31. — Let  $X$  be a complex manifold and  $\Phi : X \rightarrow \mathbf{R}$  a  $C^\infty$  strongly  $q$ -pseudoconvex function  $> 0$  such that the sets

$$X_{\varepsilon, c} = \{x \in X \mid \varepsilon < \Phi(x) < c\}$$

be relatively compact in  $X$  for every  $\varepsilon > 0, c > 0$ .

Let  $\bar{\Psi}$  be the family of closed sets  $F$  in  $X$  such that

$$\sup_F \Phi < \infty.$$

Then  $H_{\bar{\Psi}}^s(X, \Omega^r(E)) = 0$  for  $s \leq n - q$ .

A manifold  $X$  is called strongly  $q$ -pseudoconvex if there exists a  $C^\infty$  function  $\Phi : X \rightarrow \mathbf{R}$  and a compact set  $K \subset X$  such that  $\Phi$  is strongly  $q$ -pseudoconvex outside  $K$  and the sets  $\{\Phi < \text{const}\}$  are relatively compact in  $X$ .

*Corollary 32.* — For a  $q$ -pseudoconvex manifold  $X$

$$\dim_{\mathbb{C}} H_k^s(X, \Omega^r(E)) < \infty \quad \text{for } s \leq n - q.$$

*Proof.* — By the above proposition every  $\bar{\partial}$ -closed distribution of type  $(r, s)$  with compact support is  $\bar{\partial}$ -homologous to a  $\bar{\partial}$ -closed  $C^\infty$  form with compact support contained in  $\{\Phi < \sup_K \Phi + 1\}$ . This permits the application of Theorem 6 of the previous section.

**21.** *Finiteness theorem for  $q$ -pseudoconcave spaces.*

a) Let  $X$  be a complex manifold of pure complex dimension  $n$ . Let  $\Phi$  be a  $C^\infty$  strongly pseudoconvex function on  $X$  such that the sets

$$U_\varepsilon = \{x \in X \mid \Phi(x) < \varepsilon\}$$

be relatively compact, for  $0 < \varepsilon < \varepsilon_0$ . Let  $\varphi$  be a  $C^\infty$  strongly  $q$ -pseudoconvex function on  $X$ .

$$\begin{aligned} \text{We set} \quad \bar{U} &= \{x \in X \mid \Phi(x) \leq 0\}, \\ \bar{V} &= \{x \in \bar{U} \mid \varphi(x) \leq 0\}, \\ W &= \bar{U} - \bar{V} = \{x \in \bar{U} \mid \varphi(x) > 0\}. \end{aligned}$$

Let  $E$  be a holomorphic vector bundle on  $X$ .

*Lemma 33.* —  $H_k^s(W, \Omega^r(E)) = 0$  for  $s \geq q + 1$ .

*Proof.* — Letting  $V_\varepsilon = \{x \in U_\varepsilon \mid \varphi(x) < \varepsilon\}$  we see that

(i)  $\bar{U} = \bigcap_{\varepsilon} U_\varepsilon$  has a fundamental system of neighborhoods  $U$  for which  $H^i(U, \Omega^r(E)) = 0$  for  $i > 0$ .

(ii)  $\bar{V} = \bigcap_{\varepsilon} V_\varepsilon$  has a fundamental system of neighborhoods  $V$  for which  $H^i(V, \Omega^r(E)) = 0$  for  $i \geq q$  (use lemma 30 of n° 20 and Serre's duality [20]).

$$\begin{aligned} \text{Hence} \quad H^i(\bar{U}, \Omega^r(E)) &= 0 \quad \text{for } i > 0, \\ H^i(\bar{V}, \Omega^r(E)) &= 0 \quad \text{for } i \geq q. \end{aligned}$$

From the exact sequence

$$\dots \rightarrow H_k^s(W, \Omega^r(E)) \rightarrow H^s(\bar{U}, \Omega^r(E)) \rightarrow H^s(\bar{V}, \Omega^r(E)) \rightarrow H_k^{s+1}(W, \Omega^r(E)) \rightarrow \dots$$

we then deduce that

$$H_k^s(W, \Omega^r(E)) \approx H^{s-1}(\bar{V}, \Omega^r(E)) = 0 \quad \text{if } s-1 \geq q.$$

b) Let  $X$  be a complex manifold and  $B$  an open set in  $X$  such that  $\partial B$  is compact. We say that  $B$  has a strongly  $q$ -pseudoconcave boundary if we can find an open neighborhood  $U$  of  $\partial B$  in  $X$  and a  $C^\infty$  strongly  $q$ -pseudoconvex function  $\Phi$  on  $U$  such that

$$B \cap U = \{x \in U \mid \Phi(x) > 0\}.$$

Let  $(U_i)_{1 \leq i \leq t}$  be a finite covering of  $\partial B$  with coordinate balls  $U_i \subset\subset U$  and let  $\rho_i$ ,  $1 \leq i \leq t$ , be  $C^\infty$  functions on  $U$  such that

$$\rho_i \geq 0, \quad \text{supp } \rho_i \subset\subset U_i, \quad \sum \rho_i(x) > 0 \quad \forall x \in \partial B.$$

We set  $\Phi_s = \Phi + \sum_1^s \varepsilon_i \rho_i$ . If the  $\varepsilon_i > 0$  are chosen sufficiently small, then the functions  $\Phi_s$  are all strongly  $q$ -pseudoconvex in  $U$ .

We set  $B^s = \{B - U\} \cup \{x \in U \mid \Phi_s(x) > 0\}$ .

Then  $B = B^0 \supset B^1 \supset \dots \supset B^t$  since  $\Phi_s \geq \Phi_{s+1}$  ( $\Phi = \Phi_0$ ).

Moreover  $B^s - B^{s+1} \subset \subset U_{s+1}$  for  $0 \leq s \leq t-1$ .

And finally  $\bar{B}^t \subset B$ .

We thus have proved the following « bumps lemma »

*Lemma 34.* — Given on a complex manifold  $X$  an open set  $B$  with compact strongly  $q$ -pseudoconcave boundary, we can find for any finite covering  $(U_i)_{1 \leq i \leq t}$  of  $\partial B$  a sequence of open sets  $B^s$ ,  $0 \leq s \leq t$  with strongly  $q$ -pseudoconcave boundary such that

- (i)  $B = B^0 \supset B^1 \supset \dots \supset B^t$ ,
- (ii)  $B^s - B^{s+1} \subset \subset U_{s+1}$  for  $0 \leq s \leq t-1$
- (iii)  $\bar{B}^t \subset B$ .

Analogously we can construct an increasing sequence of open sets  $B^s$ ,  $0 \leq s \leq t$  with compact strongly  $q$ -pseudoconcave boundary such that

- (i)  $B = B^0 \subset B^1 \subset \dots \subset B^t$ ,
- (ii)  $B^s - B^{s-1} \subset \subset U_s$  for  $1 \leq s \leq t$ ,
- (iii)  $\bar{B}_0 \subset B^t$ .

c) Let  $X$  be a complex manifold and let  $\Phi : X \rightarrow \mathbf{R}$  be a strongly  $q$ -pseudoconvex  $C^\infty$  function on  $X$  such that the sets

$$X_{c\varepsilon} = \{x \in X \mid C > \Phi(x) > c\}$$

be relatively compact for every  $C > 0$ ,  $c > 0$ .

Let  $B_c = \{x \in X \mid \Phi(x) > c\}$

and let  $\Psi$  be the family of closed sets  $F$  of  $X$  such that  $\inf_F \Phi > 0$ .

*Proposition 35.* — For any  $c > 0$  there exists an  $\varepsilon > 0$  with  $c - \varepsilon > 0$  such that the homomorphisms

$$\begin{aligned} H_{\Psi}^s(B_c, \Omega^r(E)) &\rightarrow H_{\Psi}^s(B_{c-\varepsilon}, \Omega^r(E)) \\ H_{\Psi}^s(B_{c+\varepsilon}, \Omega^r(E)) &\rightarrow H_{\Psi}^s(B_c, \Omega^r(E)) \end{aligned}$$

are surjective for any  $s \geq q + 1$ .

*Proof.* — With the notations of lemma 34, setting  $B = B_c$ , making use of lemma 33 we see that for  $s \geq q + 1$

$$H_{\Psi}^s(B^1, \Omega^r(E)) \rightarrow H_{\Psi}^s(B^0, \Omega^r(E))$$

is surjective. Repeating the argument we see that for  $s \geq q + 1$

$$H_{\Psi}^s(B^t, \Omega^r(E)) \rightarrow H_{\Psi}^s(B, \Omega^r(E))$$

is surjective. If  $\varepsilon$  is sufficiently small  $B^c \subset B_{c+\varepsilon} \subset B$ , hence the second assertion. The first assertion is proved in the same way.

Let  $\varepsilon(c)$  be the sup. of all  $\varepsilon$  such that  $c - \varepsilon > 0$  for which the conclusions of the above proposition hold; then one verifies that  $\varepsilon(c) \geq \varepsilon(c_0) - |c - c_0|$ , i.e. that  $\varepsilon(c)$  is a lower semicontinuous function.

d) Let  $X$  be a strongly  $q$ -pseudoconcave manifold. That means that we are given a compact set  $K$  in  $X$  and  $\Phi > 0$  a  $C^\infty$  function on  $X$ , strongly  $q$ -pseudoconvex on  $X - K$ , such that the sets

$$B_c = \{x \in X \mid \Phi(x) > c\}$$

are relatively compact in  $X$ . Let  $c_0 = \inf_K \Phi$ . We then have the following

*Proposition 36.* — For any  $\sigma > 0$ ,  $c_0 - \sigma > 0$ , the natural map

$$H_k^s(B_{c_0 - \sigma}, \Omega^r(E)) \rightarrow H_k^s(X, \Omega^r(E))$$

is surjective for  $s \geq q + 1$ .

*Proof.* — Let  $\xi \in H_k^s(X, \Omega^r(E))$  and let  $\text{supp } \xi \subset B_c$ . We can find a sequence  $c_1 = c < c_2 < \dots$  with  $c_\nu \rightarrow c_0 + \sigma/2$  such that

$$H_k^s(B_{c_{\nu+1}}, \Omega^r(E)) \rightarrow H_k^s(B_{c_\nu}, \Omega^r(E))$$

is surjective. Hence  $\xi$  can be represented by a cocycle with support in any  $B_{c_\nu}$ . If  $\nu$  is large enough  $B_{c_\nu} \subset B_{c_0 - \sigma}$ . This proves our assertion.

*Corollary 37.* — For a strongly  $q$ -pseudoconcave manifold  $X$  and for any holomorphic vector bundle  $E$  on  $X$  we have

$$\dim_{\mathbb{C}} H_k^s(X, \Omega^r(E)) < \infty \quad \text{for } s \geq q + 1.$$

**22.** The groups  $H_k^{n-q+1}(X, \Omega^r(E))$  on a strongly  $q$ -pseudoconvex manifold.

a) Let  $X$  be a strongly  $q$ -pseudoconvex manifold of pure dimension  $n$ . Let  $K$  be compact in  $X$  and let  $\Phi : X \rightarrow \mathbb{R}$  a  $C^\infty$  function on  $X$  such that

(i)  $\Phi$  is strongly  $q$ -pseudoconvex on  $X - K$ ;

(ii) the sets  $B_c = \{x \in X \mid \Phi(x) < c\}$  are relatively compact in  $X$  for every  $c \in \mathbb{R}$ .

Let  $c_0 = \sup_K \Phi$ .

Using the vanishing theorems for  $q$ -complete manifolds and the bumps lemma one proves that for  $c > c_0$  we can find an  $\varepsilon > 0$  such that

$$H^s(B_{c+\varepsilon}, \Omega^r(E)) \rightarrow H^s(B_c, \Omega^r(E))$$

is a surjective map for  $s \geq q$ . From this one deduces the following [2]

*Proposition 38.* — Under the above specified assumptions one has, if  $s \geq q$ ,

$$\dim_{\mathbb{C}} H^s(B_c, \Omega^r(E)) < +\infty$$

for any  $c > c_0$ , and any holomorphic vector bundle  $E$  on  $X$ .

*Corollary 39.* — With the same assumptions the image of

$$\bar{\partial} : \mathcal{D}^{r, n-q}(B_c, E) \rightarrow \mathcal{D}^{r, n-q+1}(B_c, E)$$

is a closed subspace of  $\mathcal{D}^{r, n-q+1}(B_c, E)$ .

*Proof.* — We consider the sequence

$$C^{r,q-1}(B_c, E) \xrightarrow{\bar{\partial}} C^{r,q}(B_c, E) \xrightarrow{\bar{\partial}} C^{r,q+1}(B_c, E).$$

By the assumptions, since  $H^q(B_c, \Omega^r(E))$  is finite dimensional, the first map  $\bar{\partial}$  is a topological homomorphism. Denoting by  $K^{n-r,n-q+1}(B_c, E)$ ,  $K^{n-r,n-q}(B_c, E)$  the dual spaces of  $C^{r,q-1}(B_c, E)$ ,  $C^{r,q}(B_c, E)$  respectively, it then follows that

$$\bar{\partial} : K^{n-r,n-q}(B_c, E) \rightarrow K^{n-r,n-q+1}(B_c, E)$$

has a weakly closed image.

This holds for any  $r$  and any vector bundle  $E$ . Now we consider  $\mathcal{D}^{r,n-q}(B_c, E)$  and  $\mathcal{D}^{r,n-q+1}(B_c, E)$  as subspaces of  $K^{r,n-q}(B_c, E)$ ,  $K^{r,n-q+1}(B_c, E)$  by associating to any form  $\varphi \in \mathcal{D}^{r,*}(B_c, E)$  the distribution  $T_\varphi \in K^{r,*}(B_c, E)$

$$T_\varphi[u] = (u, \varphi).$$

To prove the corollary it is enough to show that  $\bar{\partial}\mathcal{D}^{r,n-q}(B_c, E)$  is sequentially closed (cf. n° 11 b), remark). Let  $(\varphi_\nu) \subset \bar{\partial}\mathcal{D}^{r,n-q}(B_c, E)$  with  $\varphi_\nu \rightarrow \varphi$ . Assume that  $\varphi_\nu = \bar{\partial}\eta_\nu$  with  $\eta_\nu \in \mathcal{D}^{r,n-q}(B_c, E)$ . We have to show that there exists a  $\eta \in \mathcal{D}^{r,n-q}(B_c, E)$  such that

$$\varphi = \bar{\partial}\eta.$$

By the assumption  $\varphi_\nu \rightarrow \varphi$  we have  $T_{\varphi_\nu} \rightarrow T_\varphi$ , but  $T_{\varphi_\nu} \in \bar{\partial}K^{r,n-q}(B_c, E)$ . Thus  $T_\varphi \in \bar{\partial}K^{r,n-q}(B_c, E)$  by the above argument. Thus there exists a distribution  $S$  with compact support in  $B_c$  such that

$$\varphi = \bar{\partial}S.$$

We now apply lemma 12 and we can find  $\eta \in \mathcal{D}^{r,n-q}(B_c, E)$  such that  $\varphi = \bar{\partial}\eta$ .

*Proposition 40.* — Let  $X$  be strongly  $q$ -pseudoconvex and let  $E$  be any holomorphic vector bundle on  $X$ . Then the image of

$$\bar{\partial} : \mathcal{D}^{r,n-q}(X, E) \rightarrow \mathcal{D}^{r,n-q+1}(X, E)$$

is closed. Thus  $H_k^{n-q+1}(X, \Omega^r(E))$  has a natural topology of a separated topological vector space.

*Proof.* — We have to show that  $\bar{\partial}\mathcal{D}^{r,n-q}(X, E)$  is sequentially closed. Let  $\varphi_\nu = \bar{\partial}\eta_\nu \in \bar{\partial}\mathcal{D}^{r,n-q}(X, E)$  be a convergent sequence,  $\varphi_\nu \rightarrow \varphi$ .

There exists a compact set  $K' \subset X$  such that

$$\text{supp } \varphi_\nu \subset K', \quad \text{supp } \varphi \subset K'.$$

We select  $c_1 > c_0$  such that  $B_{c_1} \supset K'$ .

Then on  $X - \bar{B}_{c_1}$ ,  $\bar{\partial}\eta_\nu = 0$ . By virtue of proposition 31, if  $n - q \geq 1$ , there exist forms  $\gamma_\nu \in C^{r,n-q-1}(X - \bar{B}_{c_1}, E)$  such that

$$\sup_{\text{supp } \gamma_\nu} \Phi < \infty, \quad \eta_\nu = \bar{\partial}\gamma_\nu \quad \text{on} \quad X - \bar{B}_{c_1}.$$

Let  $\mu$  be a  $C^\infty$  function with the properties

$$\begin{aligned}\mu(x) &= 1 & \text{if } x \in X - B_{c_1+1}, \\ \mu(x) &= 0 & \text{if } x \in B_{c_1+1/2};\end{aligned}$$

then  $\mu\gamma_\nu$  are compact supported forms and we can write

$$\varphi_\nu = \bar{\partial}(\eta_\nu - \bar{\partial}(\mu\gamma_\nu)).$$

Replacing the forms  $\eta_\nu$  by the forms  $\eta_\nu - \bar{\partial}(\mu\gamma_\nu)$  we see that we can assume that

$$\text{supp } \eta_\nu \subset B_{c_1+1}.$$

If  $n - q = 0$  the same conclusion obviously holds.

Taking  $c = c_1 + 1$  and applying corollary 39 we then conclude that there exists an  $\eta \in \mathcal{D}^{r, n-q}(X, E)$  such that

$$\varphi = \bar{\partial}\eta.$$

This achieves the proof.

**23.** a) As an application we prove now the following

*Proposition 41.* — Let  $X$  be a complex manifold of pure dimension  $n$ . Let  $E$  be a holomorphic vector bundle on  $X$ .

We assume that

$$\bar{\partial} : C^{n-r, n-s}(X, E) \rightarrow C^{n-r, n-s+1}(X, E)$$

is a topological homomorphism. Let  $T$  be a distribution of type  $(r, s)$  with values in  $E$  <sup>(1)</sup> and compact support, such that  $\bar{\partial}T = 0$ . The necessary and sufficient condition for the solvability of the equation

$$T = \bar{\partial}S$$

with a distribution  $S$ , with compact support of type  $(r, s-1)$  and with values in  $E$ , is that for any  $u \in C^{n-r, n-s}(X, E)$  with  $\bar{\partial}u = 0$  we have

$$T[u] = 0.$$

*Proof.* — Let  $\Omega_c^{n-r, n-s}(E)$  be the sheaf of germs of  $C^\infty$ ,  $\bar{\partial}$ -closed forms of type  $(n-r, n-s)$  and with values in  $E$ . We have the exact sequence

$$0 \rightarrow \Gamma(X, \Omega_c^{n-r, n-s}(E)) \rightarrow C^{n-r, n-s}(X, E) \xrightarrow{\bar{\partial}} C^{n-r, n-s+1}(X, E).$$

By the assumptions  $T[u]$  as a linear function on  $C^{n-r, n-s}(X, E)$  defines a continuous linear function on  $C^{n-r, n-s}(X, E)/\Gamma(X, \Omega_c^{n-r, n-s}(E))$  with respect to the natural Fréchet topology of this quotient space. Since  $\bar{\partial}$  is a topological homomorphism, then  $\bar{\partial}C^{n-r, n-s}(X, E)$ , with the induced topology of  $C^{n-r, n-s+1}(X, E)$ , is topologically isomorphic with the previously considered quotient space. Thus  $T$  defines a continuous

<sup>(1)</sup> i.e.  $T$  is a continuous linear function on  $C^{n-r, n-s}(X, E)$ .

linear function on  $\bar{\partial}C^{n-r, n-s}(X, E)$ . By the theorem of Hahn-Banach we can extend this linear function to a continuous linear function  $S : C^{n-r, n-s+1}(X, E) \rightarrow \mathbf{C}$ . We thus have

$$T[u] = S[\bar{\partial}u]$$

for any  $u \in C^{n-r, n-s}(X, E)$ . This means that

$$T = \bar{\partial}S.$$

Moreover  $S$  as an element of the dual of  $C^{n-r, n-s+1}(X, E)$  has compact support.

The necessity of the condition is obvious.

*Remark.* — If  $\dim H^{n-s+1}(X, \Omega^{n-r}(E)) < \infty$  then the assumption of the previous proposition is satisfied.

Let  $B$  be open and relatively compact on  $X$ . We will assume that  $\partial B$  is smooth. Let  $B_\varepsilon = \{x \in B \mid d(x, \partial B) \geq \varepsilon\}$ . If  $\varepsilon$  is sufficiently small then  $\partial B_\varepsilon$  is also smooth.

*Corollary 42.* — We assume that

$$\dim_{\mathbf{C}} H^{n-s}(B, \Omega^{n-r}(E^*)) < \infty$$

Let  $\varphi \in C^{r,s}(X-B, E)$  with  $\bar{\partial}\varphi = 0$  be defined and  $\bar{\partial}$ -closed in  $X - \bar{B}_{\varepsilon_0}$ . The necessary and sufficient condition for the existence of a form  $\hat{\varphi} \in C^{r,s}(X, E)$  such that

$$\bar{\partial}\hat{\varphi} = 0, \quad \hat{\varphi}|_{X-B} = \varphi$$

is that for any  $u \in C^{n-r, n-s-1}(B, E^*)$  with  $\bar{\partial}u = 0$  we have

$$\int_{\partial B_\varepsilon} \varphi \wedge u = 0 \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

*Proof.* — From the exact sequence

$$H^0(X, \Omega_c^{r,s}(E)) \rightarrow H^0(X-B, \Omega_c^{r,s}(E)) \xrightarrow{\bar{\partial}} H_k^1(B, \Omega_c^{r,s}(E))$$

we see that the existence of  $\hat{\varphi}$  is equivalent to the condition  $\delta\{\varphi\} = 0$ . Now  $H_k^1(B, \Omega_c^{r,s}(E)) \simeq H_k^{s+1}(B, \Omega^r(E))$  and thus  $\delta\{\varphi\}$  can be represented as follows. We take any  $\tilde{\varphi} \in C^{r,s}(X, E)$  such that  $\tilde{\varphi}|_{X-B} = \varphi$ ; then  $\bar{\partial}\tilde{\varphi}$  is compactly supported in  $B$ . The condition  $\delta\{\varphi\} = 0$  means that  $\bar{\partial}\tilde{\varphi}$  is the  $\bar{\partial}$  of a compactly supported distribution with support in  $B$ . This is equivalent to the condition

$$\int_B \bar{\partial}\tilde{\varphi} \wedge u = 0 \quad \text{for all } u \in C^{n-r, n-s-1}(B, E^*) \text{ with } \bar{\partial}u = 0.$$

But

$$\int_B \bar{\partial}\tilde{\varphi} \wedge u = \int_B \bar{\partial}(\tilde{\varphi} \wedge u) = \int_B d(\tilde{\varphi} \wedge u) = \int_{\partial B_\varepsilon} \varphi \wedge u.$$

One will recognize the analogy of this result with a classical theorem [23] which asserts that if  $f$  is a  $C^\infty$  function on the circle  $|z| = 1$  in  $\mathbf{C}$  then  $f$  is the trace of a function holomorphic in  $|z| < 1$  if and only if

$$\int_{|z|=1} f z^k dz = 0 \quad \text{for every } k \in \mathbf{N}.$$

## REFERENCES

- [1] A. ANDREOTTI, *Coomologia sulle varietà complesse*, II : Summer course on « *Funzioni e varietà complesse* » sponsored by C.I.M.E., Varenna (Italy), Summer 1963, Edizioni Cremonese, Roma.
- [2] A. ANDREOTTI et H. GRAUERT, Théorèmes de finitude pour la cohomologie des espaces complexes, *Bull. Soc. Math. France*, 90 (1962), 193-259.
- [3] A. ANDREOTTI e E. VESENTINI, Sopra un teorema di Kodaira, *Ann. Sc. Norm. Sup. Pisa* (3) 15 (1961), 283-309.
- [4] A. ANDREOTTI et E. VESENTINI, Les théorèmes fondamentaux de la théorie des espaces holomorphiquement complets, *Séminaire Ehresmann*, 4 (1962-63), 1-31, Paris, Secrétariat Mathématique.
- [5] A. ANDREOTTI e E. VESENTINI, Disuguaglianze di Carleman sopra una varietà complessa, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.* (8), 35 (1963), 431-434.
- [6] N. BOURBAKI, *Espaces vectoriels topologiques*, chap. I-IV, Hermann, Paris, 1953 et 1955.
- [7] E. CALABI and E. VESENTINI, On compact, locally symmetric Kähler manifolds, *Ann. of Math.*, 71 (1960), 472-507.
- [8] T. CARLEMAN, Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes, *Ark. Mat. Astr. och Fys.*, 26 B (1939), n° 17, 1-9 = Édition complète des articles, Malmö, 1960 497-505.
- [9] J. DIEUDONNÉ et L. SCHWARTZ, La dualité dans les espaces ( $\mathcal{F}$ ) et ( $\mathcal{L}\mathcal{F}$ ), *Ann. Inst. Fourier*, Grenoble, 1 (1950), 61-101.
- [10] K. O. FRIEDRICH, On the differentiability of the solutions of linear elliptic differential equations, *Comm. Pure Applied Math.*, 6 (1953), 299-325.
- [11] A. GROTHENDIECK, *Espaces vectoriels topologiques*, 2<sup>e</sup> éd., Publicação da Sociedade de Matemática de S. Paulo, São Paulo, 1958.
- [12] F. HIRZEBRUCH, *Neue topologische Methoden in der algebraischen Geometrie*, Berlin, Springer, 1956.
- [13] L. HÖRMANDER, On the uniqueness of Cauchy problem, *Math. Scand.*, 6 (1958), 213-225.
- [14] K. KODAIRA, On a differential geometric method in the theory of analytic stacks, *Proc. Nat. Acad. Sci. U.S.A.*, 39 (1953), 1268-1273.
- [15] G. KÖTHE, *Topologische lineare Räume*, I, Berlin, Springer, 1960.
- [16] E. MAGENES e G. STAMPACCHIA, I problemi al contorno per le equazioni differenziali di tipo ellittico, *Ann. Sc. Norm. Sup. Pisa* (3), 12 (1958), 247-358.
- [17] L. SCHWARTZ, Homomorphismes et applications complètement continues, *C. R. Acad. Sci. Paris*, 236 (1953), 2472-2473.
- [18] J. SEBASTIÃO E SILVA, Su certe classi di spazi localmente convessi importanti per le applicazioni, *Rend. Math. e Appl.* (5) 14 (1955), 398-410.
- [19] *Séminaire H. CARTAN*, 1953-1954, Paris, École Normale Supérieure ; Cambridge, Mass., Mathematics Department of M.I.T., 1955.
- [20] J.-P. SERRE, Un théorème de dualité, *Comment. Math. Helv.*, 29 (1955), 9-26.
- [21] E. VESENTINI, *Coomologia sulle varietà complesse*, I : Summer course on « *Funzioni e varietà complesse* » sponsored by C.I.M.E., Varenna (Italy), Summer 1963, Edizioni Cremonese, Roma.
- [22] A. WEIL, *Introduction à l'étude des variétés kählériennes*, Hermann, Paris, 1958.
- [23] K. YANO and S. BOCHNER, Curvature and Betti numbers, *Ann. of Math. Studies*, n° 32, Princeton University Press, 1953.
- [24] G. ZIN, Esistenza e rappresentazione di funzioni analitiche, le quali, su una curva di Jordan, si riducono a una funzione assegnata, *Ann. di Mat.* (4) 34 (1953), 365-405.

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