PHILOSOPHIA SCIENTIÆ

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Philosophia Scientiæ, tome 3, nº 2 (1998-1999), p. 43-60
<http://www.numdam.org/item?id=PHSC_1998-1999__3_2_43_0>

Poincaré on Understanding Mathematics*

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Abstract. Poincaré holds up that the varieties of formal logical theories don't express the essential proof-theoretical structure in order to understand mathematics. Intuition and æsthetic reasoning, the latter depending from the criterion "harmony by a *surprising* order-character", are other decisive proof-aspects. In what sense elements from Peirce's semiotics and Goodman's æsthetics contribute something to Poincaré's aim of mathematical reasoning besides logical inference ?

Résumé. Selon Poincaré, les différentes théories de la logique formelle n'expriment pas l'élément essentiel à la compréhension d'une démonstration mathématique. A cet égard, l'intuition et le raisonnement esthétique, ce dernier conçu en dépendance du critère «harmonie grâce au chractère surprenant d'une structure d'ordre», sont les aspects manquants. En quel sens des éléments de la sémiotique de Peirce et de l'esthétique de Goodman peuvent-ils contribuer à atteindre l'objectif de Poincaré, à savoir de comprendre le raisonnement mathématique au-delà de sa structure logique ?

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* Le titre de ma conférence tenue à ce colloque fut *Convention et observabilité. Une reprise à partir de Poincaré*. Ce texte va paraître dans E. Agazzi / M. Pauri (éds.), *Observability, Unobservability and Scientific Realism*, Dordrecht : Kluver 1999. Je remercie très vivement Antonia Soulez et Jan Sebestik d'avoir accepté que je remplace mon texte original par une conférence faite le 28 Août 1998 à Berlin au *Colloquium Logicum '98* de la *Deutsche Vereinigung für Mathematische Logik und für Grundlagen der exakten Wissenschaften.*

Philosophia Scientiae, 3 (2), 1998-1999, 43-60

> It is, as newspaper editors know, a tendency of the public to read with interest and even to accept uncritically the opinions of an eminent person on matters about which he is not an expert. [...] M. Poincaré is one of our greatest mathematicians, and centuries have proved that a man who is a great mathematician need be neither a great philosopher nor a great logician. [Jourdain 1912, 481-482]

According to Philip Jourdain, quoted here from an *Introductory* Note to a translation of a debate between Poincaré and Couturat, Poincaré's articles on philosophy and mathematical logic are based on a very superficial acquaintance with these subjects. He doesn't understand the new logic, because — I'll argue — he wants to understand mathematics.

Now, since Aristotle, understanding is connected with learning. And it is well known that, according to Poincaré, in order to teach and to learn mathematics there must be appeal to intuition and reasoning by analogy [Folina 1996, 421]. The same capacities are also necessary to create new mathematics. Jourdain's evaluation of Poincaré's articles on the foundational enterprise is not surprising once we concede that the notion of intuition concerns the psychological conviction for the former, whereas logicicians are concerned with the latter. So Poincaré and the logicians (and especially the logicists) are trying to achieve different objectives. This is Warren Goldfarb's claim in his article *Poincaré against the logicists*. He finds that even Poincaré's "more serious antilogicist arguments", namely that the program of founding number theory on logic involves a *petitio principii* (because this foundation presupposes already some number theory) and that mathematical definitions have to be predicative, reveal a continued dependence on a psychologistic conception" [Goldfarb 1988, 64].

Actually, Poincaré's use of intuition does not reduce to psychology in the modern sense of this term. This will be clear if one gives up Goldfarb's thesis that Poincaré's "papers on foundations are disconnected from his positive work in mathematics" [Goldfarb 1988, 62]. As a first example, I'll take Poincaré's view on the foundations of geometry, and especially his distinction between the so called sensible and geometrical spaces. The evolution from sensible space to geometry could be understood in two different manners : logically and psychologically. By a logical genesis of geometry I understand a systematic way of reconstruction geometrical Thinking. By a psychological genesis of geometry, I understand the description of the development of animal behaviour to grasp a spatial orientation.

I think that Poincaré very well distinguishes between the two points in question : The psychological genesis is treated by him in the spirit of an evolutionary empiricism largely influenced by Darwin's biological adaptation. It concerns the translation from the "I" to the "we"—perspective and finds its echo with Poincaré in the transfer from the Individual to the race summarised in his 1907 article *The relativity of space* [SM, 120sq.]. But in the article *On the Foundations of Geometry,* an important paper of 43 pages, written in English and published in the Monist of October 1898, this level of psychological evolution is presupposed : he begins with "Our sensations cannot give us the notion of space". There is no problem of confusing levels here, because Poincaré does not use the term "psychological" in the modern sense. In fact, he uses it in all the situations where the dimension of understanding is involved, and in particular it's genetic component which – as opposed to the logically correct linguistic exposition of the result — "is indispensable for a complete knowledge of science itself" [SH, 138 (153)]. Psychology concerns here the epistemological question of developing standards of clarity for conceptual presuppositions. In this perspective, Poincaré's refusal to distinguish psychology from logic and epistemology, expressed in 1909 in a famous reply to Russell, does by no means signify that he is confusing the questions *quid iuris* and *quid facti,* but it tells us to supplement our structural investigations in Logic and Epistemology by a genetic analysis[1].

No wonder also that, according to Poincaré, mathematics requires intuition not only in the context of discovery but also in context of justification. By the way, Poincaré's often quoted statement (for example [Resnik 1996, 459] : "Logic, which alone give certainty, is the instrument of demonstration; intuition is the instrument of invention" [VS, 23 (37)], (even [SM, (130)], resumes only a discussion about the distinction of *sensible intuition* and analytic procedures. Some pages later, he underlines that *pure intuition* gives certainty too and enables us to demonstrate and to invent [VS, 25 (39)].

Within formal mathematics, pure intuition is necessary for its justification and its proofs in order to *understand* them. The context of justification is pragmatically connected with the logical reconstruction of the genesis because "to understand" mathematics means : to learn their development. So, if S has a true belief for p, justified by a formal proof, it does not follow that he is understanding p. Justified belief may be equivalent with a abstract proof structure [Helman 1992] but not with mathematical knowledge.

In order to understand this, we should go back to Poincaré's article *Sur la nature du raisonnement mathématique*, published in 1894, where he discusses the following dilemma. On the one hand, mathematics is an exact science, that is their proofs are exactly rigorous. But if, on the other hand, "all the pro-

1 "Mr Russell will probably tell me that it is not a question of psychology but of logic and epistemology; and I should be led to reply that there are no logic or epistemology independent of psychology; this need will probably close the discussion as it will make evident an irremediable difference of opinion." [Poincaré 1909, 482].

positions it enunciates can be deduced from another by the rules of formal logic, why is mathematics not reduced to an immense tautology ?" Here is Poincaré's solution : in opposition to the logicians' declared position, analyticity or logical rigour suggested in the dilemma cannot be a general criterion for correct mathematical reasoning. Genuine mathematical knowledge is in general not the result proved by means of a series of analytical deductions, called by him *verifications.* – by "verification" Poincaré means a deduction based on syllogism, substitution and nominal definition and so he is trivially right in arguing that mathematical reasoning is to be distinguished from logic (SH, 3 (32)). Nevertheless, he adds that logic and mathematics can be distinguished *because* mathematics has itself a kind of creative virtue, exemplified by the principle of complete induction. So the question arises : is it possible to deduce the principle of complete induction from the new systems of logic invented arround 1900 and, if yes, why should we call the new theory "logic" ? I argued in my book on Poincaré [Heinzmann 1995] that all proofs of complete induction violate certain forms of Poincaré's predicativity requirement and Charles Chihara argued in his article *Poincaré and Logicism* that it is by no means evident to consider axioms expressed in purely logical terms to be laws of logic [Chihara 1996]. For example, is it really a logical procedure (of the second order) to turn predicates into names and afterwards affirm their existence, i. e. the existence of entities signified ? On the contrary, if one argues that modern formal logic has turned into a mathematical field in interpreting, for example, the logical consequences as knowledge about all models, i. e. all mathematical entities, then one should recognise that it has essentially cut the bounds to its original task of serving as a means to mathematical inference [Lorenz 1986, 42]. Its relevance is thereby limited to investigations into the foundations of mathematics.

How should we now understand the concept of mathematical proof ? If logical inference cannot give rise to understand mathematical knowledge, how can pure intuition do better. Can the concept of intuition replace symbolic logic as standard for rigorous mathematical reasoning without forcing us to commit a mystical return to Locke's connection of ideas ? To discuss these questions requires some preliminary elucidation.

It is well known, that, according to Poincaré, the reductionnist program of Logicism seems to waver already by the antinomies occurring in the new logic. Regarding the reduction to logical definitions, the surmounting of a technical problem is in the foreground and gives rise to Poincaré's widely-known onslaught on Logicism: "*Logistic is no longer barren, it engenders antinomies*" [SM, 194 (211)]. Here the sterility applies to purely logical inferences, the antinomy to the constitution of sets defined by the axiom of comprehension. According to Poincaré, this constitution of sets is based on a methododical inversion which already concerns the philosophical presuppositions, i.e. the ontological hypotheses implied by the choice of a language. The cause of the antinomies resides, according to Poincaré, in an implicit recourse to a "false" intuition regarding abstract entities; in fact, for an anti-platonist, which was Poincaré's case from 1906 on, conceptual realism makes a usurping use of intuition when it relates it to the evident mode of presentation of an abstract entity instead of relating it to the capability to follow an action-schema. A 'true' pure intuition can be distinguished from simple evidence by the fact that it refers to what can be *done* instead of merely to something that is. In this sense pure intuition has not the same object as sensible intuition or imagination [SH, 25 (39)]. It is an awareness of a mental capacity and experience gives the opportunity [occasion] of using this capacity. So, the certainty with respect to complete induction taken by Poincaré as a synthetic judgement *apriori*, derives from the fact that it is the *affirmation* of a direct intuition into the capacity of the mind to comprehend the indefinite repetition of one and the same act. We would today say that such an intuition obtains with respect to an action-schema which is apriori because it is a result of our own creation and that is called *intuitive* because it is not generated but only represented by indefinite repetition of different levels [Heinzmann 1987, 72], that means that the separation between object and symbol is not yet accomplished : it postulates a survey of a potentially reiterated stroke-concatenation or something analogous and a survey of a potentially reiterated modus ponens. This sort of intuitive survey, in a set theoretical approach guaranteed by the order type of the sequence of natural numbers, allows Poincaré to consider complete induction as the expression of an infinite number of hypothetical syllogisms *condensed,* as it were, in a single formula. — To give up the necessary character of the apriori allows us to avoid an awkward consequence advocated by Poincaré : he believed — wrongly — that it is impossible to deny the principle of complete induction. Nevertheless, we are not obliged to adopt a conventionalistic standpoint : Standard-Arithmetic has even now the privilege to be built up by an ability [Heinzmann 1988, 6]. And to have an ability means exactly to understand something[2].

One can undoubtedly agree that understanding a proof cannot be reduced to being able to checking a linguistic type, whose tokens may be printed in books [Kitcher 1984, 36], but requires the awareness of the acquisition of an action schema, awareness which Poincaré seemingly calls "pure intuition". Indeed, when a mathematical proof has been shown to conform to the explicitly formulated rules or principles of logical inference we usually considered it valid. This precisely would constitute the problem the logicists are trying to

2 This is rightly underlined by Resnik 1996, 465.

solve. Poincaré, refusing to conceive of mathematical and logical symbolism as mere systems of code notations of existing entities— this is in fact the condition to take predicativism seriously —, suspects that there may be something awry with the problem formulated by the logicians : because what it means to *follow* correctly the defined rules as, for example, for the step-by-step process of substitution which go to make up the atomic elements of proof, is only determined "within the established practices of working with" the substitution expression [Stenlund 1996, 469]. We suddenly find ourselves in the tradition of the philosophy of the later Wittgenstein, where language has lost its role of being something available on the metalevel with respect to the level of formalism [Heinzmann 1988, 4]. The awareness of a mastery of a schema (= the execution of an action in a schematic perspective) is called intuitive and cannot be itself formalised without committing a *petitio principii.* This matters are taken care of by Poincaré in his article *Mathematics and Logic* and runs as follows : in order to produce and understand definitions in a formal system, one must use a name of a number, an indefinite numeral or at least a plural. Supported by Hadamard and taken up later by Fraenkel, Wittgenstein and Bernays this argument includes, no doubt, a correct observation but depends for its conclusiveness on its exact form [Heinzmann 1988, 13-14]. Stenlund remarks rightly that "formalisation presupposes and applies the mathematical calculus of finite sequences [...] which we do not acquire until we learn elementary mathematics, [...] because the notion of a finite sequence is not the same one as the everyday notion of a list of concrete individual signs" [Stenlund 1996, 476]. This statement hides a related critique of Poincaré against a formal approach, consisting in what I called the connection problem [Heinzmann 1988, 8sqq.]. Suppose we have learned by intuition to follow the rules of a formalism. This formalism may be considered justified either because generating interesting problems within formalism or because it is considered to clarify or to simplify informal reasoning. In the later case, justification consists in connecting a presupposed practical familiarity with informal reasoning with its formal characterisation. We have to ascertain that the formal characterisation is the translation, in the sense of precision, of the former. Poincaré adds in the second part of his article *Mathematics and Logic* that this connection would be a *desideratum* even if the formalists reach a justification on the basis of their own system, that is, if they find a proof of non-contradiction [Poincaré 1906, 23]. Ironically, Frege, in his posthumous manuscript "Logik" from 1897, has already clearly diagnosed the connection problem with regard to the explicit definition of truth[3]. The trouble is that, according to Wittgenstein and Quine, the problem is quite insoluble insofar as there are simply two systems of rules without criterion to compare them. This insight constitutes, so to speak, the "conception"-day of general

3 Cf. his manuscript "Logik" from 1897, in Frege 1969, 139sq.

proof theory : the somewhat misleading metaphysical program to compare ideal language with ordinary language in hoping of explaining occult properties of the former is replaced by the studies of different formal languages with respect to their deductive connections. Then, naturally, exist proof-theoretic criteria of comparison, for example, identity of normal form for identity of proofs (Martin-Löf/Prawitz). Now, you can even give criteria for the passage by formalisation from a body of mathematics M to a formal theory T. According to Feferman, for example, every concept, argument and result of M have to be represented by a concept, proof or theorem of T. But the Non-Standard models show that such a formalisation of a body of mathematics M goes far beyond what is actually needed to represent M [Feferman 1992, 15]. This digression confirms Poincaré's feeling that one have to be sceptical about logical consequences as a sufficient guide to exhaust the domain of truth in mathematics at least as complex as elementary arithmetic.

If we are now ready to tolerate the interpretation of intuition as the awareness of the mastery of a schema — what level it may be, I am afraid that we are nowhere near having any sound insight in proof understanding transcending logical means. We just reached Poincaré's non-psychological *petitio* and connection arguments against the logicists. What intrigues us as a problem is not an adequate understanding of logical calculus but Poincaré's positive draft of a rigorous *not exclusively* logical mathematical proof. Of course, this proposal is quite different from recent authors' interest in emphasising non rigorous factors in proofs. Still, it is different from the attempt to clarify informal proofs in a customary sense (cf. e.g. [Suppes 1957, 122], that is, considered as an incomplete formal proof using logical laws covertly. Such a misleading conception of "informal proof" was already criticised by Lakatos [Lakatos 1985, 156].

As Detlefsen put it forth — Poincaré accepts as a datum that "conclusions of mathematical proofs can, and often do, constitute extensions of mathematical knowledge represented by the premises"[4]. Formulated in modern terms, Poincaré holds that the varieties of formal logical theories don't express the essential proof-theoretical structure in order to understand mathematics [SM, 149 (159)]. He insists on the non-invariance of mathematical reasoning with regards to its contents, he promotes, so to speak, a "local" conception of mathematical reasoning according to which "a 'gap' is no longer a logical gap but, rather, a gap in *mathematical understanding*. [...] The elimination of gaps thus no longer calls for the *exclusion* of topic-specific information in an inference" but for the inclusion of what Detlefsen calls a *epistemic condenser* "to fill what would otherwise be a mathematical gap between the premises and the

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4 Cf. Detlefsen 1992, 354/355, 359; SH, 4 (33/34), Poincaré 1902, 94.

conclusion" [Detlefsen 1992, 366, 360]. Indeed, what means exactly to seek for an element of condensation ? You may know Poincaré's famous confession : "I know not what is which makes the unity of the demonstration" [VS, 22 (36)]. Concerning this unity, Detlefsen goes on to say, that Poincaré's answer is in fact metaphorical in character : he suggests that the premise must be related to the conclusion by means of a "mathematical architecture". Poincaré considers complete induction as the expression of such as structure, that is to posses a survey of the mastery of an infinite sequence of *modus ponentes*. Indeed, he adds, that he does not "mean to say, as has been supposed, that all mathematical arguments can be reduced to an application of this principle". Induction is only the simplest of all "other similar principles, offering the same essential characteristics" [SM 149/150 (159/160)]. Such analogous principles are especially the awareness of our capacity to construct a continuum of any dimension, called topological intuition, or to conceive groups, called algebraic intuition. Both concepts pre-exist, according to Poincaré, in our mind as a form of reason and the awareness of them is occasioned by experience[5]. Now, these characteristics clears the air but leaves a lot to be done. Main questions not yet settled are : How groups and continua are accessible to intuitive knowledge ? How the mentioned algebraic and topological intuitions can be rigorous ? I will postpone this matter and once again return to Poincaré's explicit considerations on proofs.

It happens that (at least) two other proof-aspects, up to now neglected, can be found in his work. The first is Mach's principle of economy of thought, the second concerns æsthetic considerations. Both are interrelated together by the concept of harmony. In fact, the logicians view to close logical gaps in the representation of proofs by means of formal calculi conceals the role of understanding plays in proofs, because it violates our simple survey of the latter by its somewhat lengthy formulation. A proof seems better accessible to us if one introduces first some order into the subject domain's complexity. By introducing order we recognise classes of analogous combinations between certain elements [SM, 28 (23)] and this "enables us to see at a glance each of these elements in the place it occupies in the whole" [SM, 30 (25)]. So we obtain economy of thought by making complexity harmonious in introducing an order. This argument is made evident in the following passage from *Science and Method :*

> Mathematicians attach a great importance to the elegance of their methods and of their results, and this is not mere dilettantism. What is it that gives us the feeling of elegance in a solution or a demonstration ? It is the harmony of the different parts [...]; it is, in a word, all that introduces order, all that gives them unity, that enables us to obtain a clear

5 Cf. DP, 157, 134sq.; SH, 87sq (107).

> comprehension of the whole as well as of the parts.[...] And in fact the more we see this whole clearly at a single glance, the better we shall perceive the analogies with other neighbouring objets, and consequently the better chance we shall have of guessing the possible generalisations. [...] Briefly stated, the sentiment of mathematical elegance is nothing but the satisfaction due to *I don't know not what kind of conformity*[6] between the solution we wish to discover and the necessities of our mind, and it is on account of this very conformity that the solution can be an instrument for us. This æsthetic satisfaction is consequently connected with the economy of thought. [SM, 31 (25/26)]

You remember that Poincaré first expressed his aporia concerning the unity of demonstration in using the expression "I know not is what which makes the unity of the demonstration". In the above quotation he is using again this expression but the aporia is pushed some step further : The unity of proof is now waranted by the recognition of harmony which we will lose as far as grow our demonstration in length [SM, 33 (28) and what we seek now is the criterion of harmony entailing economy of thought. By introducing æsthetic sensibility in context of scientific justification, Poincaré attempts an important and very surprising turning.

Of course, it is quite trivial to underline the resemblance between the methods used by artists and mathematicians in context of invention. We know all nice anecdotes supporting this situation and one attributed to Hilbert is amusing enough to mention it : once exhausted, he remarks concerning one of his student: 'being not sufficiently poet to work in mathematics he should study literature'. But I think Poincaré's concerns are embedded within a larger effort at revising our historical legacy. Since Aristotle, mathematics was thought to investigate eternal truths and Arts to make (or recreate) novel forms. This way of demarcating mathematics and Arts becomes impossible once Poincaré realises that mathematical objects, too, are made and not given. Mathematics, Poincaré remarks and Goodman later argues, is equally an activity of making - that of making theoretical artefacts, or, of shaping the symbolic systems in the process of understanding. So, the difference between Arts and Mathematics do not consist in either the feature of being a artefact or that of being symbolic. One has to seek the difference, on the one hand in the ways of being made rather than in the *fact* of being made, and on the other, in the kind of symbol systems each are made up of rather than the fact of them being symbolic. The difference between Arts and Science is not ontologically founded as difference between objects, but it is semiotically justified as difference in the use of symbols. Pictures are rather samples of a lot of properties, scientific language rather refers in a non-ambiguous manner[7]. (Voir note 7 à la page suivante.)

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6 Maitland translates: "due to some conformity"

And, in fact, in a last step, Poincaré suggests in the given context of the above quoted passage, that one should attempt to find the solution of the "I know not what which makes the harmony of the demonstration" in regularities established by "happy innovations of language" :

> Mathematics is the art of giving the same name to different things. [...] When language has been well chosen, one is astonished to find that all demonstrations made for a known object apply immediately to many new objects: nothing requires to be changed, not even the terms, since the names have become the same. [SM, 34 (29)]

Language is also constitutive for mathematical thinking whereas logic is derivative for it. In this sense Poincaré is with regard to Brouwer a semi-intuitionist.

Because Poincaré counts "make no sacrifice of rigour" [SM, 33 (29)], æsthetics in mathematical reasoning cannot be the result of sensible intuition or arbitrary imagination. On the contrary, the conformity of harmony to necessities of our mind, as expressed in the first quotation above, confirms that the working basis of mathematical æsthetics is *pure intuition* as we have defined it, namely as the awareness of the mastery of an order schema such as "uniformity of convergence", complete induction, continuum- or group-constructions. In its semantic function pure intuition is directed from objects to symbols (schemes) whereas denotation goes from symbols to objects. Intuition has a picture-function, denotation a descriptive function. In this loosely Goodman-like sense, mathematical æsthetics can be redefined as a matter of density of representation-properties. Or to say it simpler : the more order-structures mathematical objects represent, the more their æsthetic value grow.

For Poincaré, group-theoretic and axiomatic foundations of Geometry are alternative. His review of Hilbert's *Foundation of Geometry* clearly illustrates how closely the French mathematician's critic on logical inferences is related to his critic on the axiomatic method. This follows obviously from Poincaré's negative solution of the connection-problem, as pointed out earlier. Hence, no wonder, that compared with the group theoretic approach of geometry, Hilbert's foundations is considered by Poincaré as æstheticaly unsatisfactory and as epistemologically incomplete. The former happens because different geometries, as non Archimedian, non Arguesian or non Pascalian Geometry, may be "ordered" in a logical sense insofar as they are the results of a choice between several formal axioms but Poincaré's criterion for harmony, their common mathematical group structure, is quite invisible. Hilbert's approach is

7 On this point I benefited from a helpful discussion with Narahari Rao on the occasion of the common organisation in Nancy of a conference on Goodman's work.

incomplete insofar as the logical exposition reveals nothing about what Poincaré calls the psychological genesis of the foundational enterprise. That means it completely neglects the perspective of understanding [Poincaré 1902, 112sq]. On the contrary, it is well known that, in his *Foundations of Geometry*, Poincaré focuses exactly on this point — I think with some success [Heinzmann 1998] — and since so early as 1880 the definition of Geometry itself is closely connected with the group concept. In a manuscript published only last year in the *Publications of the Archives Henri Poincaré* he writes:

> What is in fact Geometry ? It is the study of a *group of operations* formed by displacements applied to a figure without to deform it. In Euclidean Geometry this group reduces to *rotations and translations*. In the pseudo-Geometry of Lobatchevski it is more complicated [Gray/Walter 1997, 35].

To be sure, it may be a historically right and hence a commonly admitted observation, that in time of theory-formation the mathematical style is quite different from times were we dispose a whole theory seemingly described by an axiomatic system[8]. But once again, this observation concerns the evolution of non-rigorous procedures, as intuitive imagination or example-collection, to a rigorous proof and by no means non-logical rigour. Surely, Poincaré was writing about both problems[9] whereas my proper aim is to find out Poincaré's interpretation of non-logical exactness. Therefore, on the methodological level we must, in a systematic perspective, to waive of axiomatisability as criterion for fully developed theory and I am not even convinced that errors in factually given logical proofs are proportionally more seldom as in non-logical proofs. Indeed, for the latter the criterion is fixed, for the former we are seeking it. But by formulating this I am not at all contending that there exists only one general concept of non-logical rigour. History even may show that there are different sectorial standards. So, as Dieudonné point out it, "the proofs of algebra [having come long before "abstract" algebra] have never been challenged [and] around 1880 the canon of "Weierstrassian rigour" in classical analysis gained with acceptance among analysts, have never been modified." [Dieudonné 1989, 15] And in fact, Poincaré classifies in his *Value of Science* Weierstraß among the *geometers* : he, naturally continues to be an analyst but one guided by pure intuition [VS, 15-25 (27-39)].

Now, in order to explaining how Poincaré's, so to speak, *mathematical logic* functions in his mathematical reasoning, we ought to notice, that, according to Poincaré, æsthetic reasoning depends not only on the criterion "harmo-

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8 Cf. e.g. Eymard 1996, 25sq. and Volkert 1994, 125.

9 Cf. e.g. SM 123sq. (130sq.) for a discussion about the evolution of rigour.

ny by order" but even on an additional condition. It lies in the "surprising" order-character. This emerges from always the same passage of *Science and Method* :

> Elegance may result from the feeling of surprise caused by the unlooked-for occurrence together of objects not habitually associated. [...] In order to obtain a result having any real value, it is not enough to grind out calculations, or to have a machine for putting things in order : it is not order only, but unexpected order, that has a value. [SM, 31 sq. (26sq.)]

In closing so his explicit characterisation of creative mathematical reasoning, Poincaré's quoted dilemma concerning the compatibility of deductive rigour and extensive knowledge is in several respects very similar to a dilemma formulated by Peirce nine years sooner :

> It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in is nature [...] while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. [Peirce 1933/58, 3.363 (1885)]

Peirce's solution of the paradox is based on a distinction between corollarial reasoning and theorematic reasoning [Heinzmann 1994, 1995]. The deductive process leading to a corollarial logical reasoning corresponds to formal inferences while reasoning in mathematics is generally theorematic. A theorematic proof of a proposition is not given by a finite column of propositions, but requires a diagrammatic interpretation of the premise yielding to a procedure of elimination of possible interpretations, and so to a modal interpretation of reasoning. Iconic diagrams which can be drawn, as figures in geometry, or written, as algebraic formula, have in the Peircian theory of mathematical inferences the same function as the mathematical architecture suggested by Poincaré: both determine the possible perspectives under which conclusions can be reached from hypotheses. If, all perspectives considered, the interpretation of the diagram remains invariant with regard to experimentation, the deduction is corollarial, otherwise, theorematic.

Finally, in what sense elements from Peirce's semiotics and Goodman's æsthetics contribute something to Poincaré's aim of mathematical reasoning besides logical inference ? This point I will discuss now by analysing a concrete example of mathematical reasoning in Poincaré's work. My attention is centred upon what Poincaré calls Analysis situs. The reason for this is simple : according to the remarks in the analysis of his own scientific work, Poincaré sees in *Analysis situs* the central means for all mathematical fields where it was engaged[10]. Defining it [Poincaré 1892, 189sq.] as the science of classification

10 Cf. Poincaré 1921, 323. For further discussion cf. Dieudonné 1989 and Volkert 1994.

of closed surfaces, called later manifolds, with respect to continuous de formations, it requires geometric intuition concerning the qualitative property of a n-dimensional manifold, arithmetic intuition insofar as he introduced computing with the topological object "manifold" and, insofar as the strongest classification-criterion is the fundamental group, one needs algebraic intuition, too [DP, 135]. Through this last group theoretical determination Poincaré can extend his epistemological justification of geometry to topology. I quote :

> Geometry is first of all [...] any analytic study of a group. Therefore, nothing hinders to approach other analogous and more general groups. [...] [The group of homeomorphism] is one of the most general to be imaginable. The science whose subject is the study of this group and some other analogous, is called Analysis situs. [Poincaré 1895, 193, 198]

Still now, I have deprived you of the answer concerning the intuitive character of groups. As late as 1899 Poincaré does not seem to know or, at any rate, does not use the abstract group-definition. His concept of a group — often he uses only the properties of a groupoid — is always related to concrete representations of groups of permutation, transformation or substitution [Scholz 1980, 313-315] ; [Volkert 1994, 75]. So, ordinary Geometry is identified with the group of orthogonal substitution, Projective Geometry with the group of linear substitution and *Analysis situs* with the group of homeomorphic substitution in Poincaré's sense [Poincaré 1890, 153]. Only this restriction allows him in fact to consider the group concept as intuitive that is as awareness of a mastery occasioned by concrete samples. Therefore, it is no need for Poincaré to *proof* that a topological object as the fundamental group defined by a "substitution"-language form, in fact, a group, because substitutions are, so to speak, natural samples of groups [Volkert 1994, 74-75].

Now, in his most general definitions Poincaré characterises *Analysis situs* as "purely qualitative geometry" and draws from this the consequent conclusion to replace for qualitative investigations on n-dimensional spaces geometrical means by analogous topological means [Poincaré 1921, 286, 323]. So he obtains by the way a passage from the improper, that is sensible geometrical intuition concerning figures to the proper geometrical, that is topological intuition concerning n-manifolds [Poincaré 1895, 194] ; [DP, 134-135]. But what kind of knowledge concerning them is calling for intuition ? To see this clearer let us consider an illustration of deduction in the field of manifolds.

In his article of 1895, Poincaré gives several procedures to construct the topological subject "manifold" [G. Scholz 1989, 287 sq.]. But to facilitate his study, he uses very often representations of manifolds, and especially polyhedra [Poincaré 1895, 229] ; [Hereman 1997, 245]. A polyhedron P is a closed manifold V of dimension n subdivided into manifolds $v_n, v_{n-1}, \ldots, v_0$ of dimension n, n-1, .0 so that the boundaries of the α_n manifolds vn are formed by a finite number α_{n-1} of manifolds v_{n-1}, etc. Poincaré continues as follows :

> The figure formed by all these manifolds is called polyhedron; the reason is that the analogy with ordinary polyhedra is very evident. An ordinary polyhedron is in fact a closed 2-dimensional manifold V which is subdivided into a certain number of manifolds v2, being the sides. The sides have as boundaries a certain number of manifolds v1 being the edges and which possess themselves as boundaries a certain number of manifolds v0 called vertexes. [Poincaré 1895, 271]

Assuming now that
v_n has been divided into $a_1^n, \ldots, a_{\alpha_n}^n$ where the a_i^n design the different manifolds v_n
v_{n-1} has been divided into $a_1^{n-1}, \ldots, a_{\alpha_{n-1}}^{n-1}$ where the a_i^{n-1} design the different manifolds v_{n-1} etc.
Poincaré considers the linear combination $\sum \lambda_i a_i^q$ with integer coefficients λ_i again as a manifold and, as Dieudonné pointed out, he used it in a purely algebraic manner [Dieudonné 1989, 30]. We now can formulate the example of deduction token from a rich documentation in Alain Herreman's doctor thesis about the *Semiotic History of the Concept of Homology* : [Herreman 1996, 114sq.]
What are the conditions, asks Poincaré, for the manifold $\sum \lambda_i a_i^q$ to be closed ?
In answering this question Poincaré extends the geometrical concept of boundary to the algebraic used linear combinations. To find the boundary of $\sum \lambda_i a_i^q$ "it is sufficient to replace a_i^q by its boundary" [Poincaré 1899, 299]. The boundary of a linear combination is given by the linear combination $\sum\sum \lambda_i \varepsilon_{i,j}^q a_i^{q-1}$ of the boundaries of a_i^q. And Poincaré **deduces:**
"In order that the manifold $\sum \lambda_i a_i^q$ is closed it is thereby sufficient to write the identity $\sum\sum \lambda_i \varepsilon_{i,j}^q a_i^{q-1} = 0$ " [Poincaré 1899, 299]

What happens ? The geometrical image of the closure of an ordinary polyhedron with respect to its edges is a sample for the annulment of the arithmetical sum of their length and this arithmetical sum is again a sample for the algebraic expression of the closure of linear combinations with entires as coefficients, itself interpreted as topological object. Deduction enclosed here the non logical switch between different semiotic levels considered in part on the same level of notation. Mathematical symbols can be read with respect to different contents. There are iconic diagrams in the Peircian sense. The semiotic ambiguity involved cannot be checked on the level of notation but requires the acquisition of a practice. Such procedures concern, to be sure, not mathematical reasoning in its totality. Nevertheless they may be predominant in some fields and constitute then a lack of logical rigour.

Before ending, I will mention briefly one other point concerning Analysis situs. The qualitative geometry has for its object to study the relation of position of the different elements of a figure", after eliminating the "measure of magnitudes" [SM, 43 (39/40), SH, 33 (60), even Poincaré 1895, 194] but, naturally, it eliminates not magnitudes themselves : the concept of homology presupposes the introduction of arithmetical operation. What signifies also the term "purely qualitative" ? Understandably, the most serious supposition seems to be that it has to do with the idea of continuous deformation of manifolds by means of a "point-transformation" [VS, 40 (59)] ;[DP, 134]. Although near the concept of homeomorphy, defined as special substitutions on manifolds, the notion of continuous deformation, so Poincaré's mathematical interpreters agree, has never been precisely defined. Hence the intuitive use of deformation arguments seems to be responsible of several proof-gaps [Dieudonné 1989, 18, 25] ; [Volkert 1994, 121]. Should one also say that the arguments are not rigorous, but based on that delusive intuition which characterises a beginning of theory-formation ? Or should one say that the arguments constitute exactly a case of mathematical æsthetics without analytical rigour insofar as their expression shows (represents) as a sample the possibility of different algebraic notations of which no one may alone contain all senses of the geometrical content of "deformation" ? The weight of this last interpretation is increased when we take into account a general aspect of Poincaré's articles on Analysis situs. Many results are discussed in the context of examples, as polyhedra, introduced in a famous paragraph entitled "geometric representation". Indeed, according to Klaus Volkert, these examples should not be called examples because they don't illustrate an existing theory [Volkert 1994, 88]. The way things go is inverse : in reality, the examples are samples for theory formation. But if this is right, there is no need for a deductive proof because the samples are justified just by denotation.

Should one call the sample procedure rigorous ? This looks like a similar question asked by Poincaré and mentioned above : is the new logic analytic ? Indeed there is a important difference : As principles of logical inference, we have highly developed laws, but there are no such laws available of sample inference. All of what we dispose is the æsthetic criterion of high density, that is of the possibility to switch between different contents. But this evidently depends on our deep understanding.

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