MICHEL HACQUE

Structure of Right Artinian Algebras over a Field

Publications du Département de Mathématiques de Lyon, 1994, fascicule 1
« Structure of right artinian algebras over a field », , p. 1-121
<http://www.numdam.org/item?id=PDML_1994___1_A1_0>

STRUCTURE OF RIGHT ARTINIAN ALGEBRAS OVER A FIELD

Michel HACQUE

Institut de Mathématiques et Informatique de l'I.S.M
Université Claude Bernard LYON I
69622 VILLEURBANNE France

INTRODUCTION

This present work is an Appendix to the previous paper : "Some new invariants for right Artinian rings" [10], in which we give a "Construction Theorem" for right Artinian rings, that is a Theorem which gives the Description of a Systematic Method of Construction of any right Artinian ring, by means of a finite number of Fundamental Constructions, which are of two different kinds.

Indeed, in the case of algebras over a field F, the general results for right Artinian rings give *a more precise* Description of a Systematic Method of Construction of any right Artinian F-algebra, which constitutes, in some sense, a "Structure Theorem" for right Artinian algebras over a field.

Thus, our main objective is to give the complete proofs of some results given in [10] without proof, in order to achieve the presentation of an adaptation of the Hochschild Cohomology for Algebras.

Then, by means of the notion of F-*"Completely structured vertex set"*, we obtain the Theorem 7-4, which constitutes a *"Structure Theorem"* for *right Artinian algebras over a field.*

After this FIRST PART : PROOF OF THE STRUCTURE THEOREM (Paragraphs 1, 2, 3, 4, 5, 6 and 7), in the SECOND PART : ILLUSTRATION OF THE STRUCTURE THEOREM (Paragraph 8), we give a great number of Examples and Applications, in order to show how the notions and the theorical results of the FIRST PART are applicable in concrete situations.

FIRST PART :

PROOF OF THE STRUCTURE THEOREM

1. CANONICAL DECOMPOSITION OF THE RIGHT SOCLE.

All algebras considered are associative algebras over a *field* F.

For any F-algebra A, let J = J(A) be the Jacobson Radical of A and let S = S(A) be the *right Socle* of A, that is the sum of all minimal right ideals of A.

It is well known (See, for instance [4], [5] or [1] Ex. 9, p. 58) that S = S(A) is a semisimple right A-module and that the isotypical (or homogeneous) components of S are also two-sided ideals of A, called the right feet (pieds) of S or of A.

These notions have been introduced in [4] and used in [3] by J. Dieudonné for the study of the Structure of Hypercomplex systems.

It is obvious that the Jacobson Radical J = J(A) is very important in the study of algebras, as for instance in the statement of the Wedderburn-Dickson-Malcev Principal Theorem (See for instance [17] p. 209).

Although the Jacobson Radical J = J(A) is more frequently used that the right Socle S = S(A), as in the previous work [10], one of our aims is to show that the "Canonical Decomposition" of the right Socle gives one of the main tools for the proof of our Structure Theorem for right Artinian algebras over a field.

For any F-algebra B, a two-sided B-module (See for instance [2] p. 167) or simply a B-*bimodule* is an abelian group M on which B operates on the left and on the right in such a way that (bm)b' = b(mb') and $\alpha m = m\alpha$ for all $m \in M$, $b \in B$, $b' \in B$ and $\alpha \in F$. For example, any two-sided ideal of a F-algebra B is a B-*bimodule.*

With this Definition, it is immediate that for each statement of [10] (Definition, Lemma, Proposition, Theorem, Corollary and Remarks), it is possible to obtain the analogous statement by the replacement of "ring" by "algebra over a field F" or "F-algebra", of "ring homomorphism" by "F-algebra homomorphism", of "bimodule" by "F-algebra bimodule", of "ring extension" by "F-algebra extension", of "general ring extension" by "general F-algebra extension", of "singular ring extension" by "singular F-algebra extension", etc...

LEMMA 1-1 - *For any* F-*algebra* A, *there exists a* Canonical Decomposition:

$$S = S(A) = M(A) \oplus N(A) = M \oplus N$$

in which :

(a) *If* $l(S)$ *is the left annihilator of* S, *the* zero square *proper two-sided ideal :*

$$M = M(A) = S(A) \cap l[S(A)] = S \cap l(S)$$

called the "zero-socle" of A, *is the direct sum of the family of nilpotent (or zero square) feet of* S *or of* A.

(b) *The* idempotent *two-sided ideal :*

$$N = N(A) = [S(A)]^2 = S^2$$

called the "one-socle" of A, *is the direct sum of the family of idempotent feet of* S *or of* A.

PROOF - This is a particular case of the Lemma 1-1 of [10].

LEMMA 1-2 - *For any* F-*algebra* B *and any two-sided ideal* T *of* B, *which verifies :* $T \subset S(B)$, *the following conditions are equivalent :*

(a) *The two-sided ideal* T *of* B *is the direct sum of a family of* idempotent feet *of* B.

(b) *There exists a two-sided ideal* T' *of* B *such that :*

$$T' \subset S(B) \qquad \textit{and} \qquad T = T'^2$$

(c) *The two-sided ideal* T *of* B *is* idempotent : $T = T^2$.

(d) *The two-sided ideal* T *of* B *is* "right strongly idempotent" *in the sense of* [7] , *that is : for every* $\alpha \in T$, *there exists* $\beta \in T$, *such that :* $\alpha = \alpha\beta$.

(e) *The left annihilator* $l(T))$ *of* T *verifies :* $T \cap l(T) = (0)$.

(f) *Every right ideal* I *of* B *verifies :* $IT = I \cap T$.

(g) *The left* B-*module* B/T *is flat.*

(h) *The right* B-*module* T *is projective and* T *is the direct sum of a family of feet of* B.

Moreover, under these equivalent conditions, then : $T \subset N(B)$.

PROOF - This is a particular case of the Lemma 1-2 of [10].

DEFINITION 1-3 - *For any* F-*algebra* B, *let* $\mathscr{C}(B)$ *be the set of* proper *two-sided ideals* T *of* B, *which verify the equivalent conditions of the Lemma 1-2.*

PROPOSITION 1-4 - *Any algebra* A *over a field* F *verifies :*

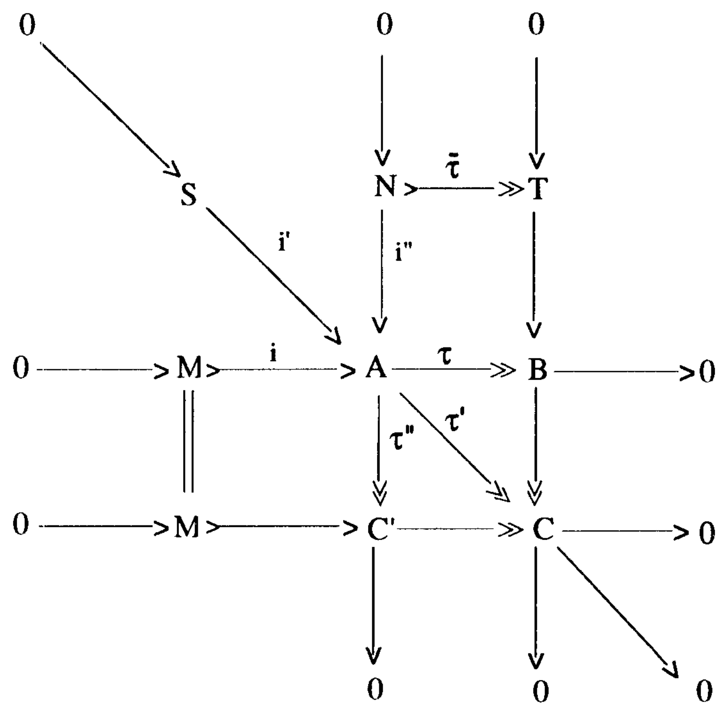
(a) *The* F*-algebra* A *is a semisimple Artinian* F*-algebra if and only if :*

$$S = S(A) = A$$

(b) *If the* F*-algebra* A *is not a semisimple Artinian* F*-algebra, that is if* S = S(A) *is a* proper *two-sided ideal of* A, *which determines the proper two-sided ideals* M = M(A) *and* N = N(A), *the factor* F*-algebras :*

$$B = A/M \qquad C' = A/N \qquad C = A/S$$

appear in the following commutative and "exact" diagram :



in which the surjective F*-algebra epimorphism :*

$$\tau : A \twoheadrightarrow B$$

induces an isomorphism of multiplicative A*-bimodules :*

$$\bar{\tau} : N \rightarrowtail\!\!\twoheadrightarrow T$$

from the idempotent proper two-sided ideal N = N(A) *of* A *onto an idempotent proper two-sided ideal* T *of* B, *which verifies :*

$$T \subset N(B) \qquad \textit{and} \qquad T \in \mathcal{C}(B)$$

PROOF - This is a particular case of the Proposition 1-4 of [10].

REMARKS 1-5 -

(a) With the classical notion of *"algebra extension"* (See for instance [16] p. 284) and whenever : $S(A) \neq A$, if $M = M(A) \neq (0)$, the Proposition 1-4 exhibits in particular the *singular* F-*algebra extension :*

$$(\tau) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} A \overset{\tau}{\twoheadrightarrow} B \longrightarrow 0$$

in which $M = M(A)$ is a non null *zero square* proper two-sided ideal of A, and if : $M = M(A) = (0)$, then the *general* F-*algebra extension :*

$$(\tau') \qquad 0 \longrightarrow S \overset{i'}{\rightarrowtail} A \overset{\tau'}{\twoheadrightarrow} C \longrightarrow 0$$

coincides with the *general* F-*algebra extension :*

$$(\tau'') \qquad 0 \longrightarrow N \overset{i''}{\rightarrowtail} A \overset{\tau''}{\twoheadrightarrow} C' \longrightarrow 0$$

in which $C = C'$ and $S = N = N(A)$ is an *idempotent* proper two-sided ideal of A.

(b) Whenever A is a right Artinian F-algebra which is not a semisimple Artinian F-algebra and which verifies the condition :

$$M = M(A) = (0)$$

an adaptation of the Theorem 2-13 of [10] gives a ***"complete characterization"*** of the ***"one link"*** $(\tau') = (\tau'')$ determined by the *general* F-*algebra extension :*

$$(\tau') \qquad 0 \longrightarrow S \overset{i'}{\rightarrowtail} A \overset{\tau'}{\twoheadrightarrow} C \longrightarrow 0$$

and in particular a description of the Structure of the *right Artinian* F-*algebra* A, which is, in this case, a (right) *almost semisimple right Artinian* F-*algebra,* in the sense of the Definition 2-1 of [10].

(c) Whenever A is a right Artinian F-algebra which is not a semisimple Artinian F-algebra and which verifies the condition :

$$M = M(A) \neq (0)$$

an adaptation of the Theorem 3-14 of [10] gives a characterization of the ***"zero-link"*** (τ) determined by the *singular* F-*algebra extension :*

$$(\tau) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} A \overset{\tau}{\twoheadrightarrow} B \longrightarrow 0$$

and in particular a description of the Structure of the *right Artinian* F-*algebra* A, which is not a (right) *almost semisimple right Artinian* F-*algebra,* by means of a T-***essential*** *singular* F-*algebra extension :*

$$(\tau, T) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} (A,N) \overset{\tau}{\twoheadrightarrow} (B, T) \longrightarrow 0$$

characterized by an unique T-***essential*** *singular class :*

$$\xi \in Ext_e(B, T, M)$$

which gives the *"characterization" :*

$$(A, N) = (B, T, M, \xi)$$

which implies :

$$S = \mathbf{S}(A) = \mathbf{M}(A) \oplus N(A) = M \oplus N$$

In order to obtain a ***"complete characterization"*** of the ***"zero-link"*** (τ), that is in order *"to calculate the space" :*

$$\mathrm{Ext}_e(B, T, M)$$

it will be sufficient to give a complete proof of the Lemma 3-6 and of the Theorem 3-10 of [10], that is of the Lemma 3-6 and of the Theorem 3-10 of this present work.

2. THE NOTION OF ALMOST SEMISIMPLE ALGEBRA.

An old problem, set in 1964 by A.W. Goldie (See [6] p. 268), was : « ... the determination of artinian rings with a zero singular ideal ».

In order to give a solution to this old problem (See [8] and [9]) it has been very useful to introduce the notion of (right) almost semisimple ring (See Def. 3-1 of [9]).

DEFINITION 2-1 - A *ring (or an algebra)* A *is a (right) almost semisimple ring (or an almost semisimple algebra) if its right Socle* S = **S**(A) *is left faithful, that is verifies :* $l(S) = (0)$.

Of course, any semisimple Artinian ring is a (right) almost semisimple ring.

More generally, the Theorem 3-3 of [9] constitutes a Structure Theorem for (right) almost semisimple rings.

In particular, it will be very useful to have the Theorem 4-3 of [9], which constitutes a Structure Theorem for (right) almost semisimple right Artinian rings (since they coincide with right Artinian rings with a zero right singular ideal, that is with *right non singular* right Artinian rings).

LEMMA 2-2 - *For any right Artinian algebra* A, *then :*

(a) *The algebra* A *is a semisimple Artinian algebra if and only if :*

$$S = \mathbf{S}(A) = A$$

(b) *The algebra* A *is a (right) almost semisimple right Artinian algebra if and only if :*

$$M = \mathbf{M}(A) = (0)$$

that is if and only if A *is a* right non singular *right Artinian algebra, that is a right Artinian algebra with a* zero right singular ideal [$Z_r(A) = l(S) = (0)$], *and under these equivalent conditions, then :*

$$S = S(A) = N(A) = N$$

is a non null idempotent two-sided ideal.

PROOF - The Proposition 1-4 implies the part (a).

In a right Artinian algebra A, since the non null right Socle $S = S(A)$ is a minimum essential right ideal, its left annihilator $l(S)$ coincides with the right singular ideal $Z_r(A)$ of A and the conditions :

$$l(S) = (0) \qquad \text{and} \qquad M = S \cap l(S) = (0)$$

are equivalent. This implies immediately the part (b).

NOTATIONS 2-3 -

(a) Let $\mathfrak{K}$ be the class of *right Artinian rings,* let $\mathfrak{K}_a$ be the class of (right) *almost semisimple right Artinian rings,* let $\mathfrak{K}_0$ be the class of *semisimple Artinian ring* and let $\mathcal{K}$ be the class of *skewfields*, which verify the relation :

$$\mathcal{K} \subset \mathfrak{K}_0 \subset \mathfrak{K}_a \subset \mathfrak{K}$$

(b) For any field F, let $\mathfrak{K}(F)$ be the class of right Artinian F-algebras, let $\mathfrak{K}_a(F)$ be the class of almost semisimple right Artinian F-algebras, let $\mathfrak{K}_0(F)$ be the class of semisimple Artinian F-algebras and let $\mathcal{K}(F)$ be the class of F-skewfields, that is of skewfields which are F-algebras, which verify the relation :

$$\mathcal{K}(F) \subset \mathfrak{K}_0(F) \subset \mathfrak{K}_a(F) \subset \mathfrak{K}(F)$$

DEFINITION 2-4 - *A* F-*"concrete vertex set"* $\widetilde{\Lambda}$ *is an object of the form :*

$$\widetilde{\Lambda} = [\Lambda\,;\,(K_\lambda), (p_\lambda)] = [\Lambda\,;\,(V_\lambda)] = [\Lambda\,;\,(V_\lambda^*)]$$

characterized by a finite and non empty set Λ *(called the underlying abstract vertex set) and by one of the three equivalent previous data, connected by the conditions :*

(a) *In the family* $(K_\lambda) = (K_\lambda)_{\lambda \in \Lambda}$, *each* K_λ *is a* F-*skewfield :*

$$K_\lambda \in \mathcal{K}(F)$$

(b) *In the family* $(p_\lambda) = (p_\lambda)_{\lambda \in \Lambda}$, *each* p_λ *is a non null integer :*

$$p_\lambda \in \mathbb{N}^*$$

(c) *In the family* $(V_\lambda) = (V_\lambda)_{\lambda \in \Lambda}$, *each* V_λ *is a non null and finite dimensional right* K_λ*-vector space of dimension* p_λ.

(d) *In the family* $(V^*_\lambda) = (V^*_\lambda)_{\lambda\in\Lambda}$, each V^*_λ *is a non null and finite dimensional left* K_λ*-vector space of dimension* p_λ , *which may be considered as the dual space* $V^*_\lambda = \mathcal{L}_{K_\lambda}(V_\lambda, K_\lambda)$ *of the right* K_λ*-vector space* V_λ.

Of course, for this kind of objects the notion of isomorphism is obvious and as in the case of *quivers,* in the sense of the Definition p. 96 of [17], it is convenient (and harmless) to call two F-*"concrete vertex sets"* $\widetilde{\Lambda}$ and $\widetilde{\Lambda}'$ **equal when they are only isomorphic.**

LEMMA 2-5 - *Any* F-"concrete vertex set" *of the form :*

$$\widetilde{\Lambda} = [\Lambda ; (K_\lambda), (p_\lambda)] = [\Lambda ; (V_\lambda)] = [\Lambda ; (V^*_\lambda)]$$

determines a semisimple Artinian F-algebra :

$$R = \mathbf{R}(\widetilde{\Lambda}) = \prod_{\lambda\in\Lambda} R^\lambda$$

characterized by the family $(R^\lambda)_{\lambda\in\Lambda}$ *of* simple Artinian F-algebras R^λ *having the* "realizations" :

$$R^\lambda = \mathcal{L}(V_\lambda) = M_{p_\lambda}(K_\lambda) = [\mathcal{L}(V^*_\lambda)]^\circ$$

which imply that the right K_λ*-vector space* V_λ *is a* $(R^\lambda\text{-}K_\lambda)$*-bimodule* $V_\lambda = {}_{R^\lambda}(V_\lambda)_{K_\lambda}$ *and also a* $(R\text{-}K_\lambda)$-bimodule $V_\lambda = {}_R(V_\lambda)_{K_\lambda}$, *and that the left* K_λ*-vector space* V^*_λ , *dual of* V_λ, *is a* $(K_\lambda\text{-}R^\lambda)$*-bimodule* $V^*_\lambda = {}_{K_\lambda}(V^*_\lambda)_{R^\lambda}$ *and also a* $(K_\lambda\text{-}R)$*-bimodule* $V^*_\lambda = {}_{K_\lambda}(V^*_\lambda)_R$, *in such a way that the families :*

$$(V_\lambda)_{\lambda\in\Lambda} \qquad \textit{and} \qquad (V^*_\lambda)_{\lambda\in\Lambda}$$

are respectively the set of isomorphism classes of simple left R*-modules* $V_\lambda = {}_R(V_\lambda)$ *and the set of isomorphism classes of simple right* R*-modules* $V^*_\lambda = (V^*_\lambda)_R$ *of* type $\lambda \in \Lambda$, *identified with the underlying* "abstract vertex set" $\Lambda = V(R)$ *of the* **quiver** :

$$\Gamma(R) = (V(R), \varnothing)$$

of the semisimple Artinian F*-algebra* $R = \mathbf{R}(\widetilde{\Lambda})$.

PROOF - The Wedderburn-Artin Structure Theorem for semisimple Artinian rings or algebras implies immediately the proof.

LEMMA 2-6 - *Any semisimple Artinian* F-*algebra :*

$$R \in \mathfrak{R}_0(F)$$

determines an unique F-*"concrete vertex set" :*

$$\widetilde{\Lambda} = \widetilde{\mathbf{\Lambda}}(R)$$

such that :

$$R = \mathbf{R}(\widetilde{\Lambda})$$

"up to an F-*isomorphism".*

PROOF - With the previous notations, this Lemma is a partial translation of the Wedderburn-Artin Structure Theorem for semisimple Artinian rings or algebras.

COROLLARY 2-7 - [Technical version of the WEDDERBURN-ARTIN STRUCTURE THEOREM] -

With the previous notations, the Structure of any semisimple Artinian F-*algebra :*

$$R \in \mathfrak{R}_0(F)$$

is characterized, "up to an F-*isomorphism", by a (or by its)* F-***"concrete vertex set"*** *:*

$$\widetilde{\Lambda} = [\Lambda \,;\, (K_\lambda), (p_\lambda)] = [\Lambda \,;\, (V_\lambda)] = [\Lambda \,;\, (V^*_\lambda)]$$

such that :

$$R = \mathbf{R}(\widetilde{\Lambda})$$

and therefore :

$$\widetilde{\Lambda} = \widetilde{\mathbf{\Lambda}}(R)$$

In other words, the correspondences :

$$\widetilde{\Lambda} \longmapsto \mathbf{R}(\widetilde{\Lambda}) \qquad \textit{and} \qquad R \longmapsto \widetilde{\mathbf{\Lambda}}(R)$$

are one-to-one reciprocal correspondences between the F-***"concrete vertex sets"*** $\widetilde{\Lambda}$ *and the isomorphism classes* [R] *of* semisimple Artinian F-algebras $R \in \mathfrak{R}_0(F)$.

PROOF - This is a complete and technical translation of the Wedderburn-Artin Structure Theorem for semisimple Artinian rings or algebras.

DEFINITION 2-8 - *A pair* (H, G) *of semisimple Artinian* F-*algebras :*

$$H \in \mathfrak{K}_0(F) \qquad \textit{and} \qquad G \in \mathfrak{K}_0(F)$$

verify the relation : <<G *dominates* H>> *or* << H *is dominated by* G>> *noted* :

$$H \lhd G$$

if and only if G *has a* F-*concrete vertex set of the form :*

$$\widetilde{\Lambda}(G) = \widetilde{\Lambda} = [\Lambda\,;(K_\lambda),(p_\lambda)] = [\Lambda\,;(V_\lambda)] = [\Lambda\,;(V_\lambda^*)]$$

and there exists a non empty subset :

$$\Lambda' \subset \Lambda$$

such H *has a* F-*concrete vertex set of the form :*

$$\widetilde{\Lambda}(H) = \widetilde{\Lambda}' = [\Lambda'\,;(K_\lambda),(q_\lambda)] = [\Lambda'\,;(U_\lambda)] = [\Lambda'\,;(U_\lambda^*)]$$

or a "generalized F-*concrete vertex set" of the form :*

$$\underline{\widetilde{\Lambda}}(H) = \underline{\widetilde{\Lambda}}' = [\Lambda\,;(K_\lambda),(q_\lambda)] = [\Lambda\,;(U_\lambda)] = [\Lambda\,;(U_\lambda^*)]$$

in which : $q_\lambda = 0$, $U_\lambda = (0)$ *and* $U_\lambda^* = (0)$ *for all* $\lambda \in (\Lambda - \Lambda')$, *in such a way that the semisimple Artinian* F-*algebra* H *has the realization :*

$$H = \mathbf{R}(\widetilde{\Lambda}') = \prod_{\lambda\in\Lambda'} \mathcal{L}(U_\lambda) = \prod_{\lambda\in\Lambda'} M_{q_\lambda}(K_\lambda) = \prod_{\lambda\in\Lambda'} [\mathcal{L}(U_\lambda^*)]^\circ$$

and the "generalized realization" :

$$H = \mathbf{R}(\underline{\widetilde{\Lambda}}') = \prod_{\lambda\in\Lambda} \mathcal{L}(U_\lambda) = \prod_{\lambda\in\Lambda} M_{q_\lambda}(K_\lambda) = \prod_{\lambda\in\Lambda} [\mathcal{L}(U_\lambda^*)]^\circ$$

LEMMA 2-9 - *With the previous notations, any pair* (H, G) *of semisimple Artinian* F-*algebras, subject to the condition :*

$$H \lhd G$$

determines the canonical (H-G)-bimodule :

$$L = {}_H L_G = \prod_{\lambda \in \Lambda} \mathcal{L}(V_\lambda, U_\lambda) = \prod_{\lambda \in \Lambda} M_{p_\lambda, q_\lambda}(K_\lambda)$$

in which $\mathcal{L}(V_\lambda, U_\lambda)$ *is the abelian group of* K_λ*-linear maps from* V_λ *into* U_λ *and in which* $M_{p_\lambda, q_\lambda}(K_\lambda)$ *is the abelian group of* $(p_\lambda \times q_\lambda)$*-matrices with coefficients in the* F*-skewfield* K_λ, *and such that for every element :*

$$l = (l_\lambda) \in L = \prod_{\lambda \in \Lambda} \mathcal{L}(V_\lambda, U_\lambda)$$

if

$$h = (h_\lambda) \in H = \prod_{\lambda \in \Lambda} \mathcal{L}(U_\lambda) \qquad \textit{and} \qquad g = (g_\lambda) \in G = \prod_{\lambda \in \Lambda} \mathcal{L}(V_\lambda)$$

then, the conditions :

$$hl = (h_\lambda l_\lambda) \qquad \textit{and} \qquad lg = (l_\lambda g_\lambda)$$

characterize the structure of (H-G)*-bimodule.*

PROOF - This is obvious.

LEMMA 2-10 - *With the previous notations, for any semisimple Artinian* F*-algebra* G, *with the* F*-concrete vertex set :*

$$\widetilde{\Lambda}(G) = \widetilde{\Lambda}$$

the semisimple Artinian F*-algebras* H, *subject to the condition :*

$$H \lhd G$$

are characterized by the families of integers :

$$(q_\lambda) = (q_\lambda)_{\lambda \in \Lambda}$$

with a non empty support :

$$\Lambda' = \{\lambda \in \Lambda \; ; q_\lambda \neq 0\} \neq \varnothing$$

in such a way that :

(a) *The* canonical (H-G)-bimodule $L = {}_H L_G$ *is a non null finitely generated right* G*-module with an isotypical decomposition of the form :*

$$L = \bigoplus_{\lambda \in \Lambda} L^\lambda \simeq \prod_{\lambda \in \Lambda} L^\lambda$$

such that q_λ *is the length of the isotypical component* L^λ *of type* $\lambda \in \Lambda$, *characterized by the condition :*

$$L^\lambda = \mathcal{L}(V_\lambda, U_\lambda) = U_\lambda \underset{K_\lambda}{\otimes} V_\lambda^* = M_{p_\lambda, q_\lambda}(K_\lambda)$$

in which U_λ *is a right* K_λ*-vector space of dimension* q_λ.

(b) *The semisimple Artinian* F*-algebra :*

$$H = \prod_{\lambda \in \Lambda'} \mathcal{L}(U_\lambda) = \prod_{\lambda \in \Lambda'} M_{q_\lambda}(K_\lambda) = \prod_{\lambda \in \Lambda} \mathcal{L}(U_\lambda) = \prod_{\lambda \in \Lambda} M_{q_\lambda}(K_\lambda)$$

is the F-*algebra of endomorphisms :*

$$H = \mathcal{L}_G(L_G)$$

of the non null finitely generated right G-*module* $L = L_G$ *characterized by the* canonical (H-G)-bimodule $L = {}_H L_G$, *or by the family of integers* (q_λ), *with a non empty support.*

PROOF - According to the Definition 2-8, the Lemma 2-5 implies immediately the part (a) and the classical properties of the finitely generated semisimple modules (See for instance the Theorem 1 p. 15 of [1]) imply easily the part (b) and complete the proof.

EXAMPLE 2-11 - Any pair (H, G) of semisimple Artinian F-algebras, subject to the condition :

$$H \lhd G$$

determines the *canonical* (H-G)-*bimodule* $L = {}_H L_G$ and therefore the *Formal triangular matrix* F-*algebra :*

$$B = \begin{pmatrix} H & L \\ 0 & G \end{pmatrix} = (H = H \lhd G)$$

with the right Socle S = S(B) defined by the Formal matrix relation :

$$S = \begin{pmatrix} 0 & L \\ 0 & G \end{pmatrix} = S(B)$$

Then, it is easy to verify that B is a (right) almost semisimple right Artinian F-algebra which is not a semisimple Artinian F-algebra, that is :

$$B \in [\mathfrak{A}_a(F) - \mathfrak{A}_0(F)]$$

FIRST FUNDAMENTAL CONSTRUCTION 2-12.

This "First Fundamental Construction" determines a F-algebra :

$$A \in [\mathfrak{A}_a(F) - \mathfrak{A}_0(F)]$$

In fact, for any pair (H, G) of semisimple Artinian F-algebras, subject to the condition : $H \lhd G$, for any right Artinian F-algebra $C \in \mathfrak{A}(F)$ and for any *"parameter"* Ψ, constituted by an injective F-algebra homomorphism :

$$\Psi : C \rightarrowtail H$$

which defines a F-subalgebra : $\Psi(C) \subset H$, simply noted : $C \subset H$, it is easy to verify that the right Artinian F-algebra A determined as the F-subalgebra of the previous F-algebra B constituted by the Formal triangular matrix F-algebra :

$$A = \begin{pmatrix} C & L \\ 0 & G \end{pmatrix} \equiv \begin{pmatrix} \Psi(C) & L \\ 0 & G \end{pmatrix} = (C \rightarrowtail^{\Psi} H \lhd G)$$

with a right Socle :

$$S = \begin{pmatrix} 0 & L \\ 0 & G \end{pmatrix} = S(A) = S(B)$$

is also a (right) almost semisimple right Artinian F-algebra which is not a semisimple F-algebra, that is :

$$A \in [\mathfrak{K}_a(F) - \mathfrak{K}_0(F)]$$

Moreover, this F-algebra A appears in the exact sequence :

$$0 \longrightarrow S \rightarrowtail A \twoheadrightarrow C \longrightarrow 0$$

which means that A is a "general F-algebra extension" of the non null idempotent ideal S by the right Artinian F-algebra C.

THEOREM 2-13 - *For any* F-*algebra* A, *the following conditions are equivalent :*

(a) *The* F-*algebra* A *is a (right) almost semisimple right Artinian* F-*algebra which is not a semisimple Artinian* F-*algebra, that is :*

$$A \in [\mathfrak{K}_a(F) - \mathfrak{K}_0(F)]$$

(b) *The* F-*algebra* A *is a right Artinian* F-*algebra which is not a semisimple Artinian* F-*algebra and which verifies the condition :*

$$M = M(A) = (0)$$

which implies that the F-*algebra* A *appears in the* general F-algebra extension :

$$(\tau') \qquad 0 \longrightarrow S \overset{i'}{\rightarrowtail} A \overset{\tau'}{\twoheadrightarrow} C \longrightarrow 0$$

in which $S = S(A) = N(A) \neq (0)$ *is a non null proper* idempotent *two-sided ideal and* $C = A/S$.

(c) *The Structure of the right Artinian* F-*algebra* A *is determined by the previous* **"First Fundamental Construction"**, *that is characterized by :*

(α') A *right Artinian* F-*algebra* C.

(β') A *pair* (H , G) *of semisimple Artinian* F-*algebras :*

$$H \in \mathfrak{K}_0(F) \qquad \textit{and} \qquad G \in \mathfrak{K}_0(F)$$

subject to the condition :

$$H \lhd G$$

and which defines S *as the* right Socle :

$$S = \begin{pmatrix} 0 & L \\ 0 & G \end{pmatrix} = S(B)$$

of the Formal triangular matrix F-algebra :

$$B = \begin{pmatrix} H & L \\ 0 & G \end{pmatrix} = (H = H \lhd G)$$

associated to the canonical (H-G)-bimodule $L = {}_H L_G$.

(γ') A "parameter" Ψ *constituted by an* injective F-algebra homomorphism :

$$\Psi : C \rightarrowtail H$$

such that the right Artinian F-*algebra* A *is the* Formal triangular matrix F-algebra :

$$\boxed{A = \begin{pmatrix} C & L \\ 0 & G \end{pmatrix} \equiv \begin{pmatrix} \Psi(C) & L \\ 0 & G \end{pmatrix} = (C \overset{\Psi}{\rightarrowtail} H \lhd G)}$$

with a right Socle :

$$S = \begin{pmatrix} 0 & L \\ 0 & G \end{pmatrix} = S(A) = S(B)$$

Moreover, under these equivalent conditions, the Jacobson Radicals **J**(A) *and* **J**(C) *verify the relation :*

$$\boxed{A/\mathbf{J}(A) = [C/\mathbf{J}(C)] \times G}$$

PROOF - Firstly, according to the Proposition 1-4, the Lemma 2-2 implies that the conditions (a) and (b) are equivalent.

Secondly, the previous "First Fundamental Construction" 2-12 shows easily that the condition (c) implies the equivalent conditions (a) and (b), and also the last relation.

Conversely, when the equivalent conditions (a) and (b) are verified, according to the Lemma 2-2, the Theorem 4-3 of [9] implies the existence of a pair (H, G) of semisimple Artinian F-algebras, subject to the condition : $H \lhd G$, which determines the Formal triangular matrix F-algebra :

$$B = \begin{pmatrix} H & L \\ 0 & G \end{pmatrix} = (H = H \lhd G)$$

with a right Socle :

$$S = \begin{pmatrix} 0 & L \\ 0 & G \end{pmatrix} = S(B)$$

such that the right Artinian F-algebra A verifies the relation :

$$S(B) = S = S(A) \subset A \subset B$$

This implies the existence of an *unique* F-*subalgebra :*

$$A' \subset H$$

such that A is the Formal triangular matrix F-algebra :

$$A = \begin{pmatrix} A' & L \\ 0 & G \end{pmatrix}$$

Thus, for the right Artinian factor F-algebra C = A/S, which is defined only "up to an F-isomorphism", there exists an *unique injective F-algebra homomorphism :*

$$\Psi : C \rightarrowtail H$$

which is a "section", such that :

$$C \xrightarrow{\sim} \Psi(C) = A' \subset H$$

and this implies the characterization of the right Artinian F-algebra A given in the condition (c).

Therefore, the equivalent conditions (a) and (b) imply the condition (c) and this completes the proof.

3. THE NOTION OF T-ESSENTIAL SINGULAR ALGEBRA EXTENSION.

It is convenient to remark that for any right Artinian F-algebra A which verifies the condition : $M = M(A) \neq (0)$, which implies automatically : $S = S(A) \neq A$ and $N = N(A) \neq A$, according to the Proposition 1-4, the right Artinian factor F-algebra B = A/M appears in the singular F-algebra extension :

$$(\tau) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} A \overset{\tau}{\twoheadrightarrow} B \longrightarrow 0$$

in which the surjective F-algebra epimorphism $\tau : A \twoheadrightarrow B$, induces an isomorphism of multiplicative A-bimodules $\bar{\tau} : N \rightarrowtail\!\!\twoheadrightarrow T$, from the idempotent proper two-sided ideal N = N(A) of A onto an idempotent proper two-sided ideal $T = \tau(N)$ of B, which verifies : $T \subset N(B)$ and $T \in \mathcal{C}(B)$, in such a way that for the right Artinian factor F-algebra C = A/S = B/T, the singular F-algebra extension (τ) induces on M a structure of C-bimodule.

This justifies the introduction of the following notions.

DEFINITION 3-1 - *For any pair* (B, T) *constituted by a* F-*algebra* B *and a proper two-sided ideal* T *of* B, *which determines the factor* F-*algebra* C = B/T *and the canonical surjective* F-*algebra epimorphism* $\varphi : B \twoheadrightarrow C$, *for any* C-*bimodule* M, *then :*

(a) A *pair* (A, N) *constituted by a* F-*algebra* A *and a proper two-sided ideal* N *of* A, *characterizes a* T-singular F-algebra extension *of* M *by* (B, T) *of the form :*

$$(\sigma, T) \qquad 0 \longrightarrow M \stackrel{i}{\rightarrowtail} (A, N) \stackrel{\sigma}{\twoheadrightarrow} (B, T) \longrightarrow 0$$

if the F-*algebra* A *characterizes a* singular F-algebra extension *of* M *by* B *of the form :*

$$(\sigma) \qquad 0 \longrightarrow M \stackrel{i}{\rightarrowtail} A \stackrel{\sigma}{\twoheadrightarrow} B \longrightarrow 0$$

in which the surjective F-*algebra epimorphism :*

$$\sigma : A \twoheadrightarrow B$$

induces an isomorphism of multiplicative A-*bimodules :*

$$\bar{\sigma} : N \rightarrowtail\!\!\!\twoheadrightarrow T$$

from the special proper two-sided ideal N *of* A, *onto the proper two-sided ideal* T *of* B, *considered as an* A-*bimodule by means of* σ, *in such a way that* (σ, T) *induces on* M *the structure of* C-*bimodule.*

(b) *Two* T-*singular* F-*algebra extensions :*

$$(\sigma, T) \qquad 0 \longrightarrow M \stackrel{i}{\rightarrowtail} (A, N) \stackrel{\sigma}{\twoheadrightarrow} (B, T) \longrightarrow 0$$

and

$$(\sigma', T) \qquad 0 \longrightarrow M \stackrel{i'}{\rightarrowtail} (A', N') \stackrel{\sigma'}{\twoheadrightarrow} (B, T) \longrightarrow 0$$

which induce on M *the structure of* C-*bimodule, are* "equivalent" *if there exists a* F-*algebra isomorphism :*

$$w : A \xrightarrow{\sim} A'$$

such that : i' = woi *and* σ = σ'ow *and* N' = w(N).

Then, they belong to the "same class" *noted :*

$$[\sigma, T] = [\sigma', T] = [A, N] = [A', N']$$

(c) *The pair* (B, T) *and the* C-*bimodule* M *determine the space :*

$$\text{Ext}\,(B, T, M)$$

of "classes" [σ, T] *of* T-singular F-algebra extensions (σ, T) *of* M *by* (B, T), *which induce on* M *the given structure of* C-*bimodule.*

(d) *In particular, for every* class :

$$\xi \in \text{Ext}(B, T, M)$$

the **"characterization"** *noted* :

$$\boxed{(A, N) = (B, T, M, \xi)}$$

means that the pair (A, N) *appears in a* T-singular F-algebra extension :

$$(\sigma, T) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} (A, N) \overset{\sigma}{\twoheadrightarrow} (B, T) \longrightarrow 0$$

unique, "up to an equivalence", such that :

$$[A, N] = [\sigma, T] = \xi \in \text{Ext}(B, T, M)$$

DEFINITION 3-2 - *With the previous notations*, a T-singular F-algebra extension:

$$(\sigma, T) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} (A, N) \overset{\sigma}{\twoheadrightarrow} (B, T) \longrightarrow 0$$

which induces on M *the given structure of* C-*bimodule, is called a* **T-essential** singular F-algebra extension *if and only if for every* right ideal $\mathfrak{a}$ *of* A, *the conditions:*

$$i(M) \cap \mathfrak{a} = (0) \qquad \text{and} \qquad N \cap \mathfrak{a} = (0)$$

imply : $\mathfrak{a} = (0)$.

LEMMA 3-3 - *With the previous notations, if two* T-*singular* F-*algebra extensions* (σ, T) *and* (σ', T) *which induce on* M *the given structure of* C-*bimodule, are equivalent, that is if :*

$$[\sigma, T] = [\sigma', T] \in \text{Ext}(B, T, M)$$

then (σ, T) *is a* T-essential *singular* F-*algebra extension if and only if* (σ', T) *is a* T-essential *singular* F-*algebra extension.*

PROOF - According to the Definition 3-1, the Definition 3-2 implies immediately the proof.

This Lemma 3-3 justifies the following Definition.

DEFINITION 3-4 - *For any pair* (B, T) *constituted by a* F-*algebra* B *and a proper two-sided ideal* T *of* B, *which determines the factor* F-*algebra* C = B/T *and the canonical surjective* F-*algebra epimorphism* $\varphi : B \twoheadrightarrow C$, *any* C-*bimodule* M *determines the space :*

$$\text{Ext}(B, T, M)$$

of "classes" $[\sigma, T]$ *of* T-singular F-algebra extensions (σ, T) *of* M *by* (B, T), *which induce on* M *the given structure of* C-*bimodule, and the* "subspace" or "subset":

$$\mathrm{Ext}_e(B, T, M)$$

of T-***essential*** singular classes $[\sigma, T]$ *of* T-***essential*** *singular* F-***algebra*** *extensions* (σ, T) *of* M *by* (B, T), *which induce on* M *the given structure of* C-*bimodule.*

LEMMA 3-5 - (G. HOCHSCHILD) - *For any algebra* B *over a* field F *and any* B-*bimodule* M, *then :*

(a) *Every* factor set *constituted by a* 2-cocycle :

$$f \in Z^2(B, M)$$

determines the F-algebra :

$$A = (B, M, f)$$

and the singular F-algebra extension :

$$(\sigma) = \sigma(f) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} A \overset{\sigma}{\twoheadrightarrow} B \longrightarrow 0$$

in which the F-*algebra* A *is defined by the* F-*vector space :*

$$A = M \times B$$

equipped with the multiplication *characterized by the condition :*

$$(m_1, b_1)(m_2, b_2) = (m_1 b_2 + b_1 m_2 + f(b_1, b_2), b_1 b_2)$$

for all $(m_1, b_1) = a_1 \in A$ *and* $(m_2, b_2) = a_2 \in A$

and in which :

$$i(m) = (m, 0) \qquad \text{and} \qquad \sigma((m, b)) = b$$

for all $m \in M$ *and* $b \in B$.

(b) *The* class *of* singular F-algebra extensions :

$$[\sigma(f)] = [\sigma] \in \mathrm{Ext}(B, M)$$

also noted :

$$[\sigma(f)] = [\sigma] = [A] = [B, M, f] = [B, M, \xi]$$

depends only of the cohomology class :

$$\hat{f} = \xi \in H^2(B, M)$$

(c) *There exists an* isomorphism *of* F-*vector spaces :*

$$\Phi : H^2(B, M) \xrightarrow{\sim} \mathrm{Ext}(B, M)$$

such that :

$$\Phi(\xi) = [\sigma(f)] = [B, M, f] = [B, M, \xi]$$

for every $f \in \xi \in H^2(B, M)$.

PROOF - Since F is a *field*, any F-algebra is *projective* and any singular F-algebra extension is a F-*split* singular F-algebra extension.

Thus, this Lemma summarizes the classical and fundamental properties of the Hochschild Cohomology Theory for associative algebras [11], [12], [13] (See also, for instance the Theorem 2-1, p. 295 of [2] or the Theorem 3-1, p. 285 of [16]).

LEMMA 3-6 - *For any pair* (B, T) *constituted by a* F-*algebra* B *and a proper two-sided ideal* T *of* B, *which determines the factor* F-*algebra* C = B/T *and the canonical surjective* F-*algebra epimorphism* :

$$\varphi : B \twoheadrightarrow C$$

and for any C-*bimodule* M, *then :*

(a) *The* F-*vector space noted :*

$$Z^2(B, T, M) \equiv Z^2(B/T, M) = Z^2(C, M)$$

of normalized 2-cocycles h *from the* F-*algebra* C *in the* C-*bimodule* M, *determines the Second Hochschild Cohomology group noted :*

$$H^2(B, T, M) \equiv H^2(C, M) = Z^2(C, M)/B^2(C, M)$$

and which is in fact a F-*vector space.*

(b) *There exists a morphism of* F-*vector spaces :*

$$\varphi_2 : Z^2(B, T, M) \equiv Z^2(C, M) \longrightarrow Z^2(B, M)$$

characterized by the condition :

$$h^* = \varphi_2(h) = h \circ (\varphi \times \varphi)$$

(c) *Every* factor set *constituted by a* 2-cocycle :

$$h \in Z^2(B, T, M) \equiv Z^2(B/T, M) = Z^2(C, M)$$

determines the "pair" :

$$\boxed{(A, N) = (B, T, M, h)}$$

and the T-singular F-algebra extension :

$$\boxed{(\sigma, T) = \underline{\sigma}(h) \quad 0 \longrightarrow M \xrightarrow{\ i\ } (A, N) \xrightarrow{\ \sigma\ } (B, T) \longrightarrow 0}$$

in which the F-algebra A *is well defined by the condition :*

$$\boxed{A = (B, M, h^*)}$$

which means that the F-algebra A *is the* F-vector space *defined by the condition :*

$$(*) \qquad \boxed{A = M \times B}$$

and equipped with the multiplication *characterized by the condition :*

(**) $\boxed{(m_1, b_1)(m_2, b_2) = (m_1 b_2 + b_1 m_2 + h^*(b_1, b_2), b_1 b_2)}$

for all $(m_1, b_1) = a_1 \in A$ *and* $(m_2, b_2) = a_2 \in A$; *in which :*

$$i(m) = (m, 0) \qquad \text{and} \qquad \sigma((m, b)) = b$$

for all $m \in M$ *and* $b \in B$; *and in which the proper two-sided ideal* N *of* A *is well defined by the condition :*

(***) $$\boxed{N = (0, T) = \{(0, t)\ ,\ t \in T\}}$$

(d) *The* class *of* T-singular F-algebra extensions :

$$[\underline{\sigma}(h)] = [\sigma, T] \in \mathrm{Ext}(B, T, M)$$

also noted :

$$[\underline{\sigma}(h)] = [\sigma, T] = [A, N] = [B, T, M, h] = [B, T, M, \xi]$$

depends only of the cohomology class :

$$\hat{h} = \xi \in H^2(B, T, M) \equiv H^2(C, M)$$

(e) *There exists an* isomorphism *of* F-vector spaces :

$$\Psi : H^2(B, T, M)\ ) \xrightarrow{\sim} \mathrm{Ext}(B, T, M)$$

such that :

$$\Psi(\xi) = [\underline{\sigma}(h)] = [B, T, M, h] = [B, T, M, \xi]$$

for every:

$$h \in \xi \in H^2(B, T, M) \equiv H^2(B/T, M) \equiv H^2(C, M).$$

PROOF - The Definition of the Hochschild Cohomology implies immediately the parts (a) and (b).

The Lemma 3-5 shows that the F-*algebra :*

$$\boxed{A = (B, M, h^*)}$$

determines the *singular* F-*algebra extension* :

$$\sigma(h^*) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} A \overset{\sigma}{\twoheadrightarrow} B \longrightarrow 0$$

in which the F-algebra A is defined by the F-vector space :

(*) $$\boxed{A = M \times B}$$

equipped with the *multiplication* characterized by the condition :

(**) $$\boxed{(m_1, b_1)(m_2, b_2) = (m_1 b_2 + b_1 m_2 + h^*(b_1, b_2), b_1 b_2)}$$

for all $(m_1, b_1) = a_1 \in A$ and $(m_2, b_2) = a_2 \in A$

and in which :

$$i(m) = (m, 0) \qquad \text{and} \qquad \sigma((m, b)) = b$$

for all $m \in M$ and $b \in B$.

For every $t_1 \in T$ and $t_2 \in T$, the Definition :

$$h^* = \varphi_2(h) = h \circ (\varphi \times \varphi)$$

implies the relations :

$$h^*(t_1, b_2) = h^*(b_1, t_2) = 0$$

and therefore, the condition (**) implies the relations :

$$(m_1, b_1)(0, t_2) = (0, b_1 t_2) \quad \text{and} \quad (0, t_1)(m_2, b_2) = (0, t_1 b_2)$$

since M is a C-bimodule, annihilated by T.

These relations show that there exists a two-sided ideal N of A defined by :

$$(***) \qquad \boxed{N = (0, T) = \{(0, t) \; ; \; t \in T\}}$$

and that the surjective F-algebra epimorphism :

$$\sigma : A \twoheadrightarrow B$$

induces an isomorphism of multiplicative A-bimodules :

$$\bar{\sigma} : N \rightarrowtail\!\!\!\twoheadrightarrow T$$

from the special two-sided ideal N of A, onto the proper two-sided ideal T of B, considered as an A-bimodule by means of σ.

This proves the existence of the T-singular F-algebra extension :

$$(\sigma, T) = \underline{\sigma}(h) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} (A, N) \overset{\sigma}{\twoheadrightarrow} (B, T) \longrightarrow 0$$

and completes the proof of the part (c).

For every cohomology class :

$$\xi \in H^2(C, M) = H^2(B/T, M) \equiv H^2(B, T, M)$$

and two *2-cocycles :*

$$h \in \xi \qquad \text{and} \qquad h' \in \xi$$

which determine the two *"pairs" :*

$$(A, N) = (B, T, M, h) \quad \text{and} \quad (A', N') = (B, T, M, h')$$

there exists a *normalized 1-cochain* $v \in C^1(C, M)$ such that :

$$h' = h + \delta^2 v$$

which implies :

$$h'^* = h^* + \delta^2 v^*$$

and it is immediate that there exists a F-algebra isomorphism :

$$w : A = (B, M, h^*) \xrightarrow{\sim} A' = (B, M, h'^*)$$

characterized by the condition :

$$w[(m, b)] = (m - v^*(b), b)$$

for all $(m, b) = a \in A$, and which verifies the relations :

$$i' = w \circ i \qquad \text{and} \qquad \sigma = \sigma' \circ w \qquad \text{and} \qquad N' = w(N)$$

Thus, w characterizes an *"equivalence"* from $(\sigma, T) = \underline{\sigma}(h)$ onto $(\sigma', T) = \underline{\sigma}(h')$, which implies the relation :

$$[A, N] = [\underline{\sigma}(h)] = [\sigma, T] = [\sigma', T] = [\underline{\sigma}(h')] = [A', N']$$

and completes the proof of the part (d).

For any T-*singular* F-*algebra extension :*

$$(\sigma', T) \qquad 0 \longrightarrow M \overset{i'}{\rightarrowtail} (A', N') \overset{\sigma'}{\twoheadrightarrow} (B, T) \longrightarrow 0$$

which induces on M the given structure of C-bimodule, there exists at least one morphism of F-vector spaces $u : B \longrightarrow A'$, *normalized* by $u(1) = 1$, which is a *section* or a *right inverse* of σ' and such that the *restriction* of u to T is an *inverse* of the isomorphism of multiplicative A'-bimodules $\bar{\sigma}' : N' \rightarrowtail\!\!\twoheadrightarrow T$, induced by σ'.

It is possible to identify each $m \in M$ with $i'(m) \in A'$ so that $i' : M \longrightarrow A'$ is the identity injection. Then, it is easy to verify that the relation :

$$\sigma'[u(b_1b_2)] = b_1b_2 = \sigma'[u(b_1)\, u(b_2)]$$

implies the existence of an unique 2-*cocycle :*

$$f \in Z^2(B, M)$$

characterized by the condition :

(1) $$f(b_1, b_2) = u(b_1)\, u(b_2) - u(b_1b_2)$$

for all $b_1 \in B$ and $b_2 \in B$.

Likewise, the description of the structure of B-bimodule on M can be written in terms of u as :

$$u(b)m = bm \qquad \text{and} \qquad m\, u(b) = mb$$

for all $b \in B$ and $m \in M$.

The existence of the isomorphism of multiplicative A'-bimodules :

$$\bar{\sigma}' : N' \rightarrowtail\!\!\twoheadrightarrow T$$

implies the relations :

$$\bar{\sigma}'[a_1u(t_2)] = \sigma'(a_1)t_2 \qquad \text{and} \qquad \bar{\sigma}'[u(t_1)a_2] = t_1\sigma'(a_2)$$

and therefore the relations :

$$a_1u(t_2) = u[\sigma'(a_1)t_2] \qquad \text{and} \qquad u(t_1)a_2 = u[t_1\, \sigma'(a_2)]$$

for all $a_1 \in A'$, $a_2 \in A'$, $t_1 \in T$ and $t_2 \in T$.

In particular, the conditions : $a_1 = u(b_1)$ and $a_2 = u(b_2)$, imply the relations :

(2) $$u(b_1)\, u(t_2) = u(b_1t_2) \qquad \text{and} \qquad u(t_1)\, u(b_2) = u(t_1b_2)$$

for all $b_1 \in B$, $b_2 \in B$, $t_1 \in T$ and $t_2 \in T$.

Thus, the condition (1) and the relations (2) imply the relation :

(3) $$f(b_1, t_2) = f(t_1, b_2) = 0$$

for all $b_1 \in B$, $b_2 \in B$, $t_1 \in T$ and $t_2 \in T$.

This relation (3) implies immediately the existence of an unique 2-*cocycle :*

$$h \in Z^2(C, M) = Z^2(B/T, M) \equiv Z^2(B, T, M)$$

characterized by the condition :

(4) $$h^* = \varphi_2(h) = f$$

and which determines the T-*singular* F-*algebra extension :*

$$(\sigma, T) = \underline{\sigma}(h) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} (A, N) \overset{\sigma}{\twoheadrightarrow} (B, T) \longrightarrow 0$$

Then, the conditions (1) and (4) imply the existence of an *"equivalence" of* T-*singular* F-*algebra extensions :*

$$w : (A, N) = (B, T, M, h) \overset{\sim}{\longrightarrow} (A', N')$$

characterized by the condition :

$$w[(m, b)] = m + u(b)$$

for all $(m, b) = a \in A$, and this implies the relation :

$$[A', N'] = [A, N] = [B, T, M, h] = [\underline{\sigma}(h)]$$

Moreover, with obvious notations, the replacement of u by a u' of the same kind, implies the replacement of f by a f' and also the replacement of h by a h'.

Then, it is easy to verify that there exists $v \in C^1(C, M)$ such that : $u' = u + v^*$, and this relation implies : $f' = f + \delta^2 v^*$, that is : $h'^* = h^* + \delta^2 v^*$, which implies the relation :

$$h' = h + \delta^2 v$$

Therefore, the T-*singular* F-*algebra extension* (σ', T) determines an *unique cohomology class :*

$$\hat{h}' = \hat{h} = \xi \in H^2(C, M) = H^2(B/T, M) \equiv H^2(B, T, M)$$

such that :

$$[\sigma', T] = [A', N'] = [B, T, M, h] = [B, T, M, h'] = [B, T, M, \xi]$$

This last property and the part (d) imply the existence of a *bijection :*

$$\Psi : H^2(C, M) \equiv H^2(B, T, M) \overset{\sim}{\longrightarrow} \mathrm{Ext}(B, T, M)$$

such that :

$$\Psi(\xi) = [\underline{\sigma}(h)] = [B, T, M, h] = [B, T, M, \xi]$$

for every $h \in \xi \in H^2(C, M) = H^2(B/T, M) \equiv H^2(B, T, M)$.

Thus, there exists on the set Ext(B, T, M) a *structure* of F-*vector space* such that this *bijection* Ψ become an *isomorphism* of F-vector spaces and this completes the proof.

DEFINITION 3-7 - *For any pair* (B, T) *constituted by a* F-*algebra* B *and a proper two-sided ideal* T *of* B, *which determines the factor* F-*algebra* C = B/T *and the canonical surjective* F-*algebra epimorphism* :

$$\varphi : B \longrightarrow\!\!\!\!\gg C$$

and for any C-*bimodule* M, *then , for any* 2-*cocycle :*

$$h \in Z^2(B, T, M) \equiv Z^2(B/T, M) = Z^2(C, M)$$

for which any element :

$$a_0 = (m_0, b_0) \in M \times B$$

determines the F-*vector space :*

$$s_B(a_0, h) = \{b \in B \; ; \; m_0 b + h^*(b_0, b) = 0\}$$

and the right annihilator :

$$r_B(b_0) = \{b' \in B \; ; \; b_0 b' = 0\}$$

which is a right ideal of the F-*algebra* B, *then, this* 2-*cocycle* h *belongs to the subset :*

$$Z_e^2(B, T, M) \equiv Z_e^2(B/T, M) = Z_e^2(C, M)$$

of **T-essential** 2-cocycles *if* h *verifies the condition :*

(E) « *For every element :*

$$a_0 = (m_0, b_0) \in M \times l_B(M) \subset M \times B$$

the conditions :

(r) $\quad r_B(b_0) \subset s_B(a_0, h)$ *and* (s) $\quad b_0 s_B(a_0, h) \cap T = (0)$

imply : $b_0 = 0$ (*which implies automatically :* $a_0 = 0$, *that is :* $m_0 = 0$ *and* $b_0 = 0$) ».

LEMMA 3-8 - *For any pair* (B, T) *constituted by a* F-*algebra* B *and a proper two-sided ideal* T *of* B, *which determines the factor* F-*algebra* C = B/T *and the canonical surjective* F-*algebra epimorphism* :

$$\varphi : B \longrightarrow\!\!\!\!\gg C$$

and for any C-*bimodule* M, *then, if two 2-cocycles :*

$$h \in Z^2(C, M) \qquad \textit{and} \qquad h' \in Z^2(C, M)$$

are cohomologous, that is if :

$$\hat{h} = \hat{h}' = \xi \in H^2(B, T, M) \equiv H^2(B/T, M) = H^2(C, M)$$

then, h *is* T-essential *if and only if* h' *is* T-essential, *that is :*

$$h \in Z_e^2(B, T, M) \text{ if and only if } h' \in Z_e^2(B, T, M).$$

PROOF - This follows easily from the Definition 3-7 and this Lemma 3-8 justifies the following Definition.

DEFINITION 3-9 - *In the case of algebras over a field* F, *for any pair* (B, T) *constituted by a* F-*algebra* B *and a proper two-sided ideal* T *of* B, *which determines the factor* F-*algebra* C = B/T *and the canonical surjective* F-*algebra epimorphism* $\varphi : B \twoheadrightarrow C$, *any* C-*bimodule* M *determines the* F-*vector space :*

$$H^2(B, T, M) \equiv H^2(B/T, M) = H^2(C, M) = Z^2(C, M)/B^2(C, M)$$

of cohomology classes $\xi = \hat{h}$ *of 2-cocycles :*

$$h \in Z^2(B, T, M) \equiv Z^2(B/T, M) = Z^2(C, M)$$

and the "subspace" *or* "subset" *:*

$$H^2_e(B, T, M) \equiv H^2_e(B/T, M) = Z^2_e(B, T, M)/B^2(C, M)$$

of T-**essential** cohomology classes $\xi = \hat{h}$ *of* T-**essential** 2-cocycles :

$$h \in Z^2_e(B, T, M) \equiv Z^2_e(B/T, M) = Z^2_e(C, M)$$

which is only ***a set.***

THEOREM 3-10 - *In the case of algebras over a field* F, *for any pair* (B, T) *constituted by a* F-*algebra* B *and a proper two-sided ideal* T *of* B, *which determines the factor* F-*algebra* C = B/T *and the canonical surjective* F-*algebra epimorphism :*

$$\varphi : B \twoheadrightarrow C$$

and for any C-*bimodule* M, *the* isomorphism *of* F-vector spaces :

$$\Psi : H^2(C, M) \equiv H^2(B, T, M) \xrightarrow{\sim} Ext(B, T, M)$$

induces a bijection :

$$\boxed{\Psi_e : H^2_e(B, T, M) \xrightarrow{\sim} Ext_e(B, T, M)}$$

such that :

$$\Psi_e(\xi) = [\underline{\sigma}(h)] = [B, T, M, h] = [B, T, M, \xi]$$

for every:

$$h \in \xi \in H^2_e(B, T, M) = Z^2_e(B, T, M)/B^2(C, M)$$

PROOF - The Lemma 3-6 characterizes the isomorphism Ψ and shows that every 2-cocycle :

$$h \in Z^2(C, M) = Z^2(B/T, M) \equiv Z^2(B, T, M)$$

determines the *"pair"* :

$$(A, N) = (B, T, M, h)$$

and the T-*singular* F-*algebra extension :*

$$(\sigma, T) = \underline{\sigma}(h) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} (A, N) \overset{\sigma}{\twoheadrightarrow} (B, T) \longrightarrow 0$$

in which the F-*algebra* A is defined by :

(5) $$A = (B, M, h^*)$$

and in which the proper two-sided ideal N of A is defined by :

$$N = (0, T) = \{(0, t) ; t \in T\}$$

and the proper two-sided ideal M' = i(M) of A is defined by :

$$M' = (M, 0) = \{(m, 0) ; m \in M\}$$

Then, according to the Lemmas 3-3 and 3-8, in order to complete the proof, it is sufficient to prove that the condition :

(6) $$h \in Z^2_e(B, T, M) \equiv Z^2_e(B/T, M) = Z^2_e(C, M)$$

is equivalent to the condition :

(7) « The T-singular F-algebra extension $(\sigma, T) = \underline{\sigma}(h)$ is a T-*essential* singular F-algebra extension » ;

that is to the condition :

(8) « For every $a_0 = (m_0, b_0) \in A$, the conditions :

(9) $M' \cap a_0A = (0)$ and (10) $N \cap a_0A = (0)$

imply : $a_0 = 0$, that is : $m_0 = 0$ and $b_0 = 0$ ».

The Lemma 3-5 and the relation (5) imply that two elements :

$$a_0 = (m_0, b_0) \in A \qquad \text{and} \qquad a = (m, b) \in A$$

verify the relation :

(11) $$a_0a = (m_0, b_0)(m, b) = (b_0m + m_0b + h^*(b_0, b), b_0b)$$

In particular, if $b_0 \notin l_B(M)$, there exist $m_1 \in M$ and $m_2 \in M$ such that : $b_0m_1 = m_2 \neq 0$, and therefore the elements $a_1 = (m_1, 0) \in A$ and $a_2 = (m_2, 0) \in A$ verify : $a_1 \neq 0$ and $a_2 \neq 0$, and also the relation :

$$a_0a_1 = (m_0, b_0)(m_1, 0) = (b_0m_1, 0) = (m_2, 0) = a_2 \in M'$$

which imply the relation :

$$M' \cap a_0 A \neq (0)$$

Thus, in the condition (8) it is sufficient to consider any element :

$$a_0 = (m_0, b_0) \in M \times l_B(M) \subset M \times B$$

for which the relation (11) gives the relation :

(12) $$a_0a = (m_0, b_0)(m, b) = (m_0b + h^*(b_0, b), b_0b)$$

which implies the relations :

(13) $$M' \cap a_0A = \{(m_0b + h^*(b_0, b), 0) ; b \in r_B(b_0)\}$$

and

(14) $$N \cap a_0A = \{(0, b_0b) ; b_0b \in T \text{ and } b \in s_B(a_0, h)\}$$

Then, it is immediate that the condition (9) is equivalent to the condition :

(r) $$r_B(b_0) \subset s_B(a_0, h)$$

and that the condition (10) is equivalent to the condition :

(s) $$b_0s_B(a_0, h) \cap T = (0)$$

Thus, the condition (8) is equivalent to the condition (E) that is to the condition (6) and this completes the proof.

NOTATIONS 3-11 - *For any pair* (B, T) *constituted by a* right Artinian F-algebra B *and a proper two-sided ideal* $T \in \mathcal{C}(B)$, *which determines the right Artinian factor* F*-algebra* C = B/T *and the canonical surjective* F*-algebra epimorphism :*

$$\varphi : B \longrightarrow\!\!\!\!\!\rightarrow C$$

the semisimple Artinian F*-algebra :*

$$B' = B/J(B)$$

and the canonical surjective F*-algebra epimorphism :*

$$\varphi' : B \longrightarrow\!\!\!\!\!\rightarrow B' = B/J(B)$$

which determine the proper two-sided ideal :

$$\varphi'(T) = T' \in \mathcal{C}(B')$$

the semisimple Artinian factor F*-algebra :*

$$C' = B'/T' = B/(T+J(B))$$

and the canonical surjective F*-algebra epimorphism :*

$$\varphi_1 : B' \longrightarrow\!\!\!\!\!\rightarrow C' = B'/T'$$

then :

(a) *Let* $\mathcal{M}$(B, T) *be the class of non null* C*-bimodules* M *such that the right* C*-module* $M = M_C$ *is* semisimple *and* Artinian (*or of* finite length, *or* finitely generated).

(b) *Let* $\mathcal{M}'(B', T') \equiv \mathcal{M}'(C')$ *be the class of non null* finitely generated right C'-modules $M' = M'_{C'}$ *which determine the semisimple Artinian* F*-algebras of endomorphisms :*

$$H' = \mathcal{L}_{C'}(M'_{C'})$$

and the canonical (H'-C')-bimodules :

$$M' = {}_{H'}M'_{C'}$$

LEMMA 3-12 - *With the previous Hypothesis and Notations, then :*

(a) *The* F*-algebra* C' = B'/T' *verifies the relation :* C' = C/**J**(C), *so that the canonical surjective* F*-algebra epimorphism :*

$$\varphi'' : C \longrightarrow\!\!\!\!\!\gg C' = C/\mathbf{J}(C)$$

verifies the relation : $\varphi_1 \circ \varphi' = \varphi'' \circ \varphi$.

(b) *Any non null* C*-bimodule :*

$$M \in \mathcal{M}(B, T)$$

is characterized by a non null right C'*-module :*

$$M' = M'_{C'} \in \mathcal{M}'(C') \equiv \mathcal{M}'(B', T')$$

and by a "parameter" Ψ' *constituted by a* F*-algebra homomorphism :*

$$\Psi' \in Mor_F[C, H']$$

such that the C*-bimodule* M *is defined by the* ***characterization*** *:*

$$\boxed{M = [M' \,;\, \Psi' : C \longrightarrow H']}$$

which means that the non null C*-bimodule* $M = {}_C M_C$ *derives from the non null* canonical (H'-C')-bimodule $M' = {}_{H'} M'_{C'}$, *by the "scalar restrictions" defined by the* F*-algebra homomorphisms :*

$$\Psi' : C \longrightarrow H' \qquad \textit{and} \qquad \varphi'' : C \longrightarrow\!\!\!\!\!\gg C'$$

PROOF - The general properties of the Jacobson Radical imply easily the part (a). Then, the part (b) follows from the general properties of semisimple modules.

SECOND FUNDAMENTAL CONSTRUCTION 3-13.

This "Second Fundamental Construction" determines a F-algebra :

$$A \in [\mathfrak{A}(F) - \mathfrak{A}_a(F)]$$

In fact, for the data constituted by :

(α) A right Artinian F-algebra B.

(β) A proper two-sided ideal $T \in \mathcal{C}(B)$, which determines the right Artinian factor F-algebra C = B/T and the canonical surjective F-algebra epimorphism :

$$\varphi : B \longrightarrow\!\!\!\!\!\gg C$$

(γ) A non null C-bimodule :

$$M \in \mathcal{M}(B, T)$$

determined by the *"characterization"* :

$$M = [M' ; \Psi' : C \longrightarrow H']$$

defined by a non null right C'-module :

$$M' = M'_{C'} \in \mathcal{M}'(C') \equiv \mathcal{M}'(B', T')$$

and by a *"parameter"* Ψ'constituted by a F-algebra homorphism :

$$\Psi' \in Mor_F[C, H']$$

(δ) A *"parameter"* $\xi = \hat{h}$ constituted by a T-***essential cohomology class*** **:**

$$\xi = \hat{h} \in H^2_e(B, T, M) \simeq Ext_e(B, T, M)$$

these data give the construction of a pair (A, N) by the ***"characterization"*** :

$$\boxed{(A, N) = (B, T, M, h) = (B, T, M, \xi)}$$

which determines, according to the following Theorem 3-14, a T-essential singular F-algebra extension :

$$\boxed{(\tau, T) = \underline{\sigma}(h) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} (A, N) \overset{\tau}{\twoheadrightarrow} (B, T) \longrightarrow 0}$$

in which A is a right Artinian F-algebra, which is not a (right) almost semisimple right Artinian F-algebra, that is :

$$A \in [\mathfrak{A}(F) - \mathfrak{A}_a(F)]$$

with a right Socle :

$$S = S(A) = M(A) \oplus N(A) = M \oplus N$$

More precisely, according to the Lemma 3-6 this ***"Charaterization"*** means that in the T-singular F-algebra extension $(\tau, T) = \underline{\sigma}(h)$, the F-algebra A *is well defined* by the condition :

$$\boxed{A = (B, M, h^*)}$$

which means that the F-algebra A is the F-vector space defined by the condition :

$$(*) \qquad \boxed{A = M \times B}$$

and equipped with the multiplication characterized by the condition :

$$(**) \quad \boxed{(m_1, b_1)\,(m_2, b_2) = (m_1 b_2 + b_1 m_2 + h^*(b_1, b_2), b_1 b_2)}$$

for all $(m_1, b_1) = a_1 \in A$ and $(m_2, b_2) = a_2 \in A$; in which :

$$i(m) = (m, 0) \qquad \text{and} \qquad \sigma((m, b)) = b$$

for all $m \in M$ and $b \in B$; and in which the proper two-sided ideal N of A *is well defined* by the condition :

$$(***) \qquad \boxed{N = (0, T) = \{(0, t) \; ; \; t \in T\}}$$

THEOREM 3-14 - *For any* F-*algebra* A, *the following conditions are equivalent :*

(a) *The* F-*algebra* A *is a right Artinian* F-*algebra which is not a (right) almost semisimple right Artinian* F-*algebra, that is :*

$$A \in [\mathfrak{A}(F) - \mathfrak{A}_a(F)]$$

(b) *The* F-*algebra* A *is a right Artinian* F-*algebra which verifies the condition :*

$$M = M(A) \neq (0)$$

which implies that the F-*algebra* A *and the proper two-sided ideal :*

$$N = N(A)$$

characterized by the canonical decomposition of the right Socle :

$$S = S(A) = M(A) \oplus N(A) = M \oplus N$$

determine the "pair" (A, N) *which characterizes a* T-***essential singular*** F-***algebra extension :***

$$(\tau, T) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} (A, N) \overset{\tau}{\twoheadrightarrow} (B, T) \longrightarrow 0$$

for the factor F-*algebras:*

$$B = A/M \qquad and \qquad C = B/T = A/S$$

and for the proper two-sided ideal $T = \tau(N) = \bar{\tau}(N)$ *of* B, *which verifies :*

$$T \in \mathcal{C}(B)$$

and which induces on M *a structure of* C-*bimodule such that :*

$$M \in \mathcal{M}(B, T)$$

(c) *The Structure of the right Artinian* F-*algebra* A *is determined by the previous* **"Second Fundamental Construction"**, *that is characterized by :*

(α) *A right Artinian* F-*algebra* B.

(β) *A proper two-sided ideal* $T \in \mathcal{C}(B)$, *which determines the right Artinian factor* F-*algebra* C = B/T *and the canonical surjective* F-*algebra epimorphism :*

$$\varphi : B \twoheadrightarrow C$$

(γ) *A non null* C-*bimodule :*

$$M \in \mathcal{M}(B, T)$$

determined by the " characterization" :

$$M = [M' \,;\, \Psi' : C \longrightarrow H']$$

defined by a non null right C'-*module :*

$$M' = M'_{C'} \in \mathcal{M}'(C') \equiv \mathcal{M}'(B', T')$$

with the semisimple Artinian F-*algebra of endomorphisms :*

$$H' = \mathfrak{L}_{C'}(M'_{C'})$$

and by a "parameter" Ψ' *constituted by a* F-*algebra homomorphism :*

$$\Psi' \in \mathrm{Mor}_F[C, H']$$

(δ) *A* "parameter" $\xi = \hat{h}$ *constituted by a* T-**essential cohomology class** *:*

$$\xi = \hat{h} \in H^2_e(B, T, M) \simeq \mathrm{Ext}_e(B, T, M)$$

which give the construction of the pair (A, N) *by the* **"characterization"** :

$$\boxed{(A, N) = (B, T, M, h) = (B, T, M, \xi)}$$

which implies :

$$\boxed{S = S(A) = M(A) \oplus N(A) = M \oplus N}$$

so that the pair (A, N) *is completely and well defined by the previous conditions* (∗), (∗∗) *and* (∗∗∗) *of the Lemma 3-6.*

Moreover, under these equivalent conditions, the Jacobson Radicals **J**(A) *and* **J**(B) *determine the same semisimple Artinian* F-*algebra :*

$$\boxed{A/\mathbf{J}(A) = B/\mathbf{J}(B)}$$

PROOF - According to the Lemma 2-2 and the Proposition 1-4, the Remarks before the Definition 3-1 show that the condition (a) is equivalent to the first part of the condition (b), which implies the existence of the T-singular F-algebra extension (τ, T), in which : $T \in \mathcal{C}(B)$, and the relation :

$$M = M(A) \subset S(A) = S$$

implies that M is a semisimple and Artinian right A-module $M = M_A$ which derives from a non null semisimple and Artinian right C'-module :

$$M' = M'_{C'} \in \mathcal{M}'(C') \equiv \mathcal{M}'(B', T')$$

and it is easy to verify that (τ, T) induces on M a structure of C-bimodule such that :

$$M \in \mathcal{M}(B, T)$$

Moreover, since in the right Artinian F-algebra A, its right Socle :

$$S = S(A) = M(A) \oplus N(A) = M \oplus N \equiv i(M) \oplus N$$

is a "minimum essential right ideal", in particular, for any *non null* right ideal $\mathfrak{a}$ of A, which verifies necessarily : $S \cap \mathfrak{a} \neq (0)$, there exists at least one minimal right ideal $\mathfrak{a}'$ of A such that : $\mathfrak{a}' \subset \mathfrak{a}$, and which verifies necessarily : $\mathfrak{a}' \subset M$ or $\mathfrak{a}' \subset N$, which imply : $M \cap \mathfrak{a} \neq (0)$ or $N \cap \mathfrak{a} \neq (0)$, and therefore the Definition 3-2 shows that (τ, T) is a T-*essential singular* F-*algebra extension.*

This proves that the condition (a) implies the condition (b).

According to the Theorem 3-10 and the Lemma 3-6, the Lemmas 3-3 and 3-12, the Definition 3-4 and the Definition 3-1 show that the condition (b) implies the condition (c).

Now, let A be a F-algebra determined by the condition (c), in which, according to the Definition 3-1, the **"characterization"** :

$$(A, N) = (B, T, M, h) = (B, T, M, \xi)$$

implies that the *pair* (A, N) appears in a T-***essential singular*** F-***algebra extension*** :

$$(\sigma, T) = \underline{\sigma}(h) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} (A, N) \overset{\sigma}{\twoheadrightarrow} (B, T) \longrightarrow 0$$

unique, "up to an equivalence", such that :

$$\xi = \hat{h} \in H_e^2(B, T, M) \simeq \mathrm{Ext}_e(B, T, M)$$

In particular, in the singular F-algebra extension :

$$(\sigma) \qquad 0 \longrightarrow M \overset{i}{\rightarrowtail} A \overset{\sigma}{\twoheadrightarrow} B \longrightarrow 0$$

since the right A-modules $B = B_A$ and $M = M_A$ are Artinian, the right A-module A_A is Artinian and therefore the F-algebra A is a right Artinian F-algebra.

The condition : $M \in \mathcal{M}(B, T)$, implies that M is a semisimple right A-module, which verifies necessarily the relation :

$$M \equiv i(M) \subset S(A)$$

According to the Definition 1-3, the Lemma 1-2 implies that the ideal : $T \in \mathcal{C}(B)$, verifies : $T \subset N(B) \subset S(B)$, which shows that T is a semisimple right B-module, and according to the Definition 3-1 which shows that the T-***essential singular*** F-***algebra extension*** (σ, T) induces an isomorphism of multiplicative A-bimodules $\bar{\sigma} : N \rightarrowtail\!\!\twoheadrightarrow T$, it follows easily that N is a semisimple right A-module, which verifies necessarily the relation :

$$N \subset S(A)$$

and therefore, it follows the relation :

$$S = M \oplus N \equiv i(M) \oplus N \subset S(A)$$

For the T-***essential singular*** F-***algebra extension*** (σ, T), the Definition 3-2 shows that for any right ideal $\mathfrak{a}$ of A, the conditions :

$$i(M) \cap \mathfrak{a} = (0) \qquad \text{and} \qquad N \cap \mathfrak{a} = (0)$$

imply : $\mathfrak{a} = (0)$, and this property implies that the two-sided ideal $S = M \oplus N$ is an essential right ideal of A, and therefore :

$$S = M \oplus N \equiv i(M) \oplus N = \mathbf{S}(A)$$

which implies easily : $M = \mathbf{M}(A) \neq (0)$ and $N = \mathbf{N}(A)$.

This proves that the condition (c) implies the condition (a), and the property of the Jacobson Radicals is obvious in any singular F-algebra extension.

PROPOSITION 3-15 - *For any pair* (B, T) *constituted by a* F-*algebra* B *and a proper two-sided ideal* T *of* B, *which determines the* F-*algebra* C = B/T *and for any* C-*bimodule* M, *if* T *verifies :*

$$T \in \mathcal{C}(B)$$

then :

(a) *The subset :*

$$Z_e^2(C, M) = Z_e^2(B/T, M) \equiv Z_e^2(B, T, M)$$

of T-**essential** 2-cocycles *is constituted by the* 2-*cocycles :*

$$h \in Z^2(C, M) = Z^2(B/T, M) \equiv Z^2(B, T, M)$$

which verify the condition :

(E') « *For every element :*

$$a_0 = (m_0, b_0) \in M \times [l_B(M) - T] \subset M \times B$$

the conditions :

(r) $$r_B(b_0) \subset s_B(a_0, h)$$

and

(s) $$b_0\, s_B(a_0, h) \cap T = (0)$$

imply : $b_0 = 0$ ».

(b) *In particular, if the* C-*bimodule* M *verifies :*

$$l_B(M) = T$$

then :

$$\hat{0} = 0 \in H_e^2(B, T, M) = H^2(B, T, M)$$

and in this case the space $H_e^2(B, T, M)$ *is non empty.*

PROOF - At first, any C-bimodule M verifies the relation :

$$T \subset \mathrm{Ann}_B(M) = l_B(M) \cap r_B(M)$$

which gives a sense to the condition (E') and which shows that the Definition 3-7 implies the relation :

$$T \subset s_B(a_0, h)$$

Secondly, for every element :

$$a_0 = (m_0, b_0) \in M \times T \subset M \times B$$

according to the Definition 1-3, the Lemma 1-2 shows that the hypothesis : $T \in \mathcal{C}(B)$, implies that there exists $\beta \in T$ such that : $b_0 = b_0\beta$, so that the condition (s) implies :

$$b_0 = b_0\beta \in b_0 s_B(a_0, h) \cap T = (0)$$

that is : $b_0 = 0$, which proves that the conditions (E) and (E') are equivalent and this proves the part (a).

Then, the part (a) and the Definition 3-9 imply immediately the part (b), and this completes the proof.

4. CLASSICAL INVARIANTS FOR RIGHT ARTINIAN ALGEBRAS.

One of the most important *classical invariants* for any right Artinian F-algebra A, is its ***quiver*** :

$$\Gamma(A) = (V, E) = (V(A), E(A))$$

in the sense of the Definition of the page 97 of [17], for which it is convenient (and harmless) to call two quivers equal when they are only isomorphic, that is, there is a bijection between their vertex sets that maps the edge sets bijectively.

PROPOSITION 4-1 - *Any right Artinian* F*-algebra :*

$$A \in \mathfrak{A}(F)$$

determines a semisimple Artinian F*-algebra:*

$$A/J(A) = R \in \mathfrak{A}_0(F)$$

and a **classical invariant** *constituted by the* F*-**"concrete vertex set"** :*

$$\widetilde{\Lambda}(A) = \widetilde{\Lambda}(R) = \widetilde{\Lambda} = [\Lambda\,; (K_\lambda), (p_\lambda)] = [\Lambda\,; (V_\lambda)] = [\Lambda\,; (V_\lambda^*)]$$

such that :

$$R = \mathbf{R}(\widetilde{\Lambda})$$

in which the underlying "abstract vertex set" Λ *coincides with the* common vertex set :

$$\Lambda = V(A) = V(R)$$

of the **quivers** :

$$\Gamma(A) = (V(A), E(A)) \qquad \textit{and} \qquad \Gamma(R) = (V(R), \varnothing)$$

of the F*-algebras* $A \in \mathfrak{A}(F)$ *and* $R \in \mathfrak{A}_0(F)$.

PROOF - The first assertion is obvious since the right Artinian F-algebra R, without Radical, is semisimple.

The fact that the mapping $P \longmapsto P/PJ(A)$ defines a bijective correspondence between V(A) and V(R) (See for instance the Proposition p. 93 of [17]) gives the relation :

$$\Lambda = V(A) = V(R)$$

according to the previous convention.

At last, if two right Artinian F-algebras A and A' are isomorphic, it is obvious that the semisimple Artinian F-algebras A/J(A) = R and A'/J(A') = R' are isomorphic, so that the Corollary 2-7 gives the relation :

$$\widetilde{\Lambda}(A) = \widetilde{\Lambda}(R) = \widetilde{\Lambda}(R') = \widetilde{\Lambda}(A')$$

which completes the proof.

5. CANONICAL RESOLUTION OF RIGHT ARTINIAN ALGEBRAS.

Now, our aim is to define some new invariants for right Artinian F-algebras.

DEFINITION 5-1 - *For any* F-*algebra* B *with a right Socle* S(B) *having the canonical decomposition :*

$$S(B) = M(B) \oplus N(B)$$

the **"special ideal"** *of* B, *noted* Q(B) *is the (two-sided) ideal defined by the conditions :*

Q(B) = M(B)	*if*	M(B) ≠ (0)
Q(B) = N(B)	*if*	M(B) = (0)

which show that Q(B) *is a* non null ideal *if and only if* S(B) *is non null.*

LEMMA 5-2 - *For any right Artinian* F-*algebra* B ∈ $\mathfrak{A}$(F), *the* "special ideal" Q = Q(B) *is a* non null ideal *of* B, *which is a* "proper ideal" *of* B *if and only if* B *is not a semisimple Artinian* F-*algebra, that is if and only if :*

$$B \in [\mathfrak{A}(F) - \mathfrak{A}_0(F)]$$

PROOF - Since the hypothesis : B ∈ $\mathfrak{A}$(F), implies : S(B) ≠ (0), this follows immediately from the Lemma 2-2 and the Definition 5-1.

THEOREM 5-3 - *For any right Artinian* F-*algebra:*

$$A \in \mathfrak{K}(F)$$

there exist an unique integer $m \in \mathbb{N}$, *a (finite right)* Canonical Resolution, *of the form :*

$$\mathfrak{R}(A) = [A = A_m \xrightarrow{\tau_m} A_{m-1} \dots A_i \xrightarrow{\tau_i} A_{i-1} \dots A_1 \xrightarrow{\tau_1} A_0]$$

and a (finite right) strictly descending Fundamental Sequence *of proper (two-sided) ideals, of the form :*

$$\mathfrak{a}(A) = (\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_i, \dots, \mathfrak{a}_{m-1}, \mathfrak{a}_m = (0))$$

associated by the conditions :

$$A_i = A/\mathfrak{a}_i \qquad \textit{for all} \qquad i \in I = \{0, 1, 2, \dots, m\}$$

and characterized by the following conditions :

(a) *If* A *is a* semisimple Artinian F-algebra : $A \in \mathfrak{K}_0(F)$, *then* : $m = 0$, *and*

$$\mathfrak{R}(A) = [A = A_0] \qquad \textit{and} \qquad \mathfrak{a}(A) = (\mathfrak{a}_0 = (0))$$

are respectively the trivial (finite right) Canonical Resolution *and the trivial* Fundamental Sequence.

(b) *If* A *is not a semisimple Artinian* F-*algebra:* $A \notin \mathfrak{K}_0(F)$, *then :* $1 \leq m$, *and the* F-*algebra* A_0 *is a semisimple Artinian* F-*algebra:*

$$A_0 \in \mathfrak{K}_0(F)$$

such that for every integer $i \in I^* = \{1, 2, \dots, m\}$, *the right Artinian* F-*algebra:*

$$A_i \in \mathfrak{K}(F)$$

is not a semisimple Artinian F-*algebra:*

$$A_i \notin \mathfrak{K}_0(F)$$

and the i-th ***"link"*** *:*

$$A_i \xrightarrow{\tau_i} A_{i-1}$$

of the Canonical Resolution $\mathfrak{R}(A)$ *is the* surjective F-algebra epimorphism τ_i *associated to the* exact sequence *or* "F-algebra extension" :

(τ_i)	$0 \longrightarrow Q_i \rightarrowtail A_i \xrightarrow{\tau_i} A_{i-1} \longrightarrow 0$

characterized by the condition :

$$\text{Ker}\, \tau_i = Q_i = \mathbf{Q}(A_i)$$

equivalent to the equivalent conditions :

$$A_{i-1} = A_i/Q_i = A_i/\mathbf{Q}(A_i) \qquad \textit{and} \qquad \mathfrak{a}_{i-1}/\mathfrak{a}_i = \mathbf{Q}(A/\mathfrak{a}_i)$$

PROOF - The Theorem 5-4 of [10], applied in the case of F-algebras, implies immediately the proof.

COROLLARY 5-4 - *For any right Artinian* F-*algebra:*

$$A \in \mathfrak{K}(F)$$

with the Notations of the Theorem 5-3, there exists a (finite right) Canonical Sequence $\rho(A)$, *characterized by the following conditions :*

(a) *If* A *is a* semisimple Artinian F-algebra : $A \in \mathfrak{K}_0(F)$, *then* $\rho(A)$ *is the empty or trivial* Canonical Sequence *:* $\rho(A) = \varnothing$.

(b) *If* A is not a semisimple Artinian F-algebra : $A \notin \mathfrak{K}_0(F)$, *then, the* Canonical Sequence :

$$\rho(A) = (\rho_1, \rho_2, ..., \rho_i, ..., \rho_m)$$

is the finite sequence of integers with values in {0, 1}, *characterized by the following conditions :*

(b_0) *For every integer* $j \in I^* = \{1, 2, ..., m\}$, *the condition :*

$$\boxed{\rho_j = 0}$$

is equivalent to the equivalent conditions :

$$Q_j^2 = [Q(A_j)]^2 = (0)$$

and

$$\mathrm{Ker}\, \tau_j = Q_j = Q(A_j) = M(A_j) = M_j \neq (0)$$

which mean that the j-th ***"link"**:*

$$A_j \xrightarrow{\tau_j} A_{j-1}$$

of the Canonical Resolution $\mathfrak{R}(A)$ *is a **"zero-link"**associated to the **singular*** F-***algebra extension*** *:*

$$\boxed{(\tau_j) \qquad 0 \longrightarrow M_j \rightarrowtail A_j \xrightarrow{\tau_j} A_{j-1} \longrightarrow 0}$$

in which the right Artinian F-*algebra :* $A_j \in \mathfrak{K}(F)$, *is not a (right) almost semisimple right Artinian* F-*algebra:*

$$A_j \in [\mathfrak{K}(F) - \mathfrak{K}_a(F)]$$

[with a Structure characterized by the Theorem 3-14].

(b_1) *For every integer* $k \in I^* = \{1, 2, ..., m\}$, *the condition :*

$$\boxed{\rho_k = 1}$$

is equivalent to the equivalent conditions :

$$Q_k^2 = [Q(A_k)]^2 = Q(A_k) = Q_k \neq (0)$$

and

$$\mathrm{Ker}\, \tau_k = Q_k = Q(A_k) = N(A_k) = S(A_k) = S_k \neq (0)$$

which mean that the k-th ***"link"*** *:*

$$A_k \xrightarrow{\tau_k} A_{k-1}$$

of the Canonical Resolution $\mathfrak{R}(A)$ *is a* ***"one-link"*** *associated to the* ***general*** F-***algebra extension*** :

(τ_k)		$0 \longrightarrow S_k \rightarrowtail A_k \overset{\tau_k}{\longrightarrow\!\!\!\!\twoheadrightarrow} A_{k-1} \longrightarrow 0$

in which the right Artinian F-*algebra* : $A_k \in \mathfrak{A}(F)$, *is a (right) almost semisimple right Artinian* F-*algebra* : $A_k \in \mathfrak{A}_a(F)$, *which is not a semisimple Artinian* F-*algebra* :

$$A_k \in [\mathfrak{A}_a(F) - \mathfrak{A}_0(F)]$$

[with a Structure characterized by the Theorem 2-13].

PROOF - This is an immediate consequence of the Proposition 1-4, of the Definition 5-1 and of the Theorem 5-3 which characterizes the Canonical Resolution $\mathfrak{R}(A)$ of a right Artinian F-algebra A.

DEFINITION 5-5 - *The right* Resolutive Dimension *of a right Artinian* F-*algebra :*

$$A \in \mathfrak{A}(F)$$

is the natural integer :

$$m = \rho \dim(A)$$

characterized in the Theorem 5-3 and in the Corollary 5-4 as :

1 - *The* length m *of the finite right* Canonical Resolution $\mathfrak{R}(A)$.

2 - *The* length m *of the finite right* Fundamental Sequence $\mathfrak{a}(A)$ *of two-sided ideals.*

3 - *The cardinal number* m *of the finite right* Canonical Sequence $\rho(A)$ *associated to the right Artinian* F-*algebra* A.

For instance, in the finite right Canonical Resolution $\mathfrak{R}(A)$, each right Artinian F-algebra : $A_i \in \mathfrak{A}(F)$, verifies the relation :

$$i = \rho \dim(A_i) \qquad \text{for all } i \in I = \{0, 1, 2, ..., m\}$$

THEOREM 5-6 - *For two right Artinian* F-*algebras* A *and* A', *if there exists a* F-*algebra isomorphism* $u : A \xrightarrow{\sim} A'$, *then :*

(a) *The right Resolutive Dimensions are equal :*

$$m = \rho \dim(A) = \rho \dim(A') = m'$$

In other words, the finite right Resolutive Dimension $\rho \dim(A)$ *is a* **numerical invariant** *for right Artinian* F-*algebras.*

(b) *The finite right Canonical Sequences are equal :*

$$\rho(A) = \rho(A')$$

In other words, the finite right Canonical Sequence $\rho(A)$ *is a* **sequential invariant** *for right Artinian* F-*algebras.*

(c) *The finite right Fundamental Sequences of two-sided ideals :*

$$\mathfrak{a}(A) = (\mathfrak{a}_0, \mathfrak{a}_1, ..., \mathfrak{a}_j, ..., \mathfrak{a}_m = (0))$$

and

$$\mathfrak{a}(A') = (\mathfrak{a}'_0, \mathfrak{a}'_1, ..., \mathfrak{a}'_j, ..., \mathfrak{a}'_m = (0))$$

verify the relations :

$$u(\mathfrak{a}_j) = \mathfrak{a}'_j \qquad\qquad \textit{for all } j \in I = \{0, 1, 2, ..., m\}$$

In other words, the finite right Fundamental Sequences $\mathfrak{a}(A)$ *and* $\mathfrak{a}(A')$ *are* **canonically isomorphic.**

(d) *There exists an unique* F-*isomorphism of Resolutions :*

$$\mathfrak{R}(u) : \mathfrak{R}(A) \xrightarrow{\sim} \mathfrak{R}(A')$$

which is an extension of the F-*algebra isomorphism* $u : A \xrightarrow{\sim} A'$.

In other words, the finite right Canonical Resolutions $\mathfrak{R}(A)$ *and* $\mathfrak{R}(A')$ *are* **canonically isomorphic.**

(e) *In particular, for each index* $i \in I^* = \{1, 2, ..., m\}$ *the* i-th ***"link"***:

$$A_i \xrightarrow{\tau_i} \!\!\!\!\!\!\twoheadrightarrow A_{i-1}$$

of the Canonical Resolution $\mathfrak{R}(A)$, *is "unique up to an* F-*isomorphism" and constitutes the* i-th **"resolutive invariant"** *of the right Artinian* F-*algebra* A.

PROOF - The Theorem 5-7 of [10], applied in the case of F-algebras, implies immediately the proof.

6. NEW INVARIANTS FOR RIGHT ARTINIAN ALGEBRAS.

The following Notions are introduced in order to describe the Structure of the Canonical Resolution $\mathfrak{R}(A)$ of any right Artinian F-algebra A.

DEFINITION 6-1 - *For any integer* $m \in \mathbb{N}$, *which determines the sets :*

$$I = \{0, 1, 2, ..., m\} \qquad \textit{and} \qquad I^* = \{1, 2, ..., m\}$$

a «Complete Decomposition of m» *or a* «Combinatorial Type of dimension m», *is an object of the form :*

$$\text{(I)} \qquad I = I_1 \coprod I_0^* = I_1 \coprod \left(\coprod_{k \in I_2} I_0^k \right)$$

characterized by a disjoint union *of the from :*

(I*) $$I^* = I_1^* \amalg I_0^*$$

so that : $I^* = I_1^* \cup I_0^*$ *and* $I_1^* \cap I_0^* = \varnothing$, *with :* $I_1 = \{0\} \amalg I_1^*$ *and* $I_1^* = I_1 \cap I^*$,

which determines also the set :

$$I_2 = \{k \in I_1 \,;\, (k+1) \in I_0^*\}$$

such that if $I_0^* \neq \varnothing$, *that is if* $I_2 \neq \varnothing$, *each integer* $j \in I_0^*$ *determines the integer*

$k(j) = k \in I_2$ *defined by the condition :*

$$k(j) = \mathrm{Sup}\,\{k' \in I_1 \,;\, k' < j\} = \mathrm{Sup}\,\{k'' \in I_2 \,;\, k'' < j\}$$

and each $k \in I_2$ *determines the non empty subset :*

$$I_0^k = \{j \in I_0^* \,;\, k = k(j)\}$$

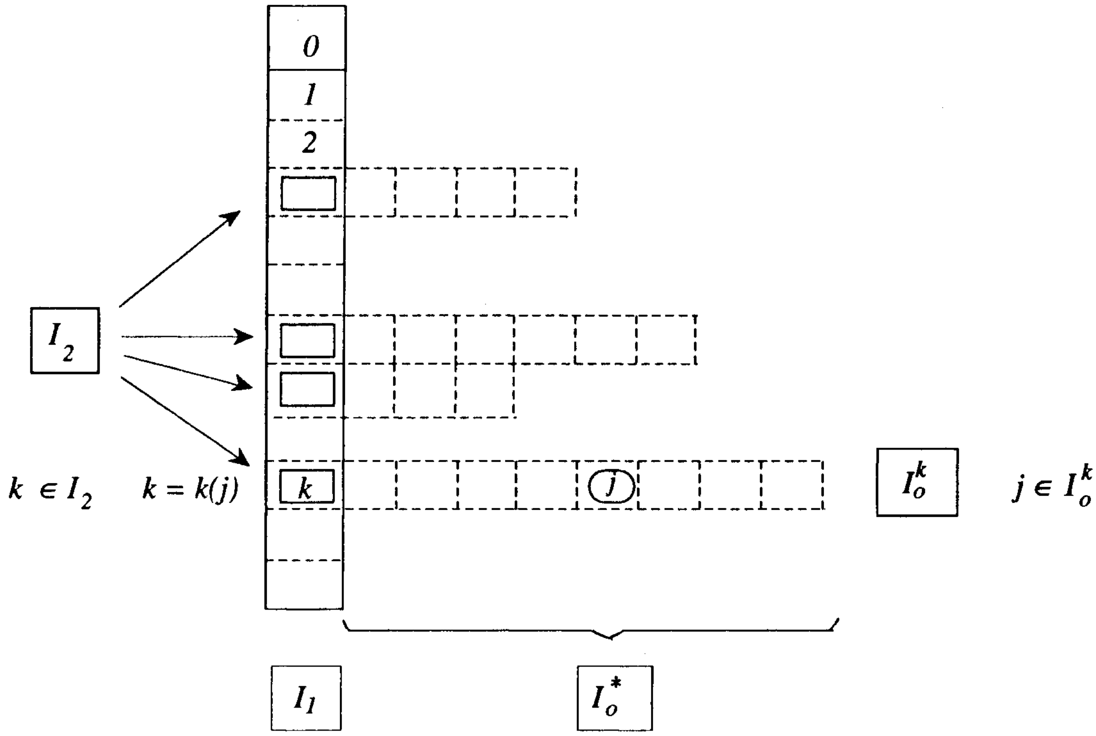
in a partition *of the form :*

$$I_0^* = \coprod_{k \in I_2} I_0^k$$

with an obvious convention whenever $I_0^* = \varnothing$ *or* $I_2 = \varnothing$ *and* $I = I_1$.

REMARKS 6-2 - A Combinatorial Type (I) of dimension m, may be represented by a Combinatorial Table, constituted by a first column and possibly by some rows, in which the ordered set $I = \{0, 1, 2, \ldots, m\}$ is described by the "lexicographic order" in such a way that the subset I_1 is represented by the first column (which always contains the integer 0) and that if $I_0^* \neq \varnothing$, that is if $I_2 \neq \varnothing$, each integer $k \in I_2$, which appears in the first column, is also the first element of the row which represents the union of $\{k\}$ and of the non empty subset I_0^k, which appears in the partition of the set I_0^*, represented by the set of integers which are not in the first column.

See, for instance, the following Combinatorial Table.



DEFINITION 6-3 - *A* **"Structured vertex set"** *is an object of the form :*

$$[\Lambda ; \Sigma]$$

in which Λ *is a* finite and non empty set *(called the underlying* "abstract vertex set"*) equipped with a* **"Combinatorial Structure"** Σ, *noted :*

$$\Sigma = [\Lambda ; m , (I) ; \{\Lambda_i\} , \quad , \Lambda' , \quad , (\Lambda''^k_j), \quad , (\Lambda'_j)]$$

or more precisely :

$$\Sigma = [\Lambda ; m , (I) ; \{\Lambda_i\}, (\Lambda^k), \Lambda', (\Lambda'^k), (\Lambda''^k_j), (\Lambda''_j), (\Lambda'_j)]$$

and characterized by the following data :

(a) *A* finite and non empty set Λ.

(b) *An* integer $m \in \mathbb{N}$ *and a* Combinatorial Type (I) of dimension m, *of the form :*

$$\text{(I)} \qquad I = I_1 \coprod I_0^* = I_1 \coprod \left(\coprod_{k \in I_2} I_0^k \right)$$

with an obvious convention whenever $I_0^* = \varnothing$ *or* $I_2 = \varnothing$ *and* $I = I_1$.

(c) *An* exhaustive and ascending filtration of Λ by non empty sets Λ_i, *of the form :*

$$\{\Lambda_i\} = \{\Lambda_i\}_{i \in I} = \{\varnothing \neq \Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_i \subset \ldots \subset \Lambda_{m-1} \subset \Lambda_m = \Lambda\}$$

which verifies the relation :

$$I_0^* = \{j \in I^* \, ; \Lambda_{j-1} = \Lambda_j\}$$

and which is equivalent to a partition *of* Λ, *of the form :*

$$\Lambda = \coprod_{k \in I_1} \Lambda^k$$

for a family of non empty subsets :

$$(\Lambda^k) = (\Lambda^k)_{k \in I_1}$$

connected by the equivalent conditions :

$$\begin{cases} \Lambda^0 = \Lambda_0 & \textit{for } 0 \in I_1 = \{0\} \coprod I_1^* \\ \Lambda^k = \Lambda_k - \Lambda_{k-1} & \textit{for all } k \in I_1^* \end{cases}$$

and

$$\Big[\, \Lambda_i = \coprod_{i \geq k \in I_1} \Lambda^k \qquad \textit{for all } i \in I$$

(d) *A* subset :

$$\Lambda' \subset \Lambda$$

subject to the conditions :

$$\begin{cases} \Lambda'^0 = \Lambda' \cap \Lambda^0 = \varnothing & \textit{for } 0 \in I_1 = \{0\} \coprod I_1^* \\ \Lambda'^k = \Lambda' \cap \Lambda^k \neq \varnothing & \textit{for all } k \in I_1^* \end{cases}$$

that is a subset : $\Lambda' \subset \Lambda$, *connected to a family* :

$$(\Lambda'^k) = (\Lambda'^k)_{k \in I_1^*}$$

of non empty subsets *:* $\Lambda'^k \subset \Lambda^k$, *by the condition :*

$$\Lambda' = \coprod_{k \in I_1^*} \Lambda'^k$$

(e) *Whenever* $I_0^* \neq \varnothing$ *or* $I_2 \neq \varnothing$, *a* double family of subsets :

$$(\Lambda''^k_j) = (\Lambda''^k_j)_{k \in I_2 \, , \, j \in I_0^k}$$

such that for each index $k \in I_2$, *the family :*

$$(\Lambda''^k_j)_{j \in I_0^k}$$

is a "descending sequence" *of subsets :*

$$\Lambda''^k_j \subset \Lambda^k = \Lambda^{k(j)}$$

subject to the condition that if $0 \in I_2$, *then :*

$$\Lambda''^{0}_{1} \neq \Lambda^0 = \Lambda_0$$

and which determines the family :

$$(\Lambda''_j) = (\Lambda''_j)_{j \in I_0^*}$$

of non empty subsets :

$$\Lambda''_j \subset \Lambda_{k(j)} = \Lambda_j$$

defined by the conditions :

$$\Lambda''_j = \Lambda_{k(j)} - \Lambda''^{k(j)}_{j} = \Lambda_j - \Lambda''^{k(j)}_{j} \qquad \textit{for all } j \in I_0^*$$

(f) *Whenever* $I_0^* \neq \varnothing$ *or* $I_2 \neq \varnothing$, *a family :*

$$(\Lambda'_j) = (\Lambda'_j)_{j \in I_0^*}$$

of non empty subsets :

$$\Lambda'_j \subset \Lambda''_j \subset \Lambda_j$$

DEFINITION 6-4 - *For any field* F, a F- **"Completely structured vertex set"** *is an object of the form :*

$$\hat{\Lambda} = \{\Lambda \, ; \Sigma \, ; (K_\lambda), (p_\lambda), (q_\lambda), (n^j_\lambda)\}$$

defined by the following data :

(a) *A* **"combinatorial data"** *constituted by a* **"Structured vertex set"** *of the form :*

$$[\Lambda \, ; \Sigma]$$

in which the underlying "abstract vertex set" Λ *is equipped with a* **"Combinatorial Structure"** Σ, *noted :*

$$\Sigma = [\Lambda \, ; m \, , (I) \, ; \{\Lambda_i\} \, , \quad , \Lambda', \quad , (\Lambda''^{k}_{j}), \quad , (\Lambda'_j)]$$

or more precisely :

$$\Sigma = [\Lambda \, ; m \, , (I) \, ; \{\Lambda_i\}, (\Lambda^k), \Lambda', (\Lambda'^k), (\Lambda''^{k}_{j}), (\Lambda''_j), (\Lambda'_j)]$$

(b) *A* **"numerical data"** :

$$\upsilon = \{(p_\lambda), (q_\lambda), (n^j_\lambda)\}$$

compatible with Σ, *in the sense that the families of integers :*

$$(p_\lambda) = (p_\lambda)_{\lambda \in \Lambda} \quad ; \quad (q_\lambda) = (q_\lambda)_{\lambda \in \Lambda}$$

and

$$(n^j_\lambda) = (n^j_\lambda)_{j \in I_0^*,\ \lambda \in \Lambda}$$

verify the conditions :

$$\Lambda = \mathrm{Supp}[(p_\lambda)] \qquad ; \qquad \Lambda' = \mathrm{Supp}[(q_\lambda)]$$

and

$$\Lambda'_j = \mathrm{Supp}\left[(n^j_\lambda)_{\lambda \in \Lambda}\right] \qquad \textit{for all } j \in I_0^*$$

(c) *An* **"algebraic data"** *constituted by a family :*

$$(K_\lambda) = (K_\lambda)_{\lambda \in \Lambda}$$

of F-*skewfields :*

$$K_\lambda \in \mathcal{K}(F) \qquad \textit{for all } \lambda \in \Lambda$$

REMARKS 6-5 - For this kind of objects, the notion of isomorphism is obvious.

Like in the case of quivers or of F-Concrete vertex sets, it is convenient (and harmless) to call two F-*"Completely structured vertex sets"* **equal when they are only isomorphic.**

This Convention is included in the previous Definition 6-4.

LEMMA 6-6 - *Any* F-*Completely structured vertex set* $\hat{\Lambda}$ *characterizes the following objects :*

(a) *A* F-*Concrete vertex set of the form :*

$$\widetilde{\Lambda} = [\Lambda ; (K_\lambda), (p_\lambda)] = [\Lambda ; (V_\lambda)] = [\Lambda ; (V^*_\lambda)]$$

(b) *A* F-*Concrete vertex set of the form :*

$$\widetilde{\Lambda}' = [\Lambda' ; (K_\lambda), (q_\lambda)] = [\Lambda' ; (U_\lambda)] = [\Lambda' ; (U^*_\lambda)]$$

and a Generalized F-*concrete vertex set of the form :*

$$\underline{\widetilde{\Lambda}}' = [\Lambda ; (K_\lambda), (q_\lambda)] = [\Lambda ; (U_\lambda)] = [\Lambda ; (U^*_\lambda)]$$

whenever $\Lambda' \neq \varnothing$ *or* $I_1^* \neq \varnothing$.

(c) *A family of* F-*Concrete vertex sets of the form :*

$$\widetilde{\Lambda}'_j = [\Lambda'_j \, ; (K_\lambda), (n^j_\lambda)] = [\Lambda'_j \, ; (W^j_\lambda)] = [\Lambda'_j \, ; (W^{*j}_\lambda)]$$

for all j ∈ I^*_0 *and a family of Generalized* F*-concrete vertex sets of the form :*

$$\underline{\widetilde{\Lambda}}'_j = [\Lambda \, ; (K_\lambda), (n^j_\lambda)] = [\Lambda \, ; (W^j_\lambda)] = [\Lambda \, ; (W^{*j}_\lambda)]$$

for all j ∈ I^*_0, *whenever* $I^*_0 \neq \varnothing$.

PROOF - According to the Definitions 6-3 and 6-4, this follows immediately from the Definitions 2-4 and 2-8.

More precisely, this means that for each $\lambda \in \Lambda$ and possibly for each $j \in I^*_0$,

the right K_λ-vector spaces :

$$V_\lambda \qquad U_\lambda \qquad W^j_\lambda$$

and the left K_λ-vector spaces :

$$V^*_\lambda \qquad U^*_\lambda \qquad W^{*j}_\lambda$$

are respectively of the finite dimensions :

$$p_\lambda \qquad q_\lambda \qquad n^j_\lambda$$

defined by the numerical data of the F-Completely structured vertex set $\hat{\Lambda}$.

COROLLARY 6-7 - *Any* F*-Completely structured vertex set* $\hat{\Lambda}$ *characterizes, "by restriction to the corresponding subsets", the* F*-Concrete vertex sets of the form :*

$$\widetilde{\Lambda}_i = (\widetilde{\Lambda}/\Lambda_i) = [\Lambda_i \, ; (K_\lambda), (p_\lambda)] = [\Lambda_i \, ; (V_\lambda)] = [\Lambda_i \, ; (V^*_\lambda)] \qquad \textit{for all } i \in I$$

$$\widetilde{\Lambda}^k = (\widetilde{\Lambda}/\Lambda^k) = [\Lambda^k \, ; (K_\lambda), (p_\lambda)] = [\Lambda^k \, ; (V_\lambda)] = [\Lambda^k \, ; (V^*_\lambda)] \qquad \textit{for all } k \in I_1$$

$$\widetilde{\Lambda}'^k = (\widetilde{\Lambda}'/\Lambda'^k) = [\Lambda'^k \, ; (K_\lambda), (q_\lambda)] = [\Lambda'^k \, ; (U_\lambda)] = [\Lambda'^k \, ; (U^*_\lambda)] \qquad \textit{for all } k \in I^*_1$$

$$\widetilde{\Lambda}''_j = (\widetilde{\Lambda}/\Lambda''_j) = [\Lambda''_j\,;(K_\lambda),(p_\lambda)] = [\Lambda''_j\,;(V_\lambda)] = [\Lambda''_j\,;(V^*_\lambda)] \qquad \textit{for all } j \in I^*_0$$

PROOF - This follows immediately from the Definitions 2-4, 6-3, 6-4 and the Lemma 6-6.

LEMMA 6-8 - *For any right Artinian* F*-algebra* $A \in \mathfrak{A}(F)$,*the data constituted by its Resolutive Dimension* $m = \rho\dim(A)$ *and by its Canonical Sequence :*

$$\rho(A) = (\rho_1, \rho_2, ..., \rho_i, ..., \rho_m) = (\rho_i)_{i\in I^*}$$

are equivalent to its Combinatorial Type :

$$I(A) = (I) \qquad I = I_1 \coprod I^*_0 = I_1 \coprod \left(\coprod_{k\in I_2} I^k_0\right)$$

of dimension m, *connected by the equivalent conditions :*

$$I^*_1 = \{k \in I^*\,;\rho_k = 1\} \qquad \text{and} \qquad I^*_0 = \{j \in I^*\,;\rho_j = 0\}$$

PROOF - This follows from the Theorem 5-3, the Corollary 5-4 and the Definitions 5-5 and 6-1.

THEOREM 6-9 - *The Structure of any* right Artinian F-algebra :

$$A \in \mathfrak{A}(F)$$

is characterized by its Combinatorial Type :

$$I(A) = (I) \qquad I = I_1 \coprod I^*_0 = I_1 \coprod \left(\coprod_{k\in I_2} I^k_0\right)$$

of dimension $m = \rho\dim(A)$ *and by its Canonical Resolution :*

$$\mathfrak{R}(A) = [A = A_m \xrightarrow{\tau_m} A_{m-1} \dots A_i \xrightarrow{\tau_i} A_{i-1} \dots A_1 \xrightarrow{\tau_1} A_0]$$

in which the Structures of the right Artinian F-algebras :

$$A_i \in \mathfrak{A}(F) \qquad \textit{for all } i \in I = \{0, 1, 2, ..., m\}$$

which are determined by an "ascending iterative contruction" *starting from the semisimple Artinian* F*-algebra :*

$$A_0 \in \mathfrak{A}_0(F)$$

are characterized by the following conditions :

(a) *The* F*-algebra* A_0 *is a semisimple Artinian* F*-algebra :*

$$A_0 = R_0 = G_0 \in \mathfrak{A}_0(F)$$

and more generally, each F*-algebra* A_i *is connected to a semisimple Artinian* F*-algebra :*

$$A_i/J(A_i) = R_i \in \mathfrak{R}_0(F)$$

by a canonical surjective F-*algebra epimorphism :*

$$\varphi'_i : A_i \twoheadrightarrow R_i = A_i/J(A_i)$$

(b) *Whenever* $I_1^* \neq \emptyset$, *for each index* $k \in I_1^*$, *the Structure of the right Artinian* F-*algebra* A_k *is characterized by the previous* **"First Fundamental Construction"**, *that is described by the* **Characterization** :

$$\boxed{A_k = \begin{pmatrix} A_{k-1} & L_k \\ 0 & G_k \end{pmatrix} \equiv \begin{pmatrix} \Psi_k(A_{k-1}) & L_k \\ 0 & G_k \end{pmatrix} = (A_{k-1} \xrightarrow{\Psi_k} H_k \lhd G_k)}$$

determined by a pair (H_k, G_k) *of semisimple Artinian* F-*algebras subject to the condition :* $H_k \lhd G_k$, *which determines the canonical* $(H_k\text{-}G_k)$-*bimodule* $L_k = {}_{H_k}L_{G_k}$, *and by a* ***"parameter"*** Ψ_k *constituted by an injective* F-*algebra homomorphism :*

$$\Psi_k : A_{k-1} \rightarrowtail H_k$$

Moreover, this Characterization implies the relation :

$$\boxed{A_k/J(A_k) = R_k = R_{k-1} \times G_k = (A_{k-1}/J(A_{k-1})) \times G_k}$$

(c) *Whenever* $I_0^* \neq \emptyset$ *or* $I_2 \neq \emptyset$, *for each index* $j \in I_0^*$, *associated to the index* $k(j) = k \in I_2$, *the Structure of the right Artinian* F-*algebra* A_j *is characterized by the previous* **"Second Fundamental Construction"**, *that is described by the* **Characterization** *:*

$$\boxed{(A_j, N_j) = (A_{j-1}, T_{j-1}, M_j, h_j) = (A_{j-1}, T_{j-1}, M_j, \xi_j)}$$

determined by an ideal $T_{j-1} \in \mathcal{C}(A_{j-1})$ *which defines the right Artinian* F-*algebra* $C_{j-1} = A_{j-1}/T_{j-1}$, *a non null* C_{j-1}-*bimodule :*

$$M_j \in \mathcal{M}(A_{j-1}, T_{j-1})$$

and a ***"parameter"*** $\xi_j = \hat{h}_j$ *constituted by an unique* T_{j-1}-***essential cohomology class*** *:*

$$\xi_j = \hat{h}_j \in H_e^2(A_{j-1}, T_{j-1}, M_j) \simeq Ext_e(A_{j-1}, T_{j-1}, M_j)$$

Moreover, this Characterization implies the relation :

$$\boxed{A_j/J(A_j) = R_j = R_{j-1} = A_{j-1}/J(A_{j-1})}$$

Furthermore, in fact the ideal $T_{j-1} \in \mathcal{C}(A_{j-1})$ *may be characterized by some ideal* $T'_{j-1} \in \mathcal{C}(R_{j-1}) \equiv \mathcal{C}(R_k)$ *connected by the conditions :*

$$T'_{j-1} = \varphi'_{j-1}(T_{j-1}) \qquad \text{and} \qquad T_{j-1} = N(A_{j-1}) \cap \varphi'^{-1}_{j-1}(T'_{j-1})$$

so that the non null C_{j-1}*-bimodule :*

$$M_j \in \mathcal{M}(A_{j-1}, T_{j-1})$$

is determined by a characterization of the form :

$$M_j = [M'_j ; \Psi'_j : C_{j-1} \longrightarrow H'^j]$$

in which the non null right C'_{j-1}*-module :*

$$M'_j \in \mathcal{M}'(C'_{j-1}) \equiv \mathcal{M}'(R_{j-1}, T'_{j-1})$$

with the semisimple Artinian F*-algebra of endomorphisms :*

$$H'^j = \mathcal{L}_{C'_{j-1}}(M'_j)$$

is also the canonical $(H'^j\text{-}C'_{j-1})$*-bimodule associated to a pair* (H'^j, C'_{j-1}) *of semisimple Artinian* F*-algebras, subject to the condition :* $H'^j \triangleleft C'_{j-1}$*, and in which the* ***"parameter"*** Ψ'_j *is constituted by a* F*-algebra homomorphism :*

$$\Psi'_j \in \mathrm{Mor}_F[C_{j-1}, H'^j]$$

PROOF - According to the Lemmas 2-10 and 6-8, the conditions (a), (b) and the first part of the condition (c) follow easily from the Theorems 2-13, 3-14, 5-3 and the Corollary 5-4.

Furthermore, whenever $I_0^* \neq \varnothing$ or $I_2 \neq \varnothing$, for each $j \in I_0^*$ and for the canonical surjective F-algebra epimorphism :

$$\varphi'_{j-1} : A_{j-1} \longrightarrow\!\!\!\!\!\rightarrow R_{j-1} = A_{j-1}/J(A_{j-1})$$

the two-sided ideal $N_{j-1} = N(A_{j-1})$ of A_{j-1} is the direct sum of the idempotent right feet P of A_{j-1}, for which the Lemma 1-2 of [8] shows that they verify the relation :

$$P \not\subset t(P) = P \cap r(P) = P \cap J(A_{j-1})$$

which implies that they are in one-to-one correspondence with the *non null* two-sided ideals :

$$P' = \varphi'_{j-1}(P) = P/P \cap J(A_{j-1})$$

of the semisimple Artinian F-algebra : $R_{j-1} = R_k$, which are exactly the idempotent right feet P' of $R_{j-1} = R_k$ which verify the condition :

$$P' \subset N'_{j-1} = \varphi'_{j-1}(N_{j-1})$$

and for which, conversely :

$$P = N(A_{j-1}) \cap \varphi'^{-1}_{j-1}(P')$$

These properties imply easily that the conditions :

$$T' = \varphi'_{j-1}(T) \qquad \text{and} \qquad T = N(A_{j-1}) \cap \varphi'_{j-1}(T')$$

determine an one-to-one correspondence between the proper two-sided ideals :

$$T \in \mathcal{C}(A_{j-1})$$

and the proper two-sided ideals :

$$T' \in \mathcal{C}(R_{j-1})$$

subject to the general condition :

$$T' \subset N'_{j-1} = \varphi'_{j-1}(N_{j-1})$$

and to the supplementary condition :

$$T' \neq N'_{j-1} = N'_k$$

whenever : $(j-1) = k = 0 \in I_2$.

According to the Notations 3-11 and the Lemma 3-12, this implies immediately the last assertion and completes the proof.

THEOREM 6-10 - *Any right Artinian* F-*algebra :*

$$A \in \mathfrak{A}(F)$$

determines an unique **"complete invariant"** $\hat{\Lambda}(A)$ *constituted by the* **F-Completely structured vertex set** *of the form :*

$$\hat{\Lambda}(A) = \hat{\Lambda} = \{\Lambda \,;\, \Sigma \,;\, (K_\lambda), (p_\lambda), (q_\lambda), (n^j_\lambda)\}$$

equipped with the **Combinatorial Structure** *:*

$$\Sigma(A) = \Sigma = [\Lambda \,;\, m, (I) \,;\, \{\Lambda_i\} \,, (\Lambda^k), \Lambda', (\Lambda'^k), (\Lambda''^k_j), (\Lambda''_j), (\Lambda'_j)]$$

and characterized by the following conditions.

(a) *The right Artinian* F-*algebra* $A \in \mathfrak{A}(F)$ *determines the semisimple Artinian* F-*algebra* $A/\mathbf{J}(A) = R \in \mathfrak{A}_0(F)$ *and the* **classical invariant** *constituted by the* **F-concrete vertex set** *:*

$$\tilde{\Lambda}(A) = \tilde{\Lambda}(R) = \tilde{\Lambda} = [\Lambda \,;\, (K_\lambda), (p_\lambda)] = [\Lambda \,;\, (V_\lambda)] = [\Lambda \,;\, (V^*_\lambda)]$$

such that :

$$R = \mathbf{R}(\tilde{\Lambda}) = \prod_{\lambda \in \Lambda} R^\lambda$$

for the family $(R^\lambda)_{\lambda \in \Lambda}$ *of simple Artinian* F-*algebras* R^λ *having the realizations :*

$$R^\lambda = \mathcal{L}(V_\lambda) = M_{p_\lambda}(K_\lambda) = [\mathcal{L}(V^*_\lambda)]^\circ$$

and in which the underlying abstract vertex set Λ *coincides with the common vertex set :*

$$\Lambda = V(A) = V(R)$$

of the quivers :

$$\Gamma(A) = (V(A), E(A)) \quad \text{and} \quad \Gamma(R) = (V(R), \varnothing)$$

of the F-*algebras* $A \in \mathfrak{K}(F)$ *and* $R \in \mathfrak{K}_0(F)$.

(b) *The Canonical Resolution* $\mathfrak{R}(A)$ *determines the Resolutive Dimension* $m = \rho\dim(A)$ *and the Combinatorial Type* $(I) = \mathbf{I}(A)$ *of dimension* m, *defined in the Lemma 6-8.*

(c) *The filtration* $\{\Lambda_i\}$ *is characterized by the conditions :*

$$\mathbf{R}(\widetilde{\Lambda}_i) = R_i = A_i/\mathbf{J}(A_i) \qquad \textit{for all } i \in I$$

equivalent to the conditions :

$$R_i = \prod_{\lambda \in \Lambda_i} R^\lambda = \prod_{\lambda \in \Lambda_i} \mathcal{L}(V_\lambda) = \prod_{\lambda \in \Lambda_i} M_{p_\lambda}(K_\lambda) \qquad \textit{for all } i \in I$$

which imply in particular the relations :

$$A_0 = R_0 = \mathbf{R}(\widetilde{\Lambda}_0) = G_0 \qquad \textit{and} \qquad A/\mathbf{J}(A) = R = \mathbf{R}(\widetilde{\Lambda})$$

(d) *The subset :* $\Lambda' \subset \Lambda$, *connected to the family* (Λ'^k), *and the families of integers* (p_λ) *and* (q_λ) *are defined, with the previous Notations of the Corollary 6-7 and of the Theorem 6-9, by the conditions :*

$$\widetilde{\Lambda}^k = \widetilde{\Lambda}(G_k) \quad \textit{for } k \in I_1 \qquad \textit{and} \qquad \widetilde{\Lambda}'^k = \widetilde{\Lambda}(H_k) \quad \textit{for } k \in I_1^*$$

(e) *Whenever* $I_0^* \neq \varnothing$ *or* $I_2 \notin \varnothing$, *the double family* (Λ''^k_j) *is defined, with the previous Notations of the Theorem 6-9, by the conditions :*

$$T'_{j-1} = \bigoplus_{\lambda \in \Lambda''^k_j} R^\lambda \qquad \textit{for all } j \in I_0^* \textit{ and } k = k(j)$$

which imply, with the previous Notations of the Corollary 6-7, the relations :

$$C'_{j-1} = \mathbf{R}(\widetilde{\Lambda}''_j) = \prod_{\lambda \in \Lambda''_j} R^\lambda \qquad \textit{for all } j \in I_0^*$$

(f) *Whenever* $I_0^* \neq \varnothing$ *or* $I_2 \neq \varnothing$, *the family* (Λ'_j) *and the family of integers* (n^j_λ) *are defined, with the previous Notations of the Lemma 6-6 and of the Theorem 6-9, by the conditions :*

$$\widetilde{\Lambda}'_j = \widetilde{\Lambda}(H'^j) \qquad \textit{for all } j \in I_0^*$$

PROOF - The Proposition 4-1 implies the part (a), which gives in particular the determinations of the "algebraic data" :

$$(K_\lambda) = (K_\lambda)_{\lambda \in \Lambda}$$

of the F-Completely structured vertex set $\hat{\Lambda}(A) = \hat{\Lambda}$ and also of the first family of integers $(p_\lambda) = (p_\lambda)_{\lambda \in \Lambda}$ of the "numerical data" : $\mathbf{\upsilon}(A) = \upsilon = \{(p_\lambda), (q_\lambda), (n_\lambda^i)\}$ of $\hat{\mathbf{\Lambda}}(A) = \hat{\Lambda}$.

The Theorem 5-3 and the Corollary 5-4 give the characterization of the Canonical Resolution $\mathfrak{R}(A)$, which is unique up to an F-isomorphism, according to the Theorem 5-6.

This implies immediately that the data described in the Theorem 6-9 are unique up to an F-isomorphism.

Firstly, according to the Lemma 6-8, this Theorem 6-9 implies the part (b).

Secondly, the Theorem 6-9 gives the relations :

$$R_k = G_k \times R_{k-1} \quad \text{for all } k \in I_1^* \qquad \text{and} \qquad R_j = R_{j-1} \qquad \text{for all } j \in I_0^*$$

which imply immediately, with the convention : $R_0 = G_0$, the relations :

$$R_i = \prod_{i \geq k \in I_1} G_k \qquad \text{for all} \quad i \in I \quad \text{and} \quad A/\mathbf{J}(A) = R = R_m = \prod_{k \in I_1} G_k$$

Then, this last relation implies the existence of a *partition* of Λ, of the form :

$$\Lambda = \coprod_{k \in I_1} \Lambda^k$$

such that :

$$G_k = \prod_{\lambda \in \Lambda^k} R^\lambda = \mathbf{R}(\widetilde{\Lambda}^k)$$

that is such that :

$$\widetilde{\mathbf{\Lambda}}(G_k) = \widetilde{\Lambda}^k = (\widetilde{\Lambda}/\Lambda^k)$$

for all $k \in I_1$.

Moreover, with the Notations and the conditions of the Definition 6-3 it is immediate that the previous conditions are equivalent to the conditions :

$$R_i = \prod_{i \geq k \in I_1} G_k = \prod_{\lambda \in \Lambda_i} R^\lambda$$

that is to the conditions :

$$\widetilde{\mathbf{\Lambda}}(R_i) = \widetilde{\Lambda}_i = (\widetilde{\Lambda}/\Lambda_i)$$

for all $i \in I$, which characterize the *exhaustive and ascending filtration* $\{\Lambda_i\} = \{\Lambda_i\}_{i \in I}$ *of* Λ, *by non empty sets* Λ_i, of the form :

$$\{\Lambda_i\} = \{\Lambda_i\}_{i \in I} = \{\emptyset \neq \Lambda_0 \subset \Lambda_1 \subset ... \subset \Lambda_i \subset ... \subset \Lambda_{m-1} \subset \Lambda_m = \Lambda\}$$

which verifies the relation :

$$I_0^* = \{j \in I^* \, ; \, \Lambda_{j-1} = \Lambda_j\}$$

equivalent to the conditions of the Lemma 6-8.

This implies the part (c) and the first conditions of the part (d).

Thirdly, according to the Theorem 6-9 which gives the conditions :

$$H_k \lhd G_k$$

for all $k \in I_1^*$, and according to the Lemma 2-10, with the Notations and the conditions of the Definition 6-3, it is immediate that the subset : $\Lambda' \subset \Lambda$, connected to the family (Λ'^k), and the family of integers $(q_\lambda) = (q_\lambda)_{\lambda \in \Lambda}$ may be defined by the conditions :

$$\widetilde{\Lambda}(H_k) = \widetilde{\Lambda}'^k = (\widetilde{\Lambda}'/\Lambda'^k)$$

for all $k \in I_1^*$, whenever $I_1^* \neq \emptyset$, that is whenever $\Lambda' \neq \emptyset$.

This completes the proof of the part (d).

Fourthly, whenever $I_0^* \neq \emptyset$ or $I_2 \neq \emptyset$, for each index $k \in I_2$ and every $j \in I_0^k$, the parts (b) and (c) imply the relations :

$$R_k = ... = R_{j-1} = R_j$$

and

$$\varphi'_{j-1} \circ \tau_j = \varphi'_j$$

in such a way that the Theorems 3-14 and 6-9 show that the T_{j-1}-essential singular F-algebra extension (τ_j, T_{j-1}) gives the relation :

$$\tau_j(N_j) = T_{j-1}$$

and the isomorphism of multiplicative A_j-bimodules :

$$\bar{\tau}_j : N_j \rightarrowtail\!\!\twoheadrightarrow T_{j-1} \subset N_{j-1}$$

which imply the relation :

$$N'_j = \varphi'_j(N_j) = \varphi'_{j-1}(T_{j-1}) \subset \varphi'_{j-1}(N_{j-1}) = N'_{j-1}$$

for all $j \in I_0^k$.

Then, the Theorem 6-9 and more precisely the last part of its proof imply that each ideal $T_{j-1} \in \mathcal{C}(A_{j-1})$ is connected by the conditions :

$$T'_{j-1} = \varphi'_{j-1}(T_{j-1}) \qquad \text{and} \qquad T_{j-1} = N(A_{j-1}) \cap \varphi'^{-1}_{j-1}(T'_{j-1})$$

to some ideal $T'_{j-1} \in \mathcal{C}(R_{j-1}) \equiv \mathcal{C}(R_k)$ of the semisimple Artinian F-algebras :

$$R_{j-1} = R_k = \mathbf{R}(\widetilde{\Lambda}_k) = \prod_{\lambda \in \Lambda_k} R^\lambda$$

subject to the general condition :

$$T'_{j-1} \subset N'_{j-1} = \varphi'_{j-1}(N_{j-1})$$

and to the supplementary condition :

$$T'_{j-1} \neq N'_{j-1} = N'_k$$

whenever : $(j-1) = k = 0 \in I_2$, in such a way that :

$$N'_j = T'_{j-1} = \varphi'_{j-1}(T_{j-1}) \subset \varphi'_{j-1}(N_{j-1}) = N'_{j-1} = T'_{j-2}$$

for all $j \in I_0^k$, such that : $k \leq (j-2) < (j-1)$.

This implies immediately that for each $k \in I_2$ and every $j \in I_0^k$, the ideal $T'_{j-1} \in \mathcal{C}(R_{j-1}) = \mathcal{C}(R_k)$ is of the form :

$$T'_{j-1} = \bigoplus_{\lambda \in \Lambda''^k_j} R^\lambda$$

for a family :

$$(\Lambda''^k_j)_{j \in I_0^k}$$

which is a "descending sequence" of subsets :

$$\Lambda''^k_j \subset \Lambda^k = \Lambda^{k(j)}$$

subject to the condition that if $0 \in I_2$, then :

$$\Lambda''^0_1 \neq \Lambda^0 = \Lambda_0$$

Thus, with the Notations and the conditions of the Definition 6-3, this implies immediately the part (e).

At last, whenever $I_0^* \neq \varnothing$ or $I_2 \neq \varnothing$, according to the Theorem 6-9 which gives the conditions :

$$H'^j \triangleleft C'_{j-1}$$

for all $j \in I_0^*$ and according to the Lemma 2-10, it is immediate that the family (Λ'_j) and the family of integers (n^j_λ) defined by the conditions :

$$(\widetilde{\Lambda}'_j) = \widetilde{\mathbf{\Lambda}}(H'^j)$$

for all $j \in I_0^*$, verifies the conditions of the Definition 6-3 and this completes the proof.

REMARKS 6-11 - The last part of the Theorem 6-9 may be completed by the following Remarks.

(a) Whenever $I_1^* \neq \varnothing$, with the Notations of the Theorems 6-9 and 6-10, for every index $k \in I_1^*$, the Formal triangular matrix F-algebra :

$$B_k = \begin{pmatrix} H_k & L_k \\ 0 & G_k \end{pmatrix} = (H_k = H_k \lhd G_k)$$

associated to the canonical $(H_k - G_k)$-bimodule $L_k = {}_{H_k}L_{G_k}$ and its right Socle :

$$S_k = \begin{pmatrix} 0 & L_k \\ 0 & G_k \end{pmatrix} = S(B_k) = S(A_k)$$

are characterized by the conditions :

$$B_k = \prod_{\lambda \in \Lambda^k} B^\lambda$$

and

$$S_k = \prod_{\lambda \in \Lambda^k} S^\lambda = \bigoplus_{\lambda \in \Lambda^k} S^\lambda$$

by means of the family of (right) almost simple right Artinian F-algebras :

$$(B^\lambda)_{\lambda \in \Lambda}$$

with a family of right Socles :

$$(S^\lambda)_{\lambda \in \Lambda}$$

characterized by the conditions :

$$B^\lambda = R^\lambda = S^\lambda \qquad \text{for all } \lambda \in (\Lambda\text{-}\Lambda')$$

and by the conditions :

$$B^\lambda = \begin{pmatrix} H^\lambda & L^\lambda \\ 0 & R^\lambda \end{pmatrix} \text{ and } S^\lambda = \begin{pmatrix} 0 & L^\lambda \\ 0 & R^\lambda \end{pmatrix} \qquad \text{for all } \lambda \in \Lambda'$$

in which H^λ is the simple Artinian F-algebra defined by :

$$H^\lambda = \mathcal{L}(U_\lambda) = M_{q_\lambda}(K_\lambda) = [\mathcal{L}(U_\lambda^*)]^\circ \qquad \text{for all } \lambda \in \Lambda'$$

and in which L^λ is the canonical $(H^\lambda\text{-}R^\lambda)$-bimodules defined by :

$$L^\lambda = \mathcal{L}(V_\lambda, U_\lambda) = M_{p_\lambda, q_\lambda}(K_\lambda) = U_\lambda \underset{K_\lambda}{\otimes} V_\lambda^*$$

for all $\lambda \in \Lambda'$, with the Notations of the Lemma 6-6.

(b) Whenever $I_0^* \neq \varnothing$ or $I_2 \neq \varnothing$, for each index $j \in I_0^*$ associated to the index $k(j) = k \in I_2$, the last part of the Theorem 6-9 shows that in fact, the ideal $T_{j-1} \in \mathcal{C}(A_{j-1})$ may be characterized by some ideal $T'_{j-1} \in \mathcal{C}(R_{j-1}) \equiv \mathcal{C}(R_k)$ connected by the conditions :

$$T'_{j-1} = \varphi'_{j-1}(T_{j-1}) \qquad \text{and} \quad T_{j-1} = N(A_{j-1}) \cap \varphi'^{-1}_{j-1}(T'_{j-1})$$

and defined, according to the Theorem 6-10, by the condition :

$$T'_{j-1} = \bigoplus_{\lambda \in \Lambda''^k_j} R^\lambda$$

Then, if we consider the surjective F-algebra epimorphisms :

$$\tau^j_k : A_{j-1} \longrightarrow\!\!\!\!\!\!\rightarrow A_k$$

defined by the conditions :

$$\tau^j_k = Id_{A_k} \qquad\qquad \text{if } (k+1) = j \in I_0^k$$

and

$$\tau^j_k = \tau_{k+1} \circ \ldots \circ \tau_{j-1} \qquad\qquad \text{if } (k+1) < j \in I_0^k$$

by means of some elements of the Canonical Resolution $\mathfrak{R}(A)$, and which verify the relations :

$$\varphi'_{j-1} = \varphi'_k \circ \tau^j_k$$

it is easy to verify that the ideal $T_{j-1} \in \mathcal{C}(A_{j-1})$ may be characterized by some ideal $T''^j_k \in \mathcal{C}(A_k) \equiv \mathcal{C}(B_k)$ connected by the conditions :

$$T''^j_k = \tau^j_k(T_{j-1}) \qquad \text{and} \quad T_{j-1} = N(A_{j-1}) \cap [\tau^j_k]^{-1}\,(T''^j_k)$$

and defined, according to the Theorem 6-10 and the previous Notations, by the condition :

$$T''^j_k = \bigoplus_{\lambda \in \Lambda''^k_j} S^\lambda$$

which constitutes the translation of the conditions :

$$T'_{j-1} = \varphi'_k(T''^j_k) \quad \text{and} \quad T''^j_k = N(A_k) \cap \varphi'^{-1}_k(T'_{j-1})$$

which follow easily from the relations : $\varphi'_{j-1} = \varphi'_k \circ \tau^j_k$, for all $k \in I_2$ and $j \in I_0^k$.

REMARKS 6-12 - Any right Artinian F-algebra :

$$A \in \mathfrak{A}(F)$$

determines its Canonical Resolution :

$$\mathfrak{R}(A) = [A = A_m \xrightarrow{\tau_m} A_{m-1} \ldots A_i \xrightarrow{\tau_i} A_{i-1} \ldots A_1 \xrightarrow{\tau_1} A_0]$$

which is unique up to an F-isomorphism, and the previous unique *"complete invariant" :*

$$\hat{\Lambda}(A) = \hat{\Lambda}$$

in which the adjective *"complete"* has been chosen in order to explain that this invariant contains, *at the same time*, three kinds of invariants, namely :

(1) The Combinatorial Invariant $\Sigma(A) = \Sigma$.

(2) The Numerical Invariant $\upsilon(A) = \upsilon = \{(p_\lambda), (q_\lambda), (n^j_\lambda)\}$ compatible with Σ.

(3) The Algebraic Invariant constituted by the family of F-skewfields $(K_\lambda) = (K_\lambda)_{\lambda \in \Lambda}$ defined by the F-Concrete vertex set $\widetilde{\Lambda}(A) = \widetilde{\Lambda}(R) = \widetilde{\Lambda}$.

Nevertheless, this adjective *"complete"* does not mean that this "complete invariant" $\hat{\Lambda}(A) = \hat{\Lambda}$ is sufficient in order to describe the Structure of the right Artinian F-algebra A.

Indeed, the Theorem 6-9 shows that the Structure of the Canonical Resolution $\mathfrak{R}(A)$ depends explicitely from the knowledge of *a set of "parameters" :*

$$\Pi = \Pi[\mathfrak{R}(A)] = \{(\Psi_k), (\Psi'_j), (\xi_j)\}$$

constituted by the three families of *"parameters" :*

$$(\Psi_k)_{k \in I_1^*} \qquad (\Psi'_j)_{j \in I_0^*} \qquad (\xi_j)_{j \in I_0^*}$$

which are only some *"semi-invariants"*, in a suitable sense, which is possible to define more precisely. [See for instance, the part 8-(E)].

The converse problem is examined in the next section.

7. STRUCTURE OF RIGHT ARTINIAN F-ALGEBRAS.

Now, our aim is to show that conversely, for any field F, in the case of F-algebras, for any F-*"Completely structured vertex set"*:

$$\hat{\Lambda}$$

and any choice of a set of "parameters" *:*

$$\Pi = \{(\Psi_k), (\Psi'_j), (\xi_j)\}$$

constituted by three families of *"parameters" :*

$$(\Psi_k)_{k \in I_1^*} \qquad (\Psi'_j)_{j \in I_0^*} \qquad (\xi_j)_{j \in I_0^*}$$

compatible with $\hat{\Lambda}$ in the sense of the following Definition 7-2, it is possible to give the Description of a Systematic Method of Construction of a right Artinian F-algebra :

$$A \in \mathfrak{A}(F)$$

by means of a finite number of *Fundamental Constructions,* which are of two different kinds, is such a way that :

$$\hat{\Lambda}(A) = \hat{\Lambda}$$

that is, having the given *"complete invariant"* and also the given *"semi-invariants"* constituted by the given *"parameters"* which determine a Canonical Resolution $\mathfrak{R}(A)$ of the right Artinian F-algebra A.

CONSTRUCTIONS 7-1 - For any F-Completely structured vertex set $\hat{\Lambda}$, the Lemma 6-6 characterizes the F-Concrete vertex set :

$$\tilde{\Lambda} = [\Lambda ; (K_\lambda), (p_\lambda)] = [\Lambda ; (V_\lambda)] = [\Lambda ; (V_\lambda^*)]$$

the F-Concrete vertex set :

$$\tilde{\Lambda}' = [\Lambda' ; (K_\lambda), (q_\lambda)] = [\Lambda' ; (U_\lambda)] = [\Lambda' ; (U_\lambda^*)]$$

and the Generalized F-concrete vertex set :

$$\underline{\tilde{\Lambda}}' = [\Lambda ; (K_\lambda), (q_\lambda)] = [\Lambda ; (U_\lambda)] = [\Lambda ; (U_\lambda^*)]$$

whenever $\Lambda' \neq \varnothing$ or $I_1^* \neq \varnothing$, and the F-Concrete vertex sets :

$$\tilde{\Lambda}'_j = [\Lambda'_j ; (K_\lambda), (n_\lambda^j)] = [\Lambda'_j ; (W_\lambda^j)] = [\Lambda'_j ; (W_\lambda^{*j})]$$

and the Generalized F-concrete vertex sets :

$$\underline{\widetilde{\Lambda}}'_j = [\Lambda \,;\, (K_\lambda),\, (n^j_\lambda)] = [\Lambda \,;\, (W^j_\lambda)] = [\Lambda \,;\, (W^{*j}_\lambda)]$$

for all $j \in I_0^*$, whenever $I_0^* \neq \varnothing$.

These data determine the *simple Artinian* F-*algebras* :

$$R^\lambda = \mathcal{L}(V_\lambda) = M_{p_\lambda}(K_\lambda) = [\mathcal{L}(V^*_\lambda)]^\circ \qquad \text{for all } \lambda \in \Lambda$$

$$H^\lambda = \mathcal{L}(U_\lambda) = M_{q_\lambda}(K_\lambda) = [\mathcal{L}(U^*_\lambda)]^\circ \qquad \text{for all } \lambda \in \Lambda'$$

$$H'^j_\lambda = \mathcal{L}(W^j_\lambda) = M_{n^j_\lambda}(K_\lambda) = [\mathcal{L}(W*^j_\lambda)]^\circ \qquad \text{for all } j \in I_0^* \text{ and } \lambda \in \Lambda'_j$$

and also the *canonical* (H^λ-R^λ)-*bimodules :*

$$L^\lambda = \mathcal{L}(V_\lambda\,,\, U_\lambda) = M_{p_\lambda\,,\, q_\lambda}(K_\lambda) = U_\lambda \underset{K_\lambda}{\otimes} V^*_\lambda$$

for all $\lambda \in \Lambda'$ and the *canonical* (H'^j_λ-R^λ)-*bimodules :*

$$L'^j_\lambda = \mathcal{L}(V_\lambda,\, W^j_\lambda) = M_{p_\lambda\,,\, n^j_\lambda}(K_\lambda) = W^j_\lambda \underset{K_\lambda}{\otimes} V^*_\lambda$$

for all $j \in I_0^*$ and $\lambda \in \Lambda'_j$.

Then, according to the Corollary 6-7, there exist the *semisimple Artinian* F-*algebras :*

$$R_i = \mathbf{R}(\widetilde{\Lambda}_i) = \prod_{\lambda \in \Lambda_i} R^\lambda \qquad \text{for all } i \in I$$

$$G_k = \mathbf{R}(\widetilde{\Lambda}^k) = \prod_{\lambda \in \Lambda^k} R^\lambda \qquad \text{for all } k \in I_1$$

$$H_k = \mathbf{R}(\widetilde{\Lambda}'^k) = \prod_{\lambda \in \Lambda'^k} H^\lambda \qquad \text{for all } k \in I_1^*$$

$$C'_{j-1} = \mathbf{R}(\widetilde{\Lambda}''_j) = \prod_{\lambda \in \Lambda''_j} R^\lambda \qquad \text{for all } j \in I_0^*$$

$$H'^j = \mathbf{R}(\widetilde{\Lambda}'_j) = \prod_{\lambda \in \Lambda'_j} H'^j_\lambda \qquad \text{for all } j \in I_0^*$$

and also the *canonical* (H_k-G_k)-*bimodules* :

$$L_k = \prod_{\lambda \in \Lambda'^k} L^\lambda = \prod_{\lambda \in \Lambda^k} \mathcal{L}(V_\lambda,\, U_\lambda) \qquad \text{for all } k \in I_1^*$$

and the *canonical* (H'^j-C'_{j-1})-*bimodules :*

$$M'_j = L'^j = \prod_{\lambda \in \Lambda'_j} L'^j_\lambda = \prod_{\lambda \in \Lambda_j} \mathcal{L}(V_\lambda, W^j_\lambda) \qquad \text{for all } j \in I_0^*$$

A last, the conditions :

$$T'_{j-1} = \bigoplus_{\lambda \in \Lambda''^k_j} R^\lambda \qquad \text{for all } j \in I_0^* \text{ and } k = k(j)$$

characterize the proper two-sided ideals :

$$T'_{j-1} \in \mathcal{C}(R_{j-1}) \equiv \mathcal{C}(R_j) \equiv \mathcal{C}(R_k)$$

such that :

$$C'_{j-1} = R_{j-1}/T'_{j-1} \qquad \text{for all } j \in I_0^*$$

Thus, these data characterize the *"geometrical objects"* defined by $\hat{\Lambda}$.

Now, with all these *"geometrical objects"* and for any choice of a *set of "parameters"* :

$$\Pi = \{(\Psi_k), (\Psi'_j), (\xi_j)\}$$

constituted by three families of *"parameters" :*

$$(\Psi_k)_{k \in I_1^*} \qquad (\Psi'_j)_{j \in I_0^*} \qquad (\xi_j)_{j \in I_0^*}$$

compatible with $\hat{\Lambda}$, it is possible to give the Description of a Systematic Method of Construction of a right Artinian F-algebra :

$$A \in \mathfrak{A}(F)$$

with the *"complete invariant"* :

$$\hat{\Lambda}(A) = \hat{\Lambda}$$

and characterized by its (finite right) Canonical Resolution :

$$\mathfrak{R}(A) = [A = A_m \xrightarrow{\tau_m} A_{m-1} \dots A_i \xrightarrow{\tau_i} A_{i-1} \dots A_1 \xrightarrow{\tau_1} A_0]$$

in which the right Artinian F-algebras A_i are constructed by a recurrence on the integer $i \in I = \{0, 1, 2, \dots, m\}$, characterized by the following conditions.

The semisimple Artinian F-algebra $A_0 \in \mathfrak{A}_0(F)$ is defined by the condition :

$$\boxed{A_0 = R_0 = \mathbf{R}(\widetilde{\Lambda}_0) = \mathbf{R}(\widetilde{\Lambda}^0) = G_0}$$

and when the right Artinian F-algebras $A_0, A_1, \dots, A_i$ are constructed, such that :

$$\hat{\Lambda}(A_0) = \hat{\Lambda}_0 = (\hat{\Lambda}/\Lambda_0),\ \hat{\Lambda}(A_1) = \hat{\Lambda}_1 = (\hat{\Lambda}/\Lambda_1) \dots \hat{\Lambda}(A_i) = \hat{\Lambda}_i = (\hat{\Lambda}/\Lambda_i)$$

if $\boxed{i < m}$, the right Artinian F-algebra : $A_{i+1} \in \mathfrak{A}(F)$, is obtained by one and only one of the two following **"Fundamental Constructions"**.

(a) **FIRST FUNDAMENTAL CONSTRUCTION.**

This "First Fundamental Construction" occurs in the case where :

$$\boxed{(i+1) = k \in I_1^*}$$

Then, the Theorem 2-13 shows easily that the existence of a *"parameter"* Ψ_k, constituted by *an injective* F-*algebra homomorphism*, of the form :

$$\Psi_k : A_i = A_{k-1} \rightarrowtail H_k$$

that is *compatible with* $\hat{\Lambda}$, implies the existence of a right Artinian F-algebra $A_k \in \mathfrak{A}(F)$, defined by the condition :

$$\boxed{A_k = A_{i+1} = \begin{pmatrix} A_i & L_k \\ 0 & G_k \end{pmatrix} \equiv \begin{pmatrix} \Psi_k(A_i) & L_k \\ 0 & G_k \end{pmatrix} = (A_i \overset{\Psi_k}{\rightarrowtail} H_k \lhd G_k)}$$

with a right Socle :

$$S_k = S_{i+1} = \begin{pmatrix} 0 & L_k \\ 0 & G_k \end{pmatrix} = S(A_k) = S(A_{i+1})$$

and which appears in the general F-algebra extension, of the form :

$$\boxed{(\tau_{i+1}) \qquad 0 \longrightarrow S_{i+1} \rightarrowtail A_{i+1} \overset{\tau_{i+1}}{\twoheadrightarrow} A_i \longrightarrow 0}$$

in which : $S_{i+1} = S(A_{i+1}) = N(A_{i+1}) = Q(A_{i+1})$ and $A_i = A_{i+1}/S_{i+1}$, so that the right Artinian F-algebra $A_k = A_{i+1}$ has the *"complete invariant" :*

$$\hat{\Lambda}(A_k) = \hat{\Lambda}_k = (\hat{\Lambda}/A_k)$$

induced by $\hat{\Lambda}$ on the subset : $A_k \subset \Lambda$.

(b) **SECOND FUNDAMENTAL CONSTRUCTION.**

This "Second Fundamental Construction" occurs in the case where :

$$\boxed{(i+1) = j \in I_0^*}$$

Then, by means of the canonical surjective F-algebra epimorphism :

$$\varphi'_i : A_i \twoheadrightarrow R_i = A_i/J(A_i) = R(\tilde{\Lambda}_i) = R_{j-1}$$

the proof of the Theorem 6-9 implies that the conditions :

$$T_{j-1} \equiv T_i = N(A_i) \cap \varphi'^{-1}_i(T'_{j-1}) \qquad \text{and} \quad T'_i \equiv T'_{j-1} = \varphi'_i(T_{j-1}) = \varphi'_i(T_i)$$

determine a proper two-sided ideal :

$$T_{j-1} \equiv T_i \in \mathcal{C}(A_i) \equiv \mathcal{C}(A_{j-1})$$

such that :

$$C_{j-1} = C_i = A_i/T_i \qquad \text{and} \qquad C'_{j-1} = R_{j-1}/T'_{j-1} = C_{j-1}/J(C_{j-1})$$

so that the existence of a *"parameter"* Ψ'_j constituted by a F-*algebra homorphism :*

$$\Psi'_j \in Mor_F[C_{j-1}, H'^j] \equiv Mor_F[C_i, H'^j]$$

that is *compatible with* $\hat{\Lambda}$, determines the non null C_i-bimodule :

$$M_{i+1} = M_j \in \mathcal{M}(A_i, T_i) \equiv \mathcal{M}(A_{j-1}, T_{j-1})$$

defined by the characterization :

$$M_{i+1} \equiv M_j = [M'_j \,;\, \Psi'_j : C_i \to H'^j]$$

and therefore, the Theorem 3-14 shows easily that the existence of a *"parameter"* $\xi_j = \hat{h}_j$ constituted by a T-***essential cohomology class :***

$$\xi_j \equiv \xi_{i+1} \in H^2_e(A_i, T_i, M_{i+1}) \equiv H^2_e(A_{j-1}, T_{j-1}, M_j)$$

that is *compatible with* $\hat{\Lambda}$, implies the existence of a right Artinian F-algebra $A_j \in \mathfrak{A}(F)$, defined by the condition :

$$\boxed{(A_{i+1}, N_{i+1}) = (A_i, T_i, M_{i+1}, h_{i+1}) = (A_i, T_i, M_{i+1}, \xi_{i+1})}$$

or

$$\boxed{(A_j, N_j) = (A_{j-1}, T_{j-1}, M_j, h_j) = (A_{j-1}, T_{j-1}, M_j, \xi_j)}$$

which determines a T_i-essential singular F-algebra extension of the form :

$$\boxed{(\tau_{i+1}, T_i) \quad 0 \longrightarrow M_{i+1} \rightarrowtail (A_{i+1}, N_{i+1}) \xrightarrow{\tau_{i+1}} \!\!\!\!\twoheadrightarrow (A_i, T_i) \longrightarrow 0}$$

with the relation :

$$S_{i+1} = S(A_{i+1}) = M(A_{i+1}) \oplus N(A_{i+1}) = M_{i+1} \oplus N_{i+1}$$

in which : $M_{i+1} = M(A_{i+1}) = Q(A_{i+1}) = Q(A_j) \neq (0)$ and $A_i = A_{i+1}/M_{i+1}$, so that the right Artinian F-algebra $A_j = A_{i+1}$ has the *"complete invariant" :*

$$\hat{\Lambda}(A_j) = \hat{\Lambda}_j = (\hat{\Lambda}/\Lambda_j)$$

induced by $\hat{\Lambda}$ on the subset : $\Lambda_j \subset \Lambda$.

DEFINITION 7-2 - *For any* F-**"Completely structured vertex set"** :

$$\hat{\Lambda}$$

a **set of "parameters"** :

$$\Pi = \{(\Psi_k), (\Psi'_j), (\xi_j)\}$$

constituted by three families of "parameters" :

$$(\Psi_k)_{k \in I_1^*} \qquad (\Psi'_j)_{j \in I_0^*} \qquad (\xi_j)_{j \in I_0^*}$$

is **compatible with** $\hat{\Lambda}$ *if the* "parameters" *have the previous characterizations used in the Constructions 7-1, which permit the Construction, by degrees, that* *is the ascending iterative construction, of a Canonical Resolution* $\mathfrak{R}(A)$ *of a right Artinian* F-*algebra* A.

DEFINITION 7-3 - *For any* F-**"Completely structured vertex set"** :

$$\hat{\Lambda}$$

and for any **set of "parameters"** :

$$\Pi = \{(\Psi_k), (\Psi'_j), (\xi_j)\}$$

compatible with $\hat{\Lambda}$, *let :*

$$\boxed{A = A_m = \mathbf{D}[\hat{\Lambda} ; \Pi] = \mathbf{D}[\hat{\Lambda} ; (\Psi_k) , (\Psi'_j) , (\xi_j)]}$$

be the right Artinian F-algebra *defined by the Constructions 7-1.*

THEOREM 7-4 ("STRUCTURE THEOREM")

For any field F, *the Structure of any* right Artinian F-algebra :

$$A \in \mathfrak{K}(F)$$

is defined, "up to an F-*isomorphism", by a (or by its)* F-**"Completely structured vertex set"** :

$$\hat{\Lambda} = \{\Lambda ; \Sigma ; (K_\lambda) , (p_\lambda) , (q_\lambda) , (n^j_\lambda)\}$$

equipped with a **"Combinatorial Structure"** Σ, *noted :*

$$\Sigma = [\Lambda ; m , (I) ; \{\Lambda_i\} , \Lambda' , (\Lambda''^k_j) , (\Lambda'_j)]$$

or more precisely :

$$\Sigma = [\Lambda ; m , (I) ; \{\Lambda_i\} , (\Lambda^k) , \Lambda' , (\Lambda'^k) , (\Lambda''^k_j) , (\Lambda''_j) , (\Lambda'_j)]$$

and by a **set of "parameters"** *:*

$$\Pi = \{(\Psi_k) , (\Psi'_j) , (\xi_j)\}$$

constituted by three families of "parameters" :

$$(\Psi_k) = (\Psi_k)_{k \in I_1} \qquad (\Psi'_j) = (\Psi'_j)_{j \in I_0^*} \qquad (\xi_j) = (\xi_j)_{j \in I_0^*}$$

compatible with $\hat{\Lambda}$, *in such a way that the* right Artinian F-algebra A *has a realization of the form :*

$$\boxed{A = \mathbf{D}[\hat{\Lambda} ; \Pi] = \mathbf{D}[\hat{\Lambda} ; (\Psi_k) , (\Psi'_j) , (\xi_j)]}$$

in which $\hat{\Lambda} = \hat{\Lambda}(A)$ *is a* "complete invariant", *escorted by some* "semi-invariants" *constituted by a* set of "parameters" Π, *which determines a Canonical Resolution* $\mathfrak{R}(A)$ *of the* right Artinian F-algebra A.

Moreover, with the previous notations, the underlying F-vector space $|A|$ *of the* right Artinian F-algebra A *is characterized by one of the following equivalent conditions :*

(15) $$|A| = \left[\bigoplus_{\lambda\in\Lambda} \mathcal{L}(V_\lambda)\right] \oplus \left[\bigoplus_{\lambda\in\Lambda'} \mathcal{L}(V_\lambda, U_\lambda)\right] \oplus \left[\bigoplus_{j\in I_0^*}\left[\bigoplus_{\lambda\in\Lambda'_j} \mathcal{L}(V_\lambda, W_\lambda^j)\right]\right]$$

or

(16) $$|A| = \left[\bigoplus_{\lambda\in\Lambda} M_{p_\lambda}(K_\lambda)\right] \oplus \left[\bigoplus_{\lambda\in\Lambda'} M_{p_\lambda, q_\lambda}(K_\lambda)\right] \oplus \left[\bigoplus_{j\in I_0^*}\left[\bigoplus_{\lambda\in\Lambda'_j} M_{p_\lambda, n_\lambda^j}(K_\lambda)\right]\right]$$

the "multiplication" *of the* right Artinian F-algebra A *being determined by the three families :*

$$(\Psi_k)_{k\in I_1} \qquad (\Psi'_j)_{j\in I_0^*} \qquad (\xi_j)_{j\in I_0^*}$$

of **"parameters"**.

PROOF - The Definition 7-3 and the Constructions 7-1 imply that any F-algebra A, which has a realization of the form :

$$A = \mathbf{D}[\hat{\Lambda}\,;\,\Pi]$$

is a right Artinian F-algebra.

Conversely, the Theorems 6-9 and 6-10 imply that any right Artinian F-algebra A has a realization of this form :

$$A = \mathbf{D}[\hat{\Lambda}\,;\,\Pi]$$

in which $\hat{\mathbf{\Lambda}}(A) = \hat{\Lambda}$ and Π is a *set of "parameters"* compatible with $\hat{\Lambda}$ and which determines a Canonical Resolution $\mathfrak{R}(A)$ of the right Artinian F-algebra A.

This completes the proof of the first assertion.

Moreover, in the case of F-algebras, for each index $k \in I_1^*$, the ***"one-link"*** :

$$0 \longrightarrow S_k \rightarrowtail A_k \xrightarrow{\tau_k} \twoheadrightarrow A_{k-1} \longrightarrow 0$$

is an exact sequence of F-vector spaces, which gives :

(17) $$|A_k| = |A_{k-1}| \oplus S_k$$

and for each index $j \in I_0^*$, the ***"zero-link"***:

$$0 \longrightarrow M_j \rightarrowtail A_j \xrightarrow{\tau_j} \twoheadrightarrow A_{j-1} \longrightarrow 0$$

is an exact sequence of F-vector spaces, which gives :

(18) $$|A_j| = |A_{j-1}| \oplus M_j$$

Then, in the Canonical Resolution $\mathfrak{R}(A)$, the relations (17) and (18) imply the relation :

(19) $$|A| = |A_m| = |A_0| \oplus \left[\bigoplus_{k \in I_1^*} S_k\right] \oplus \left[\bigoplus_{j \in I_0^*} M_j\right]$$

The Constructions 7-1 imply the relation :

(20) $$|A_0| = |R_0| = \bigoplus_{\lambda \in \Lambda_0} \mathcal{L}(V_\lambda)$$

The Constructions 7-1, the Remarks 6-11 and the Definition 6-3 imply the relations :

$$S_k = \bigoplus_{\lambda \in \Lambda^k} S^\lambda = \left[\bigoplus_{\lambda \in \Lambda^k} R^\lambda\right] \oplus \left[\bigoplus_{\lambda \in \Lambda'^k} L^\lambda\right]$$

and therefore the Constructions 7-1 imply the relations :

(21) $$S_k = \left[\bigoplus_{\lambda \in \Lambda^k} \mathcal{L}(V_\lambda)\right] \oplus \left[\bigoplus_{\lambda \in \Lambda'^k} \mathcal{L}(V_\lambda, U_\lambda)\right]$$

The Constructions 7-1 imply the relation :

(22) $$M_j = M'_j = \bigoplus_{\lambda \in \Lambda'_j} L'^j_\lambda = \bigoplus_{\lambda \in \Lambda'_j} \mathcal{L}(V_\lambda, W^j_\lambda)$$

According to the Definition 6-3 which gives the relation :

(23) $$\Lambda' = \coprod_{k \in I_1^*} \Lambda'^k$$

and the relation :

(24) $$\Lambda = \Lambda_0 \coprod \left[\coprod_{k \in I_1^*} \Lambda^k\right]$$

it is immediate that the relations (19), (20), (21) and (22) imply the relation (15) which is equivalent to the relation (16), according to the Constructions 7-1.

The last assertion is obvious and completes the proof.

REMARKS 7-5 -

(a) In the previous Theorem 7-4 the F-skewfields $K_\lambda \in \mathcal{K}(F)$ are not necessarily of finite dimension over F.

(b) In the previous Theorem 7-4, if each F-skewfield $K_\lambda \in \mathcal{K}(F)$ is of *finite dimension over* F :

(25) $$\boxed{r_\lambda = \dim_F[K_\lambda]} \qquad \text{for all } \lambda \in \Lambda$$

then the relation (16) implies the relation :

(26) $$r = \sum_{\lambda\in\Lambda} (r_\lambda \times p_\lambda \times p_\lambda) + \sum_{\lambda\in\Lambda'} (r_\lambda \times p_\lambda \times q_\lambda) + \sum_{j\in I_0^*}\left[\sum_{\lambda\in\Lambda'_j} (r_\lambda \times p_\lambda \times n_\lambda^j)\right]$$

which gives the formula :

(27) $$\boxed{r = \sum_{\lambda\in\Lambda} (r_\lambda \times p_\lambda)\left[p_\lambda + q_\lambda + \sum_{j\in I_0^*} n_\lambda^j\right]}$$

which characterizes the *finite dimension over* F :

(28) $$\boxed{r = \dim_F[A]}$$

of the right Artinian F-algebra A.

REMARKS 7-6

The Theorem 7-4 of the previous paper [10], gives a "Construction Theorem" for right Artinian rings and the previous Theorem 7-4 gives a "Structure Theorem" for right Artinian F-algebras, in the sense that any right Artinian ring or right Artinian F-algebra A has a realization of the form :

$$A = \mathbf{D}[\hat{\Lambda}\,;\Pi] = \mathbf{D}[\hat{\Lambda}\,;(\Psi_k)\,,(\Psi'_j)\,,(\xi_j)]$$

for an unique "Completely structured vertex set" or F-"Completely structured vertex set" :

$$\hat{\Lambda} = \hat{\mathbf{\Lambda}}(A)$$

These results constitute a generalization of the "Construction Theorem" for semisimple Artinian rings, given by the Wedderburn-Artin Structure Theorem, which implies that any semisimple Artinian ring R has a realization of the form :

$$R = \mathbf{R}(\tilde{\Lambda})$$

for an unique "Concrete vertex set" :

$$\tilde{\Lambda} = \tilde{\mathbf{\Lambda}}(R)$$

Likewise, according to the Theorem 6-9, the Theorem 6-10 gives a new kind of partial "Classification" of right Artinian F-algebras, by means of the "complete invariant" $\hat{\mathbf{\Lambda}}(\ \ )$, in the sense that for two right Artinian F-algebras A and A', the *existence of an isomorphism* $u : A \xrightarrow{\sim} A'$, implies :

$$\hat{\mathbf{\Lambda}}(A) = \hat{\mathbf{\Lambda}}(A')$$

or equivalently, the condition :

$$\hat{\Lambda}(A) \neq \hat{\Lambda}(A')$$

implies that the F-algebras A and A' *are not isomorphic.*

This result constitutes a generalization of the "Classification" of semisimple Artinian F-algebras, by means of the "classical invariant" $\widetilde{\Lambda}(\ )$, described in the Proposition 4-1 and which is given by the Wedderburn-Artin Structure Theorem.

SECOND PART :

ILLUSTRATION OF THE STRUCTURE THEOREM

8. EXAMPLES AND APPLICATIONS.

Let F be any *field* and let K be any F-skewfield : $K \in \mathcal{K}(F)$.

(A) CANONICAL RESOLUTION.

(a) . For any non null integer $n \in \mathbb{N}^*$, let $M_n(K)$ be the F-algebra of $(n \times n)$ square matrices with coefficients in K.

Firstly, this semisimple Artinian F-algebra :

$$A = M_n(K)$$

is characterized by its F-*"concrete vertex set"* :

$$\widetilde{\Lambda}(A) = \widetilde{\Lambda} = [\Lambda = \{1\} ; (K_1 = K) , (p_1 = n)]$$

Secondly, this *right Artinian* F-*algebra* A has the trivial finite right *Canonical Resolution :*

$$\mathfrak{R}(A) = [A = A_0]$$

and verifies the relations :

$$m = \rho \dim(A) = 0 \quad \text{and} \quad \rho(A) = \varnothing$$

which give the «*Complete Decomposition* of $m = 0$» :

$$(I) = I(A) \qquad I = I_0^* \coprod I_1 = \left(\coprod_{k \in I_2} I_0^k\right) \coprod I_1$$

in which : $I = I_1 = \{0\}$ and $I^* = I_0^* = I_1^* = I_2 = \varnothing$; and the F-*"Completely structured vertex set"* :

$$\hat{\Lambda}(A) = \hat{\Lambda} = \{\Lambda = \{1\} ; \Sigma = \Sigma(A) ; (K_1 = K), (p_1 = n), (q_1 = 0), (n_\lambda^j) = \varnothing\}$$

(since $I_0^* = \varnothing$), with the *"Combinatorial Structure"* :

$\Sigma(A) = \Sigma = [\Lambda = \{1\}; m = 0, (I); \{\Lambda_i\} = \{\Lambda_0\}, (\Lambda^k) = (\Lambda^0), \Lambda' = \varnothing, \varnothing, \varnothing, \varnothing, \varnothing]$

in which :

$$\{1\} = \Lambda = \Lambda_0 = \Lambda^0 \qquad \text{for } 0 \in I \text{ and } 0 \in I_1$$

and :

$$(\Lambda'^k)_{k \in I_1^*} = \varnothing \quad ; \quad (\Lambda''^k_j)_{k \in I_2} = \varnothing \quad ; \quad (\Lambda''_j)_{j \in I_0^*} = \varnothing \quad ; \quad (\Lambda'_j)_{j \in I_0^*} = \varnothing$$

since $I_1^* = \varnothing$, $I_2 = \varnothing$ and $I_0^* = \varnothing$.

Of course, this example is trivial since all the informations contained in $\hat{\Lambda}(A)$ are already contained in $\widetilde{\Lambda}(A)$, but it shows how the General Structure Theorem for right Artinian F-algebras is applicable, for instance, to simple Artinian F-algebras.

(b) For any non null integer $n \in \mathbb{N}^*$, let $T_n(K)$ be the F-subalgebra :

$$T_n(K) \subset M_n(K)$$

of *upper triangular matrices.*

The obvious relation :

$$T_{n+1}(K) = \left(\begin{array}{c|c} T_n(K) & \begin{matrix} K \\ \cdot \\ \cdot \\ \cdot \\ K \end{matrix} \\ \hline 0 \; . \; . \; . \; 0 & K \end{array}\right)$$

is equivalent to the relation :

$$T_{n+1}(K) = (T_n(K) \subset M_n(K) \lhd M_1(K))$$

that is, according to the Theorem 2-13, to the *general* F-*algebra extension :*

$$0 \longrightarrow S_{n+1}(K) \rightarrowtail T_{n+1}(K) \twoheadrightarrow T_n(K) \longrightarrow 0$$

in which $S_{n+1}(K)$ is the *idempotent right Socle* of the (right) *almost semisimple right Artinian* F-*algebra* $T_{n+1}(K)$.

Firstly, for any $m \in \mathbb{N}$, the right Artinian F-algebra :

$$A = T_{m+1}(K)$$

characterizes its F-*"concrete vertex set" :*

$$\widetilde{\Lambda}(A) = \widetilde{\Lambda} = [\Lambda \,;\, (K_\lambda), (p_\lambda)]$$

in which : $\Lambda = \{0, 1, 2, ..., \lambda, ..., m\}$ and :

$$(K_\lambda) = (K_\lambda)_{\lambda \in \Lambda} = (K_0, K_1, K_2, ..., K_\lambda, ..., K_m)$$

with :

$$K = K_0 = K_1 = K_2 = ... = K_\lambda = ... = K_m$$

and

$$(p_\lambda) = (p_\lambda)_{\lambda \in \Lambda} = (p_0 = 1, p_1 = 1, ..., p_\lambda = 1, ..., p_m = 1)$$

and which determines the *semisimple Artinian* F-*algebra :*

$$A_{/J(A)} = R = \mathbf{R}(\widetilde{\Lambda}) = \prod_{\lambda \in \Lambda} M_{p_\lambda}(K_\lambda) = \prod_{\lambda \in \Lambda} K_\lambda = K_0 \times K_1 \times ... \times K_m$$

Secondly, for any $m \in \mathbb{N}$, the right Artinian F-algebra :

$$A = T_{m+1}(K)$$

has a finite right *Canonical Resolution :*

$$\mathfrak{R}(A) = [A = A_m \xrightarrow{\tau_m} A_{m-1} \ldots A_k \xrightarrow{\tau_k} A_{k-1} \ldots A_1 \xrightarrow{\tau_1} A_0]$$

characterized by the following conditions :

$$A_0 = T_1(K_0) = M_1(K_0) = G_0 = K_0 \equiv K$$

$$A_1 = T_2(K_1) = (A_0 = T_1(K_0) \overset{\Psi_1}{\rightarrowtail} M_1(K_1) = H_1 \lhd G_1 = M_1(K_1))$$

$$A_2 = T_3(K_2) = (A_1 = T_2(K_1) \overset{\Psi_2}{\rightarrowtail} M_2(K_2) = H_2 \lhd G_2 = M_1(K_2))$$

$$A_k = T_{k+1}(K_k) = (A_{k-1} = T_k(K_{k-1}) \overset{\Psi_k}{\rightarrowtail} M_k(K_k) = H_k \lhd G_k = M_1(K_k))$$

in which :

$$K \equiv K_0 = K_1 = K_2 = \ldots = K_k = \ldots = K_{m-1} = K_m$$

and in which the *"parameters" :*

$$\Psi_1, \Psi_2, \ldots, \Psi_k, \ldots, \Psi_m$$

are the canonical *injective* F-*algebra homomorphisms :*

$$\Psi_k : A_{k-1} = T_k(K_{k-1}) \rightarrowtail M_k(K_k) = H_k$$

resulting from the conditions : $K_{k-1} = K_k$, for all $k \in \{1, 2, \ldots, m\}$.

Thus, this *right Artinian* F-*algebra* $A = T_{m+1}(K)$ verifies the relations :

$$m = \rho \dim(A) \quad \text{and} \quad \boldsymbol{\rho}(A) = (\rho_1 = 1, \ldots, \rho_k = 1, \ldots, \rho_m = 1)$$

which give the *«Complete Decomposition* of m» :

$$(I) = \mathbf{I}(A) \qquad I = I_0^* \coprod I_1 = \left(\coprod_{k \in I_2} I_0^k\right) \coprod I_1$$

in which : $I = I_1 = \{0, 1, 2, \ldots, m\}$, $I^* = I_1^* = \{1, 2, \ldots, m\}$ and $I_0^* = I_2 = \varnothing$;

and the F-*"Completely structured vertex set" :*

$$\hat{\Lambda}(A) = \hat{\Lambda} = \{\Lambda = \{0, 1, \ldots, m\} ; \Sigma = \boldsymbol{\Sigma}(A) ; (K_\lambda), (p_\lambda), (q_\lambda), (n_\lambda^j)\}$$

in which :

$$(p_\lambda) = (p_\lambda \equiv 1), (q_\lambda) = (q_\lambda \equiv \lambda) \text{ and } (n_\lambda^j) = \varnothing ;$$

with the *"Combinatorial Structure" :*

$$\Sigma(A) = \Sigma = \{\Lambda ; m, (I) ; \{\Lambda_i\} , (\Lambda^k), \Lambda', (\Lambda'^k), \varnothing, \varnothing, \varnothing\}$$

in which :

$$\Lambda = \{0, 1, 2, ..., \lambda, ..., m\}$$

$\Lambda_i = \{0, ..., i\}$ for all $i \in I$

$\Lambda^k = \{k\}$ for all $k \in I_1$

$\Lambda'^k = \{k\}$ for all $k \in I_1^*$

$$\Lambda' = \{1, 2, ..., m\}$$

and :

$$(\Lambda''^k_j)_{k\in I_2} = \varnothing \quad ; \quad (\Lambda''_j)_{j\in I_0^*} = \varnothing \quad ; \quad (\Lambda'_j)_{j\in I_0^*} = \varnothing$$

since $I_2 = \varnothing$ and $I_0^* = \varnothing$.

This example, shows the difference between the F-*"concrete vertex set"* $\widetilde{\Lambda}$ (A) which gives only informations about R = A/J(A) and the F-*"Completely structured vertex set"* $\hat{\Lambda}$(A) which, with the family of *"parameters"* $(\Psi_k)_{k\in I_1^*}$, characterizes the Structure of the right Artinian F-algebra $A = T_{m+1}(K)$.

Moreover, it is possible to remark that if in the family of F-skewfields :

$$(K_\lambda) = (K_\lambda)_{\lambda\in\Lambda} = (K_0, K_1, K_2 ..., K_\lambda, ..., K_m)$$

the condition :

$$K \equiv K_0 = K_1 = K_2 = ... = K_\lambda = ... = K_m$$

is replaced by the existence of F-*algebra monomorphisms :*

$$K_0 \subset K_1 \subset K_2 \subset ... \subset K_\lambda \subset ... \subset K_m$$

the previous conditions characterize a *"generalized upper triangular matrix F-algebra" :*

$$T_{m+1}(K_0 \subset K_1 \subset ... \subset K_\lambda \subset ... \subset K_m) = \begin{pmatrix} K_0 & K_1 & K_\lambda & K_{m-1} & K_m \\ 0 & K_1 & K_\lambda & K_{m-1} & K_m \\ . & . & K_\lambda & . & . \\ . & . & & . & . \\ . & . & & K_{m-1} & K_m \\ 0 & 0 & ... & 0 & K_m \end{pmatrix}$$

with the same *"Combinatorial Structure"* and with the same *"numerical invariants"*.

(c) For any integer $m \in \mathbb{N}$, let :

$$A = D_m(K) = K[X]/(X^{m+1}) = K[x] = \{a = \sum_{p=0}^{p=m} a_p x^p \; ; \; a_p \in K \; ; \; x^{m+1} = 0\}$$

be the *Artinian local* F-*algebra* factor of the F-algebra K[X] of polynomials with coefficients in K by the ideal generated by X^{m+1}, that is the F-algebra generated by K and by a central element x such that $x^{m+1} = 0$.

For m = 0, then $A = D_0(K) = K$ is a F-skewfield.

For m = 1, then $A = D_1(K)$ is the F-algebra of *"dual numbers"* over the skewfield K :

$$D_1(K) = K[X]/(X^2) = K[x] = \{a = a_0 + a_1 x \; ; \; a_i \in K \; ; \; x^2 = 0\}$$

For $m \geq 1$, it is immediate that :

$$J(A) = \mathfrak{M} = (x) \quad ; \quad A/J(A) = R = K \quad ; \quad N(A) = (0)$$

and :

$$[J(A)]^m = S(A) = M(A) = M = (x^m) = Kx^m \neq (0)$$

which give the *"exact sequence" :*

$$(\tau) \qquad 0 \longrightarrow M \rightarrowtail A \overset{\tau}{\twoheadrightarrow} B \longrightarrow 0$$

in which :

$$B = A/M = D_{m-1}(K) = \{b = \sum_{p=0}^{p=m-1} b_p x^p \; ; \; b_p \in K \; ; \; x^m = 0\}$$

and :

$$M = M_m = Kx^m = \{a_m x^m = x^m a_m \; ; \; a_m \in K\}$$

is the B-bimodule characterized by the conditions :

$$b \cdot (a_m x^m) = (b_0 a_m) x^m \qquad \text{and} \qquad (a_m x^m) b' = (a_m b'_0) x^m$$

for every $b \in B$ and every $b' \in B$.

In fact, since : $N(A) = N = (0)$, the proper two-sided ideal :

$$T \equiv (0) \in \mathcal{C}(B)$$

and the *singular* F-*algebra extension* (τ) determine the $T \equiv (0)$-***essential singular*** F-***algebra extension :***

$$(\tau, T) \qquad 0 \longrightarrow M \rightarrowtail (A, (0)) \overset{\tau}{\twoheadrightarrow} (B, T) \longrightarrow 0$$

which is characterized, according to the Theorem 3-14, by an unique $T \equiv (0)$-***essential cohomology class :***

$$\hat{h}_m = \xi_m \in H^2_e(B, T, M) \subset H^2(B, (0), M) \equiv H^2(B, M)$$

such that :

$$(A, N) = (B, T, M, \xi_m) = (B, (0), M, h_m)$$

that is :

$$A = (B, M, \xi_m) = (B, M, h_m)$$

for the $T \equiv (0)$-***essential*** 2-***cocycle*** *:*

$$h_m \in Z^2_e(B, T, M) \subset Z^2(B, (0), M) \equiv Z^2(B, M)$$

characterized by the conditions :

$$h_m(b, b') = \left(\sum_{p=1}^{p=m-1} b_p b'_{m-p} \right) x^m \qquad \text{if } m \geq 2$$

and

$$h_1(b, b') = 0.\ x = 0 \qquad \text{if } m = 1$$

for every $b \in B$ and every $b' \in B$.

Then, firstly, for any $m \in \mathbb{N}$, the right Artinian F-algebra :

$$A = D_m(K)$$

characterizes its F-*"concrete vertex set" :*

$$\widetilde{\Lambda}(A) = \widetilde{\Lambda} = [\Lambda = \{1\} ; (K_\lambda) = (K_1 \equiv K) , (p_\lambda) = (p_1 = 1)]$$

which determines the *semisimple Artinian* F-*algebra :*

$$A/J(A) = R = \mathbf{R}(\widetilde{\Lambda}) = \prod_{\lambda \in \Lambda} M_{p_\lambda}(K_\lambda) = M_1(K_1) = K_1 \equiv K$$

Secondly, with the previous notations, this right Artinian F-algebra :

$$A = D_m(K)$$

has a finite right *Canonical Resolution :*

$$\mathfrak{R}(A) = [A = A_m \xrightarrow{\tau_m} A_{m-1} \dots A_j \xrightarrow{\tau_j} A_{j-1} \dots A_1 \xrightarrow{\tau_1} A_0]$$

characterized by the following conditions :

$$(A_0, N_0) = (D_0(K), (0)) = (G_0, (0)) = (K, (0))$$

$$(A_1, N_1) = (D_1(K), (0)) = (A_0, T_0, M_1, \xi_1) = (D_0(K), (0), M_1, h_1 \equiv 0)$$

$$(A_2, N_2) = (D_2(K), (0)) = (A_1, T_1, M_2, \xi_2) = (D_1(K), (0), M_2, h_2)$$

.

.

$$(A_j, N_j) = (D_j(K), (0)) = (A_{j-1}, T_{j-1}, M_j, \xi_j) = (D_{j-1}(K), (0), M_j, h_j)$$

.

.

$$(A_m, N_m) = (D_m(K), (0)) = (A_{m-1}, T_{m-1}, M_m, \xi_m) = (D_{m-1}(K), (0), M_m, h_m)$$

for the T_{j-1}-***essential cohomology classes*** *:*

$$\hat{h}_j = \xi_j \in H^2_e(A_{j-1}, T_{j-1}, M_j) \subset H^2(A_{j-1}, M_j) = H^2(D_{j-1}(K), M_j)$$

defined by the $T_{j-1} \equiv (0)$-***essential*** 2-***cocycles*** *:*

$$h_j \in Z^2_e(A_{j-1}, T_{j-1}, M_j) \subset Z^2(A_{j-1}, M_j) = Z^2(D_{j-1}(K), M_j)$$

characterized by the conditions :

$$h_j(b, b') = \left(\sum_{p=1}^{p=j-1} b_p b'_{j-p}\right) x^j \qquad \text{if } j \geq 2$$

and

$$h_1(b, b') = 0.\ x = 0 \qquad \text{if } j = 1$$

for every $b \in B$ and every $b' \in B$, and in which, with the general notations, the non null C_{j-1}-bimodules :

$$M_j \in \mathcal{M}(A_{j-1}, T_{j-1})$$

have the characterizations :

$$M_j = [M'_j ; \Psi'_j : C_{j-1} \longrightarrow H'^j]$$

in which :

$$C_{j-1} = A_{j-1/T_{j-1}} \equiv A_{j-1} \ ; \ M'_j = K \simeq Kx^j \ ; \ H'^j = K$$

and the *"parameters"* constitued by the F-*algebra homomorphisms :*

$$\Psi'_j \in \mathrm{Mor}_F[C_{j-1}, H'^j]$$

are determined by the conditions :

$$\Psi'_j\left(c = \sum_{p=0}^{p=j-1} c_p\, x^p\right) = c_0$$

for every $c \in C_{j-1}$.

Thus, this *right Artinian* F-*algebra* $A = D_m(K)$ verifies the relations :

$$m = \rho\dim(A) \qquad \text{and} \qquad \rho(A) = (\rho_1 = 0, \ldots, \rho_j = 0, \ldots, \rho_m = 0)$$

which give the « *Complete Decomposition* of m »:

$$(I) = I(A) \qquad I = I_0^* \coprod I_1 = \left(\coprod_{k \in I_2} I_0^k\right) \coprod I_1$$

in which : $I = \{0, 1, 2, \ldots, m\}$, $I_1 = I_2 = \{0\}$, $I_1^* = \varnothing$ and $I^* = I_0^* = I_0^0 = \{1, 2, \ldots, m\}$; and the F-*"Completely structured vertex set" :*

$$\hat{\Lambda}(A) = \hat{\Lambda} = \{\Lambda = \{1\} \ ; \Sigma = \Sigma(A) \ ; (K_\lambda) = (K_1 \equiv K), (p_\lambda) = (p_1 \equiv 1),$$
$$(q_\lambda) = (q_1 \equiv 0), (n_1^j) = (n_1^j \equiv 1)\}$$

with the *"Combinatorial Structure" :*

$$\Sigma(A) = \Sigma = \{\Lambda = \{1\} \ ; m, (I) \ ; \{\Lambda_i\}, (\Lambda^k), \Lambda' = \varnothing, (\Lambda''^k_j), (\Lambda''_j), (\Lambda'_j)\}$$

in which :

$$\Lambda_i = \Lambda = \{1\} \qquad \text{for all } i \in I$$
$$\Lambda^k = \Lambda^0 = \Lambda_0 = \{1\} \qquad \text{for } k = 0 \in I_2$$
$$\Lambda' = \varnothing$$

$(\Lambda'^k) = (\Lambda'^k)_{k\in I_1^*} = \varnothing$ since $I_1^* = \varnothing$

$\Lambda''^k_j = \Lambda''^0_j = \varnothing$ for $k = 0 \in I_2$ and $j \in I_0^k = I_0^0$

and

$\Lambda''_j = \Lambda'_j = \Lambda = \{1\}$ for all $j \in I_0^0 = I^*$

This example, shows also the difference between the F-*"concrete vertex set"* $\widetilde{\Lambda}(A)$ which gives only informations about $A/J(A) = R = K = K_1$ and the F-*"Completely structured vertex set"* $\hat{\Lambda}(A)$ which, with the two families of *"parameters"* $(\Psi'_j)_{j\in I_0^*=I^*}$ and $(\xi_j)_{j\in I_0^*=I^*}$, characterizes the Structure of the right Artinian F-algebra $A = D_m(K)$.

(B) ESSENTIAL COHOMOLOGY.

(a) For the F-algebra $D_0 = D_0(F) \equiv F$, whenever F is considered as a (F-F)-bimodule or as a D_0-bimodule $F \equiv M_1$, for the proper two-sided ideal :

$$T_0 \equiv (0) \in \mathcal{C}(F) \equiv \mathcal{C}(D_0)$$

according to the relation :

$$l_{D_0}(M_1) = l_F(F) = (0) = T_0$$

the Proposition 3-15 and the obvious relation :

$$H^2(D_0, M_1) = H^2(F, F) = \{0\}$$

imply the relation :

$$\{\xi_1 = \hat{0}\} = \{0\} = H^2_e(D_0, T_0, M_1) = H^2_e(F, (0), F) = H^2(F, F)$$

in which $\xi_1 = \hat{0}$ is the unique $T_0 \equiv (0)$-***essential cohomology class*** from $D_0 \equiv F$ into $M_1 \equiv F$, which determines the unique T_0-***essential singular class :***

$$[\sigma_1, T_0] \in Ext_e(D_0, T_0, M_1) = Ext_e(F, (0), F)$$

characterized by the $T_0 \equiv (0)$-***essential singular*** F-***algebra extension :***

$$(\sigma_1, T_0)\; 0 \longrightarrow (F \equiv M_1) \overset{i_1}{\rightarrowtail} (D_1, N_1 \equiv (0)) \overset{\sigma_1}{\twoheadrightarrow} (D_0, T_0) \longrightarrow 0$$

in which the *"pair"* $(D_1, N_1 \equiv (0))$ is defined by the condition :

$$(D_1, N_1) = (D_0, T_0, M_1, \xi_1) = (F, (0), F, \hat{0})$$

which determines, in particular, the *singular* F-*algebra extension :*

$$(\sigma_1) \qquad 0 \longrightarrow (F \equiv M_1) \overset{i_1}{\rightarrowtail} (D_1 = D_1(F)) \overset{\sigma_1}{\twoheadrightarrow} (D_0 \equiv F) \longrightarrow 0$$

in which the right Artinian F-algebra D_1 is defined by the condition :

$$D_1 = (D_0, M_1, \xi_1) = (F, F, 0) = D_1(F)$$

in such a way that the non null element :

$$i_1(1) = x \in J(D_1)$$

which characterizes the isomorphism of F-vector spaces :

$$i_1 : F \equiv M_1 \xrightarrow{\sim} J(D_1) = M(D_1)$$

determines a *"realization"* of the F-algebra D_1, as the F-algebra of *"dual numbers":*

$$D_1 = D_1(F) = F[X]/(X^2) = F[x] = \{a = a_0 + a_1x \; ; x^2 = 0, a_i \in F\}$$

which has also the *"matrix realization"* :

$$D_1 = D_1(F) = \begin{bmatrix} \boxed{F} & F \\ 0 & \boxed{F} \end{bmatrix} = \left\{ a = \begin{bmatrix} a_0 & a_1 \\ 0 & a_0 \end{bmatrix} \quad a_i \in F \right\}$$

It is convenient to remark that when the F-algebra D_1 is given, the previous "realization" depends of the choice of a *non null* element $x \in J(D_1)$, which is not uniquely determined, but only "up to an F-automorphism" :

$$\omega_\mu \in Aut_F(D_1) \simeq F^*$$

such that :

$$\omega_\mu(x) = x' = \mu x \qquad \text{and} \qquad \omega_\mu(a) = \omega_\mu(a_0 + a_1x) = a_0 + a_1\mu x = a_0 + a_1x'$$

for the associated element $\mu \in F^* \simeq Aut_F(D_1)$ and for every $a \in D_1$.

(**b**) For the F-algebra $D_1 = D_1(F)$, whenever the factor F-algebra :

$$D_{1/M_1} = D_1(F)/J(D_1(F)) = \begin{bmatrix} \boxed{F} & 0 \\ 0 & \boxed{F} \end{bmatrix} = \left\{ a = \begin{bmatrix} a_0 & 0 \\ 0 & a_0 \end{bmatrix} \quad a_0 \in F \right\} \equiv F \equiv M_2$$

is considered as a $(D_1\text{-}D_1)$-bimodule or as a D_1-bimodule $F \equiv M_2$, it is easy to prove the relations :

$$B^2(D_1, M_2) = B^2(D_1(F), F) = \{0\}$$

and

$$Z^2(D_1, M_2) = Z^2(D_1, F) = \{f_\mu \in C^2(D_1, F) \; ; f_\mu(a, a') = \mu a_1 a'_1 \text{ and } \mu \in F\} \equiv F$$

which imply the relation :

$$H^2(D_1, M_2) = H^2(D_1, F) = \{f_\mu \in C^2(D_1, F) \; ; f_\mu(a, a') = \mu a_1 a'_1 \text{ and } \mu \in F\} \equiv F$$

Thus, for the proper two-sided ideal :

$$T_1 \equiv (0) \in \mathcal{C}(D_1)$$

and for any cohomology class :

$$\hat{f}_\mu = \mu \in F \equiv H^2(D_1, M_2) = H^2(D_1, F)$$

according to the obvious relations :

$$l_{D_1}(M_2) = \{b" = b"_0 + b"_1x \; ; \; b"_i \in F \text{ and } b"_0 = 0\}$$

and

$$r_{D_1}(b") = \{b' = b'_0 + b'_1x \; ; \; b'_i \in F \text{ and } b"_1b'_0 = 0\}$$

for every element :

$$a" = (a"_2, b") \in M_2 \times l_{D_1}(M_2) \subset M_2 \times D_1$$

which determines the F-vector space :

$$s_{D_1}(a", f_\mu) = \{b = (b_0 + b_1x) \in D_1 \; ; \; a"_2b + f^*_\mu(b", b) = 0\}$$

$$= \{b = (b_0 + b_1x) \in D_1 \; ; \; a"_2b_0 + \mu b"_1b_1 = 0\}$$

the Definition 3-7 considers the condition (E), in which the conditions (r) and (s) become the condition :

(r") $\qquad r_{D_1}(b") \subset s_{D_1}(a", f_\mu)$

and the condition :

(s") $\qquad b"s_{D_1}(a", f_\mu) \cap T_1 = (0)$

such that, since $T_1 = (0)$, this last condition (s") is automatically verified.

Then, whenever $\mu = 0$, since the *special* and *non null* element :

$$a" = (a"_2, b") = (0, x) \in M_2 \times l_{D_1}(M_2) \subset M_2 \times D_1$$

verifies the condition (r"), the Definition 3-7 shows that the 2-cocycle $f_0 \equiv 0$ *is not a* T_1*-essential 2-cocyle :*

$$f_0 \equiv 0 \notin Z^2_e(D_1, T_1, M_2) \subset Z^2(D_1, T_1, M_2)$$

and therefore, the Definition 3-9 shows that the cohomology class $\hat{f}_0 \equiv \hat{0} = 0$ *is not a* T_1*-essential cohomology class :*

$$\hat{f}_0 = 0 \notin H^2_e(D_1, T_1, M_2) \subset H^2(D_1, T_1, M_2)$$

On the contrary, whenever $\mu \neq 0$, that is whenever $\mu \in F^* = F - \{0\}$, since the condition : $b" \neq 0$, that is $b"_1 \neq 0$, implies the relations :

$$x \in r_{D_1}(b") \qquad \text{and} \qquad x \notin s_{D_1}(a", f_\mu)$$

the condition (r") implies necessarily : $b" = 0$, which gives :

$$r_{D_1}(b") = D_1 = s_{D_1}(a", f_\mu)$$

and in particular : $1 \in s_{D_1}(a", f)$, which implies : $a"_2 = 0$, and this proves that the condition (r") implies : $a" = (a"_2, b") = 0$, in such a way that the Definition 3-7 shows that, for $\mu \in F^*$, the 2-cocycle f_μ is a T_1*-essential cocycle :*

$$f_\mu \in Z^2_e(D_1, T_1, M_2) \subset Z^2(D_1, T_1, M_2)$$

and therefore, the Definition 3-9 shows that, for $\mu \in F^*$, the cohomology class $\hat{f}_\mu = \mu$ is a T_1-*essential cohomology class :*

$$\hat{f}_\mu = \mu \in H^2_e(D_1, T_1, M_2) \subset H^2(D_1, T_1, M_2)$$

These two last results characterize the *"space"* :

$$H^2_e(D_1, T_1, M_2)$$

of T_1-*essential cohomology classes* from D_1 into M_2, *which is only a set,* by the relation :

$$H^2_e(D_1, T_1, M_2) \equiv F^* \subset F \equiv H^2(D_1, T_1, M_2) = H^2(D_1, M_2)$$

and the Theorems 3-10 and 3-14 imply that every $T_1 \equiv (0)$-*essential cohomology class :*

$$\hat{f}_\mu = \mu \in F^* \equiv H^2_e(D_1, T_1, M_2)$$

determines the unique T_1-*essential singular class :*

$$[\sigma_2, T_1] \in \mathrm{Ext}_e(D_1, T_1, M_2) = \mathrm{Ext}_e(D_1(F), (0), F)$$

characterized by the $T_1 \equiv (0)$-*essential singular* F-*algebra extension :*

$$(\sigma_2, T_1) \qquad 0 \longrightarrow (F \equiv M_2) \overset{i_2}{\rightarrowtail} (B_\mu, N_2 \equiv (0)) \overset{\sigma_2}{\twoheadrightarrow} (D_1, T_1) \longrightarrow 0$$

in which the *"pair"* $(B_\mu, N_2 \equiv (0))$ is defined by the condition :

$$(B_\mu, N_2) = (D_1, T_1, M_2, \mu) = (D(F), (0), F, f_\mu)$$

which determines, in particular, the *singular* F-*algebra extension :*

$$(\sigma_2) \qquad 0 \longrightarrow (F \equiv M_2) \overset{i_2}{\rightarrowtail} (B_\mu = B_\mu(F)) \overset{\sigma_2}{\twoheadrightarrow} (D_1 = D_1(F)) \longrightarrow 0$$

in which the right Artinian F-algebra B_μ is defined by the condition :

$$B_\mu = (D_1, M_2, \mu) = (D_1(F), F, f_\mu) = B_\mu(F)$$

which implies easily the existence of a F-basis $(1, e_1, e_2)$, which determines a *"realization"* of the F-algebra B_μ of the form :

$$B_\mu = B_\mu(F) = \{a = a_0 + a_1e_1 + a_2e_2 \ ; \ e_1e_2 = e_2e_1 = e_2^2 = 0, e_1^2 = \mu e_2 \ ; \ a_i \in F\}$$

such that :

$$\sigma_2(a) = \sigma_2(a_0 + a_1e_1 + a_2e_2) = (a_0 + a_1x) \in D_1 = D_1(F)$$

for every $a \in B_\mu$ and such that :

$$i_2(a_2) = a_2e_2 \qquad\qquad \text{for all } a_2 \in F \equiv M_2$$

and also the *"matrix realization"* :

$$B_\mu = B_\mu(F) = \begin{bmatrix} \boxed{F} & \mu\boxed{F} & F \\ 0 & \boxed{F} & \boxed{F} \\ 0 & 0 & \boxed{F} \end{bmatrix} = \left\{ a = \begin{bmatrix} a_0 & \mu a_1 & a_2 \\ 0 & a_0 & a_1 \\ 0 & 0 & a_0 \end{bmatrix} \quad a_i \in F \right\}$$

in which appears the *"parameter"* $\mu \in F^* \equiv H^2_e(D_1, T_1, M_2)$.

(c) For the previous F-algebra $B_\mu = B_\mu(F)$, whenever the factor algebra :

$$B_\mu/J(B_\mu) = \begin{bmatrix} \boxed{F} & 0 & 0 \\ 0 & \boxed{F} & 0 \\ 0 & 0 & \boxed{F} \end{bmatrix} = \left\{ a = \begin{bmatrix} a_0 & 0 & 0 \\ 0 & a_0 & 0 \\ 0 & 0 & a_0 \end{bmatrix} \quad a_0 \in F \right\} \equiv F \equiv M_3$$

is considered as a $(B_\mu$-$B_\mu)$-bimodule or as a B_μ-bimodule $F \equiv M_3$, it is easy to prove the relation :

$$H^2(B_\mu, M_3) = H^2(B_\mu, F)$$
$$= \{g_\nu \in C^2(B_\mu, F) \; ; \; g_\nu(a, a') = \nu(a_1a'_2 + a_2a'_1) \text{ and } \nu \in F\} \equiv F$$

Thus, for the proper two-sided ideal :

$$T_2 \equiv (0) \in \mathcal{C}(B_\mu)$$

as in the previous example, it is possible to prove that the *"space"* :

$$H^2_e(B_\mu, T_2, M_3)$$

of T_2-***essential*** *cohomology classes* from B_μ into M_3, *which is only a set,* is characterized by the relation :

$$H^2_e(B_\mu, T_2, M_3) \equiv F^* \subset F \equiv H^2(B_\mu, T_2, M_3) = H^2(B_\mu, M_3)$$

Then, the Theorems 3-10 and 3-14 imply that every $T_2 \equiv (0)$-***essential*** *cohomology class :*

$$\hat{g}_\nu = \nu \in F^* \equiv H^2_e(B_\mu, T_2, M_3)$$

determines the unique T_2-***essential*** *singular class* $[\sigma_3, T_3]$, defined by :

$$(\sigma_3, T_2) \qquad 0 \longrightarrow (F \equiv M_3) \overset{i_3}{\rightarrowtail} (B_{\mu,\nu}, N_3 \equiv (0)) \overset{\sigma_3}{\twoheadrightarrow} (B_\mu, T_2) \longrightarrow 0$$

in which the *"pair"* $(B_{\mu,\nu}, N_3 \equiv (0))$ is defined by the condition :

$$(B_{\mu,\nu}, N_2) = (B_\mu, T_2, M_3, \nu) = (B_\mu(F), (0), F, g_\nu)$$

which determines, in particular, the *singular* F-*algebra extension :*

$$(\sigma_3) \qquad 0 \longrightarrow (F \equiv M_3) \overset{i_3}{\rightarrowtail} (B_{\mu,\nu} = B_{\mu,\nu}(F)) \overset{\sigma_3}{\twoheadrightarrow} (B_\mu = B_\mu(F)) \longrightarrow 0$$

in which the right Artinian F-algebra $B_{\mu,\nu}$ is defined by the condition :

$$B_{\mu,\nu} = (B_\mu, M_3, \nu) = (B_\mu(F), F, g_\nu) = B_{\mu,\nu}(F)$$

which implies easily the existence of a F-basis $(1, e_1, e_2, e_3)$, which determines a *"realization"* of the F-algebra $B_{\mu,\nu} = B_{\mu,\nu}(F)$ of the form :

$$B_{\mu,\nu} = B_{\mu,\nu}(F) = \left\{ a=a_0+a_1e_1+a_2e_2+a_3e_3 \,, \; \begin{array}{r} e_1e_3=e_3e_1=e_2^2=e_2e_3=e_3e_2=e_3^2 = 0 \\ e_1^2=\mu e_2, e_1e_2=e_2e_1=\nu e_3 \,,\; a_i\in F \end{array} \right\}$$

such that :

$$\sigma_3(a) = \sigma_3(a_0+a_1e_1+a_2e_2+a_3e_3) = (a_0+a_1e_1+a_2e_2) \in B_\mu = B_\mu(F)$$

for every $a \in B_{\mu,\nu}$ and such that :

$$i_3(a_3) = a_3e_3 \qquad\qquad \text{for all } a_3 \in F \equiv M_3$$

and also the *"matrix realization"* :

$$B_{\mu,\nu}(F) = \begin{bmatrix} \boxed{F} & \nu\boxed{F} & \nu\boxed{F} & F \\ 0 & \boxed{F} & \mu\boxed{F} & \boxed{F} \\ 0 & 0 & \boxed{F} & \boxed{F} \\ 0 & 0 & 0 & \boxed{F} \end{bmatrix} = \left\{ a = \begin{bmatrix} a_0 & \nu a_1 & \nu a_2 & a_3 \\ 0 & a_0 & \mu a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \\ 0 & 0 & 0 & a_0 \end{bmatrix} , \; a_i \in F \right\}$$

in which appear the *"parameters"* $\mu \in F^* \equiv H_e^2(D_1, T_1, M_2)$ and

$\nu \in F^* \equiv H_e^2(B_\mu, T_2, M_3)$.

(**d**) At last, for the right Artinian F-algebra :

$$B = (D_1(F) \subset M_2(F) \lhd F)$$

which has the *"matrix realization"* :

$$B = B(F) = \begin{bmatrix} \boxed{F} & F & F \\ 0 & \boxed{F} & F \\ 0 & 0 & F \end{bmatrix} = \left\{ b = \begin{bmatrix} b_0 & b_1 & l_1 \\ 0 & b_0 & l_2 \\ 0 & 0 & g_1 \end{bmatrix} , \; \begin{array}{r} b_i \in F \\ l_j \in F \\ g_1 \in F \end{array} \right\}$$

and its *idempotent right Socle :*

$$S(B) = N(B) = T \in \mathscr{C}(B)$$

with the *"matrix realization"* :

$$S(B) = N(B) = T = \begin{bmatrix} 0 & 0 & F \\ 0 & 0 & F \\ 0 & 0 & F \end{bmatrix} = \left\{ t = \begin{bmatrix} 0 & 0 & l_1 \\ 0 & 0 & l_2 \\ 0 & 0 & g_1 \end{bmatrix} , \; \begin{array}{r} b_i = 0 \\ l_j \in F \\ g_1 \in F \end{array} \right\}$$

which determines the right Artinian F-algebra :

$$B/T = C = D_1(F)$$

and the *general* F-*algebra extension :*

$$(\tau') \qquad 0 \longrightarrow T \overset{i_0}{\rightarrowtail} (B = B(F)) \overset{\tau'}{\twoheadrightarrow} (C = D_1(F)) \longrightarrow 0$$

and for the C-bimodule or D_1-bimodule $M' \equiv M_3 \equiv M_2 \equiv F$, the Example **(B)** - **(b)** gives the relation :

$$H^2(B, T, M') = H^2(C, M') = Z^2(C, M') = Z^2(D_1, M')$$
$$= \{h_\mu \in C^2(D_1, F) \; ; \; h_\mu(a, a') = \mu a_1 a'_1 \text{ and } \mu \in F\} \equiv F$$

Then, for the proper two-sided ideal :

$$T \in \mathscr{C}(B)$$

and for any cohomology class :

$$\hat{h}_\mu = \mu \in F \equiv H^2(B, T, M') = H^2(C, M')$$

according to the obvious relation :

$$l_B(M') = \{b' \in B \; ; \; b'_0 = 0\}$$

for every element :

$$a' = (m', b') \in M' \times l_B(M') \subset M' \times B$$

which determines the F-vector space :

$$s_B(a', h_\mu) = \{b \in B \; ; \; m'b + \overset{*}{h}_\mu(b', b) = 0\} = \{b \in B \; ; \; m'b_0 + \mu b'_1 b_1 = 0\}$$

the Definition 3-7 considers the condition (E) , in which the conditions (r) and (s) become the condition :

$$(r') \qquad r_B(b') \subset s_B(a', h_\mu)$$

and the condition :

$$(s') \qquad b' s_B(a', h_\mu) \cap T = (0).$$

If the element $b' \in l_B(M')$ verifies : $b'_1 \neq 0$, the element :

$$t_0 \in T \subset s_B(a', h_\mu)$$

defined by the conditions : $l_1 = g_1 = 0$ and $l_2 = 1$, verifies the relation :

$$0 \neq b't_0 \in b's_B(a', h_\mu) \cap T$$

Therefore, the condition (s') implies : $b'_1 = 0$, that is : $b' \in T$, and the Lemma 1-2 implies the existence of an element $t' \in T$ such that : $b' = b't'$, which verifies :

$$b' = b't' \in b's_B(a', h_\mu) \cap T = (0)$$

that is : $b' = 0$, which gives : $r_B(b') = B$, so that the condition (r') implies : $s_B(a', h_\mu) = B$, and in particular the relation : $b'' \in s_B(a', h_\mu)$, for the element $b'' \in B$ defined by the conditions : $b''_1 = l''_1 = l''_2 = g''_1 = 0$ and $b''_0 = 1$, which imply : $m' = 0$, that is : $a' = (m', b') = 0$, in such a way that the Definition 3-7 shows that, for $\mu \in F$, the 2-cocycle h_μ is a T-***essential*** *cocycle :*

$$h_\mu \in Z^2_e(B, T, M') \equiv Z^2(B, T, M') = Z^2(C, M')$$

and therefore, the Definition 3-9 shows that, for $\mu \in F$, the cohomology class $\hat{h}_\mu = \mu$ is a T-***essential*** *cohomology class :*

$$\hat{h}_\mu = \mu \in F \equiv H^2_e(B, T, M') \equiv H^2(B, T, M') = H^2(C, M')$$

This proves that the *"space" :*

$$H^2_e(B, T, M')$$

of T-***essential*** *cohomology classes* from B into M', *which is in general only a set,* is characterized by the relation :

$$H^2_e(B, T, M') \equiv F \equiv H^2(B, T, M') = H^2(C, M')$$

Moreover, the Theorems 3-10 and 3-14 imply that every T-***essential*** *cohomology class :*

$$\hat{h}_\mu = \mu \equiv F \equiv H^2_e(B, T, M') \equiv H^2_e(B, T, M_3)$$

determines the unique T-***essential*** *singular class :*

$$[\sigma', T] \in Ext_e(B, T, M') \equiv Ext_e(B, T, M_3)$$

characterized by the T-***essential*** *singular* F-*algebra extension :*

$$(\sigma', T) \qquad 0 \longrightarrow (M' \equiv F) \overset{i'}{\rightarrowtail} (B'_\mu, N') \overset{\sigma'}{\twoheadrightarrow} (B, T) \longrightarrow 0$$

in which the *"pair"* (B'_μ, N') is defined by the condition :

$$(B'_\mu, N') = (B, T, M', \mu) = (B(F), T, F, h_\mu)$$

which determines, in particular the *singular* F-*algebra extension :*

$$(\sigma') \qquad 0 \longrightarrow (M' \equiv F) \overset{i'}{\rightarrowtail} (B'_\mu = B'_\mu(F)) \overset{\sigma'}{\twoheadrightarrow} (B = B(F)) \longrightarrow 0$$

in which the right Artinian F-algebra B'_μ is defined by the condition :

$$B'_\mu = (B, M', \mu) = (B(F), F, h^*_\mu) = B'_\mu(F)$$

which implies easily that this F-algebra B'_μ has the *"matrix realization"* of the form :

$$B'_\mu = B'_\mu(F) = \begin{bmatrix} \boxed{F} & 0 \\ 0 & F \end{bmatrix} = \left\{ a = \begin{bmatrix} a_0 & 0 \\ 0 & g_1 \end{bmatrix} \right\}$$

completely characterized in the following $\boxed{\text{TABLE N° 1}}$, in which the coefficients a_i, l_j and g_l are in F, in such a manner that the relations :

$$N' = N(B'_\mu) = (0, T) = \{a \in B'_\mu \, ; a_0 = a_1 = a_2 = 0\}$$

and

$$M' = M(B'_\mu) = (F, 0) = \{a \in B'_\mu \, ; a_0 = a_1 = l_1 = l_2 = g_1 = 0, a_2 \in F\}$$

characterize the *right Socle :*

$$S' = S(B'_\mu) = M(B'_\mu) \oplus N(B'_\mu) = M' \oplus N' = \{a \in B'_\mu \, ; a_0 = a_1 = 0\}$$

(C) FIRST EXAMPLE.

Firstly, with the previous notations, every $\mu \in F$ determines the right Artinian F-algebra :

$$A = A^\mu = A^\mu(F) = (B'_\mu(F) \subset M_6(F) \triangleleft F)$$

which has the *"matrix realization"* of the form :

$$A = A^\mu = A^\mu(F) = \begin{bmatrix} \boxed{F} & F \\ 0 & F \end{bmatrix} = \left\{ a = \begin{bmatrix} a_0 & l_3 \\ 0 & g_2 \end{bmatrix} \right\}$$

completely characterized in the following [TABLE N° 2], in which the coefficients a_i, l_j and g_k are in F, in such a manner that its *idempotent right Socle* is characterized by the relation :

$$S = S(A) = S(A^\mu) = \{a \in A = A^\mu \, ; a_0 = a_1 = a_2 = l_1 = l_2 = g_1 = 0\}$$

and determines the right Artinian F-algebra :

$$A/S = A^\mu/S = B'_\mu = B'_\mu(F)$$

and the *general* F-*algebra extension :*

$$(\sigma) \qquad 0 \longrightarrow S \overset{i}{\rightarrowtail} (A = A^\mu = A^\mu(F)) \overset{\sigma}{\twoheadrightarrow} (B'_\mu = B'_\mu(F)) \longrightarrow 0$$

For any *field* F and every $\mu \in F$, this construction gives an example of *right Artinian* F-*algebra :*

$$A = A^\mu = A^\mu(F)$$

of *finite dimension over* F :

$$\dim_F [A] = r = 13$$

with a finite right *Canonical Resolution :*

$$\mathfrak{R}(A) = [A = (A_4 = A^\mu) \overset{\tau_4=\sigma}{\twoheadrightarrow} (A_3 = B'_\mu) \overset{\tau_3=\sigma'}{\twoheadrightarrow}$$

$$(A_2 = B) \overset{\tau_2=\tau'}{\twoheadrightarrow} (A_1 = D_1) \overset{\tau_1=\sigma_1}{\twoheadrightarrow} (A_0 = F)]$$

which gives the finite right *Resolutive Dimension :*

$$\rho \dim(A) = m = 4$$

that is :

$$I = \{0, 1, 2, 3, 4\} \qquad I^* = \{1, 2, 3, 4\}$$

and the finite right *Canonical Sequence :*

TABLES OF MATRIX REALIZATIONS

TABLE N° 1

$$
B'_{\mu} = B'_{\mu}(F) = \begin{pmatrix} \boxed{F} & \mu\boxed{F} & F & 0 & 0 & 0 \\ 0 & \boxed{F} & \boxed{F} & 0 & 0 & 0 \\ 0 & 0 & \boxed{F} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{F} & \boxed{F} & F \\ 0 & 0 & 0 & 0 & \boxed{F} & F \\ 0 & 0 & 0 & 0 & 0 & F \end{pmatrix} = \left\{ a = \begin{pmatrix} a_0 & \mu a_1 & a_2 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_0 & a_1 & l_1 \\ 0 & 0 & 0 & 0 & a_0 & l_2 \\ 0 & 0 & 0 & 0 & 0 & g_1 \end{pmatrix} \right\}
$$

TABLE N° 2

$$
A = A^{\mu} = A^{\mu}(F) = \begin{pmatrix} \boxed{F} & \mu\boxed{F} & F & 0 & 0 & 0 & F \\ 0 & \boxed{F} & \boxed{F} & 0 & 0 & 0 & F \\ 0 & 0 & \boxed{F} & 0 & 0 & 0 & F \\ 0 & 0 & 0 & \boxed{F} & \boxed{F} & F & F \\ 0 & 0 & 0 & 0 & \boxed{F} & F & F \\ 0 & 0 & 0 & 0 & 0 & F & F \\ 0 & 0 & 0 & 0 & 0 & 0 & F \end{pmatrix} = \left\{ a = \begin{pmatrix} a_0 & \mu a_1 & a_2 & 0 & 0 & 0 & l_3 \\ 0 & a_0 & a_1 & 0 & 0 & 0 & l_4 \\ 0 & 0 & a_0 & 0 & 0 & 0 & l_5 \\ 0 & 0 & 0 & a_0 & a_1 & l_1 & l_6 \\ 0 & 0 & 0 & 0 & a_0 & l_2 & l_7 \\ 0 & 0 & 0 & 0 & 0 & g_1 & l_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_2 \end{pmatrix} \right\}
$$

$$\rho(A) = (\rho_1 = 0, \rho_2 = 1, \rho_3 = 0, \rho_4 = 1)$$

equivalent to the conditions :

$$I_0^* = \{j \in I^* ; \rho_j = 0\} = \{1, 3\} \qquad I_1^* = \{k \in I^* ; \rho_k = 1\} = \{2, 4\}$$

which give a « *Complete Decomposition* of m = 4» of the form :

$$(I) = I(A) \qquad I = I_0^* \coprod I_1 = \left(\coprod_{k \in I_2} I_0^k\right) \coprod I_1$$

in which : $I_1 = \{0, 2, 4\}$, $I_2 = \{k \in I_1 ; (k+1) \in I_0^*\} = \{0, 2\}$, $I_0^0 = \{1\}$ and $I_0^2 = \{3\}$.

In order to complete the characterization of the F-*"Completely structured vertex set" :*

$$\hat{\Lambda}(A) = \hat{\Lambda} = \{\Lambda ; \Sigma ; (K_\lambda), (p_\lambda), (q_\lambda), (n_\lambda^j)\}$$

equipped with its *"Combinatorial Structure" :*

$$\mathbf{\Sigma}(A) = \Sigma = [\Lambda ; m = 4, (I) ; \{\Lambda_i\}, (\Lambda^k), \Lambda', (\Lambda'^k), (\Lambda''^k_j), (\Lambda''_j), (\Lambda'_j)]$$

in which the relations :

$$\Lambda = \{1, 2, 3\}$$

$$(K_\lambda) = (K_1 = F, K_2 = F, K_3 = F) \text{ and } (p_\lambda) = (p_1 = 1, p_2 = 1, p_3 = 1)$$

are obvious, it is possible to prove the relations :

$$\Lambda_0 = \{1\} = \Lambda_1 \subset \Lambda_2 = \{1, 2\} = \Lambda_3 \subset \Lambda_4 = \{1, 2, 3\} = \Lambda ;$$

$$\Lambda^0 = \{1\} \quad ; \quad \Lambda^2 = \{2\} \quad ; \quad \Lambda^4 = \{3\} ;$$

$$\Lambda' = \{2, 3\} \quad ; \quad \Lambda'^0 = \varnothing \quad ; \quad \Lambda'^2 = \{2\} \quad ; \quad \Lambda'^4 = \{3\} ;$$

$$\Lambda''^0_1 = \varnothing \qquad \Lambda''^2_3 = \{2\}$$

$$\Lambda''_1 = \Lambda'_1 = \Lambda_0 = \{1\} \qquad \Lambda''_3 = \Lambda'_3 = \Lambda_2 - \Lambda''^2_3 = \{1\}$$

and also the relations :

$$(q_\lambda) = (q_1 = 0, \; q_2 = 2, q_3 = 6)$$

$$(n_\lambda^1) = (n_1^1 = 1, \; n_2^1 = 0, n_3^1 = 0)$$

$$(n_\lambda^3) = (n_1^3 = 1, \; n_2^3 = 0, n_3^3 = 0)$$

which imply in particular the relation :

$$A_0 = B_0 = G_0 = R_0 = \prod_{\lambda\in \Lambda_0} R^\lambda = R^1 = K_1 = F$$

and the relation :

$$A_{/J(A)} = R = R_4 = \prod_{\lambda\in \Lambda} R^\lambda = K_1 \times K_2 \times K_3 = F \times F \times F$$

Moreover, with the Constructions 7-1, the Theorem 7-4 shows that the Structure of the right Artinian F-algebra A is characterized by its F-*"Completely structured vertex set"* $\hat{\Lambda} = \hat{\Lambda}(A)$ and by the three families of *"parameters"* :

$$(\Psi_k)_{k\in I_1^*} \qquad (\Psi'_j)_{j\in I_0^*} \qquad (\xi_j)_{j\in I_0^*}$$

characterized by the following conditions :

(c_1) For $j = 1 \in I_0^*$, the *"parameter"* Ψ'_1 is the F-*algebra homomorphism :*

$$\Psi'_1 = Id_{K_1} : C_0 = K_1 = F \longrightarrow H'^1 = K_1 = F$$

which determines the non null C_0-bimodule :

$$M_1 = [M'_1 ; \Psi'_1 : C_0 \longrightarrow H'^1] \equiv F$$

and the *"parameter"* ξ_1 is the unique $T_0 \equiv (0)$-***essential*** *cohomology class :*

$$\xi_1 = \hat{0} \in H_e^2(A_0, T_0, M_1) = H_e^2(D_0, T_0, M_1) = H_e^2(F, (0), F)$$

which give the characterization :

$$(D_1, N_1 \equiv (0)) = (A_1, N_1) = (A_0, T_0, M_1, \xi_1) = (F, (0), F, \hat{0})$$

(b_1) For $k = 2 \in I_1^*$, the *"parameter"* Ψ_2 is the *canonical injective* F-*algebra homomorphism :*

$$\Psi_2 : A_1 = D_1 = D_1(F) \rightarrowtail H_2 = M_2(K_2) = M_2(F)$$

which gives the characterization :

$$B = A_2 = (A_1 \overset{\Psi_2}{\rightarrowtail} H_2 \lhd G_2) \equiv (D_1(F) \subset M_2(F) \lhd F)$$

(c_2) For $j = 3 \in I_0^*$, which implies the relation :

$$T_2 = T''^3_2 = \bigoplus_{\lambda\in \Lambda''^2_3} S^\lambda = S^2 = T = S(A_2) = S(B)$$

the *"parameter"* Ψ'_3 is the *canonical* F-*algebra homomorphism :*

$$\Psi'_3 : C_2 = A_2/T_2 = B/T = C = D_1(F) \twoheadrightarrow H'^3 = K_1 = F$$

which determines the non null C_2-bimodule :

$$M_3 = [M'_3 ; \Psi'_3 : C_2 \twoheadrightarrow H'^3] = M' \equiv F$$

and the *"parameter"* ξ_3 is the $T_2 \equiv T$-***essential** cohomology class :*

$$\xi_3 = \hat{h}_\mu = \mu \in F \equiv H_e^2(A_2, T_2, M_3) \equiv H_e^2(B, T, M')$$

which give the characterization :

$$(B'_\mu, N') = (A_3, N_3) = (A_2, T_2, M_3, \xi_3) = (B(F), T, F, h_\mu)$$

(b_2) For $k = 4 \in I_1^*$, the *"parameter"* Ψ_4 is the *canonical injective F-algebra homomorphism :*

$$\Psi_4 : A_3 = B'_\mu = B'_\mu(F) \rightarrowtail H_4 = M_6(K_3) = M_6(F)$$

Thus, this description gives a first illustration of our Structure Theorem.

(D) SECOND EXAMPLE.

Secondly, with the previous notations, for any *field* F, every pair (μ, ν) of elements $\mu \in F^*$ and $\nu \in F^*$, determines an example of *right Artinian* F-*algebra :*

$$A = B_{\mu,\nu} = B_{\mu,\nu}(F)$$

of *finite dimension over* F :

$$\dim_F[A] = r = 4$$

with a finite right *Canonical Resolution :*

$$\mathfrak{R}(A) = [(A = A_3 = B_{\mu,\nu}) \xrightarrow{\sigma_3}\!\!\!\!\twoheadrightarrow (A_2 = B_\mu) \xrightarrow{\sigma_2}\!\!\!\!\twoheadrightarrow (A_1 = D_1) \xrightarrow{\sigma_1}\!\!\!\!\twoheadrightarrow (A_0 = F)]$$

which gives the finite right *Resolutive Dimension :*

$$\rho\dim(A) = m = 3$$

that is :

$$I = \{0, 1, 2, 3\} \qquad\qquad I^* = \{1, 2, 3\}$$

and the finite right *Canonical Sequence :*

$$\rho(A) = (\rho_1 = 0\,, \rho_2 = 0\,, \rho_3 = 0)$$

equivalent to the conditions :

$$I_0^* = \{j \in I^*\,;\, \rho_j = 0\} = I^* = \{1, 2, 3\} \qquad\qquad I_1^* = \{k \in I^*\,;\, \rho_k = 1\} = \varnothing$$

which give a «*Complete Decomposition* of m = 3» of the form :

$$(I) = I(A) \qquad\qquad I = I_0^* \coprod I_1 = \left(\coprod_{k \in I_2} I_0^k\right) \coprod I_1$$

in which : $I = \{0, 1, 2, 3\}$, $I_1 = I_2 = \{0\}$, $I_1^* = \varnothing$ and $I^* = I_0^* = I_0^0 = \{1, 2, 3\}$.

In order to complete the characterization of the F-*"Completely structured vertex set" :*

$$\hat{\Lambda}(A) = \hat{\Lambda} = \{\Lambda \,;\, \Sigma(A) \,;\, (K_\lambda), (p_\lambda), (q_\lambda), (n^j_\lambda)\}$$

equipped with its *"Combinatorial Structure" :*

$$\mathbf{\Sigma}(A) = \Sigma = [\Lambda \,;\, m = 3, (I) \,;\, \{\Lambda_i\}, (\Lambda^k), \Lambda', (\Lambda'^k), (\Lambda''^k_j), (\Lambda''_j), (\Lambda'_j)]$$

in which the relations :

$$\Lambda = \{1\}$$

$$(K_\lambda) = (K_1 = F) \qquad\qquad (p_\lambda) = (p_1 = 1)$$

are obvious, it is possible to prove the relations :

$$\Lambda_i = \Lambda = \{1\} \qquad \text{for all } i \in I$$

$$\Lambda^k = \Lambda^0 = \Lambda_0 = \{1\} \qquad \text{for } k = 0 \in I_2$$

$$\Lambda' = \varnothing$$

$$(\Lambda'^k) = (\Lambda'^k)_{k\in I_1^*} = \varnothing \qquad \text{since } I_1^* = \varnothing$$

$$\Lambda''^k_j = \Lambda''^0_j = \varnothing \qquad \text{for } k = 0 \in I_2 \text{ and } j \in I_0^k =$$

$$I_0^0 = I^* = \{1, 2, 3\}$$

$$\Lambda''_j = \Lambda'_j = \Lambda = \{1\} \qquad \text{for all } j \in I_0^0 = I^*$$

and also the relations :

$$(q_\lambda) = (q_1 = 0)$$

$$(n^j_1) = (n^1_1 = 1 \,,\, n^2_1 = 1 \,,\, n^3_1 = 1)$$

which imply in particular the relation :

$$A_0 = B_0 = G_0 = R_0 = \prod_{\lambda\in\Lambda_0} R^\lambda = R^1 = K_1 = F$$

and the relation :

$$A_{/J(A)} = R = R_3 = R_2 = R_1 = R_0 = K_1 = F$$

which shows that the *right Artinian* F-*algebras :*

$$A = A_3 \quad , \quad A_2 \quad , \quad A_1 \quad , \quad A_0 = F$$

are *local* F-*algebras.*

Moreover, with the Constructions 7-1, the Theorem 7-4 shows that the Structure of the right Artinian F-algebra A is characterized by its F-*"Completely structured vertex set"* $\hat{\Lambda} = \hat{\Lambda}(A)$ and by the two families of *"parameters" :*

$$(\Psi'_j)_{j\in I_0^*=I^*} \qquad \text{and} \qquad (\xi_j)_{j\in I_0^*=I^*}$$

characterized by the following conditions :

(c_1) For $j = 1 \in I_0^*$, the *"parameter"* Ψ'_1 is the F-algebra homomorphism :

$$\Psi'_1 = \mathrm{Id}_{K_1} : C_0 = K_1 = F \longrightarrow H'^1 = K_1 = F$$

which determines the non null C_0-bimodule :

$$M_1 = [M'_1 ; \Psi'_1 : C_0 \longrightarrow H'^1] \equiv F$$

and the parameter ξ_1 is the unique $T_0 \equiv (0)$-***essential cohomology class*** :

$$\xi_1 = \hat{0} \in H_e^2(A_0, T_0, M_1) = H_e^2(D_0, T_0, M_1) = H_e^2(F, (0), F)$$

which give the characterization :

$$(D_1, N_1 \equiv (0)) = (A_1, N_1) = (A_0, T_0, M_1, \xi_1) = (F, (0), F, \hat{0})$$

(c_2) For $j = 2 \in I_0^*$, the *"parameter"* Ψ'_2 is the *canonical surjective F-algebra epimorphism :*

$$\Psi'_2 : D_1 = A_1 = C_1 \longrightarrow\!\!\!\!\rightarrow H'^2 = K_1 = F = D_1/J(D_1) = A_1/J(A_1)$$

which determines the non null C_1-bimodule :

$$M_2 = [M'_2 ; \Psi'_2 : C_1 \longrightarrow H'^2] \equiv F$$

and the *"parameter"* $\xi_2 = \mu$ is a $T_1 \equiv (0)$-***essential cohomology class :***

$$\xi_2 = \hat{f}_\mu \in F^* \equiv H_e^2(A_1, T_1, M_2) = H_e^2(D_1, T_1, M_2) = H_e^2(D_1(F), (0), F)$$

which give the characterization :

$$(B_\mu, N_2 \equiv (0)) = (A_2, N_2) = (A_1, T_1, M_2, \xi_2) = (D_1(F), (0), F, f_\mu)$$

(c_3) For $j = 3 \in I_0^*$, the *"parameter"* Ψ'_3 is the *canonical surjective F-algebra epimorphism :*

$$\Psi'_3 : B_\mu = A_2 = C_2 \longrightarrow\!\!\!\!\rightarrow H'^3 = K_1 = F = B_\mu/J(B_\mu) = A_2/J(A_2)$$

which determines the non null C_2-bimodule :

$$M_3 = [M'_3 ; \Psi'_3 : C_2 \longrightarrow H'^3] \equiv F$$

and the *"parameter"* $\xi_3 = \nu$ is a $T_2 \equiv (0)$-***essential cohomology class :***

$$\xi_3 = \hat{g}_\nu = \nu \in F^* \equiv H_e^2(A_2, T_2, M_3) = H_e^2(B_\mu, T_2, M_3) = H_e^2(B_\mu(F), (0), F)$$

which give the characterization :

$$(B_{\mu,\nu}, N_3 \equiv (0)) = (A_3, N_3) = (A_2, T_2, M_3, \xi_3) = (B_\mu(F), (0), F, g_\nu)$$

Thus, this description gives a second illustration of our Structure Theorem.

(E) THE ROLE OF THE PARAMETERS.

The previous example shows that for any *field* F, *every pair :*

$$(\mu, \nu) \in (F^* \times F^*)$$

determines a *right Artinian* F-*algebra* :

$$A = B_{\mu,\nu} = B_{\mu,\nu}(F)$$

with a finite right *Canonical Resolution :*

$$\mathfrak{R}(A) = [(A = A_3 = B_{\mu,\nu}) \xrightarrow{\sigma_3} (A_2 = B_\mu) \xrightarrow{\sigma_2} (A_1 = D_1) \xrightarrow{\sigma_1} (A_0 = F)]$$

and a Structure characterized by its F-*"Completely structured vertex set" :*

$$\hat{\Lambda} = \hat{\Lambda}(A)$$

and by two families of *"parameters" :*

$$(\Psi'_j)_{j\in I_0^*} = (\Psi'_1, \Psi'_2, \Psi'_3) \quad \text{and} \quad (\xi_j)_{j\in I_0^*} = (\xi_1 = \hat{0}\ , \xi_2 = \hat{f}_\mu = \mu\ , \xi_3 = \hat{g}_\nu = \nu)$$

In particular, with the Notations of the Example (A) - (c), for any *field* F, the particular pair :

$$(1, 1) \in (F^* \times F^*)$$

determines the *right Artinian* F-*algebra :*

$$\bar{A} = B_{1,1} = B_{1,1}(F) = D_3 = D_3(F)$$

with a finite right *Canonical Resolution :*

$$\mathfrak{R}(\bar{A}) = [(\bar{A} = \bar{A}_3 = D_3(F)) \xrightarrow{\tau_3} (\bar{A}_2 = D_2(F)) \xrightarrow{\tau_2} (\bar{A}_1 = D_1) \xrightarrow{\tau_1} (\bar{A}_0 = F)]$$

and a Structure characterized by *the same* F-*"Completely structured vertex set" :*

$$\hat{\Lambda}(\bar{A}) = \hat{\Lambda} = \hat{\Lambda}(A)$$

and by two families of *"parameters" :*

$$(\bar{\Psi}'_j)_{j\in I_0^*} = (\bar{\Psi}'_1, \bar{\Psi}'_2, \bar{\Psi}'_3) \quad \text{and} \quad (\bar{\xi}_j)_{j\in I_0^*} = (\bar{\xi}_1 = \hat{0}, \bar{\xi}_2 = \hat{f}_1 = 1, \bar{\xi}_3 = \hat{g}_1 = 1)$$

Thus, *it is not impossible* that the right Artinian F-algebras : A and $\bar{A}$, be isomorphic.

In fact, it is easy to verify that the F-*algebra automorphism :*

$$\omega_\mu \in \mathrm{Aut}_F(D_1(F)) \simeq F^*$$

defined by the condition :

$$\omega_\mu(a = a_0 + a_1x) = \bar{a} = a_0 + a_1\mu x \qquad \text{for all } a \in D_1(F)$$

the F-*algebra isomorphism :*

$$\bar{\omega}_\mu : A_2 = B_\mu(F) \xrightarrow{\sim} \bar{A}_2 = D_2(F)$$

defined by the condition :

$$\bar{\omega}_\mu(a = a_0 + a_1e_1 + a_2e_2) = \bar{a} = a_0 + a_1\mu x + a_2\mu x^2$$

for all $a \in A_3 = B_{\mu,\nu}(F)$ and the F-*algebra isomorphism :*

$$\bar{\omega}_{\mu,\nu} : A_3 = B_{\mu,\nu}(F) \xrightarrow{\sim} \bar{A}_3 = D_3(F)$$

defined by the condition :

$$\bar{\omega}_{\mu,\nu}(a = a_0 + a_1e_1 + a_2e_2 + a_3e_3) = \bar{a} = a_0 + a_1\mu x + a_2\mu x^2 + a_3\nu^{-1}\mu^2x^3$$

for all $a \in A_3 = B_{\mu,\nu}(F)$, determine an *isomorphism of Resolutions :*

$$\bar{\omega} : R(A) \xrightarrow{\sim} R(\bar{A})$$

characterized by the following commutative diagram :

$$\begin{array}{ccccccc}
A = A_3 = B_{\mu,\nu}(F) & \xrightarrow{\sigma_3} & A_2 = B_\mu(F) & \xrightarrow{\sigma_2} & A_1 = D_1(F) & \xrightarrow{\sigma_1} & A_0 = F \\
\downarrow \bar{\omega}_{\mu,\nu} & & \downarrow \bar{\omega}_\mu & & \downarrow \omega_\mu & & \downarrow Id_F \\
\bar{A} = \bar{A}_3 = D_3(F) & \xrightarrow{\tau_3} & \bar{A}_2 = D_2(F) & \xrightarrow{\tau_2} & \bar{A}_1 = D_1(F) & \xrightarrow{\tau_1} & \bar{A}_0 = F
\end{array}$$

Now, it is possible to compare the *"iterative constructions"* of the F-algebras $A = B_{\mu,\nu}(F)$ and $\bar{A} = D_3(F) = B_{1,1}(F)$.

With obvious notations, in the first step, starting from the same F-algebras :

$$A_0 = F = \bar{A}_0$$

which give : $\Psi'_1 = \bar{\Psi}'_1$, and therefore : $M_1 = \bar{M}_1$, the same *"parameters"* :

$$\xi_1 = \bar{\xi}_1 = \hat{0} \in H^2_e(A_0, T_0, M_1) \equiv H^2_e(\bar{A}_0, \bar{T}_0, \bar{M}_1) \equiv H^2(F, (0), F)$$

determine the same algebras :

$$A_1 = D_1(F) = \bar{A}_1$$

with the same canonical F-algebra epimorphisms : $\sigma_1 = \tau_1$.

Then, in the second step, starting from the same F-algebras :

$$A_1 = D_1(F) = \bar{A}_1$$

which give : $\Psi'_2 = \sigma_1 = \tau_1 = \bar{\Psi}'_2$, and therefore : $M_2 = \bar{M}_2$, this gives the same *"spaces" :*

$$F^* \equiv H^2_e(A_1, T_1, M_2) \equiv H^2_e(\bar{A}_1, \bar{T}_1, \bar{M}_2) \equiv H^2_e(D_1(F), (0), F)$$

From here, the choice of different *"parameters" :*

$$\xi_2 = \hat{f}_\mu = \mu \in F^* \equiv H^2_e(A_1, T_1, M_2) \equiv H^2_e(D_1(F), (0), F)$$

and

$$\bar{\xi}_2 = \hat{f}_1 = 1 \in F^* \equiv H_e^2(\bar{A}_1, \bar{T}_1, \bar{M}_2) \equiv H_e^2(D_1(F), (0), F)$$

determines two (0)-***essential*** *singular* F-*algebra extensions :*

$$(\sigma_2, T_1) \qquad 0 \longrightarrow (F \equiv M_2) \rightarrowtail (B_\mu(F), N_2 \equiv (0)) \xrightarrow{\sigma_2}\!\!\!\!\twoheadrightarrow (A_1, T_1) \longrightarrow 0$$

and

$$(\tau_2, \bar{T}_1) \qquad 0 \longrightarrow (F \equiv \bar{M}_2) \rightarrowtail (D_2(F), \bar{N}_2 \equiv (0)) \xrightarrow{\tau_2}\!\!\!\!\twoheadrightarrow (\bar{A}_1, \bar{T}_1) \longrightarrow 0$$

which are ***"equivalent"*** if and only if : $\xi_2 = \bar{\xi}_2$, that is if and only if : $\mu = 1$.

Thus, whenever $1 \neq \mu \in F^*$, the two previous (0)-***essential singular*** F-*algebra extensions* (σ_2, T_2) and $(\tau_2, \bar{T}_1)$ ***are not equivalent***, but ***they are isomorphic*** by means of the following exact and commutative diagram :

$$\begin{array}{lccccccccc}
(\sigma_2) & 0 \longrightarrow & (F \equiv M_2) & \xrightarrow{i_2} & (A_2 = B_\mu(F)) & \xrightarrow{\sigma_2} & (A_1 = D_1(F)) & \longrightarrow 0 \\
 & & \downarrow \bar{\omega}'_\mu & & \downarrow \bar{\omega}_\mu & & \downarrow \omega_\mu & \\
(\tau_2) & 0 \longrightarrow & (F \equiv \bar{M}_2) & \xrightarrow{\bar{i}_2} & (\bar{A}_2 = D_2(F)) & \xrightarrow{\tau_2} & (\bar{A}_1 = D_1(F)) & \longrightarrow 0
\end{array}$$

in which the isomorphisms of F-vector spaces :

$$i_2 : (F \equiv M_2) \rightarrowtail\!\!\!\twoheadrightarrow M(A_2) = M(B_\mu(F)) = Fe_2$$

and

$$\bar{i}_2 : (F \equiv \bar{M}_2) \rightarrowtail\!\!\!\twoheadrightarrow M(\bar{A}_2) = M(D_2(F)) = Fx^2$$

are defined by :

$$i_2(a_2) = a_2 e_2 \qquad \text{and} \qquad \bar{i}_2(a_2) = a_2 x^2$$

for all $a_2 \in F \equiv M_2$ and all $a_2 \in F \equiv \bar{M}_2$, in which :

$$\bar{\omega}'_\mu : F \equiv M_2 \xrightarrow{\sim} F \equiv M_3$$

is the isomorphism of $D_1(F)$-bimodules defined by :

$$\bar{\omega}'_\mu(a_2) = \mu a_2 \qquad \text{for all } a_2 \in F \equiv M_2$$

and in which $\bar{\omega}_\mu$ and ω_μ are the previous F-*algebra isomorphisms.*

It seems that this curious phenomenon results from the fact that the *group of* F-*automorphisms :*

$$Aut_F(D_1(F)) \equiv F^*$$

operates on the space of (0)-***essential cohomology classes :***

$$H^2_e(D_1(F), (0), F)$$

by an *"action" :*

$$Aut_F(D_1(F)) \times H^2_e(D_1(F), (0), F) \longrightarrow H^2_e(D_1(F), (0), F)$$

$$(\omega_{\mu'} \quad , \quad \hat{f}) \longrightarrow \omega_{\mu'}[\hat{f}] = \hat{f}'$$

characterized by the condition :

$$f' = [\bar{\omega}'_{\mu'}]^{-1} \circ f \circ [\omega_{\mu'} \times \omega_{\mu'}]$$

for every $f \in Z^2_e(D_1(F), (0), F)$ and every $\mu' \in F^* \equiv Aut_F(D_1(F))$, in such a way that the relation :

$$\hat{f}_\mu = \omega_\mu[\hat{f}_1]$$

shows that the *"parameters"* $\xi_2 = \hat{f}_\mu = \mu$ and $\bar{\xi}_2 = \hat{f}_1 = 1$ are connected by the relation :

$$\xi_2 = \omega_\mu[\bar{\xi}_2]$$

determined by the F-*algebra automorphism :*

$$\omega_\mu \in Aut_F(D_1(F)) \equiv F^*$$

and which implies the existence of the F-algebra isomorphism :

$$\bar{\omega}_\mu : (A_2 = B_\mu(F)) \xrightarrow{\sim} (\bar{A}_2 = D_2(F))$$

Then, in the third step, starting from the isomorphic F-algebras :

$$A_2 = B_\mu(F) \qquad \text{and} \qquad \bar{A}_2 = D_2(F)$$

by means of the F-algebra isomorphism $\bar{\omega}_\mu$, which gives :

$$\Psi'_3 = \sigma_1 \circ \sigma_2 \qquad \text{and} \qquad \bar{\Psi}'_3 = \tau_1 \circ \tau_2$$

and therefore :

$$\Psi'_3 = \bar{\Psi}'_3 \circ \bar{\omega}_\mu$$

it is possible to compare the structures of the A_2-bimodule M_3 and of the $\bar{A}_2$-bimodule $\bar{M}_3$, and also the (0)-***essential*** *cohomology classes :*

$$\xi_3 = \hat{g}_\nu = \nu \in F^* \equiv H^2_e(A_2, T_2, M_3) = H^2_e(B_\mu(F), (0), F)$$

and

$$\bar{\xi}_3 = \hat{g}_1 = 1 \in F^* \equiv H^2_e(\bar{A}_2, \bar{T}_2, \bar{M}_3) = H^2_e(D_2(F), (0), F)$$

for which there exists a connection analogous to a previous relation.

This example shows that for the *right Artinian* F-*algebras :*

$$A = B_{\mu,\nu} = B_{\mu,\nu}(F) \qquad \text{and} \qquad \bar{A} = B_{1,1} = B_{1,1}(F) = D_3 = D_3(F)$$

having *the same* F-*"Completely structured vertex set" :*

$$\hat{\Lambda}(A) = \hat{\Lambda} = \hat{\Lambda}(\bar{A})$$

their *"parameters"* verify in particular the relations :

$$\Psi'_1 = \bar{\Psi}'_1 \qquad \Psi'_2 = \bar{\Psi}'_2 \qquad \Psi'_3 = \bar{\Psi}'_3 \circ \bar{\omega}_\mu$$

and

$$\xi_1 = \bar{\xi}_1 \qquad \xi_2 = \omega_\mu[\bar{\xi}_2]$$

which show how they are connected, by means of the *"iterative construction"* of the F-algebra isomorphisms :

$\omega_\mu \in \mathrm{Aut}_F(D_1(F))$ and $\bar{\omega}_\mu : A_2 \xrightarrow{\sim} \bar{A}_2$, which determine the F-algebra isomorphism : $\bar{\omega}_{\mu,\nu} : A \xrightarrow{\sim} \bar{A}$.

These observations explain the reasons for which, in the Remarks 6-11 of [10], we have said that the *"parameters"* are some *"semi-invariants"* of the Structure of right Artinian rings or F-algebras.

Moreover, it is convenient to remark that *the rôle of the* ***parameters*** *is necessary,* as this is shown by the last example following the Theorem 8-1.

(F) LOCAL RIGHT ARTINIAN F-ALGEBRAS.

For any *field* F, let $\mathfrak{A}_l(F)$ be the class of *local right Artinian* F-*algebras,* that is of right Artinian F-algebras :

$$A \in \mathfrak{A}(F)$$

which are *local* in the sense that the semisimple Artinian F-algebra $A/\mathbf{J}(A) = R$ is a F-*skewfield :*

$$A/\mathbf{J}(A) = R = K \in \mathfrak{K}(F)$$

called its *"residue class* F-*skewfield"*, this last condition being equivalent to the fact that A has a F-*"concrete vertex set"* of the form :

$$\tilde{\Lambda}(A) = \tilde{\Lambda} = [\Lambda = \{1\}; (K_\lambda) = (K_1 \equiv K), (p_\lambda) = (p_1 \equiv 1)]$$

For a finite right *Resolutive Dimension :*

$$\rho\dim(A) = m$$

which gives :

$$I = \{0, 1, 2, ..., m\} \qquad I^* = \{1, 2, ..., m\}$$

the condition : $\Lambda = \{1\}$, determines a finite right *Canonical Sequence* of the form:

$$\boldsymbol{\rho}(A) = (\rho_1 = 0, \rho_2 = 0, ..., \rho_j = 0, ..., \rho_m = 0)$$

equivalent to the conditions :

$$I_0^* = \{j \in I^* ; \rho_j = 0\} = I^* \qquad I_1^* = \{k \in I^* ; \rho_k = 1\} = \varnothing$$

which give a «*Complete Decomposition* of m» of the form :

$$(I) = \mathbf{I}(A) \qquad I = I_0^* \coprod I_1 = \left(\coprod_{k \in I_2} I_k^*\right) \coprod I_1$$

in which : $I = \{0, 1, 2, ..., m\}$, $I_1 = I_2 = \{0\}$, $I_1^* = \varnothing$

and

$$I^* = I_0^* = I_0^0 = \{1, 2, ..., m\}$$

In order to complete the characterization of the F-*"Completely structured vertex set" :*

$$\hat{\Lambda}(A) = \hat{\Lambda} = \{\Lambda ; \Sigma = \Sigma(A) ; (K_\lambda) , (p_\lambda) , (q_\lambda), (n_\lambda^j)\}$$

equipped with its *"Combinatorial Structure" :*

$$\boldsymbol{\Sigma}(A) = \Sigma = [\Lambda ; m, (I) ; \{\Lambda_i\}, (\Lambda^k) , \Lambda', (\Lambda'^k), (\Lambda''^k_j), (\Lambda''_j), (\Lambda'_j)]$$

it is possible to prove that the *conditions :*

$$\Lambda = \{1\} \qquad (K_\lambda) = (K_1 \equiv K) \qquad (p_\lambda) = (p_1 \equiv 1)$$

imply automatically the relations :

$$\Lambda_i = \Lambda = \{1\} \qquad \text{for all } i \in I$$

$$\Lambda^k = \Lambda^0 = \Lambda_0 = \{1\} \qquad \text{for } k = 0 \in I_2$$

$$\Lambda' = \varnothing$$

$$(\Lambda'^k) = (\Lambda'^k)_{k \in I_1^*} = \varnothing \qquad \text{since } I_1^* = \varnothing$$

$$\Lambda''^k_j = \Lambda''^0_j = \varnothing \qquad \text{for } k = 0 \in I_2 \text{ and } j \in I_0^k =$$

$I_0^0 = I^* = \{1, 2, ..., m\}$

$$\Lambda''_j = \Lambda'_j = \Lambda = \{1\} \qquad \text{for all } j \in I_0^0 = I^*$$

and also the relations :

$$(q_\lambda) = (q_1 \equiv 0)$$

$$(n_1^j) = (n_1^j \equiv n^j \equiv n_j \geq 1) = (n_j)_{j \in I_0^* = I^*}$$

for *non null integers* $n_j \in \mathbb{N}^*$, indexed by $j \in I^* = \{1, 2, ..., m\}$.

Morever, it is convenient to remark that with the general notations, the previous conditions imply automatically the relations :

$$T''^{j}_{0} \equiv (0) \qquad \text{and} \qquad T_{j-1} \equiv (0) \in \mathscr{C}(A_{j-1})$$

which give the relations :

$$A_{j-1}/T_{j-1} = C_{j-1} \equiv A_{j-1} \qquad \text{for all } j \in I^*$$

and that for each index $j \in I_0^* = I^*$, which determines the right A_{j-1}-module :

$$M'_j = L'^{j}_{1} = \mathscr{L}(V_1, W^{j}_{1}) = W^{j}_{1} = K^{n_j}_{1} = K^{n_j} = W^j$$

and its F-algebra of endomorphisms :

$$H'^j = \mathscr{L}_{A_{j-1}}(M'_j) = \mathscr{L}(W^j) = M_{n_j}(K)$$

the characterization of a non null C_{j-1}-bimodule :

$$M_j = [M'_j \, ; \Psi'_j : C_{j-1} \longrightarrow H'^j]$$

is determined by a *"parameter"* Ψ'_j constituted by a F-*algebra homomorphism :*

$$\Psi'_j \ : C_{j-1} \equiv A_{j-1} \longrightarrow H'^j \equiv \mathscr{L}(W^j) = M_{n_j}(K)$$

Thus, for a local right Artinian F-algebra :

$$A \in \mathfrak{A}_l(F)$$

the knowledge of its F-*"Completely structured vertex set" :*

$$\hat{\Lambda}(A) = \hat{\Lambda}$$

is equivalent to the knowledge of its F-*"Local completely structured vertex set"*, of the form :

$$\Omega(A) = \Omega = [m \, , (I) \, , K \, , (n_j)_{j \in I^*}]$$

in which m is an integer : $m \in \mathbb{N}$ which determines $I = \{0, 1, 2, ..., m\}$, in which K is a F-skewfield : $K \in \mathcal{K}(F)$ and in which $(n_j)_{j \in I^*}$ is a family of *non null integers* : $n_j \in \mathbb{N}^*$, indexed by $j \in I^* = \{1, 2, ..., m\}$.

Then, the translation of the Constructions 7-1 and of the Theorem 7-4 gives the following result.

THEOREM 8-1 (STRUCTURE THEOREM IN THE LOCAL CASE).

For any field F, *the Structure of any* local right Artinian F-algebra :

$$A \in \mathfrak{A}_l(F)$$

is characterized "up to an F-*isomorphism" by a (or by its)* F-"Local completely structured vertex set" :

$$\Omega = [m \, , (I) \, , K \, , (n_j)_{j \in I^*}]$$

and by two families of "parameters" :

$$(\Psi'_j)_{j \in I^*} \qquad\qquad (\xi_j)_{j \in I^*}$$

in such a manner that in its finite right Canonical Resolution :

$$\mathfrak{R}(A) = [A = A_m \xrightarrow{\tau_m} A_{m-1} \dots A_i \xrightarrow{\tau_i} A_{i-1} \dots A_1 \xrightarrow{\tau_1} A_0 = K]$$

the Structures of the local right Artinian F-algebras :

$$A_i \in \mathfrak{A}_l(F) \qquad \textit{for all } i \in I$$

are determined by an "ascending iterative construction" *from the* F-*skewfield* $A_0 = K \in \mathcal{K}(F)$, *to the local right Artinian* F-*algebra :* $A = A_m$, *characterized by the following conditions :*

(a) *The local right Artinian* F-*algebras* A_i *verify the relations :*

$$A_i/J(A_i) = R_i = K \qquad \textit{for all } i \in I$$

which give in particular the relation :

$$A/J(A) = R = R_m = K$$

(b) *For each index* $j \in I^*$, *the Structure of the local right Artinian* F-*algebra :*

$$A_j \in \mathfrak{A}_l(F)$$

is characterized by :

(α) *The local right Artinian* F-*algebra :*

$$A_{j-1} \in \mathfrak{A}_l(F)$$

constructed by an ascending iterative construction from $A_0 = K \in \mathcal{K}(F)$.

(β) *The proper two-sided ideal :*

$$T_{j-1} \equiv (0) \in \mathcal{T}(A_{j-1})$$

which determines the local right Artinian F-*algebra :*

$$A_{j-1}/T_{j-1} = C_{j-1} \equiv A_{j-1}$$

with the canonical surjective F-*algebra epimorphism :*

$$\varphi_{j-1} \equiv Id_{A_{j-1}} : A_{j-1} \twoheadrightarrow C_{j-1}$$

which is an isomorphism, and also the canonical surjective F-*algebra epimorphism :*

$$\varphi''_{j-1} \equiv (\tau_1 \circ \dots \circ \tau_{j-1}) : C_{j-1} \equiv A_{j-1} \twoheadrightarrow C'_{j-1} \equiv C_{j-1}/J(C_{j-1}) \equiv K$$

(γ) *A* "parameter" Ψ'_j *constituted by a* F-algebra homomorphism :

$$\Psi'_j : C_{j-1} \equiv A_{j-1} \longrightarrow H'^j = \mathcal{L}(W^j) = M_{n_j}(K)$$

which determines, by the characterization :

$$M_j = [M'_j \equiv W^j = K^{n_j} ; \Psi'_j : C_{j-1} \longrightarrow H'^j]$$

the non null C_{j-1}-*bimodule or* A_{j-1}-*bimodule :*

$$M_j \in \mathcal{M}(A_{j-1}, T_{j-1} \equiv (0))$$

(δ) *The* "parameter" ξ_j *constituted by an unique* $T_{j-1} \equiv (0)$-**essential** cohomology class :

$$\hat{h}_j = \xi_j \in H_e^2(A_{j-1}, T_{j-1}, M_j) \equiv H_e^2(A_{j-1}, (0), M_j)$$

such that :

$$(A_j, N_j \equiv (0)) = (A_{j-1}, T_{j-1}, M_j, h_j) = (A_{j-1}, T_{j-1}, M_j, \xi_j)$$

which gives the **"iterative cohomological characterization" :**

$$\boxed{A_j = (A_{j-1}, M_j, h_j) = (A_{j-1}, M_j, \xi_j)}$$

by means of the "parameters" :

$$(\Psi'_j)_{j\in I^*} \qquad \text{and} \qquad (\xi_j)_{j\in I^*}$$

(c) *Moreover, the* underlying F-vector space $|A|$ *of the local right Artinian F-algebra* A *is characterized by the conditions :*

$$|A| = K \oplus \left[\bigoplus_{j\in I^*} W^j\right] = K \oplus \left[\bigoplus_{j\in I^*} K^{n_j}\right] = K^s$$

in which :

$$s = 1 + \sum_{j\in I^*} n_j$$

the "multiplication" *of the* local right Artinian F-algebra A *being determined by the two families :*

$$(\Psi'_j)_{j\in I^*} \qquad (\xi_j)_{j\in I^*}$$

of "parameters".

PROOF - This is a particular case of the Theorem 7-4.

For instance, this general Theorem 8-1 may be illustrated by the SECOND EXAMPLE :

$$A = B_{\mu,\nu} = B_{\mu,\nu}(F)$$

and also by the following particular example.

In the case where : $m = 1$, a F-"*Local completely structured vertex set*" of the form :

$$\Omega = [m, (I), K, (n_j)_{j\in I^*}]$$

is completely determined by a F-skewfield :

$$A_0 = K \in \mathcal{K}(F)$$

and by a non null integer :

$$n_j = n_1 \in \mathbb{N}^*$$

Thus, if we choose : $n_j = n_1 = 1$, the local right Artinian F-algebra A is completely determined by the two ***"parameters"*** constituted by a F-algebra homomorphism :

$$\Psi'_1 \in \mathrm{Mor}_F[C_0, H'^1] \equiv \mathrm{Mor}_F[K, K]$$

and by a $T_0 \equiv (0)$-*essential cohomology class :*

$$\hat{h}_1 = \xi_1 \in H^2_e(A_0, T_0, M_1) \equiv H^2_e(K, (0), M_1)$$

which give the *"cohomological characterization" :*

$$A = (A_0, M_1, \xi_1)$$

In particular, for any F-automorphism :

$$\sigma \in \mathrm{Aut}_F(K) = \mathrm{Gal}[K : F]$$

the Proposition 3-15 shows that the choice of the ***"parameters" :***

$$\Psi'_1 = \sigma \in \mathrm{Aut}_F(K) = \mathrm{Gal}[K : F]$$

and :

$$\xi_1 = \hat{0} \in H^2_e(K, (0), M_1) \equiv H^2(K, (0), {}_\sigma K_1)$$

characterizes a local right Artinian F-algebra :

$$A_\sigma = (A_0, M_1, \xi_1) = (K, {}_\sigma K_1, \hat{0})$$

having the *"matrix realization" :*

$$A = A_\sigma = \begin{bmatrix} \sigma\boxed{K} & K \\ 0 & \boxed{K} \end{bmatrix} = \left\{ a = \begin{bmatrix} \sigma(a_1) & a_0 \\ 0 & a_1 \end{bmatrix}, \ a_i \in K \right\}$$

In this last example, whenever the F-skewfield $K \in \mathfrak{K}(F)$ is *commutative* and the Galois group $\mathrm{Aut}_F(K) = \mathrm{Gal}[K : F]$ *is not trivial* [for instance : $F = \mathbb{R}$ and $K = \mathbb{C}$], it is immediate that the local right Artinian F-algebra A_σ is *non commutative* if and only if :

$$\sigma \neq 1 \in \mathrm{Aut}_F(K) = \mathrm{Gal}[K : F]$$

It follows that the structures of the F-algebras A_σ with the same ***complete invariant*** : $\hat{\Lambda}(A_\sigma) = \hat{\Lambda}$, depend explicitely of the ***"parameter"*** $\Psi_1 = \sigma$, which is necessary in order to describe the "multiplication" of the F-algebra A_σ, and this gives an example of the necessity of the rôle of the ***parameters.***

(G) A LAST EXAMPLE.

For any *field* F and any F-*"Completely structured vertex set"* $\hat{\Lambda}$, a natural problem is the problem of the *existence* and of the *choice* of *"parameters"*, in order to construct a right Artinian F-algebra :

$$A \in \mathfrak{A}(F)$$

subject to the condition :

$$(\gamma) \qquad \hat{\Lambda}(A) = \hat{\Lambda}$$

Our aim is to discuss this problem on an example, in which the given F-*"Completely structured vertex set"* :

$$\hat{\Lambda} = \{\Lambda\ ;\ \Sigma\ ;\ (K_\lambda)\ ,\ (p_\lambda)\ ,\ (q_\lambda)\ ,\ (n^j_\lambda)\}$$

and its *"Combinatorial Structure"* :

$$\Sigma = [\Lambda\ ;\ m\ ,\ (I)\ ;\ \{\Lambda_i\}\ ,\ (\Lambda^k)\ ,\ \Lambda',\ (\Lambda'^k)\ ,\ (\Lambda''^k_j)\ ,\ (\Lambda''_j)\ ,\ (\Lambda'_j)]$$

are defined by the following conditions :

(α_1) $\qquad \Lambda = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

(α_2) For the integer m = 4, which gives :

$$I = \{0, 1, 2, 3, 4\} \qquad ; \qquad I^* = \{1, 2, 3, 4\}$$

the equivalent conditions :

$$I^*_0 = \{2, 3\} \qquad ; \qquad I^*_1 = \{1, 4\} \qquad ; \qquad I_1 = \{0, 1, 4\}$$

give the « *Complete Decomposition* of m = 4 », of the form :

$$(I) \qquad I = I^*_0 \coprod I_1 = \left(\coprod_{k \in I_2} I^k_0\right) \coprod I_1$$

in which :

$$I_2 = \{k \in I_1\ ;\ (k+1) \in I^*_0\} = \{1\} \quad \text{and} \quad I^k_0 = I^1_0 = I^*_0 = \{2, 3\}$$

(α_3) $\qquad \Lambda_0 = \{1, 2\}$

$$\Lambda_1 = \Lambda_2 = \Lambda_3 = \{1, 2, 3, 4, 5, 6, 7\}$$

$$\Lambda_4 = \Lambda = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

(α_4) $\qquad \Lambda^0 = \Lambda_0 = \{1, 2\}$

$$\Lambda^1 = \Lambda_1 - \Lambda_0 = \{3, 4, 5, 6, 7\}$$

$$\Lambda^4 = \Lambda_4 - \Lambda_3 = \{8, 9\}$$

(α_5) $\qquad \Lambda' = \{5, 6, 7, 8\}$

$$\Lambda'^1 = \{5, 6, 7\}$$

$$\Lambda'^4 = \{8\}$$

(α_6) $\qquad \Lambda''^1_2 = \{4, 5, 6, 7\}$

$$\Lambda''^1_3 = \{5, 6, 7\}$$

(α_7) $\qquad \Lambda''_2 = \Lambda_1 - \Lambda''^1_2 = \{1, 2, 3\}$

$$\Lambda''_3 = \Lambda_1 - \Lambda''^1_3 = \{1, 2, 3, 4\}$$

(α_8) $$\Lambda'_2 = \Lambda''_2 = \{1, 2, 3\}$$
$$\Lambda'_3 = \Lambda''_3 = \{1, 2, 3, 4\}$$

(β_1) $(K_\lambda) = (K_\lambda)_{\lambda\in\Lambda} = (K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8\ K_9)$

in which each K_λ is a F-*skewfield :* $K_\lambda \in \mathfrak{K}(F)$

(β_2) $(p_\lambda) = (p_\lambda)_{\lambda\in\Lambda} = (p_1{=}1, p_2{=}1, p_3{=}1, p_4{=}1, p_5{=}1, p_6{=}1, p_7{=}1, p_8{=}1, p_9{=}1)$

(β_3) $(q_\lambda) = (q_\lambda)_{\lambda\in\Lambda} = (q_1{=}0, q_2{=}0, q_3{=}0, q_4{=}0, q_5{=}1, q_6{=}1, q_7{=}1, q_8{=}q, q_9{=}0)$

in which $q_8 = q \in \mathbb{N}^*$, is considered as a *"numerical parameter"*.

(β_4) $(n^2_\lambda) = (n^2_\lambda)_{\lambda\in\Lambda} = (n^2_1{=}1, n^2_2{=}1, n^2_3{=}1, n^2_4{=}0, n^2_5{=}0, n^2_6{=}0, n^2_7{=}0, n^2_8{=}0, n^2_9{=}0)$

$$(n^3_\lambda) = (n^3_\lambda)_{\lambda\in\Lambda} = (n^3_1{=}1, n^3_2{=}1, n^3_3{=}1, n^3_4{=}1, n^3_5{=}0, n^3_6{=}0, n^3_7{=}0, n^3_8{=}0, n^3_9{=}0)$$

Thus, our objective is to construct a right Artinian F-algebra :

$$A \in \mathfrak{A}(F)$$

subject to the condition :

(γ) $$\hat{\Lambda}(A) = \hat{\Lambda}$$

which implies the existence of a finite right *Canonical Sequence :*

$$\rho(A) = (\rho_1 = 1, \rho_2 = 0, \rho_3 = 0, \rho_4 = 1)$$

and the existence of a finite right *Canonical Resolution* of the form :

$$\mathfrak{R}(A) = [A = A_4 \xrightarrow{\tau_4}\!\!\!\!\twoheadrightarrow A_3 \xrightarrow{\tau_3}\!\!\!\!\twoheadrightarrow A_2 \xrightarrow{\tau_2}\!\!\!\!\twoheadrightarrow A_1 \xrightarrow{\tau_1}\!\!\!\!\twoheadrightarrow A_0]$$

in which the Structures of the right Artinian F-algebras :

$$A_i \in \mathfrak{A}(F) \qquad \text{for all } i \in I$$

are determined by an *"ascending iterative construction"* from the semisimple Artinian F-algebra : $A_0 \in \mathfrak{A}_0(F)$, to the right Artinian F-algebra : $A = A_4$, characterized, by means of the *"geometrical objects"* determined by $\hat{\Lambda}$ in the Constructions 7-1, by the Theorem 7-4, that is by the following conditions :

(a) Since the conditions (β_2) imply :

$$R^\lambda = \mathfrak{L}(V_\lambda) = M_{p_\lambda}(K_\lambda) = M_1(K_\lambda) = K_\lambda \qquad \text{for all } \lambda \in \Lambda$$

for every $k \in I_1 = \{0, 1, 4\}$, the conditions :

$$G_k = \prod_{\lambda\in\Lambda^k} R^\lambda$$

and the conditions (α_4) imply the relations :

$$A_0 = B_0 = G_0 = \prod_{\lambda\in\Lambda^0} R^\lambda = \prod_{\lambda\in\Lambda^0} K_\lambda = K_1 \times K_2$$

$$G_1 = \prod_{\lambda\in\Lambda^1} R^\lambda = \prod_{\lambda\in\Lambda^1} K_\lambda = K_3 \times K_4 \times K_5 \times K_6 \times K_7$$

$$G_4 = \prod_{\lambda \in \Lambda^4} R^\lambda = \prod_{\lambda \in \Lambda^4} K_\lambda = K_8 \times K_9$$

which give in particular *the first characterization :*

$$\boxed{A_0 = K_1 \times K_2}$$

and for all $i \in I = \{0, 1, 2, 3, 4\}$, the conditions :

$$A/J(A_i) = R_i = \prod_{\lambda \in \Lambda_i} R^\lambda$$

and the conditions (α_3) imply the relations :

$$R_0 = \prod_{\lambda \in \Lambda_0} R^\lambda = \prod_{\lambda \in \Lambda_0} K^\lambda = K_1 \times K_2$$

$$R_1 = R_2 = R_3 = \prod_{\lambda \in \Lambda_3} R^\lambda = \prod_{\lambda \in \Lambda_3} K_\lambda = K_1 \times K_2 \times K_3 \times K_4 \times K_5 \times K_6 \times K_7$$

$$R = R_4 = \prod_{\lambda \in \Lambda_4} R^\lambda = \prod_{\lambda \in \Lambda_4} K_\lambda = K_1 \times K_2 \times K_3 \times K_4 \times K_5 \times K_6 \times K_7 \times K_8 \times K_9$$

which give in particular the relation :

$$A/J(A) = R = K_1 \times K_2 \times K_3 \times K_4 \times K_5 \times K_6 \times K_7 \times K_8 \times K_9$$

which means that the F-algebra A is a *"reduced right Artinian F-algebra"* in the sense that :

$$A/J(A) = R$$

is a *finite product of* F-*skewfields.*

(a') Since the conditions (α_5) imply :

$$H^\lambda = \mathcal{L}(U_\lambda) = M_{q_\lambda}(K_\lambda) \qquad \text{for all } \lambda \in \Lambda' = \{5, 6, 7, 8\}$$

the condition (β_3) implies the relations :

$$H^5 = \mathcal{L}(U_5) = M_{q_5}(K_5) = M_1(K_5) = K_5$$

$$H^6 = \mathcal{L}(U_6) = M_{q_6}(K_6) = M_1(K_6) = K_6$$

$$H^7 = \mathcal{L}(U_7) = M_{q_7}(K_7) = M_1(K_7) = K_7$$

and the relation :

$$H^8 = \mathcal{L}(U_8) = M_{q_8}(K_8) = M_q(K_8)$$

so that, for all $k \in I_1^* = \{1, 4\}$, the conditions :

$$H_k = \prod_{\lambda \in \Lambda'^k} H^\lambda$$

and the, conditions (α_5) imply the relation :

$$H_1 = \prod_{\lambda \in \Lambda'^1} H^\lambda = H^5 \times H^6 \times H^7 = K_5 \times K_6 \times K_7$$

and the relation :

$$H_4 = \prod_{\lambda\in\Lambda'_4} H^\lambda = H^8 = \mathcal{L}(U_8) = M_q(K_8)$$

Moreover, there exists a family of (right) almost simple right Artinian F-algebras :

$$(B^\lambda) = (B^\lambda)_{\lambda\in\Lambda}$$

with a family of right Socles :

$$(S^\lambda) = (S^\lambda)_{\lambda\in\Lambda}$$

characterized by the conditions :

$$B^\lambda = S^\lambda = R^\lambda = K_\lambda \qquad \text{for all } \lambda \in \Lambda\text{-}\Lambda' = \{1, 2, 3, 4, 9\}$$

and by the conditions :

$$B^\lambda = \begin{bmatrix} H^\lambda & L^\lambda \\ 0 & R^\lambda \end{bmatrix} \quad \text{and} \quad S^\lambda = \begin{bmatrix} 0 & L^\lambda \\ 0 & R^\lambda \end{bmatrix} \quad \text{for all } \lambda \in \Lambda' = \{5, 6, 7, 8\}$$

which give the relations :

$$B^5 = \begin{bmatrix} K_5 & K_5 \\ 0 & K_5 \end{bmatrix} \qquad B^6 = \begin{bmatrix} K_6 & K_6 \\ 0 & K_6 \end{bmatrix} \qquad B^7 = \begin{bmatrix} K_7 & K_7 \\ 0 & K_7 \end{bmatrix}$$

$$S^5 = \begin{bmatrix} 0 & K_5 \\ 0 & K_5 \end{bmatrix} \qquad S^6 = \begin{bmatrix} 0 & K_6 \\ 0 & K_6 \end{bmatrix} \qquad S^7 = \begin{bmatrix} 0 & K_7 \\ 0 & K_7 \end{bmatrix}$$

and the relations :

$$B^8 = \left[\begin{array}{ccc:c} & & & K_8 \\ & M_q(K_8) & & K_8 \\ & & & \vdots \\ & & & K_8 \\ \hdashline 0 & 0 \cdots & 0 & K_8 \end{array}\right] \qquad S^8 = \left[\begin{array}{ccc:c} 0 & & 0 & K_8 \\ & 0 & & K_8 \\ & & & \vdots \\ 0 & & 0 & K_8 \\ \hdashline 0 & 0 \cdots & 0 & K_8 \end{array}\right]$$

For the index $k = 0 \in I_1$, the condition (α_4) determines the semisimple Artinian F-algebra :

$$B_0 = \prod_{\lambda\in\Lambda^0} B^\lambda = \prod_{\lambda\in\Lambda^0} R^\lambda = G_0 = \prod_{\lambda\in\Lambda_0} R^\lambda = R_0 = K_1 \times K_2$$

with a right Socle :

$$S_0 = S(B_0) = \prod_{\lambda\in\Lambda^0} S^\lambda = \bigoplus_{\lambda\in\Lambda^0} S^\lambda = \bigoplus_{\lambda\in\Lambda_0} R^\lambda = K_1 \oplus K_2$$

For the index $k = 1 \in I_1$, the pair (H_1, G_1) of semisimple Artinian F-algebras :

$$H_1 = K_5 \times K_6 \times K_7 \qquad \text{and} \qquad G_1 = K_3 \times K_4 \times K_5 \times K_6 \times K_7$$

which verifies automatically the condition :

$$H_1 \lhd G_1$$

determines the (right) almost semisimple right Artinian F-algebra :

$$B_k = B_1 = \begin{pmatrix} H_1 & L_1 \\ 0 & G_1 \end{pmatrix} = (H_1 = H_1 \lhd G_1) = \prod_{\lambda \in \Lambda^1} B^\lambda = B^3 \times B^4 \times B^5 \times B^6 \times B^7$$

with a right Socle :

$$S_k = S_1 = \begin{pmatrix} 0 & L_1 \\ 0 & G_1 \end{pmatrix} = S(B_1) = \prod_{\lambda \in \Lambda^1} S^\lambda = \bigoplus_{\lambda \in \Lambda^1} S^\lambda = S^3 \oplus S^4 \oplus S^5 \oplus S^6 \oplus S^7$$

which has a *"matrix realization"* of the form :

$$B_k = B_1 = \begin{bmatrix} H_1 & L_1 \\ 0 & G_1 \end{bmatrix} = \begin{bmatrix} K_5 & 0 \\ 0 & K_7 \end{bmatrix}$$

completely characterized in the following TABLE N° 3, which gives a *"matrix realization"* of the right Socle, of the form :

$$S_k = S_1 = \begin{bmatrix} 0 & L_1 \\ 0 & G_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & K_7 \end{bmatrix}$$

completely characterized in the following TABLE N° 4.

Likewise, for the index $k = 4 \in I_1$, the pair (H_4, G_4) of semisimple Artinian F-algebras :

$$H_4 = M_q(K_8) \qquad \text{and} \qquad G_4 = K_8 \times K_9$$

which verifies automatically the condition :

$$H_4 \lhd G_4$$

determines the (right) almost semisimple right Artinian F-algebra :

$$B_k = B_4 = \begin{pmatrix} H_4 & L_4 \\ 0 & G_4 \end{pmatrix} = (H_4 = H_4 \lhd G_4) = \prod_{\lambda \in \Lambda^4} B^\lambda = B^8 \times B^9$$

with a right Socle :

$$S_k = S_4 = \begin{pmatrix} 0 & L_4 \\ 0 & G_4 \end{pmatrix} = S(B_4) = \prod_{\lambda \in \Lambda^4} S^\lambda = \bigoplus_{\lambda \in \Lambda^4} S^\lambda = S^8 \oplus S^9$$

which has a *"matrix realization"* of the form :

$$B_k = B_4 = \begin{bmatrix} H_4 & L_4 \\ 0 & G_4 \end{bmatrix} = \begin{bmatrix} M_q(K_8) & 0 \\ 0 & K_9 \end{bmatrix}$$

TABLES OF MATRIX REALIZATIONS

TABLE N° 3

$$B_k = B_1 = \left[\begin{array}{c:c} H_1 & L_1 \\ \hdashline 0 & G_1 \end{array}\right] = \left[\begin{array}{ccc:cc:ccc} K_5 & 0 & 0 & 0 & 0 & K_5 & 0 & 0 \\ 0 & K_6 & 0 & 0 & 0 & 0 & K_6 & 0 \\ 0 & 0 & K_7 & 0 & 0 & 0 & 0 & K_7 \\ \hdashline 0 & 0 & 0 & K_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_4 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & K_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & K_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_7 \end{array}\right]$$

TABLE N° 4

$$S_k = S_1 = \left[\begin{array}{c:c} 0 & L_1 \\ \hdashline 0 & G_1 \end{array}\right] = \left[\begin{array}{ccc:cc:ccc} 0 & 0 & 0 & 0 & 0 & K_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & K_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_7 \\ \hdashline 0 & 0 & 0 & K_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_4 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & K_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & K_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_7 \end{array}\right]$$

completely characterized in the following [TABLE N° 5],

which gives a *"matrix realization"* of the right Socle, of the form :

$$S_k = S_4 = \begin{bmatrix} 0 & L_4 \\ 0 & G_k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & K_9 \end{bmatrix}$$

completely characterized in the following [TABLE N° 6].

(b_1) For the index $k = 1 \in I_1^*$, which determines the (right) almost semisimple right Artinian F-algebra :

$$B_1 = \begin{pmatrix} H_1 & L_1 \\ 0 & G_1 \end{pmatrix}$$

in which the F-algebra H_1 verifies the relation :

$$H_1 = K_5 \times K_6 \times K_7$$

from the F-algebra A_0 which has *the first characterization :*

$$\boxed{A_0 = K_1 \times K_2}$$

the Structure of the right Artinian F-algebra :

$$A_k = A_1 \in \mathfrak{A}(F)$$

is characterized by *"a first parameter"* Ψ_1 constituted by a *injective* F-*algebra homomorphism :*

$$\Psi_1 : A_0 = K_1 \times K_2 \rightarrowtail H_1 = K_5 \times K_6 \times K_7$$

which defines a F-subalgebra :

$$A_0 \subset H_1$$

which gives *the second characterization :*

$$\boxed{A_1 = \begin{pmatrix} A_0 & L_1 \\ 0 & G_1 \end{pmatrix} = (A_0 \overset{\Psi_1}{\rightarrowtail} H_1 \lhd G_1)}$$

which implies :

$$S_1 = \begin{pmatrix} 0 & L_1 \\ 0 & G_1 \end{pmatrix} = S(A_1) = S(B_1)$$

It is very important to remark that there may exist several OBSTRUCTIONS to the ***existence*** of this "parameter" Ψ_1.

For instance, the ***existence*** of Ψ_1, which is in particular a monomorphism of F-vector spaces, imply that the F-dimensions verify necessarily the relation :

$$[K_1 : F] + [K_2 : F] \leq [K_5 : F] + [K_6 : F] + [K_7 : F]$$

Therefore, the condition :

$$[K_1 : F] + [K_2 : F] > [K_5 : F] + [K_6 : F] + [K_7 : F]$$

TABLES OF MATRIX REALIZATIONS

TABLE N° 5

$$B_k = B_4 = \left[\begin{array}{c|c} H_4 & L_4 \\ \hline 0 & G_4 \end{array}\right] = \left[\begin{array}{cccc|c|c} & & & & K_8 & 0 \\ & & & & K_8 & 0 \\ & M_q(K_8) & & & \vdots & \vdots \\ & & & & K_8 & 0 \\ \hline 0 & 0 & \cdots & 0 & K_8 & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & K_9 \end{array}\right]$$

TABLE N° 6

$$S_k = S_4 = \left[\begin{array}{c|c} 0 & L_4 \\ \hline 0 & G_4 \end{array}\right] = \left[\begin{array}{cccc|c|c} 0 & & & 0 & K_8 & 0 \\ & & & & K_8 & 0 \\ & & 0 & & \vdots & \vdots \\ 0 & & & 0 & K_8 & 0 \\ \hline 0 & 0 & \cdots & 0 & K_8 & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & K_9 \end{array}\right]$$

is an OBSTRUCTION to the existence of the *"parameter"* Ψ_1, and for instance, this OBSTRUCTION is realized in the particular case : $F = \mathbb{R}$; $K_5 = K_6 = K_7 = \mathbb{C}$ and $K_1 = K_2 = \mathbb{H}$ (quaternions).

Now, in the general case, *we choose the* F-*skewfields* K_λ, in such a manner that there ***exist*** three (injective) F-*algebra homomorphisms :*

$$\Psi^{1,5} : K_1 \subset K_5 \quad ; \quad \Psi^{1,6} : K_1 \subset K_6 \quad ; \quad \Psi^{2,7} : K_2 \subset K_7$$

which are some *"auxiliary parameters"*.

With this hypothesis, it is possible *to choose* the *injective* F-*algebra homomorphism :*

$$\Psi_1 : A_0 = K_1 \times K_2 \rightarrowtail H_1 = K_5 \times K_6 \times K_7$$

defined by the condition :

$$\Psi_1[(a_1, a_2)] = (a_1, a_1, a_2) \in (K_5 \times K_6 \times K_7)$$

for every $(a_1, a_2) \in (K_1 \times K_2)$.

This choice implies immediately that the right Artinian F-algebra :

$$A_k = A_1 \in \mathfrak{A}(F)$$

has a *"matrix realization"* of the form :

$$A_1 = \begin{bmatrix} A_0 & L_1 \\ 0 & G_1 \end{bmatrix}$$

completely characterized in the following TABLE N° 7 , which means that each element $a \in A_1$ is represented by a (8×8) square matrix, in which the coefficients verify the conditions :

$$a_1 \in K_1 \subset K_5 \qquad l_5 \in K_5$$

$$a_1 \in K_1 \subset K_6 \qquad l_6 \in K_6$$

$$a_2 \in K_2 \subset K_7 \qquad l_7 \in K_7$$

$$a_3 \in K_3 \qquad a_4 \in K_4 \qquad a_5 \in K_5 \qquad a_6 \in K_6 \qquad a_7 \in K_7$$

the addition and the multiplication in the F-algebra A_1 being the classical addition and multiplication of matrices.

In this *"matrix realization"*, the *idempotent right Socle* $S_1 = \mathbf{S}(A_1) = S(B_1)$ has an *isotypical decomposition :*

$$S_1 = \bigoplus_{\lambda \in \Lambda^1} S^\lambda = S^3 \oplus S^4 \oplus S^5 \oplus S^6 \oplus S^7 = N_1 = \mathbf{N}(A_1)$$

TABLES OF MATRIX REALIZATIONS

TABLE N° 7

$$
A_1 = \left[\begin{array}{c|c} A_0 & L_1 \\ \hline 0 & G_1 \end{array}\right] = \left[\begin{array}{ccc|cc|ccc} \boxed{K_1} & 0 & 0 & 0 & 0 & K_5 & 0 & 0 \\ 0 & \boxed{K_1} & 0 & 0 & 0 & 0 & K_6 & 0 \\ 0 & 0 & K_2 & 0 & 0 & 0 & 0 & K_7 \\ \hline 0 & 0 & 0 & K_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & K_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & K_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_7 \end{array}\right] = \left[\begin{array}{ccc|cc|ccc} a_1 & 0 & 0 & 0 & 0 & l_5 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 & l_6 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 & 0 & l_7 \\ \hline 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_7 \end{array}\right]
$$

in which each *idempotent foot* S^λ is represented by the elements $a \in A_1$, in which the columns, different from the $(\lambda+1)$-th column, are null.

(c_2) For the index $j = 2 \in I_0^*$, associated to the index $k(j) = k = 1 \in I_2$, the condition :

$$T''^2_1 = \bigoplus_{\lambda \in \Lambda''^1_2} S^\lambda \qquad \text{for } j = 2 \in I_0^1$$

and the condition (α_6) determine the proper two-sided ideal :

$$T''^2_1 = S^4 \oplus S^5 \oplus S^6 \oplus S^7$$

which verifies :

$$T''^2_1 \in \mathcal{C}(B_1) \equiv \mathcal{C}(A_1)$$

and for the surjective F-algebra epimorphism :

$$\tau^j_k = \tau^2_1 = Id_{A_1} : A_{j-1} = A_1 \twoheadrightarrow A_k = A_1$$

the condition :

$$T_{j-1} = T_1 = N_1 \cap \left[\tau^2_1\right]^{-1}\left(T''^2_1\right)$$

characterizes a proper two-sided ideal :

$$T_{j-1} = T_1 \in \mathcal{C}(A_1) \equiv \mathcal{C}(A_{j-1})$$

defined by the condition :

$$T_1 = S^4 \oplus S^5 \oplus S^6 \oplus S^7$$

which determines the right Artinian F-algebra :

$$A_1/T_1 = C_1 \in \mathfrak{A}(F)$$

and the canonical surjective F-algebra epimorphism :

$$\varphi_1 : A_1 \twoheadrightarrow C_1 \equiv K_1 \times K_2 \times K_3$$

which associates, to any element $a \in A_1$, the matrix :

$$\varphi_1(a) = \left[\begin{array}{ccc:c} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ \hdashline 0 & 0 & 0 & a_3 \end{array}\right] \equiv (a_1, a_2, a_3) \in (K_1 \times K_2 \times K_3) \equiv C_1$$

and also the canonical surjective F-algebra epimorphism :

$$\varphi''_1 = Id_{C_1} : C_1 \twoheadrightarrow C'_1 = C_1/J(C_1) \equiv C_1 = K_1 \times K_2 \times K_3$$

Moreover, for $j = 2 \in I_0^*$, according to the conditions (α_8), (β_2) and (β_4), the relations :

$$L'^2_\lambda = W^2_\lambda \underset{K_\lambda}{\otimes} V^*_\lambda = \mathcal{L}(V_\lambda, W^2_\lambda) = M_{p_\lambda, n^2_\lambda}(K_\lambda)$$

and

$$H'^2_\lambda = \mathcal{L}(W^2_\lambda) = M_{n^2_\lambda}(K_\lambda) = \mathcal{L}_{R\lambda}(L'^2_\lambda)$$

for all $\lambda \in \Lambda'_2 = \{1, 2, 3\}$, give the relations :

$$L'^2_\lambda = M_{1,1}(K_\lambda) = M_1(K_\lambda) = K_\lambda \qquad \text{for all } \lambda \in \Lambda'_2 = \{1, 2, 3\}$$

and

$$H'^2_\lambda = M_1(K_\lambda) = K_\lambda \qquad \text{for all } \lambda \in \Lambda'_2 = \{1, 2, 3\}$$

which imply the relations :

$$M'_2 = \bigoplus_{\lambda \in \Lambda'_2} L'^2_\lambda = K_1 \oplus K_2 \oplus K_3$$

and

$$H'^2 = \prod_{\lambda \in \Lambda'_2} H'^2_\lambda = K_1 \times K_2 \times K_3$$

which characterize M'_2 as a $(H'^2 - C_1)$-bimodule.

Thus, in order to choose *"a second parameter"* Ψ'_2 constituted by a F-*algebra homomorphism :*

$$\Psi'_2 : C_1 \equiv (K_1 \times K_2 \times K_3) \longrightarrow H'^2 = (K_1 \times K_2 \times K_3)$$

which determines, by the characterization :

$$M_2 = [M'_2 ; \Psi'_2 : C_1 \longrightarrow H'^2]$$

a non null C_1-bimodule :

$$M_2 \in \mathcal{M}(A_1, T_1)$$

we can choose for Ψ'_2 any F-algebra endomorphism :

$$\Psi'_2 \in End_F(C_1)$$

For instance, we can ***choose*** three *"auxiliary parameters"* constituted by three F-*automorphisms :*

$$\eta_1 \in Aut_F(K_1) \qquad \eta_2 \in Aut_F(K_2) \qquad \eta_3 \in Aut_F(K_3)$$

which determine *"the second parameter" :*

$$\Psi'_2 = (\eta_1, \eta_2, \eta_3) \in Aut_F(K_1) \times Aut_F(K_2) \times Aut_F(K_3) \subset End_F(C_1)$$

With this ***choice***, which gives the relation :

$$l_{A_1}(M_2) = T_1$$

according to the condition :

$$T_1 \in \mathcal{C}(A_1)$$

the Proposition 3-15 implies the relation :

$$\hat{0} = 0 \in H^2_e(A_1, T_1, M_2) = H^2(A_1, T_1, M_2)$$

and therefore, in order to choose *"a third parameter"* ξ_2 it is possible to ***choose*** the T_1-***essential*** *cohomology class :*

$$\hat{h}_2 = \xi_2 = \hat{0} = 0 \in H^2_e(A_1, T_1, M_2)$$

which gives *the characterization :*

$$\boxed{(A_2, N_2) = (A_1, T_1, M_2, 0) = (A_1, T_1, M_2, \xi_2)}$$

and in particular the *"iterative cohomological characterization" :*

$$\boxed{A_2 = (A_1, M_2, \xi_2)}$$

These ***choices*** of $\Psi'_2 = (\eta_1, \eta_2, \eta_3)$ and of $\xi_2 = \hat{0}$, imply immediately that the right Artinian F-algebra :

$$A_j = A_2 \in \mathfrak{A}(F)$$

has a *"matrix realization"* of the form :

$$A_2 = \left[\begin{array}{cc:cc|ccc|cc|c|cc:c}
\eta_1(a_1) & \alpha_1 & 0 & 0 & & & & & & & 0 & 0 & 0 \\
0 & a_1 & 0 & 0 & & & & & & & 0 & 0 & 0 \\
\hdashline
0 & 0 & a_1 & 0 & & & & & & & l_5 & 0 & 0 \\
0 & 0 & 0 & a_1 & & & & & & & 0 & l_6 & 0 \\
\hline
 & & & & \eta_2(a_2) & \alpha_2 & 0 & & & & 0 & 0 & 0 \\
 & & & & 0 & a_2 & 0 & & & & 0 & 0 & 0 \\
\hdashline
 & & & & 0 & 0 & a_2 & & & & 0 & 0 & l_7 \\
\hline
 & & & & & & & \eta_3(a_3) & \alpha_3 & & & & \\
 & & & & & & & 0 & a_3 & & & & \\
\hline
 & & & & & & & & & a_4 & & & \\
\hline
 & & & & & & & & & & a_5 & 0 & 0 \\
 & & & & & & & & & & 0 & a_6 & 0 \\
\hdashline
 & & & & & & & & & & 0 & 0 & a_7
\end{array}\right]$$

which means that each element $a \in A_2$ is represented by a (13×13) square matrix, in which the coefficients verify the conditions :

$$a_1 \in K_1 \subset K_5 \qquad \alpha_1 \in K_1 \qquad l_5 \in K_5$$

$$a_1 \in K_1 \subset K_6 \qquad \alpha_2 \in K_2 \qquad l_6 \in K_6$$

$$a_2 \in K_2 \subset K_7 \qquad \alpha_3 \in K_3 \qquad l_7 \in K_7$$

$a_3 \in K_3$ $\quad a_4 \in K_4$ $\quad a_5 \in K_5$ $\quad a_6 \in K_6$ $\quad a_7 \in K_7$

the addition and the multiplication in the F-algebra A_2 being the classical addition and multiplication of matrices.

In this *"matrix realization"*, the *"one-socle"* :

$$N(A_2) = N_2 = (0, T_1) = \{0, T_1) = \{(0, t) ; t \in T_1\}$$

is represented by the four last columns, the other columns being null.

(c_3) For the index $j = 3 \in I_0^*$, associated to the index $k(j) = k = 1 \in I_2$, the condition :

$$T''^3_1 = \bigoplus_{\lambda \in \Lambda''^1_3} S^\lambda \qquad \text{for } j = 3 \in I_0^1$$

and the condition (α_6) determine the proper two-sided ideal :

$$T''^3_1 = S^5 \oplus S^6 \oplus S^7$$

which verifies :

$$T''^3_1 \in \mathscr{C}(B_1) \equiv \mathscr{C}(A_1)$$

and for the surjective F-algebra epimorphism :

$$\tau^j_k = \tau^3_1 = \tau_2 : A_{j-1} = A_2 \twoheadrightarrow A_k = A_1$$

the condition :

$$T_{j-1} = T_2 = N_2 \cap \left[\tau^3_1\right]^{-1} \left(T''^3_1\right)$$

characterizes a proper two-sided ideal :

$$T_{j-1} = T_2 \in \mathscr{C}(A_2) \equiv \mathscr{C}(A_{j-1})$$

defined by the condition :

$$T_2 = \bigoplus_{\lambda \in \Lambda''^1_3} \overline{S}^\lambda = \overline{S}^5 \oplus \overline{S}^6 \oplus \overline{S}^7$$

in which :

$$\overline{S}^\lambda = (0, S^\lambda)$$

is characterized by the (λ+6)-th column in A_2, the other columns being null, and which determines the right Artinian F-algebra :

$$A_2/T_2 = C_2 \in \mathfrak{A}(F)$$

defined by the relation :

$$C_2 \equiv D_1^{\eta_1}(K_1) \times D_2^{\eta_2}(K_2) \times D_3^{\eta_3}(K_3) \times K_4$$

in which :

$$D_\lambda^{\eta_\lambda}(K_\lambda) = \begin{bmatrix} \eta_\lambda(a_\lambda) & \alpha_\lambda \\ 0 & a_\lambda \end{bmatrix} \qquad \text{for all } \lambda \in \{1, 2, 3\}$$

and the obvious canonical surjective F-algebra epimorphism :

$$\varphi_2 : A_2 \twoheadrightarrow C_2 \equiv D_1^{\eta_1}(K_1) \times D_2^{\eta_2}(K_2) \times D_3^{\eta_3}(K_3) \times K_4$$

and also the canonical surjective F-algebra epimorphism :

$$\varphi''_2 : C_2 \twoheadrightarrow C'_2 = C_2/J(C_2) = K_1 \times K_2 \times K_3 \times K_4$$

Morever, for $j = 3 \in I_0^*$, according to the conditions (α_8), (β_2) and (β_4), the relations :

$$L'^3_\lambda = W^3_\lambda \underset{K_\lambda}{\otimes} V^*_\lambda = \mathcal{L}(V_\lambda, W^3_\lambda) = M_{p_\lambda, n^3_\lambda}(K_\lambda)$$

and

$$H'^3_\lambda = \mathcal{L}(W^3_\lambda) = M_{n^3_\lambda}(K_\lambda) = \mathcal{L}_{R\lambda}(L'^3_\lambda)$$

for all $\lambda \in \Lambda'_3 = \{1, 2, 3, 4\}$, give the relations :

$$L'^3_\lambda = M_{1,1}(K_\lambda) = M_1(K_\lambda) = K_\lambda \quad \text{for all } \lambda \in \Lambda'_3 = \{1, 2, 3, 4\}$$

and

$$H'^3_\lambda = M_1(K_\lambda) = K_\lambda \qquad \text{for all } \lambda \in \Lambda'_3 = \{1, 2, 3, 4\}$$

which imply the relations :

$$M'_3 = \bigoplus_{\lambda \in \Lambda'_3} L'^3_\lambda = K_1 \oplus K_2 \oplus K_3 \oplus K_4$$

and

$$H'^3 = \prod_{\lambda \in \Lambda'_3} H'^3_\lambda = K_1 \times K_2 \times K_3 \times K_4$$

which characterize M'_3 as a $(H'^3\text{-}C_2)$-bimodule.

Thus, in order to choose *"a fourth parameter"* Ψ'_3 constituted by a F-*algebra homomorphism :*

$$\Psi'_3 : C_2 \longrightarrow H'^3 = K_1 \times K_2 \times K_3 \times K_4$$

which determines, by the characterization :

$$M_3 = [M'_3 \, ; \Psi'_3 : C_2 \longrightarrow H'^3]$$

a non null C_2-bimodule :

$$M_3 \in \mathcal{M}(A_2, T_2)$$

it is possible to ***choose*** for Ψ'_3 the F-algebra epimorphism :

$$\Psi'_3 : C_2 \longrightarrow H'^3 = K_1 \times K_2 \times K_3 \times K_4$$

characterized by the condition :

$$\Psi'_3(c) = (\eta_1^2(a_1), \eta_2^2(a_2), \eta_3^2(a_3), a_4)$$

for every $a \in A_2$, which determine $\varphi_2(a) = c \in C_2$.

Therefore, the C_2-bimodule M_3 is characterized by the fact that for every :

$$\beta = (\beta_1, \beta_2, \beta_3, \beta_4) \in (K_1 \oplus K_2 \oplus K_3 \oplus K_4) = M_3$$

and for every $\varphi_2(a) = c \in C_2$, then :

$$c\beta = (\eta_1^2(a_1)\, \beta_1, \eta_2^2(a_2)\, \beta_2, \eta_3^2(a_3)\, \beta_3, a_4\beta_4)$$

and

$$\beta c = (\beta_1 a_1\, , \beta_2 a_2\, , \beta_3 a_3\, , \beta_4 a_4)$$

Then , for every :

$$\mu = (\mu_1, \mu_2, \mu_3) \in (F^*)^3$$

it is easy to verify that the condition :

$$h_\mu^*(a, a') = (\mu_1\eta_1(\alpha_1)\alpha'_1, \mu_2\eta_2(\alpha_2)\alpha'_2, \mu_3\eta_3(\alpha_3)\alpha'_3, 0)$$

characterizes a 2-cocycle :

$$h = h_\mu \in Z^2(A_2, T_2, M_3) \equiv Z^2(C_2, M_3)$$

Fristly, the definition of M_3 implies the relations :

$$l_{A_2}(M_3) = \{a \in A_2\, ;\, a_1 = a_2 = a_3 = a_4 = 0\} \supset T_2$$

Secondly, in order to prove the relation :

$$h = h_\mu \in Z_e^2(A_2, T_2, M_3)$$

according to the Proposition 3-15, we must show that for every element :

$$\bar{a} = (\beta, a) \in M_3 \times [l_{A_2}(M_3) - T_2] \subset M_3 \times A_2$$

the conditions :

$(\bar{r})$ $$r_{A_2}(a) \subset s_{A_2}(\bar{a}, h) = \{a' \in A_2\, ;\, \beta a' + h^*(a, a') = 0\}$$

and

$(\bar{s})$ $$a\,.\, s_{A_2}(\bar{a}, h) \cap T_2 = (0)$$

imply $\bar{a} = 0$, that is : $\beta = (\beta_1, \beta_2, \beta_3, \beta_4) = 0$ and $a = 0$.

Then, if $a \in [l_{A_2}(M_3) - T_2]$, in the element $a \in A_2$ there exists an $\alpha_i \neq 0$ for one $i \in \{1, 2, 3\}$, and if we choose $a'' \in A_2$ such that $\alpha''_i = 1$ and the other components of a" being null, then the relations :

$$a'' \in r_{A_2}(a) \qquad \text{and} \qquad a'' \notin s_{A_2}(\bar{a}, h)$$

show that the condition $(\bar{r})$ is not verified, in such a way that the Definitions 3-7 and 3-9 imply the relations :

$$h = h_\mu \in Z_e^2(A_2, T_2, M_3)$$

and

$$\hat{h} = \hat{h}_{\mu} \in H^2_e(A_2, T_2, M_3) \subset H^2(A_2, T_2, M_3)$$

Therefore, in order to choose *"a fifth parameter"* ξ_3 it is possible to ***choose*** the T_2-*essential cohomology class :*

$$\hat{h}_{\mu} = \xi_3 \in H^2_e(A_2, T_2, M_3)$$

which gives *the fourth characterization :*

$$\boxed{(A_3, N_3) = (A_2, T_2, M_3, h_{\mu}) = (A_2, T_2, M_3, \xi_3)}$$

and in particular the *"iterative cohomological characterization" :*

$$\boxed{A_3 = (A_2, M_3, \xi_3)}$$

These ***choices*** of Ψ'_3 and of $\xi_3 = \hat{f}_{\mu}$, imply immediately that the right Artinian F-algebra :

$$A_j = A_3 \in \mathfrak{A}(F)$$

has a *"matrix realization"* of the form :

$$A_3 = \begin{bmatrix} \eta_1^2(a_1) & 0 \\ 0 & a_7 \end{bmatrix}$$

completely characterized in the following $\boxed{\text{TABLE N° 8}}$, which means that each element $a \in A_3$ is represented by a (17×17) square matrix, in which the coefficients verify the conditions :

$a_1 \in K_1 \subset K_5$	$\alpha_1 \in K_1$	$\beta_1 \in K_1$	$l_5 \in K_5$
$a_1 \in K_1 \subset K_6$	$\alpha_2 \in K_2$	$\beta_2 \in K_2$	$l_6 \in K_6$
$a_2 \in K_2 \subset K_7$	$\alpha_3 \in K_3$	$\beta_3 \in K_3$	$l_7 \in K_7$

$a_3 \in K_3$ $\quad$ $a_4 \in K_4$ $\quad$ $a_5 \in K_5$ $\quad$ $a_6 \in K_6$ $\quad$ $a_7 \in K_7$

the addition and the multiplication in the F-algebra A_3 being the classical addition and multiplication of matrices.

(b4) For the index $k = 4 \in I_1^*$, which determines the (right) almost semisimple right Artinian F-algebra :

$$B_4 = \begin{pmatrix} H_4 & L_4 \\ 0 & G_4 \end{pmatrix}$$

in which the F-algebra H_4 verifies the relation :

$$H_4 = M_q(K_8)$$

TABLES OF MATRIX REALIZATIONS

TABLE N° 8

$$A_3 = \left[\begin{array}{ccccc|cccc|ccc|cc|ccc}
\eta_1^2(a_1) & \mu_1\eta_1(\alpha_1) & \beta_1 & 0 & 0 & & & & & & & & & & 0 & 0 & 0 \\
0 & \eta_1(a_1) & \alpha_1 & 0 & 0 & & & & & & & & & & 0 & 0 & 0 \\
0 & 0 & a_1 & 0 & 0 & & & & & & & & & & 0 & 0 & 0 \\
0 & 0 & 0 & a_1 & 0 & & & & & & & & & & l_5 & 0 & 0 \\
0 & 0 & 0 & 0 & a_1 & & & & & & & & & & 0 & l_6 & 0 \\
\hline
 & & & & & \eta_2^2(a_2) & \mu_2\eta_2(\alpha_2) & \beta_2 & 0 & & & & & & 0 & 0 & 0 \\
 & & & & & 0 & \eta_2(a_2) & \alpha_2 & 0 & & & & & & 0 & 0 & 0 \\
 & & & & & 0 & 0 & a_2 & 0 & & & & & & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & a_2 & & & & & & 0 & 0 & l_7 \\
\hline
 & & & & & & & & & \eta_3^2(a_3) & \mu_3\eta_3(\alpha_3) & \beta_3 & & & & & \\
 & & & & & & & & & 0 & \eta_3(a_3) & \alpha_3 & & & & & \\
 & & & & & & & & & 0 & 0 & a_3 & & & & & \\
\hline
 & & & & & & & & & & & & a_4 & \beta_4 & & & \\
 & & & & & & & & & & & & 0 & a_4 & & & \\
\hline
 & & & & & & & & & & & & & & a_5 & 0 & 0 \\
 & & & & & & & & & & & & & & 0 & a_6 & 0 \\
 & & & & & & & & & & & & & & 0 & 0 & a_7
\end{array}\right]$$

from the F-algebra A_3, the Structure of the right Artinian F-algebra :

$$A = A_k = A_4 \in \mathfrak{K}(F)$$

is characterized by a *"sixth parameter"* Ψ_4 constituted by an *injective* F-*algebra homomorphism :*

$$\Psi_4 : A_3 \rightarrowtail\!\!\twoheadrightarrow H_4 = M_q(K_8)$$

which defines a F-subalgebra :

$$A_3 \subset H_4$$

which gives *the fifth characterization :*

$$\boxed{A = A_4 = \begin{pmatrix} A_3 & L_4 \\ 0 & G_4 \end{pmatrix} = (A_3 \rightarrowtail\!\xrightarrow{\Psi_4} H_4 \lhd Gu)}$$

which implies :

$$S(A) = S_4 = \begin{pmatrix} 0 & L_4 \\ 0 & G_4 \end{pmatrix} = S(A_4) = S(B_4)$$

For instance, according to the existence of the previous (injective) F-*algebra homomorphisms :*

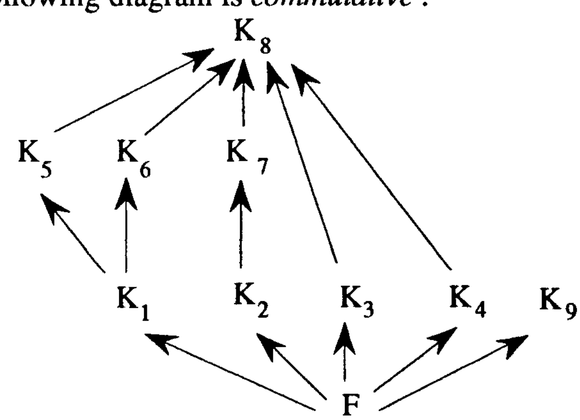
$$\Psi^{1,5} : K_1 \subset K_5 \ ; \qquad \Psi^{1,6} : K_1 \subset K_6 \ ; \qquad \Psi^{2,7} : K_2 \subset K_7$$

if *we suppose the existence* of (injective) F-*algebra homomorphisms :*

$$\Psi^{3,8} : K_3 \subset K_8 \ ; \Psi^{4,8} : K_4 \subset K_8 \ ; \Psi^{5,8} : K_5 \subset K_8 \ ;$$

$$\Psi^{6,8} \ ; K_6 \subset K_8 \ ; \Psi^{7,8} : K_7 \subset K_8$$

such that the following diagram is *commutative :*



and if the *"numerical parameter"* $q \in \mathbb{N}^*$, verifies :

$$q = 17$$

we can choose for the *injective* F-*algebra homomorphism :*

$$\Psi_4 : A_3 \rightarrowtail\!\!\twoheadrightarrow H_4 = M_q(K_8) = M_{17}(K_8)$$

the *"canonical injection"* defined by the previous *"matrix realization"* of the F-algebra A_3.

With these ***choices***, it is immediate that the right Artinian F-algebra :

$$A = A_4 \in \mathfrak{A}(F)$$

has a *"matrix realization"* of the form :

$$A = \begin{bmatrix} \eta_1^2(a_1) & & 0 \\ & & \\ & & \\ 0 & & a_9 \end{bmatrix}$$

completely characterized in the following TABLE N° 9 , which means that each element $a \in A = A_4$ is represented by a (19×19) square matrix, in which the coefficients verify the conditions :

$a_1 \in K_1 \subset K_5 \subset K_8$ $\qquad \alpha_1 \in K_1 \subset K_5 \subset K_8$

$a_1 \in K_1 \subset K_6 \subset K_8$ $\qquad \alpha_2 \in K_2 \subset K_7 \subset K_8$

$a_2 \in K_2 \subset K_7 \subset K_8$ $\qquad \alpha_3 \in K_3 \subset K_8$

$\beta_1 \in K_1 \subset K_5 \subset K_8$ $\qquad l_5 \in K_5 \subset K_8$

$\beta_2 \in K_2 \subset K_7 \subset K_8$ $\qquad l_6 \in K_6 \subset K_8$

$\beta_3 \in K_3 \subset K_8$ $\qquad l_7 \in K_7 \subset K_8$

$\beta_4 \in K_4 \subset K_8$

$a_3 \in K_3 \subset K_8$; $a_4 \in K_4 \subset K_8$; $a_5 \in K_5 \subset K_8$; $a_6 \in K_6 \subset K_8$; $a_7 \in K_7 \subset K_8$

$a_8 \in K_8$; $a_9 \in K_9$

and at last : $l_8^i \in K_8$ for $1 \leq i \leq 17$;

the addition and the multiplication in the F-algebra $A = A_4$ being the classical addition and multiplication of matrices.

With the previous *"matrix realization"* of the *right Artinian* F-*algebra* A, it is easy to determine its quiver (in the sense of [17] p. 97) :

$$\Gamma(A) = (\Lambda = V(A), E = E(A))$$

in which the *vertex set* $\Lambda = V(A)$ being defined by :

$$\Lambda = V(A) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

TABLES OF MATRIX REALIZATIONS

TABLE N° 9

$$
A = \left[\begin{array}{ccccc|cccc|ccc|cc|ccc|c|c}
\eta_1^2(a_1) & \mu_1\eta_1(\alpha_1) & \beta_1 & 0 & 0 & & & & & & & & & & 0 & 0 & 0 & l_8^1 & 0 \\
0 & \eta_1(a_1) & \alpha_1 & 0 & 0 & & & & & & & & & & 0 & 0 & 0 & l_8^2 & 0 \\
0 & 0 & a_1 & 0 & 0 & & & & & & & & & & 0 & 0 & 0 & l_8^3 & 0 \\
0 & 0 & 0 & a_1 & 0 & & & & & & & & & & l_5 & 0 & 0 & l_8^4 & 0 \\
0 & 0 & 0 & 0 & a_1 & & & & & & & & & & 0 & l_6 & 0 & l_8^5 & 0 \\
\hline
 & & & & & \eta_2^2(a_2) & \mu_2\eta_2(\alpha_2) & \beta_2 & 0 & & & & & & 0 & 0 & 0 & l_8^6 & 0 \\
 & & & & & 0 & \eta_2(a_2) & \alpha_2 & 0 & & & & & & 0 & 0 & 0 & l_8^7 & 0 \\
 & & & & & 0 & 0 & a_2 & 0 & & & & & & 0 & 0 & 0 & l_8^8 & 0 \\
 & & & & & 0 & 0 & 0 & a_2 & & & & & & 0 & 0 & l_7 & l_8^9 & 0 \\
\hline
 & & & & & & & & & \eta_3^2(a_3) & \mu_3\eta_3(\alpha_3) & \beta_3 & & & & & & l_8^{10} & 0 \\
 & & & & & & & & & 0 & \eta_3(a_3) & \alpha_3 & & & & & & l_8^{11} & 0 \\
 & & & & & & & & & 0 & 0 & a_3 & & & & & & l_8^{12} & 0 \\
\hline
 & & & & & & & & & & & & a_4 & \beta_4 & & & & l_8^{13} & 0 \\
 & & & & & & & & & & & & 0 & a_4 & & & & l_8^{14} & 0 \\
\hline
 & & & & & & & & & & & & & & a_5 & 0 & 0 & l_8^{15} & 0 \\
 & & & & & & & & & & & & & & 0 & a_6 & 0 & l_8^{16} & 0 \\
 & & & & & & & & & & & & & & 0 & 0 & a_7 & l_8^{17} & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9
\end{array}\right]
$$

the *edge set* $E = E(A) = \{(e_i, e_j) : e_i\, \mathbf{J}(A)\, e_j \neq (0)\}$ is characterized by the following table :

Λ	1	2	3	4	5	6	7	8	9
1	(1,1)				(1,5)	(1,6)		(1,8)	
2		(2,2)					(2,7)	(2,8)	
3			(3,3)					(3,8)	
4				(4,4)				(4,8)	
5								(5,8)	
6								(6,8)	
7								(7,8)	
8									
9									

This last example gives an illustration of our Structure Theorem, which shows the possibility of the existence of OBSTRUCTIONS to the choice of the *"parameters"*, which characterize the "multiplicative structure" of the right Artinian F-algebra.

Moreover, it suggests the problem of the study of the connexions between our notion of F-*"Completely structured vertex set"* and the classical notions of *quiver.*

REFERENCES

[1] N. BOURBAKI, *Algèbre Ch. 8 Modules et anneaux semi-simples*, Paris, Hermann, (Act. Sc. et Ind. 1261, Eléments de Mathématiques).

[2] H. CARTAN and S. EILENBERG, *Homological Algebra*, Princeton University Press, (1956).

[3] J. DIEUDONNE, *Sur les systèmes hypercomplexes*, J. Reine Angew. Math, 184, (1942), 178-192.

[4] J. DIEUDONNE, *Sur le socle d'un anneau et les anneaux simples infinis*, Bull.Soc. Math. France, 70, (1942), 46-75.

[5] J. DIEUDONNE, *Les idéaux minimaux dans les anneaux associatifs,* Proc. Int. Cong. Math., Cambridge, Mass., (1950), V.2, Amer. Math. Soc., Providence, (1952), 44-48.

[6] A.W. GOLDIE, *Torsion-Free Modules and Rings,* J. Algebra, 1, (1964), 268-287.

[7] M. HACQUE, *Mono-sous-catégorie d'une catégorie de modules,* Publ. Dép. Math. Sc. de Lyon, 6, fasc. 1, (1969), 13-48.

[8] M. HACQUE, *Quelques remarques sur la notion de socle,* Comm. in Algebra, 10, (1982), 1003-1025.

[9] M. HACQUE, *Anneaux fidèlement représentés sur leur socle droit,* Comm. in Algebra, 10, (1982), 1027-1072.

[10] M. HACQUE, *Some new invariants for right Artinian rings,* Comm. in Algebra, (to appear).

[11] G. HOCHSCHILD, *On the Cohomology groups of an associative algebra,* Ann. of Math, 46, (1945), 58-67.

[12] G. HOCHSCHILD, *On the Cohomology Theory for associative algebras,* Ann. of Math, 47, (1946), 568-579.

[13] G. HOCHSCHILD, *Cohomology and representations of associative algebras,* Duke Math. J, 14, (1947), 921-948.

[14] G. HOCHSCHILD and J.P. SERRE, *Cohomology of group Extensions,* Trans. Amer. Math. Soc., 74, (1953), 110-134.

[15] S. Mac LANE, *Extensions and Obstructions for Rings,* Ill. J. Math., 2, (1958), 316-345.

[16] S. Mac LANE, *Homology,* Die Grundlehren der mathematischen Wissenschaften, 114, Springer-Verlag, Berlin, (1967).

[17] R.S. PIERCE, *Associative Algebras,* Graduate Texts in Mathematics, Springer-Verlag, New York Heidelberg Berlin (1982).