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ON THE COMPLETE INTEGRAL CLOSURE OF A MORI DOMAIN

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It is well known that the complete integral closure A^* of a domain A need not be completely integrally closed (R. Gilmer and W. Heinzer). It turns out that even if A is Mori, then A^* is not necessarily completely integrally closed or Mori, thus answering a question of Professor Valentina Barucci (Università di Roma "La Sapienza"). On the positive side, for any Mori domain A, the domain A^{**} is completely integrally closed (in short c.i.c.). If A is Mori and root-closed, then A^* is c.i.c. We recall that by a result of V. Barucci, if A is Mori and $(A:A^*) \neq 0$, then A^* is Krull.

In order to construct a Mori domain A such that $A^* \neq A^{**}$, we remark that for any domain A, we have: $A^* = \bigcup_{x,y \text{ in } A} (A_0[x,y]_y \cap A)^*$, where A_0 is the prime ring contained in A. This remark leads us to consider certain subrings of $D[X,Y]_Y$ for a domain D and to define power functions. Furthermore, as being complete integral closed and the Mori property are multiplicative properties of a domain, we deal here not just with ring power functions but also with semigroup power functions.

Let x, y be elements of a cancellative semigroup S with unit. We define the function $\Psi_{S;x,y} : \mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}$ as follows: $\Psi_{S;x,y}(m) = \sup \left\{ n \in \mathbb{N} : (x^m/y^n) \in S \right\}$ (Here x^m/y^n belongs to the localization S_y of S) . A function $\Phi : \mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}$ will be called a <u>semigroup power function</u> if $\Phi = \Psi_{S;x,y}$ for some nonzero elements x,y in a cancellative semigroup S. A <u>ring power function</u> is a function of the form $\Psi_{S;x,y}$, where $S = A\setminus\{0\}$ for some domain A.

We denote by $\mathbb{M}$ the semigroup $\left\{ X^m Y^n : m \in \mathbb{N} , n \in \mathbb{Z} \right\}$, where X,Y are indeterminates. Let M be a subsemigroup of $\mathbb{M}$ containing X,Y. We denote by Ψ_M the function $\Psi_{M;X,Y}$. On the other hand, for any function $\Phi\colon \mathbb{N} \longrightarrow \mathbb{R}\cup\{\infty\}$, we denote by Λ^{Φ} the set of all elements X^m/Y^n in $\mathbb{M}$ such that $n \le \Phi(m)$; for any domain D, we denote by D^{Φ} the domain $D[\Lambda^{\Phi}]$.

It is easy to obtain the following characterization of semigroup power functions:

THEOREM 1 *Let* $\Phi\colon \mathbb{N} \longrightarrow \mathbb{N}\cup\{\infty\}$ *be a function. The following conditions are equivalent:*

1) Φ *is a ring power function.*

2) Φ *is a semigroup power function.*

3) *For all* m, n *in* $\mathbb{N}$ *it holds:* $\Phi(m+n) \ge \Phi(m)+\Phi(n)$.

4) *For any domain* D, *it holds:* $\Phi = \Psi_{D^{\Phi};X,Y}$.

5) *For any domain* D, *it holds:* $\Lambda^{\Phi} = \mathbb{M}\cap D^{\Phi}$.

EXAMPLES of power functions: Let $c \ge 0$ in $\mathbb{R}$. We denote by τ_c the function $[cn]$. For $c>0$, we define $\sigma_c(n)$ as the greatest integer which is $<cn$ for $n>0$ and set $\sigma_c(0) = 0$. We also define $\sigma_0(n) = 0$ for all $n \ge 0$, thus $\sigma_0 = \tau_0$. We denote by ℓ_c the function $\ell_c(n) = [c(n - \log(n+1))]$ $(n \in \mathbb{N})$. By Theorem 1, σ_c, τ_c and ℓ_c are power functions.

Given a class $\mathcal{C}$ of cancellative semigroups with unit, a <u>$\mathcal{C}$ - power function</u> is a power function of the form $\Psi_{S;x,y}$, where

S is a semigroup in $\mathcal{E}$. For example, we will deal here with Mori semigroup power functions, etc. (The Mori property for cancellative semigroups is defined similarly to the Mori ring property). We use a similar terminology for ring power functions. Any Mori semigroup power function which is not identically ∞ is necessarily finite.

For any function $\Phi: \mathbb{N} \longrightarrow \mathbb{R}$, we denote by $\Delta\Phi: \mathbb{N} \longrightarrow \mathbb{R}$ the function $\Delta\Phi(n) = \Phi(n+1)-\Phi(n)$ and by $\delta\Phi: \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{R}$ the function $\delta\Phi(m,n) = \Phi(m+n)-\Phi(m)-\Phi(n)$. For example, condition 3) of the Theorem 1 for the case that Φ is finite, can be stated in the form: $\delta(\Phi) \geq 0$.

We now characterize More semigroup power functions:

THEOREM 2 Let Φ be a finite power function. The following conditions are equivalent:

(i) Φ is semigroup Mori.

(ii) The function Φ has the following two properties:

(1) $\Delta\Phi$ is bounded.

(2) Any infinite set $I \subseteq \mathbb{N}$ has a finite subset F such that for all $m \in \mathbb{N}$ it holds:

$$\min \left\{ \delta\Phi(m,k) \ : \ k \in I \right\} = \min \left\{ \delta\Phi(m,k) \ : \ k \in F \right\} .$$

(iii) The semigroup Λ^{Φ} is Mori.

Condition (ii) (2) of the preceding theorem is equivalent to the $CC^{\perp}$ in the semigroup M/MY^{r} for every r .

For any finite power function Φ , we denote $\sup_n \Phi(n)/n$ by c_{Φ} . Clearly for $\Phi = \ell_c , \sigma_c$ or τ_c we have: $c = c_{\Phi}$. It is easy to show that $\lim_{n\to\infty} \Phi(n)/n = c_{\Phi}$ and $c_{\Phi} \leq \sup \Delta\Phi$ for any power function Φ . Thus, if $\Delta\Phi$ is finite, in particular if Φ is a finite semigroup Mori power function, then c_Φ is finite.

For any finite power function Φ , we define the following two functions from $\mathbb{N}$ to $\mathbb{N}\cup(\infty)$:

$$\Phi^{*}(m) := \sup_{r\in\mathbb{N}} \inf_{k\in\mathbb{N}} [\Phi(km+r)/k] \ , \quad \hat{\Phi}(m) := \sup_{k\in\mathbb{N}} [\Phi(km)/k] \ .$$

Both $\hat{\Phi}$ and Φ^{*} are power functions. Moreover, Φ is root-closed (as a semigroup or as a ring power function) if and only if $\Phi = \hat{\Phi}$. Similarly, Φ is c.i.c. if and only if $\Phi = \Phi^{*}$ (again this holds in both senses: as a semigroup or as a ring power function).

We see that we can translate semigroup or ring properties into properties of power functions and conversely. For example, we can characterize Mori root-closed or c.i.c. power functions. Indeed, let $\mathcal{S} := \left\{ \sigma_c \ : \ c\geq 0 \right\}$ and $\mathcal{T} := \left\{ \tau_c \ : \ c\geq 0 \right\}$. It can be shown that $\mathcal{S}\cup\mathcal{T}$ is the set of all root-closed power functions with c_Φ finite and $\mathcal{T}$ is the set of all c.i.c. power functions with c_Φ finite. Also, $\mathcal{S}_{\mathbb{Q}}\cup\mathcal{T}_{\mathbb{Q}}$ is the set of finite root-closed Mori power functions and $\mathcal{T}_{\mathbb{Q}}$ is the set of all finite c.i.c. Mori power functions (here $\mathcal{S}_{\mathbb{Q}}$ is the set of all σ_c with c rational and similarly for $\mathcal{T}_{\mathbb{Q}}$) . As before everything here is in both senses).

The set of all factorial finite (semigroup or ring) power functions equals $\mathcal{T}_{\mathbb{Q}}$.

For the Mori property we have a two-sided translation just for semigroup power functions by Theorem 2 above and we conject that a semigroup Mori power function is also ring Mori (the converse is clear).

We have the following

THEOREM 3 *Let* $\Phi : \mathbb{N} \longrightarrow \mathbb{R}^+$ *be a function with the following properties:*

(1) *For all* m , n *in* $\mathbb{N}$ *it holds:* $\Phi(m+n) \geq \Phi(m)+\Phi(n)$.

(2) *The sequence* $\Delta\Phi(n)$ *converges.*

(3) $\lim\limits_{\substack{m\to\infty \\ n\to\infty}} \delta\Phi(m,n) = \infty$.

Then K^{Φ} *is Mori for any field* K . *In particular,* $[\Phi]$ *is a Mori power function.*

The conditions of the last theorem are fulfilled by the function $\Phi(n) := c(n-\log(n+1))$ for any $c>0$, so all the functions ℓ_c for $c>0$ are ring Mori.

Taking into account Theorem 3 and further properties as e.g. $\hat{\ell}_c = \ell_c^{\,*} = \sigma_c$, we can obtain our counterexamples:

Let K be a field. Let $c>0$ and let $A := K^{\ell_c}$, thus A is Mori. We have:

(1) For c rational, $A^* = \bar{A}$ is Mori, but is not c.i.c.

(2) For c irrational, $A^* = \bar{A}$ is not Mori, but is c.i.c.

(3) For positive constants a and b , where a is rational and b is irrational, the domain $B := \left(K^{\ell_a}\right)^{\ell_b}$ is Mori, but B^* is neither Mori, nor c.i.c. Notice that $B \cong K^{\ell_a} \otimes_K K^{\ell_b}$.

In particular, we see that the integral closure of a Mori domain is not necessarily Mori, thus answering a question of Professor Evan G. Houston (University of North Carolina at Charlotte). We recall that by a result of V. Barucci, the integral closure of a Mori domain is not necessarily c.i.c.