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All figures are at the end of the paper (*).

(*) Advice to the reader : figures gain in readability when coloring the state-diagrams, one color for each state.

§ 1 - INTRODUCTION.

1.1. The firing squad synchronization problem was introduced by E.F.MOORE in 1962 [3]. One considers a finite -but arbitrarily long- ordered line of n finite state machines numbered from 1 to n. All machines numbered from 2 to $n-1$ are identical and called soldiers ; the machine numbered 1 is also called the general and the machine numbered n the right-end soldier. These n machines work synchroneously ; the state of a machine at time $t+1$ depends only on the states at time t of itself and its (one or two) neighbours.

For $t < 0$, all machines are in the same state, called the quiescent state.

At $t = 1$, an external intervention undergoes the general in a new state ; all other machines still being in the quiescent state. Afterward there is no external intervention ; the line evolves as an isolated system.

The problem is to define finite sets of states and transition rules (for the three types of machines) with a distinguished state, called the "fire state" so that : Whatever be the length of the line, all machines first enter the fire state at the very same time (called the synchronization time $t(n)$, which obviously depends on the length n of the line).

1.2. One can show that necessarily $t(n) > 2n-1$.

Intuitively $2n-1$ is the minimal time for the general to send a message to the right-end soldier and to get back an answer.

A minimal time solution of the synchronization problem is a family of finite sets of states and transition rules for which $t(n) = 2n-1$. An N-states solution of the synchronization problem is a solution for which the union of the sets of states of the three types of machines is of cardinality at most N.

1.3. J.MacCARTHY and M.MINSKY proved the existence of solutions (1965) ; these solutions work in times $3n-1$, $\frac{5}{2}n-1$ .... A minimal time solution was presented by A.WAKSMAN (1966) [4] ; this solution uses 16 states. R.BALZER presented an 8-states minimal time solution (1967) [1]. We present here a 6-states minimal time solution.

1.4. To study this problem, it is natural and usual to consider the set of pairs (K,t) with $1 \leq K \leq n$ and $1 \leq t \leq t(n)$. This plane set of pairs can be used in two ways.

- The state-diagram is obtained by indicating the site-values $\langle K,t \rangle$ (i.e. the state of machine K at time t). Graphically we shall represent the site (K,t) by an unit square and get an N-colored tiling of the rectangle formed by these $n \times t(n)$ unit squares.

- Geometrical diagrams are obtained by indicating the action of some distinguished transition rules which correspond intuitively to the propagation of signals. Graphically, this gives continuous lines through the portion of plane $[1,n] \times [1, t(n)]$.

1.5. To avoid the consideration of three cases of transition rules corresponding to the three kinds of machines, it is convenient to introduce :

- a new artificial state denoted X.
- two artificial machines always in state X, delimitating the line, having ranks 0 and n+1.

This trick allows us to represent the transition rules of an N-states solution by a family of N-1 matrices of states with N lines and N columns as follows :

- we denote q_o the fire state, q_1 the state X and $q_2 \dots q_N$ the remaining states.
- If at time t, the machines numbered K-1, K, K+1 (where $K \in \{1, \dots, n\}$) are in respective states q_a, q_b, q_c (where a,b,c are in $\{1, \dots, N\}$) then at time t+1, the machine K is in state q where q is the element of the b-th matrix on line a and column b.

REMARK. - Since all machines have to enter the fire state at the very same time, we do not consider transition rules where one of a,b,c is 0.

- The elements of these matrices are among $q_o, q_2, q_3, \dots, q_N$.
- Thus, all machines 1,2 ,..., n-1,n are considered as identical : they have the same set of internal states $\{q_o, q_1, q_2, q_3, \dots, q_N\}$ and the same transition rules.

The way machines 1 and n, the general and the right-end soldier, can be distinguished is as follows : machine 1 (resp. machine n) is the only one to have a left (resp. right) neighbour in state q_1 and so to make use of the related transition rules indicated in the first line (resp. column) of each matrix.

1.6. The idea of the earlier solutions (MINSKY and MacCARTHY) is the following :
- After the external intervention the general generates two waves which propagate through the line at different speeds (cf. figure 1.6).
- The fast wave is reflected by the right-end soldier.
- This reflection meets the slow wave at the middle of the line.
- By this way the initial line is broken into two new lines having equal length, which evolve independently (depending to the parity of n, these two lines are disjoint or have a common element).

- The right line evolves in a way "homothetical" to that of the initial line, so that its general is its leftest machine. The left line evolves in a way "symmetric" and "homothetical" to that of the initial line, so that its general is its rightest machine.
- This dichotomy is iterated up to the obtention of lines with length two. Then the fire state appears.

1.7. The common features of WAKSMAN and BALZER's solutions are the following :
- After the external intervention, the line generates a family of waves, all of which seem to come from the general.
- The two fastest ones ($G_1 G_2 G_3$ and $G_1 G_3$ on figure 1.7 a)) act as in MINSKY and MacCARTHY's solutions and break the initial line at G_3 creating two new lines having equal length.
- The right line (consisting of machines numbered from $\frac{n+1}{2}$ to n), which is created at time $\frac{3n}{2}-1$, has in fact begun its evolution at time n, its general being the machine n. It evolves in a way such that the trapezoïd of sites $F_3 G_3 G_2 F_2$ is symmetric and homothetical to the trapezoïd of sites $F_2 G_2 G_1 F_1$ associated to the initial line.
- The remaining portion of the initial line (machines 1 to $\frac{n+1}{2}-1$) is also iteratively broken by the meeting of $G_2 F_1$ with slower and slower waves all seemingly starting at site G_1.

This remaining portion evolves in such a way that the triangle of sites $F_1 G_3 F_3$ is "homothetical" to the triangle of sites $F_1 G_2 F_2$. The two triangles of sites $F_1 G_3 F_3$ is $F_2 G_3 F_3$ are symmetric, so that the two machines 1 and n (the first and second generals) are synchronized.

In case n is odd, figure 1.7 a) shows the first step of this iterative process.

In case n is even, figure 1.7 b) shows the needed (easy) modifications.

Figure 1.7 c), shows the iterative process in the idealized (though impossible) case where all breaks are as in figure 1.7 a).

1.8. The previous solutions seem symmetric since the initial line is broken into equal parts. However the general is alternatively at the right end or at the left end of the successive lines. This fact induces an irregular character and introduces superfluous internal states. We shall give a minimal time solution in which the general is always the left-end soldier.

§ 2 - GEOMETRY AND DELAYS.

2.1. As indicated above, in the solution presented here, all created lines will have as general their leftmost macine.

This can be done in a very convenient way by breaking lines at $^2/_3$ of their length instead of $^1/_2$.

We shall first describe our solution with continuous geometrical--diagrams : such a description applies to an idealized case where all intersections of lines have integral coordinates (in fact this ideal case does not exist !).

In this situation we have the following property shown by figure 2.1 a). The initial line is broken at site G_2 (so that $G_2\ F_1 = 2\ R_1\ G_2$). The new line consists of machines $\frac{2n}{3}$,..., n. The wave $G_2\ F_1$ (which is the progression from site G_2 of the reflection of the initial signal $G_1\ R_1$) has the same length as the wave $G_2\ R_2\ F_2$ (which is the progression of the wave created in the new line on site G_2, reflected on site R_2 and going back on F_2 to the general of the new line).

In particular these waves $G_2\ F_1$ and $G_2\ R_2\ F_2$ reach their respective generals at the very same time, thus synchronizing the two generals, i.e. machines 1 and $\frac{2n}{3}$. This is the basic step of an iterative process which leads to the synchronization of the whole line.

REMARK. This situation occurs only if the breaking point is on machine $\frac{2n}{3}$.

As in WAKSMAN and BALZER's solutions, the created line and the remaining portion of the initial line (which consists of machines $1 \ldots \frac{2n}{3} - 1$) evolve independently from time $\frac{4n}{3} - 1$ to time $2n-1$:

- The evolution of the new line from time $\frac{4n}{3} - 1$ up to time $2n-1$

is homothetical to that of the initial line from time 1 up to time 2n-1.

- The remaining portion of the initial line will be also broken at its $^2/_3$, i.e. at machine $(\frac{2}{3})^2 n$, creating a second new line consisting of machines numbered from $(\frac{2}{3})^2 n$, to $\frac{2}{3}$ n-1, and a second remaining portion of the initial line.

- This new left portion is also broken and so on.

By this way an iterative process is set up.

The creation of the new line with general $\frac{2n}{3}$ at site G_2 is the result of the meeting of the reflection $R_1 F_1$ of the initial wave with the slow wave $G_1 G_2$.

The remaining portion of the initial line (machines 1 to $\frac{2n}{3} - 1$) is also iteratively broken by the meeting of $R_1 F_1$ with slower and slower waves all (seemingly) starting at site G_1.

Figure 2.1 b), shows this iterative process (with all waves) in the idealized case.

2.2. In figures 2.1 a) and 2.1 b), we have supposed that all sites have integral coordinates. In the discrete situation where n is an integer, we have to modify the basic step (figure 2.1 a)) according to the "ternarity" (the remainder modulo 3) of n.

In the sequel we no longer consider geometrical diagrams but state-diagrams.

The initial wave fills sites (ℓ,ℓ) and its reflection fills sites $(\ell, 2n-\ell)$ (for $\ell \in \{1, \dots, n\}$). Suppose that the new general is created at site $G_2 = (K, 2n-K)$. The reflection of the initial wave reaches the old general (machine 1) at time $(2n-K) + (K-1) = 2n-1$.

If the new general (machine K) becomes active after a delay of j units of time then :

- the new initial wave starts at site $(K, 2n-K+j)$,
- it reaches the right-end soldier (machine n) at site $(n, 2n-K+j+n-K)$,
- and goes back to the new general at time $2n-K+j+(n-K)+(n-K) = 4n-2K+j$.

In this case both the new and old generals (machines K and 1) will be synchronized if $2n-1 = 4n-3K+j$, i.e. $3K = 2n+j+1$.

This equation in K and j is solvable with the constraint $j \in \{0,1,2\}$.

Suppose that the initial line has length $n = 3p+i$ with $i \in \{1,2,3\}$ and $p > 1$; then :

if $i = 1$, we get $K = 2p+1$ and $j = 0$

if $i = 2$, we get $K = 2p+2$ and $j = 1$

if $i = 3$, we get $K = 2p+3$ and $j = 2$.

The value of j will be called the "delay" for the activation of the new general. In all three cases, the new line consists of p+1 machines numbered from $K = 2p+i$ to $n = 3p+i$ and the delay is $j = i-1$.

Figure 2.2. shows these three different cases.

<u>REMARK</u>. The slow wave progresses at speed $^1/_2$ (so that it is graphically represented by a "line of sites" of slope 2). Machine K receives this slow wave at time $2K-1$ and the reflection of the initial wave at time $n+n-K = 2n-K$. The value of K determinated above is also that for which :

- the reflection of the initial wave attains machine K before (or at the same time) that the slow wave.

- the waiting delay between these two waves is minimum.

§ 3 - THE SCHEME.

We now study the generation of the family of slow waves. In fact it is more convenient to introduce the waves on which successive right-end soldiers are created (rather than those corresponding to new generals).

3.1. Define the set of sites $\tilde{S}_1$ as the set of (K,t) such that for some value of n $(n \geq 4)$ the first break of a line of n machines occurs on site $(K+1\,,\,t)$ so that machine K is the right-end soldier at site (K,t) (and the reflection of the initial wave is on machine K at time $t+1$).

As seen in 2.2., if $n = 3p+i$, then machine $2p+i$ becomes general at time $2n - (2p+i) = 4p+i$; so that :

$\tilde{S}_1 = \{(2p+i-1\,,\,4p+i)\ ;\ p \geq 1\ ;\ i \in \{1,2,3\}\}$.

Observe that two sites (K,t) and $(K'\,,\,t')$ are "related" if $K = K'$ and $|t-t'| = 1$ or if $|K-K'| = 1$ and $t = t'$. It is convenient to connect the set $\tilde{S}_1$ and introduce the set S_1 :

$S_1 = \ (2p+i-1\,,\,4p+i-1)\ ;\ p \geq 1,\ i \in \{1,2,3\}\} \cup \tilde{S}_1$

Similarly we define for every positive i, the set of sites $\tilde{S}_i$ as the set of sites (K,t) such that for some value of n, the i^{th} break of the initial line of n machines occurs on site $(K+1\,,\,t)$ (all these breaks are relative to the leftest remaining portion of the initial line) and $K \geq 2$ also S_i is defined by :

$$S_i = \tilde{S}_i \cup \{(K\,,\,t-1)\ ;\ (K,t) \in \tilde{S}_i\}.$$

Observe that if machine K transmits the initial wave at time t then it receives its reflection at time $t+2(n-K)$ which has the same parity than K. So that one unit of time out of two can be that of the arrival of its reflection.

This phenomenon will be called the "internal clock".

It allows us to use half of the time to get the synchronization either with particular constraints imposed (cf. [3]) or with fewer states.

3.2. A "connected wave" is by definition a non empty set W of sites such that if $(K,t) \in W$ then either $(K+1, t) \in W$ or $(K, t+1) \in W$ and $(K-1, t)$ and $(K+1, t)$ are not both in W.

We define the starting machine $SM(W)$ and the starting time $ST(W)$ of W as follows :

$SM(W)$ is the smallest K such that (K,t) is in W for some t and $ST(W)$ the smallest t such that $(SM(W), t)$ is in W.

To each site (K,t) with $K \geq 4$, we associate the site $NR(K,t)$ (NR for new right-end soldier) so that $NR(K,t)$ is the new right-end soldier after the first break in the evolution of the line of machines 1 to K, the external intervention occuring at time $t-K+2$ (so that the reflection occurs at time $t+1$) : (K,t) being the right-end soldier, the reflection occurs at time $t+1$).

The domain of NR is $D = \{(K,t) ; K \geq 4\}$ and, as seen in 2.2, if (K,t) in D is of the form $(K,t) = (3p+i, t)$ where $i \in \{1,2,3\}$ and $p \geq 1$, then :

[*] $NR(3p+i, t) = (2p+i-1, p+t+1)$.

We observe that if W is a connected wave, then $W \cap D$ and its image $NR(W \cap D)$ are empty or are also connected waves.

In fact [*] shows [**] :

[**]

$$\left|\begin{array}{l} \text{If } NR(K,t) = (K', t') \text{ then} \\ - NR(K, t+1) = (K', t'+1) \\ - \left|\begin{array}{ll} NR(K+1, t) = (K', t'+1) & \text{if } K \equiv 0 \mod (3) \\ NR(K+1, t) = (K'+1, t') & \text{if } K \not\equiv 0 \mod (3). \end{array}\right. \end{array}\right.$$

If the starting machine of $W \cap D$ is of the form $SM(W \cap D) = 3p+i$ with $p \geq 1$ and $i \in \{1,2,3\}$ then $NR(W \cap D)$ is non empty and its starting machine is $2p+i-1$.

We now describe a process to obtain the connected wave $NR(W \cap D)$ from $W \cap D$ (cf. figure 3.2).

This process is twofold. First we initialize the wave $NR(W \cap D)$, then we construct the whole wave from its starting site.

The initialization of $NR(W \cap D)$ cannot be done in a general setting : there is no way to get quickly the starting machine of $NR(W \cap D)$ from that of $W \cap D$.

We suppose that $SM(W \cap D) = 3p_o + i_o$ where $p_o \geq 1$ and $i_o \in \{1,2,3\}$, whence (using [*]) $SM(NR(W \cap D)) = 2p_o + i_o - 1$. This starting machine being fixed, we do construct the starting time of $NR(W \cap D)$ from that of $W \cap D$ by the following elementary process :

- from site $(SM(W \cap D), ST(W \cap D))$ a signal σ is emitted which propagates along the diagonal $\Delta(SM(W \cap D), ST(W \cap D)) = \{(SM(W \cap D)-\ell, ST(W \cap D)+\ell) \; ; \; 0 \leq \ell \leq SM(W \cap D)-1\}$.
- The starting time of $NR(W \cap D)$ is that one when this signal σ reaches machine $SM(NR(W \cap D))$.

We now describe the construction of the wave $NR(W \cap D)$ from its starting site.

First we observe using formula [*] that both sites (K,t) and

$NR(K,t)$ (for $K \geq 4$) belong to the same diagonal :

$\Delta(K,t) = \{(K-\ell , t+\ell) ; 0 \leq \ell \leq K-1\}$.

The content of [**] can be rephrased as follows :

- When one moves vertically on $W \cap D$ (i.e. from (K,t) to $(K , t+1)$), then the corresponding move by NR on $NR(W \cap D)$ is also vertical.
- When one moves horizontally on $W \cap D$ (i.e. from $(3p+i , t)$ to $(3p+i+1 , t)$), then the corresponding move by NR on $NR(W \cap D)$ is in two cases out of three also horizontal (cases $i = 1$ or $i = 2$), and in one case out of three vertical (case $i = 3$).

Similarly to BALZER [1], we introduce three kinds of distinguished states s_1 , s_2 , s_3 propagating along the diagonals $\Delta(K,t)$ for those (K,t) in $W \cap D$ such that $(K+1 , t)$ is also in $W \cap D$.

Let $r \in \{1,2,3\}$ be such that $SM(W \cap D) \equiv r \mod (3)$

We let the first such signal (that one which starts from a site $(SM(W \cap D) , t))$ be s_r and we let these signals appear successively in the order .... $s_1\ s_2\ s_3\ s_1$ .....

[+] Thus the only possible signal starting from machine K (if there is some) is s_j where $j \equiv r + (K - SM(W \cap D)) \mod (3)$

$$j \equiv K \mod (3).$$

Formula [**] can now be restated as follows :

[***] if $(K' , t') = NR(K,t)$ where (K,t) is in $W \cap D$ then
- if (K' , t') does not belong to any of these signals s_j (i.e. $(K , t+1) \in W \cap D$ and $(K+1 , t) \notin W \cap D$), then $(K' , t'+1) \in NR(W \cap D)$.

[***] - if (K', t') does belongs to such a signal s_j where $j \equiv 1$ or $j \equiv 2 \mod (3)$ (case $K \equiv 1$ or $K \equiv 2$ of [+]) then $(K'+1, t')$ is in $NR(W \cap D)$.

- if (K', t') does belong to such a signal s_j where $j \equiv 0 \mod (3)$ (case $K \equiv 3$ of [+]) then $(K', t'+1)$ is in $NR(W \cap D)$.

Figure 3.2. shows such a construction of $NR(W \cap D)$ from $W \cap D$.

3.3. As seen in 2.2. if a line has $3p+i$ machines, $i \in \{1,2,3\}$, then the new general which will be created will become active after of delay of $i-1$ units of time.

We have noticed in 3.2. that the only possible signal starting from machine $3p+i$ is s_i. This shows that s_i conveys the information : "If you become general, then be active after a delay of $i-1$ units of time".

3.4. Let S_o be $\{(K, K-1), (K, K-2)$ for $K > 4\}$.

The very definitions of S_1 and NR show that $S_1 = NR(S_o)$ (observe that $S_o \subset D$, the domain of NR). Also for all $i > 1$, we have $S_{i+1} = NR(S_i \cap D)$.

Figure 3.4. illustrates these process which give S_1 from S_o and iteratively $S_2, S_3, \ldots$ from $S_1, S_2, \ldots$.

§ 4 - STATES AND TRANSITION RULES FOR AN 8-STATES SOLUTION.

We suggest the reader to constantly refer to figure 4.10 (which gives an instance of the synchronization process) as an illustration of the material developped in this paragraph.

Besides the matrix notation for transition rules described in 1.5, it is convenient to denote $(U,V,W) \longrightarrow T$ the transition rule which asserts that if machines $K-1$, K, $K+1$ are at time t in states U,V,W, then at time $t+1$, machine K is in state T.

4.1. The very statement of the synchronization problem introduces
- two particular states : the quiescent state (denoted L), and the fire (denoted F).
- Obviously related transition rules $(L,L,L) \longrightarrow L$ and $(L,L,X) \longrightarrow L$ and $(X,L,L) \longrightarrow L$.

Clearly for $t < K$, site (K,t) is in state L.

We now introduce states and transition rules convenient to set up the process described in § 3 : first we introduce such states in a loose way, then we severely reduce the set of states.

It is essential to observe that the different notions associated to signals and waves, need not be characterized by particular states, but rather by particular situations, that is triples of states (corresponding to machines $K-1$, K, $K+1$*).*

4.2. We present a first tentative set of states and transition rules translating the inductive step of the construction of the S_i's via the signals s_j's (cf. paragraph 3). We introduce four particular states A,B,C,S :

- all sites in $\bigcup_{i > 1} S_i$ will be in state S.
- all sites "between" S_i and S_{i+1} will be in state A,B ou C.

States A,B,C will be distributed so that :

- signal s_1 (resp. s_2 , s_3) is on site (K,t) iff (K,t) is in state A (resp. B,C) and (K+1 , t) is in state B (resp. C,A).
- If s_j and $s_{j'}$ are successive signals going from S_i to S_{i+1} then all sites "between" s_j and $s_{j'}$ (and S_i and S_{i+1}) are in the same state : that one common to all sites in $s_{j'}$.

Following the different elements of the construction in § 3 we now enumerate, comment and represent on figure 4.2 (anticipating alinea 4.3, state S is indicated by L on figure 4.2) adequate transition rules.

1). S_i *emits* s_j.

We get three rules : (A,S,S) $\longrightarrow$ B, (B,S,S) $\longrightarrow$ C, (C,S,S) $\longrightarrow$ A.

The first one can be interpreted as follows :

at time t, sites (K,t) and (K+1 , t) are in S_i. Site (K-1 , t) is in state A ; this means that the last signal emitted by S_i was s_3, coded by (· C A). Site (K,t) has to emit signal s_1 : in order that site (K-1 , t+1) be in signal s_1, site (K , t+1) must be in state B (and site (K-1 , t+1) must be in state A since s_1 is coded by (· A B)).

2). S_i *does not emit any* s_j.

We get nine rules :

(A,A,S) $\longrightarrow$ A	(B,B,S) $\longrightarrow$ B	(C,C,S) $\longrightarrow$ C
(C,A,S) $\longrightarrow$ A	(A,B,S) $\longrightarrow$ B	(B,C,S) $\longrightarrow$ C
(S,A,S) $\longrightarrow$ A	(S,B,S) $\longrightarrow$ B	(S,C,S) $\longrightarrow$ C

Rules (· , A , S) $\longrightarrow$ A starts the filling of a new diagonal between two successive signals s_3 and s_1 (with the three possible left neighbors). Other rules have similar signification.

3). *Filling diagonals.*

We get two sets of rules :

(i) $(A,A,B) \longrightarrow B$ $\quad (B,B,C) \longrightarrow C$ $\quad (C,C,A) \longrightarrow A$

(ii) $\left[\begin{array}{lll} (A,B,B) \longrightarrow B & (B,C,C) \longrightarrow C & (C,A,A) \longrightarrow A \\ (B,B,B) \longrightarrow B & (C,C,C) \longrightarrow C & (A,A,A) \longrightarrow A \end{array}\right.$

Rule $(A,A,B) \longrightarrow B$ allows the propagation of signal s_1 : if (K,t) is on signal s_1 then $(K-1\ ,\ t+1)$ is also on signal s_1 so that $(K\ ,\ t+1)$ has to be in state B (and $(K-1\ ,\ t+1)$ in state A).

Rules $(A,B,B) \longrightarrow B$ and $(B,B,B) \longrightarrow B$ do insure the filling of diagonals between successive signals s_j's.

4). S_{i+1} *receives* s_j.

According to [***] in 3.5.

We get three rules : $(S,A,B) \longrightarrow S$ $\quad (S,B,C) \longrightarrow S$ $\quad (S,C,A) \longrightarrow A$.

Rule $(S,A,B) \longrightarrow S$ can be interpreted as follows : site (K,t) is on signal s_1 and has its left neighbor on S_{i+1}.

This means that S_{i+1} receives a signal s_1 on site $(K-1\ ,\ t+1)$ and has "to move right" : site $(K\ ,\ t+1)$ has to be in S_{i+1}, hence in state S.

Rule $(S,B,C) \longrightarrow S$ is similar with signal s_2. Rule $(S,C,A) \longrightarrow A$, interpreted in the same way, means that S_{i+1} receives a signal s_3 on site $(K-1\ ,\ t+1)$ hence has "to move vertically", so that site $(K\ ,\ t+1)$ is also outside S_{i+1}. Since the next signal will be s_1, site $(K\ ,\ t+1)$ has to be in state A.

5). S_{i+1} *does not receive any* s_j.

We get three rules :

$(S,A,A) \longrightarrow A$ $(S,B,B) \longrightarrow B$ $(S,C,C) \longrightarrow C$.

These rules complete the filling of diagonals between successive signals.

6). *Vertical moving of* S_i *and* S_{i+1}.

We know that if (K,t) is on S_i and $(K+1\ ,\ t)$ is not on S_i then $(K\ ,\ t+1)$ has to be on S_i. This leads to eleven rules :

$(S,S,V) \longrightarrow S$ $(U,S,W) \longrightarrow S$

where U and W are A,B or C and V is B or C.

Observe that any environment SSV corresponds to an horizontal move of some S_i hence to the reception of signals s_1 or s_2 (and not s_3). The middle S in SSV must come from A or B (and not C) with right neighbors BB or CC (recall signal s_1 and s_2 are coded par AB and BC). Thus V is obtained as the result of the rules for environments ABB and BCC : V is B or C (and not A).

7). *Construction of* S_1 *from* S_o.

We now attribute states A,B or C to sites (K,K), for $K > 3$ If $K \equiv 1 \bmod (3)$ then site $(K\ ,\ K-1)$, which is on S_o, emits a signal s_1.

Since s_1 is coded by AB and $(K-1\ ,\ K)$ is on s_1, sites $(K-1\ ,\ K)$ and (K,K) have to be in states A,B. So we put state B on site (K,K).

Similarly if $K \equiv 2 \bmod (3)$ (resp. $K \equiv 3 \pmod{3}$), then site (K,K) will have state C (resp. A).

This gives nine transition rules :

$(A,L,L) \longrightarrow B$ $(B,L,L) \longrightarrow C$ $(C,L,L) \longrightarrow A$

$(A,B,L) \longrightarrow B$ $(B,C,L) \longrightarrow C$ $(C,A,L) \longrightarrow A$

$(S,A,L) \longrightarrow A$ $\qquad$ $(S,B,L) \longrightarrow B$ $\qquad$ $(S,C,L) \longrightarrow C$

(the last three are in fact useless but not obviously so).

Observe that if (K,K) has state A (resp. B,C) then it does know : "If the reflection occurs at next unit of time, then the line has $K+1$ machines and $K+1 \equiv 1 \bmod (3)$ (resp. 2,3) so that the new general will become active after delay 0 (resp. 1,2)".

4.3. We do observe the following key fact : *replacing* S *by* L *in all previous rules* 1 *to* 7, *one gets a compatible set of rules*. This permits to eliminate state S in profit of L.

4.4.

8). *Initialization of the construction of the* S_i's $(i > 1)$.

According to 3.2, we observe that the starting machine of S_{i+1} $(i > 0)$ is machine 2, and the starting site of S_{i+1} is obtained via the signal σ emitted by machine 4 at the smallest time t for which (K,t) is in S_i (so that we have also $(3,t)$ in S_i). Observe that $t = ST(S_i \cap D)$ and $ST(S_{i+1}) = ST(S_i \cap D) + 2$ (it takes two units of time for signal σ to go from machine 4 to machine 2 along a diagonal).

We introduce two states G and H :

- G marks all sites $(1,t)$ for $t > 1$,
- H marks all sites $(2,t)$ except those already marked by L (i.e. those in $\bigcup_{i>0} S_i$).

In particular, since sites $(3 , ST(S_i \cap D))$ and $(4 , ST(S_i \cap D))$ both have state L, site $(2 , ST(S_i \cap D))$ has to be in state H. Also, since σ reaches machine 2 at time $ST(S_i \cap D) + 2$, site $(2 , ST(S_i \cap D) + 1)$

is not in S_{i+1} hence has to be in state H.

Apart from σ, the first signal emitted by S_i $(i > 0)$ starts from machine 4 at a time θ such that $(4,\theta)$ and $(5,\theta)$ are in S_i. This signal is s_1 since $4 \equiv 1 \pmod{3}$.

Since s_1 is denoted by states A,B on adjacent sites, sites $(3, \theta+1)$ and $(4, \theta+1)$ are in states A and B.

In order not to introduce new transition rules we attribute state A to all sites $(3,t')$ for $1 + ST(S_i \cap D) < t' < \theta$.

Now all sites below the first signal s_1 emitted by S_i have been atributed a state (cf. figure 4.4). Note that signal σ is coded by the environment HAL.

All this process necessitates the introduction of the following rules :

$(X,G,L) \longrightarrow G$	$(X,G,H) \longrightarrow G$	Persistency of G.
$(G,L,L) \longrightarrow H$	$(G,H,L) \longrightarrow H$	Introduction and (partial persistency) of H
$(G,H,A) \longrightarrow L$		Machine 3 is always in state A or L ; when A comes, it carries signal σ, so that machine 2 gets into S_{i+1} and has state L.
$(G,L,A) \longrightarrow L$		(Partial) persistency of L on machine 2.
$(H,L,L) \longrightarrow A$	$(H,A,L) \longrightarrow A$	Machine 3 emits signal σ.
$(H,L,B) \longrightarrow L$		(Partial) persistency of L on machine 3 (machine 4 is always in state A , B or L).

We remark that transition rules 1 to 8 (where L replaces S) attribute states to sites below the reflection of the initial wave.

Remark.

Recall the internal clock phenomenon (cf. end of 3.1) : one unit of time out of two can be, for a particular machine, that of the arrival of the reflection of the initial wave. With the above transition rules, this phenomenon is translated by the following fact : up to the arrival of the reflection of the initial wave, any machine K stays in a particular state an even number of time units.

This fact, joined to the diagonal propagation of the s_j's, implies the staircase aspect of the distribution of states on sites below the reflection (cf. figure 4.10).

4.5. Now we consider the reflection of the initial wave and its propagation up to its meeting with a signal S_i under the hypothesis that the created line is not too short (i.e. of length at least 3).

Machine K knows (at time t) that it is upon (at time t+1) to deal with the reflection of the initial wave (of the very initial line or of some of the created lines) if and only if :

- (K,t) is in state L
- (K+1 , t) is in state X or G
- (K-1 , t) is in state A,B or C.

Moreover if (K-1 , t) is in state A (resp. B,C), it does know that the next general to be created will become active-after its creation-with delay 0 (resp. 1,2).

All these informations will be transmitted by machine K to machine K-1 and will progress along diagonals up to the machine which has to become general.

In order not to introduce new states we shall code these informations as follows :

- the order of appearance of states A,B,C (which is ... ABC ..., horizontally and vertically) is broken (hence inversed since they are only three). This introduces the following transition rules :

$(A,L,X) \longrightarrow C$ $(B,L,X) \longrightarrow A$ $(C,L,X) \longrightarrow B$

$(A,L,G) \longrightarrow C$ $(B,L,G) \longrightarrow A$ $(C,L,G) \longrightarrow B$

- The previous inversion is transmitted to machines along a diagonal using the transition rules :

$(A,A,C) \longrightarrow C$ $(B,B,A) \longrightarrow A$ $(C,C,B) \longrightarrow B$

Due to the staircase distribution of states (cf. remark at the end of 4.4), we have not to consider environments different from AAC, BBA and CCB.

In this way we have set up the reflection of the initial wave. Now we deal with the propagation of the information about the delay : it is convenient to introduce an eighth state, denoted R, to mark the end of the delay-transmission process. The three possible delays and the staircase distribution of states lead to the three following sets of rules (cf. figure 4.5). Recall that delay 0 (resp. 1,2) is conveyed by s_1 (resp. s_2, s_3) which is coded by .AB (resp. .BC, .CA) and gives at reflection .AC (resp. .BA, .CB).

- Delay 0

$\left.\begin{array}{l}(A,C,X) \longrightarrow R \\ (A,C,G) \longrightarrow R\end{array}\right]$ the right-end machine enters state R after one unit of time.

$(A,C,R) \longrightarrow R$ } Propagation of the delay information.

$(C,R,X) \longrightarrow L$ $(C,R,G) \longrightarrow L$ $(C,R,L) \longrightarrow L$ $(R,L,X) \longrightarrow L$ $(R,L,G) \longrightarrow L$ $(R,L,L) \longrightarrow L$ $(L,L,G) \longrightarrow L$	After the passage of the diagonal of R'_s, any machine goes back to the quiescent state L (except in the case when the left neighbor is the general). The last rule $(L,L,G) \longrightarrow L$ gives the persistency of L after the passage of the R'_s.

- Delay 1

$(B,A,X) \longrightarrow C$ $(B,A,G) \longrightarrow C$	the right-end machine meets the conditions of delay 0 after one unit of time so that the total delay is one.
$(B,A,C) \longrightarrow C$ }	propagation of the delay-information.

Plus all the transition rules described in the delay 0 case.

- Delay 2

$(C,B,X) \longrightarrow A$ $(C,B,G) \longrightarrow A$	the right-end machine meets the conditions of delay 1 after one unit of time so that the total delay is two.
$(C,B,A) \longrightarrow A$ }	propagation of the delay information.

Plus all the transition rules described in the delay 1 case.

4.6. We now consider the meeting of the reflection of the initial wave with any signal S_i.

If at time t, machine K is in state A (resp. B,C), if its left neighbor is in state L and its right neighbor in state C (resp. A,B) instead of A or B (resp. B or C, C or A), then machine K knows :

- (K-1 , t) is on some S_i
- (K+1 , t) transmits the reflection of the initial wave.

Hence it realizes that it has to become general of a new line at time t+1. This introduces the following transition rules :

(L,A,C) $\longrightarrow$ G (L,B,A) $\longrightarrow$ G (L,C,B) $\longrightarrow$ G

This new general will stay in state G up to the firing time. We shall study in alinea 4.8 this G-stability question.

The diagonals which transmit the reflection and the delay (there are 2 or 3 or 4 such diagonals according to the value 0 or 1 or 2 of the delay) crash on the vertical line of G's sites. The new general becomes active when its right neighbor is in state R.

A simple analysis of the three types of reflection (associated to the three delays) leads to figure 4.6., and shows that only three rules are necessary :

(G,A,C) $\longrightarrow$ C (G,C,R) $\longrightarrow$ R (G,R,L) $\longrightarrow$ H

Observe that after the arrival of the R's diagonal, the new general emits the initial wave of the new right line : in fact states G and H mark the first and second machines of the new line while other machines of this new line are in the quiescent state.

4.7. We now consider the case where the created line has length two : in this case the emission of the reflected wave does interfere with its crash on the G vertical line.

We introduce four rules :

(G,A,X) $\longrightarrow$ C (G,A,G) $\longrightarrow$ C : these rules complete the previous rules
(G,A,C) $\longrightarrow$ C (B,A,G) $\longrightarrow$ C

(G,C,X) $\longrightarrow$ R (G,C,G) $\longrightarrow$ R : these rules complete the previous rules
(G,C,R) $\longrightarrow$ R (A,C,G) $\longrightarrow$ R

We do not consider environments GB. since they do not appear.

4.8. We first study the stability of G up to the creation of too short lines.

This G-stability question is twofold :

- during the crash of the reflected signal,
- after the created line is operative.

The G-stability in the reflection period leads to nine cases corresponding to

- the three possible delays conveyed by the reflected signal coming from the right up to the *activation* of the new line,
- the three possible delays conveyed by the reflected signal progressing leftwards along the remaining portion of the initial line (after the *creation* of the new line).

This leads to figure 4.6 which shows the sole environments which have to be considered, and gives the ten following rules :

$(L,G,A) \longrightarrow G$	$(L,G,R) \longrightarrow G$
$(R,G,R) \longrightarrow G$	$(A,G,R) \longrightarrow G$
$(A,G,C) \longrightarrow G$	$(B,G,R) \longrightarrow G$
$(B,G,C) \longrightarrow G$	$(C,G,R) \longrightarrow G$
$(C,G,C) \longrightarrow G$	$(L,G,C) \longrightarrow G$

After the new line is activated and up to time 2n-3,

- the right neighbor of the general is always in state L or H
- the left neighbor of the general can be in any state except state G.

This leads to the following rules :

$(A,G,L) \longrightarrow G$	$(B,G,L) \longrightarrow G$	$(C,G,L) \longrightarrow G$
$(R,G,L) \longrightarrow G$	$(L,G,L) \longrightarrow G$	$(A,G,H) \longrightarrow G$
$(B,G,H) \longrightarrow G$	$(C,G,H) \longrightarrow G$	$(R,G,H) \longrightarrow G$
$(L,G,H) \longrightarrow G$	$(H,G,L) \longrightarrow G$	$(H,G,H) \longrightarrow G$

4.9. Finally we now have to deal with short lines which cannot be broken. A priori, such lines have length 1,2 or 3.

Since we do not consider an initial line of length one and since the breaking process creates lines with length at least two (if $n = 3p+i$ where $p \geq 1$ and $i \in \{1,2,3\}$, then the new general is machine $2p+i$, so that the line has length $p+1$ and $p+1 \geq 2$), we have only to consider short lines of length two or three.

Such lines appear in three different contexts :

- The initial line has length two or three.
- A created line has length two or three.
- A remaining portion of a line has length two or three.

The firing synchronization will be obtained as follows :

- up to time $2n-3$, no two adjacent machines are both in state G,
- at time $2n-2$, all machines are in state G,
- at time $2n-1$, all machines are in state F.

The G-synchronization at time $2n-2$ leads to figures 4.9. and to the following transition rules :

$(G,L,X) \longrightarrow G$ } Rule necessary for an initial line of two machines (a_2)

$$\left.\begin{array}{l}(H,L,X) \longrightarrow H \\ (G,H,H) \longrightarrow G \\ (H,H,X) \longrightarrow G\end{array}\right]$$ Rules necessary for an initial line of three machines (a_3).

$(G,R,X) \longrightarrow G$ } Complementary rule necessary for rightmost created lines of length two (b_2).

No new rule is needed for the case of a rightmost created line of length three (b_3).

$(G,R,G) \longrightarrow G$ } complementary rule necessary for the generic created line of length two (c_2).

$(H,L,G) \longrightarrow H$
$(H,H,G) \longrightarrow G$] complementray rules necessary for the generic created line of lenth three (c_3).

$(G,L,G) \longrightarrow G$ } complementary rule necessary for the remaining portion of a line, with length two (d_2).

No new rule is needed for the case illustrated by (d_3).

Lastly we consider the stability of G within the context of short lines. Figure 4.9 shows that at time 2n-3, all machines are in state G,H,R,L. To complete the stability of G (from time 2n-3 to time 2n-2) it suffices to introduce a single new rule : $(H,G,R) \longrightarrow G$.

With these rules all short lines are G-synchronized at time 2n-2. The whole initial line, being covered by these short lines, is thus G-synchronized at time 2n-2.

It is now clear that the firing synchronization is insured by the three following rules :

$(G,G,G) \longrightarrow F$ $\qquad$ $(X,G,G) \longrightarrow F$ $\qquad$ $(G,G,X) \longrightarrow F$

4.10. Now we have all necessary transition rules which fill the seven matrices of figure 4.11.

Observe that no two rules are contradictory.

Also observe that the blanks in the matrices correspond to environments which are not met, hence for which no transition rule is necessary.

Figure 4.10 shows the synchronization of an initial line of 29 machines.

§ 5 - A SIX STATES MINIMAL TIME SOLUTION.

5.1. In this paragraph we show how to eliminate two states in the preceding solution. It is easy to eliminate state H. The matrices of the 8-states solution show that state R appears very few times. We are going first to eliminate state R and then state H.

To do this we shall introduce new rules, some of them, marked by [*], contradicting old ones. To get over the contradiction we shall also abandon some old rules and introduce some new ones. And so on up to an equilibrium.

5.2. Observe that no B-diagonal belongs to the part of the reflection which crashes on a G vertical line (cf. figure 4.10). This induces us to replace state R by state B and leads to the following transitions rules (suggested by figures of § 4) :

(B,L,L) $\longrightarrow$ L [* 1]	(B,L,X) $\longrightarrow$ L [* 1]	(B,L,G) $\longrightarrow$ L [* 1]
(C,B,X) $\longrightarrow$ L [* 2]	(C,B,G) $\longrightarrow$ L [* 2]	
(A,C,X) $\longrightarrow$ B	(A,C,G) $\longrightarrow$ B	(C,B,L) $\longrightarrow$ L
(G,B,L) $\longrightarrow$ H	(G,B,G) $\longrightarrow$ G	(G,B,X) $\longrightarrow$ G
(A,C,B) $\longrightarrow$ B	(G,C,B) $\longrightarrow$ B	(G,C,G) $\longrightarrow$ B
(G,C,X) $\longrightarrow$ B	(H,G,B) $\longrightarrow$ G	

Unfortunately

- rules marked by [* 1] contradict rules

(B,L,L) $\longrightarrow$ C (B,L,X) $\longrightarrow$ A (B,L,G) $\longrightarrow$ A

- rules marked by [* 2] contradict rules

(C,B,X) $\longrightarrow$ A (C,B,G) $\longrightarrow$ A

5.3. In order to maintain rules marked by [* 1] we suppress the environment BLL below the reflection of the initial wave. To do this we modify the attribution of state B to sites on a signal s_2 (between S_i and S_{i+1}) and to sites between s_2 and the preceeding signal s_1. Using the internal clock phenomenon we replace the portion of the state-diagram

BBL BLL	by	BCL GLL

By this way, we also suppress the environments BLX and BLG occuring on the reflection of an initial wave.

Looking at figures 4.10 and 4.2, we see that this leads to

- the suppression of :

(B,L,L) ⟶ C　　(A,L,L) ⟶ B　　(B,B,L) ⟶ B

(B,L,G) ⟶ A　　(B,L,X) ⟶ A　　(A,B,L) ⟶ B　　(L,B,L) ⟶ L

- the introduction of :

(L,G,L) ⟶ B [* 3]　　(G,L,L) ⟶ C [* 4]

(A,L,L) ⟶ G　　(B,B,L) ⟶ G　　(A,A,G) ⟶ B

(L,B,L) ⟶ G　　(B,B,G) ⟶ B　　(A,G,L) ⟶ B [* 3]

(G,L,B) ⟶ L　　(G,L,C) ⟶ L　　(L,B,G) ⟶ B

(B,G,L) ⟶ B [* 3]　　(G,L,G) ⟶ A [* 3]　　(G,L,X) ⟶ A [* 4]

Unfortunately rules marked by [* 4] and [* 3] contradict rules :

(L,G,L) ⟶ G　　(G,L,L) ⟶ H

(A,G,L) ⟶ G　　(B,G,L) ⟶ G　　(G,L,G) ⟶ G　　(G,L,X) ⟶ G

5.4. In order to maintain rules marked by [* 2] we modify the reflection of the initial wave in the case of delay 2. To do this we replace the portions of diagrams

B A C Z		B A C Z	
A Z	by	C B A Z	where Z is state X
C C B Z		C C G Z	or G.
C L Z		C L Z	

By this way we suppress the environments CBX and CBG.

Looking at these portions of diagrams and figure 4.5 (for the reflected wave), we see that this leads to.

- the suppression of :

$(C,L,X) \longrightarrow B$ $\quad$ $(C,B,X) \longrightarrow A$

$(C,L,G) \longrightarrow B$ $\quad$ $(C,B,G) \longrightarrow A$

- the introduction of :

$(C,L,X) \longrightarrow G$ $\quad$ $(C,G,X) \longrightarrow A$ $\quad$ $(C,C,G) \longrightarrow B$

$(C,L,G) \longrightarrow G$ $\quad$ $(C,G,G) \longrightarrow A$ $\quad$ $(G,H,G) \longrightarrow G$

$(G,G,C) \longrightarrow G$ $\quad$ $(G,G,B) \longrightarrow G$

In the case the line to be created is going to be short the preceding diagrams have to be modified.

To suppress the environments CBX and CBG we have to replace the portions of diagrams

L G A Z		L G A Z	
L C B Z	by	L C G Z	where Z is state X or G.
C L Z		C L Z	

This introduces the rule : $(L,C,G) \longrightarrow G$.

<u>5.5.</u> Observe that the environments of the rules marked by [* 3] occur only when machine 2 is in state L. In order to maintain these rules marked by [* 3], we always (in fact up to time 2n-2) place machine 2 in

state H. To do this, we replace the two portions of diagrams :

	G H L B
G L A	G H L B
G L A	G L L B
G H A	G L A B

by :

	G H L B
G H A	G H L B
G H A	G H L B
G H A	G H A B

This leads to :

- the suppression of rules

$(H,G,L) \longrightarrow G$ $(G,H,A) \longrightarrow L$

$(L,G,L) \longrightarrow G$ $(A,G,L) \longrightarrow G$ $(B,G,L) \longrightarrow G$

$(C,G,L) \longrightarrow G$

- the introduction of rules :

$(G,H,A) \longrightarrow H$ $(H,A,B) \longrightarrow L$

This new state-value introduced for machine 2, modifies also the G-synchronization in the case (d_2) (see figure 4.9). This leads to the suppression of the rule $(G,L,G) \longrightarrow G$ and the introduction of the rule $(G,H,G) \longrightarrow G$.

Now there remains only two contradictory rules : $(G,L,X) \longrightarrow A$ and $(G,L,L) \longrightarrow C$. We observe that the rule $(G,L,X) \longrightarrow G$ is used only when the initial line is of length two : when the rightmost new line is of length two, the rule $(G,B,X) \longrightarrow G$ introduced in 5.2 is used.

We observe that rule $(G,L,L) \longrightarrow H$ is used only to set up machine two. Thus the two remaining contradictions concern only site $(2,2)$.

5.6. Now we eliminate state H ; this suppresses rule $(G,L,L) \longrightarrow H$. Rule $(G,B,G) \longrightarrow G$ (resp. $(G,L,L) \longrightarrow A$) suggests us to replace state H by state B (resp. C). In fact using the internal clock phenomenon we shall replace state H alternatively by states C and B. Observe (look at site (3.21) in figure 5.6) that we must introduce the rule $(C,L,G) \longrightarrow L$ which contradicts the rule $(C,L,G) \longrightarrow G$ introduced in 5.4. To avoid this contradiction, we shall code states of machine 3 in a particular way : machine 3 will be alternatively in state A and in state G. To do this we replace diagrams of figure 5.6 a) by diagrams occuring in figure 5.6 b).

This leads to :

- the suppression of all the rules in which a state H occurs and of the rules :

$(L,B,L) \longrightarrow G$	$(L,G,L) \longrightarrow B$	$(L,A,L) \longrightarrow A$

- the introduction of rules :

$(G,L,L) \longrightarrow C$	$(G,C,L) \longrightarrow B$	$(G,B,A) \longrightarrow C$
$(G,C,G) \longrightarrow B$	$(B,A,L) \longrightarrow G$	$(G,B,L) \longrightarrow C$
$(B,A,B) \longrightarrow G$	$(C,G,L) \longrightarrow A$	$(G,G,L) \longrightarrow B$
$(A,B,C) \longrightarrow L$		
$(A,B,L) \longrightarrow G$	$(X,G,B) \longrightarrow G$	$(X,G,C) \longrightarrow G$

These new site values for 2 and 3 modify the G synchronization when :

- the initial line is of length three or four (cases a_3 , a_4 of figure 4.9)
- the remaining portion of the initial line is of length three of four

(cases c_3, c_4 of figure 4.9).

This leads to the introduction of the following rules :

$(A,B,A) \longrightarrow B$ $\quad$ $(G,B,C) \longrightarrow B$ $\quad$ $(B,C,G) \longrightarrow G$ $\quad$ $(B,C,X) \longrightarrow G$

$(B,G,G) \longrightarrow G$ $\quad$ $(B,G,X) \longrightarrow G$.

By this way we obtain a six-states automaton which synchronizes all lines consisting of more than two machines.

The special case of an initial line of two machines is solved by the two following diagrams :

X F F X	X G B L	the first diagram insures the synchronization of an
X A A X	X A C L	initial line of length 2 ; the second one deals with
X G L X	X G L L	the consequences brought by the first.

This leads to the introduction of the following rules :

$(A,C,L) \longrightarrow B$ $\quad$ $(X,A,C) \longrightarrow G$

$(X,G,L) \longrightarrow A$ $\quad$ $(X,A,A) \longrightarrow F$ $\quad$ $(A,A,X) \longrightarrow F$

and the suppression of the rule $(X,G,L) \longrightarrow A$.

5.7. Observe that these modified rules have the following consequence : a site (K,t) in S_1 receives state L if and only if $K > 4$ (contrary to the situation described in § 4 ; this is due to the particular coding device of machines 2 and 3).

5.8. Now we have all necessary transition rules which fill the five matrices of figure 5.7.

Observe that no two rules are contradictory.

Also observe that the blanks in the matrices correspond to environments which are not met, hence for which no transition rule is necessary. Figure 5.8 shows the synchronization of an initial line of 29 machines.

§ 6 - PROOF OF CORRECTNESS.

6.1. We now prove that the automaton defined in § 5 (whose transition rules are indicated on figure 5.7) is indeed a minimal time solution of the firing squad synchronization problem.

The proof will proceed by induction on the length n of the initial line : the synchronization of a line of length $n = 3p+i$ (with $i \in \{1,2,3\}$) is reduced to the synchronization of lines of length $2p+i-1$ and $p+1$ (corresponding to the break at the 2/3 described in paragraph 2).

In order to get such a reduction, we first prove some facts relative to :

- the behaviour of the line below the reflection (the scheme),
- the reflection itself.

The following fact will be useful :

Fact 1. Every site (K,t) with $K > t$ has state L.

It is a trivial consequence of the rules $(L,L,L) \longrightarrow L$ and $(L,L,X) \longrightarrow L$.

Finally we shall often use the fact that the distribution of states on a set X *fully determines that on a set* $\tilde{X}$ *bigger then* X *as illustrated by figure 6.1.*

6.2. It is clear (cf. the study of signals s_j in § 3) that diagonals are the essential objects to consider. Due to the internal clock phenomenon, the most pertinent object is that of double diagonals DD_m :

Notation 1.

If $7 \leq m < n$, we let DD_m be the set

$\{(m-i\ , m+i),\ (m-i\ , m+i+1)\ ;\ i \in \{-1\ , 0\ , \ldots,\ m-1\}\}$

(i is allowed to be -1 in order that DD_m captures two sites corresponding to machine m+1.

As can be seen in figure 5.8, a double diagonal DD_m consists of several "unicolor" pieces. This leads us to introduce truncated double diagonals :

Notation 2.

If $7 \leq m < n$ and $1 \leq K < j \leq m+1$ we let $DD_{m,j,K}$ be the set $DD_m \cap \{(\ell,t)\ ;\ K \leq \ell \leq j\} = \{(m-i,m+i),(m-i,m+i+1)\ ;\ i \in \{m-j\ ,\dots,\ m-K\}\}$.

REMARK.

Observe that indices m,j,K are in decreasing order : this is indeed the order of appearance of machines when one progresses upward on DD_m.

We distinguish some families of basic truncated double diagonals.

DEFINITION 1.

Suppose $j \geq K+3$ and $K \geq 2$.

1). We say that $DD_{m,j,K}$ is A-basic (resp. C-basic) if :

(α). its rightmost and leftmost sites $(j\ ,\ 2m-j)$, $(j\ ,\ 2m-j+1)$ and $(K\ ,\ 2m-K)(K\ ,\ 2m-K+1)$ have state L.

(β). its other sites $(i\ ,\ 2m-i)$, $(i\ ,\ 2m-i+1)$ have state A (resp. C) (with i in $\{m-j+1\ ,\dots,\ m-K-1\}$).

2). We say that $DD_{m,j,K}$ is B-basic is condition (α) above is satisfied and

(β') its left to rightmost sites $(j-1\ ,\ 2m-j+1)$, $(j-1\ ,\ 2m-j)$ have states B and G,

its other sites $(i\ ,\ 2m-i)$, $(i\ ,\ 2m-i+1)$ (with i in $\{m-j+2\ ,\dots,\ m-K+1\}$) have state B.

3). $DD_{m,j,K}$ is basic if it is A , B or C basic.

REMARK.

The reason of the condition $j > K+3$ is that the extreme sites are distant enough and do not directly interfer.

The next two lemmas describe some possible evolutions in time of truncated double diagonals (up to the arrival of the reflected signal).

LEMMA 1.

Let $j > K+3$ and $K > 2$.

1). If $DD_{m,j,K}$ is B-basic (resp. C-basic) and if the rightmost sites of $DD_{m+1,j,K}$ (i.e. $(j , 2m-j+2)$ and $(j , 2m-j+3)$) have state L then $DD_{m+1,j,K}$ is also B-basic (resp. C-basic).

2). If $DD_{m,j,K}$ is A-basic and the rightmost sites of $DD_{m+1,j,K}$ have state L and if

(*) sites $(K-1 , 2m-K+1)$ and $(K-1 , 2m-K+2)$ have states different from L,

then $DD_{m+1,j,K}$ is also A-basic.

Proof. We refer to figure 6.2 a).

Case C-basic.

Using rules $(C,C,L) \longrightarrow C$, $(C,C,C) \longrightarrow C$ and $(L,C,C) \longrightarrow C$ we see that all sites $(i , 2m-i+2)$ (and then all sites $(i , 2m-i+3)$) for $i \in \{K+1 , \ldots , j-1\}$ are in state C.

Observe that the environment (L,C,L) does not occur : they are at least two sites between the leftmost and rightmost sites of $DD_{m,j,K}$ since $j > K+3$.

States marked by U,V on figure 6.2 a) (on sites $(K-1 , 2m-K+1)$ and $(K-1 , 2m-K+2)$) are not X since $K \neq 1$. This fact and the transition rules $(\cdot,L,C) \longrightarrow L$ where $.$ is any state different from X, show that $(K\ ,\ 2m-K+3)$, $(K\ ,\ 2m-K+2)$ have state L.

This proves that $DD_{m+1,j,K}$ is C-basic.

Case B-basic.

We proceed in a similar way. Observe that $(B,B,L) \longrightarrow G$ puts state G to site $(j-1\ ,\ 2m-j+3)$. This imposes us to consider the case $j = K+3$ (see figure 6.2 a)) for which we use rule $(L,B,G) \longrightarrow B$.

Case A-basic.

We proceed as in the case C-basic. The lack of any transition rule with environment (L,L,A) is supplied by the hypothesis (*). □

LEMMA 2.

Let $j > K+3$ and $K > 2$ and $7 < m < n$.

1). If $DD_{m,j,K}$ is A-basic (resp. B-basic) and if the rightmost sites of $DD_{m+1,j+1,K+1}$ (i.e. sites $(j+1\ ,\ 2m-j+1)$ and $(j+1\ ,\ 2m-j+2)$) have state L then $DD_{m+1,j+1,K+1}$ is B-basic (resp. C-basic).

2). If $DD_{m,j,K}$ is C-basic, if the rightmost sites $DD_{m+1,j+1,K}$ have state L and if condition (*) of lemma 1 is satisfied then $DD_{m+1,j+1,K}$ is A-basic.

The proof is similar to that of lemma 1 and is illustrated by figure 6.2 b).

Looking at figure 5.8, we see that double diagonals consist of basic truncated double diagonals and a terminal portion somewhat different.

DEFINITION 2.

Let $7 < m < n$; a truncated double diagonal $DD_{m,4,1}$ (resp. $DD_{m,5,1}$, resp. $DD_{m,6,1}$) is called a 4-end (resp. 5-end, resp. 6-end) if its site-values are those indicated by figure 6.2 c) case 1 (resp. case 2, resp. case 3).

LEMMA 3.

Let $7 < m < n$. There exists a finite sequence of integers $(j_1^m, \dots, j_{\ell_m}^m)$ such that

$$4 < j_{\ell_m}^m < \dots < j_{\ell+1}^m < j_\ell^m < \dots < j_1^m = m+1$$

and

1). for every ℓ in $\{1, \dots, \ell_m - 1\}$ the portion of double diagonal $DD_{m, j_\ell^m, j_{\ell+1}^m}$ is basic.

2). The integer $j_{\ell_m}^m$ is 4,5 or 6 and the terminal portion $DD_{m, j_{\ell_m}^m, 1}$ is a $j_{\ell_m}^m$ - end.

Proof. The argument is an induction on the integer m in the segment $[7, \dots, n-1]$.

(i). Case $m = 7$.

Observing that the site-values of DD_7 is independent of n $(n > 8)$, it suffices to check this case with any particular value of n. This can be done using figure 5.8.

We now assume that the lemma is true up to m $(m < n-2)$ and prove the case $m+1$. The proof is quite long and occupies (ii) to (v) below.

(ii). We define ℓ_{m+1} and the sequence $j_{\ell_{m+1}}^{m+1}, \dots, j_\ell^{m+1}, \dots, j_\ell^{m+1}$ as follows :

[I] $\left[\begin{array}{l} - \ell_{m+1} \text{ is } \ell_m+1 \text{ if } j^m_{\ell_m} = 6 \text{ and all truncated double diagonals} \\ DD_{m,j^m_\ell, j^m_{\ell+1}} \text{ (with } \ell \text{ in } \{1,\dots,\ell_m-1\}) \text{ are A or B-basic ;} \\ \ell_{m+1} \text{ is } \ell_m \text{ otherwise.} \\ - \text{For } \ell \text{ in } \{1,\dots,\ell_m\} \\ j^{m+1}_\ell \text{ is } j^m_\ell+1 \text{ if all truncated double diagonals } DD_{m,j^m_s, j^m_{s+1}} \text{ with} \\ s \in \{1,\dots,\ell-1\} \text{ are A or B-basic ; } j^{m+1}_\ell \text{ is } j^m_\ell \text{ otherwise.} \\ \text{In particular for } \ell = 1 \ \ j^{m+1}_1 = j^m_1+1 = m+2. \\ - \text{If } \ell_{m+1} = \ell_m+1 \text{ then } j^{m+1}_{\ell_{m+1}} = 4. \end{array}\right.$

We observe that if $j^{m+1}_\ell = j^m_\ell$ then $j^{m+1}_s = j^m_s$ for all s in $\{\ell,\dots,\ell_m\}$ and $\ell_{m+1} = \ell_m$.

(iii). We first observe that condition (*) from lemmas 1 and 2 is always satisfied for the $DD_{m,j^m_\ell, j^m_{\ell+1}}$ with $\ell \in \{1,\dots,\ell_m-1\}$:

- Case $\ell = \ell_m-1$.

It is clear from property 2) for $DD_{m,j^m_{\ell_m},1}$ and figure 6.2 c) that sites $(j^m_{\ell_m}-1, 2m-j^m_{\ell_m}+1)$ and $(j^m_{\ell_m}-1, 2m-j^m_{\ell_m}+2)$ are not in state L.

- Case $\ell < \ell_m - 1$.

It is clear from property 1) for $DD_{m,j^m_{\ell+1}, j^m_{\ell+2}}$ and definition 1 that sites $(j^m_{\ell+1}-1, 2m-j^m_{\ell+1}+1)$ and $(j^m_{\ell+1}-1, 2m-j^m_{\ell+1}+2)$ are not in state L (they are both in state A or both in state C or in respective states G,B).

(iv). We first prove a part of property 1) for DD_{m+1} : double diagonals $DD_{m+1}, j^{m+1}_\ell, j^{m+1}_{\ell+1}$ are basic for $\ell \in \{1,\dots,\ell_m-1\}$ (the only possibly remaining $DD_{m+1}, j^{m+1}_{\ell_m}, j^{m+1}_{\ell_m+1}$ -when $\ell_{m+1} = \ell_m+1$- will be studied in point v)). The proof is by induction on ℓ in $\{1,\dots,\ell_m-1\}$.

* Initial step $\ell = 1$.

Using Fact 1, we know that sites (m+2 , m+1), (m+2 , m) are in state L. Since $m+1 = j_1^m$, this shows that $DD_{m,m+1,j_2^m}$ satisfies the hypothesis of lemma 2. Noticing that the different cases in lemma 2 correspond to the different cases in the definition of j_2^{m+1} from j_2^m, the conclusion of lemma 2 establishes that $DD_{m+1,m+2,j_2^{m+1}}$ is basic.

* Induction step.

We suppose that $DD_{m+1,j_\ell^{m+1},j_{\ell+1}^{m+1}}$ is basic (with $1 < \ell < \ell_m - 1$).

First we observe that the rightmost sites of $DD_{m+1,j_{\ell+1}^{m+1},j_{\ell+2}^{m+1}}$ are the leftmost sites of $DD_{m+1,j_\ell^{m+1},j_{\ell+1}^{m+1}}$. The induction hypothesis (over ℓ) insures that these sites are in state L.

- Case $j_{\ell+1}^{m+1} = j_{\ell+1}^m + 1$.

The very definitions of $j_{\ell+1}^{m+1}$ from $j_{\ell+1}^m$ show that every DD_{m,j_s^m,j_{s+1}^m}, with $s \in \{1, \ldots, \ell - 1\}$ is A or B-basic. Thus, the value of $j_{\ell+2}^{m+1}$ is :

$$\begin{cases} j_{\ell+2}^m + 1 & \text{if } DD_{m,j_{\ell+1}^m,j_{\ell+2}^m} \text{ is A or B-basic} \\ j_{\ell+2}^m & \text{if it is C-basic.} \end{cases}$$

Since sites $(j_{\ell+1}^m + 1 , 2m-j_{\ell+1}^m + 1)$ and $(j_{\ell+1}^m + 1 , 2m-j_{\ell+1}^m + 2)$ - being the rightmost sites of $DD_{m+1,j_{\ell+1}^{m+1},j_{\ell+2}^{m+1}}$ since $j_{\ell+1}^{m+1} = j_{\ell+1}^m + 1$ - have state L, the hypothesis of lemma 2 is satisfied for $DD_{m,j_{\ell+1}^m,j_{\ell+2}^m}$.

Observing that the different cases in lemma 2 correspond to the different cases of the definitions of $j^{m+1}_{\ell+2}$ from $j^m_{\ell+2}$, we get the basic character of $DD_{m+1, j^{m+1}_{\ell+1}, j^{m+1}_{\ell+2}}$.

- Case $j^{m+1}_{\ell+1} = j^m_{\ell+1}$.

Recall (cf. end of point ii)) that this equality implies $j^{m+1}_{\ell+2} = j^m_{\ell+2}$.

Since sites $(j^m_{\ell+1}, 2m-j^m_{\ell+1}+2)$ and $(j^m_{\ell+1}, 2m-j^m_{\ell+1}+3)$ -being the rightmost sites $DD_{m+1, j^{m+1}_{\ell+1}, j^{m+1}_{\ell+2}}$ since $j^m_{\ell+1} = j^{m+1}_{\ell+1}$- have state L, the hypothesis of lemma 1 is satisfied for $DD_{m+1, j^{m+1}_{\ell+1}, j^{m+1}_{\ell+2}}$.

Thus $DD_{m+1, j^{m+1}_{\ell+1}, j^{m+1}_{\ell+2}}$ -which is the same as $DD_{m+1, j^m_{\ell+1}, j^m_{\ell+2}}$ - is also basic.

(v). Up to now, we have shown that the double diagonals $DD_{m+1, j^{m+1}_{\ell}, j^{m+1}_{\ell+1}}$ will $\ell \in \{1, \ldots, \ell_m - 1\}$ satisfy condition 1) of lemma 3. We still have to study the remaining part $DD_{m+1, j^{m+1}_{\ell_m}, 1}$ of DD_{m+1}.

It is convenient to consider two cases corresponding to the possible values of $j^{m+1}_{\ell_m}$ from $j^m_{\ell_m}$.

- Case $j^{m+1}_{\ell_m} = j^m_{\ell_m}$.

Notice (as above in (iv)) that sites $(j^m_{\ell_m}, 2m-j^m_{\ell_m}+2)$ and $(j^m_{\ell_m}, 2m-j^m_{\ell_m}+3)$ -being the leftest sites of $DD_{m+1, j^{m+1}_{\ell_m - 1}, j^{m+1}_{\ell_m}}$ - have state L.

Using the induction hypothesis over m, $DD_{m, j^m_{\ell_m}, 1}$ is (according to property 2) of the lemma 3 at step m) a 4 (or 5 or 6)-end. As illustrated by figure 6.2 d), the state-values of sites in $DD_{m+1, j_{\ell_m}^{m+1}, 1}$ are completely determined and this truncated double diagonal is also a 4 (or 5 or 6)-end.

Note : In this case, we always have $\ell_{m+1} = \ell_m$ (which agrees with point ii).

- Case $j_{\ell_m}^{m+1} = j_{\ell_m}^m + 1$.

Notice (as above in iv)) that sites $(j_{\ell_m}^m + 1, 2m-j_{\ell_m}^m + 1)$ and $(j_{\ell_m}^m + 1, 2m-j_{\ell_m}^m + 2)$ -being the leftest sites of $DD_{m+1, j_{\ell_m}^{m+1} - 1, j_{\ell_m}^{m+1}}$ - have state L.

Using the induction hypothesis over m, $DD_{m, j_{\ell_m}^m, 1}$ is a 4 (or 5 or 6)-end.

As illustrated by figure 6.2 e) the state values of sites in $DD_{m+1, j_{\ell_m}^{m+1}, 1}$ are completely determined.

It is trivial to check that :

- if $DD_{m, j_{\ell_m}^m, 1}$ is a 4-end (resp. 5-end),

then $DD_{m+1, j_{\ell_{m+1}}^{m+1}, 1}$ is a 5-end (resp. 6-end).

In this case we have $\ell_{m+1} = \ell_m$ which agrees with point ii).

- If $DD_{m, j_{\ell_m}^m, 1}$ is a 6-end, then :

* $DD_{m+1,4,1}$ is a 4-end

* $DD_{m+1,7,4}$ is A-basic.

In this case we have $j^m_{\ell_m} = 6$, $j^{m+1}_{\ell_m} = 7$, $j^{m+1}_{\ell_{m+1}} = 4$ and $\ell_{m+1} = \ell_m+1$ which agrees with point ii).

This complete the proof of lemma 3. □

The following notion is quite convenient.

DEFINITION 3.

The sets of sites U and V included in $\{1,\dots,n\} \times \{1,\dots,2n-1\}$ are n-equivalent if there exists a translation $T : \mathbb{N}^2 \longrightarrow \mathbb{N}^2$ such that :

- $T(U) = V$,
- For every (K,t) in U, sites (K,t) and $T(K,t)$ have the same state in the state diagram of an initial line of n machines.

We shall use the following consequences of the above proof :

LEMMA 4.

1). If $j^m_\ell = j^{m+1}_\ell$ then the translation $(K,t) \longrightarrow (K, t+2)$ maps $DD_{m, j^m_\ell, 1}$ onto $DD_{m+1, j^{m+1}_\ell, 1}$ and respects the state values.

2). Let $1 < \ell < \ell_{m_o}$, the function $m \longrightarrow j^m_\ell - j^m_{\ell+1}$ with domain $\{m_o, \dots, n-1\}$ is not decreasing.

Proof. (1). From the proof of lemma 3 we see that the condition $j^m_\ell = j^{m+1}_\ell$ implies :

- $\ell_{m+1} = \ell_m$
- for s in $\{\ell,\dots,\ell_m-1\}$ the set $DD_{m+1, j^{m+1}_s, j^{m+1}_{s+1}}$ and its state values is obtained from $DD_{m, j^m_s, j^m_{s+1}}$ (and its state values) via the translation $(K,t) \longrightarrow (K, t+2)$.

- The set $DD_{m+1, j^{m+1}_{\ell_{m+1}}, 1}$ and its state values is also obtained via this translation from $DD_{m, j^m_{\ell_m}, 1}$.

(2). This is a trivial consequence of the proof of lemma 3 : in fact $(j^{m+1}_{\ell} - j^{m+1}_{\ell+1}) - (j^m_{\ell} - j^m_{\ell+1})$ is 0 or 1. □

The next lemma states an essential property of the scheme, i.e. the distribution of states below the reflection. It will be basic to prove that the remaining portion of the initial line after the break at the 2/3 evolves -for the part during and after the reflection- as an initial line of length $\frac{2n}{3}$ (cf. 6.1).

<u>LEMMA 5.</u>

Let $12 < m < n$ be of the form $m = 3p+i$ with $p > 4$ and $i \in \{0,1,2\}$.

1). $DD_{m,m+1,2p+i}$ is A (resp. B,C)-basic if $i = 0$ (resp. 1,2) ; also $j^m_2 = 2p+i$.

2). $DD_{m,2p+i,1}$ is n-equivalent to $DD_{2p+i-1} = DD_{2p+i-1,2p+i,1}$.

<u>Proof.</u> We argue by induction on the integer m. The case $m = 12$ is easily checked (cf. figure 5.8).

Suppose now that properties 1 and 2 hold for m.

(i). From Fact 1 we see that sites $(m+1 , m)$ and $(m+1 , m-1)$ have state L. Since $DD_{m,m+1,2p+i}$ is basic, its leftest sites $(2p+i , 2m-2p-i)$ and $(2p+i , 2m-2p-i+1)$ have state L. Due to the decomposition of DD_m in basic and end DD's (lemma 3), their neighbors $(2p+i-1 , 2m-2p-i+1)$ and $(2p+i-1 , 2m-2p-i+2)$ have states different from L. This shows that the hypothesis of lemma 2, including condition (*), are satisfied by $DD_{m,m+1,2p+i}$.

Applying lemma 2, we are led to the three cases :

- **Case** $i = 0$.

By induction hypothesis, $DD_{m,m+1,2p+i}$ is A-basic and lemma 2 shows that $DD_{m+1,m+2,2p+i+1}$ is B-basic. Noticing that the equality $m+1 = 3q+j$ with $j \in \{0,1,2\}$ implies $q = p$ and $j = i+1 = 1$, so that $2q+j = 2p+i+1$, we see that the first part of property 1) holds for $m+1$.

- **Case** $i = 1$. Similar.

- **Case** $i = 2$.

By induction hypothesis $DD_{m,m+1,2p+i}$ is C-basic and lemma 2 shows that $DD_{m+1,m+2,2p+i}$ is A-basic. Noticing that the equality $m+1 = 3q+j$ with $j \in \{0,1,2\}$ implies $q = p+1$ and $j = 0$ so that $2q+j = 2p+2 = 2p+i$, we see that the first part of property 1) holds for $m+1$.

(ii). In the three previous cases, the basic character of $DD_{m+1,m+2,2q+j}$ implies that $j_2^{m+1} = 2q+j$ where $m+1 = 3q+j$ with $j \in \{0,1,2\}$. We consider two cases according to the value of j_2^{m+1} from j_2^m.

- **Case** $j_2^{m+1} = j_2^m$.

Point 1 of lemma 4 and the induction hypothesis prove point 2 of this lemma for $m+1$.

- **Case** $j_2^{m+1} = j_2^m + 1$.

By induction hypothesis $DD_{m,2p+i,1}$ is equivalent to $DD_{2p+i-1,2p+i,1}$. Sites $(2p+i+1\ ,\ 2p+i)$ and $(2p+i+1\ ,\ 2p+i-1)$ have state L by Fact 1. Sites $(2p+i+1\ ,\ 2m-2p-i+1)$ and $(2p+i+1\ ,\ 2m-2p-i+2)$ have state L because $j_2^{m+1} = j_2^m + 1 = 2p+i+1$ and these sites are the leftest ones of $DD_{m+1,m+2,j_2^{m+1}}$. Thus the two sets

$U = DD_{2p+i-1,2p+i,1} \cup \{(2p+i+1\ ,\ 2p+i),\ (2p+i+1\ ,\ 2p+i-1)$ and

$V = DD_{m,2p+i,1} \cup \{(2p+i+1\ ,\ 2m-2p-i+2),\ (2p+i+1\ ,\ 2m-2p-i+1)\}$

are n-equivalent via the vertical translation of vector $(0,2m-2(2p+i-1)) = (0,2p+2)$.

Notice that the state-values of $DD_{m+1,2p+i+1,1}$ (resp. $DD_{2p+i,2p+i+1,1}$) are fully determined by those of V (resp. U) (see figure 6.2 f)). We thus deduce that $DD_{m+1,2p+i+1,1}$ and $DD_{2p+i,2p+i+1,1}$ are n-equivalent.

Observe that the equality $j_2^{m+1} = j_2^m+1$ implies that $DD_{m,m+1,j_2^m}$ is A or B-basic (cf. point ii) of the proof of lemma 3). Thus, point 1) of lemma 5 shows that if $m = 3p+i$ then $i \neq 2$.

Finally, recall that if $m+1 = 3q+j$ (where $j \in \{0,1,2\}$) then $2q+j = 2p+i+1$ (where $m = 3p+i$ with $i \in \{0,1\}$).

Thus the n-equivalent preceding sets are exactly those of point 2 of lemma 5 for $m+1$. □

<u>6.3.</u> The two next lemmas describe some properties of the reflection of the initial wave. The first one considers the emission and transmission by the right-end soldier of the reflected initial wave.

<u>LEMMA 6.</u>

Let $n > 25$, $n = 3p+i$ with $i \in \{1,2,3\}$.

Let Z and U be the sets :

$Z = \{(\ell, 2n-\ell+i+z)\ ;\ \ell \in \{j_2^{n-1}+5, \dots, n\}\ ;\ z \in \{-1,0,1,2,3\}\}$

$U = \{(\ell, 2n-\ell+u)\ ;\ \ell \in \{j_2^{n-1}+5, \dots, n\}\ ;\ u \in \{0, \dots, 6\}$ (cf. fig. 6.3 b)).

These sets are non empty and :

1). The distribution of states on Z is as indicated in figure 6.3 a), i.e. every vertical section has states CBLLL from bottom to top.

2). The distribution of states on U completes that of Z as indicated in figure 6.3 a).

Case $i = 1$: $U \setminus Z$ consists of two L-valued diagonals above Z.

Case $i = 2$: $U \setminus Z$ consists of two diagonals, one A-valued below Z and the other L-valued above Z.

Case $i=3$: $U \setminus Z$ consists of two diagonals below Z. The highest one is A-valued, the other one is B-valued except its rightmost site which has value G.

Proof. When $n = 25$, we have $j_2^{24} = 16$ and the truncated double diagonal $DD_{24,25,16}$ has sites corresponding to ten adjacent machines. Using point 2 of lemma 4, we deduce that for all $n > 25$, $DD_{n-1,n,j_2^{n-1}}$ has at least ten adjacent machines.

As illustrated in figure 6.3 b), $DD_{n-1,n,j_2^{n-1}}$ fully determines the set U. Observe that this set is non-empty since there are least six (indeed ten) adjacent machines involved in $DD_{n-1,n,j_2^{n-1}}$.

Observing that $n-1 = 3p + (i-1)$ with $(i-1) \in \{0,1,2\}$, lemma 5 -applied to $DD_{n-1,n,j_2^{n-1}}$ - gives the three distributions of states on U (as illustrated on figure 6.3 a)). □

REMARK.

Lemma 6 proves that the delay is really conveyed by the reflected wave as indicated in paragraph 4.

The following lemma considers the crash of the reflection of the initial wave and completes lemma 6.

LEMMA 7.

Let $n > 25$. The distribution of states for sites corresponding to machines $j_2^{n-1} - 2, \ldots, j_2^{n-1} + 5$ and to the arrival of the reflection of the initial wave is always that of some of the nine cases indicated in figure 6.3 c).

Proof. Let V be the set

$$DD_{n-1, j_2^{n-1}+5,\, j_2^{n-1}-2} \cup \{(j_2^{n-1}+5 ,\, 2n-j_2^{n-1}-5+\ell) \;;\; \ell \in \{0 ,\dots, 6\}\}.$$

The set $\tilde{V}$ is fully determined by V.

We first observe that $j_2^{24} > j_3^{24}+3$ and $25 = j_1^{24} > j_2^{24}+6$ (easy check, cf. figure 5.8). Point 2 of lemma 4 permits to extend these inequalities : $j_2^{n-1} > j_3^{n-1}+3$ and $n = j_1^{n-1} > j_2^{n-1}+6$. This insures that :

$$DD_{n-1, j_2^{n-1}+5,\, j_2^{n-1}-2} \subset [DD_{n-1, j_2^{n-1},\, j_3^{n-1}} \cup DD_{n-1, n, j_2^{n-1}}].$$

By lemma 3, $DD_{n-1, j_2^{n-1},\, j_3^{n-1}}$ is A or B or C-basic.

From lemmas 5 and 6 the distribution of states on

$DD_{n-1,n,j_2^{n-1}}$ and $\{(j_2^{n-1}+5 ,\, 2n-j_2^{n-1}-5+\ell),\ \ell \in \{0 ,\dots, 6\}\}$

(which is included in the set U of lemma 6) depend only on the remainder of n modulo 3. Thus there are only nine types of distribution of states on the set V.

Observing that $\tilde{V}$ is fully determined by V, we get the nine cases of figure 6.3 c). □

6.4. In order to compare states-diagrams for inital lines of different lengths, it is convenient to introduce the following notation and definition :

Notation 3.

Let Y be a set of sites included in $\{1 ,\dots, N\} \times \{1 ,\dots, 2N-1\}$ and $N < n$. By $< Y >_n$ we mean the restriction to Y of the state-diagram of the evolution of an initial line of length n. Also $< K,t >_n$ denotes $< \{(K,t)\} >_n$.

DEFINITION 4.

Let Y and Z be two sets of sites. By $< Y >_n \sim < Z >_m$ we mean

- $< Y >_n$ and $< Z >_m$ are both defined
- there exists a translation $T : (K,t) \longrightarrow (K-a\,,\,t-b)$ such that $T(Y) = Z$ and T respects state-values (i.e. $< T(K,t) >_m = < K,t >_n$ for (K,t) in Y).

REMARK.

In case $n = m$, then $< Y >_n \sim < Z >_m$ if and only if Y and Z are n-equivalent in the sense of definition 3.

Notation 4.

$H^t_{K,j}$ is the following portion of a constant-time line $\{(\ell,t)\ ;\ K < \ell < j\}$.

LEMMA 8.

Let $n > 25$, $n = 3p+i$ with $i \in \{1,2,3\}$, then

$< H^t_{2p+i,3p+i} >_{3p+i} \sim < H^3_{1,p+1} >_{p+1}$

for $t = (2n-1) - [2(p+1)-1-3] = 4p+2i+1$ (i.e. the distribution of states at time 3 for an initial line of $p+1$ machines is that at time t for the $p+1$ rightmost machines of an initial line of n machines.

Proof. It is clear that the distribution of states on $H^3_{1,p+1}$ for an initial line of length $p+1$ is GBALL ... LL.

Using lemma 7 for a line of n machines, it is easy to check that in the nine cases of figure 6.3 c)

- sites $(2p+i\ ,\ t)$, $(2p+i+1\ ,\ t)$, $(2p+i+2\ ,\ t)$ are in respective states GBA.

- sites $(2p+i+2\ ,\ t-1)$, $(2p+i+2\ ,\ t-2)$, $(2p+i+3\ ,\ t-2)$, $(2p+i+3\ ,\ t-3)$ are in state L.

By lemma 5, $j_2^{n-1} = 2p+i-1$; thus lemma 6 insures that all sites in the set
$W = \{(\ell\ ,\ 2n-\ell+i+1),\ (\ell\ ,\ 2n-\ell+i+2)\ ;\ \ell \in \{2p+i+4\ ,\ldots,\ n\}\}$ have state L (for an initial line of n machines).

Observing (cf. figure 6.4 a)) that the set
$W \cup \{(2p+i+2\ ,\ t-1),\ (2p+i+2\ ,\ t-2),\ (2p+i+3\ ,\ t-2),\ (2p+i+3\ ,\ t-3)\}$ fully dtermines a set containing $H^t_{2p+i+3,3p+i}$, we deduce that all sites in $H^t_{2p+i+3,3p+i}$ have state L (for an initial line of n machines).

This proves that the states of $< H^t_{2p+i,3p+i} >_{3p+i}$ are GBALL ... LL.

Taking $T : (h,\theta) \longrightarrow (h-2p-i+1\ ,\ \theta-t+3)$, we see that :
$< H^t_{2p+i,3p+i} >_{3p+i} \sim < H^3_{1,p+1} >_{p+1}$. □

<u>Notation 5.</u>

$LDD^t_{m,j}$ is the following set of sites (it is the union of an initial truncated double diagonal and a triangle) (cf. figure 6.4 b)) :
$LDD^t_{m,j} = DD_{m,j,1} \cup \{(h,\theta)\ ;\ h < j \text{ and } \theta < t \text{ and } h+\theta > 2m+1\}$.

<u>LEMMA 9.</u>

Let $n > 25$, $n = 3p+i$ with $i \in \{1,2,3\}$, then
$< LDD^{4p+2i+1}_{n-1,2p+i-1} >_n \sim < LDD^{2p+2i-1}_{2p+i-2,2p+i-1} >_{2p+i-1}$
(i.e. the distribution of states at time $2p+i-1+b$ (with $b \in \{0\ ,\ldots,\ i\}$)) of the b rightmost machines of an initial wave of length $2p+i-1$ is that at time $4p+i+1+b$ (with $b \in \{0\ ,\ldots,\ i\}$) of the b left neighbors of the machine $2p+i$ of an initial line of n machines.

Proof. (i). Observing that if $n = 3p+i$ with $n \in \{1,2,3\}$ then $n-1 = 3p+i-1$ with $i-1 \in \{0,1,2\}$, lemma 5 insures us that

$< DD_{n-1\ ,\ 2p+i-1,1} >_n \sim < DD_{2p+i-2\ ,\ 2p+i-1,1} >_n$.

Let I be the set $\{(h\ ,\ h-\ell)\ ;\ \text{with}\ 0 < \ell \leqslant h-1\ \text{and}\ h \leqslant 2p+i\}$, then $DD_{2p+i-2\ ,\ 2p+i-1,1}$ is fully determined by $I \cup \{(1,1)\}$.

By Fact 1 we have $< I >_n = < I >_{2p+i-1}$ (all sites have state L). Since $< 1,1 >_n = < 1,1 >_{2p+i-1} = G$, we have

$< DD_{2p+i-2\ ,\ 2p+i-1,1} >_n \sim < DD_{2p+i-2\ ,\ 2p+i-1,1} >_{2p+i-1}$.

Thus $< DD_{n-1\ ,\ 2p+i-1,1} >_n \sim < DD_{2p+i-2\ ,\ 2p+i-1,1} >_{2p+i-1}$.

(ii). States $< 2p+i-1\ ,\ 4p+i+1+b >_n$ with $b \in \{0\ ,\dots,\ i\}$ are indicated by lemma 7. These states depend on the remainder of $2p+i-1$ modulo 3 ; they are given by figure 6.3 c).

States $< 2p+i-1\ ,\ 2p+i-1+b >_{2p+i-1}$ with $b \in \{0\ ,\dots,\ i\}$ are fully determined by the states of DD_{2p+i-2} for an initial line of $2p+i-1$ machines ; they are given by figure 6.3 a) (where n is replaced by 2p+i+1). These states depend also on the remainder of $2p+i-1$ modulo 3.

Whatewer be this remainder, one checks easily on figures 6.3 c) and 6.4 c), via the translation $T : (h,t) \longrightarrow (h\ ,\ t-2p-2)$, that

$< \{(2p+i-1,4p+i+1+b)\ ;\ b \in \{0\ ,\dots,\ i\}\} >_n \sim$

$< \{(2p+i-1,2p+i-1+b),\ b \in \{0\ ,\dots,\ i\}\} >_{2p+i-1}$.

(iii). Observing that $LDD^{4p+2i+1}_{n-1,2p+i-1}$ and $LDD^{2p+2i-1}_{2p+i-2,2p+i-1}$ are fully determined by

$DD_{n-1\ ,\ 2p+i-1,1} \cup \{(2p+i-1\ ,\ 4p+i+1+b)\ ;\ 0 \leqslant b \leqslant i\}$ and

$DD_{2p+i-2\ ,\ 2p+i-1,1} \cup \{(2p+i-1\ ,\ 2p+i-1+b)\ ;\ 0 \leqslant b \leqslant i\}$, the two preceding points show that

$< LDD^{4p+2i+1}_{n-1,2p+i-1} >_n \sim < LDD^{2p+2i-1}_{2p+i-2,2p+i-1} >_{2p+i-1}$. □

6.5. Let $n = 3p+i$ with $i \in \{1,2,3\}$ and $n > 25$, we observe that :

- $LDD^{4p+2i+1}_{n-1,2p+i-1} \cup \{(2p+i , \theta) ; 4p+2i+1 < \theta < 2n-1\}$ fully determines $\{(K , 2n-1) ; 1 < K < 2p+i-1\}$ (i.e. the (wanted) synchronization step of the $2p+i-1$ leftmost machines of the initial line at time $2n-1$).
- $H^{4p+2i+1}_{2p+i,3p+i} \cup \{(2p+i , \theta) ; 4p+2i+1 < \theta < 2n-1\}$ fully determines $\{(K , 2n-1) ; 2p+i < K < 3p+i\}$ (i.e. the (wanted) synchronization step of the $p+1$ rightmost machines of the initial line at time $2n-1$).

Lemmas 8 and 9 show that the distribution of states on $LDD^{4p+2i+1}_{n-1,2p+i-1}$ and $H^{4p+2i+1}_{2p+i,3p+i}$ are those on adequate subsets of the state-diagrams of initial lines having lengths strictly shorter than n.

In order to reduce the synchronization of an initial line of n machines to that of shorter initial lines, we prove that :

(Σ)
- machine $2p+i$ stays in state G from time $4p+2i+1$ up to time $2n-2$.
- During the same time interval, the states U,V of machines $2p+i-2$ and $2p+i-1$ are such that the rules for environments (U,V,X) and (U,V,G) give the same result.

LEMMA 10.

Let $n = 3p+i$ with $i \in \{1,2,3\}$ and $n > 25$.

For every time θ in $\{1 ,\dots, 4p+2i+1\}$

$(< n-1 , \theta >, < n,\theta >)$ is an element of the following set

$\Lambda_1 = \{(C,L), (A,L), (A,C), (C,B), (B,L), (L,L), (G,L), (C,G), (B,A)\}$

Note. This proves that up to time $4p+2i+1$, the environments GGX and AAX do not occur.

Proof. From Fact 1 we know that these two states are (L,L) up to time $n-2$.

From lemma 6 one checks that these two states are in Λ_1 from time $n-1$ up to time $n+6$.

From Lemma 7 one checks that these two states are (L,L) from time $n+7$ up to time $4p+2i+1$. □

We cannot prove Σ directly. We proceed by an examination of successive horizontal lines.

LEMMA 11.

Let $n \geq 25$ and $n = 3p+i$ with $i \in \{1,2,3\}$.

Suppose that condition (**)

$$(**)\quad \left[\begin{array}{l} \text{For all } \theta,\ 1 \leq \theta \leq 2m-3\ (\langle m-1, \theta\rangle_m, \langle m,\theta\rangle_m) \in \Lambda_2 \\ \text{with } \Lambda_2 = \Lambda_1 \cup \{(B,C), (B,G), (G,C), (G,B), (G,A)\} \end{array}\right.$$

holds for all m such that $3 \leq m < n$.

Then

1). For $t \in \{0, \ldots, 2p-3\}$

(a). $\langle H^{4p+2i+1+t}_{2p+i,3p+i} \rangle_{3p+i} \sim \langle H^{3+t}_{1,p+1} \rangle_{p+1}$

(b). $\langle H^{4p+2i+1+t}_{2p-2-t,2p+i-1} \rangle_{3p+i} \sim \langle H^{2p+2i-1+t}_{2p-2-t,2p+i-1} \rangle_{2p+i-1}$

2). Condition (**) holds for $m = n$.

Proof. (i). We prove point 1) by induction over t. Observing that $H^{4p+2i+1}_{2p-2,2p+i-1}$ is included in $LDD^{4p+2i+1}_{n-1,2p+i-1}$ we see that the case $t = 0$ is answered by lemmas 8 and 9.

We suppose now that point 1 is true for some $t \leq 2p-4$ and we prove point 1 for $t+1$ in (ii) to (v) below (cf. figure 6.5).

(ii). Since $H^{4p+2i+1+(t+1)}_{2p+i+1,3p+i}$ (resp. $H^{3+(t+1)}_{2,p+1}$) is fully determined by $H^{4p+2i+1+t}_{2p+i,3p+i}$ (resp. $H^{3+t}_{1,p+1}$), point 1 a) shows that

$< H^{4p+2i+1+(t+1)}_{2p+i+1,3p+i} >_n \sim < H^{3+(t+1)}_{2,p+1} >_{p+1}$.

(iii). Observe that :

- $H^{4p+2i+1+(t+1)}_{2p-2-(t+1),2p+i-2}$ (resp. $H^{2p+2i-1+(t+1)}_{2p-2-(t+1),2p+i-2}$) is fully determined by $H^{4p+2i+1+t}_{2p-2-t,2p+i-1} \cup LDD^{4p+2i+1}_{n-1,2p+i-1}$ (resp. $H^{2p+2i-1+t}_{2p-2-t,2p+i-1} \cup LDD^{2p+2i-1}_{2p+i-2,2p+i-1}$)
- $< H^{4p+2i+1+t}_{2p-2-t,2p+i-1} >_n \sim < H^{2p+2i-1+t}_{2p-2-t,2p+i-1} >_{2p+i-1}$, by point 1) b) for t.
- $< LDD^{4p+2i+1}_{n-1,2p+i-1} >_n \sim < LDD^{2p+2i-1}_{2p+i-2,2p+i-1} >_{2p+i-1}$ by lemma 9.

Thus $< H^{4p+2i+1+(t+1)}_{2p-2-(t+1),2p+i-2} >_n \sim < H^{2p+2i-1+(t+1)}_{2p-2-(t+1),2p+i-2} >_{p+1}$.

(iv). Observe that if $3 \leqslant \theta < 2p$ then $< 1,\theta >_{p+1} = G$.

- $\theta \geqslant 14$ then $(1,\theta)$ is a leftmost site of a double diagonal DD_m with $7 \leqslant m < p+1$ and lemma 3 insures that $< 1,\theta >_{p+1} = G$.
- Cases $3 \leqslant \theta \leqslant 13$ are easily checked on figure 5.8.

Now we complete the distribution of states on $H^{4p+2i+1+(t+1)}_{2p-2-(t+1),2p+i-1}$ by determining the state value of site $(2p+i-1\ ,\ 4p+2i+1+(t+1))$.

Condition (**) shows that :

$(< 2p+i-2\ ,\ 2p+2i-1+t >_{2p+i-1}\ ,\ < 2p+i-1\ ,\ 2p+2i-1+t >_{2p+i-1}) \in \Lambda_2$.

Point 1 b) for t then shows that :

$(< 2p+i-2\ ,\ 4p+2i+1+t >_n\ ,\ < 2p+i-1\ ,\ 4p+2i+1+t >_n) \in \Lambda_2$.

It is easy to check from the transition matrices (cf. figure 5.7) that if $(U,V) \in \Lambda_2$, then (U,V,X) and (U,V,G) are environments for which there are transition rules, and these two rules have the same result.

Observing that $< 2p+i , 4p+2i+1+t >_n$ is $< 1 , 3+t >_{p+1}$ by point 1) a) for t and $< 1,3+t >_{p+1}$ is state G (cf. above), we see that the environments

$(< 2p+i-2 , 2p+2i-1+t >_{2p+i-1} , < 2p+i-1 , 2p+2i-1+t >_{2p+i-1} , X)$ and

$(< 2p+i-2 , 4p+2i+1+t >_n , < 2p+i-1 , 4p+2i+1+t >_n , G)$.

have the same result, hence

$< 2p+i-1 , 4p+2i+1+(t+1) >_n = < 2p+i-1 , 2p+2i-1+(t+1) >_{2p+i-1}$

(v). Now we complete the distribution of states on $H^{4p+2i+1+t+1}_{2p+i,3p+i}$ by determining the state value of site $(2p+i , 4p+2i+1+(t+1))$.

Observing that if $t < 2p-4$ then $3+t < 2p-1$, we note that $< 1 , 3+t >_{p+1}$ and $< 1 , t+3+1 >_{p+1}$ are in state G (cf. above).

In an initial line of $p+1$ machines, site $(2 , 3+t)$ is on double diagonal DD_s (where s is $\frac{5+t}{2}$ is t is odd, $\frac{5+t-1}{2}$ else). Since we have $s < p$, site $(2 , 3+t)$ is a site for machine 2 on a 4 (or 5 or 6)-end by lemma 3. Thus $< 2 , 3+t >_{p+1}$ is state B or C (cases $3 < t+3 < 13$ are directly checked on figure 5.8). By point 1) a) for t, $< 2p+i+1 , 4p+2i+1+t >_n$ is B or C.

Since $n > 25$ we have $p > 2$ and $2p+i-1 > 3$. Thus we can apply condition (**) to $m = 2p+i-1$ which shows that

$(< 2p+i-2 , 2p+2i-1+t >_{2p+i-1} , < 2p+i-1 , 2p+2i-1+t >_{2p+i-1}) \in \Lambda_2$

hence that

$< 2p+i-1 , 2p+2i-1+t >_{2p+i-1}$ is different from state F.

Thus by point 1 b) for t, $< 2p+i-1 , 4p+2i+1+t >_n$ is different from state F. Observing -on the transition matrices of figure 5.7- that all environments (U,G,B) and (U,G,C) (where U is different from F) give state G, we conclude that $< 2p+i , 4p+2i+1+(t+1) >_n$ is state G. and this completes the proof of point 1 for t+1.

(vi). Now we consider point 2. Lemma 10 shows that condition (**) for n holds from time 1 up to time 4p+2i+1.

By point 1 $(< n-1 , 4p+2i+1+t >_n , < n , 4p+2i+1+t >_n)$ is $(< p , 3+t >_{p+1} , < p+1 , 3+t >_{p+1})$ for $0 \leqslant t \leqslant 2p-3$.

Thus condition (**) for $p+1 < 3p+i = n$ insures us that $(< n-1 , 4p+2i+1+t >_n , < n , 4p+2i+1+t >_n)$ is in Λ_2 for $1 \leqslant 3+t \leqslant 2(p+1)-3$ i.e for $4p+2i+1 \leqslant 4p+2i+1 + t \leqslant 4p+2i+1+2p-4 = 2n-3$.

This proves that condition (**) holds for n. □

<u>6.6.</u> Now we can complete the proof of correctness.

<u>THEOREM.</u>

Let $\mathbb{A}$ be the automaton whose transition rules are indicated on figure 5.7 ; $\mathbb{A}$ is a minimal time solution of the firing squad synchronization problem.

<u>Proof.</u> We prove by induction on the length n of the initial line that :

(1). $\mathbb{A}$ synchronizes an initial line of length n in time 2n-1.

(2). $\mathbb{A}$ G-synchronizes an initial line of length n with $n \geqslant 3$ (i.e. $< K , 2n-2 >_n = G$ for $K \in \{1 ,..., n\}$).

(3). Condition (**) holds for n with $n \geqslant 3$.

We easily check the following facts :

- $\mathbb{A}$ is a minimal time solution of the firing squad for initial lines strictly shorter than 25.
- For $3 \leqslant n \leqslant 24$, $\mathbb{A}$ G-synchronizes an initial line of length n.
- Condition (**) holds for n with $3 \leqslant n \leqslant 24$.

We now suppose that facts 1,2,3 are true for $m < n-1$, and we prove them for n with $n > 25$.

The induction hypothesis 3) for $m < n-1$, shows that the hypothesis of lemma 11 holds for n. Thus, point 1 a) of lemma 11 in the case $t = 2p-3$ shows that :

$< H^{2n-2}_{2p+i\,,\,3p+i} >_n \sim < H^{2p}_{1\,,\,p+1} >_{p+1}$.

Since $p+1 < 3p+i = n$, the induction hypothesis 2) for $p+1$ shows that all sites $(K\,,\,2p)$ (with $1 < K < p+1$) are in state G if the initial line is of length $p+1$. Thus all sites $(K\,,\,2n-2)$ with $K \in \{2p+i\,,\ldots,\,3p+i\}$ are in state G if the initial line has length n.

Point 1 b) of lemma 11 in the case $t = 2p-3$ shows that

$< H^{2n-2}_{1,2p+i-1} >_n \sim < H^{4p+2i-4}_{1,2p+i-1} >_{2p+i-1}$.

Since $2p+i+1 < n$ ($n = 3p+i$ with $p > 7$ and $i \in \{1,2,3\}$), the induction hypothesis 2) for $2p+i+1$ shows that all sites $(K\,,\,2(2p+i-1)-2)$, with $1 < K < 2p+i-1$, are in state G, if the initial line has length $2p+i-1$. Thus all sites $(K\,,\,2n-2)$ -with $1 < K < 2p+i-1$- are in state G if the initial line if of length n.

By this way we have proved that all sites $(K\,,\,2n-2)$ with $K \in \{1\,,\ldots,\,n\}$ are in state G if the initial line is of length n. This proves point 2 for n.

From this G-synchronization at time $2n-2$, we get
- using rules $(G,G,G) \longrightarrow F$, $(X,G,G) \longrightarrow F$, $(G,G,X) \longrightarrow F$-
the F-synchronization at time $2n-1$ of the initial line of length n.

Finally condition (**) for n is a trivial consequence of lemma 11 (and the induction hypothesis).

□

§ 7 - CONCLUSION.

In [1], R.BALZER's has shown that :

- no minimal time solution exists with 4 states.
- no minimal time solution satisfying extra conditions exists with 5 states.

However, the solution presented here does not satisfy BALZER's four extra conditions :

- in particular his extra conditions 1 (the stability of state G) and 4 (rules (G,V,G) $\longrightarrow$ G for V $\neq$ G) are violated,
- the very idea of our solution is *not to be* an "image solution" (his condition 2),
- the only condition satisfied is condition 3 (the fire is introduced only by environments GGG, XGG, GGX).

Also it is easy to obtain a seven state solution which does not satisfy his condition 3 (introduce (GBG) $\longrightarrow$ R and (GRG) $\longrightarrow$ F and (GRX) $\longrightarrow$ F where R is a new state).

The remaining open question is : "What is the minimal number of states for a minimal time solution of the firing squad ?".

We know now that this number is 5 or 6.

REFERENCES.

[1]. BALZER R.,

"An 8-states minimal time solution to the firing squad synchronization problem",

Information and Control, 10, pp.22-42, (1967).

[2]. MAZOYER J.,

"Solutions à coupures périodiques itérées du firing squad",

To appear.

[3]. MOORE E.F.,

"Sequantial machines, Selected papers",

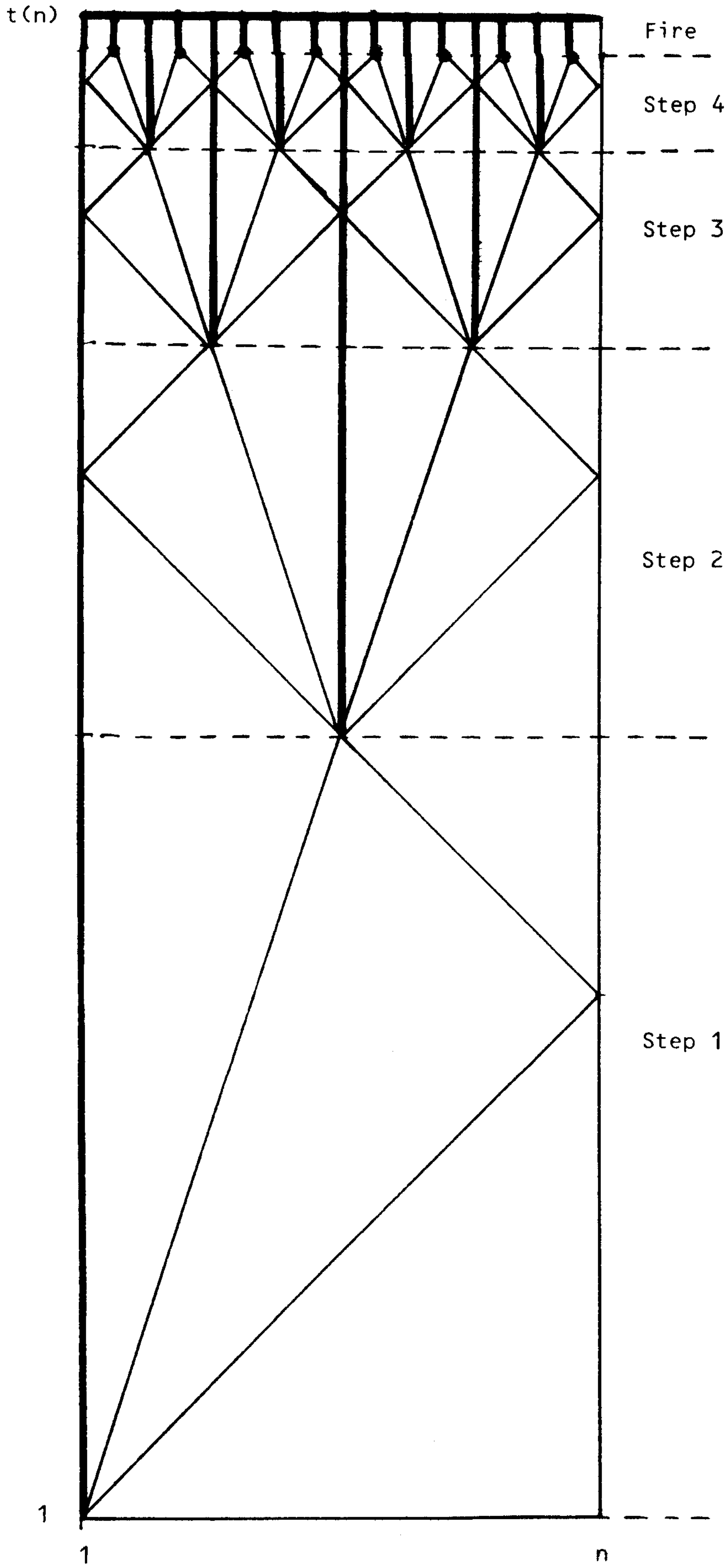
pp.213-214. Addison Wesley Reading Mass., (1964).

[4]. WAKSMAN A.,

"An optimum solution to the firing squad synchronization problem",

Information and Control, 8, pp.66-78, (1966).

Figure 1.6



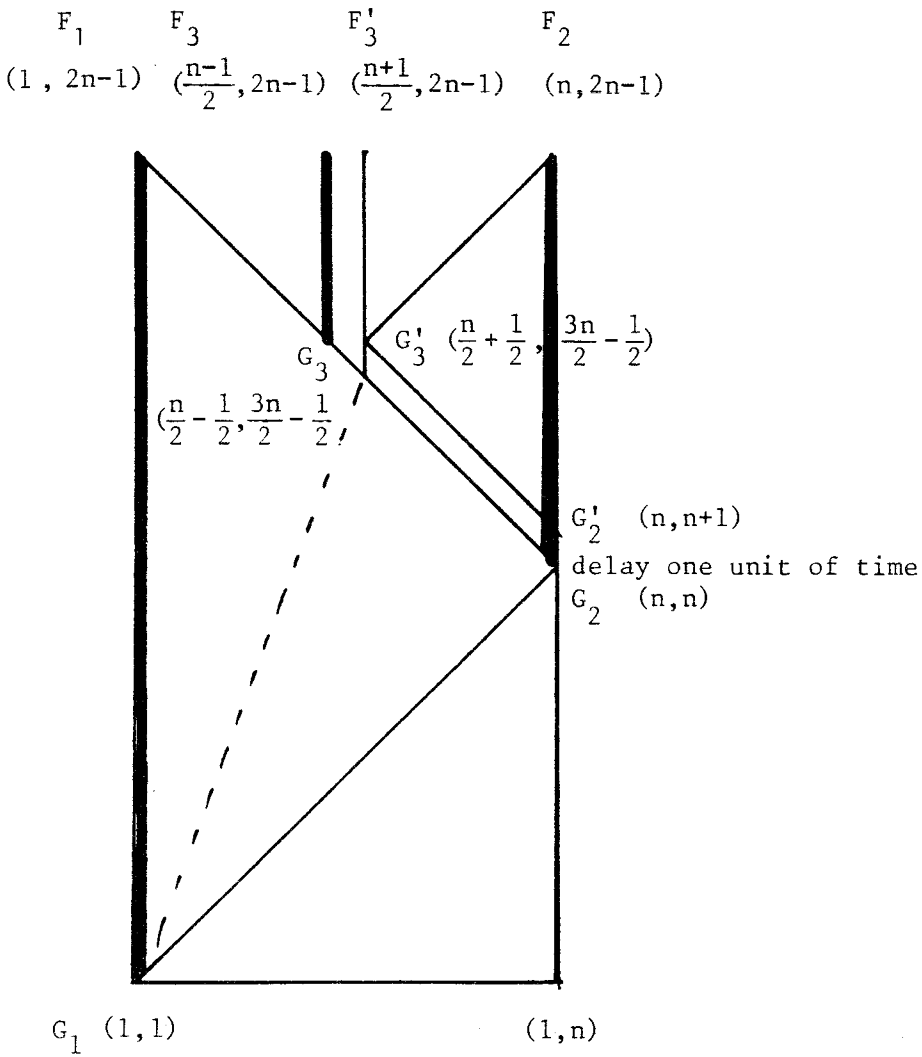


Figure 1.7 b) : n is even

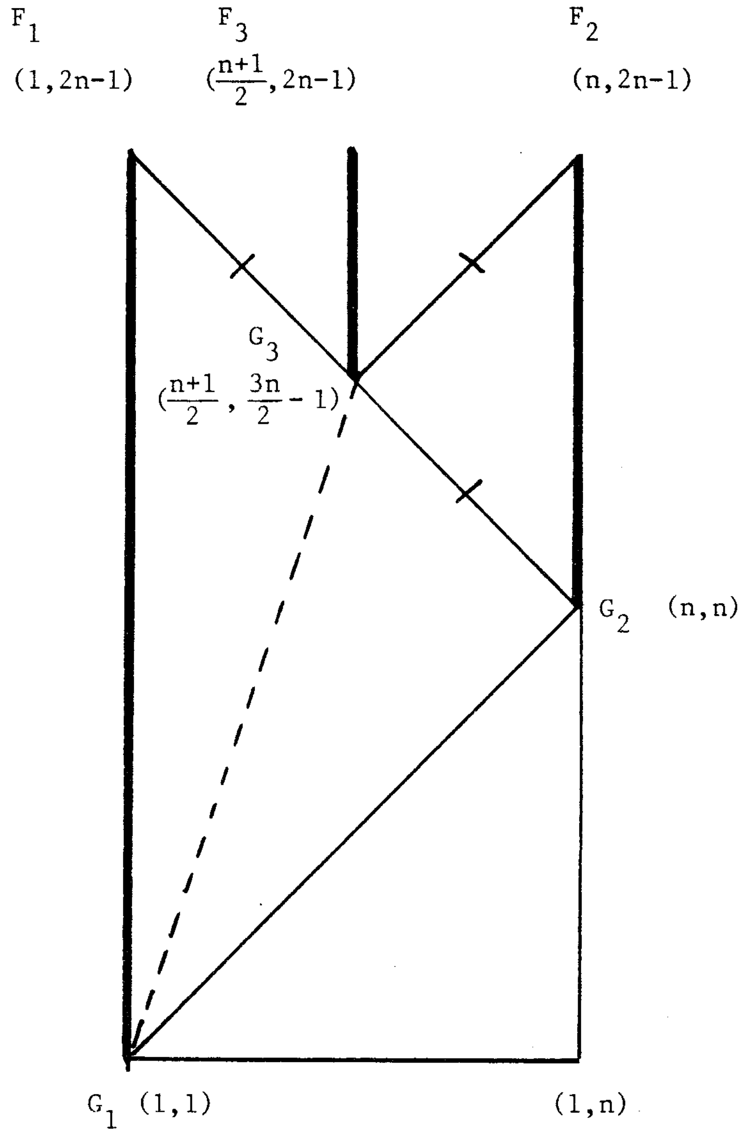


Figure 1.7 a) : n is odd

Figure 1.7 c)

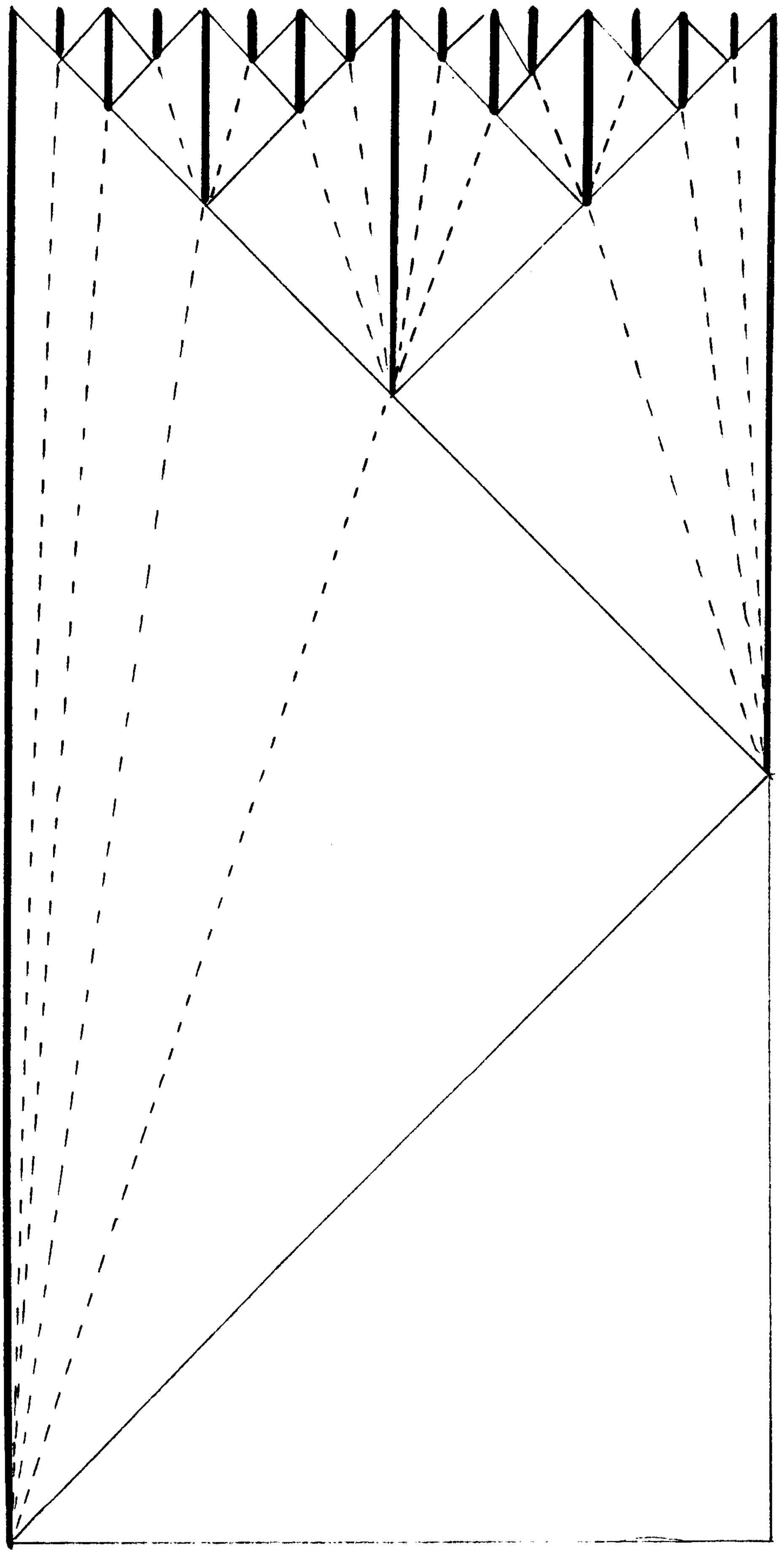
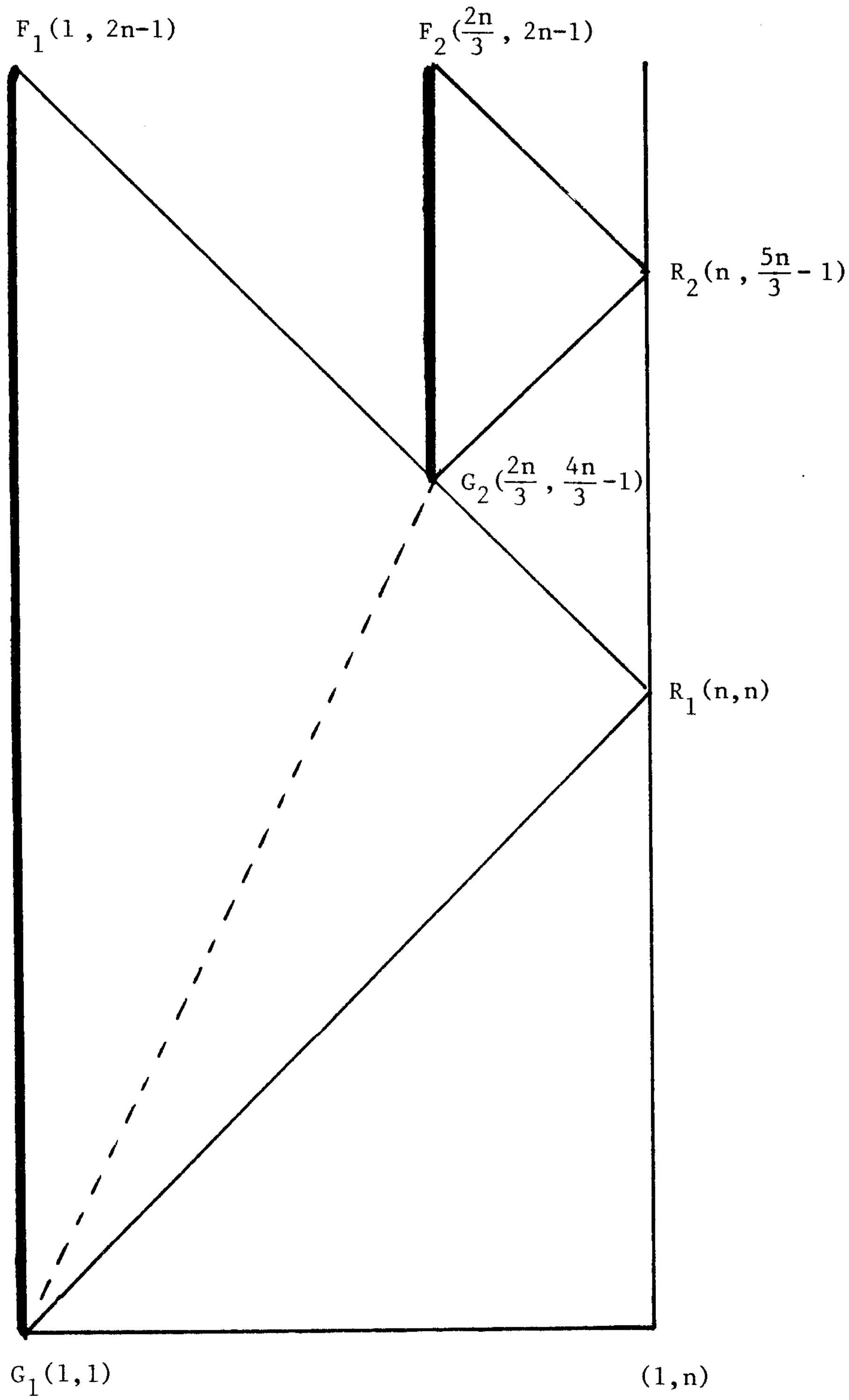


Figure 2.1 a)



$F_1(1\ ,\ 2n-1)$
$F_2(\frac{2n}{3}\ ,\ 2n-1)$
$R_2(n\ ,\ \frac{5n}{3}-1)$
$G_2(\frac{2n}{3}\ ,\ \frac{4n}{3}-1)$
$R_1(n,n)$
$G_1(1,1)$
$(1,n)$

Figure 2.1 b)

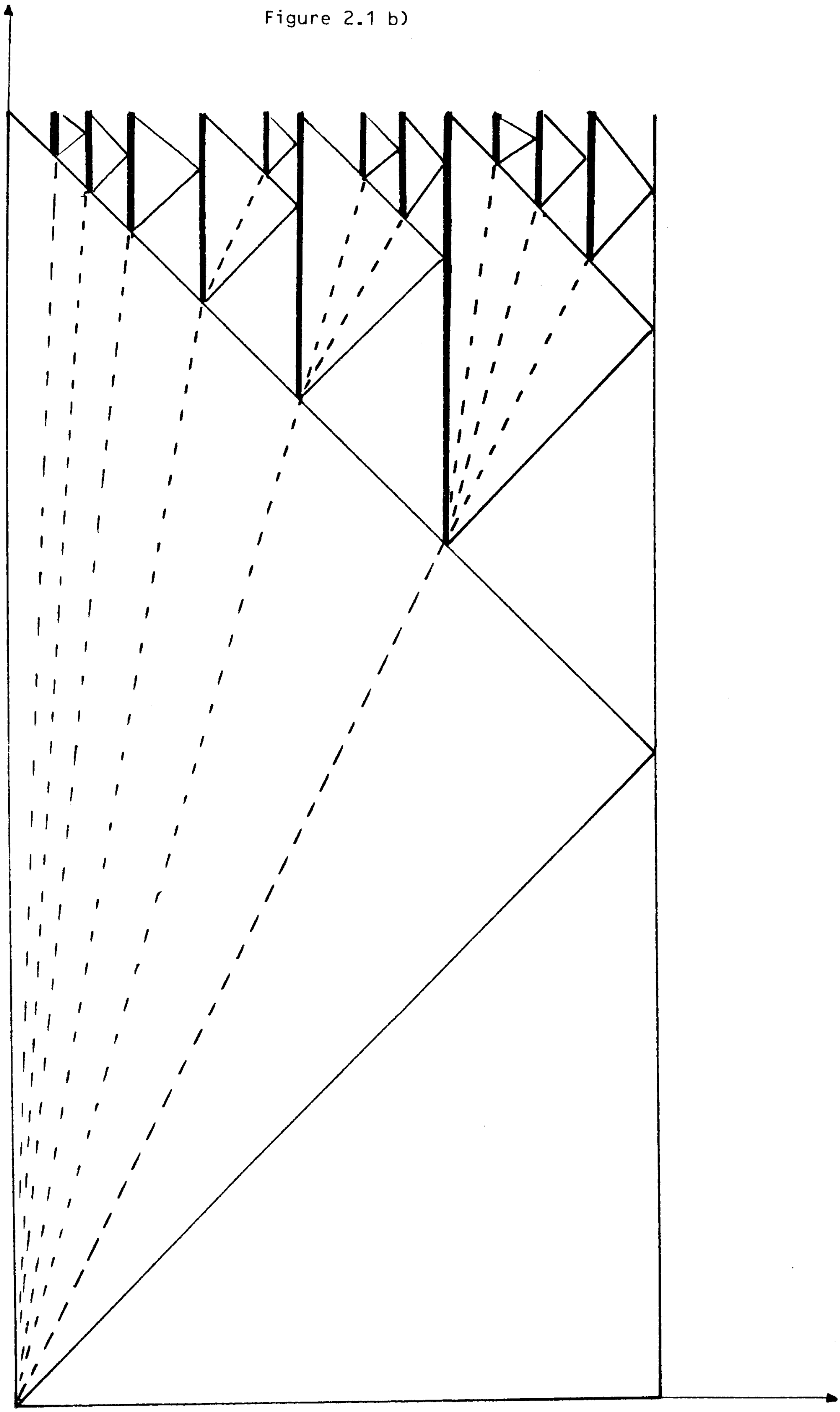


Figure 2.2 ⊠ denote initial waves and reflection of initial waves.

Case n = 3p+1 : delay 0

Case n = 2p+2 : delay 1

Case n = 3p+3 : delay 2

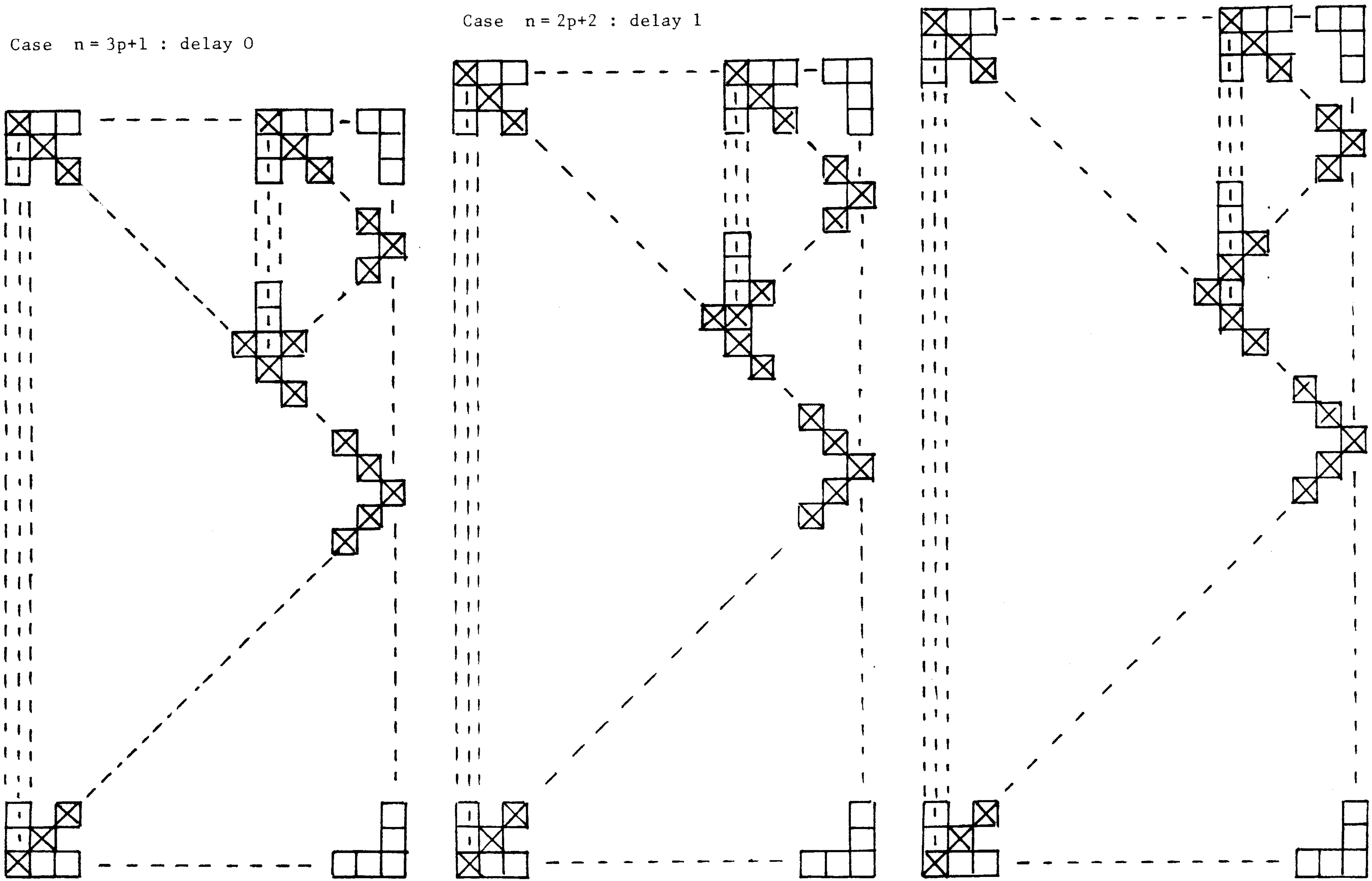


Figure 3.2

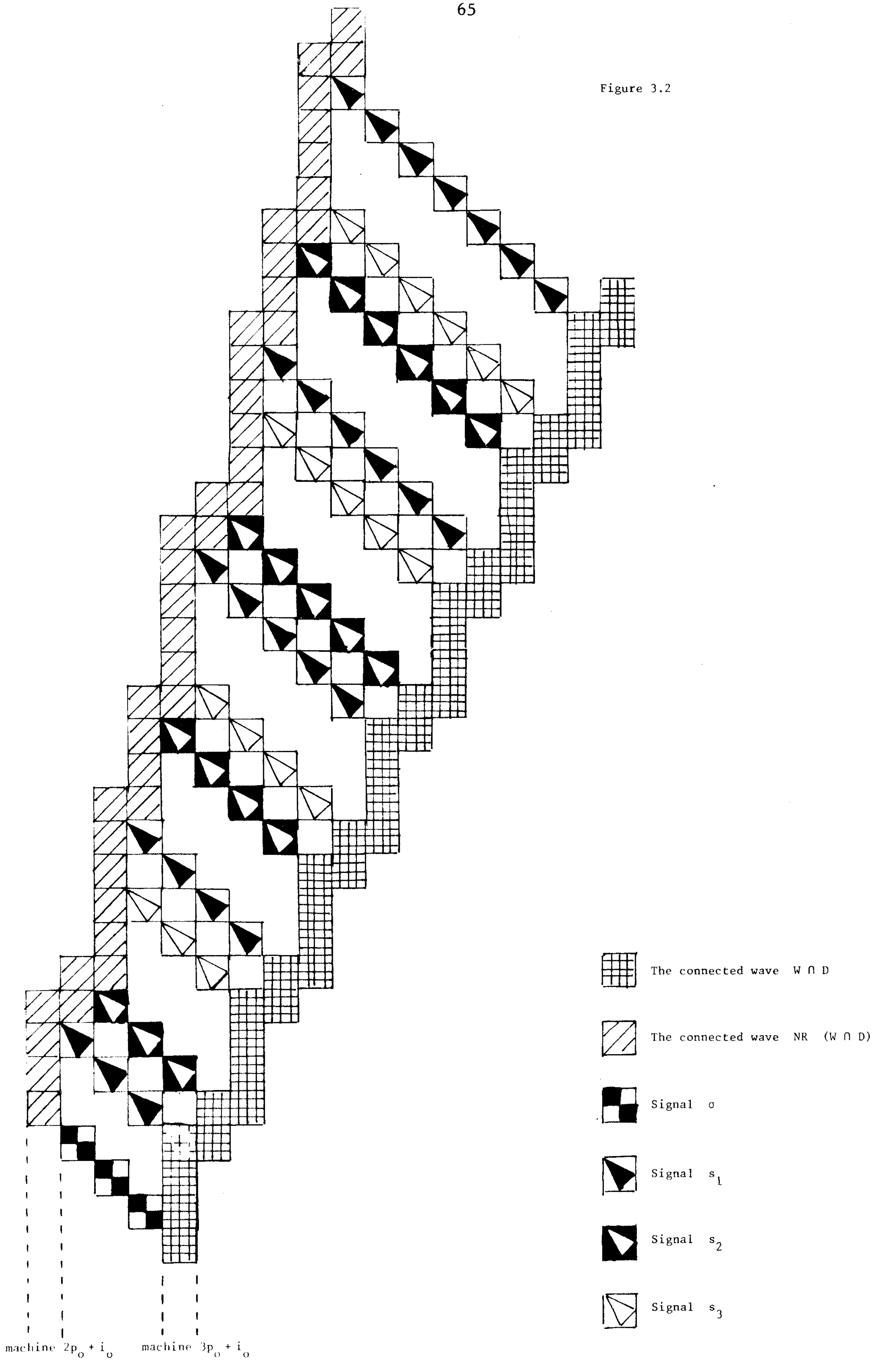
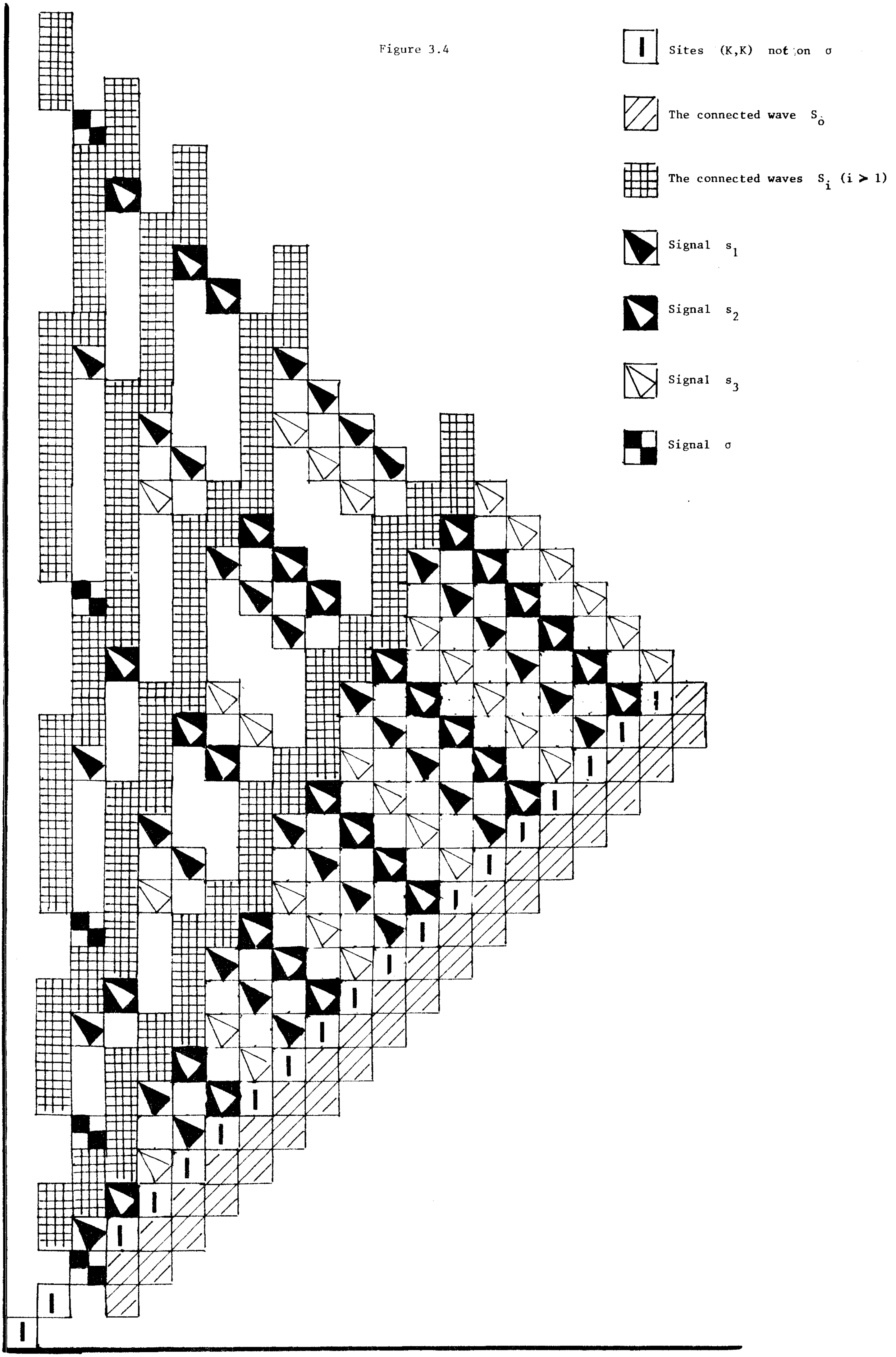
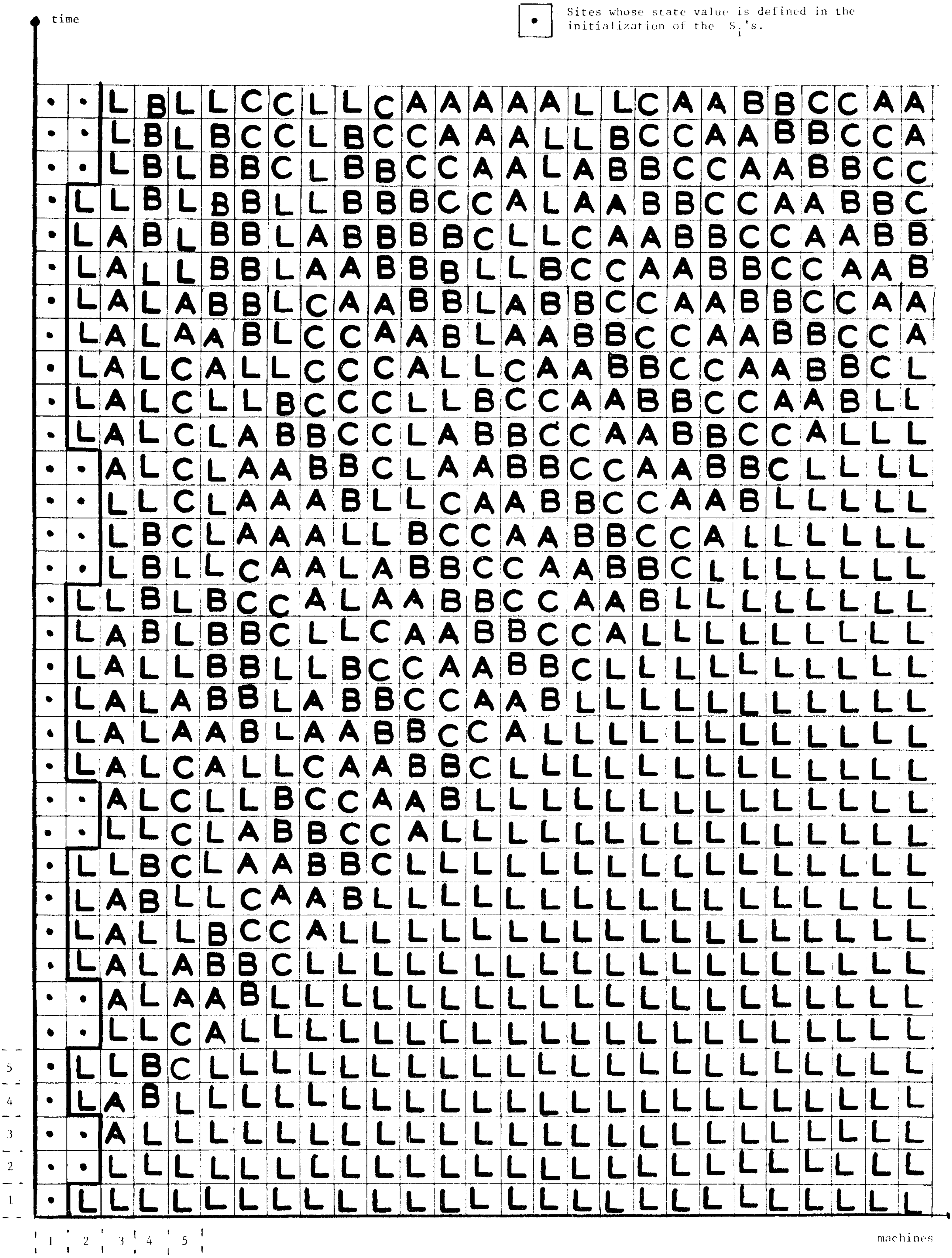

The connected wave W ∩ D
The connected wave NR (W ∩ D)
Signal σ
Signal s_1
Signal s_2
Signal s_3
machine $2p_o + i_o$
machine $3p_o + i_o$

Figure 3.4



Sites (K,K) not on σ
The connected wave S_0
The connected waves S_i (i > 1)
Signal s_1
Signal s_2
Signal s_3
Signal σ

Figure 4.2



Sites whose state value is defined in the initialization of the S_i's.
time
machines
1
2
3
4
5

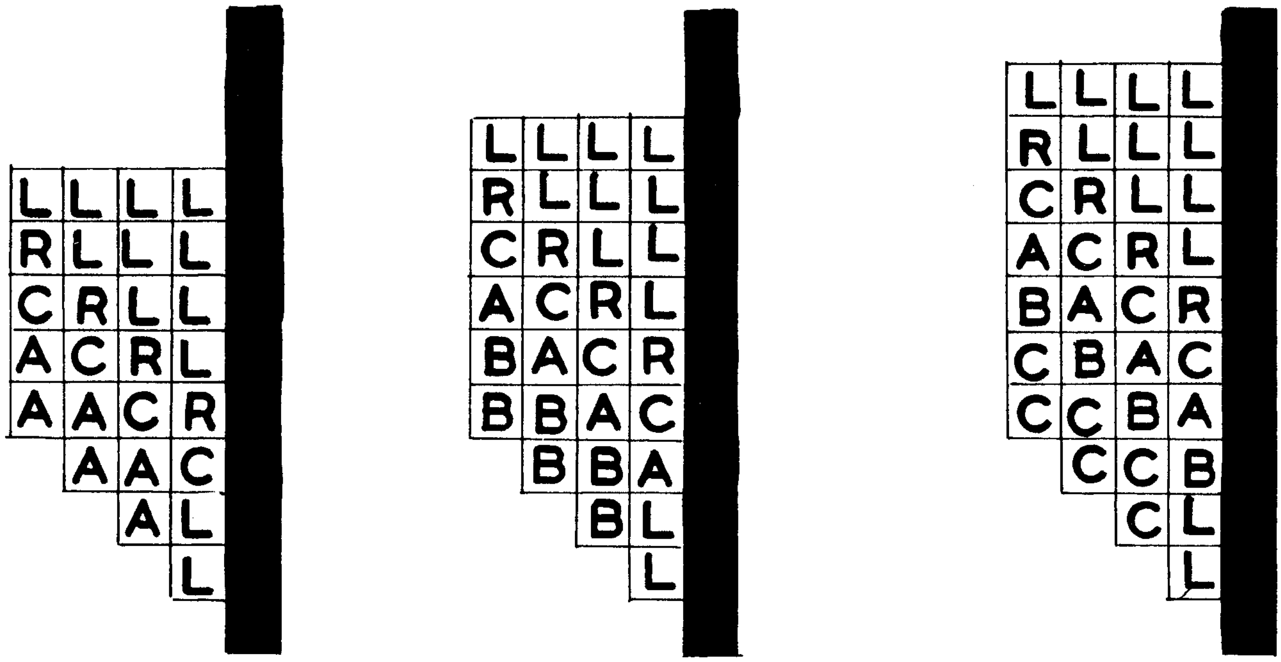
Figure 4.4

Sites and state-value of the fictive machine numbered 0 are not indicated.

time

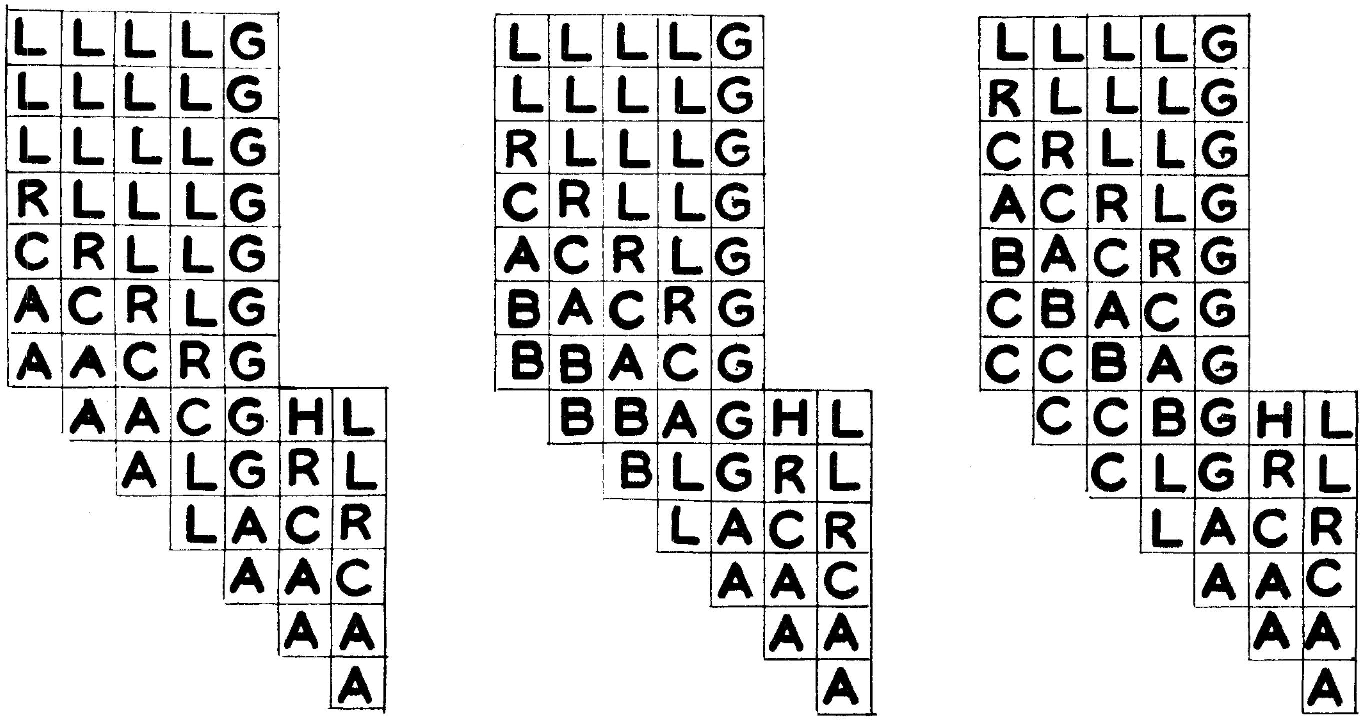
G	L	A	L
G	L	A	L
G	L	A	L
G	H	A	L
G	H	L	L
G	H	L	B
G	H	L	B
G	L	L	B
G	L	A	B
G	L	A	L
G	L	A	L
G	L	A	L
G	L	A	L
G	H	A	L
G	H	L	L
G	L	L	B
G	L	A	B
G	L	A	L
G	L	A	L
G	H	A	L
G	H	L	L
G	L	L	B
G	L	A	B
G	H	A	L
G	H	L	L
G	L	L	L
1	2	3	4

machine numbered K ($1 < K < 4$)



delay 0 delay 1 delay 2

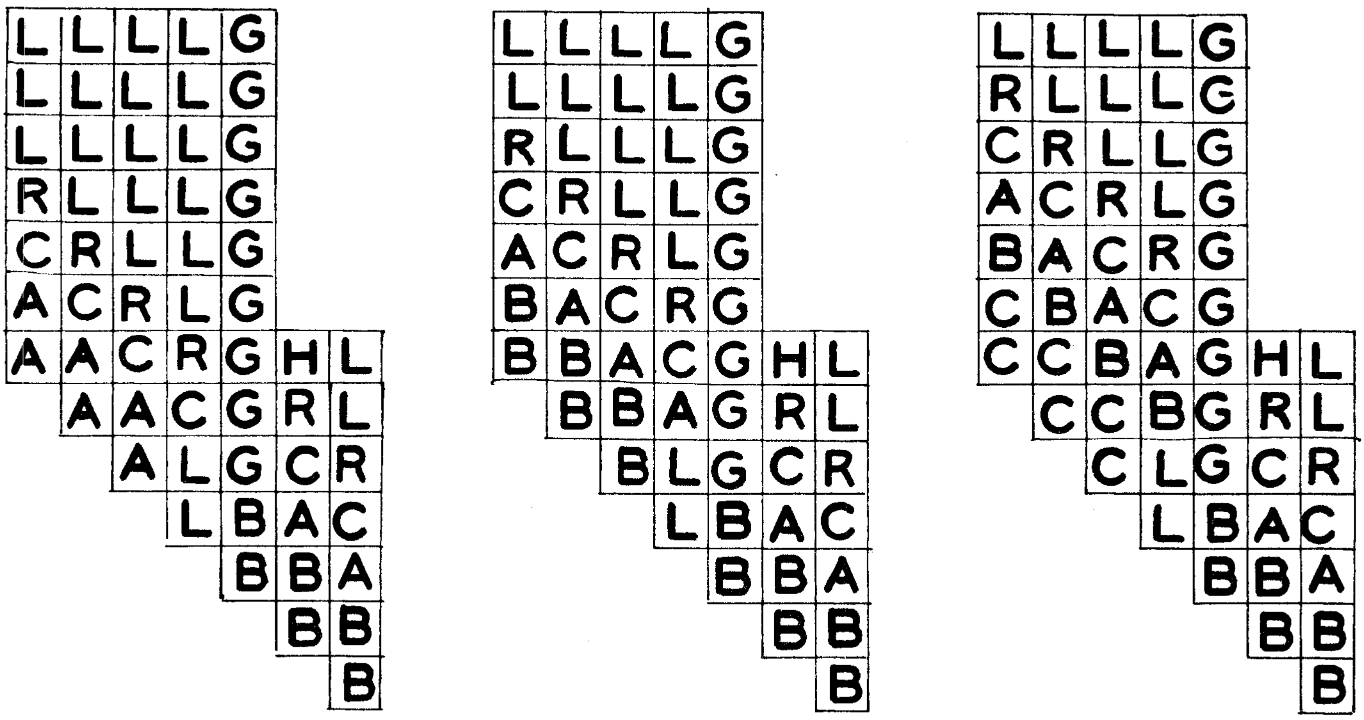
Figure 4.5 ■ State G or X.



The reflected signal progressing leftwards conveys :

delay 0 | delay 1 | delay 2

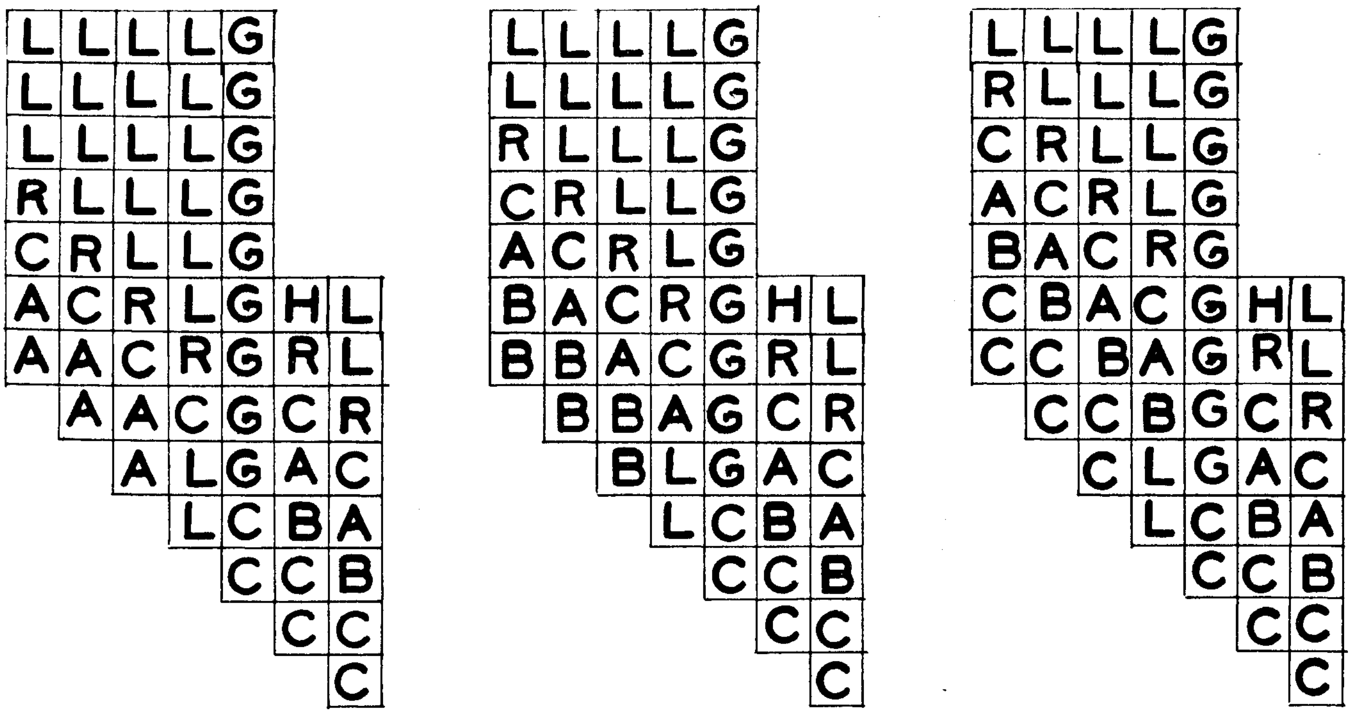
Figure 4.6 : The reflected signal coming from the right conveys delay 0



The reflected signal progressing leftwards conveys :

delay 0 | delay 1 | delay 2

Figure 4.6. : The reflected signal coming from the right conveys delay 1.



The reflected signal progressing leftwards conveys :

delay 0 delay 1 delay 2

Figure 4.6 : The reflected signal coming from the right conveys delay 2.

Figure 4.9

X	F	F	X
X	G	G	X
X	G	L	X

Case a_2

X	F	F	F	X
X	G	G	G	X
X	G	H	H	X
X	G	H	L	X
X	G	L	L	X

Case a_3

t = 2n-1 Initial line of length two or three.

•	•	F	X
•	G	G	X
•	G	R	X

Case b_2

•	•	F	F	X
•	G	G	G	X
•	G	H	H	X
•	G	H	L	X
•	G	R	L	X

Case b_3

t = 2n-1 Right-most created line of length two or three.

•	•	F	•	•
•	G	G	G	•
•	G	R	G	•

Case c_2

•	•	F	F	•	•
•	G	G	G	G	•
•	G	H	H	G	•
•	G	H	L	G	•
•	G	R	L	G	•

Case c_3

t = 2n-1 Generic-created line of length two or three.

•	•	F	•	•
•	G	G	G	•
•	G	L	G	•
•	•	L	G	•

Case d_2

•	•	F	F	•	•
•	G	G	G	G	•
•	G	H	H	G	•
•	•	H	L	G	•
•	•	•	L	G	•

Case d_3

t = 2n-1 Remaining portion of a line of length two or three.

☐	Site whose state-value is known by hypothesis
•	Site whose state-value does not interfer with these alineas.
☐ (bold)	Site whose state-value nec the introduction of a new transition rule.

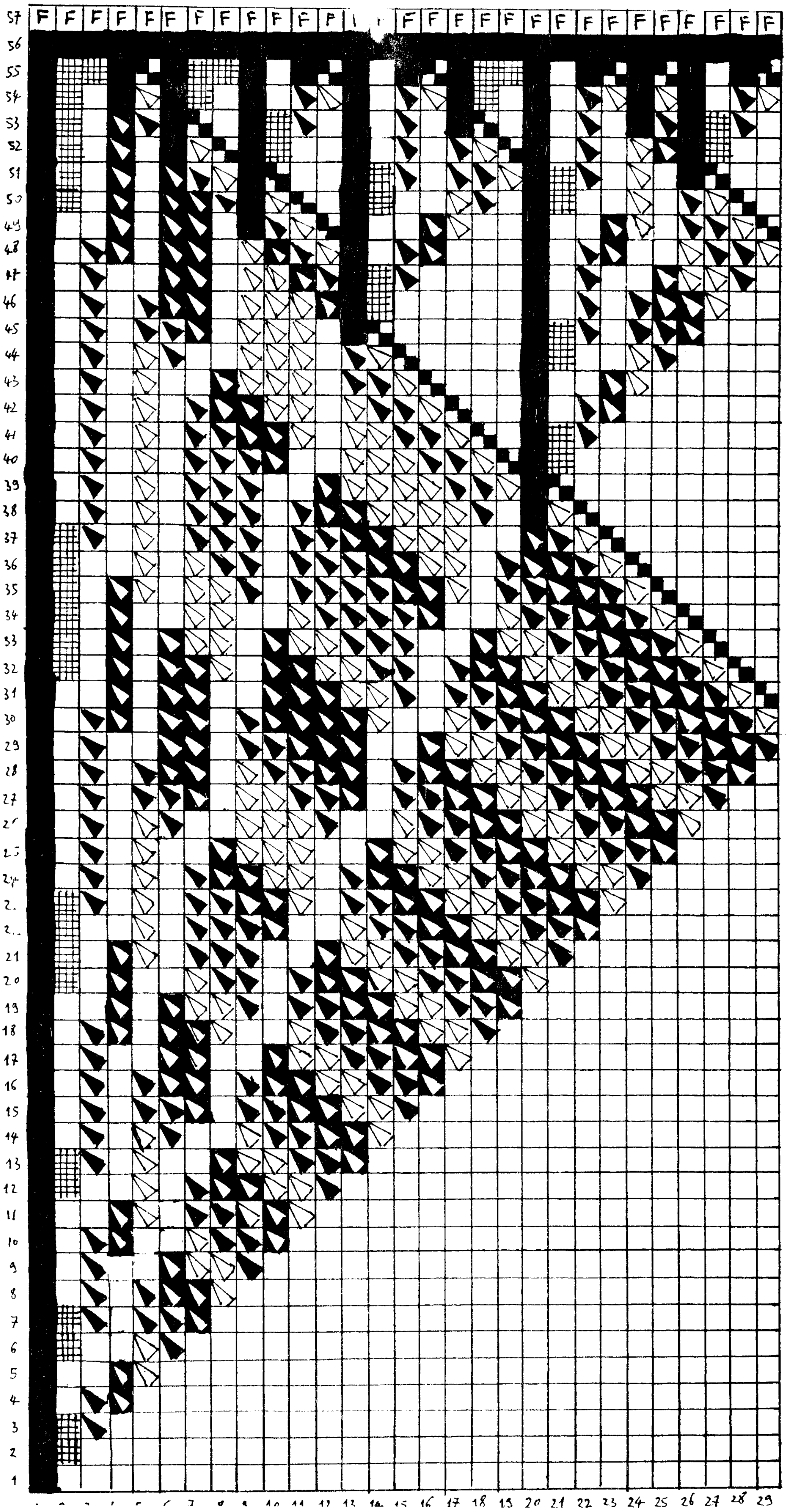


Figure 4.10

State L

State G

State H

State A

State B

State C

State R

State F

The fictive state X is not marked.

Observe that :

- machine 20 becomes general at time 38 and active at time 39 (delay 1).
- machine 13 becomes general at time 45 and active at time 45 (delay 0).
- machine 9 becomes general at time 49 and active at time 51 (delay 2).

L	X	L	A	B	C	R	H	G
X		L						
L	L	L		L	L			L
A	C	B	L	L	L			C
B	A	C	L	L	L			A
C	B	A	L	L	L			B
R	L	L						L
H	H	A		L				H
G	G	H	L					G

A	X	L	A	B	C	R	H	G
X								
L		A	A	L	G			
A		A	A	B	C			
B	C				C			C
C		A	A					
R								
H		A						
G	C				C			C

B	X	L	A	B	C	R	H	G
X								
L		B	G	B	L			
A		B		B				
B		B	A	B	C			
C	A		A					A
R								
H								
G								

C	X	L	A	B	C	R	H	G
X								
L		C	A	G	C			
A	R					R		R
B		C			C			
C		C	A	B	C			
R								
H								
G	R					R		R

R	X	L	A	B	C	R	H	G
X								
L								
A								
B								
C	L	L						L
R								
H								
G	G	H						G

H	X	L	A	B	C	R	H	G
X								
L								
A								
B								
C								
R								
H	G							G
G		H	L				G	

G	X	L	A	B	C	R	H	G
X		G					G	F
L		G	G		G	G	G	
A		G			G	G	G	
B		G			G	G	G	
C		G			G	G	G	
R		G				G	G	
H		G				G	G	
G	F							F

Figure 4.11

Figure 5.6

X	G	H	A	L	C
X	G	H	A	L	C
X	G	H	A	L	C
X	G	H	L	L	C
X	G	H	L	B	C
X	G	H	L	G	L
X	G	H	L	B	L
X	G	H	A	G	L
X	G	H	A	L	L
X	G	H	A	L	A
X	G	H	A	L	A
X	G	H	A	L	C
X	G	H	A	L	C
X	G	H	L	L	C
X	G	H	L	B	C
X	G	H	A	G	L
X	G	H	A	L	L
X	G	H	A	L	A
X	G	H	A	L	A
X	G	H	L	L	C
X	G	H	L	B	C
X	G	H	A	G	L
X	G	H	A	L	L
X	G	H	L	L	L
X	G	L	L	L	L

time 21

machine 3

Figure 5.6 a)

X	G	B	A	L	C
X	G	C	G	L	C
X	G	B	A	L	C
X	G	C	G	L	C
X	G	B	A	B	C
X	G	C	G	G	L
X	G	B	A	B	L
X	G	C	G	G	L
X	G	B	A	L	L
X	G	C	G	L	A
X	G	B	A	L	A
X	G	C	G	L	C
X	G	B	A	L	C
X	G	C	G	L	C
X	G	B	A	B	C
X	G	C	G	G	L
X	G	B	A	L	L
X	G	C	G	L	A
X	G	B	A	L	A
X	G	C	G	L	C
X	G	B	A	B	C
X	G	C	G	G	L
X	G	B	A	L	L
X	G	C	L	L	L
X	G	L	L	L	L

Figure 5.6 b)

L	X	L	A	B	C	G
X		L				
L	L	L		L	L	L
A	C	G	L	L	L	C
B	L	L	L	L	L	L
C	G	A	L	L	L	G
G	A	C	L	L	L	A

A	X	L	A	B	C	G
X			F		G	
L			A	L	G	
A	F	A	A	B	C	B
B	C	G		G	C	C
C		A	A			
G	C				C	C

B	X	L	A	B	C	G
X						
L			G	B	L	B
A		G	B	B	L	
B		G	A	B	C	B
C	L	L	A			L
G	G	C	C		B	G

C	X	L	A	B	C	G
X						
L		C	A	G	C	G
A	B	B		B		B
B	G	C			C	G
C		C	A	B	C	B
G	B	B		B		B

G	X	L	A	B	C	G
X		A		G	G	F
L			G	G	G	
A		B		G	G	
B	G	B		G	G	G
C	A	A		G	G	A
G	F	B		G	G	F

Figure 5.7

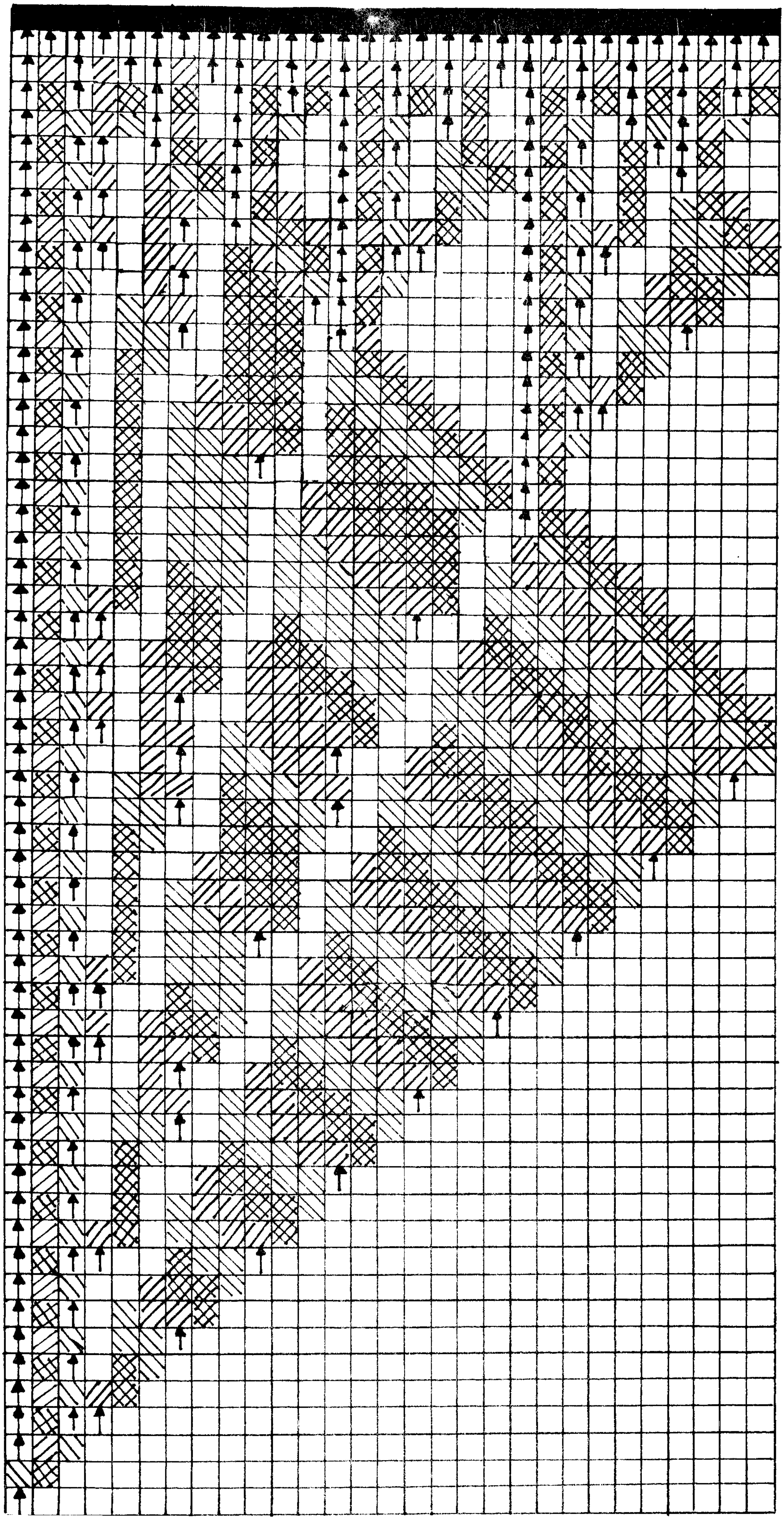


FIGURE 5.8

State A

State B

State C

State D

State E

State F

The fictive state X is not marked.

Figure 6.1.

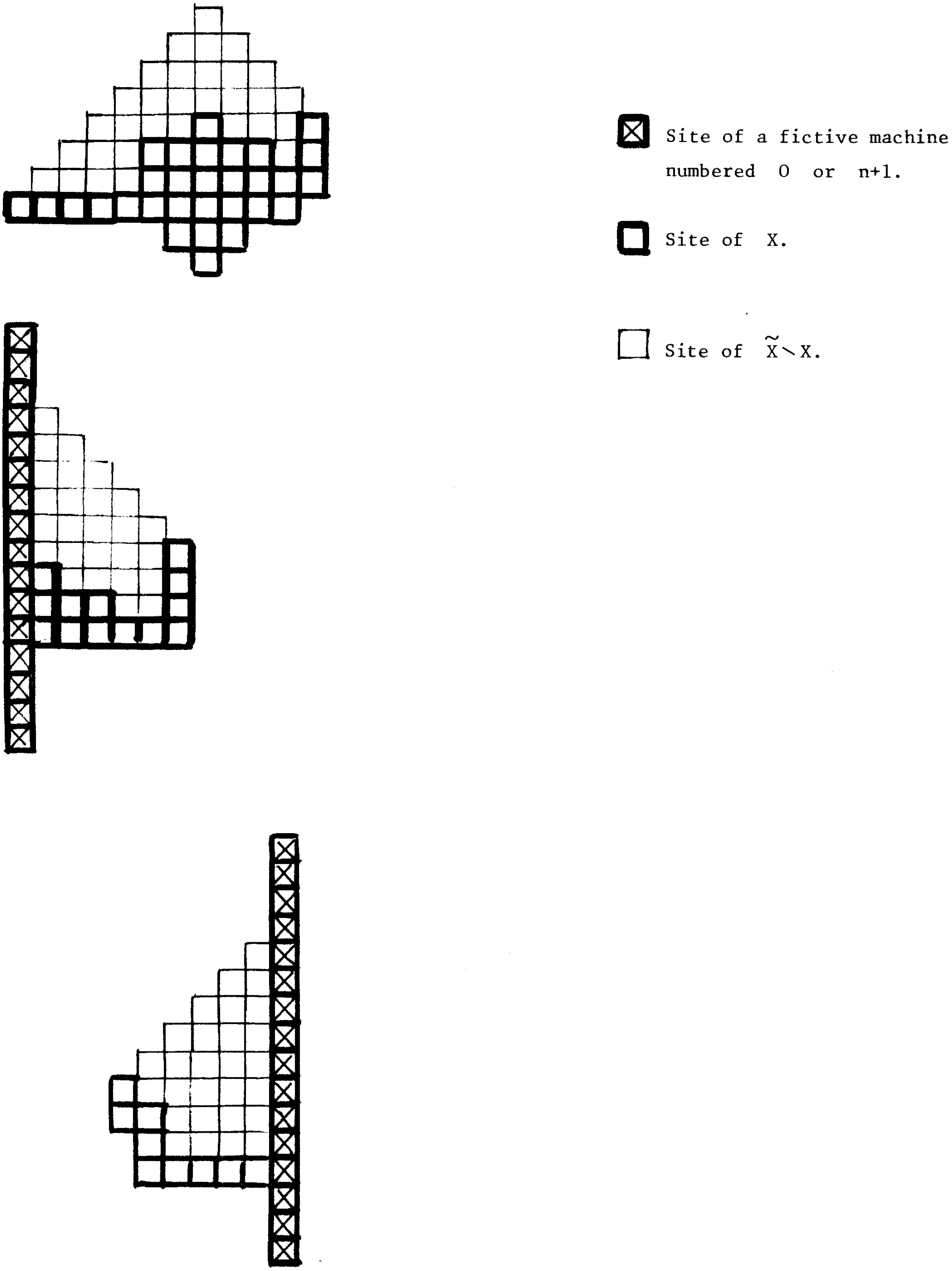
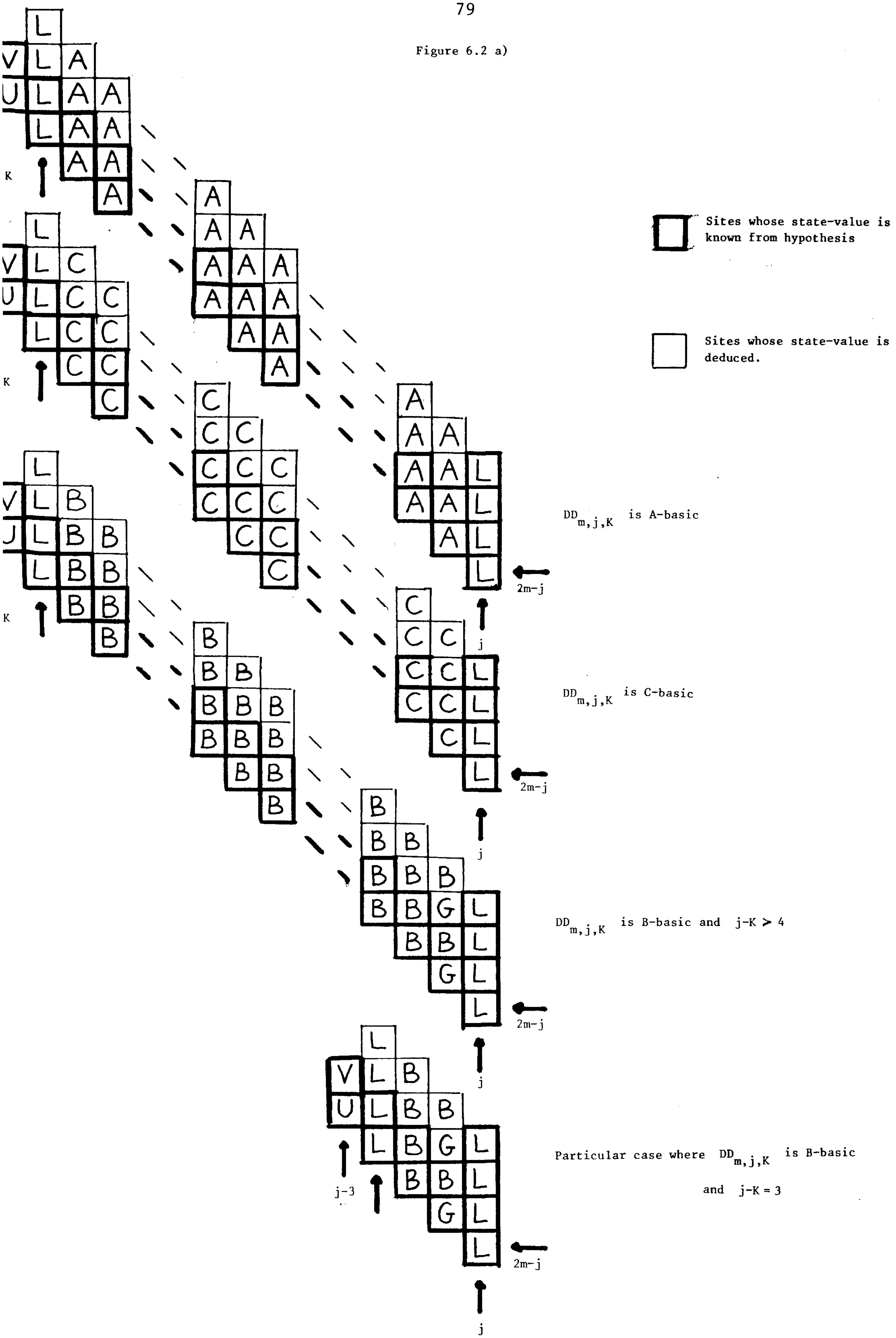

Site of a fictive machine numbered 0 or n+1.
Site of X.
Site of $\tilde{X} \setminus X$.

Figure 6.2 a)

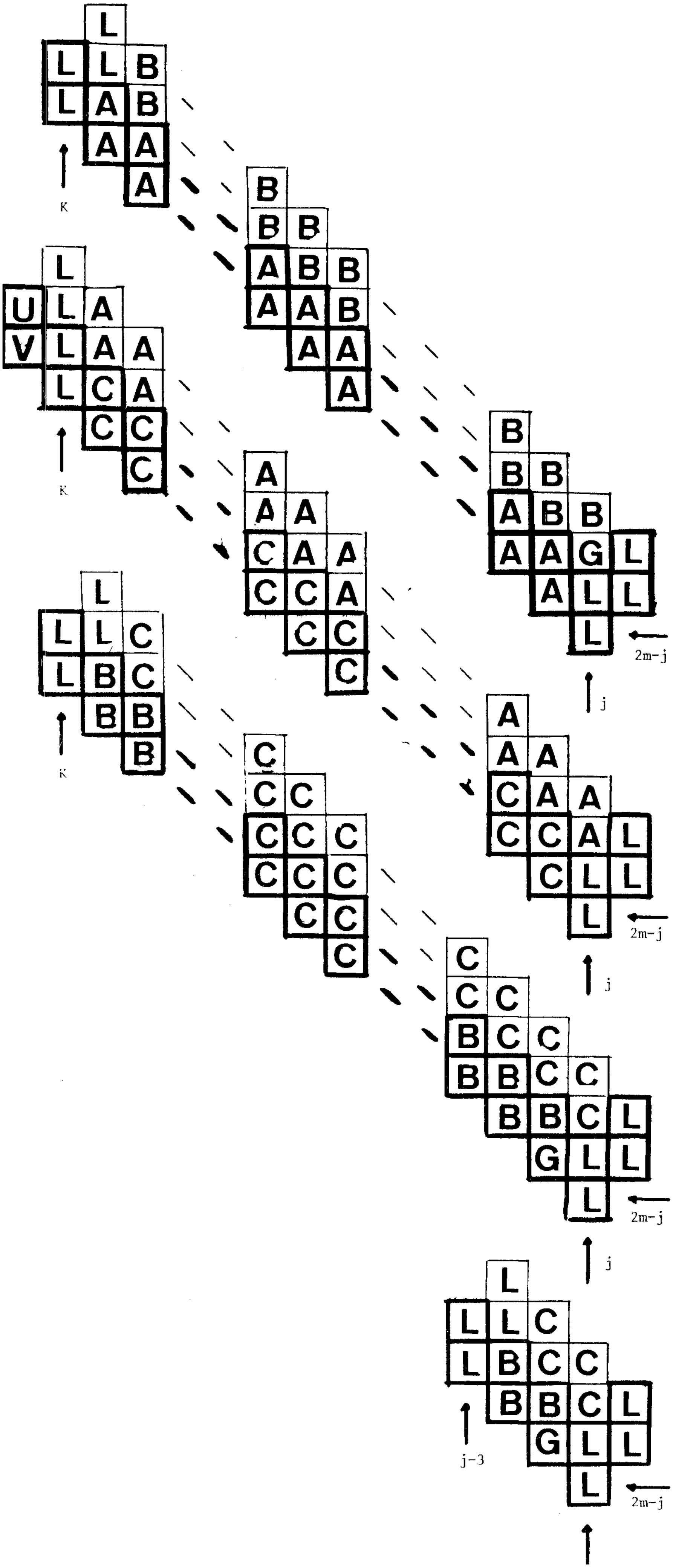


Sites whose state-value is known from hypothesis
Sites whose state-value is deduced.
$DD_{m,j,K}$ is A-basic
$DD_{m,j,K}$ is C-basic
$DD_{m,j,K}$ is B-basic and $j-K > 4$
Particular case where $DD_{m,j,K}$ is B-basic and $j-K = 3$
K
2m-j
j
j-3

Figure 6.2 b)

Sites whose state-value is known by hypothesis

Sites whose state-value is deduced.



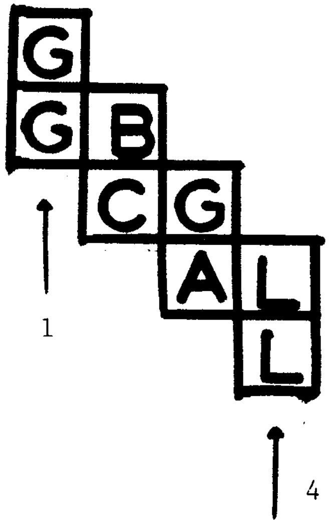
$DD_{m,j,K}$ is A-basic

$DD_{m,j,K}$ is C-basic

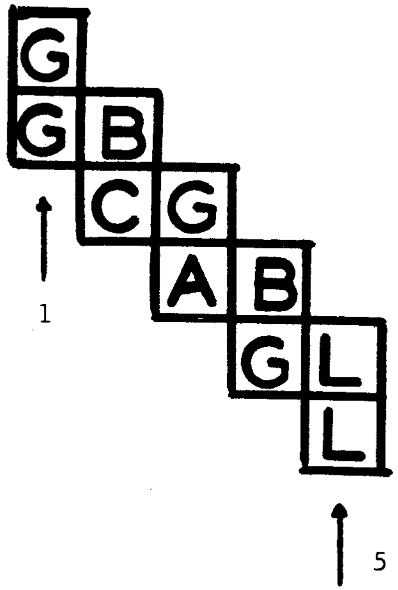
$DD_{m,j,K}$ is B-basic and $j > K+3$

Particular case : $DD_{m,j,j-3}$ is B-basic

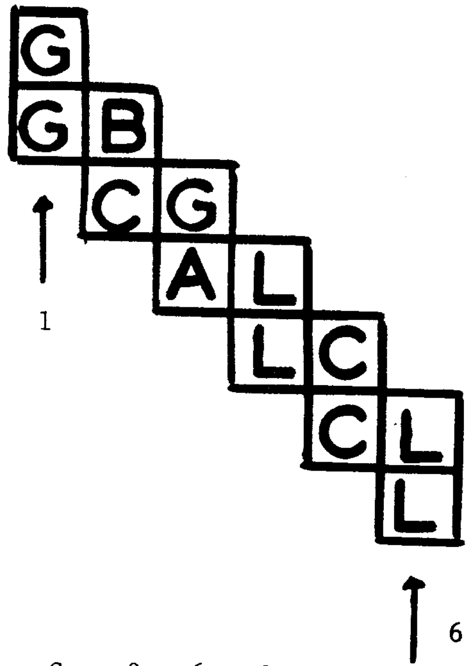
Figure 6.2 c)



Case 1 : 4-end

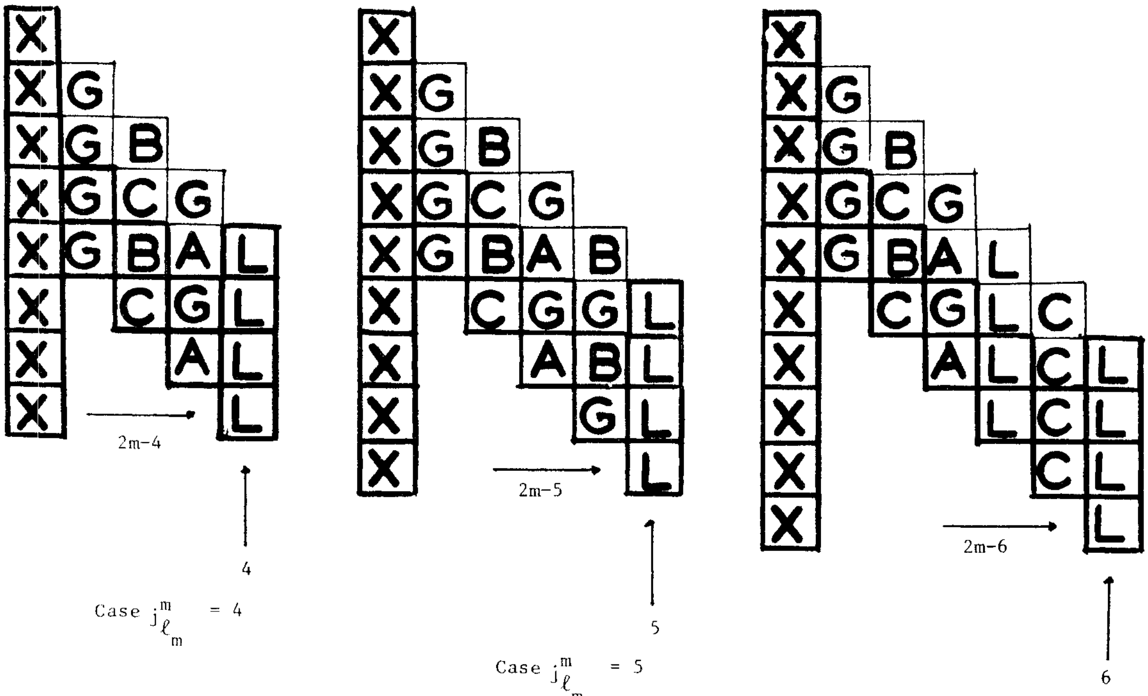


Case 2 : 5-end



Case 3 : 6-end.

Figure 6.2 d)

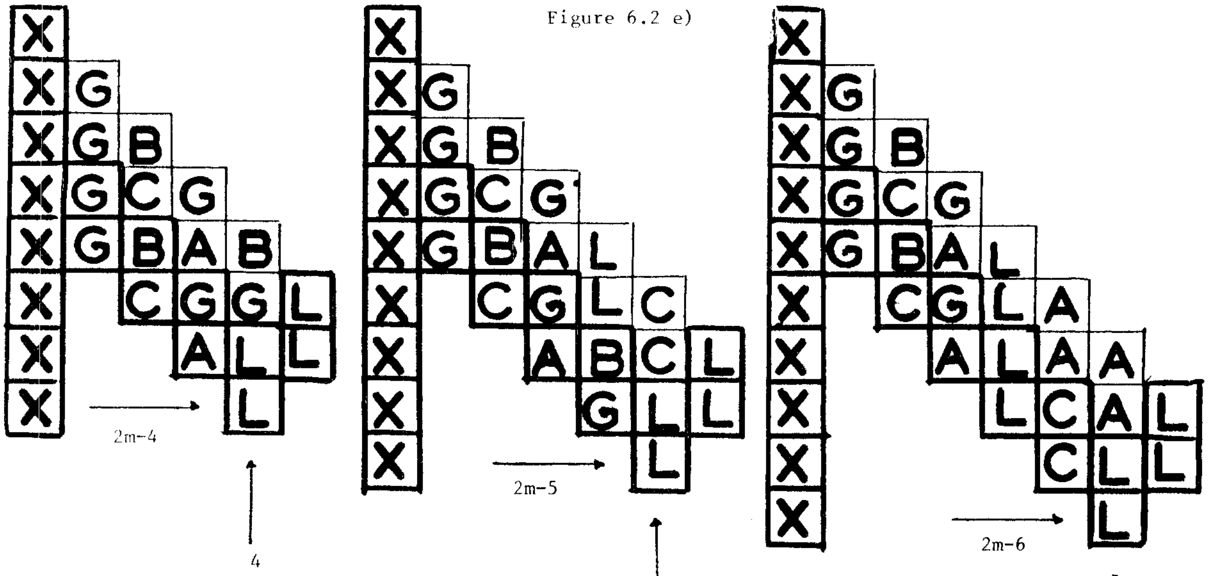


Case $j^m_{\ell_m} = 4$

Case $j^m_{\ell_m} = 5$

Case $j^m_{\ell_m} = 6$

Figure 6.2 e)



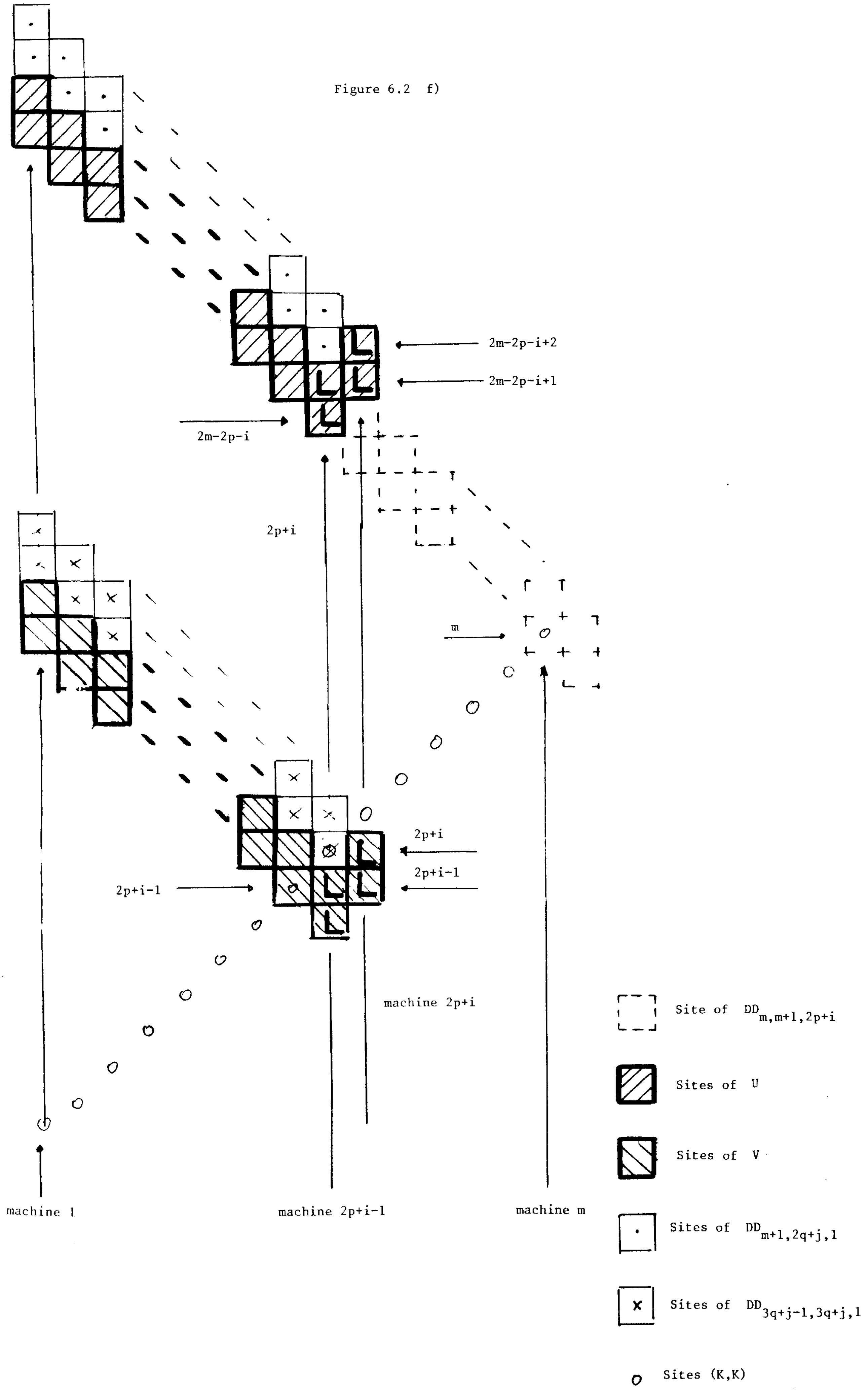
Case $j^m_{\ell_m} = 4$

Case $j^m_{\ell_m} = 5$

Case $j^m_{\ell_m} = 6$

Sites whose state-value is known by hypothesis

Sites whose state-value is deduced.



2m-2p-i+2
2m-2p-i+1
2m-2p-i
2p+i
m
2p+i
2p+i-1
2p+i-1
machine 2p+i
machine 1
machine 2p+i-1
machine m
Site of $DD_{m,m+1,2p+i}$
Sites of U
Sites of V
Sites of $DD_{m+1,2q+j,1}$
Sites of $DD_{3q+j-1,3q+j,1}$
Sites (K,K)

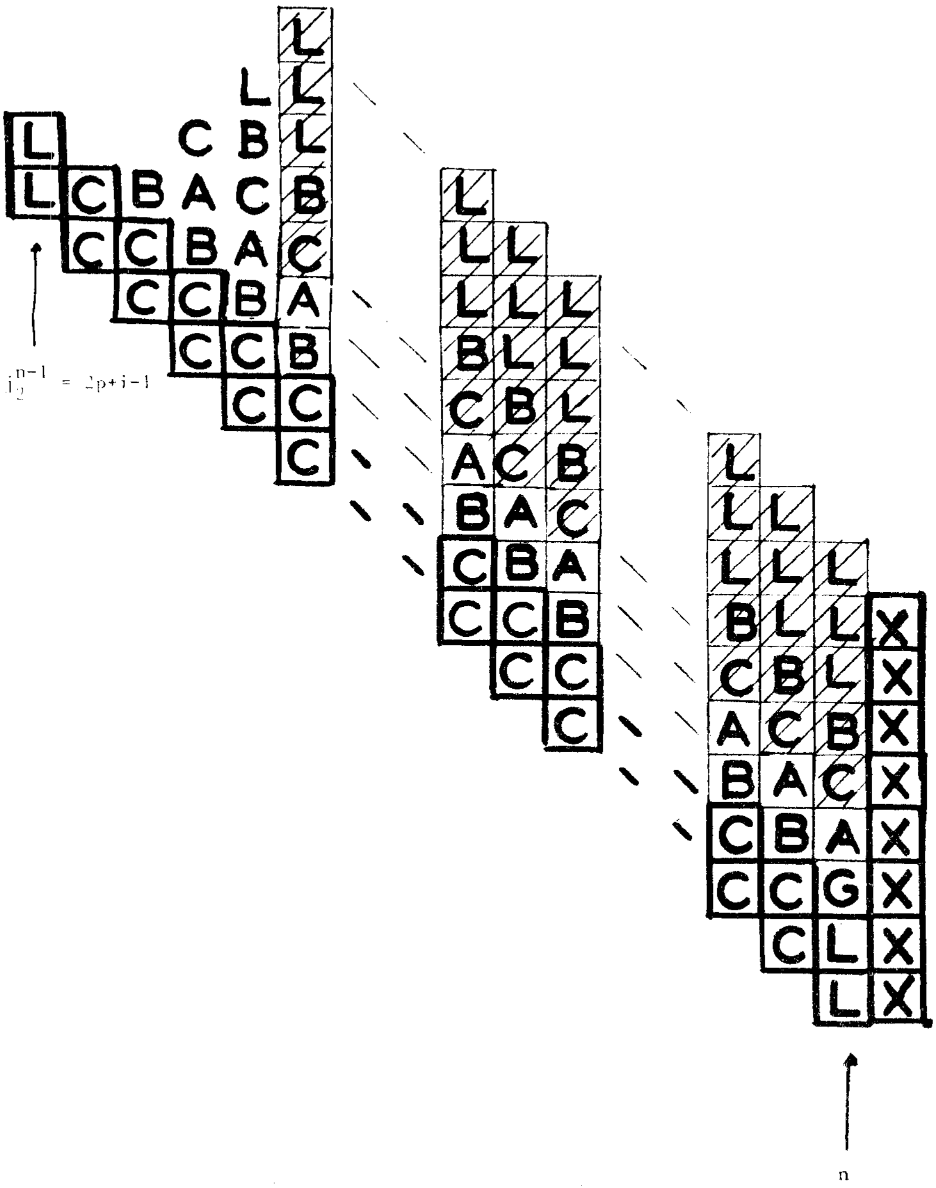
Figure 6.2 f)

Figure 6.3 a)

State whose state-value is known by hypothesis

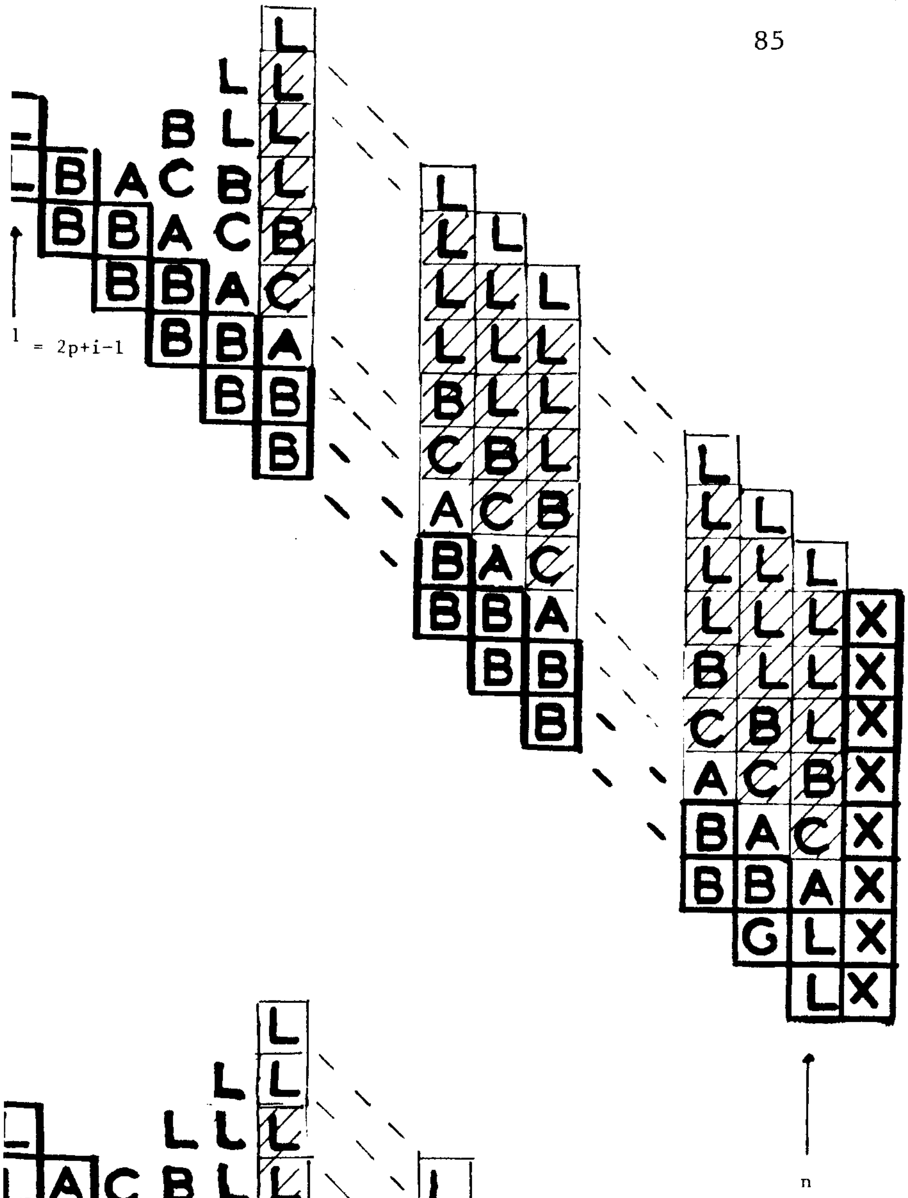
State in Z whose state-value is deduced

State in U ∖ Z whose state-value is deduced.



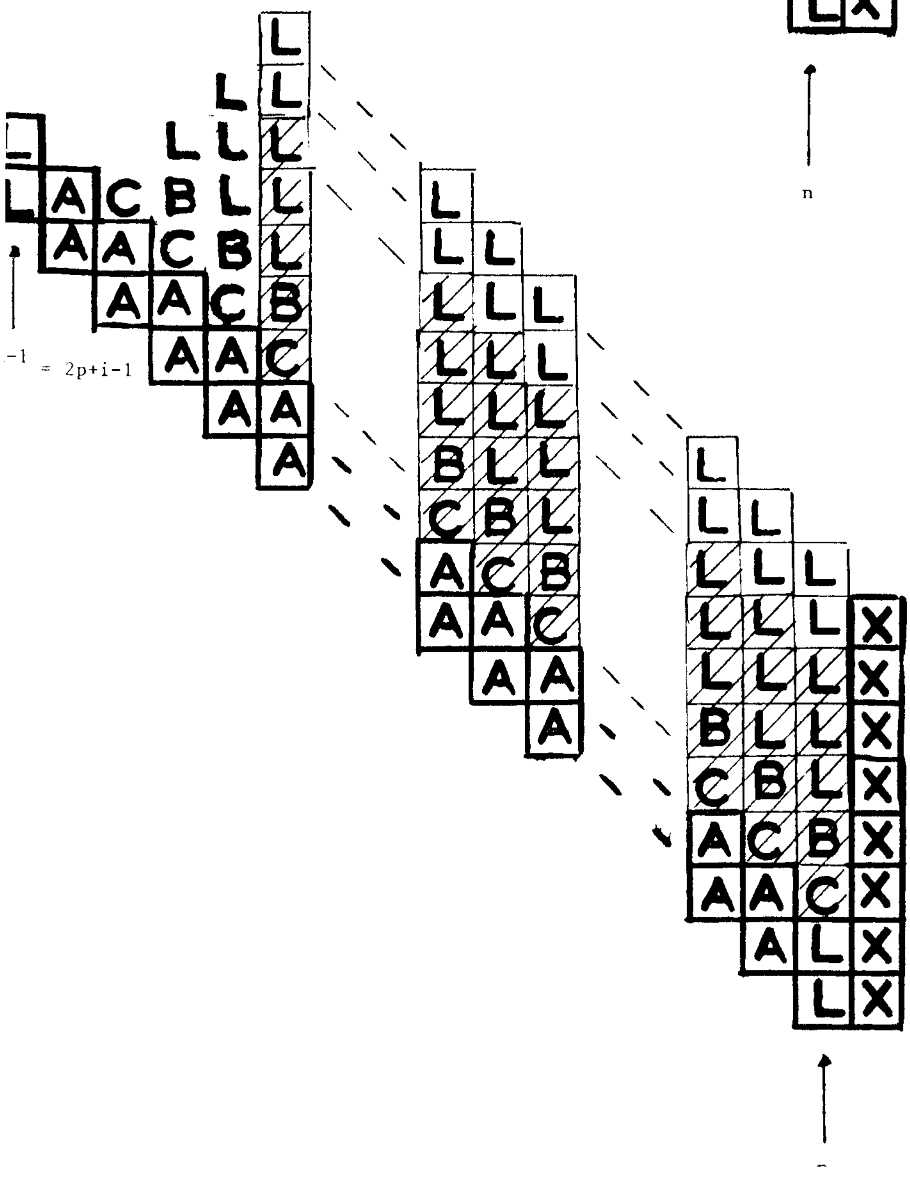
Case n ≡ 3 (mod 3)
(delay 2)

Figure 6.3 a)



Case $n \equiv 2 \pmod 3$

(delay 1)



Case $n \equiv 1 \pmod 3$

(delay 0)

Figure 6.3 b)

Sites of $DD_{n-1,n,j_2^{n-1}}$

Sites of U : their state-value is deduced from state-value of sites in $DD_{n-1,n,j_2^{n-1}}$

limit if interference between sites.

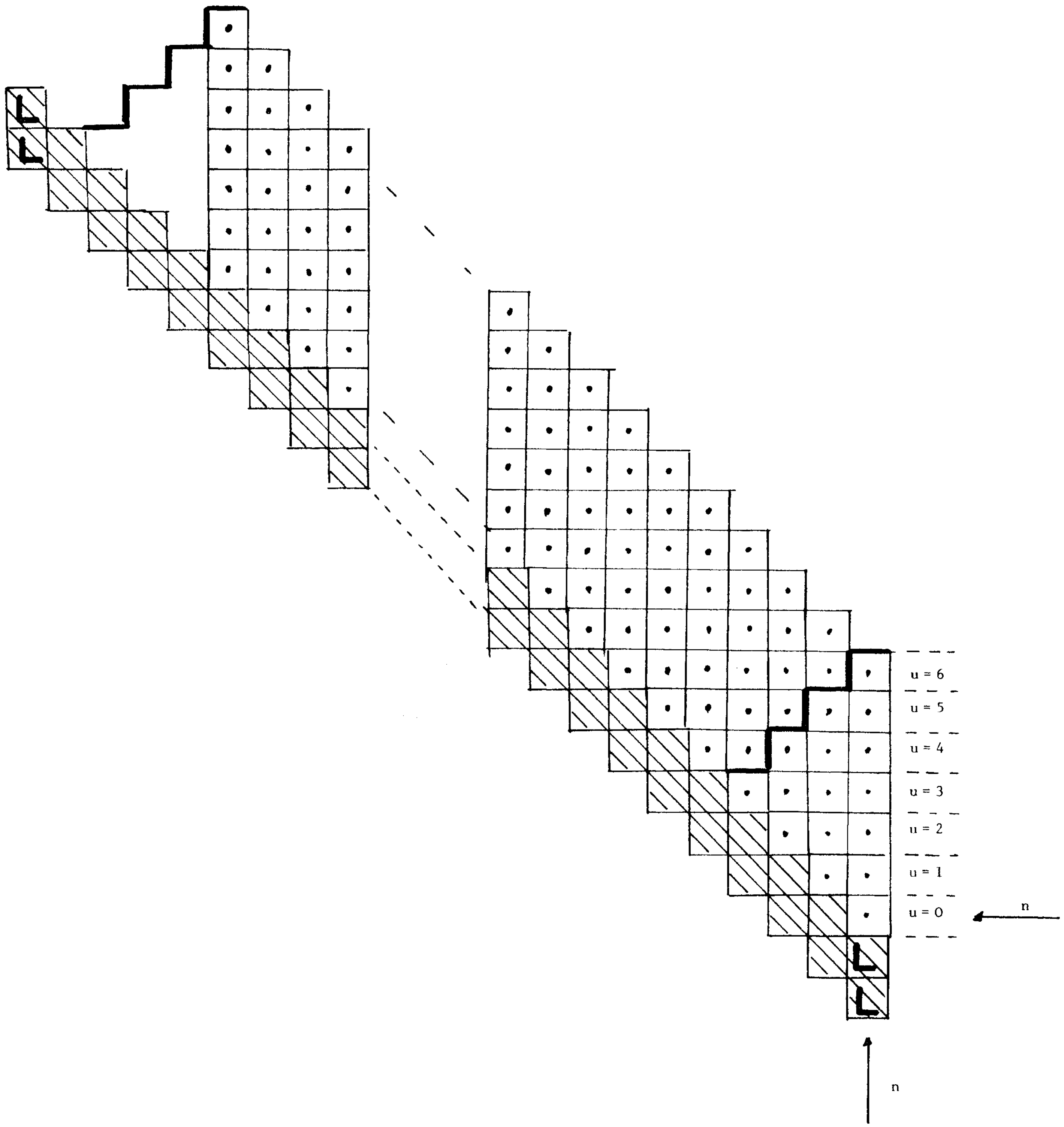
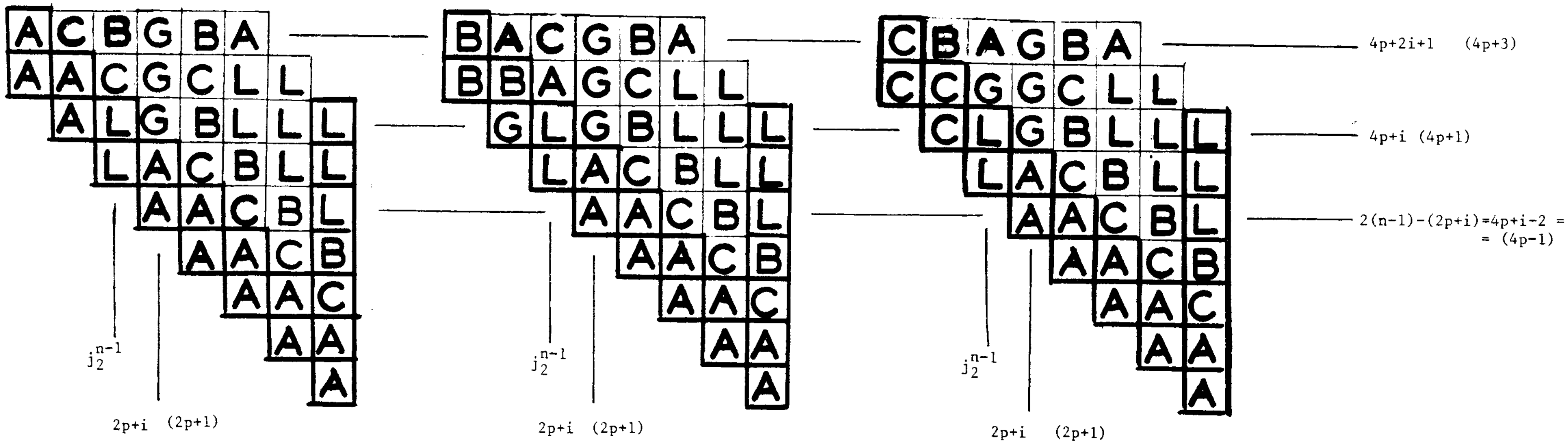


Figure 6.3 c).

Case $n = 3p+1$ (hence $i = 1$)

Sites whose state-value is known by hypothesis

Sites whose state-value is deduced.



Case : $DD_{n-1,j_2^{n-1},j_3^{n-1}}$ is A-basic

$j_2^{n-1} \equiv 1 \bmod (3)$

Case : $DD_{n-1,j_2^{n-1},j_3^{n-i}}$ is B-basic

$j_2^{n-1} \equiv 2 \bmod (3)$

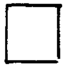
Case : $DD_{n-1,j_2^{n-1},j_3^{n-1}}$ is a C-basic

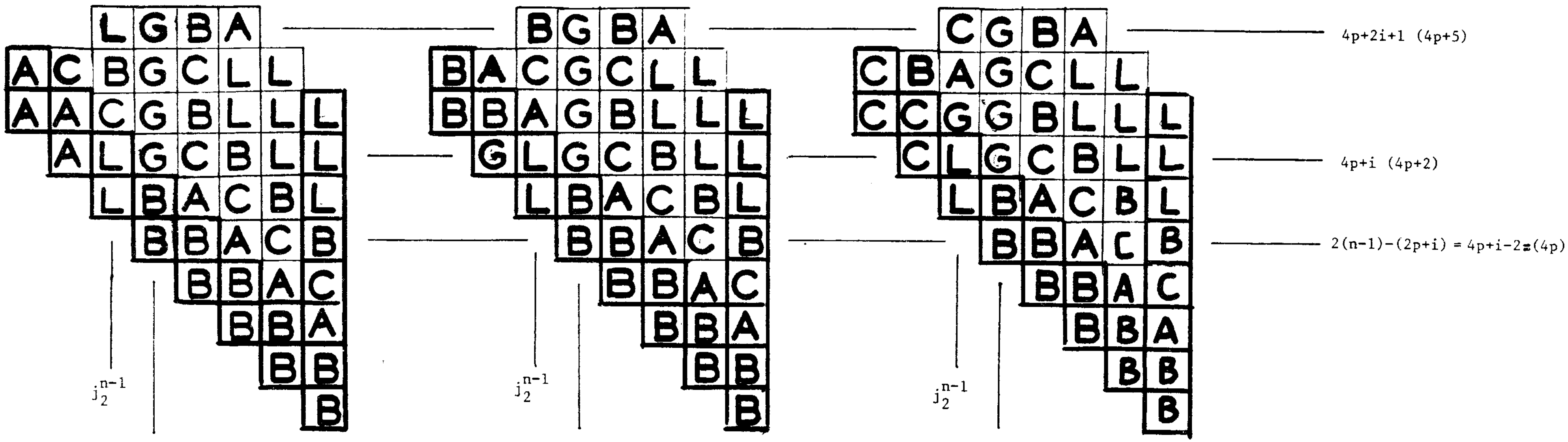
$j_2^{n-1} \equiv 3 \bmod (3)$

Case $n = 3p+2$ (hence $i = 2$).

Figure 6.3 c).


Sites whose state-value is known by hypothesis.


Sites whose state-value is deduced.



$2p+i$ $(2p+2)$

Case : DD$_{n-1,j_2^{n-1},j_3^{n-1}}$ is A-basic

$j_2^{n-1} \equiv 1 \bmod (3)$

$2p+i$ $(2p+2)$

Case : DD$_{n-1,j_2^{n-1},j_3^{n-1}}$ is B-basic

$j_2^{n-1} \equiv 2 \bmod (3)$

$2p+i$ $(2p+2)$

Case : DD$_{n-1,j_2^{n-1},j_3^{n-1}}$ is C-basic

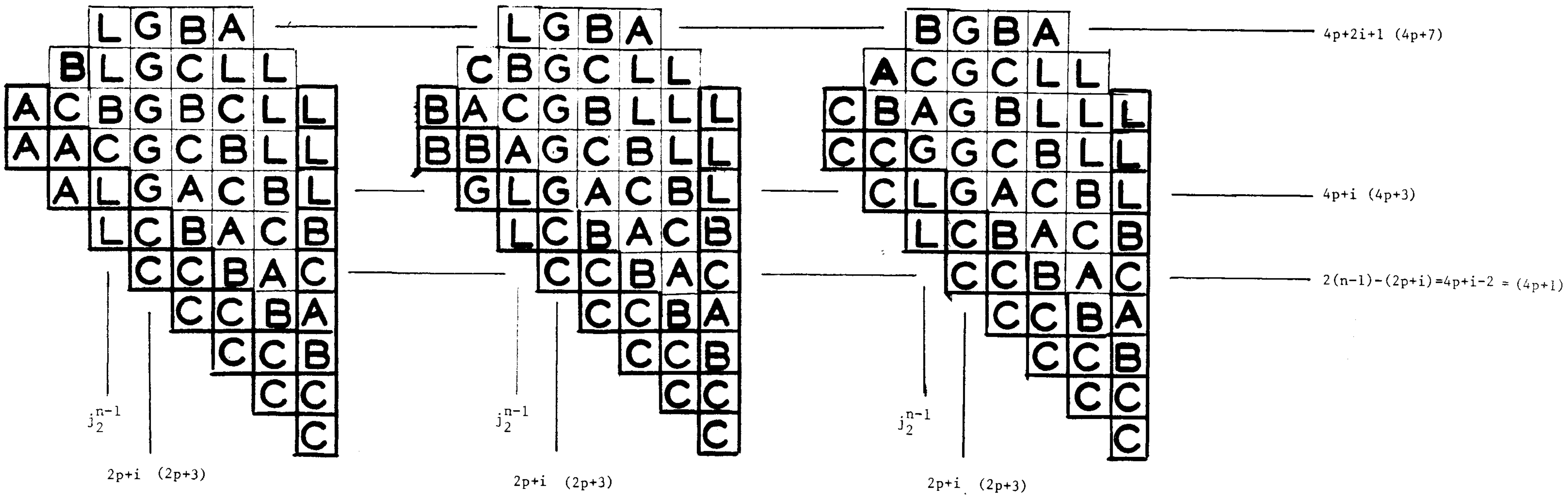
$j_2^{n-1} \equiv 3 \bmod (3)$

Figure 6.3 c)

Case $n = 3p+3$ (hence $i = 3$)

Sites whose state-value is known by hypothesis

Sites whose state-value is deduced.



Case : $DD_{n-1,j_2^{n-1},j_3^{n-1}}$ is A-basic

$j_2^{n-1} \equiv 1 \mod (3)$

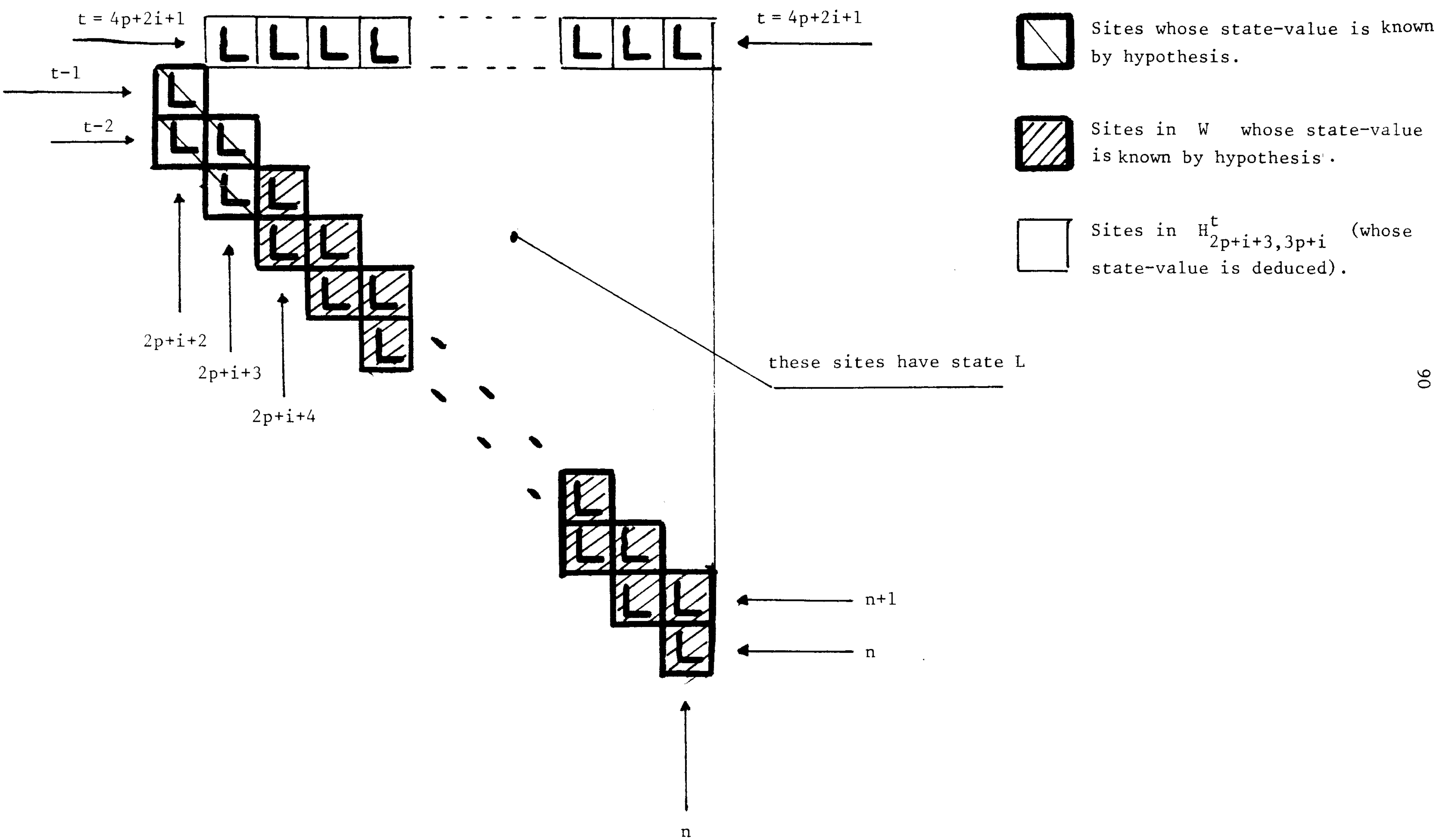
Case : $DD_{n-1,j_2^{n-1},j_3^{n-1}}$ is B-basic

$j_2^{n-1} \equiv 2 \mod (3)$

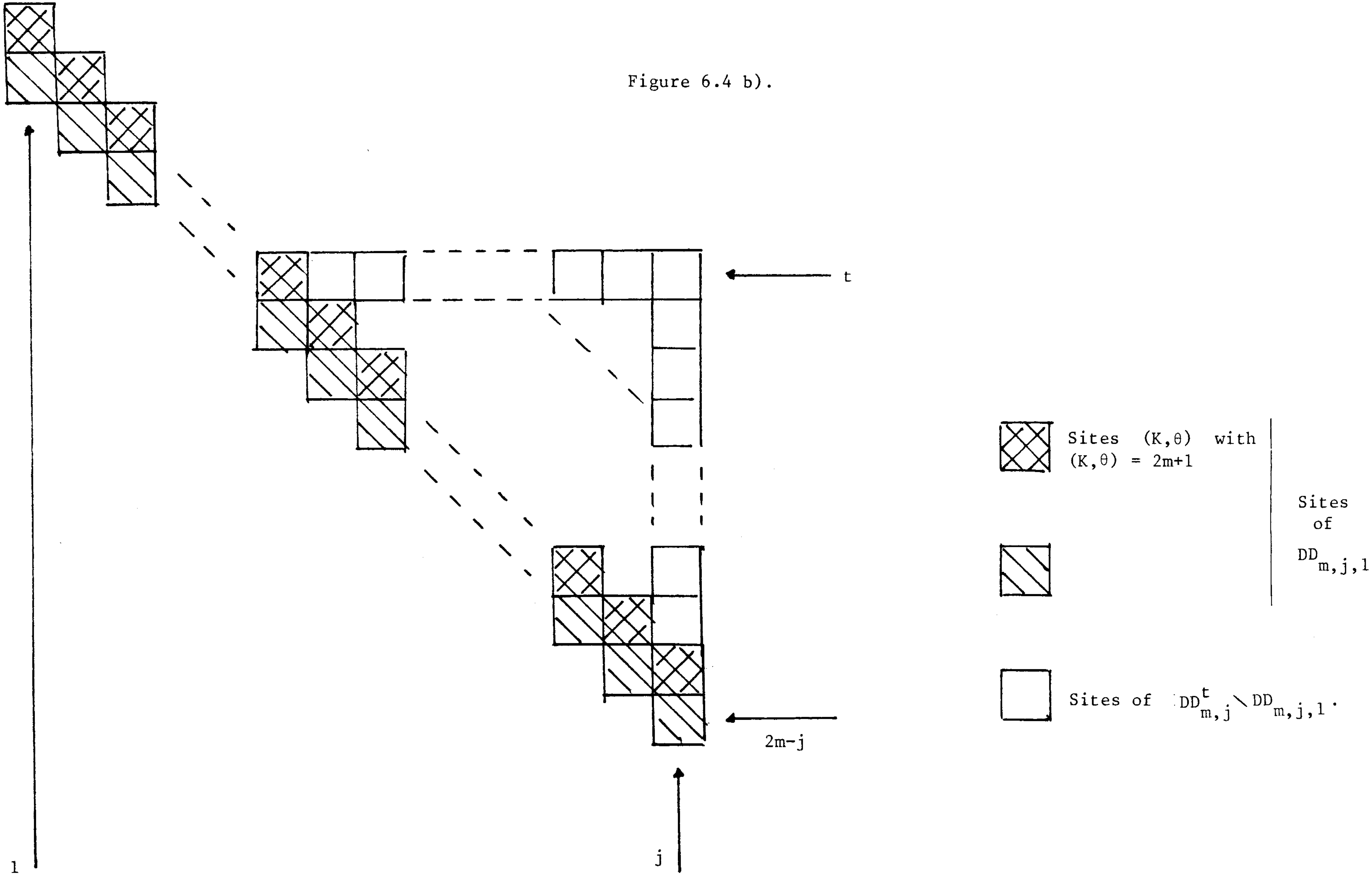
Case : $DD_{n-1,j_2^{n-1},j_3^{n-1}}$ is C-basic

$j_2^{n-1} \equiv 3 \mod (3)$

Figure 6.4 a).



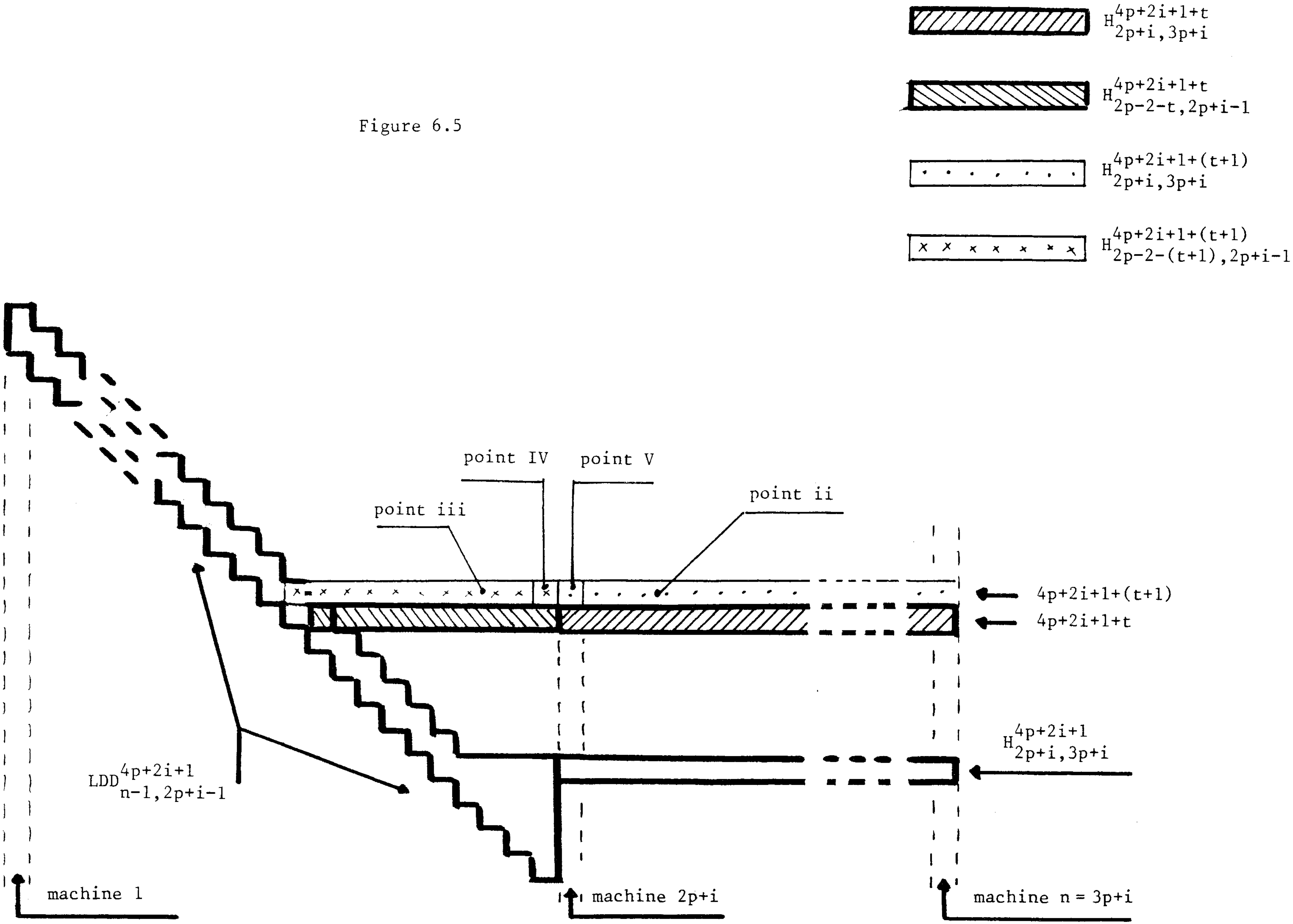
t = 4p+2i+1
t-1
t-2
L
2p+i+2
2p+i+3
2p+i+4
these sites have state L
t = 4p+2i+1
n+1
n
n
Sites whose state-value is known by hypothesis.
Sites in W whose state-value is known by hypothesis.
Sites in $H^{t}_{2p+i+3,3p+i}$ (whose state-value is deduced).



t
2m-j
j
l
Sites (K,θ) with $(K,\theta) = 2m+1$
Sites of $DD_{m,j,l}$
Sites of $DD^t_{m,j} \setminus DD_{m,j,l}$.

Figure 6.4 b).

Figure 6.5



$H^{4p+2i+1+t}_{2p+i,3p+i}$
$H^{4p+2i+1+t}_{2p-2-t,2p+i-1}$
$H^{4p+2i+1+(t+1)}_{2p+i,3p+i}$
$H^{4p+2i+1+(t+1)}_{2p-2-(t+1),2p+i-1}$
point IV
point V
point iii
point ii
$4p+2i+1+(t+1)$
$4p+2i+1+t$
$H^{4p+2i+1}_{2p+i,3p+i}$
$LDD^{4p+2i+1}_{n-1,2p+i-1}$
machine 1
machine 2p+i
machine n = 3p+i