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STATIONARY STRATEGIES IN TOPOLOGICAL GAMES

by Fred GALVIN

If X is topological space, the Banach-Mazur game $BM(X)$ is played as follows : first Black chooses a nonempty open set $B_0 \subseteq X$, then White chooses a nonempty open set $W_0 \subseteq B_0$, then Black chooses a nonempty open set $B_1 \subseteq W_0$, and so on. Thus, a play of the game $BM(X)$ is an infinite decreasing sequence $X \supseteq B_0 \supseteq W_0 \supseteq B_1 \supseteq W_1 \supseteq \dots \supseteq B_n \supseteq W_n \supseteq \dots$ of nonempty open sets ; Black wins if the intersection is empty, White wins if it is nonempty. This generalization of the classical Banach-Mazur game [5] is due to J.C. Oxtoby [6] who showed that Black has a winning strategy in $BM(X)$ if and only if X is not a Baire space. (X is a Baire space if every intersection of countably many dense open subsets of X is dense in X).

A space X is said to be "weakly α -favorable" [8] if White has a winning strategy in $BM(X)$. It follows from Oxtoby's result that every weakly α -favorable space is a Baire space. The converse is false : if X is a subset of the real line which meets every nonempty perfect set and whose complement also meets every nonempty perfect set (i.e., X is a "Bernstein set"), then neither player has a winning strategy in $BM(X)$; i.e., X is a Baire space but is not weakly α -favorable. However, the spaces satisfying the usual hypotheses of the Baire category theorem (e.g., complete metric spaces or locally compact Hausdorff spaces) are not only Baire but in fact weakly α -favorable.

In general a strategy for White in $BM(X)$ may depend on all the previous moves :

$$W_n = \tau(B_0, W_0, B_1, W_1, \dots, B_n).$$

It is easy to see that such a strategy can be reduced to an equivalent strategy which depends only on the opponent's previous moves :

$$W_n = \tau'(B_0, B_1, \dots, B_n).$$

A "stationary strategy" (also known as a "positional strategy" or a "tactic"[1]) is a strategy which depends only on the opponent's last move :

$$W_n = \sigma(B_n).$$

A space X is said to be ' α -favorable' [1] if White has a stationary winning strategy in $BM(X)$. It is easy to see that complete metric spaces and locally compact Hausdorff spaces are α -favorable, and for a long time it was an open problem whether all weakly α -favorable spaces are α -favorable. However, G. Debs has recently found a simple example [2] of a (non-regular) Hausdorff space which is weakly α -favorable but not α -favorable, as well as a more complicated example [3] of a Tychonoff space with the same properties.

Now , a "Markov strategy" for White in $BM(X)$ is one which depends only on the opponent's last move and the number of moves that have been made :

$$W_n = \tau(n, B_n).$$

If X is "pseudocomplete" in the sense of J.C. Oxtoby [7] , then it is clear that White has a Markov winning strategy in $BM(X)$. It is not so clear that he has a stationary winning strategy, but this is a consequence of a theorem of F. Galvin and R. Telgársky [4] : if White has a Markov winning strategy in $BM(X)$, then he also has a stationary winning strategy.

G. Debs [3] and F. Galvin and R. Telgársky [4] independently observed that, for every weakly σ -favorable space X , White has a winning strategy in $BM(X)$ of the form $W_n = \sigma(W_{n-1}, B_n)$; however, we do not know if there is always a winning strategy of the form $W_n = \sigma(B_{n-1}, B_n)$.

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