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Some Properties of the Bornological Spaces

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N. Bourbaki, [1, p.35] notices that it is not known if every bornological barrelled space is ultrabornological. In [2] we proved that if E is the topological product of an infinite family of bornological barrelled spaces, of non-zero dimension, there exists an infinite number of bornological barrelled subspaces, which are not ultrabornological. We also gave some examples of barrelled normable non-ultrabornological spaces. In [3] we gave an example of a bornological barrelled space E , such that E is not inductive limit of Baire spaces. We prove in this article that the example given in [3] is not inductive limit of barrelled normed spaces. Other result given here is the following: If E and F are two infinite dimensional Banach spaces, such that the conjugate of F is separable, there exists a family in E of precompact absolutely convex sets $\{B_s : s \in S\}$ such that, for every $s \in S$, $E_{B_s} = F$, $E_{\overline{B_s}}$ is the second conjugate of F , being $\overline{B_s}$ the closure of B_s in E , and E is the inductive limit of the family $\{E_{B_s} : s \in S\}$.

The vector spaces we use here are defined over the field K of the real or complex numbers. We mean under "space" a separated locally convex space. If T is the topology of a space E we shall write $E[T]$ sometimes instead of E . If A is a bounded absolutely convex set of E , then E_A denotes the normed space over the linear hull of A , with the norm associated to A . We say that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy (convergent) sequence for the Mackey convergence in E if there is a bounded closed absolutely convex set B in E such that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy (convergent) sequence in E_B . We say that E is locally complete if every Cauchy sequence for the Mackey convergence in E is convergent in E . We represent by $\hat{E}$ the completion of E . If $\mathcal{F}$ is the family of all locally complete subspaces of $\hat{E}$, which contain E , its intersection is a locally complete space $\tilde{E}$ and we call it the locally completion of E . We say that a subspace E of F is locally dense if, for every $x \in F$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of E , which converges to x in the Mackey sense. We say that a space E is a Mackey space if it is provided with the Mackey topology.

We shall need the following result, [2]: a) Let E be a locally dense subspace of a space F . If E is bornological, then F is bornological.

THEOREM 1. If E is a bornological space, then $\tilde{E}$ is bornological.

Proof: Let $\{E_i : i \in I\}$ be the family of all bornological subspaces of $\tilde{E}$, containing E . We show now that $\{E_i : i \in I\}$ with the inclusion relation is an inductive ordered set. Indeed, let $\{E_j : j \in J\}$ be a totally ordered subfamily of $\{E_i : i \in I\}$ and we set $F = \bigcup \{E_j : j \in J\}$. Since F is a Mackey space and E_j is dense in F , for every $j \in J$, we have that F is the inductive limit of the family $\{E_j : j \in J\}$ and, therefore, F is bornological. By Zorn's lemma, there exists a bornological subspace G of $\tilde{E}$

containing E , which is maximal, referred to the family $\{E_i: i \in I\}$. We shall see now that G coincides with $\tilde{E}$. Indeed, if $G \neq \tilde{E}$, G is not locally complete and, therefore, there exists a vector x in $\tilde{E}$, $x \notin G$, such that if M is the linear hull of $G \cup \{x\}$, then G is locally dense in M and, by result a), we have that M is bornological, which contradicts the maximality of G in $\{E_i: i \in I\}$. q.e.d.

For the proof of Theorem 2 we shall need the following results: b) Let E be a barrelled space. If F is a subspace of E , of finite codimension, then F is barrelled. [4]. c) If E is a metrizable barrelled space, then it is not the union of an increasing sequence of closed, nowhere dense and absolutely convex sets, [5]. d) Let E and F be two spaces so that F is a Pták space. If u is an almost continuous linear mapping from E into F and the graph of u is closed, then u is continuous, [6, p.302].

THEOREM 2. Let E be a non-complete (LB)-space. Let x_0 be a point of $\hat{E}$ which is not in E . If G is the linear hull of $E \cup \{x_0\}$ with the by $\hat{E}$ induced topology, then G is not the locally convex hull of barrelled normed spaces.

Proof: Let $\{E_n\}_{n=1}^\infty$ be an increasing sequence of subspaces of E such that $U\{E_n: n=1,2,\dots\} = E$. Let T_n be a topology on E_n finer than the topology of E_n , such that $E_n [T_n]$ is a Banach space and E is the inductive limit of $\{E_n [T_n]\}_{n=1}^\infty$. * If B_n is the unit ball in E_n let $\{\lambda_n\}_{n=1}^\infty$ be a strict increasing sequence of positive numbers such that $\lambda_n B_n \subset \lambda_{n+1} B_{n+1}$ and $U\{\lambda_n B_n: n=1,2,\dots\} = E$. We suppose that G is the locally convex hull of the family $\{G_i: i \in I\}$ of normed barrelled spaces. Since E is dense in G there exists an element i_0 in I such that $G_{i_0} \cap E$ is dense in G_{i_0} . Let j be the injective mapping of $G_{i_0} \cap E$, with the by G_{i_0} induced topology in E . If A_n is the closure of $j^{-1}(\lambda_n B_n)$ in $G_{i_0} \cap E$, then $U\{A_n: n=1,2,\dots\} = G_{i_0} \cap E$. According to result b), we have that $G_{i_0} \cap E$ is barrelled and, by result c), there exists a positive integer n_0 such that A_{n_0} is a neighbourhood of the origin in $G_{i_0} \cap E$. Let L be the linear hull of $j^{-1}(\lambda_{n_0} B_{n_0})$ with the by G_{i_0} induced topology. Let k be the canonical mapping of L into E_{n_0} . Since k , considered from L into E is the restriction of j to L we have that k is continuous from L into E and, therefore, the graph of k is closed in $L \times E_{n_0}$. Obviously, k is almost continuous and, according to result d), u is continuous since E_{n_0} is a Banach space. Since L is dense in $G_{i_0} \cap E$ we have that L is dense in G_{i_0} and, therefore, we can take a point $y_0 \in G_{i_0}$, $y_0 \notin G_{i_0} \cap E$ and a sequence $\{y_n\}_{n=1}^\infty$ in L converging to y_0 in G_{i_0} . If u is the canonical mapping of G_{i_0} in G , we have that

$$\lim_{n \rightarrow \infty} u(y_n) = \lim_{n \rightarrow \infty} k(y_n) = y_0 \notin E$$

and since E_{n_0} is a Banach space it results that

$$\lim_{n \rightarrow \infty} u(y_n) = \lim_{n \rightarrow \infty} k(y_n) = y_0 \in E_{n_0} \subset E$$

which is a contradiction.

q.e.d.

* We put now E_n instead $E_n [T_n]$.

In [7, p.434] G. Kothe gives an example of a non-complete (LB)-space which is defined by a sequence of Banach spaces such that there exists a bounded set A in E which is not a subset of E_n , $n=1,2,\dots$. If B is the closed absolutely convex hull of A , then E_B is not a Banach space, (see [3], proof of Theorem 2), and, therefore, E is not locally complete. We take $x \in E$, $x \in \hat{E}$. If G is the linear hull of $E \cup \{x_0\}$, with the by $\hat{E}$ induced topology, then G is barrelled and, by result a), G is bornological and, according to Theorem 2, G is not inductive limit of normed barrelled spaces.

In [8] Markushevich proves the existence of a generalized basis for every Banach separable space of infinite dimension, (see also [9] p. 116). In the following Lemma we give a proof of the existence of basis of Markushevich, which is valid for Fréchet spaces, and we shall need it after. Given a space E we represent by E' the topological dual of E and by $\sigma(E', E)$ and $\beta(E', E)$ the weak and strong topologies on E' , respectively.

LEMMA. Let E be a separable Fréchet space of infinite dimension. If G is a total subspace of $E' [\sigma(E', E)]$ of countable dimension, there exists a biorthogonal system $\{x_n, u_n\}_{n=1}^{\infty}$ for E , such that $\{x_n\}$ is total in E and the linear hull of $\{u_n\}_{n=1}^{\infty}$ coincides with G .

Proof: In E let $\{y_n\}_{n=1}^{\infty}$ be a convergent to the origin total sequence and let B be the closed absolutely convex hull of this sequence. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of non-zero elements of K , such that $z_n = \lambda_n y_n$, $\|z_n\| \leq 1/n$, being $\|\cdot\|$ the norm in the Banach space E_B . Let f be the mapping of ℓ^2 into E_B such that if $\{a_n\}_{n=1}^{\infty} \in \ell^2$, then

$$f(\{a_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} a_n z_n.$$

Since

$$\|f(\{a_n\}_{n=1}^{\infty})\| \leq \sum_{n=1}^{\infty} |a_n| \cdot \|z_n\| \leq \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \|z_n\|^2\right)^{1/2}$$

$$\leq \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} 1/n^2\right)^{1/2}$$

we have that f is well defined and it is continuous. Let U be the closed unit ball in ℓ^2 and we set $f(U)=A$. Then the Hilbert space $\ell^2/f^{-1}(0)$ can be identified with E_A and, therefore, E_A is a Hilbert space. Obviously, E_A is total in E and thus E' is weakly dense in $(E_A)'$. Let $\{v_n\}_{n=1}^{\infty}$ be a Hamel basis in G . In $(E_A)' [\beta((E_A)', E_A)]$ we apply the orthonormalization method of Gram-Schmidt and we obtain an orthonormal sequence $\{u_n\}_{n=1}^{\infty}$ from $\{v_n\}_{n=1}^{\infty}$. If $\{x_n\}_{n=1}^{\infty}$ is the sequence in E_A such that $\langle u_n, x_n \rangle = 1$, $\langle u_n, x_m \rangle = 0$, $n \neq m$, $n, m=1,2,\dots$, then the biorthogonal system $\{x_n, u_n\}_{n=1}^{\infty}$ verifies that $\{x_n\}_{n=1}^{\infty}$ is total in E and $\{u_n\}_{n=1}^{\infty}$ has G as linear hull.

q.e.d.

THEOREM 3. Let E and F be Banach spaces of infinite dimension. If $F' [\beta(F', F)]$ is

separable, there exists in E a family $\{B_s : s \in S\}$ of precompact absolutely convex sets such that $\bigcup \{B_s : s \in S\} = E$, $E_{B_s} = F$ and E_{B_s} is the second conjugate of F for every $s \in S$, and E is the locally convex hull of the family $\{E_{B_s} : s \in S\}$.

Proof: We shall use the symbol $|| \cdot ||$ for the norm of every normed space. By the Lemma we can choose a Markushevich basis $\{x_n, u_n\}_{n=1}^{\infty}$ for F such that $\{u_n\}_{n=1}^{\infty}$ is total in $F'[\beta(F', F)]$ and $||x_n|| = 1, n=1, 2, \dots$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers such that $||u_n|| \leq \lambda_n$. Let S be the family of all the sequences in E such that if $s = \{y_n\}_{n=1}^{\infty} \in S$ then $\{y_n\}_{n=1}^{\infty}$ is topologically free and $||y_n|| \leq 2^{-n} \lambda_n^{-2}$. Let f_s be the mapping of F into E such that if $x \in F$ then

$$f_s(x) = \sum_{n=1}^{\infty} \langle u_n, x \rangle y_n.$$

Since

$$||f_s(x)|| \leq \sum_{n=1}^{\infty} ||u_n|| \cdot ||x|| \cdot ||y_n|| \leq ||x|| \sum_{n=1}^{\infty} \lambda_n^{-1} 2^{-n}$$

we get that f_s is well defined and it is continuous. If $x \neq 0$ there exists a positive integer n_0 such that $\langle u_{n_0}, x \rangle \neq 0$ and there exists a $w \in E'$ such that $\langle w, y_{n_0} \rangle = 1$, $\langle w, y_n \rangle = 0, n \neq n_0$, since $\{y_n\}_{n=1}^{\infty}$ is topologically free. Then

$$\langle f_s(x), w \rangle = \sum_{n=1}^{\infty} \langle u_n, x \rangle \cdot \langle w, y_n \rangle = \langle u_{n_0}, x \rangle \cdot \langle w, y_{n_0} \rangle = \langle u_{n_0}, x \rangle \neq 0$$

and, thus, f_s is injective. If U is the closed unit ball in F let $f_s(U) = B_s$. We shall see that B_s is precompact. Indeed, given $x \in U$ it results that

$$f_s(x) = \sum_{n=1}^{\infty} \lambda_n^{-1} 2^{-n} \langle u_n, x \rangle \lambda_n^2 y_n = \sum_{n=1}^{\infty} \lambda_n^{-1} 2^{-n} \langle u_n, x \rangle z_n,$$

$$||z_n|| = \lambda_n^2 ||y_n|| \leq \lambda_n^2 \cdot 2^{-n} \lambda_n^{-2} = \lambda_n^{-1}$$

$$\sum_{n=1}^{\infty} |\lambda_n^{-1} 2^{-n} \langle u_n, x \rangle| \leq \sum_{n=1}^{\infty} \lambda_n^{-1} 2^{-n} ||u_n|| \cdot ||x|| \leq \sum_{n=1}^{\infty} 2^{-n} = 1$$

and, therefore, $f_s(x)$ is in the closed absolutely convex hull of the sequence $\{z_n\}_{n=1}^{\infty}$. Since $||z_n|| = \lambda_n^{-1}$, $\{z_n\}_{n=1}^{\infty}$ converges to the origin in E and, thus, B_s is precompact. Obviously F coincides with E_{B_s} . We shall show that E is the locally convex hull of the family $\{E_{B_s} : s \in S\}$. In E let V be an absorbent absolutely convex set such that $V \cap E_{B_s}$ is a neighbourhood of the origin in E_{B_s} for every $s \in S$. We suppose that V is not a neighbourhood of the origin in E. We take $t_1 \in E, w_1 \in E'$ such that $||t_1|| \leq 2^{-1} \lambda_1^{-3}, \langle w_1, t_1 \rangle = 1$. Since $V \cap w_1^{-1}(0)$ is not a neighbourhood of the origin in $w_1^{-1}(0)$, considered as subspace of E, we take $t_2 \in w_1^{-1}(0)$ and $w_2 \in E'$ such that $t_2 \notin V, ||t_2|| \leq 2^{-2} \lambda_2^{-3}, \langle w_2, t_1 \rangle = 0, \langle w_2, t_2 \rangle = 1$. We suppose we have constructed $\{t_p, w_p\}_{p=1}^n$ and, according to the fact that $V \cap w_1^{-1}(0) \cap w_2^{-1}(0) \cap \dots \cap w_n^{-1}(0)$ is not a neighbourhood of the origin in $w_1^{-1}(0) \cap w_2^{-1}(0) \cap \dots \cap w_n^{-1}(0)$, we take t_{n+1} belonging to this last subspace such that $t_{n+1} \notin V, ||t_{n+1}|| \leq 2^{-(n+1)} \lambda_{n+1}^{-3}, w_{n+1} \in E'$ such that $\langle w_{n+1}, t_p \rangle = 0, p=1, 2, \dots, n, \langle w_{n+1}, t_{n+1} \rangle = 1$. If $r_n = \lambda_n t_n$, then $||r_n|| \leq 2^{-n} \lambda_n^{-2}$ and $\{r_n\}_{n=1}^{\infty}$ is topologically free since $\langle w_n, r_n \rangle = \lambda_n, \langle w_m, r_n \rangle = 0$,

$m=n$, $m, n=1, 2, \dots$, and, thus, $r = \{r_n\}_{n=1}^{\infty} \in S \quad V \cap E_{B_r}$ is a neighbourhood of the origin in E_{B_r} and $r_n = f_r(x_n) \in B_r$ and, therefore, $\{t_n\} = \lambda_n^{-1} r_n$ converges to the origin in E_{B_r} , which is contradiction with $t_n \notin V$, $n \geq 2$. From the way we chose t_1 it results that $\bigcup \{B_s; s \in S\} = E$. We shall show now that $E_{\bar{B}_S}$ is the second conjugate of F . If F'' is the bidual of F and $s = \{y_n\}_{n=1}^{\infty} \in S$, let g_s be the mapping from F'' into E such that if $x \in F''$

$$g_s(x) = \sum_{n=1}^{\infty} \langle u_n, x \rangle y_n.$$

Since $\{y_n\}_{n=1}^{\infty}$ is topologically free and $\{u_n\}_{n=1}^{\infty}$ is total in $F'[\beta(F', F)]$ and also in $F'[\sigma(F', F'')]$, following the same patterns as we did for f_s we prove that g_s is injective. Obviously, f_s is the restriction of g_s to F . If U^* is the closure of U in $F''[\sigma(F'', F')]$ and we prove that $g_s(U^*) = \bar{B}_S$ then $E_{\bar{B}_S}$ is the second conjugate of F . Let z be a point of U^* . In $F''[\sigma(F'', F')]$, U^* is metrizable and U is dense in U^* and, therefore, there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in U $\sigma(F'', F)$ -converging to z . Given an arbitrary $\epsilon > 0$, we can find two positive integers n_0 and p_0 such that

$$\sum_{n=n_0+1}^{\infty} 2^{-n+1} \lambda_n^{-1} < \epsilon/2$$

$$|\langle u_n, z - z_p \rangle| < \epsilon / \sum_{n=1}^{n_0} 2^{-n+1} \lambda_n^{-2}, \quad n=1, 2, \dots, n_0, \quad p \geq p_0.$$

Then, it follows for $p \geq p_0$

$$\begin{aligned} |g_s(z) - g_s(z_p)| &= \left| \sum_{n=1}^{\infty} \langle u_n, z - z_p \rangle y_n \right| \\ &\leq \sum_{n=1}^{n_0} |\langle u_n, z - z_p \rangle| \cdot \|y_n\| + \sum_{n=n_0+1}^{\infty} |\langle u_n, z - z_p \rangle| \cdot \|y_n\| \\ &\leq \sum_{n=1}^{n_0} (\epsilon \|y_n\| / \sum_{n=1}^{n_0} 2^{-n+1} \lambda_n^{-2}) + \sum_{n=n_0+1}^{\infty} \|u_n\| \cdot \|z - z_p\| \cdot \|y_n\| \\ &\leq \epsilon/2 + \sum_{n=n_0+1}^{\infty} \lambda_n \cdot 2 \cdot 2^{-n} \lambda_n^{-2} < \epsilon, \end{aligned}$$

and since $g_s(z_p) = f_s(z_p) \in B_S$ we have that $g_s(z) \in \bar{B}_S$. On the other hand, if x' is a point of $\bar{B}_S$ we can find a sequence $\{x'_n\}_{n=1}^{\infty}$ in B_S converging to x' . If y'_n is the point of U such that $f_s(y'_n) = x'_n$, then, since U^* is $\sigma(F'', F')$ -compact we can choose a subsequence $\{y'_{np}\}_{p=1}^{\infty}$ of $\{y'_n\}_{n=1}^{\infty}$ $\sigma(F'', F')$ -converging to a point y' of U^* . Then $g_s(y') = x'$ and, therefore $g_s(U^*) = \bar{B}_S$. q.e.d.

In [10] we prove the following result : e) Let F be a sequentially complete infinite-dimensional space with the following properties: 1) There is in F a bounded countable total set . 2) There is in $F'[\sigma(F', F)]$ a countable total set which is equicontinuous in F . 3) If u is an injective linear mapping from F into F , with closed graph, then u is continuous. Then if E is an infinite-dimensional Banach space it results that E is the inductive limit of a family of spaces equal to F , spanning E .

According to the Lemma and using the same kind of proof as we did for result e) it is possible to prove Theorem 4.

THEOREM 4. Let E and F be two infinite-dimensional Banach spaces. If $F'[\beta(F',F)]$ is separable, there exists in E a saturated family $\{B_s : s \in S\}$, directed by inclusion, of precompact absolutely convex sets such that $\bigcup \{B_s : s \in S\} = E$, E is the locally convex hull of the family $\{E_{B_s} : s \in S\}$, $E_{B_s} = F$ and $E_{\bar{B}_s}$ is the second conjugate of F for every $s \in S$.

REFERENCES

1. Bourbaki, N.: *Eléments de Mathématiques*, Livre V: *Espaces vectoriels topologiques* (ch. III. ch.IV, ch. V), Paris (1964).
2. Valdivia, M.: A class of bornological barrelled spaces which are not ultrabornological, *Math. Ann.* 194, 43-51 (1971).
3. Valdivia, M.: Some examples on quasi-barrelled spaces, *Ann. Inst. Fourier* 22, 2, 21-26 (1972).
4. Dieudonné, J.: Sur les propriétés de permanence de certains espaces vectoriels topologiques. *Ann. Soc. Polon. Math.* 25, 50-55 (1952).
5. Amemiya, I. and Komura, Y.: Über nich-vollständige Montelräume. *Math. Ann.* 177 273-277 (1968).
6. Horváth, J.: *Topological Vector Spaces and Distributions I.* Addison-Wesley. Massachusets 1966.
7. Markushevich, A.I.: Sur les bases (au sens large) dans les espaces linéaires. *Doklady Akad. Nauk SSSR (N.S)*, 41, 227-229 (1934).
8. Marti, J.T.: *Introduction to the Theory of Bases.* Springer-Verlag, Berlin -Heidelberg-New York, 1969.
9. Kothe, G.: *Topological Vector Spaces I*, Springer-Verlag, Berlin-Heidelberg -New York, 1969.
10. Valdivia, M.: Some characterizations of ultrabornological spaces (To be published).