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The aim here is to describe briefly part of the theory of a fairly large and quite well understood class of noncommutative noetherian rings and to show where localization (or torsion theoretic) techniques have been utilised.

The class of rings concerned is hereditary noetherian rings - especially when they are prime, in which case we will call them HNP rings, for short. (Note that, throughout, we will assume that conditions are left-right symetric).

§1. GENERAL CASE.

THEOREM [4]. - *A hereditary noetherian ring is a finite direct sum of artinian hereditary rings and* HNP *rings.*

Whilst the proof in [4] is short and neat, there is a general criterion for such decompositions which has this result, and similar results for principal ideal rings and serial rings, as consequences.

THEOREM [29] . - *Let* R *be a noetherian ring with nil radical* N*, and let* $\mathscr{C}(N) = \{c \in R \mid c+N$ *is regular in* $R/N\}$*. Then* R *decomposes as the direct sum of an artinian ring and a semiprime ring precisely when* $cN = Nc = N$ *for all* $c \in \mathscr{C}(N)$.

The former result turns our attention to the two types of hereditary noetherian rings.

§2. ARTINIAN HEREDITARY RINGS

These have been well studied, and their structures fully determined [12]. Up to Morita equivalence, such a ring, R, is described via a triangular array

$$\begin{array}{llll} X_{11} & & & \\ X_{21} & X_{22} & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{array}$$

in which $X_{ii} = D_i$ is a division ring and X_{ji} is a right D_i and left D_j finite dimensional vector space. Then R has the form of a matrix ring

$$R \cong \begin{bmatrix} M_{11} & 0 & \cdots & 0 \\ M_{21} & M_{22} & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & 0 \\ M_{n1} & M_{n2} & \cdots & M_{nn} \end{bmatrix}$$

where the M_{ji} are defined recursively via the equation

$$M_{ji} = X_{ji} \oplus \coprod_{k=i+1}^{j} M_{jk} \otimes_{D_k} X_{ki} \qquad , \qquad j \geq i,$$

and the matrix multiplication is the obvious one.

§3. HNP RINGS.

First a general comment. In an HNP ring, the idempotent ideals play an important role. Of particular interest are those HNP rings which have no idempotent ideals. They are called Dedekind prime rings - basically, these are the type of maximal order which Asano studied [1] . They are easily described categorically as those rings with the property that submodules of progenerators are progenerators [26]. Whilst describing the theory of HNP rings we will point out some special features of Dedekind prime rings.

§4. OVER-RINGS.

Being a noetherian prime ring, each HNP ring R has a classical quotient ring Q which is simple artinian. Of course Q is a flat epimorphic extension of R. Since R is hereditary, every ring S such that $R \subseteq S \subseteq Q$ is a flat epimorphic extension of R. Thus S is a quotient ring of R with respect to some torsion theory, and is an HNP ring [17]. In particular, if R is Dedekind, so too is S.

Also [9] there is a 1.1 correspondence between the over-rings S, such that S_R is finitely generated, and idempotent ideals A of R via

$$A \to O_r(A) = \{q \in Q \mid Aq \subseteq A\},$$

$$S \to R.{}^{\bullet}S = \{q \in Q \mid qS \subseteq R\}.$$

§5. IDEALS.

These are described [7] in terms of two special types. An ideal A is *invertible* if AB = BA = R for some subset B of Q. And an ideal C is *eventually idempotent* if some power of C is an idempotent ideal. Every ideal is the product of an invertible ideal and an eventually idempotent ideal. Moreover the invertible ideal is a (commutative) product of ideals maximal amongst invertible ideals.

Let X be an ideal maximal amongst invertible ideals. Then either X is a maximal ideal or $X = M_1 \cap \ldots \cap M_n$ where the M_i are maximal and idempotent.

§6. SOME TORSION THEORIES.

The torsion theory with respect to which Q is the quotient ring has, as its corresponding filter, all essential right ideals of R. Let tM denote the torsion submodule of an R-module M thus given, [20].

Next [16], [18] , consider the "invertible filter" of essential right ideals I such that I contains some invertible ideal. The corresponding quotient ring we will call Q_i , and the torsion submodule iM. At the opposite extreme there is the "uninvertible filter" of essential right ideals I such that no nonzero subfactor of R/I is annihilated by an invertible ideal, with Q_u and uM. More specially we have the filter of essential right ideals I such that no nonzero subfactor of R/I is annihilated by some particular maximal invertible ideal X ; with Q_X and M_X. Finally, there is the filter of right ideals containing a power of X ; with Q_{X^*} and M_{X^*}. The fact that X is semiprime rather than prime seems of some significance for noncommutative localization theory - see [15].

§7. QUOTIENT RINGS.

We have (see [16], [18]).

$$R = Q_i \cap Q_u$$

and $Q_u = \cap Q_X$ where X ranges over the ideals maximal amongst the invertible ideals. These rings have special features. Q_i has no invertible ideals - so every ideal is eventually idempotent. Contrastingly, in Q_u every ideal contains an invertible and moreover Q_u is bounded (every essential right ideal contains a nonzero ideal). Further in Q_X there is a unique maximal invertible ideal which is the Jacobson radical.

The ring Q_i is used [18] in proving that if R is bounded then every ideal of R contains an invertible ideal.

§8. MODULES.

Next let M be a finitely generated module over R. Then $M = tM \oplus P$ where P is projective. Further

$$tM = iM \oplus uM$$

and $iM = \amalg M_{X^*}$, X ranging as above, [16] , [18] .

In the special case when R is Dedekind we can say more, [6]. For then P is the direct sum of a free module and one right ideal ; and tM is the direct sum of cyclic submodules. This leads to the result [6] that, in a Dedekind prime ring, every one-sided ideal has two generators the first of which may be chosen almost at random - the " $1^{1/2}$ generator" property. This is equivalent to asserting that every submodule of an artinian cyclic module is cyclic. Using the torsion theories above, it is shown in [16] that this property characterises Dedekind rings amongst HNP rings - but see also the next section.

§9. FINITE LOCALIZATIONS.

As mentioned in §4, there is a correspondence between idempotent ideals A and finite localizations S. It has been shown [27] that if A is a semiprime ideal then $R = \mathbb{I}_S(A) = \{s \in S \mid sA \subseteq A\}$, the idealizer of A in S and S/A is a semisimple module. This type of globalization by idealizing is easily controlled and enables results to be proved "by induction". For example [27], if B is any ideal of R then there is an ideal $C \subseteq B$ such that R/C is Morita equivalent to a finite direct sum of rings of the form

$$\begin{bmatrix} L & J & \cdots & J \\ L & L & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & J \\ L & \cdots & \cdots & L \end{bmatrix}$$

with L a local artinian principal ideal ring, J its Jacobson radical. The proof also requires the result of Asano that factor rings of Dedekind rings are principal ideal rings - but see [26] for an easy proof.

The finite localization can also be used to discuss the $1^{1/2}$ generator property. Let R be HNP-ring with an idempotent ideal A ; we may assume A to be maximal. Then $R = \mathbb{I}_S(A)$ as before, and S/A is semisimple with isomorphic simple summands. Since A is not an ideal of S, there is a right ideal B of S, $B \subsetneq A$, with S/B also a sum of copies of that simple summand. It is readily checked that S/B is a submodule of a cyclic artinian R-module, and that it has a noncyclic submodule. Thus R does not have the $1^{1/2}$ generator property.

The same argument contradicts [28] Proposition 2.7 and shows that if S is a simple HNP-ring, A a maximal right ideal of S and $R = \mathbb{I}_S(A)$ then, although R is an HNP-ring, there is no bound on the numbers of generators required by right ideals.

§10. P.I. RINGS.

Suppose that Q is a finite dimensional central simple algebra-equivalently, that R is a P.I. (polynomial identity) ring.

THEOREM [30]. - *If* R *is an* HNP-*ring and a* P.I.*ring then the centre,* Z, *of* R *is a Dedekind domain,* R *is a finitely generated* Z-*module and so* R *is a "classical" hereditary order.*

The proof entails three types of localization. Firstly, by localizing at the powers of a certain central element, one shows that R has only finitely many idempotent ideals. Finite localization and globalization enables one to reduce to the case when R is Dedekind. And in that case the result was obtained by Asano [1] using, amongst other things, the localization with respect to an invertible ideal X described in §6-§7.

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