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Induced representations and classifications for $GSp(2,F)$ and $Sp(2,F)$

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Résumé. Soit F un corps p-adique de charactéristique différente de 2. On caractérise la réducibilité des représentations de $GSp(2,F)$ et $Sp(2,F)$ qui sont induites paraboliquement par des représentations irréductibles. On donne aussi une classification (modulo les représentations cuspidales) de différentes classes de représentations irréductibles de ces groupes. Un cas spécial est la classification des représentations irréductibles unitaires.

Abstract. Let F be a p-adic field whose characteristic is different from 2. The reducibilities of the representations of $GSp(2,F)$ and $Sp(2,F)$ which are parabolically induced by the irreducible representations are described. We obtain also classifications (modulo cuspidal representations) of various classes of irreducible representations of these groups. In particular, the classification of the irreducible unitary representations is obtained.

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Introduction

Let F be a p-adic field. We shall assume that the characteristic of F is different from two. Denote by R the direct sum of the Grothendieck groups of the categories of all smooth representations of finite length of the groups $GL(n, F)$'s. The functor of the parabolic induction defines a multiplication $\times$ on R. In this way R becomes a ring ([Z1]). Obviously, one can define an additive mapping

$$m : R \otimes R \to R$$

which satisfies $m(r_1 \otimes r_2) = r_1 \times r_2$. A comultiplication

$$m^* : R \to R \otimes R.$$

is defined in [Z1]. The definition of the comultiplication involves the Jacquet modules for the maximal parabolic subgroups. In this way R becomes a Hopf algebra ([Z1]). This structure can be very helpful in the representation theory of the groups $GL(n, F)$. Some examples of the use of this structure can be found in [Z2] and [T2]. The crucial property of this structure is that the mapping $m^* : R \to R \otimes R$ is multiplicative. In the other words, we have a simple formula for the composition

$$m^* \circ m.$$

Let $R(S)$ (resp. $R(G)$) be the sum of the Grothendieck groups of the categories of the smooth representations of finite length of the groups $Sp(n, F)$'s (resp. $GSp(n, F)$'s). Using the functor of the parabolic induction one can define a structure of R-modules on $R(S)$ and $R(G)$ (see the first section). These multiplications are denoted by $\rtimes$. They induce biadditive mappings

$$\mu : R \otimes R(S) \to R(S)$$

and

$$\mu : R \otimes R(G) \to R(G).$$

Using the Jacquet modules for the maximal parabolic subgroups, one can define a comodule structures

$$\mu^* : R(S) \to R \otimes R(S)$$

and

$$\mu^* : R(G) \to R \otimes R(G)$$

(see the first section). The first question may be what is the formula for

$$\mu^* \circ \mu.$$

Formulas for these compositions were obtained in [T6]. A usefulness of such formulas could be seen from the paper [T5] where some results about the square integrable representations and the irreducibility of the parabolically induced representations were announced. An essentially new situations was treated there. We have obtained that results using the formulas for $\mu^* \circ \mu$. Examples of the use of such formulas, and outlines of proofs of some of the results announced in [T5] can be found in [T7]. A complete proofs will appear in the forthcoming papers.

In this paper we apply this type of approach to the representation theory of the groups $GSp(2, F)$ and $Sp(2, F)$. We study first the questions of the reducibility of the representations parabolically induced by the irreducible representations. Then we get the classification of various classes of irreducible representations, in particular, the classification of the irreducible unitary representations. Such questions were settled for the unramified representations by F. Rodier in [R2]. Because of that, our attention in this paper is directed more to the remaining irreducible representations and this paper completes F. Rodier's investigation. For the representations supported in the two intermediate parabolic subgroups, such questions were solved by F. Shahidi and J.-L. Waldspurger. We do not use in this paper arguments specific for the spherical representations. Also, we give very often alternative proofs to the Rodier's proofs. The main part of the paper is the analysis of the parabolically induced representations. The case corresponding to the regular characters is relatively easy. It was settled in a general setting by F. Rodier in [R1]. In the analysis of the irregular case, F. Rodier uses the explicit knowledge of the spherical functions and the connection between the matrix coefficients and the Jacquet modules. Our method uses only analysis of the composition series of the Jacquet modules. This method is able to cover all characters except one spherical case which was settled by F. Rodier (see Lemma 3.9).

The methods used in this papers were developed essentially for higher $GSp(n)$'s and $Sp(n)$'s. This is an introduction to the use of them in a relatively simpler setting. It seems that they are even more powerful for the higher ranks. The reason is simple, we have there more parabolic subgroups and we have more possibilities to compare informations coming from the Jacquet modules of various parabolic subgroups. The following example is suggestive. Let St_G and 1_G denotes the Steinberg and the trivial representation of some reductive group G. Look at $GSp(1,F)$. Then the question of the reducibility of $\chi \rtimes \mathrm{St}_{GSp(0,F)}$ (or $\chi \rtimes 1_{GSp(0,F)}$) for a character χ of $F^\times$, is the question of the reducibility of the non-unitary principal series representations of $GSp(1,F) = GL(2,F)$. As it is well known, the composition series of the Jacquet modules for the minimal parabolic subgroups imply that the reducibilities can appear only for $\chi = |\ \ |_F^\alpha, \alpha \in \mathbb{R}$ ($|\ \ |_F$ denotes the modulus character of F). No further information on α can be obtained by these considerations (we have reducibility for $\alpha = \pm 1$ for the obvious reasons). Thus, a whole line still remains to be analyzed. The following case is the case of $GSp(2,F)$. We have already noted that we can describe the reducibilities of $\chi \rtimes \mathrm{St}_{GS(1,F)}$ (or $\chi \rtimes 1_{GSp(1,F)}$) by the above methods, excluding one point. Clearly, we use the knowledge of $GSp(1,F)$ case. Now, using the knowledge of $\chi \rtimes \mathrm{St}_{GSp(1)}$, one can describe completely the reducibilities of $\chi \rtimes \mathrm{St}_{GSp(n)}$ for $n \geq 2$ (see [T5]).

This paper follows the ideas of [T4] and notation is the same as there. We give now a more detailed account of this paper.

In the first section we recall of the main notation which was introduced in [T4]. One should consult [T4] for more details concerning the notation. In the second section we present formulas for $\mu^* \circ \mu$ in the case of the group $GSp(2,F)$. These formulas are a special case of the formula obtained for

$\mu^* \circ \mu$ for any $GSp(n,F)$ in [T6]. In the third section we consider the representations of $GSp(2,F)$ parabolically induced by the irreducible representations of the Levi factors of proper parabolic subgroups, which are supported by the minimal parabolic subgroups. Note that such representations for $GSp(2,F)$ are either generalized principal series representations, or non-unitary principal series representations, or non-unitary degenerate principal series representations. We have determined in this section when these representations are irreducible. If they reduce, we find the Langlands' parameters of all irreducible subquotients. That irreducible subquotients are always of multiplicity one. A part of these results is either explicit or implicit in F. Rodier's paper [R2] where he considered the unramified case. R. Gustafson has determined the reducibility points and the length of reducible representations of the unramified non-unitary degenerate principal series of $Sp(n,F)$ for the maximal parabolic subgroup of GL-type ([Gu]). He has used a Hecke algebra method. C. Jantzen studied in [J] reducibility points of the non-unitary degenerate principal series of $Sp(n,F)$ and he has determined them for $Sp(2,F)$ and $Sp(3,F)$. He has used both the Hecke algebra and our method.

In the fourth section we apply the calculations which were done in the preceding section. We write classifications of square integrable, of tempered and of unitarizable irreducible representations of $GSp(2,F)$ which are supported in the minimal parabolic subgroups. We give Langlands' parameters of unitarizable representations. Such classifications were done by F. Rodier in the unramified case in [R2]. The ideas used in the classification in the unramified case are sufficient also for the treatment of the general case. One needs to have only the results of the proceeding section. For the sake of completeness, we include here also an analysis of representations supported in other two parabolic subgroups. These cases were settled by J.-L. Waldspurger and F. Shahidi. F. Shahidi's methods are sufficient for both cases. We want to thank F. Shahidi for computing explicitly for us in a letter the reducibility points in one of these two cases. Let us mention that A. May classified irreducible representations of $GSp(2,F)$ using the Hecke algebra isomorphisms ([Mo]). Analogous results for $Sp(2,F)$ are considered in the fifth section. We also consider the problems of the third section for $Sp(2, F)$. Note that in this case there appear representations except generalized principal series, non-unitary principal series and degenerate principal series representations.

Let us mention the following interesting situation. Take a square integrable representation of $Sp(2, F)$ supported in the minimal parabolic subgroups which is different from the Steinberg representation. Such representation always exists. Then it is a subquotient of a non-unitary principal series representation which corresponds to an irregular character. This non-unitary principal series representation has five different irreducible subquotients. Two of them are square integrable. One irreducible subquotient has multiplicity two and it is not square integrable. Other multiplicities are one. All irreducible subquotients are unitarizable. At the end, this non-unitary principal series representation is at the end of a very interesting complementary series.

The second author is thankful to the Mathematical Department of the University of Utah where this paper has got almost the final form, and where it has been typed. This paper is based on an earlier preprint "On representations of p-adic $GSp(2)$". The former preprint was profoundly revised. The case of $Sp(2)$ got a complete treatment in this new paper. Some impreciseness concerning $Sp(2)$ existing in the previous preprint, were removed in this paper. Also, the misprints that we noticed in the earlier preprint have been deleted in this paper.

1. Notation

We shall first recall of some of the notation related to the general linear groups, which was introduced in [BZ] and [Z1]. For more details about this notation, one should consult that papers.

A local non-archimedean field will be denoted by F. The topological modulus of F will be denoted by $|\ |_F$. As a homomorphism of $F^\times$, this character will be denoted by

$$\nu : F^\times \longrightarrow \mathbb{R}^\times.$$

For a two smooth representations π_1 of $GL(n_1, F)$ and π_2 of $GL(n_2, F)$, we denote by

$$\pi_1 \times \pi_2$$

the smooth representation of $GL(n_1+n_2, F)$ parabolically induced by $\pi_1 \otimes \pi_2$ from the standard parabolic subgroup (with respect to the upper triangular matrices)

$$P_{(n_1,n_2)} = M_{(n_1,n_2)} N_{(n_1,n_2)}$$

whose Levi factor $M_{(n_1,n_2)}$ is naturally isomorphic to $GL(n_1, F) \times GL(n_2, F)$ (see [BZ]). The induction that we consider is normalized.

The Grothendieck group of the category of all smooth representations of finite length of $GL(n, F)$, will be denoted by R_n. Their sum will be denoted by

$$R = \underset{n \geq 0}{\oplus} R_n.$$

Then $\times$ lifts to a multiplication in R which will be denoted by $\times$ again.

For a smooth representation π of $GL(n_1 + n_2, F)$ of finite length, we denote by

$$r_{(n_1,n_2),n_1+n_2}(\pi)$$

the Jacquet module with respect to $N_{(n_1,n_2)}$. The action of $M_{(n_1,n_2)}$ that we consider is the quotient action twisted by the modular character of $P_{(m_1,m_2)}$ to -1/2.

Let π be a smooth representation of $GL(n,F)$ of finite length. Denote by

$$m^*(\pi)$$

the sum of all semi simplifications of $r_{(p,n-p),n}(\pi)$, $0 \leq p \leq n$. One may consider $m^*(\pi) \in R \otimes R$. One lifts m^* to an additive mapping of R into $R \otimes R$. In this way R becomes a Hopf algebra ([Z1]).

Let J_n be the following $n \times n$ matrix

$$J_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

The group of all $(2n) \times (2n)$ matrices over F which satisfy

$$^tS = \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} S = \psi(S) \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}$$

for some $\psi(S) \in F^\times$, is denoted by $GSp(n,F)$ (^tS denotes the transposed matrix of S). We define formally $GSp(0,F)$ to be $F^\times$. The symplectic group is defined by

$$Sp(n,F) = \{S \in GSp(n,F);\ \psi(S) = 1\}.$$

We take formally $Sp(0,F)$ to be a trivial group.

A more detailed introduction into the notation which we shall introduce now, can be found in [T4].

We fix in $Sp(n,F)$ (resp. $GSp(n,F)$) the minimal parabolic subgroup $P_\emptyset^S$ (resp. $P_\emptyset^G$) which consists of all upper triangular matrices in the group. Let $M_\emptyset^S$(resp.$M_\emptyset^G$) be the subgroup of all diagonal matrices in $Sp(n,F)$ (resp.$GSp(n,F)$). Then $M_\emptyset^S$(resp.$M_\emptyset^G$) is a Levi factor of the standard minimal parabolic subgroup. It is also a maximal torus in $Sp(n,F)$ (resp.$GSp(n,F)$). We call them standard maximal tori.

Denote by $\operatorname{diag}(x_1, \ldots, x_m)$ the diagonal matrix which has on the diagonal entries $x_1, \ldots, x_m$. For $x_1, \ldots, x_n \in F^\times$ set

$$a(x_1, \ldots, x_n) = \operatorname{diag}(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}).$$

This is a parametrization of the standard maximal torus in $Sp(n,F)$. For $x_1,\dots,x_n,x \in F^\times$ set

$$a(x_1,\dots,x_n,x) = (x_1,x_2,\dots,x_n,xx_n^{-1},xx_{n-1}^{-1},\dots,xx_1^{-1}).$$

This is a parametrization of the standard maximal torus in $GSp(n,F)$.

Let $\chi_1,\dots,\chi_n,\chi$ be characters of $F^\times$. We define the character $\chi_1 \otimes \dots \otimes \chi_n \otimes 1$ of $M_\emptyset^S$ by

$$(\chi_1 \otimes \dots \otimes \chi_n \otimes 1)(a(x_1,\dots,x_n)) = \chi_1(x_1)\dots\chi_n(x_n).$$

The character $\chi_1 \otimes \dots \otimes \chi_n \otimes \chi$ of $M_\emptyset^G$ is defined by

$$(\chi_1,\otimes \dots \otimes \chi_n \otimes \chi)(a(x_1,\dots,x_n,x)) = \chi_1(x_1)\chi_2(x_2)\dots\chi_n(x_n)\chi(x).$$

Note that in the case of $GSp(1,F) = GL(2,F)$ this parametrization of characters of the standard maximal torus differs from the usual one.

For a smooth representation π of $GL(n,F)$ and σ of $Sp(m,F)$ we denote by

$$\pi \rtimes \sigma$$

the parabolically induced representation of $Sp(n+m,F)$ by $\pi \otimes \sigma$ from the parabolic subgroup

$$P_{(n)}^S = \left\{ \begin{bmatrix} g & * & * \\ 0 & h & * \\ 0 & 0 & {}^\tau g^{-1} \end{bmatrix} \in Sp(n+m,F); g \in GL(n,F), h \in Sp(m,F) \right\}.$$

Here ${}^\tau g$ denotes the transposed matrix of g with respect to the second diagonal. The representation $\pi \otimes \sigma$ maps

$$\begin{bmatrix} g & * & * \\ 0 & h & * \\ 0 & 0 & {}^\tau g^{-1} \end{bmatrix}$$

to $\pi(g) \otimes \sigma(h)$.

For a representation ρ of $Sp(n+m,F)$ of finite length, we denote by

$$s_{(n)}(\rho)$$

the Jacquet module of ρ with respect to the parabolic subgroup $P^S_{(n)}$. The action of the Levi factor

$$\left\{\begin{bmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & {}^{\tau}g^{-1} \end{bmatrix}; g \in GL(n,F), h \in Sp(m,F)\right\}$$

is again the quotient action twisted by the modular character to -1/2. Note that the Levi factor is naturally isomorphic to

$$GL(n,F) \times Sp(m,F).$$

We denote the Grothendieck group of the category of all smooth representations of $Sp(n,F)$ (resp. $GSp(n,F)$) of finite length by $R_n(S)$ (resp. $R_n(G)$). Set

$$R(S) = \underset{n\geq 0}{\oplus} R_n(S),$$

$$R(G) = \underset{n\geq 0}{\oplus} R_n(G).$$

One lifts $\rtimes$ to a mapping

$$\rtimes : R \times R(S) \to R(S).$$

For a smooth representation σ of $Sp(n,F)$ of finite length, we denote by

$$\mu^*(\sigma)$$

the sum of semi simplifications of $s_{(k)}(\sigma)$, $0 \leq k \leq n$. Then we can consider $\mu^*(\sigma) \in R \otimes R(S)$. We lift μ^* to an additive mapping form $R(S)$ into $R \otimes R(S)$.

For an integer $0 \leq k \leq n$ set

$$P^G_{(k)} = \left\{\begin{bmatrix} g & * & * \\ 0 & h & * \\ 0 & 0 & \psi(h)^{\tau}g^{-1} \end{bmatrix} \in GSp(n,F); g \in GL(k,F), h \in GSp(n-k,F)\right\}$$

Then $P_{(k)}^G$, $1 \leq k \leq n$, are all the standard maximal proper parabolic subgroups. Note that $P_{(0)}^G = GSp(n,F)$. The image of the homomorphism

$$(g,h) \mapsto \begin{bmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \psi(h)^\tau g^{-1} \end{bmatrix}$$

of $G(k,F) \times GSp(n-k,F)$ will be denote by $M_{(k)}^G$. The Levi factor $M_{(k)}^G$ of $P_{(k)}^G \subseteq GSp(n,F)$ is naturally isomorphic to

$$GL(k,F) \times GSp(n-k,F).$$

Therefore, one can define in the same way the multiplication $\rtimes$ of representations of $GL(n,F)$ with representations of $GSp(m,F)$. One lifts it to a biadditive mapping

$$\rtimes : R \times R(G) \to R(G).$$

The symbols $\times$ and $\rtimes$ will denote in further operations among representations, except if it is stated that they are considered as operations between Grothendieck groups. For more informations about the operation $\rtimes$ one should consult [T4] (see also [T6] and [T7]).

One defines analogously

$$\mu^* : R(G) \to R \otimes R(G).$$

There are the obvious cones of positive elements in $R, R(S)$ and $R(G)$. Therefore, we have partial orders on these groups.

Let π be a representation of $GL(n,F)$ (resp. $GSp(n,F)$), and let χ be a character of $F^\times$. Then $\chi\pi$ denotes the representation $g \mapsto \chi(g)\pi(g)$. One remark is necessary in the above definition, regarding the characters. The characters of $F^\times$ are considered also as characters of $GL(n,F)$ in a standard way, using the determinant homomorphism. We consider characters of $F^\times$ as characters of $GSp(n,F)$, using the composition with ψ.

For a representation π of $GL(n,F)$, $\tilde{\pi}$ denotes the smooth contragredient of π, while ${}^{\tau}\pi^{-1}$ denotes the representation $g \mapsto \pi({}^{\tau}g^{-1})$. If π is an irreducible smooth representation of $GL(n,F)$ and if σ is a similar representation of $GSp(m,F)$ (resp. $Sp(m,F)$), then the following equality

$$\pi \rtimes \sigma = \tilde{\pi} \rtimes \omega_\pi \sigma \qquad (\text{resp. } \pi \rtimes \sigma = \tilde{\pi} \rtimes \sigma) \tag{1.1}$$

holds in $R(G)$ (resp. $R(S)$). Here ω_π denotes the central character of π, which is considered as a character of $F^\times$ (the center of $GL(n,F)$ is identified with $F^\times$ in a standard way). If χ is a character of $F^\times$, then we have

$$\chi(\pi \rtimes \sigma) \cong \pi \rtimes (\chi\sigma) \tag{1.2}$$

when σ is a representation of $GSp(m,F)$.

We denote by D the set of all equivalence classes of the irreducible essentially square integrable representations of $GL(n,F)$'s when $n \geq 1$. The essentially square integrable representations are representations which become square integrable representations modulo center, after a twist with a suitable character of the group. For $\delta \in D$, there exists a unique real number $e(\delta)$ and there exists a unique $\delta^u \in D$ which is unitarizable, such that

$$\delta = |\det|_F^{e(\delta)}\, \delta^u.$$

Set

$$D_+ = \{\delta \in D; e(\delta) > 0\}$$

Denote by $T(S)$ the set of all equivalence classes of the irreducible tempered smooth representations of $Sp(n,F)$'s for all $n \geq 0$.

Take $t = ((\delta_1, \dots, \delta_n), \tau) \in M(D_+) \times T(S)$ where $M(D_+)$ denotes the set of all finite multisets in D_+. Choose a permutation p of the set $\{1, 2, \dots, n\}$ such that

$$e(\delta_{p(1)}) \geq e(\delta_{p(2)}) \geq \dots \geq e(\delta_{p(n)}).$$

Then the representation

$$\delta_{p(1)} \times \delta_{p(2)} \times \dots \times \delta_{p(n)} \rtimes \tau$$

has a unique irreducible quotient which will be dented by $L(t)$. This is the Langlands' classification for the symplectic groups. The mapping

$$t \mapsto L(t)$$

is a one-to-one parameterization of all irreducible representations by $M(D_+) \times T(S)$.

Denote by $T(G)$ the set of all equivalence classes of the irreducible essentially tempered smooth representations of $GSp(n, F)$'s, $n \geq 0$. Then one defines in the same way $L(t)$ for $t \in M(D_+) \times T(G)$. This is the Langlands' classification for GSp-groups.

For a reductive group G over F, $\tilde{G}$ will denote the set of all equivalence classes of the irreducible smooth representations of G. The subset of all unitarizable classes will be denoted by $\hat{G}$. The set of all cuspidal classes in $\tilde{G}$ is denoted by $C(G)$. Let $C^u(G)$ be the set of all unitarizable classes in $C(G)$. The trivial representation of G on a one dimensional vector space will be denoted by 1_G.

2. Jacquet modules of induced representations of $GSp(2)$

In this section we shall present the formulas for $\mu^*(\pi)$ where π is a parabolically induced representation of $GSp(2,F)$.

We shall first recall of the case of $GSp(1,F) = GL(2,F)$. Take an admissible representations π of $GL(1,F) = F^\times$ and σ of $GSp(0,F) = F^\times$, which are of finite length. They must be finite dimensional in this case. Suppose that π has a central character, say ω_π. Then

$$m^*(\pi) = 1 \otimes \pi + \pi \otimes 1$$

and

$$\mu^*(\sigma) = 1 \otimes \sigma.$$

Now we have the following formula

$$\mu^*(\pi \rtimes \sigma) = 1 \otimes \pi \rtimes \sigma + [\pi \otimes \sigma + \tilde{\pi} \otimes \omega_\pi \sigma]. \tag{2.1}$$

Note that $\pi \otimes \sigma$ is a quotient and $\tilde{\pi} \otimes \omega_\pi \sigma$ is a subrepresentation of $s_{(1)}(\pi \rtimes \sigma)$. In the above formulas on the right hand side, we are actually taking the semi simplifications of that representations.

We pass now to the case of $GSp(2,F)$. The following formulas follow from Theorem 5.2. of [T6], or they can be obtained, after some explicit calculations, from the Geometric Lemma from [BZ], or from [C].

We fix an admissible representations π of $GL(2,F)$ and a similar representation σ of $GSp(0,F)$. We suppose that the both representations are of finite length. We shall assume also that π has a central character. It will be denoted by ω_π. Write

$$m^*(\pi) = 1 \otimes \pi + \sum_i \pi_i^1 \otimes \pi_i^2 + \pi \otimes 1$$

and

$$\mu^*(\sigma) = 1 \otimes \sigma$$

where $\sum_i \pi_i^1 \otimes \pi_i^2$ is a decomposition into a sum of irreducible representations. Now we have

$$\mu^*(\pi \rtimes \sigma) = 1 \otimes \pi \rtimes \sigma + \tag{2.2}$$

$$\left[\sum_i \pi_i^1 \otimes \pi_i^2 \rtimes \sigma + \sum_i \tilde{\pi}_i^2 \otimes \pi_i^1 \rtimes \omega_{\pi_i^2}\sigma\right] +$$

$$\left[\pi \otimes \sigma + \tilde{\pi} \otimes \omega_\pi \sigma + \sum_i \pi_i^1 \times \tilde{\pi}_i^2 \otimes \omega_{\pi_i^2}\sigma\right].$$

Fix admissible representations π of $GL(1,F)$ and σ of $GSp(1,F)$, which are of finite length. Suppose that π has a central character, say ω_π. Write

$$m^*(\pi) = 1 \otimes \pi + \pi \otimes 1,$$

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_i \sigma_i^1 \otimes \sigma_i^2$$

We have

$$\mu^*(\pi \rtimes \sigma) = 1 \otimes \pi \rtimes \sigma + \tag{2.3}$$

$$\left[\pi \otimes \sigma + \tilde{\pi} \otimes \omega_\pi \sigma + \sum_i \sigma_i^1 \otimes \pi \rtimes \sigma_i^2\right] +$$

$$\left[\sum_i \pi \times \sigma_i^1 \otimes \sigma_i^2 + \sum_i \sigma_i^1 \times \tilde{\pi} \otimes \omega_\pi \sigma_i^2\right].$$

Analogous formulas can be written easily for $Sp(2,F)$, using [T6]. Such formulas can be obtained also directly, by "restriction" of the above formulas for $GSp(2,F)$.

3. Induced representations of $GSp(2)$

Let $P = MN$ be a proper parabolic subgroup of $GSp(2, F)$ and let σ be an irreducible smooth representation of M. If σ is a cuspidal representation and if P is not a minimal parabolic subgroup, then J.-L. Waldspurger and F. Shahidi have determined when $\mathrm{Ind}_P^G(\sigma)$ reduces (see the fourth section). If this is not the case, then σ is an irreducible subquotient of a principal series representations of M. In this section we shall see when $\mathrm{Ind}_P^G(\sigma)$ reduces in this case, what are the Langlands' parameters of the irreducible subquotients and what are the multiplicities. The analysis of the induced representations which we make in this section, was done in the unramified case by F. Rodier ([R2]). Therefore, these calculations complete the Rodier's investigations in [R2]. We shall get the answer by a detailed study of the principal series representations of $GSp(2, F)$ and their Jacquet modules for intermediate parabolic subgroups.

First we have a direct consequence of Theorem 7.5. of [T4].

Lemma 3.1. *If $\chi_1, \chi_2 \in (F^\times)\hat{}$ and $\sigma \in (F^\times)\tilde{}$, then $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible. In particular, the unitary principal series representations of $GSp(2, F)$ are irreducible.* $\square$

For a proof, one may consult [T4]. The proof in [T4] uses the Key's result in [Ke] which applies to $Sp(n, F)$. Then one gets the information about the irreducibility using the Clifford theory for the reductive p-adic groups, which was developed in [GeKn].

We have now a special case of Theorem 7.9. of [T4] which describes a necessary and sufficient conditions for the reducibility of the non-unitary principal series representations.

Lemma 3.2. *Let $\chi_1, \chi_2, \sigma \in (F^\times)\tilde{}$. The representation $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible if and only if $\chi_1 \neq \nu^{\pm 1}$, $\chi_2 \neq \nu^{\pm 1}$ and $\chi_1 \neq \nu^{\pm 1}\chi_2^{\pm 1}$.* $\square$

Let us say a few words about the proof in [T4]. If $\chi_1 = \nu^{\pm 1}$ or $\chi_2 = \nu^{\pm 1}$ or $\chi_1 = \nu^{\pm}\chi_2^{\pm 1}$, then the induction in the stages and the reducibilities for

$GL(2,F)$ or $GSp(1,F) = GL(2,F)$, imply the reducibilities of $\chi_1 \times \chi_2 \rtimes \sigma$. In the case when $\chi_1 \neq \nu^{\pm 1}, \chi_2 \neq \nu^{\pm 1}$ and $\chi_1 \neq \nu^{\pm 1}\chi_2^{\pm}$, it is enough to consider the case when χ_1 or χ_2 is not unitary. Then the properties of the Langlands' classification imply the irreducibility.

Suppose that $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible. Then using the fact that $R(G)$ is R-module, and the relation (1.1), one gets that $\chi_1 \times \chi_2 \rtimes \sigma$ is equivalent to a non-unitary principal series representation $\chi_1' \times \chi_2' \rtimes \sigma'$ where $e(\chi_1') \geq e(\chi_2') \geq 0$. Take

$$i = \begin{cases} 0 & \text{if} \quad e(\chi_1') = 0 \\ 1 & \text{if} \quad e(\chi_1') > 0 \text{ and } e(\chi_2') = 0 \\ 2 & \text{if} \quad e(\chi_2') > 0. \end{cases}$$

Let τ be the product of $\chi_j', j > i$, and of σ. Then

$$\chi_1 \times \chi_2 \rtimes \sigma = L((\chi_1', \ldots, \chi_i', \tau)).$$

In the rest of this section we shall study the non-unitary principal series representations $\chi_1 \times \chi_2 \rtimes \sigma$ when they reduce. The cases when this situation occurs are known from Lemma 3.2.

We shall consider first the case when $\chi_1 \otimes \chi_2 \otimes \sigma$ is a regular character. This case will be treated in the following four lemmas. One can prove directly that $\chi_1 \otimes \chi_2 \otimes \sigma$ is regular if and only if $\chi_1 \neq 1_{F^\times}$, $\chi_2 \neq 1_{F^\times}$ and $\chi_1 \neq \chi_2^{\pm 1}$ ([T4], Proposition 8.1., (b)).

F. Rodier attached to any regular character φ of a maximal split torus in a split reductive group over F, a non-negative integer $s(\varphi)$ (for the definition of the function s, one can consult [R1] or [T4]). The number $s(\varphi)$ is less than or equal to the semi-simple rank of the group. The length of the non-unitary principal series representation determined by φ is $2^{s(\varphi)}$.

Let $\chi_1 \otimes \chi_2 \otimes \sigma$ be a regular character of $M_\emptyset^G$. Since we consider the case when $\chi_1 \times \chi_2 \rtimes \sigma$ is reducible, we assume that $s(\chi_1 \otimes \chi_2 \otimes \sigma) \geq 1$ (if $s(\chi_1 \otimes \chi_2 \otimes \sigma) = 0$, then obviously χ_1 and χ_2 satisfy the conditions of Lemma 3.2.). If $s(\chi_1 \otimes \chi_2 \otimes \sigma) = 1$, then $\chi_1 \otimes \chi_2 \otimes \sigma$ is associate either to a character of the form $\nu^{1/2}\chi \otimes \nu^{-1/2}\chi \otimes \sigma'$ where $\chi \notin \{\xi, \nu^{\pm 1/2}\xi, \nu^{\pm 3/2}\}$ for any $\xi \in (F^\times)\hat{}$ such that $\xi^2 = 1_{F^\times}$, or it is associate to a character of the form $\chi \otimes \nu \otimes \sigma'$ where $\chi \notin \{1_{F^\times}, \nu^{\pm 1}, \nu^{\pm 2}\}$ (see [T4]). In the following two lemmas we deal with these two cases, when $s(\chi_1 \otimes \chi_2 \otimes \sigma) = 1$. They are F. Rodier's results.

For a connected reductive group $G(F)$ over F, the Steinberg representation of G will be denoted by St_G. In the rest of this paper we shall often write G instead of $G(F)$.

Lemma 3.3. *Let $\chi, \xi, \sigma \in (F^\times)^\sim$. Suppose that $\chi \notin \{\xi, \nu^{\pm 1/2}\xi, \nu^{\pm 3/2}\}$ for any ξ such that $\xi^2 = 1_{F^\times}$. Then $\chi \mathrm{St}_{GL(2)} \rtimes \sigma$ and $\chi 1_{GL(2)} \rtimes \sigma$ are irreducible representations. We have*

$$\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma = \chi 1_{GL(2)} \rtimes \sigma + \chi \mathrm{St}_{GL(2)} \rtimes \sigma$$

in $R(G)$. For the Langlands' parameters we have

$$\chi \mathrm{St}_{GL(2)} \rtimes \sigma = L((\chi \mathrm{St}_{GL(2)}, \sigma)) \quad \text{if} \quad e(\chi) > 0,$$

$$\chi \mathrm{St}_{GL(2)} \rtimes \sigma = L((\chi \mathrm{St}_{GL(2)} \rtimes \sigma)) \quad \text{if} \quad e(\chi) = 0,$$

$$\chi 1_{GL(2)} \rtimes \sigma = L((\nu^{1/2}\chi, \nu^{-1/2}\chi, \sigma)) \quad \text{if} \quad e(\chi) > 1/2,$$

$$\chi 1_{GL(2)} \rtimes \sigma = L((\nu^{1/2}\chi, \nu^{-1/2}\chi \rtimes \sigma)) \quad \text{if} \quad e(\chi) = 1/2$$

and

$$\chi 1_{GL(2)} \rtimes \sigma = L((\nu^{1/2}\chi, \nu^{1/2}\chi^{-1}, \nu^{-1/2}\chi\sigma)) \quad \text{if} \quad 1/2 > e(\chi) \geq 0.$$

Also

$$\chi \mathrm{St}_{GL(2)} \rtimes \sigma \cong \chi^{-1} \mathrm{St}_{GL(2)} \rtimes \chi^2 \sigma$$

and

$$\chi 1_{GL(2)} \rtimes \sigma \cong \chi^{-1} 1_{GL(2)} \rtimes \chi^2 \sigma.$$

Proof. Suppose that χ satisfies the assumption of the lemma. Assume that $e(\chi) \geq 0$. First, $\nu^{1/2}\chi \otimes \nu^{-1/2}\chi \otimes \sigma$ is regular and $s(\nu^{1/2}\chi \otimes \nu^{-1/2}\chi \otimes \sigma) = 1$ by Proposition 8.2 of [T4]. The length of $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$ is two ([R1]). We have

$$\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma = \chi \mathrm{St}_{GL(2)} \rtimes \sigma + \chi 1_{GL(2)} \rtimes \sigma$$

in $R(G)$. Thus, both constituents are irreducible. The Langlands' parameter of $\chi \mathrm{St}_{GL(2)} \rtimes \sigma$ is evident. Note that we have an epimorphism of

$$\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma \cong \nu^{1/2}\chi \times \nu^{1/2}\chi^{-1} \rtimes \nu^{-1/2}\chi\sigma$$

onto $\chi 1_{GL(2)} \rtimes \sigma$. From this we can read easily the Langlands' parameter of $\chi 1_{GL(2)} \rtimes \sigma$. This proves the lemma. □

Lemma 3.4. *Let $\chi, \sigma \in (F^\times)^\sim$. Suppose that $\chi \notin \{1_{F^\times}, \nu^{\pm 1}, \nu^{\pm 2}\}$. Then $\chi \rtimes \sigma \mathrm{St}_{GSp(1)}$ and $\chi \rtimes \sigma 1_{GSp(1)}$ are irreducible representations. We have*

$$\chi \times \nu \rtimes \nu^{-1/2}\sigma = \chi \rtimes \sigma \mathrm{St}_{GSp(1)} + \chi \rtimes \sigma 1_{GSp(1)}$$

in $R(G)$. For the Langlands' parameters we have

$$\chi \rtimes \sigma \mathrm{St}_{GSp(1)} = L((\chi, \sigma \mathrm{St}_{GSp(1)})) \quad \text{if} \quad e(\chi) > 0,$$

$$\chi \rtimes \sigma \mathrm{St}_{GSp(1)} = L((\chi \rtimes \sigma \mathrm{St}_{GSp(1)})) \quad \text{if} \quad e(\chi) = 0,$$

$$\chi \rtimes \sigma 1_{GSp(1)} = L((\chi, \nu, \nu^{-1/2}\sigma)) \quad \text{if} \quad e(\chi) > 0$$

and

$$\chi \rtimes \sigma 1_{GSp(1)} = L((\nu, \chi \rtimes \nu^{-1/2}\sigma)) \quad \text{if} \quad e(\chi) = 0.$$

We have also

$$\chi \rtimes \sigma \mathrm{St}_{GSp(1)} \cong \chi^{-1} \rtimes \chi\sigma \mathrm{St}_{GSp(1)},$$

$$\chi \rtimes \sigma 1_{GSp(1)} \cong \chi^{-1} \rtimes \chi\sigma 1_{GSp(1)}.$$

Proof. We prove the lemma in the same way as we proved Lemma 3.3., because $\chi \otimes \nu \otimes \nu^{-1/2}\sigma$ is regular and $s(\chi \otimes \nu \otimes \nu^{-1/2}\sigma) = 1$ when χ satisfies the assumptions of the lemma. □

We are going now to study regular $\chi_1 \otimes \chi_2 \otimes \sigma$ with $s(\chi_1 \otimes \chi_2 \otimes \sigma) = 2$. Recall that by the eight section of [T4], $\chi_1 \otimes \chi_2 \otimes \sigma$ is associated to $\nu^2 \otimes \nu \otimes \sigma'$ or $\nu\xi_o \otimes \xi_o \otimes \sigma'$ where $\xi_o \in (F^\times)\hat{}$ is of order two and $\sigma' \in (F^\times)^\sim$. This situation is the subject of the following two lemmas.

First we have a very well known situation in the following lemma ([C]).

Lemma 3.5. *For $\sigma \in (F^\times)^\sim$ the following equalities hold in $R(G)$*

$$\nu^2 \times \nu \rtimes \nu^{-1/2}\sigma = \nu^{3/2}\mathrm{St}_{GL(2)} \rtimes \nu^{-1/2}\sigma + \nu^{3/2}1_{GL(2)} \rtimes \nu^{-1/2}\sigma =$$

$$\nu^2 \rtimes \sigma\mathrm{St}_{GSp(1)} + \nu^2 \rtimes \sigma 1_{GSp(1)}$$

and

$$\nu^2 \rtimes \sigma\mathrm{St}_{GSp(1)} = \nu\sigma\mathrm{St}_{GSp(2)} + L((\nu^2, \sigma\mathrm{St}_{GSp(1)})),$$

$$\nu^2 \rtimes \sigma 1_{GSp(1)} = \nu\sigma 1_{GSp(2)} + L((\nu^{3/2}\mathrm{St}_{GL(2)}, \nu^{-1/2}\sigma)),$$

$$\nu^{3/2}\mathrm{St}_{GL(2)} \rtimes \nu^{-1/2}\sigma = \nu\sigma\mathrm{St}_{GSp(2)} + L((\nu^{3/2}\mathrm{St}_{GL(2)}, \nu^{-1/2}\sigma)),$$

$$\nu^{3/2}1_{GL(2)} \rtimes \nu^{-1/2}\sigma = \nu\sigma 1_{GSp(2)} + L((\nu^2, \sigma\mathrm{St}_{GSp(1)})).$$

Also

$$\nu^{-1/2}\sigma 1_{GSp(2)} = L((\nu^2, \nu, \nu^{-1/2}\sigma)).$$

Proof. Note that $\nu^2 \times \nu \rtimes \sigma \cong \sigma(\nu^2 \times \nu \rtimes 1_{F^\times})$ and the structure of the representation $\nu^2 \times \nu \times \nu^{-3/2}$ is well known. The length of the representation $\nu^2 \times \nu \rtimes \nu^{-1/2}\sigma$ is four and it is a multiplicity one representation.

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$$\nu\sigma\mathrm{St}_{GSp(2)}, \quad \nu\sigma 1_{GSp(2)}, \quad L((\nu^2, \sigma\mathrm{St}_{GSp(1)})) \quad \text{and} \quad L((\nu^{3/2}\mathrm{St}_{GL(2)}, \nu^{-1/2}\sigma))$$

are clearly subquotients of $\nu^2 \times \nu \rtimes \nu^{-1/2}\sigma$. Note that $L((\nu^2, \sigma\mathrm{St}_{GSp(1)}))$ is a quotient of $\nu^2 \rtimes \sigma\mathrm{St}_{GSp(1)}$ and $L((\nu^{3/2}\mathrm{St}_{GL(2)}, \nu^{-1/2}\sigma))$ is a quotient of $\nu^{3/2}\mathrm{St}_{GL(2)} \rtimes \nu^{-1/2}\sigma$. Also $\nu\sigma\mathrm{St}_{GSp(2)}$ is not a subquotient of $\nu^2 \rtimes \sigma 1_{GSp(1)}$, and it is also not a subquotient of $\nu^{3/2}1_{GL(2)} \rtimes \nu^{-1/2}\sigma$. Also we have an epimorphism from $\nu^2 \rtimes \nu \rtimes \nu^{-1/2}\sigma$ onto

$$\nu^{3/2}1_{GL(2)} \rtimes \nu^{-1/2}\sigma \text{ and } \nu^2 \rtimes \sigma 1_{GSp(1)}.$$

Therefore, both representations contain $L((\nu^2, \nu, \nu^{-1/2}\sigma)) = \nu\sigma 1_{GSp(2)}$ as quotients.

We can conclude now that in $R(G)$ we have

$$\nu\sigma\mathrm{St}_{GSp(2)} + L((\nu^2, \sigma\mathrm{St}_{GSp(1)})) \leq \nu^2 \rtimes \sigma\mathrm{St}_{GSp(1)},$$

$$\nu\sigma 1_{GSp(2)} \leq \nu^2 \rtimes \sigma 1_{GSp(1)},$$

$$\nu\sigma\mathrm{St}_{GSp(2)} + L((\nu^{3/2}\mathrm{St}_{GL(2)}, \nu^{-1/2}\sigma)) \leq \nu^{3/2}\mathrm{St}_{GL(2)} \rtimes \nu^{-1/2}\sigma$$

and

$$\nu\sigma 1_{GSp(2)} \leq \nu^{3/2}1_{GL(2)} \rtimes \nu^{-1/2}\sigma.$$

If we know that the second and the fourth inequalities are strict, we have the complete proof. We conclude it from the fact that the Jacquet modules for a minimal parabolic subgroup of the left hand sides are irreducible, while the Jacquet modules of the right hand sides are of lengths four (one can obtain it from (2.1), (2.2) and (2.3)). □

Lemma 3.6. *Let $\xi_o \in (F^\times)\hat{}$ be of order two and let $\sigma \in (F^\times)^\sim$. Then the representation $\nu\xi_o \times \xi_o \rtimes \sigma$ contains a unique essentially square integrable subquotient. This subquotient will be denoted by $\delta([\xi_o, \nu\xi_o], \sigma)$. We have in $R(G)$*

$$\nu\xi_o \times \xi_o \rtimes \sigma = \nu^{1/2}\xi_o\mathrm{St}_{GL(2)} \rtimes \sigma + \nu^{1/2}\xi_o 1_{GL(2)} \rtimes \sigma =$$

$$\nu^{1/2}\xi_o\mathrm{St}_{GL(2)} \rtimes \xi_o\sigma + \nu^{1/2}\xi_o 1_{GL(2)} \rtimes \xi_o\sigma$$

and

$$\nu^{1/2}\xi_o\mathrm{St}_{GL(2)} \rtimes \sigma = \delta([\xi_o, \nu\xi_o], \sigma) + L((\nu^{1/2}\xi_o\mathrm{St}_{GL(2)}, \sigma)),$$

$$\nu^{1/2}\xi_o 1_{GL(2)} \rtimes \sigma = L((\nu^{1/2}\xi_o\mathrm{St}_{GL(2)}, \xi_o\sigma)) + L((\nu\xi_o, \xi_o \rtimes \sigma)).$$

Proof. Note that the length of $\nu\xi_o \times \xi_o \rtimes \sigma$ is four and that $\nu\xi_o \times \xi_o \rtimes \sigma$ has a unique essentially square integrable subquotient by [R1]. Also

$$\delta([\xi_o, \nu\xi_o], \sigma),\ L((\nu^{1/2}\xi_o\mathrm{St}_{GL(2)}, \sigma)),\ L((\nu^{1/2}\xi_o\mathrm{St}_{GL(2)}, \xi_o\sigma))$$

and $L((\nu\xi_o, \xi_o \rtimes \sigma))$ are subquotients of $\nu\xi_o \times \xi_o \rtimes \sigma$ and $L((\nu^{1/2}\xi_o\mathrm{St}_{GL(2)}, \sigma))$ is a quotient of $\nu^{1/2}\xi_o\mathrm{St}_{GL(2)} \rtimes \sigma$.

Note that we have an epimorphism from $\nu\xi_o \times \xi_o \rtimes \sigma$ onto $\nu^{1/2}\xi_o 1_{GL(2)} \rtimes \sigma$. Thus $L((\nu\xi_o, \xi_o \rtimes \sigma))$ is a quotient of $\nu^{1/2}\xi_o 1_{GL(2)} \rtimes \sigma$.

We also see that we have an epimorphism

$$\xi_o \times \nu\xi_o \rtimes \xi_o\sigma \to L((\nu^{1/2}\xi_o\mathrm{St}_{GL(2)}, \xi_o\sigma)).$$

Thus we have

$$L((\nu^{1/2}\xi_o\mathrm{St}_{GL(2)}, \nu^{-1}\xi_o\tilde{\sigma})) \hookrightarrow \xi_o \times \nu^{-1}\xi_o \rtimes \xi_o\tilde{\sigma} \cong \xi_o \times \nu\xi_o \rtimes \nu^{-1}\tilde{\sigma}.$$

First, we see from the Frobenius reciprocity that the Jacquet module of the representation $L((\nu^{1/2}\xi_o\mathrm{St}_{GL(2)}, \sigma))$ for the standard minimal parabolic subgroup contains $\xi_o \otimes \nu^{-1}\xi_o \otimes \nu\sigma$ and $\xi_o \otimes \nu\xi_o \otimes \xi_o\sigma$ as subquotients.

In the same way one concludes that the other three irreducible subquotients of $\nu\xi_o \times \xi_o \rtimes \sigma$ have at least two different subquotients in the Jacquet module. Note that the length of the Jacquet module of $\nu\xi_o \times \xi_o \rtimes \sigma$ is eight. Therefore, all Jacquet modules of irreducible constituents have the lengths two.

At the end we compute the Jacquet module of $\nu^{1/2}\xi_o 1_{GL(2)} \rtimes \sigma$ using (2.2). We obtain that the semi simplification is

$$\nu\xi_o \otimes \xi_o \otimes \sigma + \xi_o \otimes \nu^{-1}\xi_o \otimes \nu\sigma + \nu\xi_o \otimes \xi_o \otimes \xi_o\sigma + \xi_o \otimes \nu\xi_o \otimes \xi_o\sigma.$$

This implies that $L((\nu^{1/2}\xi_o\mathrm{St}_{GL(2)},\sigma))$ is a subquotient of $\nu^{1/2}\xi_o 1_{GL(2)} \rtimes \sigma$ since $\nu\xi_o \otimes \xi_o \otimes \sigma$ is regular.

From the above facts one completes directly the proof. □

F. Rodier considered representations $\delta([\xi_o,\nu\xi_o],\sigma)$ in [R1].

Up to now we have analyzed the situation when $\chi_1 \otimes \chi_2 \otimes \sigma$ is regular. Suppose that $\chi_1 \otimes \chi_2 \otimes \sigma$ is not regular and that $\chi_1 \times \chi_2 \rtimes \sigma$ is not irreducible. Then $\chi_1 \otimes \chi_2 \otimes \sigma$ is associate to a character of the form $\nu \otimes 1_{F^\times} \otimes \sigma'$, or $\nu \otimes \nu \otimes \sigma'$, or $\nu^{1/2}\xi \otimes \nu^{-1/2}\xi \otimes \sigma'$ where $\xi, \sigma' \in (F^\times)^\sim$ and $\xi^2 = 1_{F^\times}$ (see the beginning of this section or the seventh section of [T4]). In the rest of this section we shall analyze these irregular cases.

Lemma 3.7. *Suppose that $\xi \in (F^\times)\hat{}$ satisfies $\xi^2 = 1_{F^\times}$. Let $\sigma \in (F^\times)^\sim$. Then we have*

$$\nu^{1/2}\xi \times \nu^{-1/2}\xi \rtimes \sigma = \xi\mathrm{St}_{GL(2)} \rtimes \sigma + \xi 1_{GL(2)} \rtimes \sigma$$

in $R(G)$. Both representations on the right hand side are irreducible and we have

$$\xi\mathrm{St}_{GL(2)} \rtimes \sigma = L((\xi\mathrm{St}_{GL(2)} \rtimes \sigma)),$$

$$\xi 1_{GL(2)} \rtimes \sigma = L((\nu^{1/2}\xi, \nu^{1/2}\xi, \nu^{-1/2}\xi\sigma)).$$

Proof. It is enough to prove that the above two representations are irreducible. From (2.2) we obtain

$$\mu^*(\xi\mathrm{St}_{GL(2)} \rtimes \sigma) = 1 \otimes \xi\mathrm{St}_{GL(2)} \rtimes \sigma +$$

$$\left[\nu^{1/2}\xi \otimes \nu^{-1/2}\xi \rtimes \sigma + \nu^{1/2}\xi \otimes \nu^{1/2}\xi \rtimes \nu^{-1/2}\xi\sigma\right] +$$

$$\left[\xi\mathrm{St}_{GL(2)} \otimes \sigma + \xi\mathrm{St}_{GL(2)} \otimes \sigma + \nu^{1/2}\xi \times \nu^{1/2}\xi \otimes \nu^{-1/2}\xi\sigma\right].$$

We see that the semi simplification of the Jacquet module for the standard minimal parabolic subgroup is

$$2(\xi\nu^{1/2} \otimes \xi\nu^{-1/2} \otimes \sigma + \nu^{1/2}\xi \otimes \nu^{1/2}\xi \otimes \nu^{-1/2}\xi\sigma).$$

Let π be an irreducible subquotient of $\xi\mathrm{St}_{GL(2)} \rtimes \sigma$ which has $\nu^{1/2}\xi \otimes \nu^{1/2}\xi \otimes \nu^{-1/2}\xi\sigma$ for a subquotient of the Jacquet module. Since $\nu^{1/2}\xi \times \nu^{1/2}\xi \otimes \nu^{-1/2}\xi\sigma$ is irreducible, we obtain that $\xi^{1/2}\xi \otimes \nu^{1/2}\xi \otimes \nu^{-1/2}\xi\sigma$ appears with the multiplicity at least two in the Jacquet module of π. Note that the Jacquet modules of $\nu^{1/2}\xi \otimes \nu^{-1/2}\xi \rtimes \sigma$ and $\nu^{1/2}\xi \otimes \nu^{1/2}\xi \rtimes \nu^{-1/2}\xi\sigma$ are the same and they are

$$\nu^{1/2}\xi \otimes \nu^{-1/2}\xi \otimes \sigma + \nu^{1/2}\xi \otimes \nu^{1/2}\xi \otimes \nu^{-1/2}\xi\sigma.$$

(more precisely, these are semi simplifications). Since $\nu^{1/2}\xi \otimes \nu^{-1/2}\xi \rtimes \sigma$ and $\nu^{1/2}\xi \otimes \nu^{1/2}\xi \rtimes \nu^{-1/2}\xi\sigma$ are irreducible, we obtain that the Jacquet module of π has the length at least three, while for any other subquotient, the length is at lest two. Since the length of the Jacquet module of $\xi\mathrm{St}_{GL(2)} \rtimes \sigma$ is four, we see that $\xi\mathrm{St}_{GL(2)} \rtimes \sigma$ is irreducible.

In the same way we prove that $\xi 1_{GL(2)} \rtimes \sigma$ is irreducible. □

The following lemma was proved by F. Rodier.

Lemma 3.8. *For $\sigma \in (F^\times)^\sim$ we have in $R(G)$*

$$\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma = \nu^{1/2}\mathrm{St}_{GL(2)} \rtimes \nu^{-1/2}\sigma + \nu^{1/2}1_{GL(2)} \rtimes \nu^{-1/2}\sigma =$$

$$1_{F^\times} \times \nu \rtimes \nu^{-1/2}\sigma = 1_{F^\times} \rtimes \sigma\mathrm{St}_{GSp(1)} + 1_{F^\times} \rtimes \sigma 1_{GSp(1)}.$$

The representations $1_{F^\times} \rtimes \sigma\mathrm{St}_{GSp(1)}$ and $\nu^{1/2}\mathrm{St}_{GL(2)} \rtimes \nu^{-1/2}\sigma$ (resp. $\nu^{1/2}1_{GL(2)} \rtimes \nu^{-1/2}\sigma$) have exactly one irreducible subquotient in common. That subquotient is essentially tempered and it will be denoted by $\tau(S, \nu^{-1/2}\sigma)$) (resp. $\tau(T, \nu^{-1/2}\sigma)$). These two essentially tempered representations are not equivalent. We have in $R(G)$

$$\nu^{1/2}\mathrm{St}_{GL(2)} \rtimes \nu^{-1/2}\sigma = \tau(S, \nu^{-1/2}\sigma) + L((\nu^{1/2}\mathrm{St}_{GL(2)}, \nu^{-1/2}\sigma)),$$

$$\nu^{1/2}1_{GL(2)} \rtimes \nu^{-1/2}\sigma = \tau(T, \nu^{-1/2}\sigma) + L((\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)),$$

$$1_{F^\times} \rtimes \sigma \mathrm{St}_{GSp(1)} = \tau(S, \nu^{-1/2}\sigma) + \tau(T, \nu^{-1/2}\sigma)$$

and

$$1_{F^\times} \rtimes \sigma 1_{GSp(1)} = L((\nu^{1/2}\mathrm{St}_{GL(2)}, \nu^{-1/2}\sigma)) + L((\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)).$$

Proof. Denote the above four representations on the left hand sides by $\pi((2), S), \pi((2), T)$, $\pi((1), S)$ and $\pi((1), T)$ respectively.

Observe that $L((\nu^{1/2}\mathrm{St}_{GL(2)}, \nu^{-1/2}\sigma))$ is a subquotient of $\nu^{1/2}\mathrm{St}_{GL(2)} \rtimes \nu^{-1/2}\sigma$ and that $\pi((1), S)$ and $\pi((1), T)$ are completely reducible representations (they are essentially unitary). Using formulas (2.2) and (2.3) we obtain

$$\mu^*(\pi((2), S)) = 1 \otimes \pi((2), S) +$$

$$\left[\nu \otimes 1_{F^\times} \rtimes \nu^{-1/2}\sigma + 1_{F^\times} \otimes \sigma \mathrm{St}_{GSp(1)} + 1_{F^\times} \otimes \sigma 1_{GSp(1)}\right] +$$

$$\left[2 \cdot (\nu^{1/2}\mathrm{St}_{GL(2)} \otimes \nu^{-1/2}\sigma) + \nu^{-1/2}\mathrm{St}_{GL(2)} \otimes \nu^{1/2}\sigma + \nu^{1/2}1_{GSp(1)} \otimes \nu^{-1/2}\sigma\right],$$

$$\mu^*(\pi((2), T)) = 1 \otimes \pi((2), T) +$$

$$\left[1_{F^\times} \otimes \sigma \mathrm{St}_{GSp(1)} + 1_{F^\times} \otimes \sigma 1_{GSp(1)} + \nu^{-1} \otimes 1_{F^\times} \rtimes \nu^{1/2}\sigma\right] +$$

$$\left[\nu^{1/2}1_{GL(2)} \otimes \nu^{-1/2}\sigma + 2 \cdot \left(\nu^{-1/2}1_{GL(2)} \otimes \nu^{1/2}\sigma\right) + \nu^{-1/2}\mathrm{St}_{GL(2)} \otimes \nu^{1/2}\sigma\right],$$

$$\mu^*(\pi((1), S)) = 1 \otimes \pi((1), S) +$$

$$\left[2 \cdot \left(1_{F^\times} \otimes \sigma \mathrm{St}_{GSp(1)}\right) + \nu \otimes 1_{F^\times} \rtimes \nu^{-1/2}\sigma\right] +$$

$$2 \cdot \left[\nu^{1/2}\mathrm{St}_{GL(2)} \otimes \nu^{-1/2}\sigma + \nu^{1/2}1_{GL(2)} \otimes \nu^{-1/2}\sigma\right]$$

and

$$\mu^*(\pi((1),T)) = 1 \otimes \pi((1),T) +$$
$$\left[2 \cdot \left(1_{F^\times} \otimes \sigma 1_{GSp(1)}\right) + \nu^{-1} \otimes 1_{F^\times} \rtimes \nu^{1/2}\sigma\right] +$$
$$2 \cdot \left[\nu^{-1/2}\mathrm{St}_{GL(2)} \otimes \nu^{1/2}\sigma + \nu^{-1/2}1_{GL(2)} \otimes \nu^{1/2}\sigma\right].$$

From the Frobenius reciprocity and the last two formulas one obtains that the intertwining algebras of $\pi((1),S)$, and of $\pi((1),T)$ are at most two dimensional. Since $\pi((1),S)$ and $\pi((1),T)$ are completely reducible, the lengths of $\pi((1),S)$ and $\pi((1),T)$ are at most two. These representations are of multiplicity one.

The above formulas imply the following relations in the Grothendieck groups

$$s_{(2)}(\pi((2),S)) + s_{(2)}(\pi((1),S)) \not\leq s_{(2)}(\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma),$$
$$s_{(2)}(\pi((2),T)) + s_{(2)}(\pi((1),S)) \not\leq s_{(2)}(\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma).$$

Also

$$s_{(2)}(\pi_1) \not\leq s_{(2)}(\pi_2)$$

for $\pi_1, \pi_2 \in \{\pi((2),S),\ \pi((2),T),\ \pi((1),S),\ \pi((1),T))\}$, when $\pi_1 \neq \pi_2$. Thus, representations $\pi((2),S)$ and $\pi((1),S)$ have exactly one irreducible subquotient in common. Both representations are of length two. The same conclusion holds for $\pi((2),T)$ and $\pi((1),S)$. Since $L((\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma))$ is a quotient of $\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma$, it is easy to conclude that the lemma holds.

□

The following unramified situation was settled by F. Rodier in [R2].

Lemma 3.9. *Let $\sigma \in (F^\times)^\sim$. Then we have*

$$\nu \times \nu \rtimes \nu^{-1/2}\sigma = \nu \rtimes \sigma\mathrm{St}_{GSp(1)} + \nu \rtimes \sigma 1_{GSp(1)}.$$

in $R(G)$. Both representations on the right hand side are irreducible and we have

$$\nu \rtimes \sigma \mathrm{St}_{GSp(1)} = L((\nu, \sigma \mathrm{St}_{GSp(1)})),$$
$$\nu \rtimes \sigma 1_{GSp(1)} = L((\nu, \nu, \nu^{-1/2}\sigma)).$$

Proof. First note that $L((\nu, \nu, \nu^{-1/2}\sigma))$ is a unique irreducible quotient of $\nu \times \nu \rtimes \nu^{-1/2}\sigma$ and thus a unique irreducible quotient of $\nu \rtimes \sigma 1_{GSp(1)}$. It has multiplicity one in $\nu \times \nu \rtimes \nu^{-1/2}\sigma$. Also $L((\nu, \sigma \mathrm{St}_{GSp(1)}))$ is a quotient of $\nu \rtimes \sigma \mathrm{St}_{GSp(1)}$, and it has multiplicity one in $\nu \rtimes \sigma \mathrm{St}_{GSp}(1)$. Since the length of $\nu \times \nu \rtimes \nu^{-1/2}\sigma$ is two by 6.3. of [R2], the lemma follows directly (actually, it is enough to prove that $\nu \rtimes \sigma 1_{GSp(1)}$ or $\nu \rtimes \sigma \mathrm{St}_{GSp(1)}$ is an irreducible representation). □

From the preceding lemmas one concludes

Corollary 3.10. *Let $\chi, \sigma, \xi \in (F^\times)^\sim$.*

(i) *The representation $\chi \mathrm{St}_{GL(2)} \rtimes \sigma$ is irreducible if and only if $\chi 1_{GL(2)} \rtimes \sigma$ is irreducible and this is the case if and only if $\chi \notin \{\nu^{\pm 1/2}\xi, \nu^{\pm 3/2}\}$ for any ξ such that $\xi^2 = 1_{F^\times}$. If we have a reducibility, then we have a multiplicity one representation of length two.*

(ii) *The representation $\chi \rtimes \sigma \mathrm{St}_{GSp(1)}$ is irreducible if and only if $\chi \rtimes \sigma 1_{GSp(1)}$ is irreducible and this is the case if and only if $\chi \notin \{1_{F^\times}, \nu^{\pm 2}\}$. If we have a reducibility, then we have a multiplicity one representation of length two.* □

4. Classifications for $GSp(2)$

The classifications formulated in this section were obtained in the unramified case already by F. Rodier in [R2].

For a reductive group G over F, the relation among parabolic subgroups of being associate in an equivalence relation. If $\pi \in \tilde{G}$, then we shall say that it is supported in a class $\mathcal{P}$, if there exists $P \in \mathcal{P}$ such that π is a composition factor of a parabolically induced representation from P by an irreducible cuspidal representation of a Levi factor of P. This is the notion that W. Casselman called the type of a representation ([C]). Each π is supported exactly in one class. In $GSp(2,F)$ (resp. $Sp(2,F)$) there are exactly four classes. They are represented by $P_{\emptyset}^G, P_{(1)}^G, P_{(2)}^G$ and $P_{(0)}^G$ (resp. $P_{\emptyset}^S, P_{(1)}^S, P_{(2)}^S$ and $P_{(0)}^S$). Therefore, we shall say that π is supported for example in $P_{(1)}$. For representations supported in $P_{\emptyset}$ we shall say that they are supported in the minimal parabolic subgroups.

Now we can summarize from the last section the following

Theorem 4.1.

(i) *The representation $\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$, $\sigma \in (F^{\times})\hat{}$, has a unique irreducible subrepresentation, which will be denoted by $\sigma \mathrm{St}_{GSp(2)}$. This subrepresentation is square integrable. For different σ's we get subrepresentations which are not equivalent.*

(ii) *For each character $\xi_o \in (F^{\times})\hat{}$ of order two and each $\sigma \in (F^{\times})\hat{}$, the representation $\nu\xi_o \times \xi_o \rtimes \nu^{-1/2}\sigma$ has a unique irreducible subrepresentation. Denote it by $\delta([\xi_o, \nu\xi_o], \nu^{-1/2}\sigma)$. This representation is square integrable. The only non-trivial equivalences among such representations are*

$$\delta([\xi_o, \nu\xi_o], \nu^{-1/2}\sigma) \cong \delta([\xi_o, \nu\xi_o], \nu^{-1/2}\xi_o\sigma).$$

The square integrable representations defined in (i) and in (ii) are disjoint groups of representations. They exhaust all square integrable representations of $GSp(2,F)$ which are supported in the minimal parabolic subgroups. □

Theorem 4.2.

(i) *Representations* $\chi_1 \times \chi_2 \rtimes \sigma$, $\chi_1, \chi_2, \sigma \in (F^\times)\hat{}$, *are irreducible and* $\chi_1 \times \chi_2 \rtimes \sigma \cong \chi_1' \times \chi_2' \rtimes \sigma'$ *if and only if* $\chi_1 \otimes \chi_2 \otimes \sigma$ *and* $\chi_1' \otimes \chi_2' \otimes \sigma'$ *are associate.*

(ii) *Representations* $\chi \rtimes \sigma \mathrm{St}_{GSp(1)}$, $\chi, \sigma \in (F^\times)\hat{}$, $\chi \neq 1_{F^\times}$, *are irreducible and the only non-trivial equivalences among them are*

$$\chi \rtimes \sigma \mathrm{St}_{GSp(1)} = \chi^{-1} \rtimes \chi\sigma \mathrm{St}_{GSp(1)}.$$

(iii) *Let* $\sigma \in (F^\times)\hat{}$. *The representation* $1_{F^\times} \rtimes \sigma \mathrm{St}_{GSp(1)}$ *is a multiplicity one representation of length two. One irreducible constituent may be characterized as the common composition factor with* $\nu^{1/2}\mathrm{St}_{GL(2)} \rtimes \nu^{-1/2}\sigma$ *(resp.* $\nu^{1/2} 1_{GL(2)} \rtimes \nu^{-1/2}$*). This representation is denoted by*
$\tau(S, \nu^{-1/2}\sigma)$ *(resp.* $\tau(T, \nu^{-1/2}\sigma)$*). Among these representations there are no non-trivial equivalences.*

(iv) *For* $\chi, \sigma \in (F^\times)\hat{}$ *the representation* $\chi \mathrm{St}_{GL(2)} \rtimes \sigma$ *is irreducible. The only non-trivial equivalences are*

$$\chi \mathrm{St}_{GL(2)} \rtimes \sigma \cong \chi^{-1} \mathrm{St}_{GL(2)} \rtimes \chi^2 \sigma.$$

The irreducible representations considered in (i)-(iv) are tempered and they are not square integrable. They form four disjoint groups of representations. Each irreducible tempered representation supported in the minimal parabolic subgroups either belongs to one of the groups in (i)-(iv), or it is square integrable. □

By $\overline{\pi}$ we denote the complex conjugate representation to a representation π. An irreducible smooth representation π is called Hermitian if $\pi \cong \overline{\tilde{\pi}}$. The formula for the contragredient representation in the Langlands' classification is

$$L((\delta_1, \ldots, \delta_n, \tau))^{\sim} = L((\delta_1, \ldots, \delta_n, \omega_{\delta_1} \ldots \omega_{\delta_n} \tilde{\tau})),$$

$\delta_i \in D_+$, $\tau \in T(G)$ (ω_{δ_i} denotes the central character of δ_i). Now we have directly the following

Lemma 4.3. *Let $\chi, \sigma, \xi \in (F^\times)\hat{}$ such that $\xi^2 = 1_{F^\times}$. Let $\beta, \beta_1, \beta_2 > 0$. The following seven groups of representations are Hermitian and they exhaust all irreducible Hermitian representations of $GSp(2, F)$ which are supported in the minimal parabolic subgroups:*

(i) *irreducible tempered representations supported in the minimal parabolic subgroups,*

(ii) $L((\nu^\beta\chi, \nu^\beta\chi^{-1}, \nu^{-\beta}\sigma))$, $\chi^2 \neq 1_{F^\times}$ (for $\chi^2 = 1_{F^\times}$ see (iv)),

(iii) $L((\nu_1^\beta, \chi \rtimes \nu^{-\beta/2}\sigma))$, $\chi \neq 1_{F^\times}$ (for $\chi = 1_{F^\times}$ see (v)),

(iv) $L((\nu^{\beta_1}\xi,\ \nu^{\beta_2}\xi, \nu^{-(\beta_1+\beta_2)/2}\sigma))$,

(v) $L((\nu^\beta\xi, \xi \rtimes \nu^{-\beta/2}\sigma))$,

(vi) $L((\nu_1^\beta, \nu^{-\beta/2}\sigma \mathrm{St}_{GSp(1)}))$,

(vii) $L((\nu^\beta\xi \mathrm{St}_{GL(2)}, \nu^{-\beta}\sigma))$.

The above groups of representations are disjoint. □

In a similar way as F. Rodier classified the unitarizable unramified representations in [R2], we get the following theorem. Clearly, the unramified part of the theorem was proved by him.

Theorem 4.4. *Denote by χ, ξ, σ unitary characters of $F^\times$ such that $\xi^2 = 1_{F^\times}$. Let $\beta, \beta_1, \beta_2 > 0$. The following groups of representations are unitarizable and they exhaust all the irreducible unitarizable representations of $GSp(2, F)$ supported in the minimal parabolic subgroups:*

(i) *irreducible tempered representations of $GSp(2, F)$ which are supported in the minimal parabolic subgroups,*

(ii) $L((\nu^2, \nu, \nu^{-3/2}\sigma)) = \sigma 1_{GSp(2)}$,

(iii) $L((\nu^\beta\chi, \nu^\beta\chi^{-1}, \nu^{-\beta}\chi\sigma))$, $\beta \leq 1/2$, $\chi^2 \neq 1_{F^\times}$ (for $\chi^2 = 1_{F^\times}$ see (v)),

(iv) $L((\nu^\beta, \chi \rtimes \nu^{-\beta/2}\sigma))$, $\beta \leq 1, \chi \neq 1_{F^\times}$ (for $\chi^2 = 1_{F^\times}$ see (vi)),

(v) $L((\nu^{\beta_1}\xi, \nu^{\beta_2}\xi, \nu^{-(\beta_1+\beta_2)/2}\sigma))$, $\beta_1 + \beta_2 \leq 1$, $\beta_1 \geq \beta_2$,

(vi) $L((\nu^\beta\xi, \xi \rtimes \nu^{-\beta/2}\sigma))$, $\beta \leq 1$,

(vii) $L((\nu^{\beta}\xi\mathrm{St}_{GL(2)}, \nu^{-\beta}\sigma))$, $\beta \leq 1/2$.

The above seven groups of representations are disjoint.

Remarks 4.5. With the notation as in the above theorem we have

(i) $L((\nu^{\beta}\chi, \nu^{\beta}\chi^{-1}, \nu^{-\beta}\chi\sigma)) = [\chi(\nu^{\beta} \times \nu^{-\beta})] \rtimes \sigma, \quad \beta < 1/2,$

(ii) $L((\nu^{\beta}, \chi \rtimes \nu^{-\beta/2}\sigma)) = \chi \times [\sigma((\nu^{\beta} \rtimes \nu^{-\beta/2})], \quad \beta < 1,$

(iii) $L((\nu^{\beta'+1/2}\xi, \nu^{-\beta'+1/2}\xi, \nu^{-1/2}\sigma)) = \nu^{\beta'}\xi 1_{GL(2)} \rtimes \nu^{-\beta'}\sigma, \quad 0 \leq \beta' < 1/2,$

(iv) $L((\nu^{\beta}\xi\mathrm{St}_{GL(2)}, \nu^{-\beta}\sigma) = \nu^{\beta}\xi\mathrm{St}_{GL(2)} \rtimes \nu^{-\beta}\sigma, \quad \beta < 1/2.$

Proof. We shall now repeat essentially the Rodier's proof from [R2].

The representations in the groups (i) and (ii) are obviously unitarizable.

We have the complementary series representations $\nu^{\beta}\chi \times \nu^{-\beta}\chi$, $0 < \beta < 1/2$, of $GL(2, F)$. Thus $\nu^{\beta}\chi \times \nu^{-\beta}\chi \rtimes \sigma$ is unitarizable. The last representation is irreducible by Lemma 3.2. Thus

$$\nu^{\beta}\chi \times \nu^{-\beta}\chi \rtimes \sigma \cong \nu^{\beta}\chi \times \nu^{\beta}\chi^{-1} \rtimes \nu^{-\beta}\chi\sigma.$$

This implies $\nu^{\beta}\chi \times \nu^{-\beta}\chi \rtimes \sigma = L((\nu^{\beta}\chi, \nu^{\beta}\chi^{-1}, \nu^{-\beta}\chi\sigma))$. Also $\nu^{1/2}\chi \times \nu^{1/2}\chi^{-1} \rtimes \nu^{1/2}\chi\sigma \cong \nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$ and $\chi 1_{GL(2)} \rtimes \sigma$ is a quotient of the former representation. Since $\chi 1_{GL(2)} \rtimes \sigma$ is irreducible by Corollary 3.10., we have $L((\nu^{1/2}\chi, \nu^{1/2}\chi^{-1}, \nu^{-1/2}\chi\sigma)) = \chi 1_{GL(2)} \rtimes \sigma$. Obviously this representation is unitarizable. Thus (iii) provides the unitarizable representations. Note that the other subquotient of $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$ is the tempered representation $\chi\mathrm{St}_{GL(2)} \rtimes \sigma$.

Now we repeat the similar construction starting from the complementary series of $GSp(1, F) = GL(2, F)$ which are $\nu^{\beta} \rtimes \nu^{-\beta/2}\sigma$, $\beta < 1$. Therefore $\chi \times \nu^{\beta} \rtimes \nu^{-\beta/2}\sigma$ is unitarizable. Since $\chi \times \nu^{\beta} \rtimes \nu^{-\beta/2}\sigma$ is irreducible for $\beta < 1$ by Lemma 3.2., we have

$$\chi \times \nu^{\beta} \rtimes \nu^{-\beta/2}\sigma \cong \nu^{\beta} \times \chi \rtimes \nu^{-\beta/2}.$$

Therefore $L((\nu^{\beta}, \chi \times \nu^{-\beta/2}\sigma))$ is unitarizable for $\beta < 1$. If $\beta = 1$, then we have in $R(G)$

$$\chi \times \nu \rtimes \nu^{-1/2}\sigma = \chi \times \sigma\mathrm{St}_{GSp(1)} + \chi \times \sigma 1_{GSp(1)}.$$

All subquotients are unitarizable. In particular, $L((\nu^{\beta}, \chi \rtimes \nu^{-\beta/2}\sigma))$ is unitarizable. So, we have seen that (iv) provides also the unitarizable representations.

We shall show now that the representations in the group (v) are unitarizable. This is a standard way how one constructs complementary series representations and we shall only outline the construction. Let us recall of some well-known intertwining operators. Define an operator

$$[(\mathcal{A}(\xi \otimes \xi \otimes \sigma, u_1 \otimes u_2 \otimes u_3))(f)](g) = \int_{N_\emptyset} f(J_4 ng)dn$$

on

$$\mathrm{Ind}_{P_\emptyset^G}^{GSp(2,F)}(u_1\xi \otimes u_2\xi \otimes u_3\sigma) = u_1\xi \times u_2\xi \rtimes u_3\sigma.$$

The operator takes values in $\mathrm{Ind}_{P_\phi^G}^{GSp(2,F)}(J_4(u_1\xi \otimes u_2\xi \otimes u_3\sigma)) =$

$$\mathrm{Ind}_{P_\phi^G}^{GSp(2,F)}(u_1^{-1}\xi \otimes u_2^{-1}\xi \otimes u_1u_2u_3\sigma) = u_1^{-1}\xi \times u_2^{-1}\xi \rtimes u_1u_2u_3\sigma.$$

Here u_1, u_2, u_3 denotes the unramified characters of $F^{\times}$. The unramified characters carry a structure of a complex algebraic variety in a natural way. The above integral is defined initially only on an non-empty open subset of unramified characters of $M_\emptyset^G$ where the above integral converges. In that region, it defines a non-trivial intertwining. Since by Lemma 3.2. the representation $u_1\xi \times u_2\xi \rtimes u_3\sigma$ is irreducible when $u_1 \otimes u_2 \otimes u_3$ is out of a proper subvariety, the operators $\mathcal{A}(\xi \otimes \xi \otimes \sigma, u_1 \otimes u_2 \otimes u_3)$ extends meromorphically to all unramified characters (actually, one can prove that it is a rational function by a method of J. Bernstein).

We consider now the unramified characters $\nu^{\alpha_1} \otimes \nu^{\alpha_2} \otimes \nu^{-(\alpha_1+\alpha_2)/2}$ where $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $|\alpha_1| + |\alpha_2| < 1$. Note that in that case

$$\nu^{\alpha_1}\xi \times \nu^{\alpha_2}\xi \rtimes \nu^{-(\alpha_1+\alpha_2)/2}\sigma$$

is irreducible by Lemma 3.2. Therefore, one can twist $\mathcal{A}(\xi \otimes \xi \otimes \sigma, u_1 \otimes u_2 \otimes u_3)$ with a rational function such that for $\nu^{\alpha_1} \otimes \nu^{\alpha_2} \otimes \nu^{-(\alpha_1+\alpha_2)/2}$ as above, the intertwining operators depend algebraically, and that they do not vanish at any above point.

Recall that we have

$$(\nu^{\alpha_1}\xi \times \nu^{\alpha_2}\xi \rtimes \nu^{-(\alpha_1+\alpha_2)/2}\sigma)^{\sim} = \nu^{-\alpha_1}\xi \times \nu^{-\alpha_2}\xi \rtimes \nu^{(\alpha_1+\alpha_2)/2}\sigma$$

A non-degenerate $GSp(2,F)$-invariant Hermitian form on the pair of representations $\nu^{\alpha_1}\xi \times \nu^{\alpha_2}\xi \rtimes \nu^{-(\alpha_1+\alpha_2)/2}\sigma$ and $\nu^{-\alpha_1}\xi \times \nu^{-\alpha_2}\xi \rtimes \nu^{(\alpha_1+\alpha_2)/2}\sigma$ is given by the formula

$$(f_1, f_2) = \int_{K_o} f_1(k)\overline{f_2(k)}dk$$

where K_o is a good maximal compact subgroup of $GSp(2,F)$ (for example, $GSp(2,F) \cap GL(2,\mathcal{O}_F)$ where $\mathcal{O}_F$ is the ring of the integers $\{x \in F; |x|_F \leq 1\}$ of F). Now the formula

$$< f_1, f_2 >= \left(f_1, \mathcal{A}\left(\xi \otimes \xi \otimes \sigma, \nu^{\alpha_1} \otimes \nu^{\alpha_2} \otimes \nu^{-(\alpha_1+\alpha_2/2)}\right) f_2\right)$$

defines a non-degenerate $GSp(2,F)$-invariant Hermitian form on $\nu^{\alpha_1}\xi \times \nu^{\alpha_2}\xi \rtimes \nu^{-(\alpha_1+\alpha_2)/2}$. This form depends continuously on α_1 and α_2. For $\alpha_1 = \alpha_2 = 0$ this form is proportional to the $GSp(2,F)$-invariant inner product which exists on $\xi \times \xi \rtimes \sigma$. Thus, it is positive definite at this point. Therefore, it is positive definite everywhere on the considered set. From this one gets that $\nu^{\alpha_1}\xi \times \nu^{\alpha_2}\xi \rtimes \nu^{-(\alpha_1+\alpha_2)/2}\sigma$ is unitarizable ($\alpha_1, \alpha_2 \in \mathbb{R}, |\alpha_1| + |\alpha_2| < 1$). In particular, this proves the unitarizability of the representations in the group (v) for $\beta_1, \beta_2 > 0$ and $\beta_1 + \beta_2 < 1$. For $\beta_1 + \beta_2 = 1$ one gets the unitarizability by a D. Mili's result from [Mi] since the corresponding representations are in the limits of the complementary series (see also Theorem 2.7. and Lemma 2.8. of [T1]). Similarly, the representations in (vi) are in the ends of the complementary series from (v). Therefore, they are unitarizable. Note that we have in $R(G)$

$$\nu^{\beta}\xi\mathrm{St}_{GL(2)} \rtimes \nu^{-\beta}\sigma \leq \nu^{\beta+1/2}\xi \times \nu^{\beta-1/2}\xi \rtimes \nu^{-\beta}\sigma = \nu^{1/2+\beta}\xi \times \nu^{1/2-\beta} \rtimes \nu^{-1/2}\xi\sigma.$$

Since each irreducible subquotient of the representation is in the limit of the complementary series from (v) must be unitarizable, the group (vii) provides the unitarizable representations.

One can check directly using the preceding section that all irreducible composition factors of $\nu^{\beta_1}\xi \times \nu^{\beta_2}\xi \rtimes \nu^{-(\beta_1+\beta_2)/2}\sigma$, $|\beta_1| + |\beta_2| < 1$, are of the types listed in (i)-(vii).

Let π be a non-tempered unitarizable representation of $GSp(2, F)$ in the sequel. Suppose that it is supported in the minimal parabolic subgroups. First of all, π is Hermitian. Therefore, π belongs to (ii)-(vii) of Lemma 4.3.

Suppose that π belongs to the group (ii) of Lemma 4.3.. Then

$$\pi = L((\nu^\beta\chi, \nu^\beta\chi^{-1}, \nu^{-\beta}\sigma))$$

with $\chi, \sigma \in (F^\times)\hat{}$, $\beta > 0$ and $\chi^2 \neq 1_{F^\times}$. Now for $\beta > 1/2$

$$L((\nu^\beta\chi, \nu^\beta\chi^{-1}, \nu^{-\beta}\sigma)) = \nu^\beta\chi \times \nu^\beta\chi^{-1} \rtimes \nu^{-\beta}\sigma = \nu^\beta\chi \times \nu^{-\beta}\chi \rtimes \chi\sigma$$

forms a continuous family of irreducible Hermitian representations (behind this fact are intertwining operators again). Thus, either all elements of the family are unitarizable, or no one element is unitarizable. Since the matrix coefficients of the representations in the family are not bounded for β large enough, we see that all representations in the family are not unitarizable. This proves that π belongs to the group (iii) of the theorem.

Suppose that $\pi = L((\nu^\beta, \chi \rtimes \nu^{-\beta/2}\sigma))$, $\beta > 0$, $\chi, \sigma \in (F^\times)\hat{}$. For $\beta > 1$ we have

$$L((\nu^\beta, \chi \rtimes \nu^{-\beta/2}\sigma)) = \nu^\beta \times \chi \rtimes \nu^{-\beta/2}\sigma.$$

This is a continuous family of Hermitian representations. We see that they are not unitarizable in the same way as it was obtained in the previous case. In particular, if π belongs to the group (iii) of Lemma 4.3., then π belongs to the group (iv) of the theorem.

Suppose that π is not one of the two previously considered types. Then by Lemma 4.3, π is a subquotient of $\nu^{\beta_1}\xi \times \nu^{\beta_2}\xi \rtimes \nu^{-(\beta_1+\beta_2)/2}\sigma$,

$\xi, \sigma \in (F^\times)\hat{}$ with $\xi^2 = 1_{F^\times}$ and $\beta_1 \beta_2 \in \mathbb{R}$. We may suppose also $\beta_1 \geq \beta_2 \geq 0$. Denote

$$\pi(\beta_1, \beta_2) = \nu^{\beta_1}\xi \times \nu^{\beta_2}\xi \rtimes \nu^{-(\beta_1+\beta_2)/2}\sigma.$$

Suppose that $\xi \neq 1_{F^\times}$. We look at the following families of representations

$$I = \{\pi(\beta_1, \beta_2); 1 > \beta_1 - \beta_2 > -1, \beta_1 + \beta_2 > 1\},$$

$$IIta_1, \beta_2); \beta_1 - \beta_2 > 1, \beta_1 + \beta_2 > 1\},$$

$$A = \{\pi(\beta_1, \beta_2); 1 = \beta_1 - \beta_2, \beta_2 > 0\}.$$

The following drawing illustrates the situation:

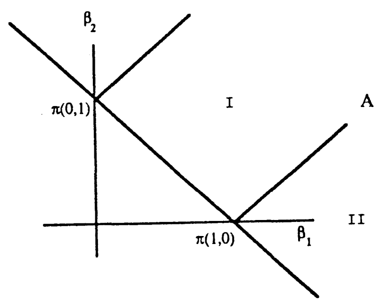

Figure 1.

Representations $\pi(\beta_1, \beta_2)$ which belong to I form a continuous family of irreducible Hermitian representations. They cannot be unitarizable by the argument which we have already used. Similarly, the region II corresponds to the non unitarizable representations. Now look at A. For $\beta > 0$ we have

$$\pi(1+\beta, \beta) = \nu^{\beta+1/2}\xi 1_{GL(2)} \rtimes \nu^{-\beta-1/2}\sigma + \nu^{\beta+1/2}\xi \mathrm{St}_{GL(2)} \rtimes \nu^{-\beta-1/2}.$$

in $R(G)$. Both representations on the right hand side are irreducible by Corollary 3.10. They are Hermitian representations. In this way we obtain two continuous families. Both of them consist of the non-unitarizable representations by the already used arguments. Thus π is a subquotient of $\pi(\beta_1, \beta_2)$with $\beta_1+\beta_2 \leq 1$. We have already noted that all such subquotients are listed in the theorem.

In the end we should consider the case of $\xi = 1_{F^\times}$.This case is already settled in [R2], so we omit the analysis of this case. The analysis is similar to the previous case expect that one need to use a Casselman's result that if a subquotient of $\nu^2 \times \nu \rtimes \nu^{-3/2}$ is unitarizable, then it is either the trivial or the Steinberg representation (see for example [HMr]). Here one has the following drawing:

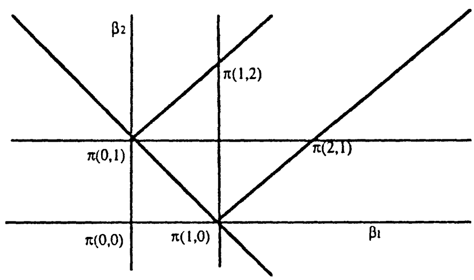

Figure 2.

□

We shall describe now the representations which are supported in the other parabolic subgroups. These representations were classified completely by F. Shahidi in [S1] (Proposition 8.4.) and [S2] (Theorem 6.1.). One case may be concluded easily from the earlier J.-L. Waldspurger's Proposition 5.1. of [Wd]. For the sake of completeness, we include these descriptions.

We shall consider first the case of the representations which are supported in $P^G_{(2)}$. Let $\rho \in C^u(GL(2,F))$ and $\sigma \in (F^\times)^\sim$. The formula (2.2) gives that $\rho \rtimes \sigma$ is irreducible if $\rho \not\cong \tilde{\rho}$ or $\omega_\rho \neq 1_{F^\times}$. Let $\beta \in \mathbb{R}^\times$. If $\nu^\beta \rho \rtimes \sigma$ is reducible for $\beta \neq 0$, then it is a multiplicity one representation of length two. One factor is essentially square integrable (see the seventh section of [C]). This implies $\rho \cong \tilde{\rho}$ and $\omega_\rho = 1_{F^\times}$.

Let $\beta \in \mathbb{R}$, $\rho \in C^u(GL(2,F))$ and $\sigma \in (F^\times)^\sim)$. By F. Shahidi (Proposition 6.1 of [S2]), the representation $\nu^\beta \rho \rtimes \sigma$ is reducible if and only if

$$\beta = \pm 1/2,\ \rho \cong \tilde{\rho} \text{ and } \omega_\rho = 1_{F^\times}.$$

We have now the following two propositions belonging F. Shahidi. Note that they are a direct consequences of the above description of the reducibilities.

Proposition 4.6.

(i) *Representations $\rho \rtimes \sigma$, $\rho \in C^u(GL(2,F))$, $\sigma \in (F^\times)\hat{}$, are irreducible tempered representations. The only non-trivial equivalences among them are*

$$\rho \rtimes \sigma \cong \tilde{\rho} \rtimes \omega_\rho \sigma.$$

These representations exhaust all irreducible tempered representations of the group $GSp(2,F)$ supported in $P_{(2)}^G$ which are not square integrable.

(ii) *Let $\rho \in C^u(GSp(1,F))$ and $\sigma \in (F^\times)\hat{}$. Suppose that $\rho = \tilde{\rho}$ and $\omega_\rho = 1_{F^\times}$. Then $\nu^{1/2}\rho \rtimes \nu^{-1/2}\sigma$ has a unique irreducible subrepresentation. That subrepresentation is square integrable. For different pairs (ρ, σ) we obtain subrepresentations which are not equivalent. These subrepresentations exhaust all irreducible square integrable representations of $GSp(2,F)$ which are supported in $P_{(2)}^G$.* □

Proposition 4.7. *An irreducible unitarizable representation of $GSp(2,F)$ supported in $P_{(2)}^G$ is either tempered, or it is one from the following series of unitarizable representations*

$$L((\nu^\beta \rho, \sigma)),$$

where $\rho \in C^u(GL(2,F))$ such that $\rho = \tilde{\rho}, \omega_\rho = 1_{F^\times}$ and $0 < \beta \leq 1/2$, $\sigma \in (F^\times)\hat{}$. □

We shall consider now the case of the representation supported in $P_{(1)}^G$. Let $\chi \in (F^\times)^\sim$ and $\rho \in C(GSp(1,F))$. Then $\chi \rtimes \rho$ is reducible in the following two cases

(i) $\chi = 1_{F^\times}$,

(ii) $\chi = \nu^{\pm 1}\xi_o$ where $\xi_o \in (F^\times)\hat{}$ is a character of order two such that $\xi_o\rho \cong \rho$.

These are the only points of the reducibility. This was proved by J.-L. Waldspurger in [Wd] and F. Shahidi proved that also in [S1] by different methods. We have now the following two propositions belonging to them. Note that they follows easily from the above description of the reducibilities.

Proposition 4.8.

(i) *The representation $1_{F^\times} \rtimes \rho$, $\rho \in C^u(GSp(1, F))$, splits into a sum of two tempered irreducible subrepresentations which are not equivalent. For different ρ's, these subrepresentations are not equivalent.*

(ii) *Representations $\chi \rtimes \in (F^\times)\hat{}$, $\chi \neq 1_{F^\times}$, $\rho \in C^u(GSp(1, F))$, are irreducible tempered. The only non-trivial equivalences among them are*

$$\chi \rtimes \rho \cong \chi^{-1} \rtimes \chi\rho.$$

(iii) *The irreducible representations listed in (i) and (ii) are disjoint groups of representations and they exhaust all irreducible tempered representations of $GSp(2, F)$ supported in $P^G_{(1)}$ which are not square integrable.*

(iv) *Let $\rho \in C^u(GSp(1, F))$ and suppose that $\xi_o \in (F^\times)\hat{}$ is a character of order two which satisfies $\xi_o\rho = \rho$. Then $\nu\xi_o \rtimes \nu^{-1/2}\rho$ has a unique irreducible subrepresentation. This subrepresentation is square integrable. For different pairs (ξ_o, ρ) we obtain subrepresentations which aruivalent. These subrepresentations exhaust all irreducible square integrable representations of $GSp(2, F)$ which are supported in $P^G_{(1)}$.*

□

Proposition 4.9. *The following two disjoint groups of representation exhaust all irreducible unitarizable representations of $GSp(2, F)$ which are supported in $P^G_{(1)}$:*

(i) *irreducible tempered representations of* $GSp(2,F)$ *which are supported in* $P^G_{(1)}$,

(ii) $L((\nu^\beta \xi_o, \nu^{-\beta/2}\rho))$ *where* $0 < \beta \leq 1$, $\xi_o \in (F^\times)\hat{}$, $\rho \in C^u(GSp(1,F))$ *which satisfy* $\xi_o^2 = 1_{F^\times}$, $\xi_o \neq 1_{F^\times}$ *and* $\xi_o\rho \cong \rho$. □

5. Consequences for $Sp(2)$

Analyzing the restriction of irreducible representations of $GSp(2,F)$ to $Sp(2,F)$, we shall derive in this section various results for $Sp(2,F)$ from the previous investigations of the case of $GSp(2,F)$. The properties of this restricting process that we need, can be find in [GeKn] and [T3]. The case of the symplectic groups was studied in [T4].

We shall recall briefly of the main properties of the restricting process. Let $\pi \in GSp(n,F)\check{}$. Then for the restriction $\pi|Sp(n,F)$ we have

$$\pi|Sp(n,F) \cong \sigma_1 \oplus \cdots \oplus \sigma_k$$

for some $\sigma_i \in Sp(n,F)\check{}$. If π is unitary (resp. tempered, square integrable, cuspidal), then each σ_i is also unitary (resp. tempered, square integrable, cuspidal). Further

$$\dim_{\mathbb{C}} \operatorname{End}_{Sp(n,F)} (\pi|Sp(n,F)) = \operatorname{card} \{\chi \in (F^{\times})\check{}; \chi\pi \cong \pi\}. \tag{5.1}$$

Take $L((\delta_1, \ldots, \delta_p, \tau)) \in GSp(n,F)\check{}$, where $\delta_i \in D_+$, and $\tau \in T(G)$ is a representation of $GSp(m,F)$. Then for $\chi \in (F^{\times})\check{}$

$$\chi L((\delta_1, \ldots, \delta_p, \tau)) = L((\delta_1, \ldots, \delta_p, \chi\tau)).$$

Write further

$$\tau|Sp(m,F) = \tau_1 \oplus \ldots \oplus \tau_k$$

where τ_i are irreducible representations. Then

$$L((\delta_1, \ldots, \delta_p, \tau))\,|Sp(n,F) \cong \bigoplus_{i=1}^{k} L((\delta_1, \ldots, \delta_p, \tau_i)). \tag{5.2}$$

Let $\sigma \in Sp(n,F)\check{}$. Then σ is isomorphic to a subrepresentation of $\pi|Sp(n,F)$ for some $\pi \in GSp(n,F)\check{}$. Moreover, if σ is unitary (resp. tempered, square integrable, cuspidal), then one may choose π to be unitary (resp. tempered, square integrable, cuspidal).

Let $\chi \in (F^\times)\check{}$ and $\sum_{i=1}^{\ell} n_i\pi_i \in R_n(G)$ where $\pi_i \in GSp(n,F)\check{}$ and $n_i \in \mathbb{Z}$. Define

$$\chi(\sum_{i=1}^{\ell} n_i\pi_i) = \sum_{i=1}^{\ell} n_i(\chi\pi_i).$$

For $\alpha \in R_n(G)$ set

$$X_{Sp(n)}(\alpha) = \{\chi \in (F^\times)\check{}; \chi\alpha = \alpha\}.$$

Theorem 5.1.

(i) *For each $\xi_o \in (F^\times)\hat{}$ of order two, the representation $\nu\xi_o \times \xi_o \rtimes 1$ has exactly two irreducible subrepresentations. They are square integrable and thquivalent. If we denote them by $\delta'(\xi_o)$ and $\delta''(\xi_o)$, then we have*

$$\delta, \nu\xi_o], \nu^{-1/2}\sigma)|Sp(2,F) = \delta'(\xi_o) \oplus \delta''(\xi_o).$$

(ii) *If δ is an irreducible square integrable representation of $Sp(2,F)$ which is supported in the minimal parabolic subgroups, then δ is either the Steinberg representation or it is a representation considered in (i). We have*

$$\sigma \mathrm{St}_{GSp(2)}|Sp(2,F) \cong \mathrm{St}_{Sp(2)}.$$

Proof.

(i) Consider $\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma \in R(G)$. Suppose that

$$\chi \in X_{Sp(2)}(\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)).$$

Then $\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma$ and

$$\chi(\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma) = \nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\chi\sigma$$

are equal in $R(G)$. Thus, $\chi \in X_{Sp(2)}(\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma)$. Conversely, if $\chi \in X_{Sp(2)}(\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma)$, then

$$\chi \in X_{Sp(2)}(\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma))$$

since $\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)$ is a unique square integrable subquotient of $\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma$. Let $\chi \in X_{Sp(2)}(\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma)$. Then $\nu\xi_0 \otimes \xi_0 \otimes \nu^{-1/2}\sigma$ and $\nu\xi_0 \otimes \xi_0 \otimes \nu^{-1/2}\chi\sigma$ are associate (i.e., in the same orbit of the action of the Weyl group). This implies $\chi \in \{1_{F^\times}, \xi_0\}$. Since $\xi_0 \rtimes \nu^{-1/2}\sigma = \xi_0 \rtimes \nu^{-1/2}\xi_0\sigma$ in the Grothendieck group, we have $X_{Sp(2)}(\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)) = \{1_{F^\times}, \xi_0\}$. This implies that $\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)|Sp(2, F)$ splits into a sum of two non-equivalent irreducible square integrable representations. The (normalized) Jacquet module for the standard minimal parabolic subgroup of each of these representations is $\nu\xi_0 \otimes \xi_0 \otimes 1$. The Frobenius reciprocity implies that these two irreducible representations are the only irreducible subrepresentation of $\nu\xi_0 \times \xi_0 \rtimes 1$.

(ii) In the same way as above one gets $X_{Sp(2)}(\sigma St_{GSp(2)}) = \{1_{F^\times}\}$. Theorem 4.1. implies the statement that the above representations exhaust all the irreducible square integrable representations supported in the minimal parabolic subgroups. $\square$

Theorem 5.2. *Let $\chi, \chi_1, \chi_2 \in (F^\times)\hat{}$, and let ξ, ξ_1, ξ_2 be characters of $F^\times$ of order two.*

(i) *$\chi_1 \times \chi_2 \rtimes 1$ is irreducible if neither χ_1, nor χ_2 is of order two. We have $\chi_1 \times \chi_2 \rtimes 1 \cong \chi_1' \times \chi_2' \rtimes 1$ if and only if $\chi_1 \otimes \chi_2 \otimes 1$ and $\chi_1' \otimes \chi_2' \otimes 1$ are associate.*

(ii) *Write $\xi \rtimes 1 = T_\xi^1 + T_\xi^2$ as a sum of irreducible representations. Suppose either $\chi = \xi$ or χ is not of order two. Then $\chi \rtimes T_\xi^1$ and $\chi \rtimes T_\xi^2$ are irreducible representations which are not equivalent. The only non-trivial equivalences among them are*

$$\chi \rtimes T_\xi^1 \cong \chi^{-1} \rtimes T_\xi^1 \quad \text{and} \quad \chi \rtimes T_\xi^2 \cong \chi^{-1} \rtimes T_\xi^2.$$

(iii) *If* $\xi_1 \neq \xi_2$, *then the representation* $\xi_1 \times \xi_2 \rtimes 1$ *is a multiplicity one representation of length four. The irreducible constituents may be characterized as common subrepresentations of* $\xi_1 \rtimes T^i_{\xi_2}$ *and* $\xi_2 \rtimes T^j_{\xi_1}$, $i, j \in \{1, 2\}$.

(iv) *Suppose that* χ *is not of order two and suppose that also* $\chi \neq 1_{F^\times}$ *(i.e.,* $\chi^2 \neq 1_{F^\times}$*). Then* $\chi \rtimes \mathrm{St}_{Sp(1)}$ *is irreducible. The only non-trivial equivalences among these representations are*

$$\chi \rtimes \mathrm{St}_{Sp(1)} = \chi^{-1} \rtimes \mathrm{St}_{Sp(1)}.$$

(v) *Suppose that* $\chi^2 = 1_{F^\times}$. *Then* $\chi \rtimes \mathrm{St}_{Sp(1)}$ *is a sum of twle subrepresentations which are not equivalent. If* $\chi \neq 1_{F^\times}$, *then one representation is a common factor with* $\nu \rtimes T^1_\chi$ *and the other one with* $\nu \rtimes T^2_\chi$. *If* $\chi = 1_{F^\times}$, *then one representation is a common factor with* $\nu^{1/2}\mathrm{St}_{GL(2)} \rtimes 1$ *and the other one with* $\nu^{1/2} 1_{GL(2)} \rtimes 1$.

(vi) *Representations* $\chi \mathrm{St}_{GL(2)} \rtimes 1$ *are irreducible. The only non-trivial equivalences among such representations are*

$$\chi \mathrm{St}_{GL(2)} \rtimes 1 \cong \chi^{-1}\mathrm{St}_{GL(2)} \rtimes 1.$$

The groups of representations (i)-(vi) are disjoint. They consist of the irreducible tempered representations of $Sp(2, F)$ *which are supported in the minimal parabolic subgroups. Each irreducible tempered representation of* $Sp(2, F)$ *which is supported in the minimal parabolic subgroups, either belongs to one of the groups (i)-(vi), or it is square integrable.*

Proof.

(i) We know that $\chi_1 \times \chi_2 \rtimes 1_{F^\times}$ is an irreducible representation of $GSp(2, F)$ by Lemma 3.1. A direct computation gives that

$$X_{Sp(2)}(\chi_1 \times \chi_2 \times 1_{F^\times}) = \{1_{F^\times}\}$$

if neither χ_1, nor χ_2 is of order two. This implies (i).

(ii) We know that $\chi \times \xi \rtimes 1_{F^\times}$ is irreducible. A simple analysis gives

$$X_{Sp(2)}(\chi \times \xi \rtimes 1_{F^\times}) = \{1_{F^\times}, \xi\}.$$

This gives the first part of (ii). We know also $\chi \rtimes T_\xi^i \cong \chi^{-1} \rtimes T_\xi^i$ for $i = 1, 2$. Suppose further that $\chi \rtimes T_\xi^i \cong \chi' \rtimes T_{\xi'}^j$ where χ' and ξ' satisfy the assumptions of (ii). Since $\chi \otimes \xi \otimes 1$ and $\chi' \otimes \xi' \otimes 1$ are associate, we get first $\xi = \xi'$ and then $\chi = \chi'$ or $\chi^{-1} = \chi'$. Suppose that $\chi = \chi'$. Then $i = j$ since $\chi \times \xi \rtimes 1$ is a multiplicity one representation. Suppose now $\chi^{-1} = \chi'$. Then $\chi \times T_\xi^i \cong \chi^{-1} \times T_\xi^j \cong \chi \times T_\xi^j$, what implies $i = j$.

(iii) One gets directly $X_{Sp(2)}(\xi_1 \times \xi_2 \rtimes 1_{F^\times}) = \{1, \xi_1, \xi_2, \xi_1\xi_2\}$. Recall that $\xi_1 \times \xi_2 \rtimes 1_{F^\times}$ is irreducible. By [Ke], $\xi_1 \times \xi_2 \rtimes 1$ is a multiplicity one representation. Looking at the Jacquet modules of $\xi_1 \rtimes T_{\xi_2}^i$ and $\xi_2 \rtimes T_{\xi_1}^j$ for $P_{(2)}^S$, one gets that these representations are of length less then or equal to two. This implies that both representations are of length two. Looking at the Jacquet modules for $P_{(1)}^S$, one gets that $\xi_1 \rtimes T_{\xi_2}^i$ and $\xi_2 \rtimes T_{\xi_1}^j$ have a non-trivial intersection, and that there is no inclusions among these representations.

(iv) Since $\chi^2 \neq 1_{F^\times}$, $\chi \rtimes \mathrm{St}_{GSp(1)}$ is irreducible by (ii) of Corollary 3.10., as well as $\chi \rtimes 1_{GSp(1)}$. The first representation is tempered, while the other is not tempered. Thus $X_{Sp(2)}(\chi \rtimes \mathrm{St}_{GSp(1)}) = X_{Sp(2)}(\chi \times \nu \rtimes \nu^{-1/2})$. A simple computation gives $X_{Sp(2)}(\chi \times \nu \rtimes \nu^{-1/2}) = \{1_{F^\times}\}$. This implies that

$$\chi \rtimes \mathrm{St}_{Sp(1)} \cong \chi \rtimes (\mathrm{St}_{GSp(1)}|Sp(1)) \cong (\chi \rtimes \mathrm{St}_{GSp(1)})|GSp(2)$$

is irreducible. The relation $\chi \rtimes \mathrm{St}_{Sp(1)} = \chi^{-1} \rtimes \mathrm{St}_{Sp(1)}$ is clear. That this is the only non-trivial equivalence among such representations follows in the same way as in the proof of (ii).

(v) Suppose that χ is of order two. Then $\chi \rtimes \mathrm{St}_{GSp(1)}$ is irreducible (Corollary 3.10) Then the some calculation as in the proof of (iv) gives $X_{Sp(2)}(\chi \rtimes \mathrm{St}_{GSp(1)}) = \{1_{F^\times}, \chi\}$. Therefore $\chi \rtimes \mathrm{St}_{Sp(1)}$ is a sum of two non-equivalent irreducible representations. From the Jacquet modules one concludes that $\chi \rtimes \mathrm{St}_{Sp(1)}$ and $\nu \rtimes T^1_\chi$(resp. $\nu \rtimes T^2_\chi$) have a non-trivial subquotient in common. The reducibility of $1_{F^\times} \rtimes \mathrm{St}_{Sp(1)}$ follows from the reducibility of $1_{F^\times} \rtimes \mathrm{St}_{GSp(1)}$. Let σ be an irreducible subrepresentation of $1_{F^\times} \rtimes \mathrm{St}_{GSp(1)}$. As before, we can conclude that $X_{Sp(2)}(\sigma) \subseteq X_{Sp(2)}(1_{F^\times} \times \nu \rtimes \nu^{-1/2})$. Since $X_{Sp(2)}(1_{F^\times} \times \nu \rtimes \nu^{-1/2}) = \{1_{F^\times}\}$, we have $X_{Sp(2)}(\sigma) = \{1_{F^\times}\}$. Thus $1_{F^\times} \rtimes \mathrm{St}_{GSp(1)}$ is of length two. From the Jacquet modules one can see that it is a multiplicity one representation. Characterization of irreducible subrepresentations one concludes in the same way as in the first part of proof of (v).

(vi) By Corollary 3.10., $\chi \mathrm{St}_{GL(2)} \rtimes 1_{F^\times}$ is irreducible. Further

$$X_{Sp(2)}(\chi \mathrm{St}_{GL(2)} \rtimes 1_{F^\times}) \subseteq X_{Sp(2)}(\nu^{1/2}\chi \times \nu^{-1/2} \rtimes 1_{F^\times}).$$

Since

$$X_{Sp(2)}(\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes 1_{F^\times}) = \{1_{F^\times}\},$$

we have the irreducibility. The statement about equivalences follows in the standard way. □

Theorem 4.4. and (5.2) imply the following theorem.

Theorem 5.3. *Let $\chi, \xi \in (F^\times)\hat{}$. Suppose that $\xi^2 = 1_{F^\times}$, and let $\beta, \beta_1, \beta_2 > 0$. The following groups of irreducible representations are unitarizable and they are supported in the minimal parabolic subgroups. They exhaust all the irreducible unitarizable representations of $Sp(2, F)$ which are supported in the minimal parabolic subgroups.*

(i) *Irreducible tempered representations described in Theorem 5.2.*

(ii) $L((\nu^2, \nu, 1)) = 1_{Sp(2)}$.

(iii) $L((\nu^\beta\chi, \nu^\beta\chi^{-1}, 1))$, $\beta \leq 1/2$, $\chi^2 \neq 1_{F^\times}$ (for $\chi^2 = 1_{F^\times}$ see (vi)).

(iv) $L((\nu^\beta, \chi \rtimes 1))$, $\beta \leq 1$, *χ is not of order two.*

(v) $L((\nu^\beta, T^i_\xi))$, $\xi \neq 1_{F^\times}$, $\beta \leq 1$, $i \in \{1, 2\}$.

(vi) $L((\nu^{\beta_1}\xi, \nu^{\beta_2}\xi, 1))$, $\beta_1 + \beta_2 \leq 1$, $\beta_1 \geq \beta_2$.

(vii) $L((\nu^{\beta}\xi, T_{\xi}^{i}))$, $\beta \leq 1$, $\xi \neq 1_{F^\times}$, $i \in \{1,2\}$.
(viii) $L((\nu^{\beta}\xi \mathrm{St}_{GL(2)}, 1))$, $\beta \leq 1/2$. $\square$

Proposition 5.4. *Let $\chi, \xi \in (F^\times)^\sim$.*

(i) *$\chi \mathrm{St}_{GL(2)} \rtimes 1$ is reducible if and only if $\chi 1_{GL(2)} \rtimes 1$ is reducible. It happens if and only if $\chi \in \{\nu^{\pm 1/2}\xi, \nu^{\pm 3/2}\}$ for some ξ such that $\xi^2 = 1_{F^\times}$. For $\chi \in \{\nu^{\pm 1/2}, \nu^{\pm 3/2}\}$, the above representations are of length two. For $\chi = \nu^{\pm 1/2}\xi$ with ξ of order two, the above representations are of length three. All above representations are of multiplicity one.*

(ii) *The representation $\chi \rtimes \mathrm{St}_{Sp(1)}$ if and only if $\chi \rtimes 1_{Sp(1)}$ is reducible. It happens if and only if $\chi \in \{\xi, \nu^{\pm 2}\}$ for some ξ which satisfies $\xi^2 = 1_{F^\times}$. If we have reducibility, then we always have a multiplicity one representation of length two.*

(iii) *Let ξ be of order two. Now $\chi \rtimes T_{\xi}^{1}$ is irreducible if and only if $\chi \rtimes T_{\xi}^{2}$ is irreducible and it happens if and only if $\chi \notin \{\nu^{\pm 1}, \xi\nu^{\pm 1}, \xi'\}$ for any character ξ' of order two which is different from ξ. If χ is a character of order two different from ξ, or if $\chi = \nu^{\pm 1}$, then the above representations are of length two while for $\chi = \nu^{\pm 1}\xi$, the above representations are of length three.*

Proof.

(i) Let $\pi \in GSp(2,F)^\sim$ be an irreducible subquotient of $\chi \mathrm{St}_{GL(2)} \rtimes 1_{F^\times}$ or $\chi 1_{GL(2)} \rtimes 1_{F^\times}$. Then $X_{Sp(2)}(\pi) \subseteq X_{Sp(2)}(\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes 1_{F^\times})$. A direct computation tells that $X_{Sp(2)}(\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes 1_{F^\times})$ is non-trivial when $\chi = \nu^{\pm 1/2}\xi$ where ξ is a character of order two. Then the above group of characters is equal to $\{1_{F^\times}, \xi\}$. In all other cases we have the trivial above group. Therefore, if $\chi \notin \{\nu^{\pm 1/2}\xi, \nu^{\pm 3/2}\}$ for any ξ which satisfies $\xi^2 = 1_{F^\times}$, then $\chi 1_{GL(2)} \rtimes 1$ and $\chi \mathrm{St}_{GL(2)} \rtimes 1$ are irreducible. If $\chi \in \{\nu^{\pm 3/2}\}$, then both $\chi 1_{GL(2)} \rtimes 1$ and $\chi \mathrm{St}_{GL(2)} \rtimes 1$ reduce. We have representations of length two. From Lemma 3.5. and (5.2) we see that in the case of $\chi = \nu^{\pm 3/2}$ the above representations are of multiplicity one. Lemma 3.6. and (5.2) imply that for $\chi = \nu^{\pm 1/2}$ we have again multiplicity one representations. The same arguments give that in the case of $\chi = \nu^{\pm 1/2}\xi$ where ξ is a character of order two, we have multiplicity one representations of length three.

(ii) Let $\pi \in GSp(2,F)\tilde{}$ be a subquotient of $\chi \rtimes \mathrm{St}_{GSp(1)}$ or $\chi \rtimes 1_{GSp(1)}$. Then

$$X_{Sp(2)}(\pi) \subseteq X_{Sp(2)}(\chi \rtimes \nu \rtimes \nu^{-1/2}).$$

If χ is a character of order two, then $X_{Sp(2)}(\chi \rtimes \nu \rtimes \nu^{-1/2}) = \{1, \chi\}$. In all the other cases, the above group is trivial. Therefore, $\chi \rtimes \mathrm{St}_{Sp(1)}$ and $\chi \rtimes 1_{Sp(1)}$ are irreducible if $\chi \neq \nu^{\pm 2}$ and if $\chi^2 \neq 1_{F^\times}$. If $\chi = \nu^{\pm 2}$ we have representations of length two. By Lemma 3.5. and (5.2) one obtains that they are multiplicity one representations. For $\chi = 1_{F^\times}$ we have a representations of length two. From Lemma 3.5. and (5.2) we see that $1_{F^\times} \rtimes 1_{Sp(1)}$ is a multiplicity one representation. The Jacquet module of $1_{F^\times} \rtimes \mathrm{St}_{Sp(1)}$ for $P_{(1)}^S$ tells that $1_{F^\times} \rtimes \mathrm{St}_{Sp(1)}$ is a multiplicity one representation. Suppose that χ is a character of order two. Then $\chi \rtimes \mathrm{St}_{GSp(1)}$ and $\chi \rtimes 1_{GSp(1)}$ are irreducible. Since the first representation is tempered, while the other one is not, we have

$$\begin{aligned} X_{Sp(2)}(\chi \rtimes \mathrm{St}_{GSp(1)}) &= X_{Sp(2)}(\chi \rtimes 1_{GSp(1)}) \\ &= X_{Sp(2)}(\chi \times \nu \rtimes \nu^{-1/2}) = \{1_{F^\times}, \chi\}. \end{aligned}$$

This finishes the proof of (ii).

(iii) Let ξ' be a character of order two. Then $\xi' \rtimes T_\xi^i$ is a multiplicity one representation of length two by (iii) of Theorem 5.2. We know that $X_{Sp(2)}(\xi \rtimes \mathrm{St}_{GSp(1)}) = X_{Sp(2)}(\xi \rtimes 1_{GSp(1)}) = \{1_{F^\times}, \xi\}$. Therefore $\xi \rtimes \mathrm{St}_{Sp(1)}$ and $\xi \rtimes 1_{Sp(1)}$ are multiplicity one representations of length two. Note that in $R(G)$ we have $\nu \times \xi \rtimes \nu^{-1/2} = \xi \times \nu \rtimes \nu^{-1/2} = \xi \rtimes \mathrm{St}_{GSp(1)}$ $+L(\nu, \xi \rtimes \nu^{-1/2})$. Therefore we have in $R(S)$ the equality $\nu \times \xi \rtimes 1 =$ $\xi \rtimes \mathrm{St}_{GSp(1)} + L(\nu, T_\xi^1) + L(\nu, T_\xi^2)$. Thus $\nu \times \xi \rtimes 1$ is a multiplicity one representation of length four. Looking at Jacquet modules it is now easy to get that $\nu^{\pm 1} \rtimes T_\xi^i$ are multiplicity one representations of length two. We have in $R(G)$ the equality $\nu\xi \times \xi \rtimes \sigma =$

$$\delta([\xi, \nu\xi], \sigma) + L\left((\nu^{1/2}\xi \mathrm{St}_{GL(2)}, \sigma)\right) + L\left((\nu^{1/2}\xi \mathrm{St}_{GL(2)}, \xi\sigma)\right) + L\left((\nu\xi, \xi \rtimes \sigma)\right).$$

Therefore in $R(S)$ holds $\nu\xi \times \xi \rtimes 1 =$

$$\delta'(\xi) + \delta''(\xi) + 2L\left((\nu^{1/2}\xi \mathrm{St}_{GL(2)}, 1)\right) + L\left((\nu\xi, T^1_\xi)\right) + L\left((\nu\xi, T^2_\xi)\right).$$

Consider $\nu\xi \rtimes T^i_\xi$. From the Jacquet modules one sees that $\delta'(\xi)$ or $\delta''(\xi)$ is a subquotient of $\nu\xi \rtimes T^i_\xi$. Further, $L\left((\nu\xi, T^i_\xi)\right)$ is a quotient of $\nu\xi \rtimes T^i_\xi$. From the Jacquet module of $\nu\xi \rtimes T^i_\xi$ for $P^S_{(1)}$ one cdes that $\nu\xi \rtimes T^i_\xi$ is a multiplicity one representation. From this one can conclude that $\chi \rtimes T^i_\xi$ are multiplicity one representations of length three. Suppose that $\chi \notin \{\nu^{\pm 1}, \xi\nu^{\pm 1}, \xi'\}$ for any ξ' of order two, different from ξ. Then $\chi \rtimes \xi \rtimes 1_{F^\times}$ is irreducible by Lemma 3.2. Now $X_{Sp(2)}(\chi \rtimes \xi \rtimes 1_{F^\times}) = \{1_{F^\times}, \xi\}$. This implies the irreducibility of $\chi \rtimes T^i_\xi$. □

Remark 5.5. Note that $\nu\xi_o \times \xi_o \rtimes 1$, $\xi_o^2 = 1_{F^\times}$, $\xi_o \neq 1_{F^\times}$, has five different irreducible subquotients, $L((\nu^{1/2}\xi_o \mathrm{St}_{GL(2)}, 1))$ is of multiplicity two while all other factors are of multiplicity one. Recall that for $GSp(2, F)$, all non-unitary principal series were of multiplicity one.

One can write down easily similar classifications for various classes of the irreducible representations which are supported in other parabolic subgroups (see [S2] and [S3]). One can get these classifications also by the "restriction" of the corresponding classifications for $GSp(2, F)$. For the sake of completeness, we shall write these classifications now.

Let ρ be an irreducible unitarizable cuspidal representation of $GL(2, F)$ and $\beta \in \mathbb{R}$. Then $\nu^\beta \rho \rtimes 1$ is reducible in the following two cases

(i) $\rho = \tilde{\rho}$, $\omega_\rho \neq 1_{F^\times}$ and $\beta = 0$,

(ii) $\rho = \tilde{\rho}$, $\omega_\rho = 1_F^\times$ and $\beta = \pm 1/2$.

These are the only points of the reducibility of $\nu^\beta \rho \rtimes 1$. This is the Shahidi's result.

We shall sketch here how the above reducibilities for $Sp(2, F)$ follow from the corresponding reducibilities for $GSp(2, F)$. Since $(\nu^\beta \rho \rtimes 1_{F^\times})|Sp(2, F)$ $\nu^\beta \rho \rtimes 1$, we have reducibility of $\nu^\beta \rho \rtimes 1$, which is a representation of $Sp(2, F)$, when $\rho = \tilde{\rho}, \omega_\rho = 1_{F^\times}$ and $\beta = \pm 1/2$. In the other cases, the representation $\nu^\beta \rho \rtimes 1_{F^\times}$ of $GSp(2, F)$ is irreducible. If we have a reducibility of $\nu^\beta\rho \rtimes 1$, then $\rho = \tilde{\rho}$. Further $X_{Sp(2)}(\nu^\beta\rho \rtimes 1_{F^\times}) \neq \{1_{F^\times}\}$ implies $\beta = 0$ and $\omega_\rho \neq 1_{F^\times}$. Then $X_{Sp(2)}(\nu^\beta\rho \rtimes 1_{F^\times}) = \{1_{F^\times}, \omega_\rho\}$. Therefore, cases (i) and (ii) above, describe the reducibilities of $\nu^\beta\rho \rtimes 1$.

The following two propositions hold now. Clearly, they belong to F. Shahidi.

Proposition 5.6.

(i) *Let $\rho \in C^u(GL(2,F))$ satisfies $\rho \cong \tilde{\rho}$ and $\omega_\rho \neq 1_{F^\times}$. Then $\rho \rtimes 1$ decomposes into a sum of two irreducible tempered representations. They are not equivalent. For different representation ρ as above, the irreducible tempered representations are not isomorphic.*

(ii) *Let $\rho \in C^u(GL(2,F))$. If $\rho \ncong \tilde{\rho}$ or if ω_ρ is not a character of the order two, then $\rho \rtimes 1_{F^\times}$ is irreducible. The only non trivial equivalences among such representations are*

$$\rho \rtimes 1_{F^\times} \cong \tilde{\rho} \rtimes 1_{F^\times}.$$

These representations are tempered

(iii) *If σ is an irreducible tempered representation of $GSp(2,F)$ which is not square integrable and which is supported in $P^S_{(2)}$, then σ belong two one of the disjoint groups of representation described (i) and (ii).*

(iv) *Let $\rho \in C^u(GL(2,F))$. Suppose that $\rho \cong \tilde{\rho}$ and $\omega_\rho = 1_{F^\times}$. Then $\nu^{1/2}\rho \times 1_{F^\times}$ contains a unique irreducible subrepresentation. This subrepresentation is square integrable. For different ρ as above, we get square inresentations which are not isomorphic. Each irreducible square integrable representation which is supported in $P^S_{(2)}$, is isomorphic to a square integrable representation as above.* $\square$

Proposition 5.7. *An irreducible unitarizable representation of $Sp(2,F)$ which is supported in $P^S_{(2)}$ is either tempered, or it belongs the following set of the unitarizable representations*

$$L((\nu^\beta\rho, 1))$$

where $\rho \in C^u(GL(2,F))$ such that $\rho \cong \tilde{\rho}, \omega_\rho = 1_{F^\times}$ and $0 < \beta \leq 1/2$. $\square$

From the Waldspurger's and the Shahidi's result for $GSp(2,F)$, we can write the reducibilities for $P^S_{(1)}$ in the following way. Let χ be a unitary character of $F^\times$, let σ be an irreducible cuspidal representation of $Sp(1,F)=SL(2,F)$ and let $\beta \in \mathbb{R}$. For $a \in F^\times$ denote by π_a the representation

$$\pi_a : g \longmapsto \pi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g \begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Put

$$F^\times_\sigma = \{a \in F^\times; \pi \cong \pi_a\}.$$

For each $\varphi \in (F^\times/F^\times_\sigma)\hat{}$ we have $\varphi^2 = 1_{F^\times}$. Now in the following cases we have reducibilities:

(i) $\chi \equiv 1_{F^\times}$ and $\beta = 0$ (i.e. $1_{F^\times} \rtimes \sigma$ is reducible),
(ii) χ is of order two, $\chi \notin (F^\times/F^\times_\sigma)\hat{}$ and $\beta = 0$,
(iii) χ is of order two, $\chi \in (F^\times/F^\times_\sigma)\hat{}$ and $\beta = \pm 1$.

These are the only cases of the reducibility of $\nu^\beta\chi \rtimes \sigma$.

For more information about the groups $(F^\times/F^\times_\sigma)\hat{}$ see [T3]. That group is denoted there by $(GL(2,F)/GL(2,F)_\sigma)\hat{}$ and it corresponds to the group which is denoted by $X_{SL(2,F)}(\pi)$ in [T3], for π an irreducible cuspidal representation of $GL(2,F)$, such that $\pi|SL(2,F)$ contains a subrepresentation isomorphic to σ.

We shall outline now how corresponding reducibilities in the case of $GSp(2,F)$ imply the above reducibilities. Take a cuspidal representation $\sigma' \in GSp(1,F)\tilde{}$ such that σ is isomorphic to a subrepresentation of $\sigma'|Sp(1,F)$. We have a decomposition into a sum of irreducible representations $\sigma'|Sp(1,F) = \sigma_1 \oplus \cdots \oplus \sigma_k$. Representations σ_i are cuspidal and $\sigma_i \not\cong \sigma_j$ if $i \neq j$. From the Jacquet modules one gets that

$$(\nu^\beta\chi \rtimes \sigma')|Sp(2,F) \cong \nu^\beta\chi \rtimes (\sigma'|Sp(1,F)) = \bigoplus_{i=1}^{k} \nu^\beta\chi \rtimes \sigma_i$$

is a multiplicity one representation. Therefore, if π is an irreducible subquotient of $\nu^\beta\chi \rtimes \sigma'$, then $X_{Sp(2)}(\pi) = X_{Sp(2)}(\nu^\beta\chi \rtimes \sigma')$. Reducibility implies $\chi^2 = 1_{F^\times}$. We shall assume that in the further analysis.

Suppose that $\beta \neq 0$. Then $X_{Sp(2)}(\nu^\beta\chi \rtimes \sigma') = X_{Sp(1)}(\sigma')$. Therefore, in this case if $\nu^\beta\chi \rtimes \sigma'$ is irreducible, since the length of $(\nu^\beta\chi \rtimes \sigma')|Sp(2,F)$ is equal to the length of $\sigma'|Sp(1,F)$, we have that all $\nu^\beta\chi \rtimes \sigma_i$ are irreducible. If $\nu^\beta\chi \rtimes \sigma'$ is reducible, then it is of length 2. Now the representation $(\nu^\beta\chi \rtimes \sigma')|Sp(2,F)$ is of length $2k$. Since $\sigma'|Sp(1,F)$ is of length k and $\nu^\beta\chi \rtimes \sigma_i$ splits into no more then two irreducible representations, we have that if $\nu^\beta\chi \rtimes \sigma'$ reduces, then all $\nu^\beta\chi \rtimes \sigma_i$ reduce.

Suppose now that $\beta = 0$. Then $X_{Sp(2)}(\chi \rtimes \sigma') = X_{Sp(1)}(\sigma') \cup \{\eta \in (F^\times)\hat{}; \eta\chi\sigma' \cong \sigma'\}$. Note that $\{\eta \in (F^\times)\hat{}; \eta\chi\sigma' \cong \sigma'\} \subseteq X_{Sp(1)}(\sigma')$ if and only if $\chi \in X_{Sp(1)}(\sigma')$. If $\chi \in X_{Sp(1)}(\sigma')$, then $\chi \rtimes \sigma'$ reduces if and only if $\chi \rtimes \sigma$ reduces. The reasons is same as had above. Thus, $1_{F^\times} \rtimes \sigma_i$ reduces and this is the only reducibility if $\chi \in X_{Sp(1)}(\sigma')$. Suppose now that $\chi \notin X_{Sp(1)}(\sigma')$. The same arguments as above give that all $\chi \rtimes \sigma_i$ reduces.

Description of $X_{Sp(1)}(\sigma')$ in terms of σ one may found in [T3].

We can conclude the following two propositions now. They belong to J.-L. Waldspurger and F. Shahidi.

Proposition 5.8.

(i) *Let $\sigma \in C^u(SL(2,F))$. Then $1_{F^\times} \rtimes \sigma$ reduces into a sum of two irreducible tempered representations. They are not isomorphic. For different representations σ one gets irreducible subrepresentations which are not isomorphic.*

(ii) *Let $\sigma \in C^u(SL(2,F))$ and let $\chi \in (F^\times)\hat{}$ be a character of order two such that $\chi \notin (F^\times/F^\times_\sigma)\hat{}$. Then $\chi \rtimes \sigma$ decomposes into a direct sum of two irreducible tempered representations. They are not isomorphic. For different pairs (σ, χ) as above, one gets irreducible subrepresentations which are not isomorphic.*

(iii) *If σ is an irreducible tempered representation of $Sp(2,F)$ which is supported in $P^S_{(1)}$, and which is not square integrable, then σ belongs to one of the two disjoint groups of the representations described in (i) and (ii).*

(iv) *Let $\sigma 2, F))$ and let χ be a character of $F^\times$ of order two which belongs* *to $(F^\times/F_\sigma^\times)\hat{}$. Then $\nu\chi \rtimes \sigma$ contains a unique irreducible subrepresentation. This subrepresentation is irreducible. For different pairs (σ, χ) as above, one gets square integrable subrepresentations which are not isomorphic. Each irreducible square integrable representation of $Sp(2, F)$ supported in $P_{(1)}^S$ is isomorphic to some square integrable representation as above.* $\square$

Proposition 5.9. *Let π be an irreducible unitarizable representation of $Sp(2, F)$ which is supported in $P_{(1)}^S$. Then π is either tempered or it belongs to the following series of the unitarizable representations*

$$L((\nu^\beta\chi, \sigma))$$

where $\sigma \in C^u(SL(2, F))$, $\quad 0 < \beta \leq 1$, and χ is a character of $F^\times$ of order two which belongs to $(F^\times/F_\sigma^\times)\hat{}$. $\square$

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