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INTERSECTION RINGS OF SPACES OF TRIANGLES

Alberto COLLINO and William FULTON

In 1880 Schubert [12] described a space which compactifies the set of (ordered) plane triangles, and described its intersection ring – giving a basis for the cycles in each dimension, and giving algorithms for computing products. In 1954 Semple [13] gave a modern construction of this space, which we denote X, as an algebraic submanifold of a product of projective and Grassmann manifolds. Tyrrell [15] verified Schubert's prescription of the cycles and their relations in codimension one, and calculated a few other intersection products. The aim of this note is to complete this analysis. We give a formula for the Chow ring (or cohomology ring) of this space : it is generated by seven classes in codimension one, with an ideal of relations generated by twelve classes. In particular we verify that Schubert's basis is correct in all dimensions, and the intersections are as he specified. It is interesting, however, that one of the defining relations for the intersection ring is independent of those given by Schubert before he lists the basis.

The proof is remarkably easy. Since the torus of diagonal matrices in $SL(3)$ acts on X with finitely many (72) fixed points, it follows from the work of Bialynicki–Birula [1], [2] that the total Chow group $A^{\cdot}(X)$ of X is free on 72 generators. We define, purely algebraically, a graded ring $A^{\cdot}$ with seven generators and certain relations, and verify that $A^{\cdot}$ has 72 generators – the same basis as given by Schubert. It is easy to verify that there is a homomorphism from the ring $A^{\cdot}$ to the Chow ring $A^{\cdot}(X)$. Since the generator of A^6 maps to the generator of $A^6(X)$, Poincaré duality implies that this homomorphism is an isomorphism.

Because the algorithms for writing any classes in terms of the basic classes are given explicitly, it becomes a simple algebraic exercise to compute any intersection products, and in particular any enumerative formula, involving the basic 72 generators.

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Although modern machinery has often been used to give rigorous proofs of classical formulas in enumerative geometry, this appears to us to be one of the rare instances where a modern framework actually simplifies the classical calculations. Only part of the first few pages of Schubert's calculations appear in this approach. Perhaps the most obscure part of Schubert's paper (pp. 167–181), which may be regarded as a calculation of the Kunneth components of the class of the diagonal on $X \times X$, can be dispensed with, since this is equivalent to knowing the intersection products of all pairs of generators in complementary dimensions.

In this paper we also compute the Chow ring of the space of triangles in a projective bundle over a given variety. This includes the space of triangles in $\mathbb{P}^n$; for $n=3$ a few equations were included at the end of Schubert's paper [Sch]. As he implies, there are few new ideas needed for this generalization ; the present framework makes it quite automatic.

Another approach to the computation of intersections on the space X of plane triangles has been developed by Roberts and Speiser [9], [10]. They show how X can be constructed by starting with $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, and forming two blowups, followed by one blowdown. This allows one to work out, although with some difficulty, any intersection products one may wish. That approach requires delving considerably deeper into the geometry of the space X, which is of independent interest. Our approach, on the other hand, gives the whole intersection theory on X all at once, with minimal knowledge needed about its geometry, and no need to verify intersection multiplicities of any but the simplest intersection products.

We were led to this idea by reading the preprint of Ellingsrud and Strømme [5], who used the Bialynicki–Birula theorem to compute the Chow groups of the Hilbert schemes of points in the plane. The simple observation of the present note is that the same theorem will yield the Chow ring of a variety, provided one can guess (say with the help of Schubert !) what the ring should be, and one can produce a suitable homomorphism from this abstract ring to the actual ring.

Le Barz [8] has used Hilbert scheme methods to construct a space of triangles in any non-singular variety. We comment on this in §5.

Schubert gives many applications, of which we discuss only one : to calculate the number of triangles which are simultaneously inscribed in a given plane curve C, and circumscribed about a given plane curve D, assuming C and D are suitably general. Here Schubert makes an error and gives an incorrect formula. This is remarkable not only because of the rarity of any errors in Schubert's formulas, but also because the correct formula had been given a decade earlier by

Caylay [3] ! Schubert's error was not in his discussion of the intersection theory of the space of triangles. Rather, he ignored the fact that the dual of a smooth curve of degree greater than two has singularities. When this is taken into account, the correct formula comes out.

The first section discusses the space X of complete triangles, reviewing that part of the work of Schubert and Semple that we need. The second section is pure algebra, describing the ring $A^\cdot$ and giving algorithms for writing any element of $A^\cdot$ as a linear combination of 72 basic classes. The proof that $A^\cdot$ is the intersection ring of X is given in §3, and the application to inscribed and circumscribed triangles in §4. The extension to higher dimensions, with a few complementary remarks occupies §5. Appendix A contains some algebraic manipulations needed for §2 (and for [12], but Schubert assumed the reader could supply them). Appendix B contains the tables of intersection products of classes of complementary dimensions. In Appendix C we prove a simple "Leray Hirsh" theorem for Chow groups of fibre bundles whose fibre is a variety such as the variety of plane triangles, or any smooth projective variety with $\mathbb{C}^*$ action with finitely many fixed points.

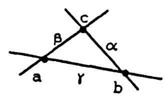
We thank Joe Harris for useful advice about the influence of plane curve singularities on enumerative formulas, and Steven Kleiman for pointing us to Cayley's paper.

Section 1. The compactified space of triangles.

We follow Schubert's notation for ordered triangles in the plane. We sketch a typical member of each type, according to dimension of the loci of such triangles.

A general triangle has vertices a, b, c, with the opposite sides being lines α, β, γ:

Dimension 6



Five–dimensional families :

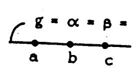
ϵ : the three lines coincide in one line denoted g, on which there are three vertices a, b, c.

τ : dually, the three vertices coincide in a point s, through which pass three lines α, β, γ.

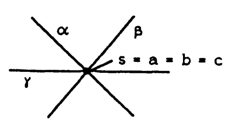
θ_a : the two lines β and γ coincide in a line g, the two points b and c coincide in a point s on g; a is another point on g, while α is another line through s.

θ_b and θ_c are defined similarly, by permuting the vertices and edges.

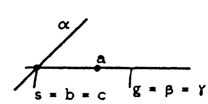
Dimension 5



Type ϵ



Type τ



Type θ_a

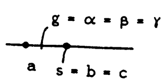
Four–dimensional families :

ω_a : one line g, with the two vertices b and c coinciding in one point s on g, with a another point on g. Similarly for ω_b and ω_c.

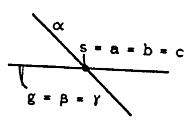
ω_α : the dual specialization of type τ, with $\beta = \gamma = g$; similarly for ω_β and ω_γ.

ψ : the three sides coincide in one line g, and the three vertices coincide in one point s on g. In addition, a net of conics is specified, which contains the pencil of conics consisting of g and an arbitrary line through s, and is contained in the web of conics consisting of all conics which are tangent to g at s. (This net is therefore a plane in the $\mathbb{P}^5$ of conics, containing a certain $\mathbb{P}^1$ and contained in a certain $\mathbb{P}^3$).

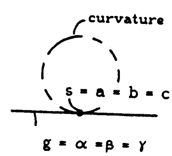
Dimension 4



Type ω_a



Type ω_α



Type ψ

Three–dimensional families :

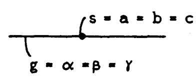
The two special nets described in the following types η and ζ should be regarded as exceptions from type ψ.

η : one line g, one point s on g; the net of conics consists of those conics which contain g as a component, i.e. consist of g and an arbitrary line.

ζ : one line g, one point s, and the net of conics consists of those which are singular at s, i.e. consist of two lines through s.

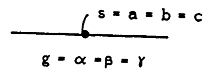
Dimension 3

curvature = 0



Type η

curvature $= \infty$



Type ζ

The set X of complete triangles is the union of the set of general triangles and the special triangles described in the above list. Schubert also described the topology of X, in the sense that he specified which triangles are to be regarded as specializations of which other types :

ω_a is a specialization of ϵ and θ_a ;
ω_α is a specialization of τ and θ_a ;
ψ is a specialization of θ_a , θ_b and θ_c , but not a specialization of ϵ or τ ;
η is a specialization of type ω_a , ω_b and ω_c and ψ , but not of ω_α , ω_β or ω_γ ;
ζ is a specialization of type ω_α , ω_β and ω_γ and ψ , but not of ω_a , ω_b or ω_c .

Each complete triangle has an associated net of conics ; except for types ψ , η and ζ it is determined by the vertices and edges.

For a general triangle, the net is the net of conics passing through the three vertices a, b, c of the triangle.

For a triangle of type ϵ or ω_a , the net consists of all conics which contain the triple line g.

For type τ or ω_a , the net consists of conics which are singular at the point s.

For a triangle of type θ_a , the net consists of all conics which are tangent to α at the point s , and which pass through the point a.

Over the real numbers a net of conics contains a unique circle – the conic which passes through the two circular points $(1: \pm i: 0)$ at infinity. With this interpretation the net of conics corresponds to a radius of curvature ; if the three vertices of a triangle lie on a curve, and approach a non–singular point of the curve, the limiting circle will be the osculating circle to the curve.

Semple [13] defined the space X to be the closure in the space

$$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \times G ,$$

where G is the Grassmannian of planes in the $\mathbb{P}^5$ of conics, of the locus consisting of all $(a, b, c, \alpha, \beta, \gamma, \Lambda)$ for which a, b, c are the vertices, and α, β, γ the sides of a general triangle as above, and Λ is the net of conics passing through the three vertices a, b, c. He showed, by straight–forward calculations in local coordinates, that X is a non–singular six–dimensional subvariety of this product variety, and that the points of X are precisely of the types described above, with the prescribed nets of conics and specialization relations. Each of the types makes up a locally closed algebraic subvariety of X, of the dimension specified with its description.

The main goal of this note is to describe the intersection ring $A^{\cdot}(X)$ of X. Following Schubert and Semple, we use the notations ϵ, τ, θ_b, θ_c, ψ, etc. to denote the classes in $A^{\cdot}(X)$ determined by the closures in X of the corresponding loci of special triangles.

There are also classes in $A^1(X)$ determined by subvarieties of X consisting of triangles in special positions :

a : the vertex "a" is required to lie on a given line. This condition defines a hypersurface in X, whose class is independent of the choice of line. In fact, a is the pull–back of the generator of $A^1(\mathbb{P}^2)$ via the projection to the first factor in the above product.

b and c are defined similarly, and are pull–backs from the second and third factors.

α : the side "α" is required to pass through a given point. This is the pull–back of the generator of $A^1(\check{\mathbb{P}}^2)$ via the projection to the fourth factor in the product. Similarly for β and γ.

d : the net of conics is required to meet a given net of conics. This is the pull–back of the standard generator of $A^1(G)$ via the projection to the last factor.

Tyrrell [15] proved that the relations among these divisor classes were as stated by Schubert. To do this he has to compute some intersection products ; we give direct proofs of these relations in §5. Among these relations are :

$$(1)\qquad \begin{aligned} b+c-\alpha &= \tau+\theta_a \\ c+a-\beta &= \tau+\theta_b \\ a+b-\gamma &= \tau+\theta_c \end{aligned}$$

$$(2)\qquad \begin{aligned} \beta+\gamma-a &= \epsilon+\theta_a \\ \gamma+\alpha-b &= \epsilon+\theta_b \\ \alpha+\beta-c &= \epsilon+\theta_c \end{aligned}$$

$$(3)\qquad d = \alpha+\beta+\gamma+\tau = a+b+c+\epsilon .$$

Some other relations are obvious from the definition, or the fact that the classes are pull–backs from divisors on surfaces :

$$(4)\qquad a^3 = b^3 = c^3 = \alpha^3 = \beta^3 = \gamma^3 = 0 .$$

From the fact that the vertex a is always contained in the side β , i.e. that the projection of X to the product of the first and fourth factors $\mathbb{P}^2 \times \check{\mathbb{P}}^2$ lies in the incidence variety gives the first of the following relations :

$$(5)\qquad \begin{aligned} &a\beta = a^2+\beta^2 \ , \ a\gamma = a^2+\gamma^2 \ , \ b\gamma = b^2+\gamma^2 \ , \\ &b\alpha = b^2+\alpha^2 \ , \ c\alpha = c^2+\alpha^2 \ , \ c\gamma = c^2+\gamma^2 \ . \end{aligned}$$

Since the points b and c (and sides β and γ) coincide on a triangle of type θ_a , we have equations

$$(6)\qquad \begin{aligned} &\theta_a b = \theta_a c \ , \ \theta_b c = \theta_b a \ , \ \theta_c a = \theta_c b \ , \\ &\theta_a \beta = \theta_a \gamma \ , \ \theta_b \gamma = \theta_b \alpha \ , \ \theta_c \alpha = \theta_c \beta \ . \end{aligned}$$

All the above, with the exception of the trivial equations (4), are among those given by Schubert. A final equation which we shall need, however,

$$(7)\qquad \epsilon\tau = 0$$

is not among those in Schubert*. It follows immediately from the definitions that the geometric loci describing the types ϵ and τ have disjoint closures, because the nets of conics can never coincide : those of type ϵ have curvature 0, while those of type τ have curvature ∞ .

* We cannot help commenting on the fact that Schubert omits such useful equations. It is now universally agreed that what Schubert was doing is exactly equivalent to the modern calculation of intersection products of cycles on manifolds, and we do not pretend to deny this. But to anyone now calculating intersection products, the first relations written down would be that products of classes determined by disjoint subvarieties are zero. In fact, Schubert only explicitly writes down products of classes where at most **one** of the factors describes figures of a special type ; all the other factors describe figures in special position with regard to given but variable objects. Of course several classes involving special type are more likely to meet improperly, and perhaps, in the absence of foundations, he wanted to avoid such dangers.

It should also be pointed out that Schubert's equations given in the beginning of his paper do **not** generate all equations in codimension ≥ 2 ; the equation $\epsilon\tau = 0$ is independent of the equations he lists.

Section 2. The algebraic intersection ring.

The ring $A^{\cdot}$ is defined to be the polynomial ring in seven variables, subject to certain relations. That is

$$A^{\cdot} = \mathbb{Z}[a,b,c,\alpha,\beta,\gamma,d)/I$$

where I is the ideal generated by the polynomials listed in (1) – (4) :

(1) a^3 , b^3 , c^3 , α^3 , β^3 , γ^3 ;

(2) $a\beta-a^2-\beta^2$, $a\gamma-a^2-\gamma^2$, $b\gamma-b^2-\gamma^2$,
$b\alpha-b^2-\alpha^2$, $c\alpha-c^2-\alpha^2$, $c\beta-c^2-\beta^2$;

(3) $(b+c+\beta+\gamma-d)(b-c)$, $(a+c+\alpha+\gamma-d)(c-a)$, $(a+b+\alpha+\beta-d)(a-b)$;
$(b+c+\beta+\gamma-d)(\beta-\gamma)$, $(a+c+\alpha+\gamma-d)(\gamma-\alpha)$, $(a+b+\alpha+\beta-d)(\alpha-\beta)$;

(4) $(d-a-b-c)(d-\alpha-\beta-\gamma)$.

Remark. This list of generators for I is not minimal. In fact, modulo relations (2), the six equations in (3) are equivalent (see equation (A.8) of the appendix) to the four equations

$$2a^2+a\alpha-\alpha^2-ad = 2b^2+b\beta-\beta^2-bd = 2c^2+c\gamma-\gamma^2-cd\ ,$$
$$2\alpha^2+a\alpha-a^2-\alpha d = 2\beta^2+b\beta-b^2-\beta d = 2\gamma^2+c\gamma-\gamma^2-\gamma d\ ;$$

so two generators, say the first and fourth of (3), could be omitted. In addition, the six generators in (1) can be replaced by any one of them (e.g., to see $a^3 \equiv \beta^3$, combine $a^2\beta \equiv a^3+a\beta^2$ with $a\beta^2 \equiv \beta^3+a^2\beta$). When this is done one has 12 generators for I which are a minimal set of generators. In fact, relations (2), (3), and (4) put 11 relations on the 28 monomials of degree 2, so all of these equations are needed to get $\dim A^2 = 17$. One may check that a relation (1) must be added to cut the dimension of A^3 from 23 to 22, or even to cut $A^{\cdot}$ down to a 0–dimensional ring.

Note that the group $G = \mathfrak{S}_3 \times \mathfrak{S}_2$ acts on $A^{\cdot}$; the symmetric group $\mathfrak{S}_3$ on three letters permutes a, b and c and simultaneously α , β and γ ; the group $\mathfrak{S}_2$ acts via the "duality" operation which interchanges a and α , b and β , c and γ ; all elements of G fix d . The generators of I are chosen so that I is clearly taken into itself by thic action of G .

For simplicity as well as to clarify the relations with geometry we define polynomials ϵ , τ , θ_a , θ_b and θ_c and express some of the generators of I in terms of them ; we set

$$\text{(i)} \qquad \begin{aligned} \epsilon &= d-a-b-c \\ \tau &= d-\alpha-\beta-\gamma \\ \theta_a &= b+c+\beta+\gamma-d \\ \theta_b &= a+c+\alpha+\gamma-d \\ \theta_c &= a+b+\alpha+\beta-d\,. \end{aligned}$$

The generators (3) and (4) for I can be written :

$$(3') \qquad \begin{array}{llll} \theta_a(b-c), & \theta_b(c-a), & \theta_c(a-b) & ; \\ \theta_a(\beta-\gamma), & \theta_b(\gamma-\alpha), & \theta_c(\alpha-\beta) & ; \end{array}$$

$$(4') \qquad \epsilon\tau\,.$$

The same notation will be used for the corresponding elements a, b, c, α, β, γ, d, ϵ, τ, θ_a, θ_b, θ_c, in $A^\cdot$. It follows immediately from the definitions that any of the elements ϵ, τ, θ_a, θ_b or θ_c could have been used instead of d as the seventh generator of $A^\cdot$, and that we have the equations :

$$\text{(ii)} \qquad \begin{aligned} \theta_a &= b+c-\alpha-\tau &&= \beta+\gamma-a-\epsilon \\ \theta_b &= c+a-\beta-\tau &&= \gamma+\alpha-b-\epsilon \\ \theta_c &= a+b-\gamma-\tau &&= \alpha+\beta-c-\epsilon\,. \end{aligned}$$

In addition, since $\theta_a b = \theta_a c$ in $A^\cdot$ by (3′), we denote this common element of $A^\cdot$ by $\theta_a s$; similarly $\theta_a g$ denotes $\theta_a\beta = \theta_a\gamma$, and the same is done for θ_b and θ_c. That is, we define :

$$\text{(iii)} \qquad \begin{array}{lllllll} \theta_a s &= \theta_a b &= \theta_a c &, & \theta_a g &= \theta_a\beta &= \theta_a\gamma \\ \theta_b s &= \theta_b c &= \theta_b a &, & \theta_b g &= \theta_b\gamma &= \theta_b\alpha \\ \theta_c s &= \theta_c a &= \theta_c b &, & \theta_c g &= \theta_c\alpha &= \theta_c\beta\,. \end{array}$$

From equation (A.3) of Appendix A follow the equations $\epsilon\alpha = \epsilon\beta = \epsilon\gamma$, which we denote by ϵg, and dually for τ; that is, we define :

$$\text{(iv)} \qquad \begin{aligned} &\epsilon g = \epsilon\alpha = \epsilon\beta = \epsilon\gamma \\ &\tau s = \tau a = \tau b = \tau c\,. \end{aligned}$$

Similarly (see (A.10)) $\theta_a b^2 = \theta_a c^2 = \theta_a bc$ is denoted $\theta_a s^2$, with analogous formulae for θ_b and θ_c. Likewise $\tau a^2 = \tau b^2 = \tau ab = \ldots$, denoted τs^2, and dually for ϵg^2 :

(v)
$$\begin{aligned}
\theta_a s^2 &= \theta_a b^2 = \theta_a c^2 = \theta_a bc\\
\theta_b s^2 &= \theta_b c^2 = \theta_b a^2 = \theta_b ca\\
\theta_c s^2 &= \theta_c a^2 = \theta_c b^2 = \theta_c ab\\
\tau s^2 &= \tau a^2 = \tau b^2 = \tau c^2 = \tau ab = \tau bc = \tau ca\\
\epsilon g^2 &= \epsilon\alpha^2 = \epsilon\beta^2 = \epsilon\gamma^2 = \epsilon\alpha\beta = \epsilon\beta\gamma = \epsilon\gamma\alpha\,.
\end{aligned}$$

Another simple calculation (A.5) shows that $\theta_a\theta_b = \theta_b\theta_c = \theta_c\theta_a$, and this element is denoted ψ:

(vi) $$\psi = \theta_a\theta_b = \theta_b\theta_c = \theta_c\theta_a\,,$$

and we have (A.19) the formula $\psi a^2 = \psi b^2 = \psi ab = \ldots$ which is denoted ψs^2, and similarly for ψg^2:

(vii)
$$\begin{aligned}
&\psi s^2 = \psi a^2 = \psi b^2 = \psi c^2 = \psi ab = \psi bc = \psi ca\\
&\psi g^2 = \psi\alpha^2 = \psi\beta^2 = \psi\gamma^2 = \psi\alpha\beta = \psi\beta\gamma = \psi\gamma\alpha\,.
\end{aligned}$$

Finally note (A.27) that $(\psi s^2)\alpha = (\psi s^2)\beta = (\psi s^2)\gamma$, which is denoted $\psi s^2 g$, and similarly (A.19) for $(\theta_a s^2)\beta = (\theta_a s^2)\gamma$:

(vii)
$$\begin{aligned}
\psi s^2 g &= (\psi s^2)\alpha = (\psi s^2)\beta = (\psi s^2)\gamma\\
\theta_a s^2 g &= (\theta_a s^2)\beta = (\theta_a s^2)\gamma\\
\theta_b s^2 g &= (\theta_b s^2)\alpha = (\theta_b s^2)\gamma\\
\theta_c s^2 g &= (\theta_c s^2)\alpha = (\theta_c s^2)\beta\,.
\end{aligned}$$

For convenience we set

(viii) $$[\star] = a^2b^2c^2$$

in A^6. We will see shortly that $[\star]$ is also equal to $\alpha^2\beta^2\gamma^2$.

PROPOSITION. *The ring* $A^{\cdot}$ *is generated as an additive group over* $\mathbb{Z}$ *by the* 72 *elements*:

1 — *in* A^0

$a, b, c, \alpha, \beta, \gamma, d$ — in A^1

$a^2, b^2, c^2, \alpha^2, \beta^2, \gamma^2, a\alpha, b\beta, c\gamma, \tau s, \epsilon g, \theta_a s, \theta_b s,$
$\theta_c s, \theta_a g, \theta_b g, \theta_c g$ — in A^2

$a^2\beta,\ a^2\gamma,\ b^2\gamma,\ b^2\alpha,\ c^2\alpha,\ c^2\beta,\ abc,\ \alpha\beta\gamma,\ \epsilon a^2,\ \epsilon b^2,\ \epsilon c^2,$
$\tau\alpha^2,\ \tau\beta^2,\ \tau\gamma^2,\ \epsilon g^2,\ \tau s^2,\ \theta_a s^2,\ \theta_b s^2,\ \theta_c s^2, \theta_a g^2,\ \theta_b g^2,\ \theta_c g^2$ in A^3

$b^2c^2,\ c^2a^2,\ a^2b^2,\ \beta^2\gamma^2,\ \gamma^2\alpha^2,\ \alpha^2\beta^2,\ \tau s^2\alpha,\ \tau s^2\beta,\ \tau s^2\gamma,$
$\epsilon g^2 a,\ \epsilon g^2 b,\ \epsilon g^2 c,\ \theta_a s^2 g,\ \theta_b s^2 g,\ \theta_c s^2 g,\ \psi s^2,\ \psi g^2$ in A^4

$\epsilon b^2c^2,\ \epsilon c^2a^2,\ \epsilon a^2b^2,\ \tau\beta^2\gamma^2,\ \tau\gamma^2\alpha^2,\ \tau\alpha^2\beta^2,\ \psi s^2 g$ in A^5

$[\star]$ in A^6.

In fact, we give recipes to write any monomials in $a,b,c,\alpha,\beta,\gamma,d$ in $A^{\cdot}$ as integral linear combinations of these 72 basic classes. Most of these rules are formulas of Schubert ; the point is simply to verify that they all follow algebraically from the basic relations (1) – (4). We list and verify those of Schubert's formulas which we need in Appendix A.

Because of the action of $G = \mathfrak{S}_3 \times \mathfrak{S}_2$ on $A^{\cdot}$, each relation that is proved to hold in $A^{\cdot}$ may give rise to up to 12 relations by applying the symmetries in G to it.

In this regard note that ϵ and τ are dual and fixed under $\mathfrak{S}_3$, that θ_a, θ_b, and θ_c are self–dual, and are permuted as the subscripts indicate by $\mathfrak{S}_3$; ψ is self–dual ; $\theta_a s$ and $\theta_a g$ are dual, as are τs and ϵg, $\theta_a s^2$ and $\theta_a g^2$, τs^2 and ϵg^2, and ψs^2 and ψg^2 ; we will see that $\theta_a s^2 g$ and $\psi s^2 g$ are self–dual.

We note also that the set of proposed generators of each A^k is closed under the action of G. Except in degrees 3 and 6, this follows immediately from the previous paragraph. For degree 3 one needs to add the equation

$$(5) \qquad a\beta^2 = (a^2+\beta^2)\beta = a^2\beta ,$$

which follows from (2) and (1). For degree 6, to show that $\alpha^2\beta^2\gamma^2 = a^2b^2c^2$, note first that

$$(6) \qquad a^2\beta^2 = a(a\beta^2) = a(a^2\beta) = 0 ,$$
$$(7) \qquad ab\gamma^2 = (a^2+\gamma^2)(b^2+\gamma^2) = a^2b^2 ,$$
$$(8) \qquad ab\alpha\beta = (a^2+\beta^2)(b^2+\alpha^2) = a^2b^2+\alpha^2\beta^2+a^2\alpha^2+b^2\beta^2 .$$

Multiplying (8) by γ^2 and applying (6) and (7) it follows that

$$(9) \qquad \alpha^2\beta^2\gamma^2 = ab\alpha\beta\gamma^2 = a^2b^2\alpha\beta .$$

Now $a^2b^2\alpha\beta = ab\alpha^2\beta^2$ by (5), and by the duals of the preceding steps, this is $a^2b^2c^2$, as required.

To show how the indicated elements generate in a given degree k, it suffices to show first how to write monomials of degree k of the form ST, where S is a monomial in a, b, c, and T is a monomial in α, β, γ, as a linear combination of the given elements. Using the action of G, one need only check one monomial in each G-orbit ; for example, by duality, it suffices to consider $\deg(S) \geq \deg(T)$. Next, for each ST as above, but of degree $k-1$, one must show how to write the product of ST with any one of the elements d, ϵ, τ, θ_a, θ_b, or θ_c, as a linear combination of the given elements. To see this one uses the equations (i). Because of (4) we have

$$(10) \qquad d^2 = (a+b+c+\alpha+\beta+\gamma)d + (a+b+c)(\alpha+\beta+\gamma),$$

so we never have to consider products of any of these last elements. The details of these computations are included in Appendix A. □

This makes the calculation of any product in $A^\cdot$ a simple algebra exercise. In particular one computes easily that the 7×7, 17×17, and 22×22 matrices obtained by multiplying the basic classes in A^1 and A^5, A^2 and A^4, A^3 and A^3, respectively and picking off coefficients of $[\star]$, are all unimodular (see Appendix B).

In the next section we will construct a homomorphism from the ring $A^\cdot$ to the Chow ring $A^\cdot(X)$. We will apply the following simple lemma to this homomorphism, to deduce that this homomorphism is an isomorphism, and that the above classes form a basis for $A^\cdot$ and $A^\cdot(X)$.

DEFINITION. *A graded ring $A^\cdot = A^0 \oplus A^1 \oplus ... \oplus A^n$ will be called an n-dimensional Poincaré duality ring if A^n has one generator $[\star]_A$ over $\mathbb{Z}$, and each A^i has a finite number of generators $\alpha_p^{(i)}$; in addition, for each i there should be integers $a_{pq}^{(i)}$ such that*

$$\alpha_p^{(i)} . \alpha_q^{(n-i)} = a_{pq}^{(i)} [\star]_A$$

and the matrices $(a_{pq}^{(i)})$ are unimodular. We will call such a ring a strong Poincaré duality ring if, in addition, the generator $[\star]_A$ is not a torsion element (or zero) ; it follows that the elements $\alpha_p^{(i)}$ form a basis for A^i over $\mathbb{Z}$, and the product $A^i \otimes A^{n-i} \longrightarrow A^n \cong \mathbb{Z}$ is a perfect pairing over $\mathbb{Z}$.

LEMMA. *Let $A^\cdot$ and $B^\cdot$ be n-dimensional Poincaré duality rings, with $B^\cdot$ assumed strong. Suppose $\varphi^\cdot : A^\cdot \longrightarrow B^\cdot$ is a homomorphism of graded rings, and that φ^n maps $[\star]_A$ onto $[\star]_B$. Then $A^\cdot$ is also strong. Suppose that the total number of generators of $A^\cdot$ is the same as the number of generators of $B^\cdot$. Then $\varphi^\cdot$ is an isomorphism.*

Proof: The first assertion is obvious, since a torsion element of A^n could not map to a torsion–free element of B^n. For the second, to see that $\varphi^{\cdot}$ is injective, suppose $x \in A^i$ and $\varphi^i(x) = 0$; choose $y \in A^{n-i}$ with $x.y = [\star]_A$. Then

$$[\star]_B = \varphi^n([\star]_A) = \varphi^i(x).\varphi^{n-i}(y) = 0 ,$$

a contradiction. Since $A^{\cdot}$ and $B^{\cdot}$ have the same ranks, each φ^i must map A^i onto a lattice in B^i. Consider the commutative diagram

$$\begin{array}{ccccc} A^i & \otimes & A^{n-i} & \longrightarrow & A^n \\ \varphi^i\downarrow & & \downarrow\varphi^{n-i} & & \downarrow\varphi^n \\ B^i & \otimes & B^{n-i} & \longrightarrow & B^n . \end{array}$$

Since the bottom pairing is perfect over $\mathbb{Z}$, the index of $\varphi^i(A^i)$ in B^i must divide the determinant of the matrix which describes the upper pairing. But this determinant is assumed to be 1, so $\varphi^i(A^i) = B^i$, as required. □

Section 3. The Chow ring for plane triangles.

Let $A^{\cdot}$ be the graded ring constructed in Section 2, and let $A^{\cdot}(X)$ be the Chow ring of the space X of complete triangles.

PROPOSITION. *The Chow ring $A^{\cdot}(X)$ is a free abelian group on 72 generators, and the canonical map from $A^{\cdot}(X)$ to the homology ring $H^{\cdot}(X)$ is an isomorphism.*

Proof : This follows from the theorem of Bialynicki–Birula [1], [2], once we prove that a torus T acts on X, with 72 fixed points ; this uses the fact that X is a non–singular projective variety. The torus T is the group of diagonal matrices in $SL(3)$, which acts on the projective plane by linear transformations, and hence acts on X. The fixed points of this action are easy to list, since the only fixed points of this action are the three points (1:0:0), (0:1:0), and (0:0:1), and the only fixed lines are the axes joining them. There are 6 honest (ordered) triangles, obtained by ordering these three points as vertices. There are 18 of type θ_a , 18 of type ω_a , 18 of type ω_α , 6 of type η , and 6 of type ζ . □

THEOREM. *There is an isomorphism from $A^{\cdot}$ to $A^{\cdot}(X)$ which takes the elements a, b, c, α, β, γ, and d to the classes described by Schubert (which are the pullbacks of the positive generators of divisor classes via the projections to the six factors). In addition, the elements ϵ, τ, θ_a, θ_b, θ_c in $A^{\cdot}$ map to the classes in $A^{\cdot}(X)$ of the closures of the corresponding loci in X. The classes listed in the proposition of §2 map to a basis for $A^{\cdot}(X)$.*

Proof : Map $\mathbb{Z}[a,b,c,\alpha,\beta,\gamma,d]$ to $A^{\cdot}(X)$, with generators going to the pullbacks of the designated hyperplane classes. To obtain a homomorphism from $A^{\cdot}$ to $A^{\cdot}(X)$ it must be verified that the generators of the ideal I map to zero, which was already proved in Section 1, the essential point being the formulae relating the divisors ϵ, τ, θ_a, θ_b, θ_c to the divisors a, b, c, α, β, γ, d proved in [15] or §5 below. We proved in Section 2 that $A^{\cdot}$ is a Poincaré duality ring with 72 generators. Since X is a smooth projective variety whose Chow ring is isomorphic to its cohomology ring, $A^{\cdot}(X)$ is a strong Poincaré duality ring, and we know it has 72 generators. The class $[\star] = a^2b^2c^2$ maps to the class of a point in X, namely that representing the unique triangle with three given general vertices. By the lemma of §2, it follows that the map from $A^{\cdot}$ to $A^{\cdot}(X)$ is an isomorphism. □

Remark. The classes of the closures of the loci described in Section 1 by the notations ψ, ω_a, ω_b, ω_c, ω_α, ω_β, ω_γ, η, and ζ also correspond to the elements in $A^{\cdot}$ specified by Schubert. These can be deduced from the formulae

$$\psi = \theta_{\mathrm{a}}\theta_{\mathrm{b}}\ ,\ \omega_{\mathrm{a}} = \epsilon\theta_{\mathrm{a}}\ ,\ \omega_{\alpha} = \tau\theta_{\alpha}\ ,\ \eta = \epsilon\psi\ ,\ \zeta = \tau\psi\ ,$$

by the algorithms of Appendix A. To verify these formulae, it suffices to show that, at a generic point of a locus on the left side of the equation, the two loci on the right meet transversally ; this can be done in local coordinates, as in [13], [15], or [8].

Section 4. Inscribed and circumscribed triangles

Among Schubert's applications is a calculation of the number of triangles which are simultaneously inscribed in one curve C and circumscribed about another curve D, i.e., the vertices lie on C and the sides are tangent to D. In this section we carry out this application, while correcting an error of Schubert's.

Let V_C be the subvariety of the space of complete triangles consisting of those which are inscribed in C. More precisely, V_C is the closure in X of the set of triangles with non–collinear vertices which lie on C. As the image of a rational map from $C\times C\times C$ to X, V_C is an irreducible three–dimensional subvariety. Let $v = v_C$ denote the class of $[V_C]$ in $A^3(X)$. To compute the coefficients for v of the 22 basic generators, it will suffice to compute the intersection of v with 22 independent elements of $A^3(X)$.

LEMMA 4.1. *If C is an irreducible curve of degree n with δ ordinary nodes, and κ ordinary cusps as its only singularities, then*

(i) *the intersection numbers of v with the following classes are* 0 : a^2b , $a^2\beta$, $a^2\alpha$, $\theta_a s^2$, $\theta_a a^2$, ϵa^2 , τs^2 , $\theta_a \alpha^2$, $\tau\alpha^2$.

(ii) $v.\alpha^2\beta = n(n-1)^2$, $v.a\alpha^2 = n^2(n-1)$, $v.\epsilon g^2 = n(n-1)(n-2)$, $v.abc = n^3$, $v.\theta_a sa = n^2$, $v.\theta_a g^2 = n(n-1)$.

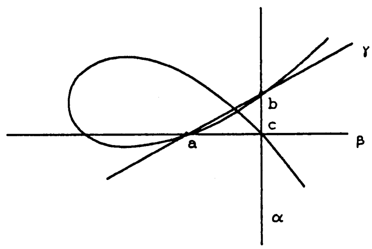
(iii) $v.\tau\alpha\beta = 2\delta+3\kappa$.

Proof: (i) all conditions but the last two require one of the vertices to be a fixed general point, which would not be on C. For the intersection with $\theta_a\alpha^2$, a degenerate triangle of the form θ_a is in V_C only if the line α is tangent to C at the point $b=c$. Since in this condition α is fixed and general, so transversal to C, the intersection is empty. A similar argument works for $\tau\alpha^2$.

(ii) For the first, the general line α meets C transversally, giving $n(n-1)$ choices for the points b and c; for each of these, the line β is determined, and there are $n-1$ choices for the point a on this line. The other cases in (ii) are similar.

(iii) For a triangle of type τ to be in V_C it is necessary that the point s of τ is the limit of three non–collinear points of C. If s is a smooth point of C, the three lines must come together in the tangent line to C at s. Two general points are fixed for the lines α and β to pass through. The only possibility for intersection therefore comes from the singular points of C. We analyze these locally. Assume $(0,0) = (0:0:1)$ is a node of C, that C has affine equation of the form $y^2 = x^2 +$ higher terms, and that α must pass through $(0:1:0)$ and β

must pass through (1:0:0). To get a triangle of type τ as a limit of honest triangles with vertices on C, two of the three vertices must move on one branch, one on the other ; the limiting triangle has sides $\alpha : x = 0$; $\beta : y = 0$, and γ : either $y = x$ or $y = -x$. We must show that each of these counts for 1 in the intersection product. It suffices to consider the first case. Let $y = x+g(x)$, $y = -x+h(x)$ define the two branches, with g and h power series vanishing to order at least two at the origin.



The four-dimensional variety Y of triangles with α passing through (0:1:0) and β passing through (1:0:0) can be parametrized by coordinates s, t, u, v, where $x = s$, $y = t$ are equations for α and β respectively, $c = (s,t)$, $a = (s-v,t)$, $b = (s,t+(1+u)v)$, and γ has equation $y = (1+u)(x-s+v) + t$. The intersection of Y with V_C is described by equations

$t = s-v+g(s-v)$	(the point b is on the first branch)
$t+(1+u)v = s+g(s)$	(the point a is on the first branch)
$t = -s+h(s)$	(the point c is on the second branch).

This curve is parametrized by s ; $t = -s +$ higher terms, $v = 2s +$ higher terms, and $u = (g(s)-g(s-v))/v = \dots$. The hypersurface of triangles of type τ is defined in Y by the equation $v=0$. Since the order of v as a function of s is 1, the intersection multiplicity is 1, as required.

Similarly for a cusp of C at the origin, say defined by $y^2 = x^3 +$ higher terms. Let Z be the locus in X of triangles such that α passes through $(1:p:0)$, and β passes through $(1:q:0)$, for p and q general constants. The point P in $Z \cap \tau$ will have sides $\alpha : Y = pX$,

$\beta: Y = qX$, $\gamma: Y = 0$. We will parametrize the curve $Z \cap V_C$, and intersect with the hypersurface τ.

Parametrize C near the origin by $t \longmapsto c_t = (t^2, t^3+\ldots)$. The point b_t is the other point on $C \cap \alpha$, which has the parametrization

$$t \longmapsto b_t = (t^2 - \frac{2}{p}t^3 + \ldots, -t^3 + \ldots).$$

Similarly for a_t, replacing p by q. One verifies easily[1] that the hypersurface $\tau \cap Z$ is defined near P by the equality of the x-coordinates of the points a and b. Pulling this hypersurface back to the t-disk, one has the equation

$$(t^2 - \frac{2}{p}t^3) - (t^2 - \frac{2}{q}t^3) + \ldots = (\frac{2}{q} - \frac{2}{p})t^3 + \ldots .$$

Since the order of vanishing in t is three, the intersection number is three[2], as required. □

PROPOSITION. *If C is an irreducible plane curve of degree n with only δ ordinary nodes and κ ordinary cusps as singularities, then*

$$[V_C] = n(n-1)(\tau\alpha^2 + \tau\beta^2 + \tau\gamma^2) + 2n(n-1)(\theta_a g^2 + \theta_b g^2 + \theta_c g^2)$$
$$+ \check{n}\tau s^2 + (3n^2 - 2n)\epsilon g^2 + n(n-1)(n-2)abc,$$

where $\check{n} = (n-1) - 2\delta - 3\kappa$ is the class of C.

Proof: It suffices to check that both sides have the same intersection numbers with the 22 basis elements of $A^3(X)$. For $[V_C]$, all but the intersection with $\alpha\beta\gamma$ are listed in the lemma, up to permutations. From Appendix A one can write $\tau\alpha\beta$ in terms of the basic elements:

$$\tau\alpha\beta = c^2\alpha + c^2\beta + 2\epsilon g^2 + 2\tau s^2 + \epsilon a^2 + \epsilon b^2 + \theta_a s^2 + \theta_b s^2 + 2\theta_c s^2 + \theta_c g^2 - \alpha\beta\gamma .$$

---

[1] Z has local coordinates (u,v,g,h), where the sides have equations $\alpha: Y = pX+u$, $\beta: Y = qX+v$, $\gamma: Y = gX+h$. One solves for the points a, b and c in terms of these coordinates and checks that an equation for τ is $(v-h)(g-p) = (u-h)(g-q)$, which is equivalent to equating the x-coordinates of a and b.

[2] This follows from the projection formula for the parametrization from the disk to the curve.

From the lemma, we derive

$$2\delta+3\kappa = 2n(n-1)(n-2) + n(n-1) - [V_C].\alpha\beta\gamma .$$

One then checks the table (Appendix B) to see that the coefficients agree. □

By duality we have :

PROPOSTION. *If D is an irreducible plane curve of degree m with only τ ordinary bitangents and ι ordinary inflections (as singularities on the dual curve), and W_D is the locus of triangles circumscribed about D, then*

$$[W_D] = \check{m}(\check{m}-1)(\epsilon a^2+\epsilon b^2+\epsilon c^2) + 2\check{m}(\check{m}-1)(\theta_a s^2+\theta_b s^2+\theta_c s^2)$$
$$+ m\epsilon g^2 + (3\check{m}^2-2\check{m})\tau s^2 + \check{m}(\check{m}-1)(\check{m}-2)\alpha\beta\gamma ,$$

where $\check{m}$ is the class of D. Note that $m = \check{m}(\check{m}-1)-2\tau-3\iota$.

COROLLARY. *With C and D as in the propositions, and in general position in the plane, the number of triangles simultaneously inscribed in C and circumscribed about D is one-sixth of*

$$2n(n-1)(n-2)\check{m}(\check{m}-1)(\check{m}-2) + n(n-1)(n-2)m + \check{m}(\check{m}-1)(\check{m}-2)\check{n} .$$

Proof: The fact that $[V_C].[W_D]$ is equal to the displayed number follows from the two propositions and the tables for intersecting basic 3–cycles with each other. One must also verify that V_C meets W_D transversally at points which correspond to honest triangles ; this follows as usual from the transitive action of the projective linear group. One must use the actual description of X to see that there are no others. For example, a triangle of type θ_a is in V_C if the line α is tangent to the curve at the point $b=c$, and the point a is another point on C. Dually, this triangle will belong to W_D if a is on D and the line $\beta=\gamma$ is tangent do D at a, and α is another tangent to D. Thus, if θ_a is in $V_C \cap W_D$, a is one of the mn points on $C \cap D$, α is one of the $\check{m}\check{n}$ common tangents to C and D, and the tangent to D at a meets α at its point of tangency to C; this does not happen if C and D are in general position. Similar arguments apply to other types of degenerate triangles. □

Remarks. (1) Schubert's formula $n(n-1)\check{m}(\check{m}-1)(2n\check{m}-3n-3\check{m}+4)$ differs from the correct answer by the quantity

$$(2\delta+3\kappa)\check{m}(\check{m}-1)(\check{m}-2) + (2\tau+3\iota)n(n-1)(n-2)\ .$$

If C is smooth – which Schubert presumably assumed – the first term can be ignored, but the second term is non–zero when D is a smooth curve of degree ≥ 3 (if degree $C \geq 3$). Schubert gives the intersection of $[V_C]$ with $\tau\alpha\beta$ as zero, which is only correct if C is smooth ; the dual formula for $[W_D].\epsilon ab = 0$ is false when D has flexes and bitangents, even if D is smooth.

(2) The formula of the corollary depends only on the degrees and classes of the two curves. From this one might expect that the same formula is valid for curves with arbitrary singularities, as is the case in the contact formula [7]. However, this is not the case. A singularity of the form $y^p = x^q$, with $p < q$ coprime, contributes $q(p-1)^2$ to the intersection product $v.\tau\alpha\beta$, while its contribution to the class number formula – intersecting the curve with a polar curve – is only $q(p-1)$. For a discussion of variations of numerical invariants of singularities in families see the article of Diaz and Harris [4].

Section 5. Triangles in a projective bundle.

Let E be a vector bundle on a smooth quasiprojective variety S. Let $Y = G_3(E) = G_2(\mathbb{P}(E))$ be the Grassmann bundle of 2–planes in the projective bundle $\mathbb{P}(E)$ of lines in E. Let U be the universal 3–plane bundle on Y, and $\mathbb{P}(U)$ the bundle of projective planes. The space of triangles of $\mathbb{P}(E)$ is defined to be the fiber bundle X over Y whose fiber over a plane is the space of (complete) triangles in that plane. In this section we determine the Chow ring $A^{\cdot}(X)$ as an algebra over $A^{\cdot}(Y)$, and hence as an algebra over $A^{\cdot}(S)$.

Take three copies of $\mathbb{P}(U)$, with tautological sub–line bundles of U denoted A, B, and C. Take three copies of $G_2(U)$, with tautological sub–plane bundles of U denoted $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. We can construct X globally as the closure in

$$\mathbb{P}(U)\times_Y\mathbb{P}(U)\times_Y\mathbb{P}(U)\times_Y G_2(U)\times_Y G_2(U)\times_Y G_2(U)\times_Y G_3(\mathrm{Sym}^2(U)$$

of the set of honest triangles. Over any open set Y^o of Y where U is trivial, X is the product of Y^o by the triangle space discussed in §1. In particular we have the loci of triangles of special type θ_a, ϵ, τ, etc., and we denote the classes in $A^{\cdot}(X)$ of such subvarieties by the same Greek letters.

On X we have inclusions of vector bundles (denotes by the same letters)

$$A \subset \mathcal{B} \subset U,\ A \subset \mathcal{C} \subset U,\ B \subset \mathcal{A} \subset U,\ B \subset \mathcal{C} \subset U,\ C \subset \mathcal{A} \subset U,\ C \subset \mathcal{B} \subset U,$$

corresponding to the inclusions of points in lines. We define classes in $A^1(X)$ by :

$$\begin{aligned} \mu_1 &= c_1(U^{\vee}) \\ a &= c_1(A^{\vee}), \quad b = c_1(B^{\vee}), \quad c = c_1(C^{\vee}), \\ \alpha &= c_1(\mathcal{A}^{\vee}), \quad \beta = c_1(\mathcal{B}^{\vee}), \quad \gamma = c_1(\mathcal{C}^{\vee}), \end{aligned}$$

and define $\mu_2 = c_2(U^{\vee}) \in A^2(X)$, $\mu_3 = c_3(U^{\vee}) \in A^3(X)$.

When S is a point, so Y is a Grassmann variety, μ_1 is represented by the condition that the plane of the triangle meet a given linear space of codimension three, μ_2 the condition that the plane meets a given codimension two space in at least a line, and μ_3 the condition for the plane to be contained in a given hyperplane. For $Y = \check{\mathbb{P}}^3$, we have $\mu_i = \mu^i$, with μ the condition for the plane to pass through a point ; this is the notation used by Schubert [12], §11.

The class a is represented by the condition for the vertex "a" to lie on a given hyperplane, while α by the condition that the side "a" meet a given space of codimension two. The notations for special triangles θ_a, ϵ, τ, ψ, have the same meaning as before.

LEMMA. *The following equations are valid in* $A^{\cdot}(X)$:

(i) $\theta_c = a+b-\gamma-\tau$

(ii) $\theta_c = \alpha+\beta-c-\epsilon-\mu_1$

(iii) $a^3 = \mu_1 a^2-\mu_2 a+\mu_3$

(iv) $\alpha^3 = 2\mu_1\alpha^2-\mu_1^2\alpha-\mu_2\alpha+\mu_1\mu_2-\mu_3$

(v) $a\beta = a^2+\beta^2-\mu_1\beta+\mu_2$.

Proof : For (i), we look at the locus where the points "a" and "b" coincide. This is given by the vanishing of the composite map of line bundles

$$A \longrightarrow \mathscr{C} \longrightarrow \mathscr{C}/\mathrm{B} .$$

Thus this locus represents the class $c_1(\mathscr{C}/\mathrm{B}) - c_1(A) = -\gamma+b+a$. On the other hand, the locus where these two points coincide consists of all triangles of type θ_c or type τ (or the closure of these two types) ; one checks easily, say in local coordinates (cf. [13]), that the map of line bundles vanishes to order 1 along each of these divisors, which proves (i).

The proof of (ii) is similar, the locus where "α" and "β" coincide being the zero scheme of the composite

$$\mathscr{A}/C \longrightarrow U/C \longrightarrow U/\mathscr{B} .$$

This locus, which is $\theta_c+\epsilon$, therefore represents the class

$$c_1(U/\mathscr{B}) - c_1(\mathscr{A}/C) = (-\mu_1+\beta) - (-\alpha+c) = \alpha+\beta-c-\mu_1 .$$

Equation (iii) is just the universal equation for the tautological bundle A on the first copy of $\mathbb{P}(U)$; i.e., $U \otimes A^{\vee}$ has a nowhere vanishing section. Likewise (iv) is from the universal line bundle $U/\mathscr{A}$ on the first copy of $G_2(U) = \mathbb{P}(U^{\vee})$; i.e., $U^{\vee} \otimes (U/\mathscr{A})$ has a nowhere vanishing section. Finally $A \subset \mathscr{B}$ gives a nowhere vanishing section of $\mathscr{B} \otimes A^{\vee}$, which gives

$$c_2(\mathscr{B}) - \beta a + a^2 = 0 .$$

Since $0 = c_2(U/\mathscr{B}) = \mu_2 - \mu_1\beta + \beta^2 - c_2(\mathscr{B})$, (v) follows. □

From (i) and (ii) we may set

$$d = \alpha+\beta+\gamma+\tau = a+b+c+\epsilon+\mu_1 .$$

Hence

(vi) $\tau = d-\alpha-\beta-\gamma$, and $\epsilon = d-a-b-c-\mu_1$,

and from the lemma, $\theta_c = a+b-\gamma-(d-\alpha-\beta-\gamma)$, or

(vii) $\theta_c = a+b+\alpha+\beta-d$.

Although we do not need this, in fact one has $d = c_1(D^{\vee})$, where D is the universal subbundle on the Grassmann bundle $G_3(\mathrm{Sym}^2(U))$. To see this, note that the canonical map of rank three bundles

$$A^{\otimes 2} \oplus B^{\otimes 2} \oplus C^{\otimes 2} \longrightarrow D$$

(which determines the map to the Grassmann bundle over the locus of honest triangles) vanishes on the loci θ_a, θ_b, θ_c, and τ , the latter to order two. Hence

$$c_1(D) - (2c_1(A)+2c_1(B)+2c_1(C)) = \theta_a+\theta_b+\theta_c+2\tau ,$$

so $c_1(D^{\vee}) = \theta_a+\theta_b+\theta_c+2\tau-2a-2b-2c$, which, by (i) and (vi), is d .

There is a **duality** map from X to $X' =$ space of triangles in $\mathbb{P}(U^{\vee})$. A triangle Δ in X determines a triangle Δ^2 in X^2. In terms of bundles, given $A, B, C, \mathcal{A}, \mathcal{B}, \mathcal{C} \subset U$, the dual is determined by $A^6, B^6, C^6, \mathcal{A}^6, \mathcal{B}^6, \mathcal{C}^6 \subset U^{\vee}$, where

$$A^2 = \mathrm{Ker}(U^{\vee} \longrightarrow \mathcal{A}^{\vee}) \ , \ \mathcal{A}^2 = \mathrm{Ker}(U^{\vee} \longrightarrow \mathcal{A}^{\vee}) \ , \text{ etc.}$$

It follows that the duality map acts as follows on the classes :

$$a^5 = \alpha-\mu \ , \ \alpha^5 = a-\mu \ , \ \epsilon^5 = \tau \ , \ \tau^5 = \epsilon \ , \ \theta_a{}^5 = \theta_a \ ,$$
$$\mu^4 = -\mu \ , \ \mu_2^4 = \mu_2 \ , \ \mu_3^4 = -\mu_3 \ , \text{ and } \ d^4 = d-4\mu \ .$$

THEOREM. *We have*

$$A^{\cdot}(X) = A^{\cdot}(Y)[a,b,c,\alpha,\beta,\gamma,d]/I\,,$$

where I *is the ideal generated by the polynomials listed in* (1)–(4) *below ; for each polynomial listed, it is to be understood that the polynomials obtained by the action of the symmetric group* $\mathfrak{S}_3$ *on* a, b, c *and simultaneously* α, β, γ *are included*:

(1) $$a^3-\mu_1 a^2+\mu_2 a-\mu_3\,,$$
$$\alpha^3-2\mu_1\alpha^2+\mu_1^2\alpha+\mu_2\alpha-\mu_1\mu_2+\mu_3\,;$$

(2) $$a\beta-a^2-\beta^2+\mu_1\beta-\mu_2$$

(3) $$(b+c+\beta+\gamma-d)(b-c)\,,$$
$$(b+c+\beta+\gamma-d)(\beta-\gamma)\,;$$

(4) $$(d-a-b-c-\mu_1)(d-\alpha-\beta-\gamma)\,.$$

Proof: Let $B^{\cdot} = A^{\cdot}(Y)$, and define $A^{\cdot}$ to be the graded algebra over $B^{\cdot}$ with generators $a,b,c,\alpha,\beta,\gamma,d$, and relations specified in the theorem. We have a canonical homomorphism of graded $B^{\cdot}$–algebras from $A^{\cdot}$ to $A^{\cdot}(X)$. Indeed, the lemma shows that relations (1) and (2) map to zero ; by equation (vii), (3) follows from the fact that $a=b$ and $\alpha=\beta$ on the locus θ_c. Relation (4) follows from (vi), as before, and the fact that the loci ϵ and τ are disjoint.

Define elements τ, ϵ, θ_a, θ_b, θ_c in $A^{\cdot}$ by formulas (vi) and (vii). The formulas (iii)–(viii) of §2 can be used to construct classes $\theta_a s,\ldots,\psi,\ldots,\theta_c s^2 g$, $[\star]$ in $A^{\cdot}$ contains the 72 elements named in the Proposition of §2. We call these 72 elements the **basic classes**. While completing the proof of the theorem, we prove :

PROPOSITION. $A^{\cdot}(X)$ *is a free module over* $A^{\cdot}(Y)$ *on the* 72 *basic classes.*

Proof: If $\zeta_1,\ldots,\zeta_{72}$ are the basic classes in $A^{\cdot}(X)$, the proposition follows from the "Leray–Hirsh" theorem proved in Appendix C : the map

$$\bigoplus_1^{72} A.(Y) \longrightarrow A.(X)\,,\ \oplus\,\alpha_i \longmapsto \Sigma\,\zeta_i \cap f^*(\alpha_i),$$

is an isomorphism, where $f : X \longrightarrow Y$ is the projection. □

The theorem now follows. Indeed, from Appendix A it follows that the 72 basic classes generate the algebra $A^{\cdot}$ as a module over the ring $B^{\cdot}$. From the proposition we have a surjection $A^{\cdot} \longrightarrow A^{\cdot}(X)$ of $B^{\cdot}$–modules, and the second is free over $B^{\cdot}$ on the images of these 72 generators. Hence the map is an isomorphism. □

Remarks. (1) Note that if E has rank e over S, and $Y = G_3(E)$, then $A^{\cdot}(Y) = A^{\cdot}(S)[\mu_1,\mu_2,\mu_3]/J$ where J is the ideal generated by three universal homogeneous polynomials $P_i(\mu_1,\mu_2,\mu_3,c_1(E),\ldots,c_e(E))$ of degrees $i = e-2, e-1$, and e, which express the vanishing of $c_i(E/U)$ for these indices, as Grothendieck showed (cf. [6], Ex. 14.6.6). Therefore $A^{\cdot}(X)$ is a polynomial ring in ten variables $a,b,c,\alpha,\beta,\gamma,d,\mu_1,\mu_2,\mu_3$ over $A^{\cdot}(S)$, modulo the ideal generated by these three polynomials P_i together with the nineteen polynomials specified in the theorem.

If S is allowed to be a singular variety, similar arguments show that the Chow group $A.(X)$ is a direct sum of $12e(e-1)(e-2)$ copies of $A.(S)$.

The same results hold when $A^{\cdot}$ and $A.$ are replaced by cohomology $H^{\cdot}$ and (Borel–Moore) homology $H.$; this version of the theorem follows from the standard Leray–Hirsch theorem for fibre bundles.

(2) The full working out of intersection products in higher dimensions can be tedious, but Appendix A contains a complete recipe for computing all such products. For a simple application to triangles in three space, one can verify that *the number of triangles each of whose sides meets three given space curves is* 8 *times the product of the degrees of the nine given curves.* To see this ; note that the condition for the side α to meet a curve of degree n is the class $n\alpha$. We are reduced to showing that $\alpha^3\beta^3\gamma^3 = 8$. But $\alpha^3 = 2\mu\alpha^2 - 2\mu^2\alpha$, so

$$\alpha^3\beta^3\gamma^3 = 8\mu^3\alpha\beta\gamma(\alpha-\mu)(\beta-\mu)(\gamma-\mu) = 8\mu^3\alpha^2\beta^2\gamma^2 = 8 .$$

(3) The method of this paper can be used in other situations where the intersection rings are rather simple. For example, it can be applied to the space Ψ of "infinitely small triangles", which is the four–dimensional locus in the triangle space X whose class is denoted ψ above. The pull–backs of divisors on X give the basic divisors on Ψ : if i is the inclusion of Ψ in X, and we define

$$\begin{aligned} s &= i^*(a) = i^*(b) = i^*(c), \\ g &= i^*(\alpha) = i^*(\beta) = i^*(\gamma), \\ \eta &= i^*(\epsilon), \\ \zeta &= i^*(\tau), \end{aligned}$$

while the pull–back of d is still denoted d. Equation (3) of §1 pulls back to the equation

$$3s+\eta = 3g+\zeta = d\,.$$

PROPOSITION. (1) $A^{\cdot}(\Psi) = \mathbb{Z}[s,g,d]/K$, *where* K *is the ideal generated by* $sg\text{-}s^2\text{-}g^2$, $(d-3s)(d-3g)$, s^3, *and* g^3.

(2) *The ranks of its cycles are* 1, 3, 4, 3, 1 ; *a basis for* $A^{\cdot}(\Psi)$ *is*

$$1\ ;\ s\ ,\ g\ ;\ s^2\ ,\ g^2\ ,\ \eta g\ ,\ \zeta s\ ;\ \eta g^2\ ,\ \zeta s^2\ ,\ s^2 g\ ;\ \textit{and}\ \ ds^2 g\,.$$

The intersection tables are

Products of A^1 and A^3

	s	g	d
ηg^2	1		
ζs^2		1	
$s^2 g$			1

Products of A^2 and A^2

	s	g	ηg	ζs
s^2			1	
g^2				1
ηg	1			
ζs		1		

Proof : The relations in K pull back from basic equations on X, so one has a homomorphism from $\mathbb{Z}[s,g,d]/K$ to $A^{\cdot}(\Psi)$. Since Ψ has twelve fixed points by the torus action, and the stated classes clearly generate $\mathbb{Z}[s,g,d]/K$, and ds^2g maps to the class of a point, the map is an isomorphism. □

More generally, for the space Ψ of infinitely small triangles in a varying plane $\mathbb{P}(U)$, with U as at the beginning of this section,

$$A^{\cdot}(\Psi) = A^{\cdot}(Y)[s,g,d]/K\,,$$

where K is generated by $sg-s^2-g^2+\mu_1 g-\mu_2$, $(d-3s-\mu)(d-3g)$, $s^3-\mu_1 s^2+\mu_2 s-\mu_3$, and $g^3-2\mu_1 g^2+\mu_1^2 g+\mu_2 g-\mu_1\mu_2+\mu_3$.

(4) The results of this paper extend to arbitrary charateristic, avoiding only characteristics 2 and 3 for the discussion of inscribed and circumscribed triangles (§4). The theorem of Bialynicki–Birula is still valid for the Chow groups, as discussed in [14].

(5) Le Barz [8] has given another construction of the space of plane triangles which generalizes to give a space of triples of points in any smooth variety V. This space is the closure of the space of honest triangles in the product of three copies of V, three copies of the Hilbert scheme $\mathrm{Hilb}_2 V$ of length two subschemes of V, and one copy of $\mathrm{Hilb}_3 V$. Le Barz shows by calculations in local coordinates that this closure is smooth.

In fact the variety constructed by Le Barz represents a natural functor. From this fact the smoothness of the variety follows from a simpler calculation of its deformations. To use notation which agrees with Schubert, the data given by a family of triangles in V parametrized by a scheme S is a collection of subschemes $a,b,c,\alpha,\beta,\gamma,d$ of $V\times S$, finite and flat over S of degrees 1, 1, 1, 2, 2, 2, 3, with inclusions $a\subset\beta\subset d$, $a\subset\gamma\subset d$, and similarly for the permutations by $\mathfrak{S}_3$. The key condition of Le Barz is that the corresponding ideal sheaves satisfy :

$$\mathcal{I}(a).\mathcal{I}(\alpha)\subset I(d)$$

for each "vertex" a and its opposite "side" α.

This construction can be used for plane triangles in place of that of Semple, and can be generalized to smooth families. It would be interesting to describe the cohomology of this space in terms of the cohomology of V.

Appendix A. Algebra

In this appendix we use the relations (1)–(4) of the main theorem of §5 to deduce formulas that are valid in the ring $A^{\cdot}$, and use these to give recipes to write any element of $A^{\cdot}$ as a $B^{\cdot}$–linear combination of the 72 basic elements. When $B^{\cdot} = \mathbb{Z}$ this specializes to the assertions needed in §2. We put the terms involving the classes μ_i in braces $\{ \}$; these terms are to be ignored for the case of plane triangles. For formulae in low degree that are used frequently, we have written out the term in braces : the recipes we give determine them in all degrees, but the expressions become rather long to write out.

The equations labelled with a star $\star$ have all terms (outside braces) on the right appearing in the proposed list of 72 generators ; they are also, modulo the terms in braces, equations which appear in Schubert [12]. They give an effective algorithm for computing all products in these intersection rings ; note that the terms in braces involved only elements of $A^{\cdot}$ of lower degree.

Recall that the ring $A^{\cdot}$ is defined as a polynomial ring in variables a, b, c, α, β, γ, d over $B^{\cdot}$ modulo relations (1)–(4). We sometimes denote the element μ_1 of $B^{\cdot}$ by μ for brevity.

We use freely the symmetry under the group $\mathfrak{S}_3$, and usually write only one equation to represent the 1, 2, 3, or 6 equations resulting from the action of $\mathfrak{S}_3$. In fact $B^{\cdot}$ has an involution which takes

$$\mu_1 \longmapsto -\mu_1 \, , \; \mu_2 \longmapsto \mu_2 \, , \text{ and } \; \mu_3 \longmapsto -\mu_3 \, .$$

Then $G = \mathfrak{S}_3 \times \mathfrak{S}_2$ acts on $A^{\cdot}$, compatibly with this involution ; the dual of an equation is obtained by the substitutions :

$$a \longmapsto \alpha - \mu \, , \; \alpha \longmapsto a - \mu \, , \; d \longmapsto d - 4\mu \, .$$

It is easy to verify that the defining equations I are preserved by this duality operation, so G acts as automorphisms of $A^{\cdot}$. We include a few of the most useful dual equations, labelled with a prime $'$; since dual statements follow formally, proofs will be omitted.

In the ring $A^{\cdot}$ we have defined

$$\begin{aligned} \tau &= d-\alpha-\beta-\gamma \\ \epsilon &= d-a-b-c-\mu \\ \theta_c &= a+b+\alpha+\beta-d\,. \end{aligned}$$

These are preserved by the duality map : $\epsilon \longmapsto \tau$, $\tau \longmapsto \epsilon$, $\theta_a \longmapsto \theta_a$.

By equation (3) we have $\theta_c a = \theta_c b$, which we call $\theta_c s$, and similarly $\theta_c \alpha = \theta_c \beta$ is called $\theta_c g$. All the equations (i)–(vii) in and after the Lemma in §5 are valid in the ring $A^{\cdot}$.

We shall need one more equation in degree one :

(A.1)* $$\epsilon+\tau+\theta_b+\theta_c = a+\alpha+\{-\mu\}\,.$$

Proof : By (i) and (ii), $(\theta_c+\tau) + (\theta_b+\epsilon) = (\alpha+\gamma-b) + (a+b-\gamma-\mu) = \alpha+a-\mu$.

We turn to degree two :

(A.2)* $$bc = \alpha^2+\tau s+\theta_a s+\{-\mu\alpha+\mu_2\}\,.$$

(A.2′) $$\beta\gamma = a^2+\epsilon g+\theta_a g+\{\mu_2\}\,.$$

Proof : By (i) and (v), $\theta_a s+\tau s = (b+c-\alpha)b = b^2+bc-(\alpha^2+b^2-\mu\alpha+\mu_2) = bc-\alpha^2+\mu\alpha-\mu_2$.

(A.3) $$\epsilon\alpha = \epsilon\beta\,.$$

(A.3′) $$\tau a = \tau b\,.$$

Proof : By (ii), $\epsilon = \alpha+\beta-c-\mu-\theta_c$, so $\epsilon\alpha = \alpha^2+\alpha\beta-c\alpha-\mu\alpha-\theta_c\alpha = \alpha^2+\alpha\beta-(c^2+\alpha^2-\mu\alpha+\mu_2)-\mu\alpha-\theta_c\alpha = \alpha\beta-c^2-\mu_2-\theta_c\alpha$. Interchanging the roles of α and β , $\epsilon\beta = \beta\alpha-c^2-\mu_2-\theta_c\beta$. The equality of $\epsilon\alpha$ and $\epsilon\beta$ then follows from the equation $\theta_c\alpha = \theta_c\beta$.

By symmetry we therefore have $\epsilon\alpha = \epsilon\beta = \epsilon\gamma$ and $\tau a = \tau b = \tau c$; these elements are denoted ϵg and τs respectively.

(A.4)* $$\epsilon a = a^2+a\alpha-\tau s-\theta_b s-\theta_c s+\{-\mu a\}\,.$$

(A.4′) $$\tau\alpha = \alpha^2+a\alpha-\epsilon g-\theta_b g-\theta_c g+\{-\mu\alpha\}\,.$$

Proof : $\epsilon a+\tau s+\theta_b s+\theta_c s = (\epsilon+\tau+\theta_b+\theta_c)a = a^2+a\alpha-\mu a$ by (A6).

We now have more that enough equations to write any element of degree 2 in $A^\cdot$ as a linear combination of the 17 basic elements of A^2 of degree two, following the discussion of Section 2. For monomials S in a, b and c, use (A.2). For products ST of linear monomials use the defining equation (2). For products $S\epsilon$ use (A.4). As shown in Section 2, this finishes the proof for degree 2.

It will be useful to make a few more calculations in degree two :

(A.5) $$\theta_a\theta_b = \theta_a\theta_c\ .$$

Proof : $\theta_b-\theta_c = (a+c-\beta-\tau)-(a+b-\gamma-\tau) = c-b+\gamma-\beta$. The result follows by multiplying by θ_a and using the equations $\theta_a b = \theta_a c$ and $\theta_a\beta = \theta_a\gamma$.

The element $\theta_a\theta_b = \theta_a\theta_c = \theta_b\theta_c$ is denoted ψ. We list the equation for ψ which follows from the preceding rules, although this is not needed here ; the equation (4), that $\epsilon\tau = 0$, is needed :

(A.6)*
$$\psi = \theta_a g+\theta_b g+\theta_c g+\theta_a s+\theta_b s+\theta_c s-a\alpha-b\beta-c\gamma+2\tau s+2\epsilon g+\{-\mu\alpha-\mu\beta-\mu\gamma+\mu d+2\mu_2\}$$

(A.7)* $$\theta_a a = \beta^2+\gamma^2-a\alpha+\theta_b s+\theta_c s+\tau s+\{-\mu\beta-\mu\gamma+2\mu_2\}\ .$$
(A.7′) $$\theta_a\alpha = b^2+c^2-a\alpha+\theta_b g+\theta_c g+\epsilon g+\{-\mu\alpha+2\mu_2\}\ .$$

Proof : $\theta_a a = (\beta+\gamma-a-\epsilon-\mu)a = (a^2+\beta^2-\mu\beta+\mu_2)+(a^2+\gamma^2-\mu\gamma+\mu_2)-a^2-\epsilon a-\mu a =$ $\beta^2+\gamma^2+a^2-\epsilon a+\{-\mu\beta-\mu\gamma-\mu a+2\mu_2\}$ by (ii) and (2). Substituting for ϵa from (A.4) finishes the proof.

(A.8) $$2b^2+b\beta-\beta^2-bd+\{\mu\beta\} = 2c^2+c\gamma-\gamma^2-cd+\{\mu\gamma\}\ .$$

Proof : $\theta_a b = (b+c+\beta+\gamma-d)b = b^2+bc+b\beta+(b^2+\gamma^2-\mu\gamma+\mu_2)-bd$, so (A.8) is equivalent to the equation $\theta_a b = \theta_a c$.

Next we deduce some equations of degree three :

(A.9) $$a\beta^2 = a^2\beta+\{\mu\beta^2-\mu^2\beta+\mu\mu_2-\mu_3\}\ .$$

Proof : $a\beta^2 = \beta(a^2+\beta^2-\mu\beta+\mu_2) = a^2\beta+(2\mu\beta^2-\mu^2\beta-\mu_2\beta+\mu\mu_2-\mu_3)-\mu\beta^2+\mu_2\beta\ .$

(A.10) $\theta_a b^2 = \theta_a c^2 = \theta_a bc$, $\tau a^2 = \tau b^2 = \tau ab$.

(A.10′) $\theta_a \beta^2 = \theta_a \gamma^2 = \theta_a \beta\gamma$, $\epsilon\alpha^2 = \epsilon\beta^2 = \epsilon\alpha\beta$.

Proof : $\theta_a bb = \theta_a cb$ by (3), from which the first formula follows. The second is similar, using (A.3) in place of (3).

These elements are denoted $\theta_a s^2$, τs^2 , $\theta_a g^2$ and ϵg^2 respectively.

(A.11)* $b^2c = b^2\alpha+\tau s^2+\theta_a s^2+\{-\mu b^2+\mu_2 b-\mu_3\}$.

(A.11′) $\beta^2\gamma = a^2\beta+\epsilon g^2+\theta_a g^2+\{\mu_2\beta\}$.

Proof. Multiply (A.2) by b , getting $b^2c = b\alpha^2+\tau sb+\theta_a sb-\mu\alpha b+\mu_2 b$, and by (A.9) this is $(b^2\alpha+\{\mu\alpha^2-\mu^2\alpha+\mu\mu_2-\mu_3\})+\tau s^2+\theta_a s^2-\mu(b^2+\alpha^2-\mu\alpha+\mu_2)+\mu_2 b$.

(A.12)* $a^2\alpha = \tau s^2+\epsilon a^2+\theta_b s^2+\theta_c s^2+\{\mu_2 a-\mu_3\})$.

(A.12′) $a\alpha^2 = \epsilon g^2+\tau\alpha^2+\theta_b g^2+\theta_c g^2+\{-\mu\alpha^2+\mu^2\alpha+\mu_2\alpha-\mu\mu_2+\mu_3\}$.

Proof : Multiply (A.1) by a^2 .

(A.13)* $\theta_a sa = abc-\epsilon g^2-\tau s^2-\tau\alpha^2-\theta_b g^2-\theta_c g^2+\{\mu\alpha^2+\mu a\alpha-\mu^2\alpha-\mu_2 a-\mu_2\alpha+\mu\mu_2-\mu_3\}$.

Proof : Multiply the equation $\theta_a = b+c-\alpha-\tau$ by ab , getting
$\theta_a sa = ab^2+abc-ab\alpha-\tau ab = ab^2+abc-a(b^2+\alpha^2-\mu\alpha+\mu_2)-\tau s^2 = abc-a\alpha^2-\tau s^2+\{\mu a\alpha-\mu_2 a\}$.
Applying (A.12′) to replace $a\alpha^2$ in the right side of this equation gives (A.13).

(A.14)* $\theta_a a^2 = a^2\beta+a^2\gamma-\epsilon a^2+\{-2\mu a^2+\mu_2 a-\mu_3\}$.

Proof. Multiply $\theta_a = \beta+\gamma-a-\epsilon-\mu$ by a^2, and use (iii) of the lemma.

(A.15)*
$$\theta_a a\alpha = b^2\gamma+c^2\beta+\tau s^2+\epsilon g^2+\theta_b s^2+\theta_c s^2+\theta_b g^2+\theta_c g^2 + \{-\mu(b^2+c^2+a\alpha+\epsilon g+\theta_b g+\theta_c g) + \mu_2(2a+2\alpha+b+c+\beta+\gamma-d)-\mu\mu_2-\mu_3\} .$$

Proof : Multiply (A.7) by α , yielding
$\theta_a a\alpha = \alpha\beta^2+\alpha\gamma^2-a\alpha^2+\theta_b s\alpha+\theta_c s\alpha+\tau s\alpha+\{-\mu\alpha\beta-\mu\alpha\gamma+2\mu_2\alpha\}$. The first two terms on the right of this equation are known by (A.11′), the third by (A.12′). For the fourth and fifth terms we have by (v), $\theta_b s\alpha = \theta_b c\alpha = \theta_b c^2+\theta_b\alpha^2-\mu\theta_b\alpha+\mu_2\theta_b = \theta_b s^2+\theta_b g^2-\mu\theta_b g+\mu_2\theta_b$, and similarly for the sixth, $\tau s\alpha = \tau c\alpha = \tau s^2+\tau\alpha^2-\mu\tau\alpha+\mu_2\tau$. (A15) follows by substituting these six expressions.

(A.16) $$ab\gamma = a^2\gamma+b^2\gamma+\tau s^2+\theta_c s^2+\{-\mu a^2-\mu b^2+\mu_2 a+\mu_2 b-2\mu_3\} .$$

Proof : From (v) we have $ab\gamma = a(b^2+\gamma^2-\mu\gamma+\mu_2)$. Using (A.11), (A.9), and (v) this becomes $(b^2\gamma+\tau s^2+\theta_c s^2-\mu b^2+\mu_2 b-\mu_3)+(a^2\gamma+\mu\gamma^2-\mu^2\gamma+\mu\mu_2-\mu_3) - \mu(a^2+\gamma^2-\mu\gamma+\mu_2) + \mu_2 a$, which simplifies as required.

These formulae suffice to write any element of A^3 in terms of the 22 basic elements of degree three. Indeed, from (A.11) and (1) we obtain any monomial S . From (A.16) and (A.12) come all products ST . To obtain the products $S\theta_a$, one has $\theta_a b^2 = \theta_a bc = \theta_a s^2$, $\theta_a a^2$ by (A.14), $\theta_a ab = \theta_a sa$ by (A.13). Finally, to obtain $ST\theta_a$, one has $\theta_a a\alpha$ by (A.15), $\theta_a b\beta = \theta_a c\beta = \theta_a\beta^2-\mu\theta_a\beta+\mu_2\theta_a$ from (v), and the remaining follow similarly using (v).

We also have

(A.17) $$\psi a = \psi b .$$

(A.17′) $$\psi\alpha = \psi\beta .$$

Proof : $\psi a = \theta_b\theta_c a = \theta_b\theta_c b = \psi b$ by (3).

These elements are denoted ψs and ψg .

For later use we record another equation of degree 3, which follows from the preceding prescription. The notation $\{...\}$ indicates an expression involving μ_i's and lower degree terms in the basic classes. In the proofs we write $\equiv$ to denote that two expressions differ by a class of the form $\{...\}$.

(A.18)* $$\psi s = abc-\tau\alpha^2-\tau\beta^2-\tau\gamma^2-2\tau s^2-\epsilon g^2-\theta_a g^2-\theta_b g^2-\theta_c g^2+\{...\} .$$

Now for equations of degree four :

(A.19) $$\theta_a s^2\beta = \theta_a s^2\gamma ,\ \psi a^2 = \psi ab = \psi b^2 \text{, and } \psi\alpha^2 = \psi\alpha\beta = \psi\beta^2 .$$

Proof : The first equation follows immediately from the equation $\theta_a\beta = \theta_a\gamma$. The others follow from the first, and the definition of ψ .

These elements are denoted $\theta_a s^2 g$, ψs^2 and ψg^2 . Note that from (v) we have $\theta_a g^2 b = \theta_a g^2 c = \theta_a s^2 g + \{...\}$. Similarly $\epsilon g a^2 = \epsilon g^2 a + \{...\}$, and $\tau s\alpha^2 = \tau s^2\alpha + \{...\}$.

(A.20)* $$a^2\alpha^2 = \epsilon g^2a+\tau s^2\alpha+\theta_b s^2g+\theta_c s^2g+\{...\} .$$

Proof: Multiply (A.1) by $a^2\alpha$, getting $a^2\alpha^2 \equiv (a+\alpha)a^2\alpha \equiv \epsilon a^2\alpha+\tau s^2\alpha+\theta_b s^2g+\theta_c s^2g$, which suffices, since $\epsilon a^2\alpha \equiv \epsilon g^2 a$.

(A.21)* $$\theta_a ga^2 = \beta^2\gamma^2-\epsilon g^2a+\{...\} .$$

Proof: Multiply (A.14) by β, getting $\theta_a ga^2 \equiv a^2\beta^2+a^2\beta\gamma-\epsilon ga^2$. But $a^2\beta^2 \equiv 0$ and $a^2\beta\gamma \equiv \beta^2\gamma^2$, as in equations (6) and (7) of §2, and the result follows.

(A.22)* $$a^2bc = \beta^2\gamma^2+\theta_b s^2g+\theta_c s^2g+\tau s^2\alpha+\tau s^2\beta+\tau s^2\gamma+\psi s^2+\{...\} .$$

Proof: By (A.18),
$\psi s^2 \equiv \psi sa = a^2bc-\tau s\alpha^2-\tau s\beta^2-\tau s\gamma^2-0-\epsilon g^2a-\theta_a g^2a-\theta_b g^2a-\theta_c g^2a \equiv$
$a^2bc-\tau s^2\alpha-\tau s^2\beta-\tau s^2\gamma-\epsilon g^2a-\theta_a ga^2-\theta_b s^2g-\theta_c s^2g$. Subtituting $\beta^2\gamma^2$ for $\theta_a ga^2+\epsilon g^2a$ by (A.21), one obtains (A.22).

(A.23)* $$\theta_a s^2\alpha = \theta_a s^2g+\tau s^2\beta+\tau s^2\gamma+\psi s^2+\{...\} .$$

Proof: Multiplying (A.13) by b, one has
$\theta_a s^2\alpha = (\theta_a sa)b \equiv ab^2c-\epsilon g^2b-0-\tau s\alpha^2-\theta_b g^2b-\theta_c g^2b \equiv ab^2c-\epsilon g^2b-\tau s^2\alpha-\theta_b gb^2-\theta_c s^2g$, and one concludes by substituting from (A.22) and (A.21).

The next three sets of equations are essentially equations (6), (7) and (8) of §2; the equalities there become congruences here:

(A.24) $$a^2\beta^2 = \{...\} .$$

(A.25) $$ab\gamma^2 = a^2b^2+\{...\} ,\ c^2\alpha\beta = \alpha^2\beta^2+\{...\} .$$

(A.26) $$ab\alpha\beta = a^2b^2+\alpha^2\beta^2+a^2\alpha^2+b^2\beta^2 .$$

To verify that the 17 basic elements generate in degree 4, one has all monomials S by (A.22) and (1). For monomials ST we have $a^2\alpha^2$ by (A.20), $a^2b\alpha \equiv a^2b^2+a^2\alpha^2$ by (v), $abc\alpha \equiv abc^2+ab\alpha^2$, which one has by duals of preceding equations; using (A.12) one has

$$a^2b\beta \equiv ab(a^2+\beta^2) \equiv a(\epsilon g^2+\tau\beta^2+\theta_a g^2+\theta_c g^2) \equiv \epsilon g^2a+\tau s^2\beta+\theta_a ga^2+\theta_a s^2g ,$$

and therefore also $a^2\alpha\beta \equiv a\alpha(\alpha^2+\beta^2) \equiv a\alpha\beta^2$; likewise, using (A.24), $a^2b\gamma \equiv a^2b^2+a^2\gamma^2 \equiv a^2b^2$, and similarly $a^2\beta\gamma \equiv \beta^2\gamma^2$, $ab\alpha\beta \equiv a^2\alpha^2+b^2\beta^2$, and $ab\alpha\gamma \equiv a^2b^2+a^2\gamma^2$. It remains to show how to obtain the product of each ST times one of the elements θ_a , θ_b or θ_c . For a^2b or abc use (A.23) ; for $a^2\beta$ use (A.21) ; one then has $ab\gamma \equiv ab^2+a\gamma^2$; for $a^2\alpha$ one has $\theta_b a^2\alpha = \theta_b s^2 g$, which completes the proof.

Among the equations in degree 5 we need

(A.27) $$\psi s^2\beta = \psi s^2\gamma .$$

Proof : This follows from (A.17).

We denote this class by $\psi s^2 g$. Note that $\psi g^2 a = \psi g^2 b = \psi s^2 g+\{...\}$.

(A.28)* $$\theta_b s^2 b^2 = \tau\alpha^2\gamma^2+\psi s^2 g+\{...\} .$$

Proof : By (A.23), $(\theta_b s^2 b)\alpha \equiv \theta_b s^2 g\alpha+\tau s^2\alpha^2+\tau s^2\alpha\gamma+\psi s^2\alpha \equiv \tau\alpha^2\gamma^2+\psi s^2 g$ by (A.24) and (A.25). And $(\theta_b s^2 b)\alpha \equiv \theta_b cb\alpha^2 \equiv \theta_b c^2 b^2 = \theta_b s^2 b^2$.

(A.29)* $$b^2c^2\beta = \epsilon b^2c^2+\tau\alpha^2\beta^2+\psi s^2 g+\{...\} .$$

Proof : By (A.9) and (A.20), $b^2c^2\beta \equiv (b^2\beta^2)c \equiv \epsilon g^2 bc+\theta_c s^2 cg$. Now $\epsilon g^2 bc \equiv \epsilon b^2c^2$ by (A.25), and, by (A.23), $\theta_c s^2 cg \equiv (\theta_c s^2 g+\tau s^2\alpha+\tau s^2\beta+\psi s^2)\alpha \equiv \tau\alpha^2\beta^2+\psi s^2 g$, using (A.24) and (A.25) again.

(A.30)* $$ab^2c^2 = \epsilon b^2c^2+\tau\alpha^2\gamma^2+\tau\alpha^2\beta^2+2\psi s^2 g+\{...\} .$$

Proof : By (vii), $ab^2c^2 \equiv b^2c^2(\epsilon+\tau+\theta_b+\theta_c-\alpha) \equiv \epsilon b^2c^2+\theta_b s^2 b^2+\theta_c s^2 c^2$, and one concludes by (A.28).

To verify that A^5 is generated by the required 7 classes, consider first products ST . Note that we have a^2b^2c and $a^2b^2\alpha$ by (A.30) and (A.29), and $a^2b^2\gamma \equiv a^2b\gamma^2 \equiv 0$, $a^2bc\alpha \equiv a^2b^2c+a^2c\alpha^2 \equiv a^2b^2c+a^2c^2\alpha$, and similarly $a^2bc\beta \equiv a^2bc^2+a^2b^2\beta$. This gives any product where $\deg(S) = 5$ or 4. When S is a cubic, the use of equation (2) and the resulting $a\beta^2 \equiv a^2\beta$ and $a^2\beta^2 \equiv 0$ reduce to the previous cases : $a^2b\gamma^2 \equiv a^2b^2\gamma$, $a^2b\beta^2 \equiv 0$, $a^2b\alpha^2 \equiv a^2b^2\alpha$, $a^2b\beta\gamma \equiv a^2b^2\beta+a^2\beta\gamma^2 \equiv a^2b^2\beta$, $a^2b\alpha\gamma \equiv a^2b^2\alpha+a^2\alpha\gamma^2 \equiv a^2b^2\alpha$, $a^2b\alpha\beta \equiv a^2b^2\beta+a^2\alpha^2\beta \equiv a^2b^2\beta+a\alpha^2\beta^2$, $abc\alpha^2 \equiv abc^2\alpha$, and $abc\alpha\beta \equiv abc^2\beta+ab\alpha^2\beta$. Finally consider $ST\tau$ for S and T as above. This is zero if $\deg(S) \geq 3$, and if $\deg(S) = 2$, it is $\tau s^2 T$; for $T = \alpha\beta$, one gets $\tau c^2\alpha\beta \equiv \tau\alpha^2\beta^2$ by

(A.25) ; for $T = \alpha\beta$, one gets $\tau c^2\alpha\beta \equiv \tau\alpha^2\beta^2$ by (A.25) ; for $T = \alpha^2$ one similarly gets zero ; this completes the proof in degree 5.

To see that $[\star] = a^2b^2c^2$ generates A^6, we have $a^2b^2c\alpha \equiv a^2b^2c^2 + a^2b^2\alpha^2 \equiv [\star]$, and similarly $a^2b^2c\gamma \equiv 0$. For ST with $\deg(S) = 4$, for $S = a^2b^2$, the products are zero if $T = \alpha^2$, β^2 or γ^2. Equation (A.26) yields $a^2b^2\alpha\beta \equiv [\star]$, and $a^2b^2\alpha\gamma \equiv a^2b\alpha\gamma^2 \equiv 0$. For $S = a^2bc$, the product : with α^2 is $a^2b^2c\alpha \equiv [\star]$; with β^2 is $\equiv 0$; with $\alpha\beta$ is $a^2b^2c\beta + a^2c\alpha^2\beta \equiv [\star]$; with $\beta\gamma$ is $\equiv [\star]$ by (A.24), (A.25) and (A.26). For products ST of cubic monomials, consider first $S = a^2b$. The product with T is $\equiv 0$ if T contains β^2 or γ^2 , the product with $\alpha^2\gamma$ is $a^2b^2\alpha^2 + a^2\alpha^2\gamma^2 \equiv 0$, with $\alpha^2\beta$ is $a^2b^2\alpha\beta \equiv [\star]$, and with $\alpha\beta\gamma$ is $a^2b^2\alpha\beta + a^2\alpha\beta\gamma^2 \equiv [\star]$. Finally, the product of abc with $\alpha\beta\gamma$ is $(a^2+\beta^2)(b^2+\gamma^2)(c^2+\alpha^2) \equiv a^2b^2c^2 + \alpha^2\beta^2\gamma^2 \equiv 2[\star]$. Lastly, we must consider products of ST with either ϵ or τ . If $\deg(S) \geq 3$, the product with τ is $\equiv 0$, while if $\deg(T) \geq 3$, the product with ϵ is $\equiv 0$.

Remark. This includes proofs of many of the formulas in Schubert [12], pp. 153–164. Most of those not listed above are obtained by symmetry, i.e. the action of G . The two remaining equations involving the above classes.

(A.31)* $$\theta_a s a^2 = \beta^2\gamma^2 - \epsilon g^2 a + \tau s^2\beta + \tau s^2\gamma + \psi s^2 + \{\ldots\} ,$$

(A.32)* $$\tau s\alpha\gamma = \gamma^2\alpha^2 - \epsilon g^2 b + \tau s^2\alpha + \theta_b s^2\gamma - \psi g^2 + \{\ldots\} ,$$

are obtained from the others by using the above prescription. In addition, these pages of Schubert contain formulae for the classes ω_a, ω_b, ω_c, ω_α,ω_β, ω_γ, η and ζ . These follow similarly, starting from

$$\omega_a = \epsilon\theta_a , \omega_\alpha = \tau\theta_\alpha , \eta = \epsilon\psi , \zeta = \tau\psi .$$

For example, one deduces easily the formula

(A.33)* $$3\psi s + \eta = 3\psi g + \zeta + \{-\mu\psi\} ,$$

used for infinitely small triangles. In fact $3\psi s + \eta = \psi(a+b+c+\epsilon) = \psi(d-\mu) = \psi(\alpha+\beta+\gamma+\tau) - \mu\psi = 3\psi g + \zeta - \mu\psi$.

Appendix B. Intersection tables of basic classes

By the preceding appendix, all products in the ring $A^{\cdot}$ of §2 can be derived. In particular, given two elements of complementary dimension we may compute an integer such that the product of these elements is that integer times $[\star]$. (The uniqueness of this coefficient then follows from the theorem in §2).

Each entry in a table denotes the coefficient of $[\star]$ in the product of the entry labelling the row and column.

Table for products of A^1 and A^5

	a	b	c	α	β	γ	d
εb^2c^2	1						
εc^2a^2		1					
εa^2b^2			1				
$\tau\beta^2\gamma^2$				1			
$\tau\gamma^2\alpha^2$					1		
$\tau\alpha^2\beta^2$						1	
ψs^2g							1

Table for products of A^2 and A^4

	a^2	b^2	c^2	α^2	β^2	γ^2	$a\alpha$	$b\beta$	$c\gamma$	τs	εg	$\theta_a s$	$\theta_b s$	$\theta_c s$	$\theta_a g$	$\theta_b g$	$\theta_c g$
b^2c^2	1																
c^2a^2		1															
a^2b^2			1														
$\beta^2\gamma^2$				1													
$\gamma^2\alpha^2$					1												
$\alpha^2\beta^2$						1											
$\tau s^2\alpha$															1		
$\tau s^2\beta$																1	
$\tau s^2\gamma$																	1
εg^2a												1					
εg^2b													1				
εg^2c														1			
$\theta_a s^2g$							1										
$\theta_b s^2g$								1									
$\theta_c s^2g$									1								
ψs^2											1				-1	-1	-1
ψg^2										1		-1	-1	-1			

Table for products of A^3 and A^3

	$a^2\beta$	$a^2\gamma$	$b^2\gamma$	$b^2\alpha$	$c^2\alpha$	$c^2\beta$	abc	$\alpha\beta\gamma$	εa^2	εb^2	εc^2	$\tau\alpha^2$	$\tau\beta^2$	$\tau\gamma^2$	εg^2	τs^2	$\theta_a s^2$	$\theta_b s^2$	$\theta_c s^2$	$\theta_a g^2$	$\theta_b g^2$	$\theta_c g^2$
$a^2\beta$				1																		
$c^2\gamma$					1																	
$b^2\gamma$						1																
$b^2\alpha$	1																					
$c^2\alpha$		1																				
$c^2\beta$			1																			
abc							1	2							1							
$\alpha\beta\gamma$							2	1								1						
εa^2										-1	-1						1					
εb^2									-1		-1							1				
εc^2									-1	-1									1			
$\tau\alpha^2$													-1	-1						1		
$\tau\beta^2$												-1		-1							1	
$\tau\gamma^2$												-1	-1									1
εg^2							1															
τs^2								1														
$\theta_a s^2$									1													
$\theta_b s^2$										1												
$\theta_c s^2$											1											
$\theta_a g^2$												1										
$\theta_b g^2$													1									
$\theta_c g^2$														1								

Appendix C. A Leray–Hirsh theorem for Chow groups

We consider a smooth proper morphism $f : X \longrightarrow Y$ which is locally trivial in the Zariski topology, with fibre F. As in topology, we will assume that there are elements in the Chow ring of the total space X which restrict to a basis for the Chow ring of the fibre, and we want to conclude that these elements give a basis for $A^{\cdot}X$ over $A^{\cdot}Y$, at least when Y is non–singular. Unlike the situation in topology, however, there is no Künneth theorem for Chow groups, so we need to make rather strong assumptions on the fibre. We will assume that F has a filtration by closed subschemes

$$F = F_0 \supset F_1 \supset \ldots \supset F_r = 0$$

such that each $F_i \setminus F_{i+1}$ is a disjoint union of affine spaces. This assumption guarantees that $A^{\cdot}F$ is a free abelian group generated by the closures of these affine spaces (cf. [6] and [11]). We will also assume that *satisfies Poincaré duality*, i.e., the degree map from A_0F to $\mathbb{Z}$ is an isomorphism, and, if $d = \dim(F)$, the intersection pairings

$$A^iF \otimes A^{d-i}F \longrightarrow A^dF \cong A_0F \cong \mathbb{Z}$$

are perfect pairings for all i. Any smooth projective variety with a action by the multiplicative group $\mathbb{G}_m$ with a finite number of fixed points satisfies these conditions ([1], [2], [14]). If the ground field k has characteristic zero, Poincaré duality is automatic from the existence of a filtration, since, when $k = \mathbb{C}$, the map from $A^{\cdot}F$ to cohomology $H^{\cdot}F$ is an isomorphism. In positive characteristic one may use ℓ–adic homology and cohomology to prove analogous statements, although one may need to take coefficients in a field. At any rate, these assumptions are verified for many varieties which occur in enumerative geometry.

It follows easily from the definitions that if F is a variety over k which satisfies the above conditions, then $F_K = F \otimes_k K$ is a variety over K which also satisfies the conditions.

Let $f : X \longrightarrow Y$ be a proper smooth morphism of relative dimension d, locally trivial in the Zariski topology, with fibre F satisfying the above conditions. We assume that for all fibres X_y, the restriction map from $A^{\cdot}(X)$ to $A^{\cdot}(X_y)$ is surjective ; this will be true for all fibres if it holds for one point y in each component of Y, for example the generic point of each irreducible component.

PROPOSITION. *Let $\zeta_1 \ldots \zeta_m$ be homogeneous elements of $A^{\cdot}X$ whose restrictions to fibres form a basis over $\mathbb{Z}$. Then every element in $A.X$ has a unique expression of the form*

$$\sum_{i=1}^{m} \zeta_i \cap f^*\alpha_i \ , \quad \alpha_i \in A.Y .$$

Equivalently, the homomorphism

$$\varphi : \bigoplus_{i=1}^{m} A.Y \longrightarrow A.X \ , \quad \varphi(\oplus\, \alpha_i) = \Sigma\, \zeta_i \cap f^*\alpha_i \ ,$$

is an isomorphism.

When Y is non-singular, this says that the ζ_i form a free basis for $A^{\cdot}X$ as a module over $A^{\cdot}Y$.

Proof : The proof of surjectivity of φ is the standard argument by Noetherian induction on the dimension of Y. One can assume Y is a variety, with function field K; regard the generic fibre $X_K \cong F \otimes_k K$ as a variety over K. Since $A.(X_K)$ is generated by the images of the ζ_i, one need only consider classes in $A.X$ whose restriction to the generic fibre are zero. Such classes will restrict to zero in $A.(f^{-1}U)$ for some open U in Y, hence will be in the image of $A.(f^{-1}Z)$, where Z is the complement of U in Y. By induction one knows the result for $f^{-1}Z \longrightarrow Z$, and the proof concludes as usual (cf. [6], §1.9).

For the injectivity of φ, let $[\star] \in A^dF$ be the generator corresponding to $1 \in \mathbb{Z}$ by the degree isomorphism. We may assume Y is connected. We first verify that if η is any element of A^dX, and the restriction of η to a fibre is $n[\star]$, for some integer n, then $f_*(\eta \cap f^*\alpha) = n\alpha$ for all $\alpha \in A.Y$. This too is standard. To prove it one may assume $\alpha = [V]$, with V a variety, then replace Y by V, in which case $f_*(\eta \cap f^*\alpha)$ must be $n'[Y]$ for some integer n'; one sees that n' equals n by restricting to a fibre. Similarly one sees that $f_*(\eta \cap f^*\alpha) = 0$ if $\eta \in A^pX$ with $p < d$.

We relabel the elements ζ_i with double subscripts, so that ζ_{pj} are the elements which are in A^pX. Since the restriction from $A^{\cdot}X$ to fibres is assumed to be surjective, we may choose elements ω_{pj} in $A^{d-p}X$ whose restrictions to fibres give the dual basis of the restrictions of ζ_{pj}, i.e., the restriction of $\omega_{pj}\zeta_{pk}$ to a fibre is $\delta_{jk}[\star]$. Now if $\Sigma_{ij}\zeta_{ij} \cap f^*\alpha_{ij} = 0$, consider the maximum p for which some $\alpha_{pk} \neq 0$. By the previous assertions

$$0 = f_*(\omega_{pk}(\Sigma\, \zeta_{ij} \cap f^*\alpha_{ij})) = f_*(\omega_{pk}\zeta_{pk} \cap f^*\alpha_{pk}) + 0 = \alpha_{pk} \ ,$$

which concludes the proof. □

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