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THE HOMOLOGICAL THEORY OF MAXIMAL COHEN-MACAULAY APPROXIMATIONS

by

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Summary: Let R be a commutative noetherian Cohen-Macaulay ring which admits a dualizing module. We show that for any finitely generated R-module N there exists a maximal Cohen-Macaulay R-module M which surjects onto N and such that any other surjection from a maximal Cohen-Macaulay module onto N factors over it. Dually, there is a finitely generated R-module I of finite injective dimension into which N embeds, universal for such embeddings. We prove and investigate these results in the broader context of abelian categories with a suitable subcategory of "maximal Cohen-Macaulay objects" extracting for this purpose those ingredients of Grothendieck-Serre duality theory which are needed.

Résumé: Soit R un anneau commutatif, noethérien et de Cohen-Macaulay, tel que un module dualisant existe pour R. On démontre que pour chaque R-module N de type fini il existe un R-module M de profondeur maximale et un homomorphisme surjectif de M sur N, tel que toute autre surjection d'un tel module sur N s'en factorise. De manière duale, il existe aussi un plongement de N dans un R-module I de type fini et de dimension injective finie, universelle pour telles plongements. Nous démontrons et examinons ces résultats dans le cadre des catégories abéliennes avec une sous-catégorie convenable des "objets de Cohen-Macaulay maximaux", à cet effet mettant en évidence les propriétés de la théorie de dualité de Grothendieck-Serre dont on a besoin.

§0. *A Commutative Introduction*

The aim of this work is to analyze the framework in which the theory of *maximal Cohen-Macaulay approximations* can be developed. Instead of outlining right away the abstract results, we want to start by describing the situation in the classical case of a commutative local noetherian ring R with maximal ideal $\mathbf{m}$ and residue class field $k = R/\mathbf{m}$.

Assume that R admits a *dualizing module* ω. Then R is Cohen-Macaulay, and the finitely generated R-modules M which are maximal Cohen-Macaulay in the sense that $\mathrm{depth}_{\mathbf{m}} M = \dim R$ can be characterized homologically as those modules for which $\mathrm{Ext}^i_R(M, \omega) = 0$ for $i \neq 0$.

Our main result can then be paraphrased as saying that **R-mod**, the category of finitely generated R-modules, is obtained by glueing together the orthogonal subcategories of *modules of finite injective dimension* over R and the category of *maximal Cohen-Macaulay modules* along their common intersection which is spanned by ω.

More precisely, let us recall that ω is *dualizing* for the local ring $(R,\mathfrak{m},k)$ if and only if it satisfies the following three conditions:

(i) ω is finitely generated and of finite injective dimension over R.

(ii) The natural ring homomorphism which is given by multiplication with scalars from R on ω, $R \to \mathrm{Hom}_R(\omega,\omega)$ is an isomorphism.

(iii) For any integer $i \neq 0$, one has $\mathrm{Ext}^i_R(\omega,\omega) = 0$.

Now our main results in this context are

<u>Theorem A:</u> *(Existence of the decomposition).* Let $(R,\mathfrak{m},k)$ be a commutative, local noetherian ring with dualizing module ω. For any finitely generated R-module N there exist finitely generated R-modules M_N and I^N together with an R-linear map

$$d_N : M_N \to I^N$$

such that

(a) The image of d_N is isomorphic to N.

(b) M_N is maximal Cohen-Macaulay and $I_N = \mathrm{Ker}\, d_N$ is an R-module of finite injective dimension.

(c) I^N is of finite injective dimension and $M^N = \mathrm{Cok}\, d_N$ is maximal Cohen-Macaulay.

(d) There exists an integer $n \geq 0$ such that d_N can be factored into an injection $j: M_N \to \omega^{\oplus n}$ and a surjection $p: \omega^{\oplus n} \to I^N$. ∎

If $d_N = \iota^N \circ \pi_N$ denotes the factorization of d_N over its image N, we can arrange the data given in the theorem into the following exact commutative diagram of R-modules:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & N \arrow[u] \arrow[r, "\iota^N"] & I^N \arrow[u] \arrow[r] & M^N \arrow[r] & 0 \\
0 \arrow[r] & M_N \arrow[u, "\pi_N"] \arrow[ur, "d_N"] \arrow[r, "j"'] & \omega^{\oplus n} \arrow[u, "p"'] \arrow[r] & M^N \arrow[u, equal] \arrow[r] & 0 \\
 & I_N \arrow[u] \arrow[r, equal] & I_N \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Theorem B: (Essential Uniqueness)

(a) Assume given a second homomorphism $d'_N: M'_N \to {}'I^N$ satisfying Theorem A for the same module N. If the image factorization of d'_N is given as

$$M'_N \xrightarrow{\pi'_N} N \xrightarrow{\iota'_N} {}'I^N,$$

there exist modules P, P' and Q, 'Q which are each finite direct sums of copies of ω, and R-module isomorphisms μ, κ so that the following diagram commutes:

$$\begin{array}{ccccc} M'_N \oplus P & \xrightarrow{\pi'_N \oplus 0} & N & \xrightarrow{'\iota^N \oplus 0} & {}'I^N \oplus Q \\ \mu\downarrow & & \| & & \downarrow\kappa \\ M_N \oplus P' & \xrightarrow{\pi_N \oplus 0} & N & \xrightarrow{\iota^N \oplus 0} & I^N \oplus {}'Q \end{array}$$

(b) If $f: M \to N$ is any homomorphism from a maximal Cohen-Macaulay R-module M into N, it factors over π_N. If $g: N \to J$ is any homomorphism from N into an R-module J of finite injective dimension, it factors over ι^N. ∎

These results suggest to call $0 \to I_N \to M_N \xrightarrow{\pi_N} N \to 0$ a *maximal Cohen-Macaulay approximation of* N and $0 \to N \xrightarrow{\iota^N} I^N \to M^N \to 0$ a *hull of finite injective dimension* for N.

To give a simple illustration, consider the case where N itself is a Cohen-Macaulay R-module, hence satisfying $\operatorname{depth}_{\mathfrak{m}} N = \dim N$.
Set $n = \operatorname{codepth}_{\mathfrak{m}} N = \dim R - \dim N$. Then local duality theory implies:

(i) $\operatorname{Ext}^i_R(N, \omega) = 0$ for $i \neq n$.

(ii) $N^\vee = \operatorname{Ext}^n_R(N, \omega)$ is again Cohen-Macaulay of codepth n.

(iii) $N = \operatorname{Ext}^n_R(N^\vee, \omega) = N^{\vee\vee}$.

Using this information, let

$$0 \to \Omega_n(N) \to R^{\oplus b_{n-1}} \xrightarrow{d_{n-2}} \dots \xrightarrow{d_0} R^{\oplus b_0} \longrightarrow N^\vee \to 0$$

be an exact sequence obtained by truncating a free resolution of $N^\vee$. It follows that $\Omega_n(N)$ is maximal Cohen-Macaulay and that dualizing with respect to ω results in an exact sequence

$$0 \longrightarrow \omega^{\oplus b_0} \longrightarrow \dots \xrightarrow{d^\vee_{n-2}} \omega^{\oplus b_{n-1}} \longrightarrow \operatorname{Hom}_R(\Omega_n(N), \omega) \xrightarrow{\pi} N^{\vee\vee} = N \longrightarrow 0.$$

Then $M_n = \operatorname{Hom}_R(\Omega_n(N), \omega) \xrightarrow{\pi} N$ is a desired maximal Cohen-Macaulay approximation of N, and $I_N = \operatorname{Cok} d^\vee_{n-2}$ admits a finite resolution "by ω", which shows that I_N is of finite injective dimension. The hull of finite injective dimension I^N is then simply the cokernel of the ω-dual of the next differential in the resolution of $N^\vee$, namely
$I^N = \operatorname{Cok} \operatorname{Hom}_R(d_{n-1}, \omega)$.

If R is a domain, for example, we get even more precise information:

(i) The rank of M_N equals the alternating sum

$$\sum_{i=1}^{n}(-1)^{i+1}b_{n-i} + (-1)^n \text{ rk } N,$$

(ii) $M_N^{\vee} = \mathrm{Hom}_R(M_N, \omega) = \Omega_n(N)$ embeds into $R^{\oplus b_{n-1}}$,

(iii) M_N contains no copy of ω as a direct summand if and only if $\Omega_n(N)$, the n-th syzygy module of N, contains no free summand.

It follows that one can attach new numerical invariants to an R-module N in this way. The minimum number of copies of ω necessarily contained in M_N or I^N, the rank of the ω-free summand of either M_N or I^N, their minimum number of generators and so forth.

Here, we are not concerned with these more detailed consequences of the theory but rather with its general framework.

The first author first proved an essentially equivalent version of Theorem A but for the category of additive functors on **R-mod**, see [Ausl], where the result was phrased by saying that the category of maximal Cohen-Macaulay modules is "coherently (co-)finite". The essential step then was to establish the representability of the functors involved.

This background illuminates our approach here. Although the primary applications of the theory might be within the classical theory of rings and algebras, to a large extent it can be developed in any abelian category **C** which admits a suitable subcategory **X** of "maximal Cohen-Macaulay objects".

Here we establish sufficient conditions on **X** to guarantee the categorical analogues of Theorems A and B. Section 1 deals with the decomposition theorem and section 2 addresses the uniqueness question. Sections 3 and 4 investigate the circumstances under which - in the terminology of the above example - the category of modules with "finite ω-resolution" are *all* the modules of finite injective dimension. Section 5 assembles a few remarks on finiteness conditions and section 6 contains more examples, among other purposes highlighting the differences in the theory when applied to either commutative or non-commutative rings.

§1. *The Basic Decomposition Theorem*

In this section we prove the basic decomposition theorem on which this paper rests. Before stating the result, we give some definitions and notations.

Throughout, **C** will be an *abelian* category. By a *subcategory* **A** of **C** we will always mean a *full, additive* and *essential* subcategory of **C**, so that **A** is closed under finite direct sums in **C** and such that any object C in **C** which is isomorphic to an object in **A** is already an object in **A**.

A subcategory of **C** is said to be *additively closed* (or *karoubian* in the terminology of [SGA IV] or [Qu]), if it is closed under direct summands in **C**, or, equivalently, if any projector ($p = p^2$) in the subcategory admits an image in that subcategory. Any subcategory **A** of **C** admits an *additive closure* **add A** in **C**, consisting of all those objects C in **C** which are isomorphic to a direct summand (in **C**) of an object in **A**. Clearly **A** is additively closed in **C** if and only if **A** = **add A**.

More generally, given any collection $\{C_\iota\}_{\iota\in I}$ of objects in **C**, there is a unique smallest additively closed subcategory **add** $\{C_\iota\}_{\iota\in I}$ containing each object C_ι, $\iota\in I$. It can be described by the following "universal mapping property": If $F:\mathbf{C} \to \mathbf{D}$ is any additive functor from **C** into another additive category **D** such that $F(C_\iota)$ is a zero-object in **D** for each $\iota\in I$, then $F(\mathbf{add}\ \{C_\iota\}_{\iota\in I})$ consists entirely of zero-objects.

In particular, (cf. also [He]), there exists the *additive quotient category* $\pi:\mathbf{C} \to \mathbf{C}/\mathbf{add}\ \{C_\iota\}_{\iota\in I}$, where $\mathbf{C}/\mathbf{add}\ \{C_\iota\}_{\iota\in I}$ has the same objects as **C** and π is a full, additive functor which is the identity on objects.

The projection functor π is characterized by the property that any additive functor F as before factors uniquely over π. Of course, even if **C** is assumed to be abelian, as here, $\mathbf{C}/\mathbf{add}\ \{C_\iota\}_{\iota\in I}$ need not to be so.

If **A** is an additively closed subcategory of **C**, the morphism groups in **C/A** are given by

$$\mathrm{Hom}_{\mathbf{C/A}}(C_1, C_2) = \frac{\mathrm{Hom}_{\mathbf{C}}(C_1, C_2)}{\{\phi:C_1 \to C_2 \mid \phi \text{ factors over an object in } \mathbf{A}\}} .$$

Now suppose again that **A** is any subcategory of **C** in the sense fixed above. We say that a sequence of morphisms $\ldots \to A_{i+1} \to A_i \to A_{i-1} \to \ldots$ in **A** is *exact*, if when viewed as a sequence in **C** it is exact.

Suppose C is an object in **C**. We define **A**–resol.dimC, the **A**-*resolution dimension* of C, to be the smallest nonnegative integer n such that there exists an exact sequence $0 \to A_n \to A_{n-1} \to \ldots \to A_0 \to C \to 0$, with each A_i in **A**, if such an integer exists. We say that **A**–resol.dimC $< \infty$ if **A**–resol.dimC $= n$ for some non-negative integer n. The subcategory of **C** consisting of all C in **C** such that **A**–resol.dimC $< \infty$ will be denoted $\hat{\mathbf{A}}$.

Finally, we say a subcategory **B** of **A** is a *cogenerator* for **A** if for each object A in **A** there is an exact sequence $0 \to A \to B \to A' \to 0$ in **A** with B in **B**.

With these notations, we fix throughout the rest of this paper an additively closed subcategory **X** of **C** which is furthermore *closed under extensions*, i.e. if $0 \to C_1 \to C_2 \to C_3 \to 0$ is exact in **C** with C_1 and C_3 in **X**, then also C_2 is in **X**. (In the terminology of [Qu], for example, **X** is a *karoubian exact* subcategory of **C**.) Also we assume given an additively closed subcategory ω of **X** which is a cogenerator of **X**.

The paper is now devoted to studying how the categories **X**, ω, $\hat{\mathbf{X}}$ and $\hat{\omega}$ are related.

All of our results depend on the following

Theorem 1.1. For each C in $\hat{\mathbf{X}}$ there are exact sequences

$$0 \to Y_C \to X_C \to C \to 0 \quad \text{and}$$

$$0 \to C \to Y^C \to X^C \to 0$$

with Y_C and Y^C in $\hat{\omega}$ and X_C and X^C in $\mathbf{X}$.

Proof. The proof proceeds by induction on $\mathbf{X}$–resol.dimC and is based on the following two easily proven observations.

Lemma 1.2. Suppose given exact sequences $0 \to K \to X \to C \to 0$ and $0 \to K \to Y^K \to X^K \to 0$ with X and X^K in $\mathbf{X}$ and Y^K in $\hat{\omega}$. Then in the pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y^K \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & X^K \arrow[r, equal] \arrow[d] & X^K \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

the exact sequence

$$0 \to Y^K \to U \to C \to 0$$

has the property that Y^K is in $\hat{\omega}$ and U is in $\mathbf{X}$.

Proof. As Y^K is in $\hat{\omega}$ by assumption, it remains to be seen that U is in $\mathbf{X}$. This follows from the fact that both X and X^K are in $\mathbf{X}$ and $\mathbf{X}$ is closed under extensions. ∎

The other observation we need is the following.

Lemma 1.3. Suppose that we have an exact sequence $0 \to Y_C \to X_C \to C \to 0$ with Y_C in $\hat{\omega}$ and X_C in $\mathbf{X}$. Let $0 \to X_C \to W \to X \to 0$ be exact with X in $\mathbf{X}$ and W in ω. Then in the pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Y_C \arrow[r, equal] \arrow[d] & Y_C \arrow[d] & & \\
0 \arrow[r] & X_C \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

the exact sequence

$$0 \to C \to Z \to X \to 0$$

has the property that Z is in $\hat{\omega}$ and X is in $\mathbf{X}$.

Proof: As again X is in $\mathbf{X}$ by assumption, it is only required to prove that Z is in $\hat{\omega}$. But in the exact sequence $0 \to Y_C \to W \to Z \to 0$, we have Y_C in $\hat{\omega}$ and W in ω, so that Z is in $\hat{\omega}$ by definition of that category. ■

The proof of theorem 1.1. follows now easily from these lemmas.

Suppose $\mathbf{X}$–resol.dim$C = n$ and let $0 \to X_n \to \dots \to X_1 \xrightarrow{d_0} X_0 \to C \to 0$ be exact with each X_i in $\mathbf{X}$. If $n = 0$, we have that C is already in $\mathbf{X}$. Since ω is a cogenerator for $\mathbf{X}$, there is an exact sequence $0 \to C \to W \to X \to 0$ in $\mathbf{X}$ with W in ω which is one of our desired exact sequences. The other one is $0 \to 0 \to C \xrightarrow{=} C \to 0$. Now suppose that $n > 0$ and set $K = \operatorname{Im} d_0$, so that we have exact sequences $0 \to K \to X_0 \to C \to 0$ and $0 \to X_n \to \dots \to X_1 \to K \to 0$ with each X_i in $\mathbf{X}$. By the inductive hypothesis we know there is an exact sequence $0 \to K \to Y^K \to X^K \to 0$ with Y^K in $\hat{\omega}$ and X^K in $\mathbf{X}$. Therefore, by Lemma 1.2, the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y^K \arrow[r] & U \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

has the property that U is in $\mathbf{X}$. Hence we may choose $0 \to Y^K \to U \to C \to 0$ as one of our desired sequences for C. From the existence of this exact sequence, it follows by Lemma 1.3 that we also have an exact sequence $0 \to C \to Y^C \to X^C \to 0$ with Y^C in $\hat{\omega}$ and X^C in $\mathbf{X}$. This finishes the proof of theorem 1.1. ■

For ease of reference, we call an exact sequence $0 \to Y_C \to X_C \xrightarrow{\pi_C} C \to 0$ with X_C in $\mathbf{X}$ and Y_C in $\hat{\omega}$ an *$\mathbf{X}$-approximation* of C. Dually, we call an exact sequence $0 \to C \xrightarrow{\iota^C} Y^C \to X^C \to 0$ with Y^C in $\hat{\omega}$ and X^C in $\mathbf{X}$ an ***$\hat{\omega}$-hull*** of C.

From now on, we assume that $\mathbf{X}$ has the property that if $0 \to X_0 \to X_1 \to X_2 \to 0$ is an exact sequence with X_1 and X_2 in $\mathbf{X}$, then X_0 is also in

$\mathbf{X}$, in addition to $\mathbf{X}$ being an additively closed subcategory of $\mathbf{C}$ which is closed under extensions. (In D.Quillen's terminology, (loc. cit.), all epimorphisms from $\mathbf{C}$ in $\mathbf{X}$ are *admissible.*)

It should be noted that in all our examples the categories $\mathbf{X}$ satisfy this additional condition. As a consequence of this further hypothesis on $\mathbf{X}$, we get the following

Lemma 1.4. Suppose C in $\mathbf{C}$ has an $\hat{\omega}$-hull $0 \to C \to Y^C \to X^C \to 0$. Then it also admits an $\mathbf{X}$-approximation $0 \to Y_C \to X_C \to C \to 0$. Furthermore, Y_C can be chosen such that ω–resol.dimY_C < ω–resol.dim Y^C if Y^C is not already in ω.

Proof. Let $0 \to W_n \to W_{n-1} \to \dots \xrightarrow{d_0} W_0 \to Y^C \to 0$ be exact with the W_i in ω. Then we obtain the following pullback diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & W_0 \arrow[r] \arrow[d] & X^C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d] & Y^C \arrow[r] \arrow[d] & X^C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where $K = \mathrm{Im}\, d_0$. Since X^C and W_0 are in $\mathbf{X}$, the additional assumption yields that L is in $\mathbf{X}$ too. By definition, K is in $\hat{\omega}$ and so $0 \to K \to L \to C \to 0$ is an $\mathbf{X}$-approximation of C. Now set $Y_C = K$ and $X_C = L$. ∎

As a consequence of this lemma, we obtain the following characterization of the objects in $\hat{\mathbf{X}}$.

Proposition 1.5. Let $\mathbf{X}$ be an additively closed and exact subcategory of $\mathbf{C}$ in which every epimorphism is admissible. If ω is a cogenerator of $\mathbf{X}$, the following are equivalent for an object C in $\mathbf{C}$:

(a) C is in $\hat{\mathbf{X}}$.

(b) There exists an $\mathbf{X}$-approximation $0 \to Y_C \to X_C \to C \to 0$ of C.

(c) There is an $\hat{\omega}$-hull $0 \to C \to Y^C \to X^C \to 0$ of C.

Proof. Since (a) implies (b) and (c) by theorem 1.1, it is only required to show that (b) implies (a) and (c) implies (b).

(b) $\Rightarrow$ (a): Since Y_C is in $\hat{\omega}$ by assumption, there is an exact sequence $0 \to W_n \to \dots \to W_0 \to Y_C \to 0$ with each W_i in ω. Since ω is a subcategory of $\mathbf{X}$, it follows from the exact sequence $0 \to W_n \to \dots \to W_0 \to X_C \to C \to 0$ that C is in $\hat{\mathbf{X}}$.

(c) $\Rightarrow$ (b): This is just a restatement of Lemma 1.4. ∎

We end this section with three examples, illustrating the theory developed so far.

Example 1. Let $X \to \operatorname{Spec} k$ be a scheme of finite type over a field k. Assume that X is equidimensional of dimension d and locally Cohen-Macaulay in the sense that $\mathcal{O}_{X,x}$ is a local Cohen-Macaulay ring for each x in X. Let $\mathbf{C}$ be the category of coherent sheaves of $\mathcal{O}_X$*-modules* and define $\mathbf{X}$ to be the subcategory of maximal Cohen-Macaulay coherent sheaves, where a coherent $\mathcal{O}_X$-module $\mathcal{M}$ is said to be maximal Cohen-Macaulay if for every $x \in X$ one has $\operatorname{depth}_{\mathbf{m}_x} \mathcal{M}_x = \dim \mathcal{O}_{X,x}$; $\mathbf{m}_x$ the unique maximal ideal of $\mathcal{O}_{X,x}$.

It is then clear that if $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ is an exact sequence in $\mathbf{C}$, then

(a) $\mathcal{M}_2$ is in $\mathbf{X}$ if $\mathcal{M}_1$ and $\mathcal{M}_3$ are in $\mathbf{X}$, and

(b) $\mathcal{M}_1$ is in $\mathbf{X}$ if $\mathcal{M}_2$ and $\mathcal{M}_3$ are in $\mathbf{X}$.

Remark also that, by hypothesis, the structure sheaf $\mathcal{O}_X$ is in $\mathbf{X}$ and that consequently $\mathbf{X}$ contains all locally free sheaves of $\mathcal{O}_X$-modules. Conversely, a maximal Cohen-Macaulay $\mathcal{O}_X$-module is locally free on the regular locus $X_{reg} \subseteq X$. Moreover, $\mathbf{C} = \hat{\mathbf{X}}$, and if $\mathcal{C} \neq 0$ is in $\mathbf{C}$, then $\mathbf{X}$-resol.dim$\,\mathcal{C} = n$ if and only if n is the largest integer such that $\mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{C}, \omega_X) \neq 0$, where ω_X is a dualizing sheaf for X.

Now assume that either X admits a very ample invertible sheaf $\mathcal{L}$ or that X is affine (in which case $\mathcal{L} = \mathcal{O}_X$ in the following). Then X can be embedded into a projective space over k, say $i:X \to \mathbf{P}^N_k$, with $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^N}(1)$.

Denoting by $\omega_{\mathcal{L}}$ the smallest additively closed subcategory which contains the family of objects $\{\omega_X \otimes \mathcal{L}^{\otimes n}\}_{n \in \mathbf{Z}}$, it follows easily from Grothendieck-Serre duality theory that $\omega_{\mathcal{L}}$ is a cogenerator for $\mathbf{X}$.

Proposition 1.6. For each coherent sheaf $\mathcal{C}$ of $\mathcal{O}_X$-modules we have both an $\mathbf{X}$-approximation with respect to $\omega_{\mathcal{L}}$ of the form $0 \to \mathcal{Y}_{\mathcal{C}} \to \mathcal{X}_{\mathcal{C}} \to \mathcal{C} \to 0$ and an $\hat{\omega}_{\mathcal{L}}$-hull $0 \to \mathcal{C} \to \mathcal{Y}^{\mathcal{C}} \to \mathcal{X}^{\mathcal{C}} \to 0$. ∎

Remark that in this example the category $\mathbf{X}$ depends only on the scheme X whereas its cogenerator depends on the choice of both a dualizing module ω_X and a very ample sheaf $\mathcal{L}$. Also the $\mathbf{X}$-approximations and $\hat{\omega}_{\mathcal{L}}$-hulls will vary with these choices.

Next consider the following modified version of Example 1.

Example 2. As in Example 1, we let $X \to \operatorname{Spec} k$ be an equidimensional Cohen-Macaulay scheme over a field k. Let $X' \subset X$ be the *Gorenstein locus* of X, which is the set of all points x in X for which $\mathcal{O}_{X,x}$ is a Gorenstein local ring. Let $\mathbf{X}'$ be the subcategory of $\mathbf{C}$, the category of coherent sheaves of $\mathcal{O}_X$-modules, consisting of those Cohen-Macaulay sheaves $\mathcal{M}$ such that $\mathcal{M}_x$ is $\mathcal{O}_{X,x}$-free for all x in X'. It is clear again that $\mathbf{X}'$ is an exact subcategory of $\mathbf{C}$ in which every epimorphism is admissible. Also $\hat{\mathbf{X}}'$ consists of all those

$\mathcal{M}$ in **C** for which $\mathcal{M}_x$ is of finite projective dimension over $\mathcal{O}_{X,x}$ for each x in X'. This implies that an exact sequence $0 \to A \to B \to C \to 0$ in **C** is in $\hat{\mathbf{X}}'$ if any two of A,B or C are in $\hat{\mathbf{X}}'$.

Again, any invertible sheaf of $\mathcal{O}_X$-modules is in **X'**, and in particular for any dualizing sheaf ω_X and each very ample invertible sheaf $\mathcal{L}$ on X, the category $\omega_{\mathcal{L}}$ defined above is a cogenerator of **X'**. We leave it to the reader to give in this case the analogue of Proposition 1.6.

Our final example in this section treats a not necessarily commutative version of Gorenstein rings of finite Krull dimension.

Example 3. Let R be a ring with unit which is noetherian on both sides and such that the *injective dimension of* R *as a right module over itself is finite*, say equal to d.

Take **C** = **R-mod**, the category of finitely generated left R-modules, and let **X** be the subcategory consisting of all modules M in **R-mod** which satisfy $\mathrm{Ext}^i_R(M,R) = 0$ for $i \neq 0$.

Then **X** is certainly additively closed and has the property that an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is in **X** as soon as either M_1 and M_3 or M_2 and M_3 are in **X**. Hence **X** satisfies our general assumptions.

For ω, take the subcategory of all finitely generated *projective* left R-modules. Then ω is by definition a subcategory of **X** which is additively closed.

For our theory to apply, we have hence to show that ω constitutes a cogenerator for **X**. To obtain this result we need our assumption on R. Namely, let $\dots \xrightarrow{d_j} P_j \longrightarrow \dots \longrightarrow P_1 \xrightarrow{d_0} P_0 \longrightarrow M \longrightarrow 0$ be a projective resolution of a module M in **X**. By definition of **X**, the dualized complex

$$0 \longrightarrow M^* \longrightarrow P_0^* \xrightarrow{d_0^*} P_1^* \longrightarrow \dots \longrightarrow P_j^* \xrightarrow{d_j^*} \dots$$

is acyclic. But then our hypothesis furnishes the following more precise information.

Lemma 1.7. With notations and assumptions as above, for every module M in **X** one has

(a) For all integers $j \geq 0$, the right R-modules $K_j = \mathrm{Ker}\, d_j^*$ satisfy $\mathrm{Ext}^i_R(K_j,R) = 0$ for $i \neq 0$.

(b) M is *reflexive*, that is, the natural morphism of left R-modules $M \to M^{**}$ is an isomorphism.

(c) If $0 \longrightarrow L \longrightarrow Q \xrightarrow{p} M^* \longrightarrow 0$ is an exact sequence of right R-modules with Q finitely generated projective, then L^* satisfies $\mathrm{Ext}^i_R(L^*,R) = 0$ for $i \neq 0$.

Proof. (a) As all the modules P_j^* are finitely generated projective right R-modules, they satisfy necessarily $\mathrm{Ext}^i_R(P_j^*,R) = 0$ for $i \neq 0$. But this implies that for any integer $n \geq 0$ one has natural isomorphisms $\mathrm{Ext}^i_R(K_{j-n},R) \xrightarrow{\sim} \mathrm{Ext}^{i+n}_R(K_j,R)$ for any $i > 0$. Since by

assumption $\mathrm{Ext}^k_R(-,R) = 0$ as soon as $k > d$, it suffices to take $n \geq d$ above to conclude $\mathrm{Ext}^i_R(K_j,R) = 0$ for all $i > 0$ and $j \geq 0$.
(b) This is a consequence of (a). By [A-B; 2.1.], for any left R-module M the natural morphism $M \to M^{**}$ fits into an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_R(D(M),R) \longrightarrow M \longrightarrow M^{**} \longrightarrow \mathrm{Ext}^2_R(D(M),R) \longrightarrow 0,$$

where $D(M) = \mathrm{Cok}\, d_0^*$. But if M is in **X**, we have $\mathrm{Cok}\, d_0^* = \mathrm{Ker}\, d_2^*$ and (a) shows that the extreme terms of this exact sequence vanish, establishing (b).
(c) As $M^* = K_0$, part (a) implies that the sequence

$$(*)\ 0 \longrightarrow M^{**} \xrightarrow{p^*} Q^* \longrightarrow L^* \longrightarrow 0$$

is exact. From (b) we have $M \cong M^{**}$ and as M is in **X**, it follows already that $\mathrm{Ext}^i_R(L^*,R) = 0$ for $i > 1$. It hence only remains to be seen that $\mathrm{Ext}^1_R(L^*,R) = 0$, or, equivalently, that the dual sequence of (*):

$$0 \longrightarrow L^{**} \longrightarrow Q^{**} \xrightarrow{p^{**}} M^{***} \longrightarrow 0$$

is again exact. But this is obvious as both Q and M^* are reflexive right R-modules and $p^{**} = p$. ∎

Combining (b) and (c) of this lemma, we have now that any module M in **X** embeds into the finitely generated projective module $\mathrm{Hom}_R(Q,R)$ and that the cokernel, isomorphic to L^*, is again in **X**. This shows that ω is indeed a cogenerator for **X**.

Finally observe that $\hat{\mathbf{X}}$ consists of all left R-modules N in **C** satisfying $\mathrm{Ext}^i_R(N,R) = 0$ for all sufficiently large i, and that $\hat{\omega}$ is the category of all finitely generated left R-modules *of finite projective dimension.*

Now Theorem 1.1 yields in this context the following.

Theorem 1.8. Let R be a ring which is noetherian on both sides and of finite injective dimension as a right module over itself. Then for any finitely generated left R-module N satisfying $\mathrm{Ext}^i_R(N,R) = 0$ for all sufficiently large i, there are modules Y_N and Y^N in **R-mod** of finite projective dimension and modules X_N and X^N in **R-mod** with $\mathrm{Ext}^i_R(X_N,R) = \mathrm{Ext}^i_R(X^N,R) = 0$ for $i \neq 0$ which fit into exact sequences

$$0 \longrightarrow Y_N \longrightarrow X_N \longrightarrow N \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow N \longrightarrow Y^N \longrightarrow X^N \longrightarrow 0.$$ ∎

§2. *Injective Cogenerators*

Having established the *existence* of **X**-approximations and $\hat{\omega}$–hulls for a pair $(\mathbf{X},\omega)$ of subcategories as in the preceding section, the important question which remains is their *uniqueness.*

To see which conditions ought to be imposed, assume given two $\mathbf{X}$-approximations for the same object C in $\hat{\mathbf{X}}$, say $0 \to Y_C \to X_C \to C \to 0$ and $0 \to Y'_C \to X'_C \to C \to 0$. Then the least one should ask for is that these $\mathbf{X}$-approximations can be *compared* in the sense that there exists a morphism $\phi : X_C \to X'_C$ making the following diagram commutative

$$\begin{tikzcd}
0 \arrow[r] & Y_C \arrow[r] \arrow[d] & X_C \arrow[r] \arrow[d, "\phi"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y'_C \arrow[r] & X'_C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Apparently, the existence of such a comparison morphism is guaranteed as soon as $\operatorname{Ext}^1_{\mathbf{C}}(X_C, Y'_C) = 0$.

Hence, for comparisons to exist and to yield an equivalence relation, it certainly suffices to have that $\operatorname{Ext}^1_{\mathbf{C}}(X, Y) = 0$ for *all* X in $\mathbf{X}$ and Y in $\hat{\omega}$. This section is devoted to a study of this condition and its consequences. First, once again, some general remarks and notations.

Let A and C be objects in $\mathbf{C}$. As $\mathbf{C}$ is supposed to be abelian, the groups $\operatorname{Ext}^i_{\mathbf{C}}(A, C)$ are defined for all $i \geq 0$. If there is an integer n such that $\operatorname{Ext}^i_{\mathbf{C}}(A, C) = 0$ for all $i > n$, then the smallest nonnegative such integer n is called the A-*injective dimension of* C, (notation: A−inj.dimC), or the C-*projective dimension of* A, (notation: C−proj.dimA).

Otherwise we set A−inj.dimC $= \infty =$ C−proj.dimA . If $\mathbf{B}$ is a subcategory of $\mathbf{C}$, for each A in $\mathbf{C}$ we define A−inj.dim$\mathbf{B}$ to be the maximum (in $\mathbf{Z} \cup \{\infty\}$) of A−inj.dimB for all B in $\mathbf{B}$. Dually, for each C in $\mathbf{C}$, we define C−proj.dim$\mathbf{B}$ to be the maximum of C−proj.dimB for all B in $\mathbf{B}$.

Clearly A−inj.dim$\mathbf{B}$ = $\mathbf{B}$−proj.dimA.

Suppose now that $\mathbf{A}$ and $\mathbf{B}$ are subcategories of $\mathbf{C}$. Then define $\mathbf{A}$−proj.dim$\mathbf{B}$ to be the maximum of A−proj.dimB for all A in $\mathbf{A}$ and B in $\mathbf{B}$. We define dually $\mathbf{A}$−inj.dim$\mathbf{B}$ to be the maximum of A−inj.dimB for all A in $\mathbf{A}$ and B in $\mathbf{B}$. Again, one has clearly $\mathbf{A}$−inj.dim$\mathbf{B}$ = $\mathbf{B}$−proj.dim$\mathbf{A}$.

If for two such subcategories $\mathbf{A}$−inj.dim$\mathbf{B}$ = 0 = $\mathbf{B}$−proj.dim$\mathbf{A}$, we follow J.L. Verdier, [SGA 4½, C.D.; I.2.6.1.], and say that $\mathbf{A}$ is *left orthogonal* to $\mathbf{B}$ and $\mathbf{B}$ is *right orthogonal* to $\mathbf{A}$ - with respect to the "augmented" bilinear $\mathbf{Z}$-graded pairing induced by $(\operatorname{Ext}^i_{\mathbf{C}}(-,-))_{i>0}$ on the monoid of isomorphism classes of objects of $\mathbf{C}$.

Consequently, if $\mathbf{A}$ consists precisely of those objects A in $\mathbf{C}$ for which A−inj.dim$\mathbf{B}$ = 0, we call $\mathbf{A}$ the *left orthogonal complement* of $\mathbf{B}$ in $\mathbf{C}$, denoted $\mathbf{A} = {}^{\perp}\mathbf{B}$. Dually again, $\mathbf{A}^{\perp}$, the *right orthogonal complement* of $\mathbf{A}$ in $\mathbf{C}$, is the subcategory $\mathbf{B}$ consisting of all objects B in $\mathbf{C}$ for which $\mathbf{A}$−inj.dimB = 0.

One has obviously $\mathbf{A} \subseteq {}^{\perp}(\mathbf{A}^{\perp})$ and $\mathbf{A} \subseteq ({}^{\perp}\mathbf{A})^{\perp}$, but not necessarily ${}^{\perp}(\mathbf{A}^{\perp}) = ({}^{\perp}\mathbf{A})^{\perp}$. If $\mathbf{B}'$ is a subcategory of $\mathbf{B}$ in $\mathbf{C}$, then ${}^{\perp}\mathbf{B}$ is contained in ${}^{\perp}\mathbf{B}'$ and similary for right orthogonal complements. Remark also that by definition ${}^{\perp}\mathbf{C}$, the *left radical* of $\mathbf{C}$ with respect to the pairing $(\operatorname{Ext}^i_{\mathbf{C}}(-,-))_{i>0}$, consists precisely of all *projective* objects of $\mathbf{C}$, whereas $\mathbf{C}^{\perp}$, the *right radical* of $\mathbf{C}$, is given by all *injective* objects of $\mathbf{C}$.

Furthermore, it is obvious that orthogonal complements are additively closed and exact subcategories of $\mathbf{C}$ and that in a left orthogonal complement ${}^{\perp}\mathbf{B}$ all epimorphisms are admissible, whereas in a right orthogonal complement $\mathbf{A}^{\perp}$ all monomorphisms are admissible.

Returning to our subcategories $\mathbf{X}$ and ω of $\mathbf{C}$ from the previous section, we say that ω is an *injective cogenerator* for $\mathbf{X}$ if $\mathbf{X}$–inj.dim$\,\omega = 0$, that is, $\omega \subseteq \mathbf{X}^{\perp}$. If there is a cogenerator for $\mathbf{X}$ in $\mathbf{X} \cap \mathbf{X}^{\perp}$, we say also that the exact category $\mathbf{X}$ has *enough relatively injective objects.*

Unless stated to the contrary, we assume from now on that ω is an injective cogenerator for $\mathbf{X}$. Our next aim is to explore some important properties of $\mathbf{X}$-approximations and $\hat{\omega}$–hulls implied by this additional assumption.

We begin with the following relations between some of the dimensions we have just introduced for an object C in $\hat{\mathbf{X}}$. These relations do not require that any epimorphism in $\mathbf{X}$ is admissible.

Proposition 2.1. Given an object C in $\hat{\mathbf{X}}$, where $\mathbf{X}$ is an additively closed exact subcategory of $\mathbf{C}$ and ω is an injective cogenerator for $\mathbf{X}$, the following are equivalent for any integer $n \geq 0$.

(a) $\mathbf{X}$–resol.dim$\,C = n$,

(b) C–inj.dim$\,\omega = n$,

(c) C–inj.dim$\,\hat{\omega} = n$,

(d) $\mathrm{Ext}_{\mathbf{C}}^{n+1}(C, Y) = 0$ for all Y in $\hat{\omega}$.

Proof: Proceed by induction on $n = \mathbf{X}$–resol.dim$\,C$, the case $n = 0$ being settled as follows.

(a) $\Rightarrow$ (b) is true because ω is contained in $\mathbf{X}^{\perp}$ by assumption.

(b) $\Rightarrow$ (c) follows from the usual dimension shift argument.

(c) $\Rightarrow$ (d) is the definition of C–inj.dim$\,\hat{\omega}$.

(d) $\Rightarrow$ (a): Since C is in $\hat{\mathbf{X}}$ by the general hypothesis, there is an $\mathbf{X}$-approximation $0 \to Y_C \to X_C \to C \to 0$ which splits by (d). Hence C is a direct summand of X_C in $\mathbf{X}$ and so C is in $\mathbf{X}$.

The proof of the inductive step follows easily from what we have just shown and is left to the reader. ∎

As an obvious consequence of this proposition we have

Corollary 2.2. $\mathbf{X}$–inj.dim$\,\hat{\omega} = 0$. ∎

This corollary yields the following important properties of $\mathbf{X}$-approximations and $\hat{\omega}$-hulls.

Theorem 2.3. Let $0 \to Y_C \to X_C \xrightarrow{\pi_C} C \to 0$ be an $\mathbf{X}$-approximation for C in $\hat{\mathbf{X}}$. Then for each X in $\mathbf{X}$ we have

(a) $0 \to \mathrm{Hom}_{\mathbf{C}}(X, Y_C) \to \mathrm{Hom}_{\mathbf{C}}(X, X_C) \to \mathrm{Hom}_{\mathbf{C}}(X, C) \to 0$ is exact,

(b) π_C induces isomorphisms $\mathrm{Ext}^i_{\mathbf{C}}(X, X_C) \to \mathrm{Ext}^i_{\mathbf{C}}(X, C)$ for all $i > 0$.

Proof: As $\mathbf{X}$–inj.dim$\hat{\omega} = 0$, one has $\mathrm{Ext}^i_{\mathbf{C}}(X, Y_C) = 0$ for all $i > 0$. ∎

The exact sequence $0 \to Y_C \to X_C \to C \to 0$ is called an $\mathbf{X}$-approximation precisely because $\mathrm{Hom}_{\mathbf{C}}(X, X_C) \to \mathrm{Hom}_{\mathbf{C}}(X, C) \to 0$ is exact for all X in $\mathbf{X}$. This property of $\mathbf{X}$-approximations of C gives rise to a weak sort of uniqueness for such approximations as we now explain.

Let us call two morphisms $f{:}B \to C$ and $f'{:}B' \to C$ in $\mathbf{C}$ *equivalent* if there are morphisms $g{:}B \to B'$ and $h{:}B' \to B$ such that $f = f'g$ and $f' = fh$. Also, we say that two exact sequences $0 \to A \to B \xrightarrow{f} C$ and $0 \to A' \to B' \xrightarrow{f'} C$ are (*right*) *equivalent*, if f and f' are equivalent, which amounts to the same as saying that there is a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & A' & \longrightarrow & B' & \xrightarrow{f'} & C \\
 & & \downarrow & & \downarrow{\scriptstyle h} & & \| \\
0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{f} & C \\
 & & \downarrow & & \downarrow{\scriptstyle g} & & \| \\
0 & \longrightarrow & A' & \longrightarrow & B' & \xrightarrow{f'} & C
\end{array}$$

In particular, $\mathrm{id}_B - hg$ factors over A and $\mathrm{id}_{B'} - gh$ factors over A', so that h,g become inverse isomorphisms in $\mathbf{C}/\mathbf{add}\{A, A'\}$.

As an immediate consequence of theorem 2.3 we obtain the following uniqueness result.

Corollary 2.4. $\mathbf{X}$-approximations for an object C in $\hat{\mathbf{X}}$ are unique up to equivalence, that is, any two $\mathbf{X}$-approximations for C are (right) equivalent exact sequences. ∎

There are also similar results for $\hat{\omega}$-hulls of an object C in $\hat{\mathbf{X}}$ as we now point out.

Theorem 2.5. Let $0 \to C \xrightarrow{\iota^C} Y^C \to X^C \to 0$ be an $\hat{\omega}$-hull for C in $\hat{X}$. Then for each Y in $\hat{\omega}$ we have the following

(a) $0 \to \mathrm{Hom}_{\mathbf{C}}(X^C, Y) \to \mathrm{Hom}_{\mathbf{C}}(Y^C, Y) \to \mathrm{Hom}_{\mathbf{C}}(C, Y) \to 0$ is exact,

(b) ι^C induces isomorphisms $\mathrm{Ext}^i_{\mathbf{C}}(Y^C, Y) \to \mathrm{Ext}^i_{\mathbf{C}}(C, Y)$ for all $i > 0$.

Proof: This follows again from the fact that $\mathbf{X}$–inj.dim$\hat{\omega} = 0$. ∎

The exact sequence $0 \to C \xrightarrow{\iota^C} Y^C \to X^C \to 0$ is called an $\hat{\omega}$-hull precisely because $\mathrm{Hom}_{\mathbf{C}}(Y^C, Y) \to \mathrm{Hom}_{\mathbf{C}}(C, Y) \to 0$ is exact for all Y in $\hat{\omega}$. Again, this property gives rise to a weak sort of uniqueness for $\hat{\omega}$-hulls, similar to that already discussed for $\mathbf{X}$-approximations, as we now explain.

Dually to the above, we say two morphisms $f:C \to D$ and $f':C \to D'$ are equivalent if there are morphisms $g:D \to D'$ and $h:D' \to D$ such that $f' = gf$ and $f = hf'$. Also, we say that two exact sequences $C \xrightarrow{f} D \to E \to 0$ and $C' \xrightarrow{f'} D' \to E' \to 0$ are (*left*) *equivalent* if f and f' are equivalent, which is the same thing as saying that there is a commutative diagram

$$\begin{array}{ccccccc} C & \xrightarrow{f} & D & \longrightarrow & E & \longrightarrow & 0 \\ \| & & \downarrow g & & \downarrow & & \\ C & \xrightarrow{f'} & D' & \longrightarrow & E' & \longrightarrow & 0 \\ \| & & \downarrow h & & \downarrow & & \\ C & \xrightarrow{f} & D & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

In particular, $\mathrm{id}_D - hg$ factors over E and $\mathrm{id}_{D'} - gh$ factors over E', so that h and g become inverse isomorphisms in $\mathbf{C}/_{\mathbf{add}\{E,E'\}}$.

As an immediate consequence of theorem 2.5 we have the following uniqueness theorem.

Corollary 2.6. $\hat{\omega}$-hulls for an object C in $\hat{\mathbf{X}}$ are unique up to equivalence, that is, any two $\hat{\omega}$-hulls are (left) equivalent exact sequences. ∎

We may reformulate and sharpen these uniqueness results slightly by considering the situation "modulo ω". This depends on the following simple observation.

Lemma 2.7. Let $f:X \to C$ be a morphism in $\mathbf{C}$ with X in $\mathbf{X}$ and C in $\hat{\mathbf{X}}$. Then the following conditions on f are equivalent.

(a) f factors through an object in $\hat{\omega}$.

(b) f factors through an object in ω.

Proof: As (b) is a priori a special case of (a), we need only to show that in fact (a) implies (b). Hence assume that $f = gh$ where $h:X \to Y$ and $g:Y \to C$ are morphisms in $\mathbf{C}$ and Y is in $\hat{\omega}$. By definition of $\hat{\omega}$, there is an exact sequence $0 \to K \to W \to Y \to 0$ with W in ω and K again in $\hat{\omega}$. By corollary 2.2, $\mathbf{X}$–inj.dim $\hat{\omega} = 0$ and so $\mathrm{Ext}^1_{\mathbf{C}}(X,K) = 0$. This shows that h, and then also f, factor over W in ω. ∎

Now choose for any object C in $\hat{\mathbf{X}}$ an $\mathbf{X}$-approximation $0 \to Y_C \to X_C \xrightarrow{\pi_C} C \to 0$ and an $\hat{\omega}$-hull $0 \to C \xrightarrow{\iota^C} Y^C \to X^C \to 0$, as well as for any morphism $f:C \to D$ in $\hat{\mathbf{X}}$ liftings $f_*:X_C \to X_D$ and $f^*:Y^C \to Y^D$ which exist by the above.

By the uniqueness results just established, it follows that given a second morphism $g:D \to E$ in $\hat{\mathbf{X}}$, the differences $g^*f^* - (gf)^*$ and $g_*f_* - (gf)_*$ factor over objects in ω, hence become zero-morphisms in $\hat{\mathbf{X}}/\omega$, the full subcategory spanned by $\hat{\mathbf{X}}$ in $\mathbf{C}/\omega$.

From this we obtain immediately the following

Theorem 2.8. Denote $i: \hat{\omega} \to \hat{\mathbf{X}}$ and $j: \mathbf{X} \to \hat{\mathbf{X}}$ the natural inclusion functors. Then

(a) The induced functor $j_!: \mathbf{X}/\omega \to \hat{\mathbf{X}}/\omega$ is fully faithful and admits a *right adjoint* $j^!: \hat{\mathbf{X}}/\omega \to \mathbf{X}/\omega$ which associates to an object C in $\hat{\mathbf{X}}$ the chosen $\mathbf{X}$-approximation X_C. The adjunction morphism $j_!j^!C \to C$ is given by the class of $\pi_C: X_C \to C$ in $\mathrm{Hom}_{\hat{\mathbf{X}}/\omega}(X_C, C)$.

(b) The induced functor $i_*: \hat{\omega}/\omega \to \hat{\mathbf{X}}/\omega$ is fully faithful and admits a *left adjoint* $i^*: \hat{\mathbf{X}}/\omega \to \hat{\omega}/\omega$ which associates to an object C in $\hat{\mathbf{X}}$ the chosen $\hat{\omega}$-hull Y^C. The adjunction morphism $C \to i_*i^*C$ is given by the class of $\iota^C: C \to Y^C$ in $\mathrm{Hom}_{\hat{\mathbf{X}}/\omega}(C, Y^C)$.

(c) One has $j^!i_* = 0$ and $i^*j_! = 0$.

(d) The composition of the adjunction morphisms

$$j_!j^! \xrightarrow{\pi} \mathrm{id}_{\hat{\mathbf{X}}/\omega} \xrightarrow{\iota} i_*i^*$$

is zero in $\hat{\mathbf{X}}/\omega$.

Proof: The remarks preceding the theorem show that X_- and Y^- define functors from $\hat{\mathbf{X}}$ into $\mathbf{X}/\omega$ and $\hat{\omega}/\omega$ respectively. By the universal property of quotient categories these functors factor over $\hat{\mathbf{X}}/\omega$, yielding $j_!$ and i^*. To prove that $j_!$ is indeed right adjoint to the inclusion functor $j^!: \mathbf{X}/\omega \to \hat{\mathbf{X}}/\omega$, it suffices to give the natural isomorphisms $\phi_{X,C}: \mathrm{Hom}_{\mathbf{X}/\omega}(X, j_!C) \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbf{X}}/\omega}(j^!X, C)$. Now composition with $\pi_C: X_C = j_!C \to C$ defines the natural map $\mathrm{Hom}_{\mathbf{X}}(X, X_C) \to \mathrm{Hom}_{\hat{\mathbf{X}}}(X, C)$ which is surjective by theorem 2.3.(a). Let $\phi_{X,C}$ be the induced map on the quotient groups, which is hence still surjective. To prove that it is injective, let f in $\mathrm{Hom}_{\mathbf{X}}(X, X_C)$ be a morphism such that $\pi_C f: X \to C$ factors over some object W in ω. This means that there is a commutative diagram in $\mathbf{C}$

$$\begin{tikzcd} 0 \arrow[r] & Y_C \arrow[r] & X_C \arrow[r, "\pi_C"] & C \arrow[r] & 0 \\ & & X \arrow[u, "f"] \arrow[r, "h"'] & W \arrow[u, "g"'] & \end{tikzcd}$$

As W is a priori in $\mathbf{X}$ and Y_C is in $\hat{\omega}$, corollary 2.2 applies once again to yield $\mathrm{Ext}^1_{\mathbf{C}}(W, Y_C) = 0$ and hence to establish the existence of a morphism $g': W \to X_C$ such that $\pi_C g' = g$. But then $f - g'h$ satisfies $\pi_C(f - g'h) = \pi_C f - (\pi_C g')h = gh - gh = 0$, so that $f - g'h$ factors over Y_C. Then lemma 2.7 shows that $f - g'h$ factors already over some object W' in ω and hence the class of $f - g'h$ in $\mathrm{Hom}_{\mathbf{X}/\omega}(X, X_C)$ is the zero-morphism. As $g'h$ factors over W in ω, its class is zero as well, which shows that f and $f - g'h$ define the same morphism in $\mathrm{Hom}_{\mathbf{X}/\omega}(X, X_C) = \mathrm{Hom}_{\mathbf{X}/\omega}(X, j_!C)$. Hence already the class of f is the zero-morphism and $\phi_{X,C}$ is injective as claimed.

The definition of $\phi_{X,C}$ is natural in both arguments, so that the adjointness of $j_!$ and $j^!$ is established. Furthermore, the construction of $\phi_{X,C}$ shows that π_C induces the adjunction morphism $j^!j_!C \to C$.

This proves (a).
As the proof of (b) is completely analogous, it is left to the reader.
For (c), just remark again that by definition of $\hat{\omega}$, any object Y in $\hat{\omega}$ appears in an exact sequence $0 \to K \to W \to Y \to 0$ with W in ω and K in $\hat{\omega}$. But this sequence serves as an **X**-approximation for Y whence $j^! i_* Y$, the chosen **X**-approximation of Y, is isomorphic to W in $\mathbf{X}/\omega$, i.e. it is a zero object. This shows $j^! i_* = 0$ and $i^* j_! = 0$ follows then by adjunction.
(d) follows now from (c), as one has by naturality the commutative diagram of morphisms of functors

$$\begin{tikzcd} j_! j^! \arrow[r, "\pi"] \arrow[d, "\iota * (j_! j^!)"'] & \mathrm{id}_{\hat{\mathbf{X}}/\omega} \arrow[d, "\iota"] \\ i_* i^* j_! j^! \arrow[r, "(i_* i^*) * \pi"'] & i_* i^* \end{tikzcd}$$

in which the lower left corner is zero by (c). (In more concrete terms, (d) says that for any object C in $\hat{\mathbf{X}}$ there is a commutative diagram

$$\begin{tikzcd} X_C \arrow[r, "\pi_C"] \arrow[d, "j"'] & C \arrow[d, "\iota^C"] \\ W \arrow[r] & Y^C \end{tikzcd}$$

with W in ω, and we have seen indeed in lemma 1.3 and the proof of theorem 1.1 that ι^C can be obtained as the push-out of such a morphism j along π_C.) This finishes the proof of theorem 2.8. ∎

The reader puzzled by the notations used in the preceding theorem should compare it with the treatment of the "glueing of categories" in [BBD; 1.4]. It shows that in our situation one should think of $\hat{\mathbf{X}}$ as being obtained by "glueing together the open subcategory **X** and the closed subcategory $\hat{\omega}$ along ω ". What is missing for a complete glueing in the sense of (loc. cit.) is the existence of the other adjoints j_* and $i^!$.

The statements (c) and (d) in theorem 2.8 also explain why we think of theorem 1.1 as a "decomposition theorem": an object C in $\hat{\mathbf{X}}$ is *decomposed* - at least in $\hat{\mathbf{X}}/\omega$ - into its **X**-approximation X_C and its $\hat{\omega}$-hull Y^C, which belong to "orthogonal" subcategories of $\hat{\mathbf{X}}/\omega$.

The property which is desirable but missing yet is that **X** and $\hat{\omega}$ should have ω as their common intersection. This will be addressed later on in Proposition 3.6.

For now, we return to the examples 1 and 2 of the previous section. As soon as $X \to \operatorname{Spec} k$ is *projective*, $\omega_{\mathcal{L}}$ is not an injective generator in either **X** or **X'**, as $\operatorname{Ext}^d_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X \otimes \mathcal{L}^{\otimes n}) = H^d(X, \omega_X \otimes \mathcal{L}^{\otimes n})$ does not vanish for all integers n. None the less, the following analogues of the results for injective cogenerators are valid for these examples if one substitutes $\mathcal{E}xt^i_{\mathcal{O}_X}(A, B)$ for $\operatorname{Ext}^i_{\mathcal{O}_X}(A, B)$.

Lemma 2.9. With notations as in examples 1 and 2, the following are equivalent for a sheaf $\mathcal{M}$ in $\mathbf{C}$.

(a) $\mathcal{M}$ is in $\mathbf{X}$.

(b) $\mathcal{E}xt^i_{\mathcal{O}X}(\mathcal{M}, \omega_X) = 0$ for all $i > 0$.

(c) $\mathcal{E}xt^i_{\mathcal{O}X}(\mathcal{M}, \omega_X \otimes \mathcal{L}^{\otimes n}) = 0$ for all $i > 0$ and all n in $\mathbf{Z}$.

(d) $\mathcal{E}xt^i_{\mathcal{O}X}(\mathcal{M}, Y) = 0$ for all $i > 0$ and Y in $\omega_{\mathcal{L}}$.

Proof: Easy consequence of the fact that the corresponding statements hold for Cohen-Macaulay local rings with a dualizing module. ■

Proposition 2.10. With the same assumptions and notations as above, let $0 \to Y_C \to X_C \to C \to 0$ be an $\mathbf{X}$-approximation for C in $\mathbf{C}$.
Then we have for any $\mathcal{M}$ in $\mathbf{X}$:

(a) $0 \to Hom_{\mathcal{O}X}(\mathcal{M}, Y_C) \to Hom_{\mathcal{O}X}(\mathcal{M}, X_C) \to Hom_{\mathcal{O}X}(\mathcal{M}, C) \to 0$ is exact.

(b) The induced morphisms $\mathcal{E}xt^i_{\mathcal{O}X}(\mathcal{M}, X_C) \to \mathcal{E}xt^i_{\mathcal{O}X}(\mathcal{M}, C)$ are isomorphisms for $i > 0$.

Proof: Immediate consequence of lemma 2.7. ■

Remark that in Example 3 the category ω is in fact an injective cogenerator as by definition there $\mathbf{X} = {}^{\perp}\omega$. Furthermore, in that example $\mathbf{X}/\omega$ is the category of left *maximal Cohen-Macaulay R-modules* - in the sense that $\mathrm{Ext}^i_R(M, R) = 0$ for $i \neq 0$ - *modulo stable equivalence*: two modules M and M' from $\mathbf{X}$ become isomorphic in $\mathbf{X}/\omega$ if and only if there are finitely generated projective left R-modules P and P' such that $M \oplus P'$ is isomorphic to $M' \oplus P$ in **R-mod**.

We end this section with two more illustrations of situations where ω is an injective cogenerator for $\mathbf{X}$.

Example 4. Suppose R is a commutative noetherian Cohen-Macaulay ring in the sense that all its localizations $R_{\mathbf{p}}$ at primes $\mathbf{p}$ are local Cohen-Macaulay rings. We say that a finitely generated R-module M is *maximal Cohen-Macaulay* (MCM for short), if $M_{\mathbf{p}}$ satisfies depth $M_{\mathbf{p}} = \dim R_{\mathbf{p}}$ for all primes $\mathbf{p}$.

Now suppose that R is a *Gorenstein* ring and that S is a commutative R-algebra which is MCM as an R-module. Let $\mathbf{C} = $ **S-mod** be the category of finitely generated S-modules and let $\mathbf{X}$ be the subcategory of $\mathbf{C}$ consisting of those S-modules M which are maximal Cohen-Macaulay as R-modules. Then $\mathbf{X}$ satisfies the usual properties. Set $\omega_{S/R} = \mathrm{Hom}_R(S, R)$, which is a relative dualizing module for the algebra $R \to S$. Then $\omega = \mathbf{add}\{\omega_{S/R}\}$ consists of all S-modules of the form $\mathrm{Hom}_R(P, R)$ with P finitely generated projective over S. It is easily seen - and well-known - that ω is an injective cogenerator for $\mathbf{X}$. Also, if the Krull dimension of R is finite, then $\hat{\mathbf{X}} = \mathbf{C}$.

To acknowledge the scope of this example and to emphasize its relevance for Grothendieck duality theory, we quote the following from [FGR; Cor. 5.9.].

Theorem. Suppose S is a commutative ring with finite Krull dimension and with connected prime spectrum. Then S *admits a canonical module* if and only if S is a homomorphic image of a Gorenstein ring R such that S is maximal Cohen-Macaulay as an R-module. ∎

Our final illustration of this section is the following variant of Examples 4 and 2.

Example 5. Maintain the hypotheses on S and R from Example 4. Let $X \subset \mathrm{Spec}\, R$ have the property that if $\mathbf{p}$ is in X, then $S_{\mathbf{p}}$ is a Gorenstein ring, or equivalently, $(\omega_{S/R})_{\mathbf{p}}$ is $S_{\mathbf{p}}$-free.

Set again $\mathbf{C} = \mathbf{S\text{-}mod}$ and let $\mathbf{X'}$ consist of those S-modules M which are MCM over R and satisfy furthermore that $M_{\mathbf{p}}$ is $S_{\mathbf{p}}$-projective for all $\mathbf{p}$ in X. Then $\mathbf{X'}$ satisfies the usual properties and contains $\boldsymbol{\omega} = \mathbf{add}\{\omega_{S/R}\}$. Again, $\boldsymbol{\omega}$ is an injective cogenerator for $\mathbf{X'}$ and $\hat{\mathbf{X}}'$ consists of all S-modules C such that $\mathrm{proj.dim}_{S_{\mathbf{p}}} C_{\mathbf{p}} < \infty$ for all $\mathbf{p}$ in X.

§3. *Exactness properties of $\hat{\mathbf{X}}$ and $\hat{\boldsymbol{\omega}}$.*

We maintain our general assumption that $\mathbf{X}$ is an additively closed and exact subcategory of $\mathbf{C}$ in which every epimorphism is admissible, and that $\boldsymbol{\omega}$ is an injective cogenerator for $\mathbf{X}$.

In this situation, we show that $\hat{\mathbf{X}}$ is an additively closed subcategory of $\mathbf{C}$ which has the property that an exact sequence $0 \to A \to B \to C \to 0$ is in $\hat{\mathbf{X}}$ whenever two of A, B and C are in $\hat{\mathbf{X}}$. This result is then used to prove that $\hat{\boldsymbol{\omega}}$ is an additively closed subcategory of $\mathbf{C}$ having the property that an exact sequence $0 \to A \to B \to C \to 0$ in $\mathbf{C}$ is already in $\hat{\boldsymbol{\omega}}$ if either A and C are in $\hat{\boldsymbol{\omega}}$ or A and B are in $\hat{\boldsymbol{\omega}}$. Hence $\hat{\boldsymbol{\omega}}$ is seen to be an *additively closed exact subcategory* of $\mathbf{C}$ in which *every monomorphism is admissible.*

We begin with the following

Lemma 3.1. The category $\hat{\mathbf{X}}$ is closed under extensions.

Proof: Suppose $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathbf{C}$ with A and C in $\hat{\mathbf{X}}$. Proceed by induction on $n = \mathbf{X}\text{-resol.dim}C$. Suppose $n = 0$, which means that C is in $\mathbf{X}$. As A is in $\hat{\mathbf{X}}$, there is an $\mathbf{X}$-approximation $0 \to Y_A \to X_A \to A \to 0$ of A. Since C is in $\mathbf{X}$, we know by theorem 2.3, that the induced map $\mathrm{Ext}^1_{\mathbf{C}}(C, X_A) \to \mathrm{Ext}^1_{\mathbf{C}}(C, A)$ is an isomorphism. Hence there exists an exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Y_A \arrow[r, equal] \arrow[d] & Y_A \arrow[d] & & \\
0 \arrow[r] & X_A \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since X_A and C are in $\mathbf{X}$, the object Z is also in $\mathbf{X}$, as that category is closed under extensions. Now Y_A is in $\hat{\omega}$, hence in $\hat{\mathbf{X}}$, and it follows that B is in $\hat{\mathbf{X}}$ as required.

Suppose now that $n > 0$ and let $0 \to L \to X_0 \to C \to 0$ be exact with $\mathbf{X}$–resol.dim$L = n-1$. Since X_0 is in $\mathbf{X}$, we have that $\mathrm{Ext}^1_{\mathbf{C}}(X_0, X_A) \to \mathrm{Ext}^1_{\mathbf{C}}(X_0, A)$ is an isomorphism by theorem 2.3, and so there exists an exact commutative diagram in $\mathbf{C}$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Y_A \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_A \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

This shows that B is in $\hat{\mathbf{X}}$ since V is necessarily in $\mathbf{X}$ and U is in $\hat{\mathbf{X}}$ by the inductive hypothesis. ∎

We now use the fact that $\hat{\mathbf{X}}$ is closed under extensions to prove the following

Lemma 3.2. Let $0 \to K \to X \to C \to 0$ be an exact sequence in $\mathbf{C}$ with X in $\mathbf{X}$. Then C is in $\hat{\mathbf{X}}$ if and only if K is in $\hat{\mathbf{X}}$.

Proof: By definition, if K is in $\hat{\mathbf{X}}$ then also C is in $\hat{\mathbf{X}}$. Hence assume that C is in $\hat{\mathbf{X}}$ and let $0 \to Y_C \to X_C \to C \to 0$ be an $\mathbf{X}$-approximation of C. Since X is in $\mathbf{X}$, we have by theorem 2.3 that $\mathrm{Hom}_{\mathbf{C}}(X, X_C) \to \mathrm{Hom}_{\mathbf{C}}(X, C)$ is surjective. Therefore we obtain a commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y_C \arrow[r] & X_C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Since Y_C is in $\hat{\omega}$, we know there is an epimorphism $W \to Y_C$ with W in ω. Adding this epimorphism to the foregoing diagram, we obtain the following commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & V \arrow[r, equal] \arrow[d] & V \arrow[d] & & \\
0 \arrow[r] & K\oplus W \arrow[r] \arrow[d] & X\oplus W \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y_C \arrow[r] \arrow[d] & X_C \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where $K\oplus W \to X\oplus W$ is the sum of $K \to X$ and the identity on W. Since W and X are both in $\mathbf{X}$, we have that V is in $\mathbf{X}$, as any epimorphism of $\mathbf{C}$ in $\mathbf{X}$ is admissible by assumption. Therefore $K\oplus W$ is in $\hat{\mathbf{X}}$, since Y_C and V are in $\hat{\mathbf{X}}$ and $\hat{\mathbf{X}}$ is closed under extensions. We now show that this implies that K is in $\hat{\mathbf{X}}$. Since $K\oplus W$ is in $\hat{\mathbf{X}}$, we obtain the following exact commutative diagram from an $\mathbf{X}$-approximation of $K\oplus W$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & Y_{K\oplus W} \arrow[r] \arrow[d] & Z \arrow[d] & \\
 & & X_{K\oplus W} \arrow[r, equal] \arrow[d] & X_{K\oplus W} \arrow[d] & \\
0 \arrow[r] & W \arrow[r] & K\oplus W \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Hence we have the exact sequence $0 \to Y_{K\oplus W} \to Z \to W \to 0$. Since W and $Y_{K\oplus W}$ are in $\hat{\mathbf{X}}$, (in fact already in $\hat{\omega}$), and $\hat{\mathbf{X}}$ is closed under extensions, we have that Z is in $\hat{\mathbf{X}}$ as well. This implies that K is in $\hat{\mathbf{X}}$ since $X_{K\oplus W}$ is in $\mathbf{X}$. This completes the proof of the lemma. ∎

We now apply the foregoing lemma to prove

Proposition 3.3. Suppose C is an object in $\hat{\mathbf{X}}$. Then the following are equivalent for any integer $n \geq 0$:

(a) $\mathbf{X}$–resol.dim$C \leq n$,

(b) If $0 \to U \to X_{n-1} \to \dots \to X_0 \to C \to 0$ is exact with X_i in $\mathbf{X}$ for $i = 0,\dots,n-1$, then U is in $\mathbf{X}$.

Proof: For $n = 0$ there is nothing to prove. So suppose $n > 0$. Assuming (a), repeated application of lemma 3.2 shows that U is in $\hat{\mathbf{X}}$. Also we have $\mathrm{Ext}^1_{\mathbf{C}}(U,W) = \mathrm{Ext}^{n+1}_{\mathbf{C}}(C,W) = 0$ for all W in ω since $\mathbf{X}$–inj.dim$\,\omega = 0$ and $\mathbf{X}$–resol.dim$C \leq n$. Therefore by proposition 2.1, it follows that U is in $\mathbf{X}$ proving that (a) implies (b). As (a) follows from (b) by definition of $\mathbf{X}$–resol.dimC, we are done. ■

As a first application of this proposition we prove the following

Proposition 3.4. $\hat{\mathbf{X}}$ is an additively closed subcategory of $\mathbf{C}$, that is $\hat{\mathbf{X}} = \mathbf{add}\, \hat{\mathbf{X}}$.

Proof: Suppose $C_1 \oplus C_2$ is in $\hat{\mathbf{X}}$ for two objects C_1 and C_2 in $\mathbf{C}$. Proceed by induction on $n = \mathbf{X}$–resol.dim $(C_1 \oplus C_2)$. If $n = 0$, the summands C_1 and C_2 are in $\mathbf{X}$ as $\mathbf{X}$ is an additively closed subcategory of $\mathbf{C}$. Suppose $n > 0$. Since $C_1 \oplus C_2$ is in $\hat{\mathbf{X}}$, there is an epimorphism $X \to C_1 \oplus C_2 \to 0$ with X in $\mathbf{X}$. Therefore we obtain exact sequences $0 \to L_i \to X \to C_i \to 0$ for $i = 1,2$ which yield the exact sequence $0 \to L_1 \oplus L_2 \to X \oplus X \to C_1 \oplus C_2 \to 0$. Now by Lemma 3.2, we know that $L_1 \oplus L_2$ is in $\hat{\mathbf{X}}$ and proposition 3.3 shows that $\mathbf{X}$–resol.dim$(L_1 \oplus L_2) \leq n-1$. By the inductive hypothesis L_1 and L_2 are in $\hat{\mathbf{X}}$ and another application of lemma 3.2 shows that then also C_1 and C_2 are in $\hat{\mathbf{X}}$. ■

We are now in position to establish one of the results promised in the beginning of this section.

Proposition 3.5. An exact sequence $0 \to A \to B \to C \to 0$ from $\mathbf{C}$ is in $\hat{\mathbf{X}}$ if any two of A, B and C are in $\hat{\mathbf{X}}$.

Proof: Since we already know that $\hat{\mathbf{X}}$ is closed under extensions by lemma 3.1, it suffices to show that if B is in $\hat{\mathbf{X}}$ then A is in $\hat{\mathbf{X}}$ if and only if C is in $\hat{\mathbf{X}}$. We first show that if A and B are in $\hat{\mathbf{X}}$ then C is in $\hat{\mathbf{X}}$. Choose an $\mathbf{X}$-approximation $0 \to Y_B \to X_B \to B \to 0$ for B. It gives rise to an exact commutative diagram

(*)
$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & Y_B & \longrightarrow & L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X_B & = & X_B & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

from which we get an exact sequence $0 \to Y_B \to L \to A \to 0$. It follows that L is in $\hat{\mathbf{X}}$ since Y_B and A are in $\hat{\mathbf{X}}$ and $\hat{\mathbf{X}}$ is closed under extensions. Therefore C is in $\hat{\mathbf{X}}$ since X_B is in $\mathbf{X}$.

Suppose now that B and C are in $\hat{\mathbf{X}}$. Using lemma 3.2, the exact sequence $0 \to L \to X_B \to C \to 0$ from (*) shows that L is in $\hat{\mathbf{X}}$. Applying the just established result to the exact sequence $0 \to Y_B \to L \to A \to 0$, it follows that A is in $\hat{\mathbf{X}}$. This completes the proof of the proposition. ∎

We now turn our attention to $\hat{\omega}$. We begin with the characterization of $\hat{\omega}$ as a subcategory of $\hat{\mathbf{X}}$, proving that $\hat{\omega} = \mathbf{X}^{\perp} \cap \hat{\mathbf{X}}$ in $\mathbf{C}$.

Proposition 3.6. The following statements are equivalent for an object C in $\hat{\mathbf{X}}$.

(a) C is in $\hat{\omega}$.

(b) $\mathbf{X}$-inj.dim $C = 0$, that is: C is in $\mathbf{X}^{\perp} \cap \hat{\mathbf{X}}$.

(c) If $0 \to Y_C \to X_C \to C \to 0$ is any $\mathbf{X}$-approximation of C, then X_C is in ω.

Proof: That (a) implies (b) was seen in corollary 2.2, and it is obvious that (c) implies (a). Hence we only need to show that (b) implies (c).
Since $\mathbf{X}$-inj.dim $C = 0 = \mathbf{X}$-inj.dim Y_C, it follows that $\mathbf{X}$-inj.dim $X_C = 0$. Our desired result is therefore a trivial consequence of the following, which proves $\omega = \mathbf{X} \cap \hat{\omega}$.

Lemma 3.7. The following are equivalent for an object X in $\mathbf{X}$.

(a) X is in ω.

(b) X is in $\hat{\omega}$.

(c) $\mathbf{X}$-inj.dim $X = 0$.

Proof: Again it is obvious that (a) implies (b), and Corollary 2.2 shows that (b) implies (c). It remains to prove
(c) $\Rightarrow$ (a): Let $0 \to X \to W \to X' \to 0$ be an exact sequence in $\mathbf{X}$ with W in ω which exists as ω is a cogenerator for $\mathbf{X}$. Then by (c) this sequence splits. Therefore X is a direct summand of W which implies that X is in ω as that category is assumed to be additively closed.
This establishes lemma 3.7 and finishes the proof of proposition 3.6. ∎

These results prove the following fact, already announced in the introduction to this section.

Proposition 3.8. $\hat{\omega}$ is an additively closed and exact subcategory of $\mathbf{C}$ in which any monomorphism is admissible. In more detail, let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathbf{C}$. Then

(a) B is in $\hat{\omega}$ if A and C are in $\hat{\omega}$, and

(b) C is in $\hat{\omega}$ if A and B are in $\hat{\omega}$.

Proof: We know by now that $\hat{\omega} = \mathbf{X}^{\perp} \cap \hat{\mathbf{X}}$ in $\mathbf{C}$. But the statements hold for $\hat{\mathbf{X}}$ by propositions 3.4 and 3.5, and as $\mathbf{X}^{\perp}$ is a right orthogonal complement, it also is an additively closed and exact subcategory of $\mathbf{C}$ in which every monomorphism is admissible. As all these properties are stable under intersection in $\mathbf{C}$, the result for $\hat{\omega}$ follows. ■

We sum up the foregoing results as

Theorem 3.9. Let $\mathbf{X}$ be an additively closed and exact subcategory of an abelian category $\mathbf{C}$. Assume that

(i) all epimorphisms from $\mathbf{C}$ in $\mathbf{X}$ are admissible, and

(ii) $\mathbf{X}$ has enough relatively injective objects.

Let ω be an injective cogenerator for $\mathbf{X}$. Then there results a diagram of additively closed and exact subcategories of $\mathbf{C}$

$$\begin{tikzcd} \mathbf{X} \arrow[r] & \hat{\mathbf{X}} \arrow[r] & \mathbf{C} \\ \omega \arrow[r] \arrow[u] & \hat{\omega} \arrow[r] \arrow[u] & \mathbf{X}^{\perp} \arrow[u] \end{tikzcd}$$

such that

(a) each square is *cartesian*, i.e.: $\hat{\omega} = \hat{\mathbf{X}} \cap \mathbf{X}^{\perp}$ and $\omega = \mathbf{X} \cap \mathbf{X}^{\perp}$,

(b) in $\hat{\mathbf{X}}$ all *mono- or epimorphisms* from $\mathbf{C}$ *are admissible*,

(c) in $\mathbf{X}^{\perp}$ and $\hat{\omega}$ *all monomorphisms* from $\mathbf{C}$ *are admissible.*

In particular, there is a *unique* injective cogenerator ω for $\mathbf{X}$ in $\mathbf{C}$, namely $\omega = \mathbf{X} \cap \mathbf{X}^{\perp}$. ■

To reformulate it once again modulo $\omega = \mathbf{X} \cap \mathbf{X}^{\perp}$, let us say that a sequence $0 \to \mathbf{A} \to \mathbf{B} \xrightarrow{p} \mathbf{C} \to 0$ of additive functors between additive categories is *exact* if and only if $\mathbf{A}$ is a full, essential and additively closed subcategory of $\mathbf{B}$ and p is equivalent to the projection functor $\pi : \mathbf{B} \to \mathbf{B}/\mathbf{A}$.

With the notations of theorem 2.8 we have then the following

Corollary 3.10. The adjoint pairs of functors (i^*, i_*) and $(j_!, j^!)$ fit into the commutative diagram of exact sequences of additive categories

$$\begin{tikzcd}
0 \arrow[r] & \hat{\omega}/\omega \arrow[r, "i_*"] \arrow[d, equal] & \hat{\mathbf{X}}/\omega \arrow[r, "j^!"] \arrow[d, equal] & \mathbf{X}/\omega \arrow[r] & 0 \\
0 & \hat{\omega}/\omega \arrow[l] & \hat{\mathbf{X}}/\omega \arrow[l, "i^*"'] & \mathbf{X}/\omega \arrow[l, "j_!"'] \arrow[u, equal] & 0 \arrow[l]
\end{tikzcd}$$

§4. *The category $\hat{\omega}$.*

Our aim in this section is to describe under which assumptions $\hat{\omega}$ has the further property that an exact sequence $0 \to A \to B \to C \to 0$ is in $\hat{\omega}$ if B and C are in $\hat{\omega}$.

To investigate this problem, we first define $\tilde{\omega}$ to be the subcategory of $\mathbf{C}$ consisting of all objects C in $\mathbf{C}$ which appear in an exact sequence $0 \to C \to Y_0 \to \dots \to Y_n \to 0$ with each Y_i in $\hat{\omega}$. Clearly such an object C is in $\hat{\mathbf{X}}$ since the Y_i are in $\hat{\omega} \subseteq \hat{\mathbf{X}}$ and the kernel of an epimorphism in $\hat{\mathbf{X}}$ is again in $\hat{\mathbf{X}}$ by proposition 3.5. Also it is obvious that $\hat{\omega}$ is a subcategory of $\tilde{\omega}$.

Lemma 4.1. The following statements are equivalent:

(a) An exact sequence $0 \to A \to B \to C \to 0$ is in $\hat{\omega}$ if B and C are in $\hat{\omega}$.

(b) $\hat{\omega} = \tilde{\omega}$.

The proof is trivial. ∎

This simple observation explains the relevance of the category $\tilde{\omega}$ to our problem about $\hat{\omega}$. The following description of $\tilde{\omega}$ is basic to the results in this paragraph.

Proposition 4.2. For an object C in $\hat{\mathbf{X}}$ the following are equivalent:

(a) C is in $\tilde{\omega}$,

(b) $\mathbf{X}$-inj.dim $C < \infty$.

Proof: (a) $\Rightarrow$ (b). Let $0 \to C \to Y_0 \to \dots \to Y_n \to 0$ be exact with each Y_i in $\hat{\omega}$. Since $\mathbf{X}$-inj.dim $\hat{\omega} = 0$, it follows by induction on n that $\mathbf{X}$-inj.dim $C \leq n < \infty$.

(b) $\Rightarrow$ (a): Since C is in $\hat{\mathbf{X}}$, it admits an $\hat{\omega}$-hull $0 \to C \to Y^C \to X^C \to 0$. The assumption that $\mathbf{X}$-inj.dim $C < \infty$ and the fact that $\mathbf{X}$-inj.dim $Y^C = 0$ imply that $\mathbf{X}$-inj.dim $X^C < \infty$. Therefore if we show that an object X from $\mathbf{X}$ which satisfies $\mathbf{X}$-inj.dim $X < \infty$ is necessarily in $\tilde{\omega}$, we will be done. Indeed we have the following more specific result.

Lemma 4.3. Let X be an object in $\mathbf{X}$, n a nonnegative integer. Then $\mathbf{X}$-inj.dim $X \leq n$ if and only if there is an exact sequence $0 \to X \to W_0 \to W_1 \to \dots \to W_n \to 0$ with W_i in ω for $i = 0, \dots, n$.

Proof: The if-part follows as before from **X**-inj.dim $\omega = 0$. Hence suppose that **X**-inj.dim $X = n$. Since ω is a cogenerator for **X**, we can construct an exact sequence $0 \to X \to W_0 \to \dots \to W_{n-1} \to X' \to 0$ in **X** such that each W_i is in ω for $i = 0, \dots, n-1$. As **X**-inj.dim $\omega = 0$ and **X**-inj.dim $X \leq n$ by assumption, it follows that for any integer $i > 0$ and all objects Z in **X** one has $\mathrm{Ext}^i_{\mathbf{C}}(Z, X') \cong \mathrm{Ext}^{n+i}_{\mathbf{C}}(Z, X) = 0$. But by lemma 3.7 this shows that X' is already in ω as desired.
This concludes the proof of lemma 4.3 and proposition 4.2. ∎

As a first application we get the following.

Corollary 4.4. $\check{\omega}$ is an additively closed subcategory of **C** with the property that an exact sequence $0 \to A \to B \to C \to 0$ is in $\check{\omega}$ if any two of A, B and C are in $\check{\omega}$.

Proof: Since $\hat{\mathbf{X}}$ is additively closed, it contains with $\check{\omega}$ also **add** $\check{\omega}$. It then follows from proposition 4.2 that $\check{\omega} = \mathbf{add}\ \check{\omega}$. Also if $0 \to A \to B \to C \to 0$ is an exact sequence in **C** with two of A, B and C in $\check{\omega}$, then all of A, B and C are in $\hat{\mathbf{X}}$ by Proposition 3.5. It then follows from proposition 4.2 that they are all in $\check{\omega}$. ∎

As another immediate consequence of proposition 4.2 we get the following.

Corollary 4.5. The following are equivalent:

(a) $\hat{\omega} = \check{\omega}$,

(b) If C is an object in $\hat{\mathbf{X}}$ with **X**-inj.dim $C < \infty$, then **X**-inj.dim $C = 0$.

Proof: Let C be in $\hat{\mathbf{X}}$. By proposition 4.2 we have that C is in $\check{\omega}$ if and only if **X**-inj.dim $C < \infty$. By Proposition 3.6 we have that C is in $\hat{\omega}$ if and only if **X**-inj.dim $C = 0$.
Hence the equivalence of (a) and (b). ∎

We now give criteria for the property $\hat{\omega} = \check{\omega}$ in terms of the categories ω and **X** themselves.

Proposition 4.6. The following are equivalent:

(a) $\hat{\omega} = \check{\omega}$.

(b) If $0 \to C \to W_0 \to W_1 \to 0$ is exact in **C** with W_0 and W_1 in ω, then C is in ω.

(c) If $0 \to C \to W_0 \to W_1 \to \dots \to W_n \to 0$ is exact with each W_i in ω for $i = 0, \dots, n$, then C is in ω.

(d) If X is in **X** and **X**-inj.dim $X < \infty$, then X is in ω.

Proof: (a) $\Rightarrow$ (b). Since W_0 and W_1 are objects in ω, they are in **X**, so C is in **X**. Clearly C is in $\check{\omega}$ which means by the assumption that it is in $\hat{\omega}$. Therefore C is in $\mathbf{X} \cap \hat{\omega}$ which category equals ω by Lemma 3.7.
(b) $\Rightarrow$ (c) by induction on n.
(c) $\Rightarrow$ (d). Suppose X is in **X** with **X**-inj.dim $X < \infty$. Then by Lemma 4.3, we know there

is an exact sequence $0 \to X \to W_0 \to \dots \to W_n \to 0$ with W_i in ω for $i = 0,\dots,n$. Therefore by (c) we have that X is in ω.
(d) $\Rightarrow$ (a). Let C be an object in $\check{\omega}$. Then we can choose an $\mathbf{X}$-approximation $0 \to Y_C \to X_C \to C \to 0$ for C and proposition 4.2 shows that $\mathbf{X}$-inj.dim $C < \infty$. But $\mathbf{X}$-inj.dim $Y_C = 0$ so that $\mathbf{X}$-inj.dim $X_C < \infty$. Therefore X_C is in ω by the hypothesis (d), which shows that C is in $\hat{\omega}$. ■

Now we establish the following.

Proposition 4.7. Let C be an object in $\check{\omega}$. Then ω-inj.dim C = $\mathbf{X}$-inj.dim C.

Proof: Since ω is a subcategory of $\mathbf{X}$, we have that always ω-inj.dim C $\leq$ $\mathbf{X}$-inj.dim C . So it suffices to show that ω-inj.dim C $\geq$ $\mathbf{X}$-inj.dim C. As C is in $\check{\omega}$ by assumption, we also know from proposition 4.2 that $\mathbf{X}$-inj.dim C is finite.

To begin with, we prove the proposition when C = X is an object in $\mathbf{X} \cap \check{\omega}$. By lemma 4.3, we have that then there is an exact sequence $0 \to X \to W_0 \to \dots \to W_n \to 0$ with each W_i in ω for $i = 0,\dots,n$. Assume that ω-inj.dim X = 0. Since ω-inj.dim ω = 0, it follows by induction on n that the exact sequence $0 \to X \to W_0 \to \dots \to W_n \to 0$ splits. Hence $\mathbf{X}$ is already in ω which implies $\mathbf{X}$-inj.dim X = 0. This result shows furthermore that ω-inj.dim X $\leq$ n if and only if there is an exact sequence $0 \to X \to W_0 \to \dots \to W_n \to 0$ with each W_i in ω. But we have seen in lemma 4.3 that the existence of such an exact sequence is equivalent to $\mathbf{X}$-inj.dim X $\leq$ n. Hence we have shown that ω-inj.dim C = $\mathbf{X}$-inj.dim C when C is in $\mathbf{X} \cap \check{\omega}$.

Assume now that C is an arbitrary object in $\check{\omega}$. Let $0 \to C \to Y^C \to X^C \to 0$ be an $\hat{\omega}$-hull of C. Since C and Y^C are in $\check{\omega}$ by assumption, we get that X^C is in $\check{\omega}$ by corollary 4.4. Suppose now ω-inj.dim C = 0. Then also ω-inj.dim X^C = 0 which implies that X^C is in ω by our previous result. But our current hypothesis then implies that $\mathrm{Ext}^1_{\mathbf{C}}(X^C, C) = 0$, which means that the chosen $\hat{\omega}$-hull of C splits. So C is a direct summand of Y^C in $\hat{\omega}$ and is hence itself in $\hat{\omega}$, as $\hat{\omega}$ is additively closed by proposition 3.8. Therefore $\mathbf{X}$-inj.dim C = 0 by corollary 2.2 and we are done in this case.

Finally suppose ω-inj.dim C = n > 0. Since ω-inj.dim Y^C = 0, it follows that ω-inj.dim X^C = n−1. Therefore $\mathbf{X}$-inj.dim X^C = n−1 by our first result, which implies that $\mathbf{X}$-inj.dim C $\leq$ n. Hence ω-inj.dim C $\geq$ $\mathbf{X}$-inj.dim C for all C in $\check{\omega}$, which completes the proof of the proposition. ■

The following is an immediate consequence of our earlier results and summarizes sufficient conditions for $\hat{\omega} = \check{\omega}$ to hold.

Corollary 4.8. Consider the following conditions:
(a) ω-inj.dim $\mathbf{X}$ = 0,
(b) ω-inj.dim $\hat{\mathbf{X}}$ = 0,
(c) ω-inj.dim $\check{\omega}$ = 0,
(d) Every epimorphism $W' \to W \to 0$ in $\mathbf{C}$ with W and W' in ω admits a section.
(e) $\hat{\omega} = \check{\omega}$.

Then one has (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e).

Proof. As **X** and $\check{\omega}$ are subcategories of $\hat{\mathbf{X}}$, the implications (b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c) are trivial. That (a) $\Rightarrow$ (b) follows from the existence of an **X**-approximation $0 \to Y_C \to X_C \to C \to 0$ for any object C in $\hat{\mathbf{X}}$ and the fact that ω-inj.dim $\hat{\omega}$ = **X**-inj.dim $\hat{\omega} = 0$. That (c) $\Rightarrow$ (d) follows from the fact that the kernel K of any epimorphism $W' \to W \to 0$ between objects from ω is by definition an object in $\check{\omega}$. But (c) implies $\mathrm{Ext}^1_{\mathbf{C}}(W, K) = 0$, whence the exact sequence $0 \to K \to W' \to W \to 0$ splits. The remaining implication (d) $\Rightarrow$ (e) is a special case of proposition 4.2. ∎

Example 6. A special case in which $\hat{\omega} = \check{\omega}$ holds, has been investigated already by A.Heller [He]. Following him let us say that **X** in **C** is a *Frobenius category* if it satisfies our usual assumptions of being additively closed and exact in **C** with every epimorphism from **C** in **X** being admissible and if furthermore $\omega = \mathbf{X} \cap \mathbf{X}^{\perp}$ is also a *projective generator* for **X**, which is equivalent to ω^{op} being an injective cogenerator of $\mathbf{X}^{op}$. This means hence that

(i) **X**-inj.dim ω = **X**-proj.dim ω = 0 and

(ii) for every object X in **X** there exists both a monomorphism $i : X \to W$ as well as an epimorphism $p : W' \to X$ with W, W' in ω such that the objects Ker p and Cok i, calculated in **C**, are again objects in **X**.

A.Heller himself gave already some examples of such categories and further such categories are discussed in [Ha]. Also, it is clear from the definitions that in Example 3 the category **X** of maximal Cohen-Macaulay R-modules is Frobenius.

§5. *Some remarks on the* **X**-*resolution dimension of* $\hat{\mathbf{X}}$.

We define **X**-resol.dim $\hat{\mathbf{X}}$ to be the maximum (including ∞) of **X**-resol.dimC for all objects C in $\hat{\mathbf{X}}$. This paragraph is devoted to interpreting some of our previous results when **X**-resol.dim $\hat{\mathbf{X}}$ is *finite*. So for the remainder of this section we assume **X**-resol.dim $\hat{\mathbf{X}} = d < \infty$.

Our remarks are based on the following observation.

Lemma 5.1. The following statements are equivalent for an object C in **C**.

(a) **X**-inj.dimC $< \infty$,

(b) $\hat{\mathbf{X}}$-inj.dimC $< \infty$.

Moreover, if **X**-inj.dimC $= m < \infty$, then $\hat{\mathbf{X}}$–inj.dimC $\leq d+m$.

Proof: Usual dimension shift argument. ∎

This lemma implies immediately the following.

Proposition 5.2. Suppose C is an object in $\hat{\mathbf{X}}$.
(a) $\hat{\mathbf{X}}$-inj.dimC $< \infty$ if and only if C is in $\check{\omega}$.
(b) If C is an object in $\hat{\omega}$, then $\hat{\mathbf{X}}$-inj.dimC $\leq$ d .

Proof: (a): By lemma 5.1, we know that $\hat{\mathbf{X}}$-inj.dimC $< \infty$ if and only if $\mathbf{X}$-inj.dimC $< \infty$. But by proposition 4.2, we know that $\mathbf{X}$-inj.dimC $< \infty$ if and only if C is in $\check{\omega}$.
(b): Since $\mathbf{X}$-inj.dim $\hat{\omega} = 0$ by corollary 2.2, the result follows from lemma 5.1. ∎

As a special case we obtain the following consequence.

Corollary 5.3. If $\hat{\mathbf{X}} = \mathbf{C}$, then we have:
(a) C is in $\check{\omega}$ if and only if inj.dimC $< \infty$.
(b) If C is in $\hat{\omega}$, then inj.dimC $\leq$ d. ∎

Remark that the injective dimension of an object C in $\mathbf{C}$ is defined here in terms of vanishing of the functors $\mathrm{Ext}^*_{\mathbf{C}}(-,C)$. As soon as $\mathbf{C}$ itself has enough injective objects it coincides with the notion obtained from the length of a shortest injective resolution.

Applying the foregoing result to our decomposition into $\mathbf{X}$-approximations and $\hat{\omega}$-hulls we have the following.

Corollary 5.4. Suppose again $\hat{\mathbf{X}} = \mathbf{C}$ and let $0 \to Y_C \to X_C \to C \to 0$ and $0 \to C \to Y^C \to X^C \to 0$ be an $\mathbf{X}$-approximation and an $\hat{\omega}$-hull of an object C in $\mathbf{C}$ respectively. Then inj.dim $Y_C <$ inj.dim $Y^C \leq$ d or both Y_C and Y^C are already injective. ∎

Finally, consider the case where $\hat{\omega} = \check{\omega}$. Then we have first the following consequence of lemma 5.1.

Proposition 5.5. Suppose $\hat{\omega} = \check{\omega}$. Then the following statements are equivalent for an object C in $\hat{X}$.
(a) C is in $\hat{\omega}$.
(b) $\hat{\mathbf{X}}$-inj.dim C $\leq$ d.
(c) $\hat{\mathbf{X}}$-inj.dim C $< \infty$.

Proof: (a) $\Rightarrow$ (b) by proposition 5.2.
(b) $\Rightarrow$ (c) is trivial.
(c) $\Rightarrow$ (a): Since $\hat{\mathbf{X}}$-inj.dim C $< \infty$, we have that $\mathbf{X}$-inj.dim C $< \infty$.
Therefore C is in $\check{\omega}$ by proposition 4.2. Hence C is in $\hat{\omega}$ since $\check{\omega} = \hat{\omega}$ by assumption. ∎

As an obvious consequence of this proposition we have the following.

Corollary 5.6. Suppose $\hat{\mathbf{X}} = \mathbf{C}$ and $\check{\omega} = \hat{\omega}$. Then the following conditions are equivalent for an object C in **C**.

(a) C is in $\hat{\omega}$,

(b) inj.dim $C \leq d$,

(c) inj.dim $C < \infty$. ∎

§6. *More Examples*

In this section we describe various situations where the theory we have developed is applicable.

First we consider a generalization of Example 4.

Example 7. Let R be a commutative noetherian Gorenstein ring of finite dimension d. Let Λ be an R-algebra, not necessarily commutative, which is a *maximal Cohen-Macaulay R-module.* Set $\mathbf{C} = \mathbf{mod}\text{-}\Lambda$, the category of finitely generated right Λ-modules, and let **X** be the full subcategory of **C** whose objects are the Λ-modules which are MCM if considered as R-modules. Then **X** is again additively closed, exact and has all its epimorphisms admissible. Also we have that $\hat{\mathbf{X}} = \mathbf{mod}\text{-}\Lambda$ and that **X**-resol.dim $\hat{\mathbf{X}} = d < \infty$.

As in Example 4, we let ω consist of all Λ-modules isomorphic to $\mathrm{Hom}_R(P, R)$ for some finitely generated projective Λ^{op}–module P. Again, ω is just the additive closure of $\omega_{\Lambda/R} = \mathrm{Hom}_R(\Lambda, R)$, and it is an injective cogenerator for **X**.

Applying the results in section 5, we have the following.

Proposition 6.1. Let C be in $\mathbf{mod}\text{-}\Lambda$.

(a) inj.dim $C < \infty$ if and only if C is in $\check{\omega}$.

(b) If C is in $\hat{\omega}$ then inj.dim $C \leq d$.

Proof: See Corollary 5.3. ∎

As a consequence of this we obtain hence the following.

Corollary 6.2. Let C be in $\mathbf{mod}\text{-}\Lambda$. Then $0 \to Y_C \to X_C \to C \to 0$, the **X**–approximation of C, and $0 \to C \to Y^C \to X^C \to 0$, the $\hat{\omega}$-hull of C, have the property that X_C and X^C are maximal Cohen-Macaulay R-modules and that inj.dim $Y_C \leq d-1$ and inj.dim $Y^C \leq d$. ∎

We now turn our attention to the question of when $\hat{\omega} = \check{\omega}$ in this context.

Proposition 6.3. The following statements are equivalent for Λ.

(a) $\hat{\omega} = \check{\omega}$.

(b) If X is a Λ^{op}–module which is MCM as an R-module and such that $\mathrm{proj.dim}_{\Lambda^{op}} X < \infty$, then X is a projective Λ^{op}-module.

Proof: We know by Proposition 4.6 that $\tilde{\omega} = \hat{\omega}$ if and only if an exact sequence $0 \to C \to W_0 \to \dots \to W_n \to 0$ is in ω as soon as each W_i is in ω for $i = 0,\dots n$.
(a) $\Rightarrow$ (b): Suppose $0 \to P_m \to \dots \to P_0 \to X \to 0$ is a projective resolution for a given Λ^{op}-module X which is MCM over R. Then

$$0 \to \mathrm{Hom}_R(X,R) \to \mathrm{Hom}_R(P_0,R) \to \dots \to \mathrm{Hom}_R(P_m,R) \to 0$$

is exact in **mod**-Λ with each $\mathrm{Hom}_R(P_i,R)$ in ω for $i = 0,\dots,m$. By (a) we have that $\mathrm{Hom}_R(X,R)$ is necessarily in ω. Therefore $\mathrm{Hom}_R(X,R) \cong \mathrm{Hom}_R(P,R)$ for some projective Λ^{op}-module P, which then yields $X \cong P$.
(b) $\Rightarrow$ (a): Suppose that X in **mod**-Λ is MCM over R and that $0 \to X \to W_0 \to \dots \to W_m \to 0$ is an exact sequence with W_i in ω for all $i = 0,\dots,m$. Then $0 \to \mathrm{Hom}_R(W_m,R) \to \dots \to \mathrm{Hom}_R(W_0,R) \to \mathrm{Hom}_R(X,R) \to 0$ is exact and $\mathrm{Hom}_R(W_i,R)$ is a projective Λ^{op}-module for each i. The Λ^{op}-module $\mathrm{Hom}_R(X,R)$ is still MCM as an R-module and hence $\mathrm{Hom}_R(X,R) \cong P$ for some projective Λ^{op}-module by our assumption. As MCM's are reflexive, $X \cong \mathrm{Hom}_R(P,R)$ and X is therefore in ω. ∎

This proposition gives the following generalization of a result of R. Sharp [Sh].

Corollary 6.4. Suppose Λ is a commutative ring. Then the following are equivalent for a finitely generated Λ-module M.

(a) $\mathrm{inj.dim}_\Lambda\ M < \infty$.

(b) There is an exact sequence $0 \to W_m \to \dots \to W_0 \to M \to 0$ with W_i in ω for all $i = 0,\dots,m$.

Proof: The equivalence of (a) and (b) is nothing more than the statement that $\tilde{\omega} = \hat{\omega}$. But this follows from Proposition 6.3 since it is well-known for commutative rings, that a maximal Cohen-Macaulay module of finite projective dimension is projective. ∎

However, if Λ is not commutative, it is not necessarily true that a Λ^{op}-module which is MCM over R and of finite projective dimension over Λ^{op} is necessarily projective. For example, let R be a regular local ring of dimension $d > 0$ and let Λ be the algebra of lower triangular $n\times n$ matrices over R with $n \geq 2$. Then Λ is a free and finitely generated R-module and $\mathrm{gl.dim}\ \Lambda^{op} = d+1$. Let

$$0 \to P_{d+1} \to P_d \to \dots \to P_0 \to M \to 0$$

be a projective Λ^{op}-resolution of a Λ^{op}-module M with $\mathrm{proj.dim}_{\Lambda^{op}}\ M = d+1$. Then $\mathrm{Im}(P_d \to P_{d-1})$ is an MCM over R which is of projective dimension one over Λ^{op} and is hence not Λ^{op}-projective.

Example 8. Let k be a field and $P = k[x_0,\dots,x_n]$ a polynomial ring over k in n+1 variables which we grade by assigning arbitrary positive integral weights to the variables. Let I be a homogeneous ideal in P and set $S = P/I$ which is hence a positively graded k-algebra.

We assume that S is a Cohen-Macaulay ring. It is known then that there exists a sequence $y_1, \dots, y_m$ of homogeneous elements of strictly positive degrees in I which is a regular $k[x_0, \dots, x_n]$-sequence, and such that $R = P/(y_1, \dots, y_m)$ has the same dimension as S. As R is a complete intersection, it is a Gorenstein ring and by construction the natural surjection $R \to S \to 0$ is a degree-preserving homomorphism of rings. Let $\mathbf{C}$ = **S-grmod** be the category of finitely generated graded S-modules with degree zero graded maps as morphisms. Also let $\mathbf{X}$ be the subcategory of $\mathbf{C}$ consisting of all maximal Cohen-Macaulay modules. In addition to the usual properties, $\mathbf{X}$ also satisfies $\hat{\mathbf{X}} = \mathbf{C}$ and $\mathbf{X}$-resol.dim $\hat{\mathbf{X}} = n+1-m = d$, the dimension of S.

Set $\omega_{S/R} = \mathrm{Hom}_R(S,R)$, which is a dualizing module of S over R, and define ω to be the subcategory of $\mathbf{C}$ consisting of all $\omega_{S/R}(n)$ for n in $\mathbf{Z}$. Then ω is an injective cogenerator for $\mathbf{X}$. Moreover we know that X in $\mathbf{X}$ is of finite projective dimension if and only if it is isomorphic to a direct sum $\bigoplus_i S(a_i)$.

As in the previous example, this implies $\hat{\omega} = \check{\omega}$. We leave it to the reader to write down in detail what this means for $\mathbf{X}$-approximations, $\hat{\omega}$-hulls and modules of finite injective dimension.

We now give our last example.

Example 9. Let $\Lambda \to \Gamma$ be a ring homomorphism with Λ both left and right noetherian and Γ a finitely generated projective Λ-module on both left and right. Let $\mathbf{C}$ = Γ-**mod** be the category of all finitely generated left Γ-modules and let $\mathbf{X}$ consist of all M in $\mathbf{C}$ such that M is a projective Λ-module. In addition to the usual properties, we have that $\hat{\mathbf{X}}$ consists of all N in $\mathbf{C}$ such that $\mathrm{proj.dim}_\Lambda\ N < \infty$.

Define ω to be the category of all modules isomorphic to $\mathrm{Hom}_\Lambda(P,\Lambda)$ for some finitely generated projective Γ^{op}-module P. Then ω is an injective cogenerator for $\mathbf{X}$. In general $\hat{\omega} \neq \check{\omega}$, but if all the modules in ω are projective Γ-modules, then we do have $\hat{\omega} = \check{\omega}$ by Corollary 4.8 and in fact $\mathbf{X}$ becomes a Frobenius category, see Example 6.

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