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ABSTRACT : We define an invariant associated to a homomorphism of free modules and show, first, that this generalizes the multiplicity in the sense of Samuel and, second, that in the situation we are considering, the local Chern character of a perfect complex can be defined in terms of this invariant. Some questions are raised as to the positivity of these numbers and connections with mixed multiplicities are described.

One of the common methods in studying ideals and modules over a commutative ring has been to define numerical invariants which reflect their properties. In this paper we look at a few of these invariants, which have been defined in various contexts, and describe some relations between them. Let A be a local ring with maximal ideal $\mathfrak{m}$, and let I be an ideal of A primary to the maximal ideal, so that A/I is a module of finite length. This length is the simplest invariant associated to the ideal, and it could be considered to be the most important one, but Samuel [7] defined a somewhat more complicated one, called the multiplicity of I, and showed that it was often more fundamental in studying both algebraic and geometric questions; since then, of course, this has become a standard part of Commutative Algebra.

The comparison of invariants we discuss in this paper is analogous to the comparison of length and multiplicity of an $\mathfrak{m}$–primary ideal. Take now a bounded complex of free A–modules, which we denote F_*. In place of the assumption that I be primary, we assume that the homology of F_* is of finite length. Again, there are two invariants one can associate to F_*. The first is the Euler characteristic, denoted $\chi(F_*)$, which is the alternating sum of lengths of the homology modules. The second was defined by Baum, Fulton and MacPherson and is defined in terms of the local Chern character. This theory has been extended by Fulton [2], and certain applications have made it appear that here also this more complicated invariant may be more fundamental in studying homological questions in Commutative Algebra (see Roberts [5] [6]).

We give here an alternative construction of this invariant. More precisely, we define an invariant of a map of free modules (or of a matrix, if one chooses to look at it that way) with certain properties (corresponding to finite length, specified below). On the one hand, if this map goes to a rank one free module, the image is a primary ideal, and this is the multiplicity of Samuel. On the other hand, the alternating sum of these is the local Chern character in the second example. We define this, which we call the *multiplicity* of the homomorphism, in section 1, and, in the process, we show that the connection with multiplicities is more that simply an analogy, since the definition itself is in terms of the so–called *mixed multiplicities* of ideals of minors of the matrix. In section 2 we show that it does agree with the other invariants mentioned above. In the third section we consider homomorphisms which can be put into a complex of length equal to the dimension of the ring with homology of finite length and ask some questions concerning the properties of this invariant in that case. Finally, in the last section, we work out a couple of special cases to explain how one step of the construction works in practice.

We remark that one motivation behind this work was to investigate the contributions of the individual boundary homomorphisms of a perfect complex (i.e. a bounded complex of free modules) to the local Chern character. The fact that a complex can be divided up in this way was proven in a construction of Fulton ([2], ex. 18.3.12) to prove his local Riemann–Roch theorem. The construction we give here carries this out explicitly, specifies which locally free sheaves occur in the decomposition in terms of determinants, and gives a formula for each contribution in terms of mixed multiplicities. In addition, it is applied to an independent map of free modules, so that, in particular, it is defined whether the map fits into a perfect complex or not. What this number means when the map does not fit into a perfect complex is not clear, but it is interesting that an invariant like this can be defined in this generality.

1. The multiplicity of a homomorphism of free modules.

Let A be a local ring of dimension d and maximal ideal $\mathfrak{m}$, and let $\phi : E \longrightarrow F$ be a homomorphism of free A–modules. We wish to assume that ϕ is generically of constant rank, and, to simplify the situation here, we assume that A is an integral domain. Let r be the generic rank of ϕ. We define the support of ϕ to be the set of prime ideals of A for which the localization at P is not split of rank r, by which we mean that it is not of the form

$$A^s \otimes A^r \longrightarrow A^r \otimes A^t$$

where the map is $\begin{pmatrix} 0 & 1_{A^r} \\ 0 & 0 \end{pmatrix}$. Let e denote the rank of E and f the rank of F. We assume that the support of ϕ is the maximal ideal of A. We wish to define a number associated to ϕ which satisfy the properties outlined in the introduction.

Let M denote the matrix which defines ϕ. We assume that the bases are chosen so that both the first r rows and the first r columns of M have rank r.

We first define two sequences of ideals associated to the matrix M. We note that these are not canonically defined by the map itself, but depend on the bases chosen for E and F (or, more precisely, on filtrations by free direct summands defined by them). First, for $k = 0,1,...,r$ we let $\mathfrak{e}_k$ denote the ideal generated by the k by k minors of the first k rows of M (for $k = 0$ this is defined to be the unit ideal, i.e. A itself; we include this to avoid special cases in later notation). Next, for $k = 0,1,...,r$ we let $\mathfrak{f}_k$ denote the ideal generated by the r by r minors of the first r columns of M which include the first k rows. Note that these ideals are not necessarily $\mathfrak{m}$–primary. We also note that $\mathfrak{e}_r$ and $\mathfrak{f}_0$ are, respectively, the ideals generated by the r by r minors of the first r rows and the first r columns of M.

The invariant we define is in terms of mixed multiplicities, so we next recall some facts on mixed multiplicities of sets of ideals. These were introduced for two ideals by Bhattacharya [1] and later also by Teissier [8], and more recently the definition was extended to a set of d ideals, where d is the dimension of the ring by Rees (see [3]). We briefly recall the situation we need for our construction. This appears to by slightly different than that considered by Rees; he considered d ideals (not necessarily distinct) such that it is possible to choose one element from each of the ideals to form a system of parameters for the ring A. We require instead that at least one of the ideals be $\mathfrak{m}$–primary. So let $\mathfrak{a}_1,...,\mathfrak{a}_n$ be n ideals of A such that $\mathfrak{a}_1$ is $\mathfrak{m}$–primary. If all of the ideals were $\mathfrak{m}$–primary, there would be a polynomial P in n variables of degree d such that we would have

$$P(s_1,...,s_n) = \text{length}(A/\mathfrak{a}_1^{s_1}\mathfrak{a}_2^{s_2}...\mathfrak{a}_n^{s_n})$$

for large values of $s_1,...,s_n$. In our case these lengths are not finite, so this does not make sense. However, since $\mathfrak{a}_1$ is $\mathfrak{m}$–primary, there is still a polynomial P' in n variables of degree $d-1$ such that we have

$$P'(s_1,...,s_n) = \text{length}(\mathfrak{a}_1^{s_1}\mathfrak{a}_2^{s_2}...\mathfrak{a}_n^{s_n}/\mathfrak{a}_1^{s_1+1}\mathfrak{a}_2^{s_2}...\mathfrak{a}_2^{s_2}...\mathfrak{a}_n^{s_n})$$

for large values of $s_1,...,s_n$. In the case in which all ideals are $\mathfrak{m}$–primary, this is the difference $P(s_1+1,...,s_n) - P(s_1,...,s_n)$ and one can recover those coefficients of P which invove at least one factor of $\mathfrak{a}_1$. In our case, this gives a well–defined coefficient for each term of the polynomial for which at least one $\mathfrak{m}$–primary factor occurs. We summarize this in the following definition :

Definition. *Let* $\mathfrak{a}_1,\dots,\mathfrak{a}_n$ *be* n *ideals of* A *such that* $\mathfrak{a}_1,\dots,\mathfrak{a}_k$ *are* $\mathfrak{m}$*–primary. We call the mixed multiplicity polynomial of* $\mathfrak{a}_1,\dots,\mathfrak{a}_k$; $\mathfrak{a}_{k+1},\dots,\mathfrak{a}_n$ *the homogeneous polynomial* P *in* n *variables of degree* d *such that*

(1) *for* $i = 1,\dots,k$ *we have* $P(s_1,\dots,s_i + 1,\dots,s_n) - P(s_1,\dots,s_i,\dots,s_n) =$ *the part of degree* $d-1$ *of the polynomial which gives the length of* $\mathfrak{a}_1^{s_1}\dots\mathfrak{a}_i^{s_i}\dots\mathfrak{a}_n^{s_n}/\mathfrak{a}_1^{s_1}\dots\mathfrak{a}_i^{s_i+1}\dots\mathfrak{a}_n^{s_n}$. *For large* $s_1,\dots,s_n$,

(2) *all coefficients involving only the last* n - k *variables are zero.*

We make two remarks on this definition. First, it might seem reasonable to call it the Hilbert–Samuel polynomial in analogy with the case of one ideal; the terminology we have chosen is because we have taken only the part of degree d, and these coefficients are (up to certain multinomial coefficients) the mixed multiplicities of the ideals. The second is that the last condition, letting those coefficients which are not well–defined be zero, may seem arbitrary, but it turns out to be exactly what is needed in our formula.

We give an alternative description of the coefficients of the polynomial which will be useful later. We begin by taking the multigraded ring whose $s_1,\dots,s_n$ component is $\mathfrak{a}_1^{s_1}\mathfrak{a}_2^{s_2}\dots\mathfrak{a}_n^{s_n}$. In conformity with the usual terminology for one ideal, we call this the Rees ring associated to $\mathfrak{a}_1,\dots,\mathfrak{a}_n$. By taking the projective scheme associated to this, one gets a scheme X proper over Spec A with an imbedding into the product of projective space over Spec(A); this imbedding is defined by choosing a set of generators for each of the ideals. Finally, on X there are invertible sheaves of ideals $O(-A_1),\dots,O(-A_n)$ associated to divisors $A_1,\dots,A_n$ defined by the ideals $\mathfrak{a}_1,\dots,\mathfrak{a}_n$. The coefficients of the mixed multiplicity polynomial can then be defined as the degrees of the intersections of these divisors. More precisely, one has coefficient of

$$s_1^{k_1}s_2^{k_2}\dots s_n^{k_n} = (-1)^{d-1}\Big(\frac{1}{k_1!\dots k_n!}\Big)A_1^{k_1}A_2^{k_2},\dots,A_n^{k_n}.$$

In this intersection product one must first take the exceptional divisor corresponding to an ideal which is $\mathfrak{m}$–primary, which reduces the situation to a subscheme which lies over the closed point of Spec(A), and then intersect with the other divisors. In ring–theoretic terms, this can be done by first dividing the Rees ring by the image of one of the ideals which is $\mathfrak{m}$–primary, which reduces the situation to a multigraded ring over an Artinian ring, and then dividing by generic enough elements in appropriate graded pieces of the Rees ring (this works at least if the residue field of A is infinite). The sign occurs because every intersection after the first is with one of the hyperplanes coming from the embedding into a product of projective spaces, and this is the negative of the corresponding exceptional divisor. The mixed multiplicity polynomial can thus be expressed more simply as

$$(-1)^{d-1}(\frac{1}{d!})((A_1s_1 + A_2s_2 +...+ A_ns_n)^d - (A_{k+1}s_{k+1} +...+ A_ns_n)^d).$$

We remark that this expression is simpler, but that to actually compute the polynomial it is necessary to compute the individual mixed multiplicities. On the other hand, sometimes some of the divisors can be combined and this can be used to simplify the computations.

We now define the invariant of the homomorphism ϕ in terms of mixed multiplicities of the ideals $\mathfrak{e}_k$ and $\mathfrak{f}_k$ defined above plus some other ones defined in terms of these. Let k be an integer between 1 and r. We consider the four ideals $\mathfrak{e}_{k-1}, \mathfrak{e}_k, \mathfrak{f}_{k-1}$, and $\mathfrak{f}_k$. As described above, there is a Rees ring associated to these ideals, as well as an associated projective scheme X proper over $\mathrm{Spec}(A)$ with four divisors which we denote E_{k-1}, E_k, F_{k-1}, and F_k. Take the map :

$$O(E_{k-1} - E_k) \longrightarrow O(F_{k-1} - F_k)$$

defined locally as follows : the scheme X is covered by affine pieces corresponding to choices of one generator of each of the ideals $\mathfrak{e}_{k-1}, \mathfrak{e}_k, \mathfrak{f}_{k-1}$, and $\mathfrak{f}_k$. Choose four such generators to be the determinants $\Delta^E_{k-1}, \Delta^E_k, \Delta^F_{k-1}$, and Δ^F_k. The local expression for the map above is then multiplication by the element $\dfrac{\Delta^E_k\Delta^F_{k-1}}{\Delta^E_{k-1}\Delta^F_k}$.

LEMMA. *The element* $\dfrac{\Delta^E_k\Delta^F_{k-1}}{\Delta^E_{k-1}\Delta^F_k}$ *is in the coordinate ring defined by the generators* $\Delta^E_{k-1}, \Delta^E_k, \Delta^F_{k-1}$, *and* Δ^F_k *of* $\mathfrak{e}_{k-1}, \mathfrak{e}_k, \mathfrak{f}_{k-1}$, and $\mathfrak{f}_k$.

Proof : What must be shown is that the element in question can be written as a sum of quotients with denominators $\Delta^E_{k-1}\Delta^F_k$ and with numerators products of elements in the original ring times elements in $\mathfrak{e}_{k-1}$ and $\mathfrak{f}_k$. If the minor of M defining Δ^F_{k-1} happens to included the k^{th} row, this is easy to show by expanding Δ^E_k along the k^{th} row. If not, one must first expand Δ^E_k along the k^{th} row and then, for each element α of the k^{th} row of the minor defining Δ^E_k, add the corresponding row and column of this entry to the minor defining Δ^F_{k-1}, and, using the fact that this $r+1$ by $r+1$ determinant must be zero, expand it along the column of α to express it as a sum of other entries in that column multiplied by the corresponding cofactors. When this is all worked out, all terms drop out except those for minors including the first k rows, which are of the desired form.

We assume next that there are $\mathfrak{m}$–primary ideals $\mathfrak{g}_k$ and $\mathfrak{g}_{k-1}$ such that, with notation as above, we have

$$O(E_{k-1} - E_k) \longrightarrow O(F_{k-1} - F_k) \cong O(E_{k-1} - E_k) \otimes (O \longrightarrow O(G_k - G_{k-1})).$$

In many cases these ideals can be calculated explicitly – we give some examples below to show how this works out in practice. We now let P_k be the mixed multiplicity polynomial associated to $\mathfrak{g}_k, \mathfrak{g}_{k-1}; \mathfrak{e}_k, \mathfrak{e}_{k-1}$ (we note that the first two of these are $\mathfrak{m}$–primary but the last two might not be). Then our formula is :

$$m(\phi) = (d!) \sum_{k=1}^{r} P_k(1,-1,1,-1).$$

Actual computation of this number is fairly complicated, but we give some examples later to show that it can be done. We note also that using the last form of the mixed multiplicity polynomial this becomes

$$((G_k - G_{k-1} + E_{k-1} - E_k)^d - (E_{k-1} - E_k)^d).$$

2. Relationships with other invariants.

First, to justify the term multiplicity given to this number, we must show that it agrees with the definition of multiplicity of an ideal. We first do a more general case where the formula simplifies considerably; this is the case of a homomorphism of maximal rank. Recall that $\phi : E \longrightarrow F$ is a homomorphism of free A–modules of ranks e and f respectively. We now assume that r, the rank of ϕ, is equal to f, the rank of F. In this case the matrix defining ϕ has r rows, and the ideals $\mathfrak{f}_k$ defined in the last section are all principal (generated by the same element, the first r by r minor) and this term cancels out in the formulas. Hence we can omit this in the discussion and we left with $O(E_{k-1} - E_k) \longrightarrow O$. In this case we can clearly let $\mathfrak{g}_k = \mathfrak{e}_k$; these ideals are all $\mathfrak{m}$–primary in this case. We note that the formulas give $(G_k - G_{k-1} + E_{k-1} - E_k)^d - (E_{k-1} - E_k)^d$, and in this case the first term is zero so that we are left with $-(E_{k-1} - E_k)^d$, thus if Q_k represents the mixed multiplicity polynomial of $\mathfrak{e}_k, \mathfrak{e}_{k-1}$ then $m(\phi) = Q_1(1,-1) + ... + Q_r(1,-1)$.

Now we return to the multiplicity of an $\mathfrak{m}$–primary ideal in the sense of Samuel. In this case we are in the above situation with $r = f = 1$; that is, we have a map from A^e to A defined by a 1 by e matrix whose entries are a set of generators for the ideal. Thus the only determinantal ideal which occurs is $\mathfrak{e}_1$, which is simply the ideal we started with. Hence there we are left with $Q_1(1,-1)$, and since the first ideal is trivial (this is just $\mathfrak{e}_0$) this is just the usual multiplicity of the ideal.

The other connection is with the local Chern character as defined by Fulton. We let now F_* denote a bounded complex of free modules with support the maximal ideal of A (i.e. for every prime ideal other than $\mathfrak{m}$, the localization is (split) exact). In this case one has a number associated to the complex, and if we let $[A]$ denote the fundamental class of Spec(A) in the part of the Chow group of Spec(A) of dimension d, this is $ch(F_*)([A])$. We refer to Fulton ([2]) Chapter 18) for a description of what this is well as the properties this invariant satisfies. It is mostly these properties which we need in the proof we give below.

We first note that the condition on the support of F_* implies that the individual homomorphisms of the complex, which we denote δ_i, (δ_i will be the map from F_i to F_{i-1}) satisfy the hypotheses to make $m(\delta_i)$ defined. The formula we wish to prove is :

$$ch(F_*)([A]) = \Sigma(-1)^i(m(\delta_i)).$$

There are three main steps in this proof. Let r_i be the generic rank of δ_i. The first step is to blow up the ideals of r_i by r_i minors of the matrices defining δ_i to split the complex up into maps of rank r_i locally free sheaves on the blown up scheme X. Next, we show that, by blowing up further, each of these pieces can be filtered with quotients maps of invertible sheaves defined locally by determinants in the ideals $\mathfrak{e}_k$ and $\mathfrak{f}_k$. Finally, we put this together and derive the formula given in the first section. This is similar to the process used by Fulton ([2], Example 18.3.12) to prove his local Riemann–Roch theorem; he shows there that this can be done, at least in theory, and we show here how to carry it out.

We first introduce some notation. We wish to construct a rank r_i locally free quotient Q_i of F_i and a rank r_i locally free subsheaf R_i of F_{i-1} (such that the inclusion of R_i into F_{i-1} is locally split) such that the map δ_i factors through a map ρ_i from Q_i to R_i. The first step, as mentioned above, is to blow up the ideal of r_i by r_i minors of the matrices defining each of the maps δ_i. Call the resulting scheme X and denote the proper map from X to Spec(A) by π. If the quotients Q_i and the subsheaves R_i as above exist, we have a short exact sequence for each i :

$$0 \longrightarrow R_{i+1} \longrightarrow F_i \longrightarrow Q_i \longrightarrow 0.$$

Thus the complex can be broken up over X into the maps $Q_i \xrightarrow{\rho_i} R_i$ and it follows from the additivity of local Chern characters on short exact sequences and the compatibility with proper maps that we have

$$ch(F_*)([A]) = \pi_*(ch(\pi^*(F_*))([X])) = \Sigma\ \pi_*(ch(Q_i \xrightarrow{\rho_i} R_i)([X])).$$

To show that this decomposition does exist it suffices to do it for each i separately, and we now return to our previous notation, replacing F_i, F_{i-1}, Q_i, R_i, ρ_i and δ_i by E, F, Q, R, ρ and ϕ. Let M, as above, be the matrix defining ϕ and let I denote a set of r rows and J a set of r columns of M. For any choice of a set of rows K and a set of columns L we let $M_{K,L}$ denote the submatrix with entries from those rows and columns. We recall that we have blown up the ideal of r by r minors of M. For each set of columns J we take the map $A^e \longrightarrow A^r$ defined by $M^{-1}_{I,J}M_{I,e}$ (where e denotes all e columns; similarly for f below).

LEMMA. *This matrix does not depend on the row I chosen.*

Proof : We note that the matrix $M^{-1}_{I,J}M_{I,e}$ has an identity matrix in the J columns, no matter which I is chosen. Since the entire matrix M had rank r, if I' is another set of rows, there is a matrix N at least with entries in the quotient field of A such that

$$M^{-1}_{I,J}M_{I,e} = N(M^{-1}_{I',J}M_{I',e}).$$

But these are the same in the J columns, so N is the identity matrix and these two matrices are the same.

It follows from this lemma that we can take I to be the first r rows of M. Recall that $\mathfrak{e}_r$ is the ideal of r by r minors of the first r rows of M. It then follows from Cramer's rule that the matrix $M^{-1}_{I,J}M_{I,e}$ has entries in the part of the blow up of $\mathfrak{e}_r$ corresponding to the determinant in the J columns. Also, since the matrix $M^{-1}_{I,J}M_{I,e}$ contains an identity matrix in the J columns this map is surjective. Thus we have a quotient onto a rank r locally free sheaf over the blow up of $\mathfrak{e}_r$; this locally free sheaf has transition matrices from J to J' given by $M^{-1}_{I,J'}M_{I,J}$ (as above, this does not depend on I).

We remark here that for this part it was only necessary to blow up $\mathfrak{e}_r$, and not the entire ideal of r by r minors of M. On the other hand, the ideal of all r by r minors is isomorphic to the product $\mathfrak{e}_r\mathfrak{f}_0$, so it would have amounted to the same thing to blow up $\mathfrak{e}_r$ and $\mathfrak{f}_0$ (which we need to do in the next step) instead.

We next define a rank r vector bundle over the blow up of $\mathfrak{f}_0$ and a map which is locally split into $A^f = F$. The maps are indexed by sets I of r rows and the maps are given by $M_{f,J}M^{-1}_{I,J}$. The transition matrices are $M_{I',J}M^{-1}_{I,J}$. As before, it does not depend on which column J is chosen. Furthermore, for each I and J, we can take these maps and put them into a commutative diagram

$$\begin{array}{ccccc} & A^e & \xrightarrow{M} & A^f & \\ M^{-1}_{I,J}M_{I,e} & \downarrow & & \uparrow & M_{f,J}M^{-1}_{I,J} \\ & A^r & \xrightarrow{M_{I,J}} & A^r. & \end{array}$$

Denote the rank r quotient of E by Q and the rank r locally free subsheaf of F by R. This diagram says that we have a map from Q to R defined locally by $M_{I,J}$. The support of this map is a closed subscheme of X lying over the maximal ideal of A.

This construction shows that we can split up the complex F_* as claimed above, so we have the formula

$$ch(F_*)([A]) = \pi_*(ch(\pi^*(F_*))([X])) = \Sigma\, \pi_*(ch(Q_i \xrightarrow{\rho_i} R_i)([X]).$$

We note here that this would also give a definition of the multiplicity of a homomorphism of free modules in terms of MacPherson's graph construction for morphisms of locally free sheaves on a blown up scheme; we refer to Fulton ([2], Example 18.1.6) for this construction. In addition, it follows from this part of the proof that the number we define does not depend on choice of basis, since up to now we have blown up only the ideal of all r by r minors of M, and this does not depend on the bases chosen.

We now come to the main part of this section, the fact that the formula we gave in section 1 is the right one. To accomplish this we examine in detail a filtration of the map $\rho : Q \longrightarrow R$ with quotients maps of invertible sheaves.

We define a sequence of quotients Q_k and R_k of Q and R respectively of rank k for each $k = 1,\ldots,r-1$ together with compatible maps from Q_k to R_k induced by the map ρ. There will also be maps from Q_k to Q_{k-1} and from R_k to R_{k-1}; their kernels will be invertible sheaves which we denote $\mathscr{L}_k$ and $\mathscr{N}_k$. We then express $\pi_*(ch(Q_i \xrightarrow{\rho_i} R_i)([X]))$ in terms of the induced maps from $\mathscr{L}_k$ to $\mathscr{N}_k$ and this will give the formula.

We first define the Q_k's and the maps between them. This will be done by specifying the transition maps between the local pieces of each locally free sheaf and the local expressions for the maps between the different ones. First, these are defined on the scheme obtained by blowing up certain determinantal ideals, and a local affine piece is defined by choosing one of these, say Δ , and taking the ring generated by all Δ'/Δ, where Δ' is also one of the generators of the ideal. The matrices we define below will have entries which are quotients of determinants of this form (this usually follows directly from Cramer's rule) and we will not go over this point again at each point in the construction.

We first give the local expression for the map from Q to Q_k. Choose a set L of k columns of the matrix M. We denote the k by k matrix with entries the first k rows and the columns in L by $M_{k,L}$. Choose also an r by r submatrix $M_{I,J}$ of M. The local expression for the projection of Q onto Q_k is then $M_{k,L}^{-1}M_{k,J}$. The transition matrix on Q_k from L to L' is $M_{k,L'}^{-1}M_{k,L}$. The transition matrix from I,J to I',J' is the identity map. We verify this last statement : it must be shown that for I,J and I',J' as above, the diagram

$$\begin{array}{ccccc} & A^r & \xrightarrow{M^{-1}_{I,J'}M_{I,J}} & A^r & \\ M^{-1}_{k,L}M_{k,J} & \downarrow & & \downarrow & M^{-1}_{k,L}M_{k,J'} \\ & A^k & \xrightarrow{\quad I \quad} & A^k & \end{array}$$

commutes. Since the map in the top row does not depend on which set of rows I is chosen, we can choose the first r rows, where the commutativity is clear.

It must also be verified that the projections are (locally) surjective; if J contains L, the projection matrix contains a k by k identity matrix, so this is obvious, and the general case can be deduced by using the compatibility in the above diagram to change J.

To define the map from Q_k to Q_{k-1}, we choose sets L_k of k columns and L_{k-1} of $k-1$ columns of M respectively and define the map locally to be given by $(M_{k-1,L_{k-1}})^{-1}M_{k,L_k}$. The fact that the required diagrams commute and the maps are locally surjective follows as above.

We next define the rank k quotients R_k of R and the corresponding maps in this case. It is more convenient here to construct the rank $r-k$ locally free subsheaves which are the kernels of the projections from R to R_k instead, so we do this. We denote this kernel T_{r-k}.

Blow up the ideal $\mathfrak{f}_k$. The r by r determinants generating this ideal have their entries in the first r columns and a set of rows containing the first k; we index this by the set N of r rows. Choose one of these, and an r by r submatrix $M_{I,J}$, and define the imbedding of T_{r-k} into R locally by letting it be given by the matrix $M_{I,J}M^{-1}_{N,J}\begin{pmatrix}0\\ I\end{pmatrix}$, where the last factor is an r by $r-k$ matrix with an identity matrix in the last $r-k$ rows. This is, of course, the same as taking the last $r-k$ columns of $M_{I,J}M^{-1}_{N,J}$.

We next define the transition matrices for T_{r-k}. Take N and N' choices of r rows containing the first k rows, and choose r by r submatrices $M_{I,J}$ and $M_{I',J'}$. We must then find a matrix P such that the following diagram commutes :

$$\begin{array}{ccccc} & A^{r-k} & \xrightarrow{\quad P \quad} & A^{r-k} & \\ M_{I',J}M^{-1}_{N',J}\begin{pmatrix}0\\ I\end{pmatrix} & \downarrow & & \downarrow & M_{I,J}M^{-1}_{N,J}\begin{pmatrix}0\\ I\end{pmatrix} \\ & A^r & \xrightarrow{M_{I,J}M^{-1}_{I',J}} & A^r. & \end{array}$$

We define P to be the $r-k$ by $r-k$ submatrix of $M_{N,J}M^{-1}_{N',J}$ defined by choosing the last $r-k$ rows and the last $r-k$ columns. Since the first k rows of $M_{N,J}$ and $M_{N',J}$ are the same $M_{N,J}M^{-1}_{N',J}$ is of the form

$$\begin{pmatrix} I & 0 \\ * & P \end{pmatrix}.$$

Hence one has

$$M_{\mathrm{I,J}}M^{-1}_{\mathrm{N,J}}\begin{pmatrix} 0 \\ I \end{pmatrix}P = M_{\mathrm{I,J}}M^{-1}_{\mathrm{N,J}}\begin{pmatrix} 0 \\ P \end{pmatrix} = M_{\mathrm{I,J}}M^{-1}_{\mathrm{N,J}}(M_{\mathrm{N,J}}M^{-1}_{\mathrm{N',J}})\begin{pmatrix} 0 \\ I \end{pmatrix} = M_{\mathrm{I,J}}M^{-1}_{\mathrm{N',J}}\begin{pmatrix} 0 \\ I \end{pmatrix}.$$

Hence the diagram commutes.

We note also from this that the determinant of P is $\Delta_{\mathrm{N}}/\Delta_{\mathrm{N'}}$.

The maps from T_{r-k} to are (locally) split injections – this can be seen by comparing with the case in which the rows of N other than the first k are contained in I using the above compatibility.

The maps from T_{r-k} to T_{r-k+1} are defined locally by matrices defined analogously to the transition matrices just described : fixing N_k and N_{k-1}, the map from T_{r-k} to T_{r-k+1} is given by the lower right $r-k+1$ by $r-k$ submatrix of the matrix $M_{\mathrm{N}_{k-1},\mathrm{J}}M^{-1}_{\mathrm{N}_k,\mathrm{J}}$. The commutativity of the required diagrams is proven as above. Thus we have locally free sheaves R_k together with compatible maps from R_k to R_{k-1} for $k=1,...,r$, and we denote the invertible kernels of these maps by $\mathcal{N}_k$. We note that $\mathcal{N}_k$ can also be described as the cokernel of the map from T_{r-k} to T_{r-k+1}.

We must next show that the original map defined by M defines compatible maps from Q_k to R_k, and hence also from $\mathcal{L}_k$ to $\mathcal{N}_k$. We use the following lemma, which simplifies the situation :

Lemma. *Consider the Rees ring of* $\mathfrak{f}_0$, $\mathfrak{f}_k$, *and let* X *denote the corresponding blow up. Then* X *is covered by distinguished open sets corresponding to* (I,N) *where the rows of* N *other than* $1,...,k$ *are in* I.

Proof : Fix I. This part of the blow up is covered by all (I,N) if we put no condition on N other than that it contain the first k rows. Thus it suffices to show that if a bigraded prime ideal of the Rees ring which does not contain Δ_{I} (the Δ_{I} in degree $(1,0)$) contains Δ_{N} for those N satisfying the condition of the hypothesis it contains all of them. If N has at least one row which is neither one of the first k nor in I, we can use the Plücker identities to write $\Delta_{\mathrm{N}}\Delta_{\mathrm{I}}$ as a sum of products $\Delta_{\mathrm{N'}}\Delta_{\mathrm{I'}}$ where each N' has one more row in common with I than N does. Thus, using induction, one has that $\Delta_{\mathrm{N}}\Delta_{\mathrm{I}}$ is in the prime ideal, and since Δ_{I} is not, Δ_{N} must be in the ideal. This proves the lemma.

We now fix I, N, and L with I and N as in the lemma. Let $\tilde{N}$ denote the set of rows of N other that $1,\dots,k$; our assumption then states that $\tilde{N}$ is contained in I. We define two matrices U and V as follows :

$U=$ the k by k submatrix of $M_{\mathrm{I,J}}^{-1}$ obtained by taking rows in $I-\tilde{N}$ and columns $1,\dots,k$.

$V=$ the k by $r-k$ submatrix of $M_{\mathrm{I,J}}M_{\mathrm{N,J}}^{-1}$ obtained by taking rows in $I-\tilde{N}$ and columns $\tilde{N}$.

We claim that the map from Q_k to R_k is locally defined by $UM_{k,\mathrm{L}}$. The factor $M_{k,\mathrm{L}}$ cancels $M_{k,\mathrm{L}}^{-1}$ in the projection from Q to Q_k, and what must be proven is the commutativity of the following diagram :

$$\begin{array}{ccc} A^r & \xrightarrow{M_{\mathrm{I,J}}} & A^r \\ {\scriptstyle M_{k,\mathrm{J}}}\downarrow & & \downarrow {\scriptstyle (-I\ V)} \\ A^k & \xrightarrow{U} & A^k. \end{array}$$

In representing the local projection of R to R_k by $(-I\ V)$ we have grouped the columns in $\tilde{N}$ at the end. Doing the same for $M_{\mathrm{I,J}}M_{\mathrm{N,J}}^{-1}$ we represent the $I-\tilde{N}$ rows of $M_{\mathrm{I,J}}M_{\mathrm{N,J}}^{-1}$ as $(U\ V)$. We then have :

the $I-\tilde{N}$ rows of $M_{\mathrm{I,J}} = (U\ V)\begin{pmatrix} M_{k,\mathrm{J}} \\ M_{\tilde{N},\mathrm{J}} \end{pmatrix} = UM_{k,\mathrm{J}} + VM_{\tilde{\mathrm{N}},\mathrm{J}}$, or $UM_{k,\mathrm{J}} =$ (the $I-\tilde{N}$ rows of $M_{\mathrm{I,J}}) - VM_{\tilde{\mathrm{N}},\mathrm{J}}$. This says that the above diagram commutes.

Since the rows and columns omitted from to get U correspond to rows of $\tilde{N}$, which are common to both of them, the determinant of U is $\Delta_{\mathrm{I}}/\Delta_{\mathrm{N}}$. Hence the determinant of the map from Q_k to R_k is given locally by $\Delta_{\mathrm{L}}\Delta_{\mathrm{I,J}}/\Delta_{\mathrm{N,J}}$.

We are now in a position to verify the formula we have for the mixed multiplicities. To do this we list first the determinants of the transition maps for Q_k and R_k and of the local expression for the map from Q_k to R_k. We give the determinants of maps which go from local coordinates corresponding to L and N to those corresponding to L' and N'. From the above discussion, these are, respectively :

For $Q_k : \Delta_{\mathrm{L}}/\Delta_{\mathrm{L}'}$.

For $T_{r-k} : \Delta_{\mathrm{N}'}/\Delta_{\mathrm{N}}$.

For $Q_k \longrightarrow R_k : \Delta_{\mathrm{L}}\Delta_{\mathrm{I,J}}/\Delta_{\mathrm{N,J}}$.

We have defined the invertible sheaves $\mathscr{L}_k$ and $\mathscr{N}_k$ as the kernel of the map $Q_k \longrightarrow Q_{k-1}$ and the cokernel of the map $T_{r-k} \longrightarrow T_{r-k+1}$ respectively. Using the multiplicativity of the determinant on short exact sequences gives us the transition maps on $\mathscr{L}_k$ and $\mathscr{N}_k$ and the map from $\mathscr{L}_k$ to $\mathscr{N}_k$; these are :

For $\mathscr{L}_k : \Delta_{L_k}\Delta_{L'_{k-1}}/\Delta_{L'_k}\Delta_{L_{k-1}}$.

For $\mathscr{N}_k : \Delta_{N_k}\Delta_{N'_{k-1}}/\Delta_{N'_k}\Delta_{N_{k-1}}$.

For $\mathscr{L}_k \longrightarrow \mathscr{N}_k : \Delta_{L_k}\Delta_{N_{k-1}}/\Delta_{N_k}\Delta_{L_{k-1}}$.

Now these determinants also define the transition maps for the invertible sheaves E_k and F_k defined in the previous section; more precisely, the transition matrix for coordinates on $O(-E_k)$ are given by $\Delta_L\Delta_{L'}$ (since the local generator at L is Δ_L and at L' is $\Delta_{L'}$ and we have $r\Delta_L = ((\Delta_L/\Delta_{L'})r)\Delta_{L'}$), and similarly for F_k. Putting this together, we have that the map of invertible sheaves from $\mathscr{L}_k$ to $\mathscr{N}_k$ is :

$$O(-E_k + E_{k-1}) \longrightarrow O(-F_k + F_{k-1}).$$

Under the assumptions of section 1 this can also be represented :

$$O(-E_k + E_{k-1}) \otimes (O \longrightarrow O(G_k - G_{k-1})).$$

Using the formula for the local Chern character of a map of invertible sheaves in terms of the exponential map : this is, the local Chern character of a map of invertible sheaves $O(D_1) \otimes (O \longrightarrow O(D_2)$ is $(\sum_{n\geq 0} \frac{D_1^n}{n!})(\sum_{n\geq 1} \frac{D_2^n}{n!})$ (see Fulton [2], Ch. 18), the additivity of local Chern characters and the fact we have proven, that the original map from Q to R has a filtration with given subquotients gives the required formula.

3. Homomorphisms which can be extended to a perfect complex of minimal length.

As mentioned in the introduction, one of the motivations behind this work was to study the contributions to the local Chern character of a perfect complex from the individual boundary maps of the complex. In particular, this was of interest for a perfect complex of length d, where d is the dimension of the ring. It was shown in Roberts [6] that the number one obtains from the local Chern character is positive when the local ring has positive characteristic (and some cases which can be deduced from this one). The question which arises is whether the contributions of the individual boundary maps are positive. We first show that this set of maps of free modules can be described explicitly.

Proposition. *Let ϕ be a homomorphism of free modules with support $\mathfrak{m}$. Let i be the smallest integer such that there exists a complex $0 \longrightarrow E_i \longrightarrow \dots \longrightarrow E_0 \longrightarrow E \longrightarrow F$ with homology supported at $\mathfrak{m}$, and let j be the integer defined in the same way for the dual of ϕ. Then*

1. $i + j \geq d - 1$.

2. ϕ *is a boundary map of a complex with homology of finite length and of length* d *if and only if* $i + j = d - 1$.

Proof : If both ϕ and its dual have resolutions as in the hypothesis, the resolution of ϕ and the dual of the resolution of the dual of ϕ can be put together to give a complex with homology of finite length and of length $i + j + 1$. Thus one direction of statement 2 is clear, and the other direction and the inequality of statement 1 are easy consequences of the Peskine–Szpiro Intersection Theorem.

If A is Cohen–Macaulay, there is only one possibility for the complexes of the hypothesis of this Proposition, and that is to take free resolutions of the cokernels. It is also easy to see that in these cases the complex is unique. Is this true in general ? In any case, if there are two resolutions, there cannot exist a map from one to the other lifting the identity, since the mapping cone would again violate the intersection theorem. Another question along the same line is whether, as in the Cohen–Macaulay case, there is a unique best choice for the resolution which can be determined at each stage; that is, for example, whether one can give a criterion for what E_0 must be in terms of ϕ whithout extending the resolution further.

The other questions we raise here concern the positivity of the multiplicity in this case. This is a question even for the case of a map of maximal rank considered in the second section; this should be positive even though the formula involves negative terms. We have seen that there the expression in terms of mixed multiplicities is particularly simple; it is $\Sigma\, Q(1,-1)$ for certain polynomials Q. One could ask if even these components are positive. For dimension 1 this is easy since it is the difference of multiplicities and one ideal is contained in the other. For $d = 2$ it is deeper : in this case it follows from an inequality of Teissier [9] (proven in the general case by Rees and Sharp [4]) which implies that for any two $\mathfrak{m}$–primary ideals this number must be positive. It could be asked whether these numbers are always positive for any two $\mathfrak{m}$–primary ideals where one is contained in the other, but Rees has given some examples (not determinantal ideals of the kind which arise here, however) where they are negative. One could also ask, if ϕ can be extended to a perfect complex of length d, say

$$0 \longrightarrow F_d \longrightarrow \dots \longrightarrow F_0 \longrightarrow 0,$$

and if it occurs as the map from F_{i+1} to F_i, whether $(-1)^i m(\phi)$ must be positive. We note that it follows from the above proposition that the integer i is uniquely determined by ϕ.

4. The ideals $\mathfrak{g}_k$ in some special cases.

We work out here two special cases. The first is the opposite extreme from the first one we discussed in section 2; we here look at a homomorphism of rank one. In this case there is only one map of invertible sheaves to consider, which we denote $\mathscr{L} \longrightarrow \mathscr{N}$. The map is defined after blowing up the ideals generated by the first row and the first column of the matrix defining ϕ, and denoting the matrix as (m_{ij}), the map $\mathscr{L} \longrightarrow \mathscr{N}$ is locally defined (see the formula above) by $m_{1j}m_{i1}m_{11}$, and, since the matrix has rank one, $m_{1j}m_{i1}m_{11} = m_{ij}$. Thus if we let $\mathfrak{g}$ be the ideal generated by all entries of M and $\mathfrak{e} = \mathfrak{e}_1$ the ideal generated by the entries in the first row, we have, using notation as above,

$$\mathscr{L} \longrightarrow \mathscr{N} \cong O(-E) \otimes (O \longrightarrow O(G)).$$

Hence the multiplicity is defined in terms of the mixed multiplicities of $\mathfrak{e}$ and $\mathfrak{g}$. If one works out the formula, for example, for the matrix $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ over $k[[X,Y,Z,W]]/(XW-YZ)$, one finds

$$\begin{aligned} m(\phi) &= (G^3 - 3G^2E + 3GE^2) \\ &= (2 - 3(2) - 3(0)) \quad \text{(this is shown by looking at the Rees ring)} \\ &= -4. \end{aligned}$$

We next show a simple example of a Koszul complex; we do the case of the middle morphism in of the Koszul complex on three elements, and we take these elements to be a regular system of parameters, denoted X, Y, Z, for regular local ring. In this case, since each end gives the multiplicity of the maximal ideal of a regular local ring, which is 1, the total alternating sum must be 6, and this term occurs in odd degree, the answer must come out to be -4. This map has rank two and there are two terms in the formula. The matrix is :

$$\begin{pmatrix} -Y & -Z & 0 \\ X & 0 & -Z \\ 0 & X & Y \end{pmatrix}.$$

The ideals are as follows :

$\mathfrak{e}_0$: the unit ideal. $\mathfrak{f}_0 : (X^2, XY, XZ)$

$\mathfrak{e}_1 : (Y,Z)$ $\mathfrak{f}_1 : (XY, XZ)$

$\mathfrak{e}_2 : (XY, Y^2, YZ)$ $\mathfrak{f}_2 : (XZ)$.

In this case the ideals $\mathfrak{g}_k$ can be found easily; they are :

$\mathfrak{g}_0$: the unit ideal
$\mathfrak{g}_1$: (X, Y, Z)
$\mathfrak{g}_2$: $(X, Y, Z)^2$.

The formula can be simplified a little since $O(G_2 - G_1) \cong O(G_1) \cong O(E_2)$, and we get :

$$\begin{aligned} m(\phi) &= ch(O(-E_1) \otimes (O \longrightarrow O(G_1)) + ch(O(-E_2+E_1) \otimes (O \longrightarrow O(G_1)) \\ &= (G_1^3 - 3G_1^2E_1 + 3G_1E_1^2) + (G_1^3 - 3(G_1^2E_1 - G_1^2E_2) + 3(G_1E_1^2 - 2G_1E_1E_2 + G_1E_2^2)) \\ &= (1 - 3(1) + 3(0)) + (1 - 3(1-1) + 3(0-2+1)) \\ &= -4. \end{aligned}$$

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