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TIGHT CLOSURE AND STRONG F-REGULARITY

by Melvin HOCHSTER[1] and Craig HUNEKE[1]

This paper is written in celebration of the contributions of Pierre Samuel to commutative algebra.

1. Introduction.

Throughout this paper all rings are commutative, with identity, and Noetherian, unless otherwise specified. In [HH1] and [HH2] the authors introduced the notion of the tight closure of an ideal and the tight closure of a submodule of a finitely generated module for Noetherian rings which are either of positive prime characteristic p or else are algebras essentially of finite type over a field of characteristic 0. This notion enabled us to give new proofs, which are especially simple in characteristic p, of a number of results (not all of which were perceived to be particularly related) : that rings of invariants of linearly reductive groups acting on regular rings are Cohen–Macaulay, that the integral closure of the n^{th} power of an n generator ideal of a regular ring is contained in the ideal (the Briançon–Skoda theorem), of the monomial conjecture, and of the syzygy theorem. The new proofs yield much more general theorems. For example, we can show by these methods that if S is any Noetherian regular ring containing a field and R is a direct summand of S as an R–module (we shall sometimes say, briefly, that R is a *summand* of S to describe this situation : we always mean $R \longrightarrow S$ is R–split) then R is Cohen–Macaulay. This result was not previously known in this generality. Moreover, this illustrates the general principle that results proved using tight closure techniques but which do not refer specifically to tight closure can be extended to the general equicharacteristic case by using Artin approximation to reduce to a situation in which tight closure is defined.

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One of the most important characteristics of tight closure is that in a regular ring every ideal is tightly closed. We call the Noetherian rings all of whose localizations have this property "F–regular". (The "F" in "F–regular stands for "Frobenius" : the reason for this usage will become clear later). This is an important class of rings which includes the rings of invariants of linearly reductive groups acting on regular rings. A key point is that if S is F–regular and R is a direct summand of S as an R–module then R is F–regular. It turns out that, under mild conditions (like being a homomorphic image of a Cohen–Macaulay ring or a weakening of the requirements for excellence), F–regular rings, which are always normal, are Cohen–Macaulay as well. This is the basis for our new proof that direct summands of regular rings are Cohen–Macaulay in the equicharacteristic case.

Our objectives in this paper are, first, to recap briefly some of the features of tight closure, and then to focus on the notion of a "strongly F–regular" ring. It turns out that rings of invariants of reductive groups have, in fact, this stronger property, and that the stronger property has numerous apparent advantages over F–regularity. We should point out right away that we do not know whether the notions of F–regularity and strong F–regularity are really different in good cases. It would be very worthwhile if it could be proved that the two notions coincide.

2. A survey of tight closure.

Unless otherwise specified A, R, and S denote Noetherian commutative rings with 1. By a "local ring" we always mean a Noetherian ring with a unique maximal ideal. R^0 denotes the complement of the union of the minimal primes of R. I and J always denote ideals. Unless otherwise specified given modules M and N are assumed to be finitely generated.

We make the following notational conventions for discussing "characteristic p". We shall always use p to denote a positive prime integer. We shall use e for a variable element of $\mathbb{N}$, the set of nonnegative integers, and q for a variable element of the set $\{p^e : e \in \mathbb{N}\}$.

If R is reduced of characteristic p we write $R^{1/q}$ for the ring obtained by adjoining all q^{th} roots of elements of R: the inclusion map $R \subseteq R^{1/q}$ is isomorphic with the map $F^e : R \longrightarrow R$, where $q = p^e$, F is the Frobenius endomorphism of R and F^e is the e^{th} iteration of F, i.e. $F^e(r) = r^q$. When R is reduced we write R^∞ for the R–algebra $\cup_q R^{1/q}$. Note that R^∞ is an exception to the rule that the rings we consider be Noetherian.

If $I \subseteq R$ and $q = p^e$ then $I^{[q]}$ denotes $(i^q : i \in I) = F^e(I)R$. If S generates I then $\{i^q : i \in S\}$ generates $I^{[q]}$.

We are now ready to define tight closure for ideals in the characteristic p case.

DEFINITION. *Let* $I \subseteq R$ *of characteristic* p *be given. We say that* $x \in I^*$, *the* ***tight closure*** *of* I, *if there exists* $c \in R^o$ *such that* $cx^q \in I^{[q]}$ *for all* $q \gg 0$, i.e. *for all sufficiently large* q *of the form* p^e. *If* $I = I^*$ *we say that* I *is* ***tightly closed***.

Remarks. Note that if R is a domain, which is the most important case, the condition that $c \in R^0$ is simply the condition that $c \neq 0$. Note also that if R is reduced then $cx^q \in I^{[q]}$ iff $c^{1/q}x \in IR^{1/q}$. Thus, if $x \in I^*$ then for some $c \in R^0$ we have that $c^{1/q}x \in IR^\infty$ for all q (this condition gets stronger as q gets larger). This gives a heuristic argument for regarding x as being "nearly" in I or, at least, IR^∞: it is multiplied into IR^∞ by elements which, in a formal sense, are getting "closer and closer" to 1 (since $1/q \longrightarrow 0$ as $q \longrightarrow \infty$).

We also note that if R is reduced or if I has positive height it is not hard to show that $x \in I^*$ iff there exists $c \in R^0$ such that $cx^q \in I^{[q]}$ for all q.

We extend this notion to finitely generated algebras over a field of characteristic 0 as follows :

DEFINITION. *Let* R *be a finitely generated algebra over a field* K *of characteristic* 0, $I \subseteq R$, *and* $x \in R$. *We say that* x *is in the* ***tight closure*** I^* *of* I *if there exist an element* $c \in R^0$, *a finitely generated* $\mathbb{Z}$*–subalgebra* D *of* K, *a finitely generated* D*–subalgebra* R_D *of* R *containing* x *and* c, *and an ideal* I_D *of* R_D *such that* I_D *and* R_D/I_D *are* D*–free, the canonical map* $K \otimes_D R_D \longrightarrow R$ *induced by the inclusions of* K *and* R_D *in* R *is a* K*–algebra isomorphism,* $I = I_D R$, *and for every maximal ideal* m *of* D, *if* $\kappa = D/m$ *and* p *denotes the characteristic of* κ, *then* $c_\kappa x_\kappa^q \in I_\kappa^{[q]}$ *in* $R_\kappa \cong R_D/mR_D$ *for every* $q = p^e \gg 0$, *where the subscript* κ *denotes images after applying* $\kappa \otimes_D$. *If* $I = I^*$ *we say that* I *is* ***tightly closed***.

It is not even completely clear from this definition that I^* is an ideal, although it is not difficult to establish. Our attitude in this survey is as follows : we give a number of proofs in characteristic p to illustrate how easy many arguments are while in characteristic 0 we state results but omit discussion of the proofs (generally speaking, the arguments are rather technical but hold few surprises).

We also note that if R is an algebra essentially of finite type over a field K of characteristic 0, and $I \subseteq R$, we can define the ***tight closure***, I^*, of I as $\cup_B (I \cap B)^*$, where the union is extended over all finitely generated K–subalgebras B of R such that R is a localization of B. However, we shall not discuss the situation for algebras essentially of finite type over a field in any detail in this paper.

The next result shows, among other things, how one uses tight closure to prove that direct summands of regular rings are Cohen–Macaulay (C–M).

2.1. THEOREM. *Let R, S denote Noetherian rings which are either of characteristic p or else essentially of finite type over a field.*

a) *If R is regular, every ideal of R is tightly closed.*

b) *If $R \subseteq S$ are domains and J is tightly closed in S then $J \cap R$ is tightly closed in R. (When R and S are not necessarily domains we may assume instead that $R^0 \subseteq S^0$).*

c) *Let $R \subseteq S$ be domains such that every ideal of R is contracted from S (this holds, in particular, if R is a direct summand of S as an R–module). If every ideal of S is tightly closed then every ideal of R is tightly closed.*

d) *The tight closure of an ideal I of R is contained in the integral closure I^- of I.*

e) *If R is a locally unmixed homomorphic image of a C–M ring and $x_1,\dots,x_n \in R$ have the property that any t of the $x's$ generate an ideal of height $\geq t$, then $(x_1,\dots,x_{n-1}) :_R x_n R \subseteq (x_1,\dots,x_{n-1})R^*$, where $I :_R J = \{r \in R : rJ \subset I\}$.*

Sketch of the proof in characteristic p. a) Suppose that $I \subseteq R$, that R is regular, and that $x \in I^* - I$. By localizing at a prime containing $I :_R xR$ we may assume that (R,m) is local as well. If $cx^q \in I^{[q]}$ for all $q > q'$ then $c \in \cap_{q>q'}(I^{[q]}:x^q) = \cap_{q>q'}(I:x)^{[q]}$ (the flatness of the Frobenius endomorphism for regular rings implies that $(I^{[q]}:x^q) = (I:x)^{[q]}) \subseteq \cap_{q>q'} m^q = (0)$, so that $c = 0$.

b) is immediate from the definition of tight closure and c) is immediate from b).

d) (The reader may want to look at the discussion of integral closure given in (2.8) below before going through this argument). We may use a). Suppose $x \in I^*$ and $c \in R^0$ is such that $cx^q \in I^{[q]}$ for all $q \gg 0$. Let $h : R \longrightarrow V$ with $\ker h$ a minimal prime of R, where V is a DVR. Then $h(c)h(x)^q \in (IV)^{[q]}$ for all $q \gg 0$ and $h(c) \neq 0$, and so $h(x) \in (IV)^* = IV$ (since V is regular), and we are done. On the other hand, we may argue directly as follows : Let $I = (x_1,\dots,x_h)$. Applying the discrete valuation v to the equation $cx^q = \Sigma_{t=1}^h r_{qt}\, x_t^q$ yields $v(c) + qv(x) \geq q \min\{v(x_t):t\}$. Dividing by q and taking the limite as $q \longrightarrow \infty$ yields the result.

e) We shall not prove the result stated in full generality here : we refer the reader to [HH2]. However, we shall give the argument in the special case where the x_i are contained in a regular ring $A \subseteq R$ and R is module–finite over A. In many good cases it is possible to reduce to this case by localizing and completing R and then choosing A properly. In the interesting case (where the x_i do not generate the unit ideal) we may reduce to the situation where A is local, the x_i are part of a system of parameters for A, and R is module–finite over A. The unmixedness hypothesis then translates into the condition that R be torsion–free as an A–module. The result stated then follows from Lemma 2.2 below. QED.

2.2. LEMMA. *Let R be a Noetherian ring of characteristic p module-finite and torsion-free over regular domain A. Let I, J be ideals of A. Then $IR :_R JR \subseteq ((I :_A J)R)^*$ and $IR \cap JR \subseteq ((I \cap J)R)^*$.*

Proof: Let $F \cong A^t$ be an A–free submodule of R whose rank t is equal to the torsion–free rank of R is an A–module. Then R/F is a torsion A–module, and we can choose a nonzero element $c \in A$ such that $cR \subseteq F$. Let $x \in IR :_R JR$ (resp. $IR \cap JR$). Then, for all q, $x^q \in I^{[q]}R :_R J^{[q]}R$ (resp. $I^{[q]}R \cap J^{[q]}R$), whence $cx^q \in I^{[q]}F :_A J^{[q]}$ (resp. $I^{[q]}F \cap J^{[q]}F$). Since F is A–free, we see $cx^q \in (I^{[q]} : J^{[q]})F$ (resp. $I^{[q]} \cap J^{[q]})F$), and by the flatness of the Frobenius endomorphism of A we then have $cx^q \in (I :_A J)^{[q]}F \subseteq ((I :_A J)R)^{[q]}$ (resp. $(I \cap J)^{[q]}F \subseteq ((I \cap J)R)^{[q]}$) for all q, which yields the desired result. QED

This also completes the proof of (2.1e).

We next give a number of corollaries of (2.1) as well as some remarks about how it is used.

We first recall that a ring for which tight closure is defined is called *weakly F-regular* if every ideal is tightly closed and *F-regular* if this is true in all localizations as well. The authors do not know at present whether every weakly F–regular ring is F–regular. With this terminology we have the following corollaries of Theorem 2.1.

2.3. COROLLARY. *Every regular ring is F-regular.*

2.4. COROLLARY. *If $R \subseteq S$ is a direct summand as an R-module and S is F-regular then R is F-regular.*

2.5. COROLLARY. *A weakly F-regular ring which is a homomorphic image of a C–M ring is C–M.*

2.6. THEOREM. *In the equicharacteristic case, a direct summand of a regular ring (as in 2.3) is C–M.*

2.3 is immediate from 2.1a and 2.4 from 2.1c. 2.5 is then clear from 2.1e. 2.6 is obvious from 2.4 and 2.5 in the case where tight closure is defined. In the general case one reduces to the case of complete local rings and then uses Artin approximation to prove a subtle generalization of 2.1e which yields the result. We refer to [HH2] for details.

2.6 includes the result that rings of invariants of linearly reductive groups G over a field K acting K–rationally on a regular K–algebra R have rings of invariants R^G which are C–M. See [HR1], [K], and [B], as well as the discussion of rational singularities following the statement of the Briançon–Skoda theorem below. It is worth noting that in many cases in characteristic p where the group is reductive but not linearly reductive and where, in fact, R^G is not a direct summand of R, it is nonetheless true that R^G is F–regular : for example, the rings defined by the vanishing of the minors of a given size of a matrix of indeterminates are F–regular. See [HH2].

Remark. The theory of tight closure permits a very substantial generalization of 2.1e. Under the same hypothesis one may perform a sequence of operations including sum, product, colon, and intersection on ideals generated by monomials in the x_i. If the ring is C–M (or, more generally, if the x_i form a permutable regular sequence) then it is easy to compute the result of these iterated operations : the x_i behave as though they were indeterminates. It turns out to be extremely useful to be able to restrict the possibilitities for the answer in the general case, when the x_i are parameters but not necessarily a regular sequence. The key point is that the actual ideal resulting from the iterated operations is in the tight closure of what one gets when the x_i form a permutable regular sequence. We should note that there are some restrictions on the use of colon in doing iterated operations : we are not giving the precise statement here. One very special case of this is that $(x_1^t,\ldots,x_n^t):(x_1\ldots x_n)^{t+1} \subseteq (x_1,\ldots,x_n)^*$, which implies the monomial conjecture (and, hence, the direct summand, canonical element, and new improved intersection conjectures, all of which are equivalent : see [H3]).

Remark. If $I \subseteq R$, tight closure is defined in R, and $R \subseteq S$, where S is regular and, for simplicity, a domain, then $I^*S = IS$. This is a remarkable consequence of the theory of tight closure. For example, it implies in the situation of 2.1e that $(x_1,\ldots,x_{n-1})R:_R x_nR \subseteq (x_1,\ldots,x_{n-1})S$, and similar remarks apply to the iterated operations discussed in the preceding remark. Moreover, one can prove extremely powerful theorems parallel to this one in a somewhat different direction as follows :

2.7. THEOREM (Vanishing theorem). *Let* $A \subseteq R \xrightarrow{f} S$ *be excellent equicharacteristic rings such that* A, S *are regular domains,* f *is injective, and* R *is module-finite over* A. *Let* M *be a finitely generated* A*–module. Then the map*

$$\mathrm{Tor}_i^A(M,R) \longrightarrow \mathrm{Tor}_i^A(M,S)$$

is 0 *for all* $i \geq 1$.

More general statements are given in [HH2], but (2.7) is already quite strong : the case where $M = A/(x_1,\ldots,x_i)$, where the x_t are s.o.p. in A, and R is a direct summand of S implies that summands of regular rings are Cohen–Macaulay, while the case where $M = A/J$ and S is a DVR dominating the local ring R implies the canonical element conjecture [H3]. The proof of 2.7 uses the notion of the tight closure of a submodule of a module : one shows that certain cycles are in the tight closure of the boundaries and hence are boundaries once one passes to the regular ring S, where every submodule is tightly closed. (In char. p, if $N \subseteq M$ we say $y \in M$ is in N^* if there exists $c \in R^0$ such that for all $e \in \mathbb{N}$, $c(1\otimes y)$ maps to 0 in $F^e(M/N)$, where F is the Peskine–Szpiro functor [PS, p. 330, Def. 1.2].)

To prove 2.7 in characteristic 0 a statement is needed which can be preserved while applying Artin approximation: in consequence, we prove a more general result in which the condition that f be injective is weakened to the condition that the image of Spec(S) in Spec(R) meet the Cohen–Macaulay locus in Spec(R).

2.8. Discussion. We recall that an element x of a ring R is *integral* over an ideal I provided that there is an integer $k > 0$ and an equation $x^k + i_1x^{k-1} + \ldots + i_jx^j + \ldots + i_{k-1}x + i_k = 0$ where $i_j \in I^j$ for $1 \leq j \leq k$. This is easily seen to be equivalent to the assertion that there is an integer $k \geq 1$ such that $x^k \in I(I+Rx)^{k-1}$, and this holds iff $(I+Rx)^k = I(I+Rx)^{k-1}$. From this it is trivial to prove by induction on m that

$$(\#)\ (I+Rx)^{k+m} = I^{m+1}(I+Rx)^{k-1}$$

for every integer $m \in \mathbb{N}$. Thus, x is integral over I iff there exists a positive integer k such that (#) holds for all $m \in \mathbb{N}$.

The integral closure I^- of I is simply the set of elements integral over I, and is an ideal.

Another characterization of integral closure for ideals is given by valuations : let R be a ring with finitely many minimal prime ideals (this is, of course, automatic when R is Noetherian) and $I \subseteq R$. Then x is integral over I iff for every homomorphism h of R into a valuation domain V such that Ker h is a minimal prime of R, $hx \in IV$. If R is Noetherian the same result holds with V restricted to be a discrete valuation ring (by which we always mean a rank one discrete valuation ring).

It is instructive to compare integral closure of ideals with tight closure. If x and y are any two elements of a ring R then $(x^n,y^n)^- \supseteq (x^n,x^{n-1}y,\ldots,x^{n-i}y^i,\ldots,xy^{n-1},y^n) = (x,y)^n$ since the monomial $x^{n-i}y^i$ satisfies $z^n - (x^n)^{n-i}(y^n)^i = 0$. On the other hand, if R is regular or

F–regular, e.g. if $R = K[x,y]$ where K is a field, then $(x^n,y^n)^* = (x^n,y^n)$, since every ideal is tightly closed. Thus, the tight closure is, in general, much smaller than the integral closure. The tight closure is a "tight fit" for the original ideal, which is the reason for the choice of the term.

Suppose we define the regular closure I^{reg} of an ideal I in a Noetherian ring R as follows : $x \in I^{\mathrm{reg}}$ precisely if for every homomorphism $h : R \longrightarrow S$ with S regular such that Ker h is a minimal prime of R, $h(x) \in h(I)S$. Roughly speaking, I^{reg} is the largest ideal which cannot be distinguished from I by maps to regular rings whose kernel is a minimal prime, just as I^- plays this role for maps to valuation rings.

A crucial observation is that $I^* \subseteq I^{\mathrm{reg}}$ whenever tight closure is defined : this is immediate from the definition of tight closure and the fact that every ideal in a regular ring is tightly closed. This explains much of the usefulness of I^*. The trouble with working with I^{reg} itself is that it appears to be very difficult to prove anything interesting about its behavior directly. We have many useful results about I^{reg} all of which are proved by studying I^*. While I^{reg} is defined in mixed characteristic (where I^* is not), we cannot prove anything really useful about it. So far as we know it is possible that $I^* = I^{\mathrm{reg}}$ when I^* is defined : this appears to be a difficult question.

We next note that the theory of tight closure provides an easy proof of the Briançon–Skoda theorem, and, in fact, generalizes it. The theorem was first proved by analytic methods ([BrS]) and later by algebraic techniques ([LT], [LS]), but the argument below is simpler, and, in a certain direction, improves the result.

2.9. THEOREM (generalized Briançon–Skoda theorem). *Let R be a Noetherian ring for which tight closure is defined and let I be an ideal of positive height generated by n elements. Then* $(I^n)^- \subseteq I^*$.

Hence, if R is weakly F–regular and, in particular, if R is regular, then $(I^n)^- \subseteq I$.

Sketch of the proof in characteristic p. Let $a = I^n$. If a is contained in the union of I^* and the minimal primes of R it must be contained in I^*. If not choose y in a not in any minimal prime of R. From (#) in Discussion 2.8 concerning definitions of integral closure we have $y^{k+m} \in a^{m+1}(a+yR)^{k+1} \subseteq a^m$ for a certain integer $k \geq 1$ and all $m \in \mathbb{N}$. Let $c = y^k$, $m = q = p^e$ and note that $a^q = I^{nq} \subseteq I^{[q]}$ (since I has n generators), i.e. $cy^q \in I^{[q]}$ for all q. QED

It is easy to deduce the result for arbitrary equicharacteristic regular rings from the characteristic p case using Artin approximation.

We have already observed that direct summands of regular rings are F–regular and so C–M. This was the basis for our new proof that rings of invariants of linearly reductive groups acting rationally on regular K–algebras are C–M: see [HR1], [K], and [B]. The result of [B] is actually that in the case of algebras finitely generated over a field of characteristic 0 and in the analytic case, direct summands of rings with rational singularities have rational singularities. This suggests a connection between F–regularity and rational singularity. We observe :

2.10. THEOREM. *Suppose that* R *is of finite type over a field* K *of char.* 0 *and that* $I = I^*$ *for all* $I \subseteq R$. *If either*: a) R *has isolated singularities or* b) R *is* $\mathbb{N}$*–graded with* $R_0 = K$, $m = \bigoplus_{i=1}^{\infty} R_i$, *and* R *has rational singularities except possibly at* m, *then* R *has rational singularities.*

In particular, F–regular surfaces have rational singularities. The authors conjecture that all F–regular rings of finite type over a field of char. 0 have rational singularities. The converse is not true : an example of [W] shows that a surface in char. 0 may have rational singularities without even being of F–pure type in the sense of [HR2]. The proof of part a) depends on the fact 2.9 that the conclusion of the Briançon–Skoda theorem is valid in a weakly F–regular ring. A key point in the proof of 2.10a is that an n–dimensional isolated singularity of a local ring (R,m) of an algebra of finite type over a field of char. 0 is rational iff for every m–primary ideal I, $\overline{I^n} \subseteq I$. (One may replace I by an ideal generated by a system of parameters on which it is integrally dependent). An alternative equivalent condition for the isolated singularity to be rational is that for some s.o.p. $x_1,\dots,x_n$ whose normalized blow–up is regular, $\overline{(x_1^{tn},\dots,x_n^{tn})} \subseteq (x_1^t,\dots,x_n^t)$ for all t.

Remark. As we shall see in the next section, a Gorenstein local ring has the property that $I = I^*$ for all I iff the ideal generated by a single s.o.p. is tightly closed. If a Gorenstein local ring has an isolated nonrational singularity it follows that the ideal generated by any s.o.p. is not tightly closed. E.g. in $K[[X,Y,Z]]/(X^2+Y^3+Z^7) = K[[x,y,z]]$ the ideal generated by any two of the elements x, y, z fails to be tightly closed : $x \in (y,z)^* - (y,z)$, $y^2 \in (x,z)^* - (x,z)$, and $z^6 \in (x,y)^* - (x,y)$.

3. Strongly F–regular rings.

The notion of weak F–regularity for a ring R, that every ideal be tightly closed, is clearly a valuable one, but has annoying technical drawbacks. One of the worst of these is that we have not been able to prove that it passes, in general, to localizations. The situation is as follows : the property of weak F–regularity passes to localizations at maximal ideals, and, in the case of algebras of finite type over a field, to localizations at an element. In the Gorenstein case it passes to all localizations. However we have not, in general, been able to obtain the result for localizations at an arbitrary prime, even for algebras of finite type over a field. For this reason, we made the property of passing to localizations part of the definition of F–regularity.

In this section we shall study an *a priori* stronger property in characteristic p that may be equivalent to F–regularity or even to weak F–regularity : we simply do not know. It is only defined for reduced rings R such that $R^{1/p}$ is module–finite over R : however, this class contains finitely generated algebras over a perfect field K and complete local rings R with perfect residue class field K (one only needs that $K^{1/p}$ be finite over K), and so is not too restrictive.

Throughout the rest of this section R denotes a reduced Noetherian ring of positive prime characteristic p such that $R^{1/p}$ is module–finite over R (although we often reiterate this hypothesis in stating theorems). Of course, $R^{1/q}$ is then module–finite over R for all $q = p^e$.

In this note we shall restrict attention to the domain case : very little is lost in doing so, since, in general, a strongly F–regular ring is a finite product of strongly F–regular domains.

DEFINITION. ***We say that a domain*** R ***as above is*** ***<u>strongly</u>*** F-***<u>regular</u>*** ***if for every*** $c \in R^0$ ***there exists*** q ***such that the*** R-***linear map*** $R \longrightarrow R^{1/q}$ ***which sends*** 1 ***to*** $c^{1/q}$ ***splits as a map of*** R-***modules,*** i.e. ***iff*** $Rc^{1/q} \subseteq R^{1/q}$ ***splits over*** R .

Remarks. a) The issue of whether a homomorphism of finitely generated modules over a Noetherian ring splits is local and is unaffected by a faithfully flat extension of the base ring (since the question can be translated into whether a certain map of Hom′s is onto : see [H1]).

b) If $R \subseteq S$ and $f : R \longrightarrow M$ is split by g, where M is an S–module, then $R \subseteq S$ splits : send s to $g(sf(1))$.

c) In the definition above, if a splitting exists for one choice of $c \in R^0$ and q then $R \subseteq R^{1/q'}$ splits for every q'. (It suffices to split $R \subseteq R^{1/p}$ and hence $R \subseteq R^{1/q}$: now use b)).

d) Note also that if $R \longrightarrow R^{1/q}$ sending 1 to $c^{1/q}$ splits for one choice of q, the map $R \subseteq R^{1/q'}$ sending 1 to $c^{1/q'}$ splits for every $q' \geq q$: the map $R \longrightarrow R^{1/q}$ described is isomorphic to the map $R^{q/q'} \longrightarrow R^{1/q'}$ sending 1 to $c^{1/q'}$ and so that map splits over $R^{q/q'}$, and this splitting may be composed with the R–splitting $R \subseteq R^{q/q'}$ whose existence we showed in c).

The following result exhibits a number of the good properties of strong F–regularity.

3.1. THEOREM. *Let R be a Noetherian domain of positive prime characteristic p such that $R^{1/p}$ is module-finite over R.*

a) *R is strongly F-regular iff R_P is strongly F-regular for every prime (respectively, for every maximal) ideal P of R. Hence, if R is strongly F-regular, so is $S^{-1}R$ for every multiplicative system S.*

b) *If S is faithfully flat over R and strongly F-regular then so is R.*

c) *If R is regular, then R is strongly F-regular.*

d) *If R is strongly F-regular, then R is F-regular.*

e) *If R' is strongly F-regular and R is a direct summand of R' as an R-module, then R is strongly F-regular. In particular, a direct summand (as a module over itself) of a regular ring is strongly F-regular.*

f) *If R is weakly F-regular and Gorenstein, then R is strongly F-regular.*

Proof: a) First suppose that R is strongly F–regular. We show that $S^{-1}R$ is strongly F–regular for every S. Let a nonzero element $c = c'/s$ in $S^{-1}R$ be given, where $c' \in R^0$ and $s \in S$. Choose q such that $h : R \longrightarrow R^{1/q}$ sending 1 to $c'^{1/q}$ has a splitting g. Then $(1/s^{1/q})(S^{-1}h)$ has as a splitting the map which sends x to $(S^{-1}g)(s^{1/q}x)$. (Note that $(S^{-1}R)^{1/q} \cong S^{-1}(R^{1/q})$ canonically).

On the other hand, suppose that R_m is strongly F–regular for every maximal ideal m and let $c \in R^0$ be given. For each maximal ideal m choose $q(m)$ such that the map $R_m \longrightarrow R_m^{1/q(m)} \cong (R^{1/q(m)})_m$ which sends 1 to $c^{1/q(m)}$ splits. One then gets a splitting for the same $q(m)$ on a Zariski open neighborhood. Taking a finite subcover and the supremum q of the finite set of values of $q(m)$ used in constructing it, we obtain a splitting of h_m, where h is the map $R \longrightarrow R^{1/q}$ sending 1 to $c^{1/q}$, for every maximal ideal m of R, and this implies that h has a splitting.

b) Suppose $c \in R^0$. Choose q such that $Sc^{1/q} \subseteq S^{1/q}$ splits over S. Since S is faithfully flat over R, $Rc^{1/q} \subseteq R^{1/q}$ splits iff it splits after applying $S \otimes_R$. But we get a splitting by composing $S \otimes_R R^{1/q} \longrightarrow S^{1/q}$ with the map $S^{1/q} \longrightarrow Sc^{1/q}$ which splits the inclusion $Sc^{1/q} \subseteq S^{1/q}$.

c) Suppose that R is regular. To prove F–regularity, it suffices to do so locally, by a). We may assume without loss of generality that (R,m) is local. Since R is regular, $R^{1/q}$ is free over R. Let $c \in R^0$ be given and choose q so large that $c \notin m^q$. Then $c^{1/q} \notin m(R^{1/q})$ and so is part of a free basis for $R^{1/q}$ over R. The existence of the required map is then obvious.

d) Suppose $x \in I^*$ and that $c \in R^0$ is such that $cx^q \in I^{[q]}$ for all $q \geq q'$. Then $c^{1/q}x \in IR^{1/q}$ and we can choose q so large that there is an R–linear map $R^{1/q} \longrightarrow R$ sending $c^{1/q}$ to 1. Applying this map yields that $x \in I$. Thus, $I^* = I$ for every ideal I.

Let $c \in R^0$ and an R'–splitting of the map $R' \longrightarrow R'^{1/q}$ which sends $c^{1/q}$ to 1 be given. Simply compose with an R–linear map splitting $R \subseteq R'$ and restrict the composition to $R^{1/q}$ to obtain the desired splitting.

f) The issue is local on the maximal ideals of R. Thus, it suffices to prove that if (R,m) is a local Gorenstein domain which is weakly F–regular and $R^{1/q}$ is module–finite over R then for every $c \in R^0$, $Rc^{1/q} \subseteq R^{1/q}$ splits for sufficiently large q. Let $x_1,\ldots,x_d$ be a system of parameters for R and let the image of $u \in R$ generate the socle in $R/(x_1,\ldots,x_d)$. By [H1, Remark 2, pp. 30 and 31], since R is Gorenstein the map $R \longrightarrow M$ sending 1 to m splits iff $(x_1^t \ldots x_d^t)um \notin (x_1^{t+1},\ldots,x_d^{t+1})M$ for all nonnegative integers t, and when M has depth d this simply is the condition that $um \notin (x_1,\ldots,x_d)M$. We see that it will suffice to choose q such that $uc^{1/q} \notin (x_1,\ldots,x_d)R^{1/q}$, i.e. such that $cu^q \notin ((x_1,\ldots,x_d)R)^{[q]}$. It is possible to do this, since $u \notin (x_1,\ldots,x_d)R = I$ and I is tightly closed. QED

Remark. The argument for part f) actually shows that a Gorenstein local ring is strongly F–regular provided the ideal generated by a single system of parameters is tightly closed. We remark that, quite generally, a Gorenstein local ring is F–regular (not just weakly F–regular) if the ideal generated by one system of parameters is tightly closed : it is not necessary that $R^{1/p}$ be module–finite over R. See [HH2].

Remark. When $R^{1/p}$ is module–finite over the domain R we can always choose $c \in R^0$ such that $(R^{1/p})_c \cong (R_c)^{1/p}$ is free over R_c : for such a c, R_c is regular and, hence, strongly F–regular.

Remark 3.2. Suppose that $R^{1/p}$ is module–finite over the domain R and that $c \in R^0$ is such that R_c is strongly F–regular. Then for every $d \in R^0$ there is an integer $q = p^e$, an integer $t \geq 0$ and an R–linear map $R^{1/q} \longrightarrow R$ which sends $d^{1/q}$ to c^t. To see this, choose q sufficiently large that there is an R_c–linear map $g : (R^{1/q})_c \longrightarrow R_c$ such that $g(d^{1/q}) = 1$. Since $R^{1/q}$ is module–finite over R, $c^t g(R^{1/q}) \subseteq R$ for sufficiently large t, and then $c^t g$ restricted to $R^{1/q}$, has the required property. Notice that we may replace t by any larger integer : in particular, we may assume that it is a power of p.

3.3. THEOREM. *Let R be a Noetherian domain of positive prime characteristic p such that $R^{1/p}$ is module-finite over R.*

a) *Let c be any element of R^0 such that R_c is strongly F-regular (such elements always exist). Then R is strongly F-regular if and only if there exists $q = p^e$ such that $R[c^{1/q}] \subseteq R^{1/q}$ splits over R.*

b) *The set $\{P \in \text{Spec}\, R : R_P$ is strongly F-regular$\}$ is Zariski open in* Spec R.

Proof : a) The existence of such a c is proved in the first of the two remarks just preceding the Theorem. Let $d \in R^0$ be given. Since R_c is strongly F–regular, by Remark 3.2 we can choose an R–linear map $R^{1/q'}$ to R taking $d^{1/q'}$ to $c^{q''}$. The inverse of the iterated Frobenius endomorphism gives an isomorphism of the map $R \subseteq R^{1/q'}$ with the map $R^{1/qq''} \subseteq R^{1/qq'q''}$ and it follows that there is an $R^{1/qq''}$–linear map of $R^{1/qq'q''}$ to $R^{1/qq''}$ which sends $d^{1/qq'q''}$ to $c^{q''/qq''} = c^{1/q}$, taking $(qq'')^{th}$ roots. We may compose this with an $R^{1/q}$–linear map of $R^{1/qq''}$ to $R^{1/q}$ which sends 1 to 1 (and, hence, $c^{1/q}$ to $c^{1/q}$), and then with the R–linear map from $R^{1/q}$ to R which sends $c^{1/q}$ to 1 guaranteed by the hypothesis. This establishes part a).

b) Choose $c \in R^0$ such that R_c is strongly F–regular. Suppose R_P is strongly F–regular for a certain prime P. Then we can choose q and a splitting of $R_P c^{1/q} \subseteq (R_P)^{1/q}$ or, equivalently, of $(Rc^{1/q} \subseteq R^{1/q})_P$, and this splitting extends to a Zariski neighborhood, so that for a certain $d \notin P$, we have a splitting of $(Rc^{1/q} \subseteq R^{1/q})_d$, and it then follows from part a) that R_Q is strongly F–regular for all primes Q with $d \notin Q$. QED

One of the apparently mysterious aspects of tight closure is the nature of the element c such that $cx^q \in I^{[q]}$ for all sufficiently large q. In the definition c is allowed to vary with both x and I. As mentioned earlier, if R is reduced or if I has positive height one can replace the condition "for all sufficiently large q" by the condition "for all q".

It is natural to ask whether, for "good" choices of R, there is an element $c \in R^0$ such that for every ideal $I \subset R$ and every element $x \in R$, $x \in I^*$ iff $cx^q \in I^{[q]}$ for all q. We refer to such an element as a *test element.* It is proved in [HH2, §6] that if R is module–finite, torsion–free and generically smooth over a regular domain A then R has a test element, as does every localization of R. We note that, in particular, every algebra essentially of finite type over a field has a test element.

The constructions of test elements given in [HH2] provide only a very limited class. One of the pleasant consequences of the theory of strong F–regularity is that one can use it to show that every R such that $R^{1/p}$ is module–finite over R has a test element and, in fact, an abundance of test elements : every element in the ideal which defines the locus of primes P where R_P is not strongly F–regular has a power which is a test element. In particular, in the case of an isolated singularity, there is a power of the maximal ideal defining the singular point all of whose elements not in R^0 are test elements. This follows from :

3.4. THEOREM. *Let R be a Noetherian domain of positive prime characteristic p such that $R^{1/p}$ is module-finite over R. Then every element c' of R^0 such that $R_{c'}$ is strongly F-regular has a power which is a test element.*

More precisely, c' has a power c such that there is an R-linear map h of $R^{1/p}$ to R which sends 1 to c, *and for* $c \in R^0$ *with this property such that R_c is strongly F-regular, c^3 is a test element.*

Proof : The existence of h follows from Remark 3.2 applied with $d = 1$. We next observe that the existence of h implies that for all q, there is an R–linear map $R^{1/q} \to R$ which sends 1 to c^2 : ch works if $q = p$, while given such a map $g : R^{1/q} \to R$ we get a map $g' : R^{1/pq} \to R^{1/p}$ by taking p^{th} roots which is $R^{1/p}$–linear and sends 1 to $c^{2/p}$. Multiplying by $c^{(p-2)/p}$ yields a map which sends 1 to c and we may then apply h, for $h(c) = c^2$.

Now suppose $x \in I^*$. Then there is a $d \in R^0$ such that $dx^q \in I^{[q]}$ for all q. We must show that c^3 has the same property. As in Remark 3.2 we may choose q' and q'' such that there is an R–linear map of $R^{1/q'}$ to R sending $d^{1/q'}$ to $c^{q''}$. Let q be a varying power of p. Taking qq''^{th} roots we obtain an $R^{1/qq''}$–linear map f of $R^{1/qq'q''}$ to $R^{1/qq''}$ sending $d^{1/qq'q''}$ to $c^{1/q}$. Since $dx^{qq'q''} \in I^{[qq'q'']}$ taking $qq'q''^{th}$ roots yields that $d^{1/qq'q''}x \in IR^{1/qq'q''}$. Applying the map f we find that $c^{1/q}x \in IR^{1/qq''}$. From the first paragraph we know that there is an R–linear map $g : R^{1/q''} \to R$ sending 1 to c^2 and hence c to c^3. It follows that there is an $R^{1/q}$–linear map $R^{1/qq''} \to R^{1/q}$ sending $c^{1/q}$ to $(c^3)^{1/q}$. Applying this map we see that $(c^3)^{1/q}x \in IR^{1/q}$, and taking q^{th} powers yields exactly the fact we need. QED

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