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SOME RESULTS ABOUT EXPONENTIAL FIELDS (SURVEY)

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Summary
In the present paper, a survey about some results from the theory of exponential fields is given. The investigations are motivated by Tarski's decidability problem of the field of real numbers with an additional exponential function. For solving Tarski's problem it seems to be useful to have more information about special exponential fields and classes of such structures. So different axiomatic classes of exponential fields and their theories are investigated. Especially, the solution of the "dominance problem" and the "problem of the last root" for exponential terms are given here.

§ 1 Introduction

In the present paper, a survey about some results from the theory of exponential fields is given. The investigations of this theory are motivated by A. Tarski's decidability problem of the field of real numbers with an additional exponential function. In recent years several people have been concerned with exponential fields and rings and obtained interesting results (see e.g. [Dr], [HR], [M], [R], [Wi], [DW1], [DW2], [Da], [Wo]), but Tarski's problem is still open and a solution is not in sight for the time being. However independent of the mentioned problem, the class of exponential fields is a very interesting subject of investigation. Only the interplay of analytical and algebraic means yields fundamental results, where the algebraic methods have often to be developed first.

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In the papers [DW1], [DW2], [Da], [Wo] B.I. Dahn and I investigated different classes of exponential fields with the intention to get more information on such structures and classes and their theories in order to give perhaps a contribution to the solution of Tarski's decidability problem. The most important results from our papers are presented here in a survey and without proofs.

Definition. If F is a field and E a unary function from F into F, then (F,E) is said to be an exponential field if for all $x,y \in F$ it holds that $E(x+y) = E(x)E(y)$ and $E(o) = 1$, $E(1) \neq 1$. In this case E is said to be an exponential function on F.

In the following let L be a language for exponential fields, i.e. L contains the usual symbols $+, -, \cdot, ^{-1}$ and an additional unary function symbol E for an exponential function. Further, let E_{ax} be the set of axioms $E(x+y) = E(x)E(y)$, $E(o) = 1$, $E(1) \neq 1$ and let EF be an $\forall$-axiom system for fields of characteristic o augmented by E_{ax}. Then EF determines the theory of exponential fields. The most important models of EF are (R,e) and (C,e), where R and C are the fields of real and complex numbers, respectively, and e is the usual exponential function in these fields.
We could also regard exponential fields of characteristic p, p a
$1 = E(o) = E(px) = E(x)^p$ and finally we get $E(x) = 1$ for all x.
in the set of the p-th roots of 1.
In the following Q denotes the field of rational numbers, Z the set of integers, F an arbitrary field of characteristic o and unless stated otherwise m,n,k,l,i,j denote natural numbers.
i can also be $\sqrt{-1}$, the actual meaning of i will be clear from the context. If F is an ordered field and $a,b \in F$, then $|a|$ is the absolute value of a and $a \sim b$ means that $|a-b|$ is smaller than all positive rational numbers. Notions and denotations not specially explained in this paper are used as usual.
Our aim is now to give a contribution to finding a recursive and complete axiom system of $Th(R,e)$ if such a system exists. So we try to approximate this theory by appropriate and natural axioms.

§ 2 Unordered exponential fields

First of all we want to provide some easy, well-known facts.

Fact 1. In (F,E) E is not uniquely determined by F and EF. Indeed, if f is an additive function from F into F and $E(f(1)) \neq 1$, then $E^*(x) = E(f(x))$ is an exponential function on F, too.

Fact 2. (C,e) is strongly undecidable.

The field of rationals is definable in (C,e) by the formula $\varphi(x) := \exists y \exists z (E(y) = E(z) = 1 \wedge z \neq o \wedge x = y/z)$. In fact, $(C,e) \models e^y = 1$ iff $y = 2q\pi i$, where $q \in Z$ and $i = \sqrt{-1}$. Since Q is strongly undecidable (see e.g. [Sh]), we have the claim and, moreover, we obtain

Fact 3. EF is undecidable.

The next lemma shows that the range of the exponential function in every EF-existentially complete model is the whole field, excepting o.

Lemma 4. [DW1]

Let $\mathbf{F} = (F,E)$.

(i). If $a \in F$ and $a \neq o$, then there is an extension $\mathbf{F}^* = (F^*,E^*)$ of $\mathbf{F}$ such that $\mathbf{F}^* \models EF$ and $\mathbf{F}^* \models \exists x(E^*(x) = a)$.

(ii). If $\mathbf{F}$ is EF-existentially complete, then $\mathbf{F} \models \exists x(E(x) = a)$ for all $a \in F$, $a \neq o$.

Similar as for (C,e), there exists a formula $\Psi(x)$ which defines the field of rationals in all EF-existentially complete models.

Theorem 5. [DW1]

Let $\mathbf{F} = (F,E)$ be EF-existentially complete. Then, for all $a \in F$, $a \notin Q$ iff $\mathbf{F} \models \exists x(E(x) = 1 \wedge E(ax) = 2) := \neg\Psi(a)$.

Since $\Psi(x)$ does not define Q in (C,e), we get

Corollary 6. [DW1]

(C,e) is not existentially complete.

By compactness arguments and the strong undecidability of Q we finally obtain from the above theorem:

Corollary 7. [DW1]

(i). EF is not companionable (and hence EF has no model completion).

(ii). Every existentially complete exponential field is strongly undecidable.

Theorem 8. [DW1]

(R,e) has no existential closure, i.e. there is no EF-existentially complete extension of (R,e) that is embeddable in every existentially complete extension of (R,e).

Our results show that the theory of EF is rather complicated and since EF has models with quite different properties, EF is not a good approximation of Th(R,e). Therefore, in the following, we confine ourselves to more special classes of such fields, namely to ordered exponential fields.

§ 3 Ordered exponential fields

Now we are going to study some parts of the universal theory of the ordered field of real numbers with exponentiation.
Let OF be an $\forall$-axiom system for ordered fields and
$T = OF \cup E_{ax} \cup \{(1 + 1/n)^n \leq E(1) \leq (1 + 1/n)^{n+1} : n > o\}$.
Since the statement $\forall x > o \forall y (E(y) = 1 + 1/x \longrightarrow E(xy) < E(1))$ is true in (R,e) but not in some non-archimedean T-models, the $\forall$-theory of T is weaker than $Th_{\forall}(R,e)$.
Hence we regard the better approximation
$OEF = OF \cup E_{ax} \cup \{E(x) \geq 1 + x\}$.
The following theorem, which can be proved by standard arguments, shows that the theory of ordered exponential fields OEF is sufficiently strong to characterize the exponential function uniquely in the standard model (R,e).

Theorem 9. [DW1]

In OEF the following formulas can be proved.

(i). $E(o) = 1$, $E(x) \geq o$.

(ii). $x \neq o \longrightarrow E(x) > 1 + x$, and hence E is strictly monotonously increasing.

(iii). $x > o \wedge E(y) = 1 + 1/x \longrightarrow E(xy) < E(1) < E((x+1)y)$.
(iv). E is continuous.
(v). E is differentiable and $E'(x) = E(x)$.

Here, the derivation is defined by means of the $\varepsilon - \delta$-technique. For proving the next results some special algebraic tools were necessary, especially we need so-called partial exponential fields. These are fields with a partial exponential function. Suitable extensions of the fields and the corresponding exponential functions finally yield

Theorem 1o. [DW1]
(i). OEF-existentially complete models are real closed fields.
(ii). In every OEF-existentially complete model the statement $\forall x > o \exists y(E(y) = x)$ is true, i.e. in such models E has the intermediate value property.

OEF is not sufficiently strong to prove the $\forall$-theory of (R,e).

Theroem 11. [DW1]
$OEF \nvdash \forall x > o(E(x) \geqslant 1 + x + x^2/2)$.

On the other hand, $OEF \vdash \forall x > 1/n(E(x) \geqslant 1 + x + x^2/2)$ for all $n > o$. Now we regard a stronger axiom system OEF'.
For this let $E_k(x) = \sum_{i \leq k} x^i/i!$ and
$OEF' = OEF \cup \{E(x) \geqslant E_k(x)\colon\ k \text{ odd}\}$.
Similar as above, OEF'-existentially complete models are real closed fields. Furthermore, in such models the intermediate value property is true for all terms without iterated exponential function. It is an open question whether this property is true for all terms and it is also open whether OEF' proves $Th_{\forall}(R,e)$.

Remark. One can prove that $Th(OEF') =$
$Th(OEF \cup \{\forall x(|x| < 1/n \longrightarrow E(x) \geqslant E_k(x)\colon$ for arbitrary fixed $n > o$ and all odd $k \geqslant 3\}$.

§ 4 A method for constructing new exponential functions

Now we want to investigate how well OEF' describes the exponential function in exponential fields. First we are going to show that in archimedean ordered OEF'-models the exponential function is uniquely determined. For this purpose let L_2 be the language L augmented by a symbol E^* for a second exponential function and $OEF_2' = OEF'(E) \cup OEF'(E^*)$ be the union of the theories OEF' formulated with E and E^*, respectively.

Theorem 12. [DW2]

Let (F,E,E^*) be a model of OEF_2' .

(i). For all $a \in F$, if $|a|$ is bounded by some natural number, then $E(a)$, $E^*(a)$ are bounded and $E(a) \sim E^*(a)$.

(ii). If F is archimedean, then $E^* = E$.

Now we regard an arbitrary model (F,E,E^*) of OEF_2' and investigate the connections between E and E^*. Theorem 9 implies that E, E^* are continuous, strictly monotonously increasing (hence injective), and that E, E^* take only positive but arbitrarily small and large values. Moreover, let E take all positive values in F. Then, for every $a \in F$ there is exactly one $b \in F$ such that $E^*(a) = E(b)$.
Defining $h(a) = b-a$ we obtain a function h from F into F such that $E^*(a) = E(a + h(a))$.

Lemma 13. (partially contained in [DW2])

h is additive and differentiable (hence continuous) and the derivation h' is o everywhere.

Of course, if F is non-archimedean, then h has not to be constant. Now let h be an arbitrary additive map from F into F and E an exponential function on F.
If $E^*(x) = E(x + h(x))$ and $E^*(x) \geq E_k(x)$ for all $x \in F$ and all k odd, then E^* is an exponential function on F in the sense of OEF' too.

Theorem 14. [DW2]

Let $(F,E) \models OEF' \cup \{\forall x > o\, \exists y(E(y) = x)\}$.

Then $E^*(x) = E(x + h(x))$ is an exponential function on F in the sense of OEF' if h is an additive map from F into F and h has the following properties:

(i). If $x \sim o$, then $|h(x)| < |x|^n$ for all n.
(ii). If x is finite, then $h(x) \sim o$.
(iii). If x is infinite and $x > o$, then $h(x) \geq o$ arbitrary.

Corollary 15.
(i). If OEF' has a prime model (in the sense of A. Robinson), then e^e is trancendental where e^e is $E(E(1))$ in the standard model.
(ii). There is a model (F,E) of OEF' such that $R \subseteq F$ and $E(a)$ is transcendental for each $a \in F \cap R$ with $a \neq o$.

If we regard the additive group of a non-archimedean exponential field (F,E) as a Q-vector space with a base B, then we can define, by means of Theorem 14, at least card(F) different functions $h: B \longrightarrow B$ with the desired properties. Hence, these functions h yield card(F) different exponential functions on the same field F.
By some suitable variations of a given exponential function (in the sense of Theorem 14) one can prove

Theorem 16. [DW2]
In every OEF'_2-existentially complete model the rationals are definable by the formuly
$\varphi(x) := \forall y(E(y) = E^*(y) \longrightarrow E(xy) = E^*(xy))$.

Corollary 17. [DW2]
(i). OEF'_2 is not companionable.
(ii). Every OEF'_2-existentially complete model is strongly undecidable.
(iii). The theory of all OEF'_2-existentially complete models is undecidable.
(iv). OEF'_2 is undecidable.

Now we do not regard $\forall$-axiom systems any longer, because we need stronger axioms if we want to investigate more interesting analytical properties of exponential fields.
Let $\mathrm{OEF}^* = \mathrm{OEF} \cup \{$Intermediate value property for terms with one variable$\} \cup \{$Rolle's Theorem for terms with one variable$\}$.
By OEF^* the inequalities $E(x) \geq E_k(x)$ can be proved if k is

odd and $k \geq 3$.
By means of Wilkie's and Richardson's results, B.I. Dahn was able to solve the following dominance problem for terms.

Theorem 18. [Da]
Let $\mathbf{F} \models \mathrm{OEF}$, $\mathbf{F} \subseteq \mathbf{F}^* \models \mathrm{OEF}^*$ and let $t(x)$, $t'(x)$ be terms with parameters from $\mathbf{F}$. Then
$\mathbf{F}^* \models \exists y \forall x (x \geq y \longrightarrow t(x) \geq t'(x))$ iff
$\mathrm{Diagram}(\mathbf{F}) \cup \mathrm{OEF}^* \vdash \exists y \forall x (x \geq y \longrightarrow t(x) \geq t'(x))$.

This theorem finally implies

Theorem 19. [Da]
If $\mathbf{F}, \mathbf{F}^* \models \mathrm{OEF}^*$, $\mathbf{F} \subseteq \mathbf{F}^*$ and $\varphi(x)$ is a quantifier-free formula with one variable and parameters from F, then
$\mathbf{F} \models \exists x \varphi(x)$ iff $\mathbf{F}^* \models \exists x \varphi(x)$.

This result is a little hint that OEF^* could be model complete.

Theorem 2o. [Da]
Let $\mathbf{F} \models \mathrm{OEF}^*$, $a \in \mathbf{F}$ and let $t(x)$ be a term with one variable and parameters from $\mathbf{F}$.
If $\mathbf{F} \models \lim_{x \to \infty} t(x) = a$, then there is a constant term t^* (with the same parameters and the same number of iteration steps of E as t) such that $\mathbf{F} \models t^* = a$.

The latter theorem implies that the limit of a term t belongs already to the exponential field generated by the parameters from t.
We now want to investigate the "Problem of the last root" for exponential terms, which is induced by the following question of A. Macintyre (see [Dr]).
Let $p(x)$ be a non-zero exponential polynomial over R.
Is there an intelligible function which depends only on the real parameters of $p(x)$ and which bounds the absolute values of the real roots of $p(x)$?
The next theorems answer this question positively not only for exponential polynomials in the standard model but also for all non-zero exponential terms with one variable in all OEF^*-models.
Let $\mathbf{F} \models \mathrm{OEF}^*$ and let T be the theory OEF^* augmented by the

diagram of **F**.

Theorem 21. [Wo]

If $t(x)$ is a non-zero term with one variable and with parameters from **F**, then there exists a c in **F** such that:

(i). If $\mathbf{F} \models \exists y \forall x > y(t(x) > o)$, then $T \vdash \forall x(x > c \rightarrow t(x) > o)$.

(ii). If $\mathbf{F} \models \exists y \forall x > y(t(x) < o)$, then $T \vdash \forall x(x > c \rightarrow t(x) < o)$.

(iii). $T \vdash \forall x(t(x) = o \rightarrow |x| \leq c)$.

Now we want to sharpen this result in some sense.

Theorem 22. [Wo]

If $t(x)$ is a non-zero term with one variable and with parameters from **F**, then one can compute a constant term t^* (depending only on the parameters of $t(x)$) such that $\mathbf{F} \models \forall x(t(x) = o \rightarrow |x| \leq t^*)$.

Finally I want to present some problems that have arisen in discussions with B.I. Dahn and which are still open in my opinion.

1. Is $Th_\forall(OEF') = Th_\forall(R,e)$?
2. Is the intermediate value property for terms with one variable true in all OEF'-existentially complete models ?
3. Is $E(E(1)) = E^*(E^*(1))$ if (F,E), (F,E^*) are models of OEF^* ?
4. Is there a prime model (in the sense of A. Robinson) for one of the regarded theories ?
5. Is one of the theories model complete ?
6. Is OEF' ∪ {Intermediate value property for terms with one variable} complete (analogous to the theory of ordered fields) ?

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