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WEIGHTED ORBITAL INTEGRALS ON SL(2,ℝ)

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Weighted orbital integrals appear in the adelic version of the Selberg trace formula. They give tempered, but non-invariant, distributions on the local groups. In this paper the general notion of Fourier transform for a non-invariant distribution is discussed. In the case when the local group is SL(2,ℝ) the full Fourier transform of the weighted orbital integral is given. The formula is then interpreted in terms of known properties of weighted orbital integrals.

Les intégrales orbitales à poids figurent dans la formula des traces de Selberg au cas global. Ils donnent des distributions tempérées, mais non invariantes, des groupes locals. L'objet du ce travail est de donner des formules explicites pour la transformée de Fourier des integrales orbitales à poids lorsque le groupe local est SL(2,ℝ) . Il faut d'abord préciser la notion de transformée de Fourier d'une distribution non invariante. Enfin on démontre que la formule vérifie les propiétés connues des integrales orbitales à poids.

§1. Introduction.

The adelic version of the Selberg trace formula for rank one groups involves a variety of terms which yield interesting tempered distributions on the various local groups, in particular, on real reductive Lie groups. The calculation of the Fourier transforms of these distributions is an important aspect of the use of the trace formula in the theory of automorphic forms.

There are two main types of distributions which must be studied. The Fourier transforms of the first type, ordinary orbital integrals, were calculated for semisimple Lie groups of real rank one by Sally and Warner [6], and for groups of arbitrary rank by the author [5b]. The second, and less understood, type of distributions are the so-called weighted orbital integrals. For real groups, Arthur computed the Fourier transforms of these weighted orbital integrals restricted to the space of cusp forms [1b,c]. His methods can be generalized to include a larger class of functions in the case that the weighting is not as severe as possible [5a]. Finally, in the real rank one case, Warner has computed the Fourier transform on K-biinvariant functions for a certain limit of weighted orbital integrals [7].

The results presented in this paper give the complete Fourier transform of the weighted orbital integral and its associated singular counterpart for the case of $SL(2,\mathbb{R})$ and represent joint work with J. Arthur and P. Sally. In §2 notation and background information are given and the Fourier inversion formula is stated for regular elements. A sketch of the proof is given, but details will appear elsewhere. In §3 the inversion formula is interpreted in terms of the general properties of weighted orbital integrals proved by Arthur. Also, the Fourier transform of the associated singular distribution is given.

§2. The Fourier Inversion Formula.

Let $G = SL(2,\mathbb{R})$, the group of two-by-two matrices with real entries and determinant one. We will need to consider the following subgroups of G :

$$K = \{ t_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} : 0 \leq \theta \leq 2\pi \} ;$$

$$A = \{ h_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} : t \in \mathbb{R} \} ;$$

$$A_I = \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \} ;$$

$$N = \{ n_y = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} : y \in \mathbb{R} \} ;$$

$$\bar{N} = \{ \bar{n}_y = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} : y \in \mathbb{R} \}.$$

If $x \in G$ is decomposed according to the Iwasawa decomposition as $x = k\bar{n}a$ where $k \in K$, $\bar{n} \in \bar{N}$, and $a \in A$, define $v(x) = v(\bar{n}) = \frac{1}{2}\log(1+y^2)$ if $\bar{n} = \bar{n}_y$ as above. Then v is left K and right A-invariant and is the weighting function used to define the weighted orbital integral which occurs in the Selberg trace formula for $SL(2,\mathbb{R})$. Thus for $f \in C_c^\infty(G)$, the weighted orbital integral of f is the function on the Cartan subgroup $H = A_I A$ defined by

$$(2.1) \quad T_f(\gamma h_t) = |e^t - e^{-t}| \int_{G/A} f(x\gamma h_t x^{-1}) v(x) d\dot{x}, \quad \gamma \in A_I,\ t \neq 0 .$$

Here $d\dot{x}$ is a suitably normalized G-invariant measure on the quotient space G/A. It will be useful also to consider the (unweighted) orbital integral or "invariant integral"

$$(2.2) \quad F_f(\gamma h_t) = |e^t - e^{-t}| \int_{G/A} f(x\gamma h_t x^{-1}) d\dot{x}, \quad f \in C_c^\infty(G). \quad \gamma \in A_I \quad t \neq 0.$$

The invariant integral was studied by Harish-Chandra and shown to have the following properties [4a,b,c].

(I-1) For fixed $h \in H' = \{\gamma h_t : \gamma \in A_I,\ t \neq 0\}$, the distribution $F(h): f \longrightarrow \langle F(h),f\rangle = F_f(h)$, $f \in C_c^\infty(G)$, is tempered. That is, it extends continuously to the Schwartz space $\mathcal{C}(G)$.

(I-2) The distributions $F(h)$, $h \in H'$, are invariant. That is, for any $f \in \mathcal{C}(G)$ and $y \in G$, define $f^y \in \mathcal{C}(G)$ by $f^y(x) = f(y\,x\,y^{-1})$, $x \in G$. Then $\langle F(h),f\rangle = \langle F(h),f^y\rangle$.

(I-3) For fixed $f \in \mathcal{C}(G)$, $\gamma \in A_I$, consider the function on $\mathbb{R}\setminus\{0\}$ defined by $t \longmapsto F_f(\gamma:t) = F_f(\gamma h_t)$. It is an even function and tends to zero as $|t| \longrightarrow \infty$. Although initially defined only for $t \neq 0$, it extends to a smooth function on all of $\mathbb{R}$.

(I-4) There is a left and right invariant differential operator z on G so that

$$\frac{d^2}{dt^2} F_f(\gamma h_t) = F_{zf}(\gamma h_t).$$

Arthur has established the following properties of the weighted orbital integral which are similar to those of the invariant integral [1a,b,c,d].

(W-1) For fixed $h \in H'$, the distribution $T(h): f \longrightarrow \langle T(h),f\rangle = T_f(h)$ is tempered.

(W-2) The distributions $T(h)$, $h \in H'$, are <u>not</u> invariant. However, they are K-central. That is, for $k \in K$, $\langle T(h),f\rangle = \langle T(h),f^k\rangle$ for all $f \in \mathcal{C}(G)$. Further, there is a specific non-invariant distribution $T_N(h)$, defined in terms of its Fourier transform, such that $T(h) - T_N(h)$ is invariant.

(W-3) For fixed $f \in \mathcal{C}(G)$, $\gamma \in A_I$, $t \longmapsto T_f(\gamma:t) = T_f(\gamma h_t)$ is an even function, tending to zero at infinity. Although smooth for $t \neq 0$, it is badly behaved at $t = 0$. To describe more exactly its behavior at zero, define $S_f(\gamma:t) = T_f(\gamma:t) + \log(1-e^{-2t})F_f(\gamma:t)$. Then $S_f(\gamma:t)$ is continuous at $t = 0$ and its first derivative has well-defined one-sided limits at zero satisfying $\lim_{t\downarrow 0} \frac{d}{dt} S_f(\gamma:t) - \lim_{t\uparrow 0} \frac{d}{dt} S_f(\gamma:t) = cf(\gamma)$ where c is a constant. The singular weighted orbital integral associated to $T_f(\gamma h_t)$ is defined by

(2.3) $$T_f(\gamma) = \lim_{t \to 0} S_f(\gamma:t).$$

(W-4) $\frac{d^2}{dt^2} T_f(\gamma:t) = T_{zf}(\gamma:t) + (\sinh t)^{-2} F_f(\gamma:t)$ where z is the differential operator on G appearing in (I-4).

For $f \in C_c^\infty(G)$, there are two possible definitions of the Fourier transform of f, namely the operator-valued and scalar-valued Fourier transforms. Let $\hat{G}$ denote the set of equivalence classes of irreducible unitary representations of G. (We will make no distinction between an equivalence class and a representation of that class.) For $\pi \in \hat{G}$, $f \in C_c^\infty(G)$, define $\pi(f) = \int_G f(x)\pi(x)\,dx$. Then $\pi(f)$ is a trace class operator on $\mathcal{H}_\pi$, the Hilbert space of the representation π. The operator-valued Fourier transform of f is the operator-valued function on $\hat{G}$ defined by $\mathcal{F}f(\pi) = \pi(f)$. The scalar-valued Fourier transform is the complex-valued function on $\hat{G}$ given by $\hat{f}(\pi) = \mathrm{tr}\,\pi(f)$.

Now if Λ is an invariant distribution on G, by the Fourier transform $\hat{\Lambda}$ of Λ we mean a "distribution" on $\hat{G}$ satisfying $\hat{\Lambda}(\hat{f}) = \Lambda(f)$, $f \in C_c^\infty(G)$. A formula describing the Fourier transform $\hat{\Lambda}$ is a Fourier inversion formula for $\Lambda(f)$, that is an expansion of $\Lambda(f)$ in terms of the distributional characters $f \longrightarrow \mathrm{tr}\,\pi(f)$. It typically would have the form $\Lambda(f) = \int_{\hat{G}} \mathrm{tr}\,\pi(f)\, d_\Lambda(\pi)$ where d_Λ is some measure on $\hat{G}$. If Λ is a tempered distribution, then d_Λ should be supported on the tempered spectrum of G.

For example, consider the tempered invariant distribution given by the invariant integral $F(h)$, $h \in H'$. Then the Fourier transform of $F(h)$ is supported on the unitary principal series of representations induced from the parabolic subgroup $P = A_I AN$. For $\chi \in \hat{A}_I$, $\lambda \in \mathbb{R}$, define $\pi^{\chi,\lambda} = \mathrm{Ind}_P^G(\chi \otimes e^{i\lambda} \otimes 1)$. Then

(2.4) $$F_f(\gamma h_t) = \sum_{\chi \in \hat{A}_I} \chi(\gamma) \int_{-\infty}^{\infty} \mathrm{tr}\,\pi^{\chi,\lambda}(f) \cos \lambda t \, d\lambda.$$

(See [6] for details.)

On the other hand, if Λ is a non-invariant distribution, it is no longer possible to expand $\Lambda(f)$ in terms of the invariant distributions $\operatorname{tr}\pi(f)$, $\pi \in \hat{G}$. Thus we must work with the operator-valued Fourier transform of f and look for a "distribution" on $\hat{G}$ so that $\mathcal{F}\Lambda(\mathcal{F}f) = \Lambda(f)$. The Fourier inversion formula in this case would be expected to take the form $\Lambda(f) = \int_{\hat{G}} \operatorname{tr}\{A_\Lambda(\pi)\pi(f)\} d_\Lambda(\pi)$ where for each $\pi \in \hat{G}$, $A_\Lambda(\pi)$ is an operator on $\mathcal{H}_\pi$. If Λ is invariant, $A_\Lambda(\pi)$ would be scalar so that it pulls outside the trace. Of course the operator $A_\Lambda(\pi)$ and the measure $d_\Lambda(\pi)$ are not well-defined independently of each other.

The Fourier transform of $T(h)$, $h \in H'$, is supported on both the unitary principal series and on $\hat{G}_d$, the discrete series representations of G.

THEOREM. Let f be a K-finite function in $\mathcal{C}(G)$, $\gamma \in A_I$, $t \neq 0$. Then

$$T_f(\gamma h_t) = -|e^t - e^{-t}| \sum_{\pi \in \hat{G}_d} \Theta_\pi(\gamma h_t) \operatorname{tr}\pi(f)$$

$$+ \sum_{\chi \in \hat{A}_I} \chi(\gamma) \int_{-\infty}^{\infty} \phi_\lambda(t) \operatorname{tr}\pi^{\chi,\lambda}(f)\, d\lambda$$

$$+ i \sum_{\chi \in \hat{A}_I} \chi(\gamma) \int_{-\infty}^{\infty} e^{-i\lambda|t|} \operatorname{tr}\{M_\chi'(\lambda) M_\chi(\lambda)^{-1} \pi^{\chi,\lambda}(f)\}\, d\lambda$$

$$+ \pi \operatorname{tr}\pi^{+,0}(f).$$

The first term in the formula is the discrete series contribution computed by Arthur in [1b,c]. For $\pi \in \hat{G}_d$, Θ_π denotes the character of π as a function on the regular set of G. This term is invariant since it involves only $\operatorname{tr}\pi(f)$.

The remaining three terms correspond to principal series representations. The first and last of these are also invariant. The (scalar-valued) function $\phi_\lambda(t)$ is the solution of an inhomogeneous second order differential equation coming from (W-4). It is given by the integral formulas

(2.5)
$$\phi_\lambda(t) = \frac{1}{\lambda}\int_{|t|}^{\infty} \sin\lambda(u-|t|)\cos\lambda u\,(\sinh u)^{-2}\,du, \quad t \neq 0,\ \lambda \neq 0$$
$$\phi_0(t) = \lim_{\lambda\to 0}\phi_\lambda(t) = \int_{|t|}^{\infty}(u-|t|)(\sinh u)^{-2}\,du, \quad t \neq 0.$$

The last term is a point mass at the principal series representation corresponding to the trivial character of A_I and $\lambda = 0$. Note that the "π" preceding the trace is the number $\pi \sim 3.14$.

The remaining term is the only non-invariant part of the formula. The induced representations $\pi^{\chi,\lambda}$ are viewed in the compact realization as acting on a fixed Hilbert space $\mathcal{H}^\chi$ for all $\lambda \in \mathbb{R}$. Then $M_\chi(\lambda)$ denotes the operator on $\mathcal{H}^\chi$ which intertwines $\pi^{\chi,\lambda}$ with the equivalent representation $\bar{\pi}^{\chi,\lambda} = \mathrm{Ind}_{\bar{P}}^{G}(\chi \otimes e^{i\lambda} \otimes 1)$ where $\bar{P} = A_I A\bar{N}$. (The normalization of $M_\chi(\lambda)$ will be specified later.) $M_\chi'(\lambda)$ denotes the derivative with respect to λ of this family of operators.

Although the Fourier transforms of orbital integrals can be computed directly for arbitrary $f \in C_c^\infty(G)$ by using character formulas on G and abelian harmonic analysis, the non-invariance of the weighted orbital integrals requires a different approach. The Schwartz function f is specialized to a matrix coefficient of a discrete series representation or a wave packet corresponding to matrix coefficients of principal series representations. For such an f, the differential equation and boundary conditions can be used to find an expression for $T_f(h)$, which is then interpreted in terms of representations.

For example, suppose f is a matrix coefficient of a discrete series representation π. Then f is an eigenfunction of the differential operator z and $F_f = 0$ so that the differential equation (W-4) satisfied by T_f becomes $\frac{d^2}{dt^2}T_f(\gamma:t) = n^2 T_f(\gamma:t)$. Here n is an integer corresponding to the Harish-Chandra parameter for the discrete series representation π. Since $T_f(\gamma:t)$ is smooth for $t \neq 0$, even, and decaying at infinity, there is a constant $c(\gamma,f)$ so that $T_f(\gamma:t) = c(\gamma:f)e^{-|nt|}$. Finally $S_f(\gamma:t) = T_f(\gamma:t)$ since $F_f = 0$ and we use the formula for the jump of the derivative at $t = 0$ of $S_f(\gamma:t)$ given in (W-3) to conclude that $-2|n|c(\gamma:f) = 2f(\gamma) = 2\chi_\pi(\gamma)f(e)$. Here $\chi_\pi(\gamma)$ is the character of A_I which takes the value $(-1)^{n+1}$ on the non-trivial element of A_I. But by the Plancherel formula, $f(e) = |n|\,\mathrm{tr}\,\pi(f)$. Thus $c(\gamma:f) = -\chi_\pi(\gamma)\mathrm{tr}\,\pi(f)$. But $\Theta_\pi(\gamma h_t) = |e^t - e^{-t}|^{-1}\chi_\pi(\gamma)e^{-|nt|}$ so that we have

$$T_f(\gamma h_t) = - \left| e^t - e^{-t} \right| \Theta_\pi(\gamma h_t) \text{ tr } \pi(f).$$

This is the formula proved by Arthur. It agrees with the theorem in this situation since, because of orthogonality, all other terms are zero.

The matrix coefficients of discrete series representations span the space of cusp forms of G. It remains to check the theorem for functions in the subspace of the Schwartz space orthogonal to the cusp forms. This is the space spanned by wave packets corresponding to principal series representations.

For $\chi \in \hat{A}_I$, $\lambda \in \mathbb{R}$, $\pi^{\chi,\lambda} = \text{Ind}_P^G(\chi \otimes e^{i\lambda} \otimes 1)$ can be realized on the Hilbert space $\mathcal{H}^\chi = \{g \in L^2(K) : g(k\gamma) = \chi(\gamma)g(k)$ for all $\gamma \in A_I$, $k \in K\}$. For $n \in Z$, let ω_n be the character of K given by $\omega_n(t_\theta) = e^{in\theta}$, $t_\theta \in K$. Let $Z_\chi = \{n \in Z : \omega_n(\gamma) = \chi(\gamma)$ for all $\gamma \in A_I\}$. For χ the trivial character of A_I, Z_χ is the set of even integers. For χ non-trivial, Z_χ is the set of odd integers. Then $\mathcal{H}^\chi$ has as basis $\{\omega_n : n \in Z_\chi\}$. We wish to consider matrix coefficients of $\pi^{\chi,\lambda}$ with respect to this basis of $\mathcal{H}^\chi$. Because the distribution is K-central, only the diagonal entries are needed. Thus for $n \in Z_\chi$, we define $E(n:\lambda:x) = \langle \pi^{\chi,\lambda}(x)\omega_n, \omega_n \rangle$.

Unfortunately, $E(n:\lambda) \notin C(G)$. In order to get an analogue of a matrix coefficient which can be plugged into the distribution $T(h)$, it is necessary to form wave packets. Thus for $\alpha \in C(\mathbb{R})$, the Schwartz space on $\mathbb{R}$, $n \in Z_\chi$, we define

$$f(x) = f(\alpha:n:x) = \int_{-\infty}^{\infty} \alpha(\lambda)E(n:\lambda:x)\mu_\chi(\lambda)d\lambda. \tag{2.6}$$

Here $\mu_\chi(\lambda)d\lambda$ is the Plancherel measure corresponding to the representation $\pi^{\chi,\lambda}$. Then $f \in C(G)$ and has the following properties [4c].

(P-1) For all $k_1, k_2 \in K$, $x \in G$, $f(k_1 x k_2) = \omega_n(k_1k_2)f(x)$.

In particular $f(\gamma x) = \chi(\gamma)f(x)$ for all $\gamma \in A_I$, $x \in G$.

(P-2) The matrix of $\pi^{\chi,\lambda}(f)$, with respect to the basis $\{\omega_n : n \in Z_\chi\}$, has only one non-zero entry. For $n \in Z_\chi$,

$$\pi^{\chi,\lambda}(f)\omega_m = \begin{cases} 0 & \text{if} \quad m \neq n ; \\ [a(\lambda) + a(-\lambda)]\omega_n & \text{if} \quad n = m. \end{cases}$$

(P-3) For $\chi' \in \hat{A}_I$, $\chi' \neq \chi$. $\pi^{\chi',\lambda}(f) \equiv 0$.

(P-4) Combining (P-2) and (P-3) above,

$$\operatorname{tr} \pi^{\chi',\lambda}(f) = \begin{cases} 0 & \chi' \neq \chi ; \\ a(\lambda) + a(-\lambda), & \chi' = \chi . \end{cases}$$

(P-5) Since $E(n:\lambda)$ is an eigenfunction of the differential operator z with eigenvalue $p(\lambda) = -\lambda^2$. f satisfies $(zf)(a:n) = f(pa:n)$.

For $f = f(a:n)$ we will evaluate $T_f(h)$, $h \in H'$. As a result of (P-1), since A_I is central in G, it is clear that $T_f(\gamma h_t) = \chi(\gamma)T_f(h_t)$ for $\gamma \in A_I$, $t \neq 0$. Thus it is enough to study $T(t:a:n) = T_f(h_t)$, $t \neq 0$. Further, because T is an even function of t, it suffices to compute $T(t:a:n)$, $t > 0$. The first step is to obtain an asymptotic formula for $T(t:a:n)$ as $t \longrightarrow +\infty$.

Writing G/A as $K\bar{N}$, using the invariance of f under K implied by (P-1), and a standard change of variables for $\bar{N}$, we write

$$\begin{aligned} T(t:a:n) &= |e^t - e^{-t}| \int_K \int_{\bar{N}} f(k\bar{n}h_t\bar{n}^{-1}k^{-1})v(\bar{n})\, d\bar{n}\, dk \\ &= |e^t - e^{-t}| \int_{-\infty}^{\infty} f(\bar{n}_y h_t \bar{n}_y^{-1}) \tfrac{1}{2} \log(1 + y^2)\, dy \\ &= e^t \int_{-\infty}^{\infty} f(\bar{n}_y h_t) \tfrac{1}{2} \log(1 + (1 - e^{-2t})^{-2} y^2)\, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\lambda)\mu_\chi(\lambda) e^t E(n:\lambda:\bar{n}_y h_t) \tfrac{1}{2} \log(1+(1-e^{-2t})^{-2}y^2)\, d\lambda\, dy. \end{aligned}$$

To evaluate this expression as $t \longrightarrow +\infty$ we will use Harish-Chandra's asymptotic formula for the Eisenstein integral [4a] and the following observations.

(A-1) As $t \longrightarrow +\infty$, $\frac{1}{2}\log(1 + (1 - e^{-2t})^{-2}y^2) \longrightarrow \frac{1}{2}\log(1 + y^2)$.

Write $H(y) = \frac{1}{2}\log(1 + y^2)$ and let $h(y)$ denote the element of A with diagonal entries $(1 + y^2)^{1/2}$ and $1/(1 + y^2)^{1/2}$. Then $\bar{n}_y h_t = k(y)h(y)h_t h_t^{-1} n(y) h_t$ where $k(y) \in K$ and $n(y) \in N$. Note that $h_t^{-1} n(y) h_t \longrightarrow 1$ as $t \longrightarrow +\infty$.

(A-2) As $t \to +\infty$, $E(n:\lambda:\bar{n}_y h_t) \longrightarrow E(n:\lambda:k(y)h(y)h_t) = \omega_n(k(y))E(n:\lambda:h(y)h_t)$.

(A-3) As $t \to +\infty$,

$$e^{(t+H(y))}E(n:\lambda:h(y)h_t) \longrightarrow c_n(\lambda)e^{i\lambda(t+H(y))} + c_n(-\lambda)e^{-i\lambda(t+H(y))}.$$

Here $c_n(\lambda)$ is the c-function which is the meromorphic continuation to the real axis of the analytic function defined for $\mu \in \mathbb{C}$ with $\operatorname{Im}\mu < 0$ by

$$c_n(\mu) = \int_{-\infty}^{\infty} \omega_n(k(y))e^{(-1-i\mu)H(y)}\,dy. \tag{2.7}$$

Combining the above observations, we find that as $t \longrightarrow +\infty$,

$$T(t:\alpha:n) \longrightarrow R(t:\alpha:n) =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha(\lambda)\mu_\chi(\lambda)e^{i\lambda t}c_n(\lambda)\omega_n(k(y))e^{(-1+i\lambda)H(y)}\,H(y)\,d\lambda\,dy$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha(\lambda)\mu_\chi(\lambda)e^{-i\lambda t}c_n(-\lambda)\omega_n(k(y))e^{(-1-i\lambda)H(y)}\,H(y)\,d\lambda\,dy.$$

In order to simplify the expression for $R(t:\alpha:n)$ we change the order of integration and note that $\int_{-\infty}^{\infty} \omega_n(k(y))e^{(-1\pm i\lambda)H(y)}\,H(y)\,dy = ic_n'(\pm\lambda)$. Of course the integral is not convergent for real λ, so that we must shift the λ-integration into the appropriate half-plane in each term to get convergence of the integral in y, and then shift back. In doing this we pick up a residue at $\lambda=0$. We also use the fact that for any $n \in Z_\chi$, $\mu_\chi(\lambda) = c_n(\lambda)^{-1}c_n(-\lambda)^{-1}$ to obtain the formula

$$(2.8)\qquad R(t:a:n) = i\int_{-\infty}^{\infty} a(\lambda)\{e^{i\lambda t}c_n'(-\lambda)c_n(-\lambda)^{-1} + e^{-i\lambda t}c_n'(\lambda)c_n(\lambda)^{-1}\}\,d\lambda + 2\pi a(0)\mathrm{Res}_{\lambda=0}(c_n'(\lambda)c_n(\lambda)^{-1}).$$

The residue is computed to be $\begin{cases} -1 & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$ (See details in §3.)

The above asymptotic arguments are clearly heuristic only. More precisely what we actually prove is that

$$(2.9)\qquad T(t:a:n) - R(t:a:n) = 2\int_{-\infty}^{\infty} a(\lambda)\phi(t:\lambda:n)\,d\lambda$$

where $\phi(\lambda:n)$ is a function of t satisfying $\lim_{t\to+\infty} \phi(t:\lambda:n) = 0$ uniformly on compacta of λ.

By (W-4), T satisfies the differential equation

$$\frac{d^2}{dt^2} T(t:a:n) = T(t:pa:n) + (\sinh t)^{-2}F(t:a:n)$$

$$= T(t:pa:n) + 2(\sinh t)^{-2}\int_{-\infty}^{\infty} a(\lambda)\cos\lambda t\,d\lambda.$$

Also, R clearly satisfies the differential equation $\frac{d^2}{dt^2} R(t:a:n) = R(t:pa:n)$. Thus $\phi(\lambda:n)$ must satisfy the equation

$$(2.10)\qquad \frac{d^2}{dt^2}\phi(t:\lambda:n) = -\lambda^2\phi(t:\lambda:n) + \cos\lambda t\,(\sinh t)^{-2}.$$

But $\phi_\lambda(t)$ as defined in (2.5) is the unique solution of this equation satisfying $\lim_{t\to+\infty} \phi_\lambda(t) = 0$. Thus for $t > 0$,

$$T(t:a:n) = R(t:a:n) + 2\int_{-\infty}^{\infty} a(\lambda)\phi_\lambda(t)\,d\lambda.$$

To interpret this as a Fourier transform of $T_f(h_t)$ we note first that $\phi_\lambda(t)$ is an even function of λ so that

$$2\int_{-\infty}^{\infty} a(\lambda)\phi_\lambda(t)d\lambda = \int_{-\infty}^{\infty} [a(\lambda)+a(-\lambda)]\phi_\lambda(t)d\lambda = \int_{-\infty}^{\infty} \operatorname{tr}\, \pi^{\chi,\lambda}(f)\phi_\lambda(t)d\lambda.$$

Also using a change of variables, we get

$$R(t:a:n) = i\int_{-\infty}^{\infty} [a(\lambda)+a(-\lambda)]\, e^{-i\lambda t}\, c_n'(\lambda)c_n(\lambda)^{-1}d\lambda + \begin{cases} -2\pi a(0) & n \text{ even} \\ 0 & n \text{ odd}. \end{cases}$$

The c-functions $c_n(\lambda)$ are the eigenvalues of the intertwining operators $M_\chi(\lambda)$. In fact, $M_\chi(\lambda)\omega_n = c_n(\lambda)\omega_n$ for $n \in Z_\chi$. But combining this with (P-2) we see that

$$c_n'(\lambda)c_n(\lambda)^{-1}[a(\lambda) + a(-\lambda)] = \operatorname{tr}\{M_\chi'(\lambda)M_\chi(\lambda)^{-1}\pi^{\chi,\lambda}(f)\}.$$

Finally, in the case that n is even, $2a(0) = \operatorname{tr}\, \pi^{+,0}(f)$. This gives all the non-zero terms in the theorem for $t > 0$.

Observations

Recall the list (W-1) - (W-4) of general properties of weighted orbital integrals. While some were used to derive the formula for $T_f(\gamma h_t)$, f a wave packet, several of these properties are independent of the proof. It is interesting to check that the formula obtained does satisfy these conditions. In the course of doing this we will also derive a formula for the Fourier transform of the singular distribution defined by (2.3). In particular, we will look at:

(1) the genuinely non-invariant part of the distribution;

(2) the behavior of $T_f(\gamma h_t)$ as $|t| \longrightarrow \infty$;

(3) the behavior of $T_f(\gamma h_t)$ as $t \longrightarrow 0$.

As in §2, let $f = f(\alpha:n)$ be a wave packet corresponding to $n \in Z_\chi$ and $\alpha \in C(\mathbb{R})$. Then the Fourier inversion formula can be written for $t \neq 0$ as

(3.1) $$\begin{aligned} T_f(\gamma h_t) &= 2\chi(\gamma)\int_{-\infty}^{\infty} \alpha(\lambda)\phi_\lambda(t)\,d\lambda \\ &+ i\chi(\gamma)\int_{-\infty}^{\infty} \alpha(\lambda)\cos\lambda t\,[c_n'(\lambda)c_n(\lambda)^{-1} + c_n'(-\lambda)c_n(-\lambda)^{-1}]d\lambda \\ &+ \chi(\gamma)\int_{-\infty}^{\infty} \alpha(\lambda)\sin\lambda|t|\,[c_n'(\lambda)c_n(\lambda)^{-1} - c_n'(-\lambda)c_n(-\lambda)^{-1}]d\lambda \\ &+ \begin{cases} 2\pi\chi(\gamma)\alpha(0) & n \text{ even} \\ 0 & n \text{ odd.} \end{cases} \end{aligned}$$

In order to understand the properties of the Fourier transform, it is necessary to study the c-functions. Using formulas of Cohn [2]

(3.2) $$c_n(\lambda) = \frac{1}{\sqrt{\pi}}\,\frac{\Gamma(\frac{i\lambda}{2})\Gamma(\frac{i\lambda+1}{2})}{\Gamma(\frac{i\lambda+1+n}{2})\Gamma(\frac{i\lambda+1-n}{2})}.$$

Write $d_n(\lambda) = c_n'(\lambda)c_n(\lambda)^{-1}$. Then differentiating, we find that $d_n(\lambda) = \frac{i}{2}\,[\psi(\frac{i\lambda}{2}) + \psi(\frac{i\lambda+1}{2}) - \psi(\frac{i\lambda+1+n}{2}) - \psi(\frac{i\lambda+1-n}{2})]$ where $\psi(z) := \Gamma'(z)\Gamma(z)^{-1}$. The properties of ψ

can be found in [3]. Like the gamma function, it has simple poles at $z = 0,-1,-2,\ldots$ It also satisfies identities

$(\psi-1)$ $$\psi(z+1) = \psi(z) + \frac{1}{z},$$

$(\psi-2)$ $$\psi(\tfrac{1}{2} + z) - \psi(\tfrac{1}{2} - z) = \pi \tan(\pi z),$$

$(\psi-3)$ $$\psi(z) - \psi(-z) = - \pi \cot(\pi z) - \frac{1}{z}.$$

Thus if n is even, the term $\frac{1}{2}\psi(\frac{i\lambda}{2})$ contributes a simple pole at $\lambda = 0$ with residue -1. The other terms are holomorphic for all $\lambda \in \mathbb{R}$. If n is odd, we find using $(\psi-1)$ that all poles at $\lambda = 0$ cancel.

In any case, the even combination $d_n(\lambda) + d_n(-\lambda)$ will be well-behaved for all $\lambda \in \mathbb{R}$. Using the identities $(\psi-2)$ and $(\psi-3)$ we find that

$$[d_n(\lambda) - d_n(-\lambda)] =$$

$$\frac{1}{2}[-\pi \cot(\frac{i\pi\lambda}{2}) - \frac{2}{i\lambda} + \pi \tan(\frac{i\pi\lambda}{2}) - \pi \tan(\frac{i\pi\lambda+\pi n}{2}) - \pi \tan(\frac{i\pi\lambda-\pi n}{2})].$$

Using elementary trigonometric identities we find that

(3.3) $$\lambda[d_n(\lambda) - d_n(-\lambda)] = -1 + (-1)^{n+1} \frac{\pi\lambda}{\sinh \pi\lambda}.$$

Note that this shows that $d_n(\lambda) - d_n(-\lambda)$ depends only on whether n is even or odd. This means that for $\chi \in \hat{A}_I$, $M_\chi'(\lambda)M_\chi(\lambda)^{-1} - M_\chi'(-\lambda)M_\chi(-\lambda)^{-1}$ is a scalar matrix. Thus the only genuinely non-invariant part of the distribtuion $T_f(\gamma h_t)$ is the term

$$T_N(\gamma h_t) = i \sum_{\chi \in \hat{A}_I} \chi(\gamma) \int_{-\infty}^{\infty} \cos \lambda t \ \ \mathrm{tr}\{M'(\lambda)M(\lambda)^{-1}\pi^{\chi,\lambda}(f)\}\, d\lambda.$$

This is exactly the distribution defined by Arthur in [1d] and referred to in (W-2).

We now check that $\lim_{|t|\to\infty} T(t:\alpha:n) = 0$. It is easy to verify using the integral formula (2.5) for $\phi_\lambda(t)$ that there is a constant c such that for all $t \neq 0$, $|\phi_\lambda(t)| \le \frac{c}{|t|}$ uniformly in λ. Thus it is clear that for all $\alpha \in C(\mathbb{R})$, $\int_{-\infty}^{\infty} \alpha(\lambda)\phi_\lambda(t)\, d\lambda \longrightarrow 0$ as $|t| \longrightarrow \infty$.

Since $\alpha(\lambda)[d_n(\lambda) + d_n(-\lambda)] \in L^1(\mathbb{R})$, the second term in (3.1) also tends to zero as $|t| \longrightarrow \infty$. The same holds for the third term if n is odd. However, for n even the pole of $d_n(\lambda)$ at $\lambda = 0$ causes

$$\lim_{|t|\to\infty} \int_{-\infty}^{\infty} \alpha(\lambda) \sin \lambda|t|\ [d_n(\lambda) - d_n(-\lambda)]\, d\lambda = -2\pi\alpha(0).$$

This cancels the fourth term so that in any case $\lim_{|t|\to\infty} T_f(\gamma h_t) = 0$.

Finally, we look at the behavior of $T_f(\gamma h_t)$ as $t \longrightarrow 0$. As in (W-3) we define $S(t:\alpha:n) = T(t:\alpha:n) + \log(1-e^{-2t})F(t:\alpha:n)$. We wish to show that S is continuous at $t = 0$ and that its first derivative has jump $2f(e)$ at $t = 0$.

The second and fourth terms in (3.1) are smooth at $t = 0$. The third is continuous, but because of the $\sin \lambda|t|$ term has a jump in its first derivative equal to

$$2\int_{-\infty}^{\infty} \alpha(\lambda)\ \lambda\ [d_n(\lambda)-d_n(-\lambda)]d\lambda = -2\int_{-\infty}^{\infty} \alpha(\lambda)d\lambda + (-1)^{n+1}2\pi \int_{-\infty}^{\infty} \frac{\lambda\alpha(\lambda)}{\sinh \pi\lambda}\, d\lambda$$

using (3.3).

To study the first term we combine it with the $\log(1-e^{-2t})F(t:\alpha:n) = 2\log(1-e^{-2t}) \int_{-\infty}^{\infty} \alpha(\lambda) \cos \lambda t\, d\lambda$ term to obtain $2\int_{-\infty}^{\infty} \alpha(\lambda)\, h_\lambda(t)\, d\lambda$ where $h_\lambda(t) = \phi_\lambda(t) + \log(1-e^{-2t}) \cos \lambda t$. Using the differential equation and boundary behavior at infinity of $\phi_\lambda(t)$, we find that $\frac{d^2}{dt^2} h_\lambda(t) = -\lambda^2 h_\lambda(t) - \frac{4\lambda \sin \lambda t}{e^{2t} - 1}$ and $\lim_{t\to+\infty} h_\lambda(t) = 0$. Thus we can write, for $t > 0$, $h_\lambda(t) = -4\int_t^{\infty} \frac{\sin \lambda(u-t)\ \sin \lambda u}{e^{2u} - 1}\, du$.

It is now easy to check that $\lim_{t \downarrow 0} h_\lambda(t) = -4 \int_0^\infty \frac{\sin^2 \lambda u}{e^{2u} - 1} du$ is a convergent integral, and that $\lim_{t \downarrow 0} \frac{d}{dt} h_\lambda(t) = \lambda \int_0^\infty \frac{\sin \lambda u}{e^u - 1} du = \frac{\pi\lambda \cosh \pi\lambda}{\sinh \pi\lambda} - 1$.

Although $\phi_\lambda(t)$ is even, $h_\lambda(t)$ is not because of the $\log(1 - e^{-2t})$ term. However, $h_\lambda(-t) = h_\lambda(t) + 4t \cos \lambda t$. Thus $\lim_{t \uparrow 0} h_\lambda(t) = \lim_{t \downarrow 0} h_\lambda(t)$ so that $h_\lambda(t)$ is continuous at $t = 0$. Also,

$\frac{d}{dt} h_\lambda(-t) = -\frac{d}{dt} h_\lambda(t) - 4 \cos \lambda t + 4\lambda t \sin \lambda t$ so that

$\lim_{t \downarrow 0} \frac{d}{dt} h_\lambda(t) - \lim_{t \uparrow 0} \frac{d}{dt} h_\lambda(t) = 2 \lim_{t \downarrow 0} \frac{d}{dt} h_\lambda(t) + 4 = \frac{2\pi\lambda \cosh \pi\lambda}{\sinh \pi\lambda} + 2.$

Combining this with the jump from the third term we find that

$$\lim_{t \downarrow 0} \frac{d}{dt} S(t:a:n) - \lim_{t \uparrow 0} \frac{d}{dt} S(t:a:n) = 2\pi \int_{-\infty}^{\infty} \lambda a(\lambda) \frac{\cosh \pi\lambda + (-1)^{n+1}}{\sinh \pi\lambda} d\lambda$$

$$= \begin{cases} 2\pi \int_{-\infty}^{\infty} \lambda a(\lambda) \tanh \frac{\pi\lambda}{2} d\lambda & n \text{ even} \\ 2\pi \int_{-\infty}^{\infty} \lambda a(\lambda) \coth \frac{\pi\lambda}{2} d\lambda & n \text{ odd.} \end{cases}$$

This is exactly the Fourier transform of $f(e)$ given by the Plancherel theorem.

Finally, the singular weighted orbital integral studied by Arthur in [1b] is $T_f(\gamma) = \lim_{t \to 0} S_f(\gamma h_t)$, $\gamma \in A_I$. Using the above formulas we can write, for $f \in C(G)$,

$$(3.4) \qquad T_f(\gamma) = -\sum_{\pi \in \hat{G}_d} \chi_\pi(\gamma) \operatorname{tr} \pi(f) + \sum_{\chi \in \hat{A}_I} \chi(\gamma) \int_{-\infty}^{\infty} \operatorname{tr} \pi^{\chi,\lambda}(f)\, h(\lambda)\, d\lambda$$

$$+ \sum_{\chi \in \hat{A}_I} \chi(\gamma) \int_{-\infty}^{\infty} \operatorname{tr}\{M_\chi'(\lambda) M_\chi(\lambda)^{-1} \pi^{\chi,\lambda}(f)\}\, d\lambda + \pi \operatorname{tr} \pi^{+,0}(f)$$

where $h(\lambda) = 2 \int_0^\infty \frac{\sin^2 \lambda u}{1 - e^{2u}} du$.

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