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CONTINUATION OF UNITARY DERIVED FUNCTOR MODULES OUT OF THE CANONICAL CHAMBER

Thomas J. Enright[1] and Joseph A. Wolf[2]

Résumé

On décrit une méthode qui permet de suivre l'unitarité lors de la continuation cohérente des représentations des séries discretes quand le parametre sort de la chambre de Weyl de Borel-de Siebenthal. Au cas où les modules de représentations des séries discretes sont obtenus en appliquant le foncteur dérivé de Zuckerman à un module de Verma généralisé qui est construit à partir d'une représentation unidimensionelle, la methode est décrite explicitement. Des programmes d'ordinateurs traitant les cas E_6, E_7, E_8 sont présentés en appendix. Un certain nombre de nouvelles représensations singulières unitaires résultent de cette méthode.

Abstract

A method is described for following unitarity during coherent continuation of discrete series representations as the parameter passes out of the Borel-de Siebenthal Weyl chamber. In the case where the discrete series representations are derived functor modules, obtained from generalized Verma modules which in turn are induced from one-dimensional representations, the computation is carried out explicitly, and computer programs are appended which treat the E_6, E_7 and E_8 cases. A number of new singular unitary representations are produced.

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Let G be a real semisimple Lie group, let θ be a Cartan involution of G, and let K denote the maximal compactly embedded subgroup of G which is the fixed point set of θ. We write $\mathfrak{g}_0, \mathfrak{k}_0$ for the real Lie algebras, $\mathfrak{g}, \mathfrak{k}$ for their complexifications, and θ for the automorphisms induced on $\mathfrak{g}_0$ and $\mathfrak{g}$. In the work of Wallach [7] and Enright-Howe-Wallach [3], G is a simple group of hermitian type and one varies a character on the center of K in order to continue the holomorphic discrete series. In [4,5], using the Zuckerman derived functors, these results were extended to cover certain non highest weight modules. Among other results is a proof of unitarity for certain coherent continuations of discrete series representations out of the Borel-de Siebenthal [1] Weyl chamber. These results are tabulated in Appendix 1. They are based on

THEOREM. Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$, say $\mathfrak{q} = \mathfrak{m} + \mathfrak{n}$ where $\mathfrak{n}$ is the nilradical and $\mathfrak{m}$ is a θ-stable Levi component, such that $[\mathfrak{n} \cap \mathfrak{k}, \mathfrak{n}] = 0$ and $\mathfrak{m} = \mathfrak{m}_0 \otimes_{\mathbb{R}} \mathbb{C}$ where $\mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{g}_0$. Fix a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{m}_0$ of $\mathfrak{g}_0$ and $\lambda \in \mathfrak{h}^*$ such that the irreducible $\mathfrak{m}$-modules $F(\lambda)$ of highest weight λ is finite dimensional and unitarizable. Choose $\zeta \in \mathfrak{h}^*$ corresponding to a central element of $\mathfrak{m}$ and normalized so that (i) the one-dimensional $\mathfrak{m}$-module $F(\zeta)$ is unitarizable, and (ii) $\langle \zeta, \alpha \rangle > 0$ for all roots $\alpha \in \Delta(\mathfrak{n})$. Assume that the relative Verma module

$$N(\lambda + z\zeta) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} F(\lambda + z\zeta)$$

is irreducible for $z \leqq c$. Let

$$\Gamma^i : C(\mathfrak{g}, \mathfrak{m} \cap \mathfrak{k}) \to C(\mathfrak{g}, \mathfrak{k})$$

denote the i^{th} right derived functor, of the $\mathfrak{k}$-finite submodule functor, from the category of $\mathfrak{g}$-modules that are completely reducible and locally finite for $\mathfrak{m} \cap \mathfrak{k}$. Let $s = \dim \mathfrak{k}/\mathfrak{m} \cap \mathfrak{k}$. Then $\Gamma^s N(\lambda + c\zeta)$ is zero or unitarizable.

See [5] for the proof of this theorem.

This situation arises when rank K = rank G, the root ordering is such that there is just one noncompact simple root (say α_0), necessarily of coefficient 1 or 2 in the maximal root, and $\mathfrak{q}$ is the maximal parabolic subalgebra of $\mathfrak{g}$ defined by α_0. The hermitian case, coefficient 1, is the case of Enright-Howe-Wallach [3]. Now suppose that α_0 has coefficient 2 in the maximal root and denote

$$\Delta(i) = \{\text{roots } \beta : \alpha_0 \text{ has coefficient } i \text{ in } \beta\}$$

so the full root system $\Delta = \Delta(-2) \cup \Delta(-1) \cup \Delta(0) \cup \Delta(1) \cup \Delta(2)$. Then, if

$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ as usual under θ,

$$\mathfrak{m} = \mathfrak{h} + \sum_{\Delta(0)} \mathfrak{g}_\beta \quad , \qquad \mathfrak{u} \cap \mathfrak{p} = \sum_{\Delta(1)} \mathfrak{g}_\beta \qquad \text{and} \qquad \mathfrak{u} \cap \mathfrak{k} = \sum_{\Delta(2)} \mathfrak{g}_\beta$$

$\zeta \in \mathfrak{h}^*$ is normalized by

$$2\langle \zeta, \alpha_0 \rangle / \langle \alpha_0, \alpha_0 \rangle = 1 \qquad \text{and} \qquad \langle \zeta, \beta \rangle = 0 \ \text{ for all simple } \ \beta \neq \alpha_0 .$$

Given λ, we normalize it on the line $\ell = \mathbb{R}\zeta$ by adding a multiple of ζ, so that

$$\lambda + z\zeta + \rho \in C \iff z < 0$$

where C is the $\mathfrak{q}$-antidominant Weyl chamber and ρ as usual is half the sum of the positive roots. The question, now, for applying the theorem is to find the first reduction point, i.e. the smallest number $a = a(\lambda)$ such that $N(\lambda + a\zeta)$ is reducible. Then the theorem says that

> $\Gamma^s N(\lambda + z\zeta)$ is a unitarizable $(\mathfrak{g},K)$-module whenever $z < a$ and $\lambda + z\zeta$ is a $\Delta(\mathfrak{k})$-integral.

That gives quite a number of new irreducible unitary representations, as one sees from the tabulation in Appendix 1, for the case $\dim F(\lambda) = 1$. There, for example, it gives 30 new unitary representations of E_{8,E_7A_1} with singular infinitesimal character, it gives 17 for E_{7,D_6A_1}, and it gives 11 for E_{6,D_5A_1}.

The key to finding the first reduction point is, of course, Jantzen's irreducibility criterion [6] for relative Verma modules. Jantzen's contravariant form — or, equivalently, its hermitian analog — has formal determinant on $N(\lambda')$ given by

$$\prod_{\substack{\text{weights}\\ \text{of } M(\lambda')}} (\text{nonzero}) \prod_{\alpha \in \Delta(\mathfrak{u})} \prod_{\substack{n \text{ positive}\\ \text{integer}}} \left\{ \frac{1}{n} \left(\frac{2\langle \lambda' + \rho, \alpha \rangle}{\langle \alpha, \alpha \rangle} - n \right) \right\}^{\chi'(\lambda' - n\alpha)}$$

where $M(\lambda')$ is the ordinary Verma module and where

$$\chi'(\nu) = \sum_{w \in W(\mathfrak{m},\mathfrak{h})} \det(w)\, e^{w(\nu+\rho)} .$$

Here $N(\lambda')$ is irreducible if and only if the determinant is nonzero. In other words, if we define

$$\chi(\lambda', \alpha) = \sum_{w \in W(\mathfrak{m},\mathfrak{h})} \det(w)\, e^{w(s_\alpha(\lambda' + \rho))}$$

then $N(\lambda')$ can only reduce when

a) there is at least one root $\alpha \in \Delta(u)$ such that $2\langle \lambda'+\rho,\alpha\rangle / \langle \alpha,\alpha \rangle$ is a positive integer, and

b) the sum over all such roots α, of the $\chi(\lambda',\alpha)$, does not vanish.

Thus we can test $N(\lambda + z\zeta)$ for reduction in a systematic manner, as follows:

1. Compute $2\langle\lambda+\rho,\alpha\rangle/\langle\alpha,\alpha\rangle$ for every $\alpha \in \Delta(\mathfrak{n})$. Note that $2\langle\lambda+z\zeta+\rho,\alpha\rangle/\langle\alpha,\alpha\rangle$ is obtained from it by adding $z \times$ (coefficient of α_0 in α). Note that z must be an integer or half-integer in order for $\lambda+z\zeta+\rho$ to be $\Delta(\mathfrak{k})$-integral, thus in order for $\Gamma^s N(\lambda+z\zeta)$ to be nonzero. Start with z the smallest integer or half-integer such that some $\langle\lambda+z\zeta+\rho,\alpha\rangle/\langle\alpha,\alpha\rangle$, $\alpha \in \Delta(\mathfrak{n})$, is a positive integer.

2. Let $A = \{\alpha\in\Delta(\mathfrak{n})\colon 2\langle\lambda+z\zeta+\rho,\alpha\rangle/\langle\alpha,\alpha\rangle$ is a positive integer$\}$. If A is empty then $N(\lambda+z\zeta)$ is irreducible.

3. Suppose A is not empty. Given $\alpha\in A$, suppose there is some $\beta\in\Delta(\mathfrak{n})$ with $\langle\lambda+z\zeta+\rho,\beta\rangle = 0$ and with $s_\alpha(\beta) \in \Delta(\mathfrak{m})$, i.e. with $\langle s_\alpha(\beta),\zeta\rangle = 0$. Then $\langle s_\alpha(\lambda+z\zeta+\rho), s_\alpha(\beta)\rangle = \langle\lambda+z\zeta+\rho,\beta\rangle = 0$ shows $s_\alpha(\lambda+z\zeta+\rho)$ to be $\Delta(\mathfrak{m})$-singular, so $\chi(\lambda+z\zeta,\alpha) = 0$.

4. Let $B = \{\alpha\in A$: there is no β as in (3) above$\}$. If B is empty then $N(\lambda+z\zeta)$ is irreducible.

5. Suppose B is not empty. If $\alpha\in B$ then $\chi(\lambda+z\zeta,\alpha) \neq 0$. Thus the only way that we can have $\sum_{\alpha\in B}\chi(\lambda+z\zeta,\alpha) = 0$ is if the set B decomposes into pairs $\{\alpha,\alpha'\}$ such that $s_\alpha(\lambda+z\zeta+\rho)$ and $s_{\alpha'}(\lambda+z\zeta+\rho)$ differ by an element $w \in W(\mathfrak{m},\mathfrak{h})$ which has $\det(w) = -1$. Note that, for such a pair, $\langle s_\alpha(\lambda+z\zeta+\rho),\zeta\rangle = \langle ws_\alpha(\lambda+z\zeta+\rho),\zeta\rangle = \langle s_{\alpha'}(\lambda+z\zeta+\rho),\zeta\rangle$, so we need only try to pair off elements $\alpha,\alpha' \in B$ for which $\langle s_\alpha(\lambda+z\zeta+\rho),\zeta\rangle = \langle s_{\alpha'}(\lambda+z\zeta+\rho),\zeta\rangle$.

6. Let $C = \{\alpha\in B$: there is no $\alpha'\in B$ forming a pair $\{\alpha,\alpha'\}$ as in (5) above$\}$. Then $N(\lambda+z\zeta)$ is irreducible if and only if C is empty.

7. If $N(\lambda+z\zeta)$ is reducible then $\lambda+z\zeta$ is the first reduction point. If $N(\lambda+z\zeta)$ is irreducible, increase z by $\frac{1}{2}$ and start over again at (2).

In the case $\dim F(\lambda) = 1$, we recently carried out this computational program by hand for the classical simple groups G with rank K = rank G, by arranging the "matrix" $(2\langle\lambda+z\zeta+\rho,\alpha\rangle/\langle\alpha,\alpha\rangle)_{\alpha\in\Delta(\mathfrak{n})}$ in a way that made the various tests relatively straightforward and that made induction arguments convenient. This doesn't work for the exceptional groups, but the structures of types G_2 and F_4 are sufficiently small so that hand calculation was not difficult.

We illustrate the test for reduction by doing the case $G = Sp(1,n-1)$. Let α_i, $1\le i\le n$, be the simple roots with $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $1\le i<n$, and $\alpha_n = 2\varepsilon_n$. For

our example let $\alpha_0 = \alpha_1$. Then $\Delta(\mathfrak{n}) = \{2\varepsilon_1\} \cup \{\varepsilon_1 \pm \varepsilon_i | 2 \le i \le n\}$. Our normalizations give $\zeta = \varepsilon_1$, $\lambda = (-2n+1)\varepsilon_1$ and $\rho = (n, n-1, \ldots, 1)$. So

$$\lambda + z\zeta + \rho = (z-n+1, n-1, n-2, \ldots, 1) \quad .$$

Consider the array

$$\begin{matrix} & a_2 & a_3 & \cdots & a_n \\ c & & & & \\ & b_2 & b_3 & \cdots & b_n \end{matrix}$$

with $c = 2\langle\lambda + z\zeta + \rho, \alpha\rangle / \langle\alpha, \alpha\rangle$ with $\alpha = 2\varepsilon_1$, and a_i and b_i given by the same formula for $\alpha = \varepsilon_1 + \varepsilon_i$ and $\varepsilon_1 - \varepsilon_i$ respectively $2 \le i \le n$. Evaluating these inner products gives:

$$\begin{matrix} & z & z-1 & \cdots & z-n+2 \\ z-n+1 & & & & \\ & z-2n+2 & z-2n+3 & \cdots & z-n \end{matrix}$$

Now $N(\lambda + 2\zeta)$ is irreducible if $z \notin \mathbb{N}$. For $z \in \mathbb{N}$, $0 \le z \le n-1$, the set A is not empty. But B is empty and so $N(\lambda + z\zeta)$ is irreducible. For $n \le z \le 2n-2$ the array has a positive integer value for c, positive integer values in the top row and to the right of the $\varepsilon_1 - \varepsilon_j$ entry which is a zero, $j = 2n-z$. In this case $A = \{2\varepsilon_1\} \cup \{\varepsilon_1 + \varepsilon_\ell | 2 \le \ell \le n\} \cup \{\varepsilon_1 - \varepsilon_k | 2n-z+1 \le k \le n\}$. However, $B = \{2\varepsilon_1, \varepsilon_1 + \varepsilon_j\}$. Then $\lambda + z\zeta + \rho = (n-j+1, n-1, \ldots, 1)$ where $n-j+1$ occurs both as the first and j^{th} coordinate. Therefore, as described in (5) above, the two elements of B form a pair. Thus C is empty and $N(\lambda + z\zeta)$ is irreducible. Finally, at $z = 2n-1$, the highest weight is zero; and so, $N(\lambda + z\zeta)$ reduces with the trivial representation as quotient. This proves that the first reduction point is $z = 2n-1$.

As a second example we consider the case of the split real form of G . Let α_1 and α_2 be the two simple roots with α_2 long. Assume α_1 is compact and α_2 is noncompact. So in the notation above $\alpha_0 = \alpha_2$. Following Bourbaki we write these vectors in coordinates:

$$\alpha_1 = \varepsilon_1 - \varepsilon_2 \quad , \qquad \alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \quad .$$

Now

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1 + 2\alpha_2\}$$

$$\Delta(\mathfrak{n}) = \Delta^+ \setminus \{\alpha_1\}$$

Our normalizations give

$$\zeta = (-1,-1,2) \ , \qquad \lambda = -2\zeta = (2,2,-4) \ , \qquad \rho = (-1,-2,3)$$

$$\lambda + z\zeta + \rho = (-z+1, \ -z, \ 2z-1) \quad .$$

Now consider the array of numbers $2\langle\lambda+z\zeta+\rho,\alpha\rangle/\langle\alpha,\alpha\rangle$ where α runs through $\Delta(\mathfrak{u})$ in order

$$\begin{pmatrix} & \alpha_1+\alpha_2 = (-1,0,1) & 2\alpha_1+\alpha_2 = (0,-1,1) \\ \alpha_2 = (-2,1,1) & 3\alpha_1+\alpha_2 = (1,-2,1) & 3\alpha_1+2\alpha_2 = (-1,-1,2) \end{pmatrix}$$

The roots in the first row are short while those in the second row are long. Evaluating we obtain:

$$\begin{pmatrix} & 3z-2 & 3z-1 \\ z-1 & z & 2z-1 \end{pmatrix}$$

The element $\lambda+z\zeta$ is $\Delta(\mathfrak{k})$ integral precisely when $2z$ is an integer. Also $z=2$ corresponds to the trivial representation. $N(\lambda+z\zeta)$ is irreducible unless either $3z$ or $2z$ is an integer.

In the cases $z = 1/3$ and $z = 1/2$, the set A is empty. So $N(\lambda+z\zeta)$ is irreducible. For $z = 2/3$, $A = \{2\alpha_1+\alpha_2\}$, $\alpha_1+\alpha_2$ is singular at $\lambda+z\zeta+\rho$ and $s_{2\alpha_1+\alpha_2}(\alpha_1+\alpha_2) = -\alpha_1 \in \Delta(\mathfrak{m})$. So B is empty and $N(\lambda+z\zeta)$ is irreducible.

The next value is $z=1$, with array

$$\begin{pmatrix} & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} .$$

The singular root α_2 is long so $s_\beta(\alpha_2) \notin \Delta(\mathfrak{m})$ for any β. So $B = \{\alpha_1+\alpha_2,\ 2\alpha_1+\alpha_2,\ 3\alpha_1+\alpha_2,\ 3\alpha_1+2\alpha_2\}$. Here $\lambda+z\zeta+\rho = (0,-1,-1)$. Let $\nu = \lambda+z\zeta+\rho$. Then

$$s_{\alpha_1+\alpha_2}(\nu) = (1,-1,0), \qquad s_{3\alpha_1+\alpha_2}(\nu) = (-1,1,0),$$
$$s_{2\alpha_1+\alpha_2}(\nu) = (0,1,-1), \qquad s_{3\alpha_1+2\alpha_2}(\nu) = (1,0,-1).$$

But these cancel in pairs as in (5) so C is empty and $N(\lambda+z\zeta)$ is irreducible.

The next possible value for $N(\lambda+z\zeta)$ to reduce is $z=4/3$. Here the array is

$$\begin{pmatrix} & 2 & 3 \\ 1/3 & 4/3 & 5/3 \end{pmatrix} .$$

So $\lambda+z\zeta+\rho$ is regular and A is not empty. This implies directly that $N(\lambda+z\zeta)$ is reducible. Thus the first reduction point is $z = 4/3$.

We did the calculation for E_6, E_7 and E_8 using a simple BASIC program, listed in Appendix 2, on a home computer. As is clear from the listing, efficiency was not a consideration because it only had to be run once. In particular it left the search through $W(\mathfrak{m},\mathfrak{h})$, in (5) above, to be done by hand. Since there is no essential change, we wrote the program for $\dim F(\lambda) < \infty$ rather than just $\dim F(\lambda) = 1$.

The results above hold in a more general setting, if we add a hypothesis of $\mathfrak{k}$-semisimplicity for the $N(\lambda + z\zeta)$.

PROPOSITION. Let $\mathfrak{q} = \mathfrak{m} + \mathfrak{u}$ be a maximal θ-stable parabolic subalgebra of $\mathfrak{g}$, whose complementary simple root α_0 has coefficient 2 in the maximal root (as above). Suppose that $\mathfrak{q}$ is quasi-abelian, i.e. that $\langle\alpha,\beta\rangle \geqq 0$ for $\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{k})$ and $\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p})$. Let $F(\zeta)$ and $F(\lambda)$ be unitarizable $\mathfrak{m}_0$-modules, $\dim F(\zeta) = 1$ and $\dim F(\lambda) < \infty$, normalized by

$$2\langle\zeta,\alpha_0\rangle/\langle\alpha_0,\alpha_0\rangle = 1 \qquad \text{and} \qquad \lambda + z\zeta + \rho \in C \Longleftrightarrow z < 0 \quad .$$

Let $a = a(\lambda)$ be the first reduction point for $N(\lambda + z\zeta)$ and suppose that $N(\lambda + z\zeta)$ is semisimple as a $\mathfrak{k}$-module whenever $z < a$. Then $\Gamma^S N(\lambda + z\zeta)$ is zero or unitarizable for $z < a$.

In the case $\dim F(\lambda) = 1$ of the more general setting, the last two columns of the chart in Appendix 1 need not apply, for $\Gamma^S N(\lambda + z\zeta) = 0$ is a real possibility. So it is essential to have some *à priori* information on the $\mathfrak{k}$-spectrum of $\Gamma^S N(\lambda + z\zeta)$.

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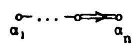
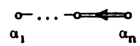
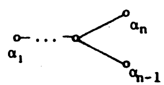
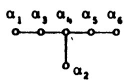
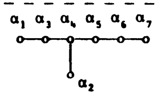
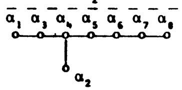
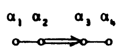
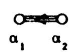
Appendix 1

Root system	Diagram	Complementary simple root	First reduction point $z = a$	Condition for $\Delta(k)$ integrality	Number of unitary $\Gamma^S N(\lambda)$, $0 \le z < a$	Unitarity of $\Gamma^S L(\lambda)$ at $z = a$
B_n		$2 \le \ell \le n-1$	$n - \frac{\ell}{2} - \frac{1}{2}$ if ℓ is odd and $n - \ell \le \frac{(\ell-1)}{2}$;	$2z \in \mathbb{Z}$	$2n - \ell - 1$	In both cases, yes if $a \le 2n - 2\ell$;
			$n - \frac{\ell}{2}$ otherwise		$2n - \ell$	? otherwise
		$\ell = n$	$2\left[\frac{n}{2}\right] + 1$	$z \in \mathbb{Z}$	$2\left[\frac{n}{2}\right] + 1$	?
C_n		$2 \le \ell \le n-1$	$n - \frac{\ell}{2}$ if ℓ is even and $n - \ell \ge \frac{\ell}{2}$; $n - \frac{\ell}{2} + \frac{1}{2}$ otherwise	$z \in \mathbb{Z}$	$n - \frac{\ell}{2}$ if ℓ is even and $n - \ell \ge \frac{\ell}{2}$; $n - [(\ell-1)/2]$ otherwise	If $a \notin \mathbb{Z}$ then $\Gamma^S L(a) = 0$. $\Gamma^S L(a)$ is unitary if $a \in \mathbb{Z}$ and $a < 2n - 2\ell$; ? otherwise
		$\ell = 1$	$2n-1$	$z \in \mathbb{Z}$	$2n-1$	$\Gamma^S L(a) = 0$
D_n		$2 \le \ell \le n-2$	$n - \frac{\ell}{2} - 1$ if ℓ is even and $n - \ell \le \frac{\ell}{2}$;	$2z \in \mathbb{Z}$	$2n - 2 - \ell$	In both cases, yes if $a \le 2n - 2\ell - 1$;
			$n - \frac{\ell}{2} - \frac{1}{2}$ otherwise		$2n - 1 - \ell$	? otherwise
E_6		3 or 5	5	$2z \in \mathbb{Z}$	10	yes
		2	11/2		11	
E_7		1	17/2	$2z \in \mathbb{Z}$	17	yes
		2	7		14	
		6	6		12	
E_8		1	23/2	$2z \in \mathbb{Z}$	23	yes
		8	29/2		29	
F_4		1	4	$2z \in \mathbb{Z}$	8	yes
		4	5	$z \in \mathbb{Z}$	5	?
G_2		2	4/3	$2z \in \mathbb{Z}$	3	$\Gamma^S L(a) = 0$

Appendix 2

These programs are written in Microsoft Basic (MBASIC), Revision 5.21, for CP/M

Run this program to create Cartan matrix and inverse Cartan matrix data files:

```
10 '*****************************************************************
20 'Program to write the Cartan matrices and their inverses for E6, E7, E8
30 'as sequential MBASIC files. Shortens data programs from March 1983. Uses
40 'Bourbaki order (1)-(3)-(4)-(5)-(6)-(7)-(8)        E6: (1) thru (6)
50 '                        :                         E7: (1) thru (7)
60 '                       (2)                        E8: (1) thru (8)
70 'for the simple roots.                 Joseph Wolf, 30 September, 1983
80 '*****************************************************************
90 DIM Z(8,8): FOR I=1 TO 8: FOR J=1 TO 8: READ Z(I,J): NEXT J: NEXT I
100 OPEN "o", #1, "CARTE6.DAT": OPEN "o", #2, "CARTE7.DAT"
110 OPEN "o", #3, "CARTE8.DAT"
120 FOR I=1 TO 6: FOR J=1 TO 6: WRITE #1, Z(I,J): NEXT J: NEXT I: CLOSE #1
130 FOR I=1 TO 7: FOR J=1 TO 7: WRITE #2, Z(I,J): NEXT J: NEXT I: CLOSE #2
140 FOR I=1 TO 8: FOR J=1 TO 8: WRITE #3, Z(I,J): NEXT J: NEXT I: CLOSE #3
150 DATA 2, 0, -1, 0, 0, 0, 0, 0, 0, 2, 0, -1, 0, 0, 0, 0
160 DATA -1, 0, 2, -1, 0, 0, 0, 0, 0, -1, -1, 2, -1, 0, 0, 0
170 DATA 0, 0, 0, -1, 2, -1, 0, 0, 0, 0, 0, 0, -1, 2, -1, 0
180 DATA 0, 0, 0, 0, 0, -1, 2, -1, 0, 0, 0, 0, 0, 0, -1, 2
190 '
200 DIM C(6,6): FOR I=1 TO 6: FOR J=1 TO 6: READ C(I,J): NEXT J: NEXT I
210 OPEN "o", #1, "INVCART6.DAT"
220 FOR I=1 TO 6: FOR J=1 TO 6: WRITE #1, C(I,J): NEXT J: NEXT I: CLOSE #1
230 DATA 1.33333, 1, 1.66667, 2, 1.33333, 0.666667, 1, 2, 2, 3, 2, 1, 1.66667
240 DATA 2, 3.33333, 4, 2.66667, 1.33333, 2, 3, 4, 6, 4, 2, 1.33333, 2, 2.66667
250 DATA 4, 3.33333, 1.66667, 0.666667, 1, 1.33333, 2, 1.66667, 1.33333
260 '
270 DIM D(7,7): FOR I=1 TO 7: FOR J=1 TO 7: READ D(I,J): NEXT J: NEXT I
280 OPEN "o", #2, "INVCART7.DAT"
290 FOR I=1 TO 7: FOR J=1 TO 7: WRITE #2, D(I,J): NEXT J: NEXT I: CLOSE #2
300 DATA 2, 2, 3, 4, 3, 2, 1, 2, 3.5, 4, 6, 4.5, 3, 1.5, 3, 4, 6, 8, 6, 4, 2
310 DATA 4, 6, 8, 12, 9, 6, 3, 3, 4.5, 6, 9, 7.5, 5, 2.5
320 DATA 2, 3, 4, 6, 5, 4, 2, 1, 1.5, 2, 3, 2.5, 2, 1.5
330 '
340 DIM E(8,8): FOR I=1 TO 8: FOR J=1 TO 8: READ E(I,J): NEXT J: NEXT I
350 OPEN "o", #3, "INVCART8.DAT"
360 FOR I=1 TO 8: FOR J=1 TO 8: WRITE #3, E(I,J): NEXT J: NEXT I: CLOSE #3
370 DATA 4, 5, 7, 10, 8, 6, 4, 2, 5, 8, 10, 15, 12, 9, 6, 3
380 DATA 7, 10, 14, 20, 16, 12, 8, 4, 10, 15, 20, 30, 24, 18, 12, 6
390 DATA 8, 12, 16, 24, 20, 15, 10, 5, 6, 9, 12, 18, 15, 12, 8, 4
400 DATA 4, 6, 8, 12, 10, 8, 6, 3, 2, 3, 4, 6, 5, 4, 3, 2
410 END
```

Run this program to create root matrix data files. It uses the Cartan matrices.

```
10 ' ***********************************************************
20 '     program to write roots of e6, e7 and e8 in bourbaki order
30 '     (1)--(3)--(4)--(5)--(6)--(7)--(8)     e6: (1) thru (6)
40 '                 :                          e7: (1) thru (7)
50 '                (2)                         e8: (1) thru (8)
60 '     onto files ROOTSE6.DAT, ROOTSE7.DAT AND ROOTSE8.DAT
70 '
80 '     Joseph Wolf  12 March 1983
90 ' ***********************************************************
```

```
100 DIM C(8,8)
110 OPEN "i", #1, "CARTE8.DAT"
120 FOR I = 1 TO 8: FOR J = 1 TO 8: INPUT #1, C(I,J): NEXT J: NEXT I
130 CLOSE #1
140 '
150 DIM R(120,8)        'matrix of roots, not sorted
160 DIM S(8,8)          'matrix of simple roots
170 FOR I = 1 TO 8: FOR J = 1 TO 8
180 IF I = J THEN R(I,J) = 1 ELSE R(I,J) = 0
190 S(I,J) = R(I,J)     'simple roots set up here for reference
200 NEXT J: NEXT I
210 '
220 DIM P(120,8)       'r(i,*)-p(i,j)s(j,*) thru r(i,*)+q(i,j)s(j,*)
230 DIM Q(120,8)       'is the maximal s(j,*)-root-string thru r(i,*)
240 FOR I = 1 TO 120: FOR J = 1 TO 8
250 P(I,J) = 0: Q(I,J) = 0
260 NEXT J: NEXT I
270 FOR I = 1 TO 8: P(I,I) = 2
280 FOR J = 1 TO 8: IF I <> J THEN Q(I,J) = -C(I,J)
290 NEXT J: NEXT I
300 '
310 DIM STARTLEV(30) 'the roots of level l are r(startlev(l),*)
320 DIM STOPLEV(30)   'through r(stoplev(l),*)
330 STARTLEV(1) = 1: STOPLEV(1) = 8
340 B = 9               'pointer for next root to go into r(*,*)
350 LEV = 1             'current level
360 '
370 LEV = LEV + 1       'next level
380 PRINT "Starting level "; LEV; ":"
390 STARTLEV(LEV) = B
400 '
410 U = STARTLEV(LEV - 1): V = STOPLEV(LEV - 1)
420 FOR I = U TO V: FOR J = 1 TO 8
430 IF Q(I,J) = 0 THEN GOTO 890
440 '
450 FOR N = 1 TO 8
460 IF N = J THEN T(N) = R(I,N) + 1 ELSE T(N) = R(I,N)
470 NEXT N
480 '
490 IF STARTLEV(LEV) = B GOTO 580
500 FOR N = STARTLEV(LEV) TO B-1
510 DUPETEST = 0
520 FOR M = 1 TO 8
530 DUPETEST = DUPETEST + (T(M) - R(N,M))^2
540 NEXT M
550 IF DUPETEST = 0 GOTO 890       't(*) already on list of roots
560 NEXT N
570 '
580 FOR N = 1 TO 8
590 R(B,N) = T(N)      'add t(*) to the list of roots
600 NEXT N
610 PRINT "Just added root";B ;": (";T(1);T(2);T(3);T(4);T(5);T(6);T(7);T(8);")"
620 '
630 FOR M = 1 TO 8                            'simple root s(m,*)
640 HITFLAG = 0                    'see whether r(b,*) - s(m,*) is root
650 FOR N = U TO V                            'r(n,*) of previous level
```

```
660 DIFFLAG = 0
670 FOR L = 1 TO 8
680 DIFFLAG = DIFFLAG + (R(B,L) - S(M,L) - R(N,L))^2
690 NEXT L
700 '    if difflag =  0 then r(b,*) - s(m,*) =  r(n,*)
710 '    if difflag <> 0 then r(b,*) - s(m,*) <> r(n,*)
720 IF DIFFLAG <> 0 GOTO 770            'next n
730 HITFLAG = 1
740 P(B,M) = P(N,M) + 1
750 Q(B,M) = Q(N,M) - 1
760 GOTO 860                            'next m
770 NEXT N
780 '    if hitflag = 0 then r(b,*) - s(m,*) not a root so
790 '            R(B,*) AT BOTTOM OF THE S(M,*)-STRING THROUGH IT
800 '    if hitflag <> 0 then p(b,m), q(b,m) have already been set
810 IF HITFLAG <> 0 GOTO 860            'next m
820 INNER = 0                           '<r(b,*),s(m,*)>
830 FOR L = 1 TO 8: INNER = INNER + R(B,L)*C(L,M): NEXT L
840 Q(B,M) = -INNER
850 P(B,M) = 0
860 NEXT M
870 '    analysis of r(b,*) complete now
880 B = B+1
890 NEXT J: NEXT I
900 '    stop when no new roots produced on current level
910 IF B = STARTLEV(LEV) GOTO 950
920 STOPLEV(LEV) = B-1
930 GOTO 370
940 '
950 PRINT "Number of roots calculated"; B-1
960 '
970 '    WRITE ROOTS TO FILES ROOTSE6.DAT, ROOTSE7.DAT AND ROOTSE8.DAT
980 '
990 OPEN "o", #1, "ROOTSE6.DAT"
1000 OPEN "o", #2, "ROOTSE7.DAT"
1010 OPEN "o", #3, "ROOTSE8.DAT"
1020 FOR I = 1 TO 120                               'run through roots r(i,*)
1030 IF R(I,8) <> 0 GOTO 1080
1040 IF R(I,7) <> 0 GOTO 1080
1050 FOR J=1 TO 6: PRINT #1, R(I,J): NEXT J     'roots of E6
1060 FOR J=1 TO 7: PRINT #2, R(I,J): NEXT J
1070 FOR J=1 TO 8: PRINT #3, R(I,J): NEXT J
1080 NEXT I
1090 FOR I = 1 TO 120                               'run through roots again
1100 IF R(I,8) <> 0 GOTO 1140
1110 IF R(I,7) =  0 GOTO 1140
1120 FOR J=1 TO 7: PRINT #2, R(I,J): NEXT J     'roots of E7 which are
1130 FOR J=1 TO 8: PRINT #3, R(I,J): NEXT J     'not roots of E6
1140 NEXT I
1150 FOR I = 1 TO 120                               'final run through roots
1160 IF R(I,8) =  0 GOTO 1180
1170 FOR J = 1 TO 8: PRINT #3, R(I,J): NEXT J   'roots of E8 but not E6 or E7
1180 NEXT I
1190 CLOSE #1: CLOSE #2: CLOSE #3
1200 END
```

Main reducibility program. It uses all data files written by programs above.

```
10 ' ************************************************************
20 ' Generalized Verma Module Reducibility Program  23 March 1983
30 '          Joseph A. Wolf and Thomas J. Enright
40 '
50 ' A maximal parabolic P is chosen in E6, E7 or E8 by means of the
60 ' choice of the complementary simple root (NONC), which is assumed
70 ' to have coefficient 2 in the expression of the maximal root in
80 ' terms of simple roots.  A real form is envisioned in which (NONC)
90 ' is the noncompact simple root in Borel-DeSiebenthal root ordering.
100 '
110 ' This program finds the first reduction point for the generalized
120 ' Verma modules associated to a finite dimensional representation of
130 ' the reductive part M of the parabolic P = MN.
140 ' ************************************************************
150 ' INITIALIZATION:
160 '   Choose algebra and noncompact simple root.
170 '    Set up Cartan matrix, RHO and ZETA.
180 '   Read in the root system.
190 ' ************************************************************
200 PRINT "ROOTS IN BOURBAKI ORDER (1)--(3)--(4)--(5)--(6)--(7)--(8)"
210 PRINT "WHERE E6 INVOLVES 1-6 ONLY          !"
220 PRINT "E7 INVOLVES 1-7, ETC.              (2)"
230 PRINT
240 INPUT "Which algebra E?  Enter 6, 7 or 8. ", RANK
250 IF RANK <> 6 AND RANK <> 7 AND RANK <> 8 THEN 310
260 DIM C(RANK, RANK)            'Cartan matrix
270 DIM CINV(RANK, RANK)         'inverse of Cartan matrix
280 DIM ZETA(RANK)               'relevant row of cinv
290 DIM RHO(RANK)
300 ON RANK - 5 GOTO 330, 440, 550
310 PRINT "Rank";RANK;"not valid here; must be 6, 7 or 8."
320 GOTO 230
330 PRINT
340 PRINT "Coefficients of the simple roots in the   1    2    3    2    1"
350 PRINT "maximal root are indicated here.  Enter  (1)--(3)--(4)--(5)--(6)"
360 PRINT "the number of a root of coefficient 2              !"
370 PRINT "which will be the noncompact simple root.         (2) 2"
380 PRINT
390 INPUT NONC: IF NONC = 2 OR NONC = 3 OR NONC = 5 THEN 410
400 PRINT "Root";NONC;"not appropriate here; check instructions.": GOTO 330
410 OPEN "i", #1, "CARTE6.DAT": OPEN "i", #2, "INVCART6.DAT"
420 RHO(1)=8:RHO(2)=11:RHO(3)=15:RHO(4)=21:RHO(5)=15:RHO(6)=8
430 GOTO 680
440 PRINT
450 PRINT "Coefficients of the simple roots in   2    3    4    3    2    1"
460 PRINT "the maximal root are indicated here. (1)--(3)--(4)--(5)--(6)--(7)"
470 PRINT "Enter the number of a root of coef.             !"
480 PRINT "2, for the noncompact simple root.             (2) 2"
490 INPUT NONC
500 IF NONC = 1 OR NONC = 2 OR NONC = 6 THEN 520
510 PRINT "Root";NONC;"not appropriate here; check instructions.": GOTO 440
520 OPEN "i", #1, "CARTE7.DAT": OPEN "i", #2, "INVCART7.DAT"
530 RHO(1)=17:RHO(2)=49/2:RHO(3)=33:RHO(4)=48:RHO(5)=75/2:RHO(6)=26:RHO(7)=27/2
540 GOTO 680
```

```
550 PRINT
560 PRINT "Coefficients of the simple roots  2    4    6    5    4    3    2"
570 PRINT "in the maximal root indicated.   (1)--(3)--(4)--(5)--(6)--(7)--(8)"
580 PRINT "Enter number of a root of coef.             !"
590 PRINT "2, for noncompact simple root.            (2) 3"
600 PRINT
610 INPUT NONC
620 IF NONC = 1 OR NONC = 8 THEN 640
630 PRINT "Root";NONC;"not appropriate here; check instructions.": GOTO 550
640 OPEN "i", #1, "CARTE8.DAT": OPEN "i", #2, "INVCART8.DAT"
650 RHO(1)=46: RHO(2)=68: RHO(3)=91: RHO(4)=135
660 RHO(5)=110: RHO(6)=84: RHO(7)=57: RHO(8)=29
670 '
680 FOR I=1 TO RANK: FOR J=1 TO RANK
690 INPUT #1, C(I,J): INPUT #2, CINV(I,J): NEXT J: NEXT I
700 CLOSE #1: CLOSE #2
710 FOR J=1 TO RANK: ZETA(J) = CINV(NONC,J): NEXT J
720 ON RANK - 6 GOTO 740, 750
730 DIM ROOTS(36,6) : ROOTNUM = 36: OPEN "i", #1, "ROOTSE6.DAT": GOTO 760
740 DIM ROOTS(63,7) : ROOTNUM = 63: OPEN "i", #1, "ROOTSE7.DAT": GOTO 760
750 DIM ROOTS(120,8): ROOTNUM = 120: OPEN "i", #1, "ROOTSE8.DAT"
760 FOR I=1 TO ROOTNUM: FOR J=1 TO RANK: INPUT #1, ROOTS(I,J): NEXT J: NEXT I
770 CLOSE #1
780 ' ***********************************************************
790 ' Count and arrange roots of reductive part M, nilradical N, of the
800 ' maximal parabolic subalgebra of E6, E7 or E8, whose complementary
810 ' root is the noncompact simple root (NONC).  Then print out the
820 ' groups of roots to verify that the last one is the highest one
830 ' (needed later) in the groups for coef 1 and coef 2 on (NONC).
840 ' ***********************************************************
850 NUM0 = 0: NUM1 = 0: NUM2 = 0:                      'counters
860 FOR I = 1 TO ROOTNUM
870 IF ROOTS(I,NONC) = 0 THEN NUM0 = NUM0 + 1
880 IF ROOTS(I,NONC) = 1 THEN NUM1 = NUM1 + 1
890 IF ROOTS(I,NONC) = 2 THEN NUM2 = NUM2 + 1
900 NEXT I
910 NUM12 = NUM1 + NUM2                    'counter
920 DIM R0(NUM0 ,RANK)                     'roots of M (all are compact)
930 DIM R1(NUM1 ,RANK)                     'noncompact roots of N
940 DIM R2(NUM2 ,RANK)                     'compact    roots of N
950 DIM R12(NUM12 ,RANK)
960 S0 = 1: S1 = 1: S2 = 1: S12 = 1        'pointers for R0, R1, R2, R12
970 FOR I = 1 TO ROOTNUM
980 ON ROOTS(I,NONC) + 1 GOTO 990, 1020, 1050
990 FOR J=1 TO RANK: R0(S0,J) = ROOTS(I,J): NEXT J
1000 FOR J=1 TO RANK: R0(S0,0) = R0(S0,0) + R0(S0,J): NEXT J
1010 S0 = S0 + 1: GOTO 1080
1020 FOR J=1 TO RANK: R1(S1,J) = ROOTS(I,J): R12(S12,J) = ROOTS(I,J):NEXT J
1030 FOR J=1 TO RANK: R1(S1,0) = R1(S1,0) + R1(S1,J): NEXT J
1040 S1 = S1 + 1: S12 = S12 + 1: GOTO 1080
1050 FOR J=1 TO RANK: R2(S2,J) = ROOTS(I,J): R12(S12,J) = ROOTS(I,J):NEXT J
1060 FOR J=1 TO RANK: R2(S2,0) = R2(S2,0) + R2(S2,J): NEXT J
1070 S2 = S2 + 1: S12 = S12 + 1
1080 NEXT I
1090 ERASE ROOTS
1100 PRINT
```

```
1110 PRINT "root matrix R0 (coef 0 for (NONC)):":PRINT
1120 PRINT "level:       ","root:":PRINT
1130 FOR I=1 TO NUM0
1140 PRINT R0(I,0),
1150 FOR J=1 TO RANK: PRINT "      ";R0(I,J);:NEXT J:PRINT
1160 NEXT I
1170 PRINT: PRINT
1180 PRINT "root matrix R1 (coef 1 for (NONC)):":PRINT
1190 PRINT "level:       ","root:":PRINT
1200 FOR I=1 TO NUM1
1210 PRINT R1(I,0),
1220 FOR J=1 TO RANK: PRINT "      ";R1(I,J);:NEXT J:PRINT
1230 NEXT I
1240 PRINT: PRINT
1250 PRINT "root matrix R2 (coef 2 for (NONC)):":PRINT
1260 PRINT "level:       ","root:":PRINT
1270 FOR I=1 TO NUM2
1280 PRINT R2(I,0),
1290 FOR J=1 TO RANK: PRINT "      ";R2(I,J);:NEXT J:PRINT
1300 NEXT I
1310 PRINT
1320 ' ***********************************************************
1330 ' Compute RHOM (RHO for M) and RHOK (RHO for K)
1340 ' Print out RHO, ZETA, RHOM and RHOK to verify plausibility.
1350 ' *********************************************************
1360 DIM RHOM(RANK): DIM RHOK(RANK): DIM TEMP(RANK)
1370 FOR I=1 TO NUM0
1380 FOR J=1 TO RANK: TEMP(J) = TEMP(J) + R0(I,J): NEXT J
1390 NEXT I
1400 FOR J=1 TO RANK: RHOM(J) = TEMP(J)/2: NEXT J
1410 FOR I=1 TO NUM2
1420 FOR J=1 TO RANK: TEMP(J) = TEMP(J) + R2(I,J): NEXT J
1430 NEXT I
1440 FOR J=1 TO RANK: RHOK(J) = TEMP(J)/2: NEXT J
1450 PRINT: PRINT
1460 PRINT "ZETA: ";:FOR J=1 TO RANK:PRINT USING "###.##";ZETA(J);:
                        PRINT "  ";:NEXT J:PRINT
1470 PRINT "RHO : ";:FOR J=1 TO RANK:PRINT USING "###.##";RHO(J) ;:
                        PRINT "  ";:NEXT J:PRINT
1480 PRINT "RHOM: ";:FOR J=1 TO RANK:PRINT USING "###.##";RHOM(J);:
                        PRINT "  ";:NEXT J:PRINT
1490 PRINT "RHOK: ";:FOR J=1 TO RANK:PRINT USING "###.##";RHOK(J);:
                        PRINT "  ";:NEXT J:PRINT
1500 PRINT: PRINT
1510 ' ***********************************************************
1520 '  Input LAMBDA, compute LAMBDA0 on line LAMBDA + (Reals)*ZETA
1530 ' ***********************************************************
1540 PRINT "Now about to start with a finite dimensional M-module and continue"
1550 PRINT "it along the line (highest weight) + (Real numbers)*ZETA.  Enter"
1560 PRINT "the L's where the highest weight is SUM( L(I)*(I'th basic weight) )"
1570 PRINT
1580 DIM ELL(RANK): DIM LAMBDA(RANK): DIM LAMBDA0(RANK)
1590 FOR J=1 TO RANK: INPUT ELL(J): NEXT J
1600 PRINT: PRINT "The highest M-weight entered was"
1610 PRINT "(";:FOR J=1 TO RANK: PRINT ELL(J);: NEXT J: PRINT ")";
1620 PRINT "   as linear combination of the fundamental weights;"
```

```
1630 T = 0
1640 FOR J=1 TO RANK: FOR K=1 TO RANK
1650 T = T + ZETA(J)*C(J,K)*R1(NUM1,K)
1660 NEXT K: NEXT J
1670 U = 0
1680 FOR J=1 TO RANK: FOR K=1 TO RANK
1690 U = U + RHO(J)*C(J,K)*R1(NUM1,K)
1700 NEXT K: NEXT J
1710 FOR J=1 TO RANK: LAMBDA(J) = 0
1720 FOR K=1 TO RANK: LAMBDA(J) = LAMBDA(J) + ELL(K)*CINV(K,J): NEXT K
1730 NEXT J
1740 PRINT "(";:FOR J=1 TO RANK: PRINT LAMBDA(J);: NEXT J: PRINT ")";
1750 PRINT "   as linear combination of the simple roots."
1760 PRINT
1770 V = 0
1780 FOR J=1 TO RANK: V = V + ELL(J)*R1(NUM1,J): NEXT J
1790 FOR J=1 TO RANK: LAMBDA0(J) = LAMBDA(J) - ((U+V)/T)*ZETA(J): NEXT J
1800 PRINT "<ZETA   , w0(NONC)> ="; T
1810 PRINT "<RHO    , w0(NONC)> ="; U
1820 PRINT "<LAMBDA, w0(NONC)> ="; V
1830 PRINT "LAMBDA0 = (";:FOR J=1 TO RANK: PRINT LAMBDA0(J);:NEXT J: PRINT ")"
1840 PRINT
1850 ' ***************************************************************
1860 '           Calculate and display reduction matrix
1870 ' ***************************************************************
1880 DIM LAMBDA0RHO(RANK): DIM LAMBDARHO(RANK)
1890 DIM REDUCT0(NUM12): DIM REDUCT(NUM12)
1900 FOR J=1 TO RANK: LAMBDA0RHO(J) = LAMBDA0(J) + RHO(J): NEXT J
1910 FOR I=1 TO NUM12: REDUCT0(I) = 0
1920 FOR J=1 TO RANK: FOR K=1 TO RANK
1930 REDUCT0(I) = REDUCT0(I) + LAMBDA0RHO(J)*C(J,K)*R12(I,K)
1940 NEXT K: NEXT J: REDUCT0(I) = CSNG(CINT(REDUCT0(I))): NEXT I
1950 '
1960 PRINT
1970 PRINT "Next value of Z (the critical increment is 0.5 )";: INPUT Z
1980 '
1990 FOR I=1 TO NUM12: REDUCT(I) = REDUCT0(I) + Z*R12(I,NONC): NEXT I
2000 FOR J=1 TO RANK: LAMBDARHO(J) = LAMBDA0RHO(J) + Z*ZETA(J): NEXT J
2010 '
2020 PRINT
2030 PRINT "Reduction matrix for E";RANK;" and root";NONC;" at Z = ";Z;
2040 PRINT "with LAMBDA0 = (";:FOR J=1 TO RANK:PRINT LAMBDA0(J);:NEXT J:PRINT ")"
2050 PRINT
2060 PRINT "value",, "root"
2070 PRINT
2080 FOR I=1 TO NUM12: PRINT REDUCT(I),
2090 FOR J=1 TO RANK: PRINT R12(I,J);: NEXT J: PRINT: NEXT I
2100 ' ***********************************************************
2110 '    A root A(I) = R12(I,*) of N causes reduction iff
2120 '   (1)  REDUCT(I) = 2<LAMBDA0+Z*ZETA+RHO,A(I)>/<A(I),A(I)>
2130 '        is an integer > 0, and
2140 '   (2)  SUM <> 0 where SUM is the sum over the Weyl group of
2150 '        M of det(s)*exp( s( sA(I)(LAMBDA0+Z*ZETA+RHO) ) ).
2160 '
2170 '  TEST 1: If there is a root B(J) = R12(J,*) such that the Weyl
2180 '    reflection sA(I)( B(J) ) is a root of M, then SUM = 0.
```

```
2190 '     To do this, we calculate <sA(I)(B(J)) , ZETA>, since
2200 '     a root C is a root of M iff <C,ZETA> = 0.
2210 '
2220 '   TEST 2: We collect all the A(I) that satisfy (1), but are not
2230 '     shown by Test 1 to satisfy (2), and print out the
2240 '     corresponding sA(I)(LAMBDA0+Z*ZETA+RHO) to see whether
2250 '     their sum, summed over the Weyl group of M, = 0.  The
2260 '     result is not definitive: this part has to be done by
2270 '     hand.  But we do compute each
2280 '           <sA(I)(LAMBDA0 + Z*ZETA + RHO) , ZETA>
2290 '     since those are invariant under the action of the Weyl
2300 '     group of M on the left hand term of <,>, just in case
2310 '     lack of cancellation is forced by those numbers, and
2320 '     if those numbers don't pair off we conclude reduction.
2330 '
2340 '   REDUCTION CRITERION: If Test 2 is reached and does not produce
2350 '     0 then we are at a reduction point
2360 ' ***********************************************************
2370 '   Collect the A's and the B's
2380 '
2390 SA = 1: SB = 1                'pointers
2400 NUMA = 0: NUMB = 0            'number of A's, B's
2410 FOR I=1 TO NUM12
2420 IF REDUCT(I) > .99 AND REDUCT(I) - INT(REDUCT(I)) < .01 THEN NUMA = NUMA + 1
2430 IF ABS(REDUCT(I)) < .01 THEN NUMB = NUMB + 1
2440 NEXT I
2450 DIM A(NUMA,RANK): DIM B(NUMB,RANK)
2460 DIM TEST2(NUMA)               'flags for proceeding from Test 1 to Test 2
2470 IF NUMA > 0 THEN 2500
2480 DIM LASTHOPE(1,RANK)          'fake - to prevent error on array erasure
2490 GOTO 3470
2500 FOR I=1 TO NUM12
2510 IF REDUCT(I) > .99 AND REDUCT(I) - INT(REDUCT(I)) < .01 THEN 2540
2520 IF ABS(REDUCT(I)) < .01 THEN 2590
2530 GOTO 2620
2540 '
2550 FOR J=1 TO RANK: A(SA,J) = R12(I,J): NEXT J
2560 A(SA,0) = REDUCT(I)                   'used in Test 2
2570 SA = SA + 1
2580 GOTO 2620
2590 '
2600 FOR J=1 TO RANK: B(SB,J) = R12(I,J): NEXT J
2610 SB = SB + 1
2620 NEXT I
2630 '
2640 IF NUMB = 0 THEN 3150
2650 '
2660 '   Compute sA(I)(B(J))
2670 '
2680 FOR I=1 TO NUMA
2690 FOR J=1 TO NUMB
2700 '
2710 INNERAB = 0
2720 FOR K=1 TO RANK: FOR L=1 TO RANK
2730 INNERAB = INNERAB + A(I,K)*C(K,L)*B(J,L)
2740 NEXT L: NEXT K
```

```
2750 '
2760 FOR K=1 TO RANK: SAB(K) = B(J,K) - INNERAB*A(I,K): NEXT K
2770 '
2780 ' Compute <sA(I)(B(J)),ZETA> and eliminate floating point error
2790 '
2800 MROOTFLAG = 0
2810 FOR K=1 TO RANK: FOR L=1 TO RANK
2820 MROOTFLAG = MROOTFLAG + SAB(K)*C(K,L)*ZETA(L)
2830 NEXT L: NEXT K
2840 MROOTFLAG = (CSNG(CINT(100*MROOTFLAG)))/100
2850 '
2860 IF MROOTFLAG <> 0 THEN 2950                    'A(I) not cancelled by this B(J)
2870 '
2880 ' Arrival here means that A(I) is eliminated before Test 2
2890 TEST2(I) = 0                                   'A(I) can skip Test 2
2900 PRINT
2910 PRINT "A(";I;") = (";:FOR K=1 TO RANK: PRINT A(I,K);: NEXT K: PRINT ") ";
2920 PRINT "cancelled by B(";J;") = (";:FOR K=1 TO RANK: PRINT B(J,K);: NEXT K
2930 PRINT ")": PRINT
2940 GOTO 3040
2950 NEXT J
2960 '
2970 ' Arrival here means that no B(J) cancelled A(I) using Test 1
2980 TEST2(I) = 1                                   'A(I) survives to Test 2
2990 PRINT
3000 PRINT "A(";I;") = (";: FOR K=1 TO RANK: PRINT A(I,K);: NEXT K: PRINT ") ";
3010 PRINT "not cancelled by any B(J), survives to second test."
3020 PRINT
3030 '
3040 NEXT I
3050 '
3060 '   See whether all A(I) were eliminated in Test 1
3070 NUMTEST2 = 0
3080 FOR I=1 TO NUMA: NUMTEST2 = NUMTEST2 + TEST2(I): NEXT I
3090 IF NUMTEST2 > 0 THEN 3150
3100 DIM LASTHOPE(2,2)
3110 GOTO 3470
3120 '
3130 '  Test 2
3140 '
3150 IF NUMB = 0 THEN NUMTEST2 = NUMA
3160 DIM LASTHOPE(NUMTEST2,RANK)
3170 SL = 1                         'pointer
3180 FOR I=1 TO NUMA
3190 IF NUMB > 0 AND TEST2(I) = 0 THEN 3320
3200 '
3210 ' Recall that A(L,0) = REDUCT(I) where A(L) = R12(I),
3220 ' so A(I,0) = <A(I),LAMBDARHO = LAMBDA0 + Z*ZETA + RHO>,
3230 ' so sA(I)(LAMBDA+Z*ZETA+RHO) = LAMBDARHO - A(I,0)*A(I).
3240 '
3250 FOR J=1 TO RANK: LASTHOPE(SL,J) = LAMBDARHO(J) - A(I,0)*A(I,J): NEXT J
3260 LASTHOPE(SL,0) = 0
3270 FOR J=1 TO RANK: FOR K=1 TO RANK
3280 LASTHOPE(SL,0) = LASTHOPE(SL,0) + LASTHOPE(SL,J)*C(J,K)*ZETA(K)
3290 NEXT K: NEXT J
3300 LASTHOPE(SL,0) = (CSNG(CINT(100*LASTHOPE(SL,0))))/100  'correct f.p. error
```

```
3310 SL = SL + 1
3320 NEXT I
3330 PRINT
3340 PRINT "There is reduction for E";RANK;"at root";NONC;"and Z =";Z
3350 PRINT "UNLESS the alternating sum over the Weyl group of M, of the "
3360 PRINT "following weights, is zero.  Have fun trying to check that."
3370 PRINT "NOTE: If weights U(I) cancel that way the <U(I),ZETA> are equal."
3380 PRINT
3390 FOR I=1 TO NUMTEST2
3400 PRINT"      (";
3410 FOR J=1 TO RANK:PRINT USING "###.##";LASTHOPE(I,J);
3420 IF J < RANK THEN PRINT ",";: NEXT J
3430 PRINT ")","< ,ZETA> =";LASTHOPE(I,0): PRINT
3440 NEXT I
3450 GOTO 3540
3460 '
3470 PRINT
3480 PRINT "There was no reduction at z = ";Z
3490 PRINT
3500 GOTO 3540
3510 ' ************************************************************
3520 '                  Keep trucking?
3530 ' ************************************************************
3540 PRINT
3550 ERASE A: ERASE B: ERASE TEST2: ERASE LASTHOPE
3560 INPUT "Try another value of Z [yes/break/quit]"; S$
3570 IF LEFT$(S$,1) = "y" THEN 1960
3580 IF LEFT$(S$,1) = "q" THEN END
3590 IF LEFT$(S$,1) = "b" THEN 3620
3600 PRINT: PRINT "What?": GOTO 3560
3610 PRINT
3620 PRINT "Pause.  All parameter values preserved.  Type CONT to continue."
3630 PRINT
3640 STOP
3650 GOTO 3560
3660 END
```

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